First Isomorphism Theorem

Let $f : G \to H$ be a group homomorphism. The

- $K = \ker(f) := g \in G | f(g) = 1_H$ is a normal subgroup of G,
- $\mathrm{Im}(f)$ is a subgroup of H, and
- G/K is isomorphic to $\mathrm{Im}(f)$.

In fact, the isomorphism is given by

$$\bar{f} : G/K \to \mathrm{Im}(f), \quad \bar{f}(\bar{g}) = \bar{f}(gK) = f(g)$$

$\bar{f}$ is well defined

Let $\bar{g}_1, \bar{g}_2 \in G/K$, the statement is true iff

$$\forall \bar{g}_1, \bar{g}_2 \in G/K, \quad \bar{g}_1 = \bar{g}_2 \to \bar{f}(\bar{g}_1) = \bar{f}(\bar{g}_2)$$

The proof is given by

$$\begin{aligned} \bar{g}_1 = \bar{g}_2 &\implies g_1 K = g_2 K \implies g_1^{-1} g_2 \in K \\ &\implies f(g_1^{-1} g_2) = 1_H \implies f(g_1^{-1}) f(g_2) = 1_H \\ &\implies f(g_1) = f(g_2) \end{aligned}$$

K is a subgroup of G

It's equivalent to

$$\forall k_1, k_2 \in K, \quad k_1 k_2 \in K$$

Given any such k_1, k_2, since f is an isomorphism, we have

$$\begin{aligned} &f(k_1 k_2) \\ =&f(k_1) f(k_2) \\ =&1_H 1_H \\ =&1_H \end{aligned}$$

Therefore $k_1 k_2 \in K$.

K is normal

By definition, K is a normal subgroup of G iff

$$\forall k \in K, g \in G, \quad gkg^{-1} \in K$$

Given any k, g, since f is an isomorphism, we have

$$\begin{aligned} &f(gkg^{-1}) \\ =&f(g) f(k) f(g^{-1}) \\ =&f(k) \end{aligned}$$

Therefore $gkg^{-1} \in K$.

$\mathrm{Im}(f)$ is a subgroup of H

It's equivalent to

$$\forall h_1, h_2 \in \mathrm{Im}(f), \quad h_1 h_2 \in \mathrm{Im}(f)$$

Let $h_1 = f(g_1)$ and $h_2 = f(g_2)$, since f is an isomorphism,

$$h_1 h_2 = f(g_1)f(g_2) = f(g_1 g_2)$$

As shown above, there $\exists$ something $\in G$ that maps to $h_1 h_2$ by f.

$\bar{f}$ is an homomorphism

What we need to prove to prove the statement is

$$\forall \bar{g}_1, \bar{g}_2 \in G/K, \quad \bar{f}(\bar{g}_1 \bar{g}_2) = \bar{f}(\bar{g}_1)\bar{f}(\bar{g}_2)$$

The proof is given by

$$\begin{aligned}
&\bar{f}(\bar{g}_1\bar{g}_2) \\
=&\bar{f}((g_1K)(g_2K)) \\
=&\bar{f}((g_1g_2K)) \\
=&f(g_1g_2) \\
=&f(g_1)f(g) \\
=&\bar{f}(\bar{g}_1)\bar{f}(\bar{g}_2)
\end{aligned}$$

$\bar{f}$ is surjective

$\mathrm{Im}(\bar{f}) = \{\bar{f}(\bar{g}) | \bar{g} \in G/K\} = \{\bar{f}(gK) | g \in G\} = \{f(g) | g \in G\} = \mathrm{Im}(f)$
If $h \in \mathrm{Im}(f)$ then $h \in \mathrm{Im}(\bar{f})$.

$\bar{f}$ is injective

Let $\bar{g}_1, \bar{g}_2 \in G/K$, the proof is given by

$$\begin{aligned}
&\bar{f}(\bar{g}_1) = \bar{f}(\bar{g}_2) \\
\rightarrow&\bar{f}(\bar{g}_1^{-1})\bar{f}(\bar{g}_1) = \bar{f}(\bar{g}_1^{-1})\bar{f}(\bar{g}_2) \\
\rightarrow&\bar{f}(\bar{g}_1^{-1})\bar{f}(\bar{g}_2) = 1 \\
\rightarrow&f(g^{-1}g) = 1 \\
\rightarrow&g^{-1}g \in K \\
\rightarrow&g_1K = g_2K \\
\rightarrow&\bar{g}_1 = \bar{g}_2
\end{aligned}$$