First Isomorphism Theorem

Let $f : G \to H$ be a group homomorphism. The

- $K = \ker(f) := g \in G | f(g) = 1_H$ is a normal subgroup of G,
- $\mathrm{Im}(f)$ is a subgroup of H, and
- G/K is isomorphic to $\mathrm{Im}(f)$.

In fact, the isomorphism is given by

$$\bar{f} : G/K \to \mathrm{Im}(f), \quad \bar{f}(\bar{g}) = \bar{f}(gK) = f(g)$$

$\bar{f}$ is well defined

Let $\bar{g}_1, \bar{g}_2 \in G/K$, the statement is true iff

$$\forall \bar{g}_1, \bar{g}_2 \in G/K, \quad \bar{g}_1 = \bar{g}_2 \to \bar{f}(\bar{g}_1) = \bar{f}(\bar{g}_2)$$

The proof is given by

$$\begin{aligned} \bar{g}_1 = \bar{g}_2 &\implies g_1K = g_2K \implies g_1^{-1}g_2 \in K \\ \implies f(g_1^{-1}g_2) = 1_H &\implies f(g_1^{-1})f(g_2) = 1_H \\ \implies f(g_1) = f(g_2) & \end{aligned}$$

K is a subgroup of G

Identity

Since f is a homomorphism, $f(1_G) = 1_H$, therefore $1_G \in \ker(f)$

Inverse

To prove: $\forall k \in K, \ k^{-1} \in K$
Let $k \in K$, $f(kk^{-1}) = f(1_G) = 1_H = f(k)f(k^{-1}) = 1_H f(k^{-1})$
As $1_H = 1_H f(k^{-1})$,$f(k^{-1}) = 1_H$

Operation

To prove: $\forall k_1, k_2 \in K, \ k_1k_2 \in K$
Given any such k_1, k_2, since f is an isomorphism, we have

$$\begin{aligned} & f(k_1k_2) \\ =& f(k_1)f(k_2) \\ =& 1_H 1_H \\ =& 1_H \end{aligned}$$

Therefore $k_1k_2 \in K$.

K is normal

By definition, K is a normal subgroup of G iff

$$\forall k \in K, g \in G, \quad gkg^{-1} \in K$$

Given any k, g, since f is an isomorphism, we have

$$\begin{aligned}
&f(gkg^{-1}) \\
=&f(g)f(k)f(g^{-1}) \\
=&f(k)
\end{aligned}$$

Therefore $gkg^{-1} \in K$.

$\mathrm{Im}(f)$ is a subgroup of H

Identity

Since G is a group, it has identity. Since f is a homomorphism, $f(1_G) = 1_H$. Therefore $1_H \in \mathrm{Im}(f)$

Inverse

To prove: $\forall h \in \mathrm{Im}(f), h^{-1} \in \mathrm{Im}(f)$
Let $h \in \mathrm{Im}(f)$, then $\exists g \in G$ s.t. $f(g) = h$ and because f is a homomorphism, $f(g^{-1}) = h^{-1}$

Operation

To prove: $\forall h_1, h_2 \in \mathrm{Im}(f),\ h_1h_2 \in \mathrm{Im}(f)$
Let $h_1 = f(g_1)$ and $h_2 = f(g_2)$, since f is an isomorphism,

$$h_1h_2 = f(g_1)f(g_2) = f(g_1g_2)$$

As shown above, there $\exists$ something $\in G$ that maps to h_1h_2 by f.

$\bar{f}$ is an homomorphism

What we need to prove to prove the statement is

$$\forall \bar{g}_1, \bar{g}_2 \in G/K, \quad \bar{f}(\bar{g}_1\bar{g}_2) = \bar{f}(\bar{g}_1)\bar{f}(\bar{g}_2)$$

The proof is given by

$$\begin{aligned}
&\bar{f}(\bar{g}_1\bar{g}_2) \\
=&\bar{f}((g_1K)(g_2K)) \\
=&\bar{f}((g_1g_2K)) \\
=&f(g_1g_2) \\
=&f(g_1)f(g) \\
=&\bar{f}(\bar{g}_1)\bar{f}(\bar{g}_2)
\end{aligned}$$

$\bar{f}$ is surjective

$\mathrm{Im}(\bar{f}) = \{\bar{f}(\bar{g}) | \bar{g} \in G/K\} = \{\bar{f}(gK) | g \in G\} = \{f(g) | g \in G\} = \mathrm{Im}(f)$
If $h \in \mathrm{Im}(f)$ then $h \in \mathrm{Im}(\bar{f})$.

$\bar{f}$ is injective

Let $\bar{g}_1, \bar{g}_2 \in G/K$, the proof is given by

$$
\begin{aligned}
&\bar{f}(\bar{g}_1) = \bar{f}(\bar{g}_2) \\
\Longrightarrow\ &\bar{f}(\bar{g}_1^{-1})\bar{f}(\bar{g}_1) = \bar{f}(\bar{g}_1^{-1})\bar{f}(\bar{g}_2) \\
\Longrightarrow\ &\bar{f}(\bar{g}_1^{-1})\bar{f}(\bar{g}_2) = 1 \\
\Longrightarrow\ &f(g^{-1}g) = 1 \\
\Longrightarrow\ &g^{-1}g \in K \\
\Longrightarrow\ &g_1K = g_2K \\
\Longrightarrow\ &\bar{g}_1 = \bar{g}_2
\end{aligned}
$$