Second Isomorphism Theorem

If $N \trianglelefteq G$ and $S \leq G$, then

1. $N \cap S \trianglelefteq S$
2. $NS = \{ns | n \in N, s \in S\} \leq G$
3. $S/N \cap S \cong NS/N = SN/N$

$N \cap S$ is a subgroup of S

Since both N and S are subgroups of G, $1 \in N$ and $1 \in S$, therefore $1 \in N \cap S$.

$\forall x_1, x_2 \in N \cap S$,

- Since $x_1, x_2 \in N$ and N is a group, $x_1 x_2 \in N$
- Since $x_1, x_2 \in S$ and S is a group, $x_1 x_2 \in S$

Therefore $x_1 x_2 \in N \cap S$

$N \cap S$ is normal

$\forall x \in N \cap S$, $\forall s \in S$

- Since $x \in N$ and $N \trianglelefteq G$ and $s \in G$, $\ sxs^{-1} \in N$
- Since $x \in S$, and S is a group, $sxs^{-1} \in S$

Therefore $sxs^{-1} \in N \cap S$

NS is a subset of G

$\forall n \in N, s \in S$, since both $n, s \in G$ and G is a group, $ns \in G$

NS is a group

Obviously, $1_{NS} = 1_N 1_S \in NS$.
Let $x_1, x_2 \in NS$, then exists $n_1 s_1 = x_1$ and $n_2 s_2 = x_2$. Thus $x_1 x_2 = n_1 s_1 n_2 s_2$.
Since $N \trianglelefteq G$, any $n \in N$ has some $n' \in N$ such that $gng^{-1} = n'$ for all $g \in G$ as well as $s \in S \leq G$.
So let $n_2 = s_1^{-1} n_2' s_1$, we get $x_1 x_2 = n_1 n_2' s_1 s_2 \in NS$

N is a normal subgroup of NS

Because $N \trianglelefteq G$ and $NS \leq G$

$S/N \cap S$ is isomorphic to NS/N

Let $f : S \to NS/N$ be $f(s) = sN$. Then

$$\begin{aligned}\ker(f) =&\{s \in S | f(s) = 1_{NS/N}\}\\ =&\{s \in S | sN = N\}\\ =&\{s \in S | s \in N\}\\ =&S \cap N\end{aligned}$$

The statement is true by first isomorphism theorem.