

First Isomorphism Theorem

Let $f : G \rightarrow H$ be a group homomorphism. The

- $K = \ker(f) := \{g \in G \mid f(g) = 1_H\}$ is a normal subgroup of G ,
- $\text{Im}(f)$ is a subgroup of H , and
- G/K is isomorphic to $\text{Im}(f)$.

In fact, the isomorphism is given by

$$\bar{f} : G/K \rightarrow \text{Im}(f), \quad \bar{f}(\bar{g}) = \bar{f}(gK) = f(g)$$

\bar{f} is well defined

Let $\bar{g}_1, \bar{g}_2 \in G/K$, the statement is true iff

$$\forall \bar{g}_1, \bar{g}_2 \in G/K, \quad \bar{g}_1 = \bar{g}_2 \rightarrow \bar{f}(\bar{g}_1) = \bar{f}(\bar{g}_2)$$

The proof is given by

$$\begin{aligned} \bar{g}_1 = \bar{g}_2 &\implies g_1K = g_2K \implies g_1^{-1}g_2 \in K \\ &\implies f(g_1^{-1}g_2) = 1_H \implies f(g_1^{-1})f(g_2) = 1_H \\ &\implies f(g_1) = f(g_2) \end{aligned}$$

K is a subgroup of G

It's equivalent to

$$\forall k_1, k_2 \in K, \quad k_1k_2 \in K$$

Given any such k_1, k_2 , since f is an isomorphism, we have

$$\begin{aligned} &f(k_1k_2) \\ &= f(k_1)f(k_2) \\ &= 1_H 1_H \\ &= 1_H \end{aligned}$$

Therefore $k_1k_2 \in K$.

K is normal

By definition, K is a normal subgroup of G iff

$$\forall k \in K, g \in G, \quad gkg^{-1} \in K$$

Given any k, g , since f is an isomorphism, we have

$$\begin{aligned} &f(gkg^{-1}) \\ &= f(g)f(k)f(g^{-1}) \\ &= f(k) \end{aligned}$$

Therefore $gkg^{-1} \in K$.

$\text{Im}(f)$ is a subgroup of H

It's equivalent to

$$\forall h_1, h_2 \in \text{Im}(f), \quad h_1 h_2 \in \text{Im}(f)$$

Let $h_1 = f(g_1)$ and $h_2 = f(g_2)$, since f is an isomorphism,

$$h_1 h_2 = f(g_1) f(g_2) = f(g_1 g_2)$$

As shown above, there \exists something $\in G$ that maps to $h_1 h_2$ by f .

\bar{f} is an homomorphism

What we need to prove to prove the statement is

$$\forall \bar{g}_1, \bar{g}_2 \in G/K, \quad \bar{f}(\bar{g}_1 \bar{g}_2) = \bar{f}(\bar{g}_1) \bar{f}(\bar{g}_2)$$

The proof is given by

$$\begin{aligned} & \bar{f}(\bar{g}_1 \bar{g}_2) \\ &= \bar{f}((g_1 K)(g_2 K)) \\ &= \bar{f}(g_1 g_2 K) \\ &= f(g_1 g_2) \\ &= f(g_1) f(g_2) \\ &= \bar{f}(\bar{g}_1) \bar{f}(\bar{g}_2) \end{aligned}$$

\bar{f} is surjective

$$\text{Im}(\bar{f}) = \{\bar{f}(\bar{g}) | \bar{g} \in G/K\} = \{\bar{f}(gK) | g \in G\} = \{f(g) | g \in G\} = \text{Im}(f)$$

If $h \in \text{Im}(f)$ then $h \in \text{Im}(\bar{f})$.

\bar{f} is injective

Let $\bar{g}_1, \bar{g}_2 \in G/K$, the proof is given by

$$\begin{aligned} & \bar{f}(\bar{g}_1) = \bar{f}(\bar{g}_2) \\ \implies & \bar{f}(\bar{g}_1^{-1}) \bar{f}(\bar{g}_1) = \bar{f}(\bar{g}_1^{-1}) \bar{f}(\bar{g}_2) \\ \implies & \bar{f}(\bar{g}_1^{-1}) \bar{f}(\bar{g}_2) = 1 \\ \implies & f(g^{-1}g) = 1 \\ \implies & g^{-1}g \in K \\ \implies & g_1 K = g_2 K \\ \implies & \bar{g}_1 = \bar{g}_2 \end{aligned}$$