A LARGE DEFORMATION-SMALL STRAIN FORMULATION FOR THE MECHANICS OF GEOMETRICALLY EXACT THIN-WALLED COMPOSITE BEAMS

C. Martín Saravia

Centro de Investigación en Mecánica Teórica y Aplicada, CONICET-Universidad Tecnológica Nacional, Facultad Regional Bahía Blanca, 11 de Abril 461, 8000 Bahía Blanca, Argentina.

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Abstract. This work presents a new formulation of the geometrically exact thin walled composite beam theory. The formulation assumes that the beam can undergo arbitrary kinematical changes while the strains remain small, thus compatibilizing the hypotheses of the strain measure and the constitutive law of the composite material. A key point of the formulation is the development of a pure small strain measure written solely in terms of scalar products of position and director vectors; the latter is accomplished through the obtention of a generalized small strain vector by decomposition of the deformation gradient. The resulting small strain measure is objective under rigid body motion. The finite element implementation of the proposed formulation is simpler than the finite strain theory implementation previously developed by the author. Numerical experiments show that the present formulation has a very good accuracy and should be used in most practical applications. In terms of computational cost, numerical experiments show that the proposed approach is computationally more efficient than the finite strain formulation.

† Corresponding Author.
E-mail adress: msaravia@conicet.gov.ar (C.M. Saravia)

1 INTRODUCTION

The use of composite beams for modeling structural components is a common practice; the behavior of slender parts of modern machines such as wind turbines, satellites, cars, etc. is often predicted using the thin-walled composite beam (TWCB) approach. Good modeling practices normally imply the use of geometrically nonlinear TWCB theories, which are capable of describing not only large kinematical changes of the beam configuration but also nonlinear interactions between different components of mechanisms or multibody systems.

The thin–walled beam formulation is due to Vlasov [1]; remarkably, it has survived fifty years without drastic changes. One of the principal extensions of the theory was the inclusion of the mechanics of composite materials; several approaches that deal with the elastic behavior of TWCBs can be found in the literature and they are generally derived from Vlasov's thin walled beam theory. Although most works introduce novel aspects in their formulations, very often their hypotheses lead to geometrical or constitutive inconsistencies, or both.

The vast majority of the thin-walled beam formulations that can be found in the literature rely in the assumption of a displacement field which is introduced into a Green strain expression to obtain generalized strain measures in terms of the kinematic variables and its derivatives. Commonly the kinematic variables are taken as three displacements and three rotations, sometimes also a warping degree of freedom is used.

At least one of the following four inconsistencies can be found in almost all the works regarding TWCB, i.e. *i)* the displacements field is said to describe moderate or large kinematical changes while the non vectorial nature of the rotation variables is disregarded, *ii)* a linear or second order nonlinear displacement field is assumed, but then it is introduced into an arbitrary large strain expression, *iii)* some terms of the Green strain regarded as nonlinear strain measures are eliminated causing the objectivity of the resulting "linear" strain measures to be lost and *iv)* the kinematic description of the formulation admits large strains while the constitutive law is only valid for small strains.

Taking, for instance, the developments by Librescu et al. [2], it can be found that they suffer from inconsistencies i, ii and iv. Also, the works by Pi et al. [3-5] suffer from inconsistencies i and iii. Analyzing their works [3, 4] it can be seen that the rotation matrix is said to be second order accurate while its components are treated as vectors, thus ignoring the non-commutativity of rotations. Also, non pure-strain (higher order) terms of the Green strain measure are eliminated, thus destroying its objectivity. In [5] an exact rotation matrix is used, but again the elimination of non-pure strain terms leads to a loss of objectivity of the formulation; also, the rotation matrix is said to belong to the Special Orthogonal Group (SO3) while it is linearized as it belonged to a vector space. The theories developed by Cortínez, Piován and Machado in works [6-10] for the study of the dynamic stability, vibration, buckling and postbuckling of both open and cross section TWCBs suffer from inconsistencies i, ii and iv.

Regarding geometrically exact TWCB formulations, Saravia et. al. [11, 12] presented Eulerian, Total Lagrangian and Updated Lagrangian formulations using a parameterization in terms of director vectors.

These formulations can describe kinematical and strain changes of arbitrary magnitude consistently; however, the constitutive law of composite laminates is only valid for small strains. A similar problem affects most of the geometrically exact formulations [13-16] developed for isotropic beams.

The mentioned works are only a few of the many that present the mentioned inconsistencies. Although it can be asked if the errors that arise from these issues are of great influence in practical situations or not, the uncertainty about the limit of application of these hypotheses strongly motivates the development of a consistent approach in which a validity assessment of the theory is not needed. It is true that due to the accuracy of the modern Variational Asymptotic Methods, the use of the TWCB approach shall probably be reduced in the future. However, a vast amount of efforts are being done by researchers to improve the theory, and thus I consider that it is worth to develop a consistent large deformation-small strain formulation for thin walled composite thin-beams.

In this context, this paper presents the derivation of mathematical aspects of the finite deformation-small strain TWCB formulation. In the present approach the kinematic changes of the beam are assumed to be arbitrary, thus allowing finite rotations. Then, extended decomposition of the deformation gradient in terms of the director field is used to obtain a pure small strain measure. Finally, the discrete version of the small strain measures are expressed in terms of the current director and displacement fields and its derivatives; the obtained relations are remarkably simple and do not involve derivatives of the reference triads. Also, the discrete generalized strain measure results to be objective under rigid body motions and independent of the integration path. Numerical results show that the proposed approach gives excellent accuracy compared to the finite strain implementation previously developed by the author.

2 KINEMATICS

The kinematic description of the beam is extracted from the relations between two states of a beam, an undeformed reference state (denoted as $\mathcal{B}_0$) and a deformed state (denoted as $\mathcal{B}$), as it is shown in Fig. 1. Being $\boldsymbol{a}_i$ a spatial frame of reference, two orthonormal frames are defined: a reference frame $\boldsymbol{E}_i$ and a current frame $\boldsymbol{e}_i$.

Figure 1. 3D beam kinematics.

The displacement of a point in the deformed beam measured with respect to the undeformed reference state can be expressed in the global coordinate system $\boldsymbol{a}_i$ in terms of a vector $\boldsymbol{u} = (u_1, u_2, u_3)$.

The current frame $\boldsymbol{e}_i$ is a function of a running length coordinate along the reference line of the beam, denoted as x, and is fixed to the beam cross-section. For convenience, it is choosen the reference curve

C to be the locus of cross-sectional inertia centroids. The origin of $\boldsymbol{e}_i$ is located on the reference line of the beam and is called *pole*. The cross-section of the beam is arbitrary and initially normal to the reference line.

The relations between the orthonormal frames are given by the linear transformations:

$$\boldsymbol{E}_i = \boldsymbol{\Lambda_0}(x)\boldsymbol{a}_i, \qquad \boldsymbol{e}_i = \boldsymbol{\Lambda}(x)\boldsymbol{E}_i, \tag{1}$$

where $\boldsymbol{\Lambda}_0(x)$ and $\boldsymbol{\Lambda}(x)$ are two-point tensor fields $\in$ SO(3); the special orthogonal (Lie) group. Thus, it is satisfied that $\boldsymbol{\Lambda_0}^T\boldsymbol{\Lambda_0} = \boldsymbol{I}$, $\boldsymbol{\Lambda}^T\boldsymbol{\Lambda} = \boldsymbol{I}$. It will be considered that the beam element is straight, so we set $\boldsymbol{\Lambda_0} = \boldsymbol{I}$.

Recalling the relations (1), we can express the position vectors of a point in the beam in the undeformed and deformed configuration respectively as:

$$\boldsymbol{X}(s, X_2, X_3) = \boldsymbol{X}_0(x) + \sum_{i=2}^{3} X_i\, \boldsymbol{E}_i, \qquad \boldsymbol{x}(s, X_2, X_3, t) = \boldsymbol{x}_0(s,t) + \sum_{i=2}^{3} X_i\, \boldsymbol{e}_i. \tag{2}$$

Where in both equations the first term stands for the position of the pole and the second term stands for the position of a point in the cross section relative to the pole. Note that x is the running length coordinate and ξ_2 and ξ_3 are cross section coordinates. At this point we note that since the present formulation is thought to be used for modeling high aspect ratio composite beams, the warping displacement is not included. As it is widely known, for such type of beams the warping effect is negligible [17].

Also, it is possible to express the displacement field as:

$$\boldsymbol{u}(s, X_2, X_3, t) = \boldsymbol{x} - \boldsymbol{X} = \boldsymbol{u}_0(s,t) + (\boldsymbol{\Lambda} - \mathbf{I}) \sum_{2}^{3} X_i\, \boldsymbol{E}_i, \tag{3}$$

where $\boldsymbol{u}_0$ represents the displacement of the kinematic center of reduction, i.e. the pole. The nonlinear manifold of 3D rotation transformations $\boldsymbol{\Lambda}(\boldsymbol{\theta})$ (belonging to the special orthogonal Lie Group SO(3)) is described mathematically via the exponential map [13]. The rotation tensor component form can be written as

$$\boldsymbol{\Lambda} = \sum_{i,j=1}^{3} \Lambda_{ij} \boldsymbol{E}_i \otimes \boldsymbol{E}_j, \tag{4}$$

Where the components Λ_{ij} of the rotation tensor can be obtained in the following form

$$\Lambda_{ij} = \boldsymbol{E}_i \cdot \boldsymbol{\Lambda}\boldsymbol{E}_j = \boldsymbol{E}_i \cdot \boldsymbol{e}_j; \tag{5}$$

then it is possible to express the rotation tensor as

$$\boldsymbol{\Lambda} = \sum_{i,j=1}^{3} \left(\boldsymbol{E}_i \cdot \boldsymbol{e}_j\right) \boldsymbol{E}_i \otimes \boldsymbol{E}_j. \tag{6}$$

Now, using the tensor product property: $(\boldsymbol{a}\otimes\boldsymbol{b})\boldsymbol{c} = (\boldsymbol{c}\cdot\boldsymbol{b})\boldsymbol{a}$, we can operate as

$$\boldsymbol{\Lambda} = \sum_{i,j=1}^{3} (\boldsymbol{E}_i\otimes\boldsymbol{E}_i)\,\boldsymbol{e}_j\otimes\boldsymbol{E}_j = \sum_{j=1}^{3} \boldsymbol{I}\,\boldsymbol{e}_j\otimes\boldsymbol{E}_j, \tag{7}$$

Finally, with summation from 1 to 3 implicitly assumed, we can obtain the following expression for the rotation tensor:

$$\boldsymbol{\Lambda} = \boldsymbol{e}_j\otimes\boldsymbol{E}_j, \tag{8}$$

which will be a very useful expression for the derivation of a pure vectorial measure of strain.

3 SMALL STRAIN TENSOR

3.1 The Green strain measure

The main motivation for the development of a large deformation-small strain formulation is to give consistency to the constitutive formulation of the geometrically exact composite thin-walled beam theory [11, 12]; since the constitutive equations are only valid for small strains, it is important to derive a strain measure consistent with this assumption.

As it was stated, most of the geometrically exact beam formulations presented in the literature assume a linear elastic constitutive law which is accurate only for small strains, but the constitutive equations are fed with a large strain deformation tensor. It is not trivial to transform a large strain tensor in a small strain tensor without losing its objectivity under rigid body motions [18], so an important part of the section is devoted to show the mathematical procedures that lead to an objective small strain measure.

The Green strain tensor is commonly written in three different forms:

$$\boldsymbol{E} = \tfrac{1}{2}\left(\boldsymbol{x}_{,i}\cdot\boldsymbol{x}_{,j} - \boldsymbol{X}_{,i}\cdot\boldsymbol{X}_{,j}\right), \tag{9}$$

$$\boldsymbol{E} = \tfrac{1}{2}((\nabla_{\boldsymbol{X}}\otimes\boldsymbol{u})^{S} + (\nabla_{\boldsymbol{X}}\otimes\boldsymbol{u})^{T}\nabla_{\boldsymbol{X}}\otimes\boldsymbol{u}), \tag{10}$$

$$\boldsymbol{E} = \tfrac{1}{2}(\boldsymbol{F}^{T}\boldsymbol{F} - \boldsymbol{I}), \tag{11}$$

Where $\boldsymbol{F}$ is the deformation gradient and the displacement gradient is given by

$$\nabla_{\boldsymbol{X}}\otimes\boldsymbol{u} = \frac{\partial\boldsymbol{u}}{\partial\boldsymbol{X}} \tag{12}$$

None of these forms can initially be seen as a linear plus a nonlinear pure strain measure; elimination of any term in the above expressions does not guaranties that the resulting formulation is objective.

The first form of strain given in Eq. (9) was used in [11, 12, 19, 20], it is an attractive expression because it only requires the derivatives of the position vector in the reference and current configurations for its evaluation; however, after taking the dot product we get a translational-rotational coupled expression that has mixed kinematic and strain information. So, being the derivatives of the position vectors

$$\begin{aligned} \boldsymbol{X}_{,1} &= \boldsymbol{X}_0' + \xi_2 \boldsymbol{E}_2' + \xi_3 \boldsymbol{E}_3', & \boldsymbol{x}_{,1} &= \boldsymbol{x}_0' + \xi_2 \boldsymbol{e}_2' + \xi_3 \boldsymbol{e}_3', \\ \boldsymbol{X}_{,2} &= \boldsymbol{E}_2, & \boldsymbol{x}_{,2} &= \boldsymbol{e}_2, \\ \boldsymbol{X}_{,3} &= \boldsymbol{E}_3, & \boldsymbol{x}_{,3} &= \boldsymbol{e}_3, \end{aligned} \tag{13}$$

the insertion of these expressions in Eq. (9) gives a strain vector $\boldsymbol{E} = [E_{11} \;\; 2E_{12} \;\; 2E_{13}]^T$ such that

$$\begin{aligned} E_{11} &= \frac{1}{2}\left(\boldsymbol{x}_0'^{\,2} - \boldsymbol{X}_0'^{\,2}\right) + \xi_2(\boldsymbol{x}_0' \cdot \boldsymbol{e}_3' - \boldsymbol{X}_0' \cdot \boldsymbol{E}_3') + \xi_3(\boldsymbol{x}_0' \cdot \boldsymbol{e}_2' - \boldsymbol{X}_0' \cdot \boldsymbol{E}_2') \\ &\quad + \frac{1}{2}\xi_2^2\left(\boldsymbol{e}_2'^{\,2} - \boldsymbol{E}_2'^{\,2}\right) + \frac{1}{2}\xi_3^2\left(\boldsymbol{e}_3'^{\,2} - \boldsymbol{E}_3'^{\,2}\right) + \xi_2\xi_3(\boldsymbol{e}_2' \cdot \boldsymbol{e}_3' - \boldsymbol{E}_2' \cdot \boldsymbol{E}_3'), \\ E_{12} &= \frac{1}{2}\left[\boldsymbol{x}_0' \cdot \boldsymbol{e}_2 - \boldsymbol{X}_0' \cdot \boldsymbol{E}_2 - \xi_3(\boldsymbol{e}_3' \cdot \boldsymbol{e}_2 - \boldsymbol{E}_3' \cdot \boldsymbol{E}_2)\right], \\ E_{13} &= \frac{1}{2}\left[\boldsymbol{x}_0' \cdot \boldsymbol{e}_3 - \boldsymbol{X}_0' \cdot \boldsymbol{E}_3 + \xi_2(\boldsymbol{e}_2' \cdot \boldsymbol{e}_3 - \boldsymbol{E}_2' \cdot \boldsymbol{E}_3)\right]. \end{aligned} \tag{14}$$

As the reader may see, ensuring that any of the above terms is either a linear strain or a nonlinear strain measure is at least not trivial.

For the case of Eq. (10), which is sometimes understood as the sum of a linear plus a nonlinear measure of strain, it is known that the gradient of the displacement field is not objective under rigid body motion, and thus it is not a pure measure of strain, i.e. it contains both strain and kinematic information. Exploiting Eq. (11) will be the approach used in this paper to obtain a pure small strain measure.

3.2 The deformation gradient

As it was said before, to obtain a pure linear strain measure without losing the capability of describing a large deformation behavior it is necessary to derive a pure strain measure from one of the expressions of the Green strain. Particularly the expression of the Green strain in terms of the deformation gradient, i.e. Eq. (11), has resulted useful for deriving pure strain measures [18].

The deformation gradient is a two point tensor given by the derivatives of the current positions with respect to the reference configuration as

$$\boldsymbol{F} = \nabla_X \otimes \boldsymbol{x} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}}. \tag{15}$$

Thus, we can say that the deformation gradient it relates quantities in the current configuration with quantities in the reference configuration. Eventually, we could also write the deformation gradient as

$$\boldsymbol{F} = f_{ij}\boldsymbol{e}_i \otimes \boldsymbol{E}_j, \qquad f_{ij} = \frac{\partial \bar{x}_i}{\partial X_j}. \tag{16}$$

It must be mentioned that order to exploit the above expression it should be necessary to express the current position vector as $\boldsymbol{x} = \bar{x}_i \boldsymbol{e}_i$ and this is certainly not convenient since the translational part of $\boldsymbol{x}$, i.e. $\boldsymbol{x}_0$, is naturally expressed as $\boldsymbol{x}_0 = x_{0i}\boldsymbol{E}_i$. Push forwarding $\boldsymbol{x}_0$ to express it in the current frame is not recommended since the rotation vector would appear explicitly; so it is preferable to avoid thinking of $\boldsymbol{e}_i$ as a current reference frame and consider it as just a triad attached to the cross section. Then, the expression of the deformation gradient must be written as

$$\boldsymbol{F} = F_{ij}\boldsymbol{E}_i \otimes \boldsymbol{E}_j, \qquad F_{ij} = \frac{\partial x_i}{\partial X_j}, \tag{17}$$

being $x_i = \boldsymbol{x} \cdot \boldsymbol{E}_i$. It is possible to operate over the deformation gradient so that

$$\boldsymbol{F} = \frac{\partial x_i}{\partial X_j}\boldsymbol{E}_i \otimes \boldsymbol{E}_j = \frac{\partial (x_i \boldsymbol{E}_i)}{\partial X_j} \otimes \boldsymbol{E}_j = \frac{\partial \boldsymbol{x}}{\partial X_j} \otimes \boldsymbol{E}_j, \tag{18}$$

and find a suitable expression for the explicit obtention of the deformation gradient tensor.

Since the materials derivatives of the position vector are easily obtained in the following way

$$\begin{aligned} \frac{\partial \boldsymbol{x}}{\partial X_1} &= \boldsymbol{x}_{,s} = \boldsymbol{x}_0' + X_2 \boldsymbol{e}_2' + X_3 \boldsymbol{e}_3', \\ \frac{\partial \boldsymbol{x}}{\partial X_2} &= \boldsymbol{x}_{,X_2} = \boldsymbol{e}_2, \\ \frac{\partial \boldsymbol{x}}{\partial X_3} &= \boldsymbol{x}_{,X_3} = \boldsymbol{e}_2, \end{aligned} \tag{19}$$

we can insert these tangent vectors in Eq. (18) and obtain a pure vectorial expression for the deformation gradient; this is

$$\boldsymbol{F} = (\boldsymbol{x}_0' + X_\alpha \boldsymbol{e}_\alpha') \otimes \boldsymbol{E}_1 + \boldsymbol{e}_\alpha \otimes \boldsymbol{E}_\alpha, \tag{20}$$

where summation over $\alpha = 2,3$ has been implicitly assumed.

Eq. (20) contains all the necessary information to describe the finite deformation-finite strain behavior of the beam.

3.3 The small strain tensor

Recalling the distributive property of the tensor product we can write de deformation gradient expression in Eq. (20) as

$$\boldsymbol{F} = (\boldsymbol{x}_0' + X_\alpha \boldsymbol{e}_\alpha') \otimes \boldsymbol{E}_1 + \left(\boldsymbol{e}_j \otimes \boldsymbol{E}_j - \boldsymbol{e}_1 \otimes \boldsymbol{E}_1\right). \tag{21}$$

Rearranging some terms we have

$$\boldsymbol{F} = \boldsymbol{\Lambda} + (\boldsymbol{x}_0' - \boldsymbol{e}_1 + X_\alpha \boldsymbol{e}_\alpha') \otimes \boldsymbol{E}_1. \tag{22}$$

If we define a pure current strain vector $\boldsymbol{\epsilon}$ as

$$\boldsymbol{\epsilon} = \boldsymbol{x}_0' - \boldsymbol{e}_1 + X_\alpha \boldsymbol{e}_\alpha', \tag{23}$$

then

$$\boldsymbol{F} = \boldsymbol{\Lambda} + \boldsymbol{\epsilon} \otimes \boldsymbol{E}_1. \tag{24}$$

The last expression has an interesting meaning since the deformation gradient can be understood as a pure rotation imposed by $\boldsymbol{\Lambda}$ plus a pure deformation measured by $\boldsymbol{\epsilon}$. This is remarkable since for a finite deformation-finite strain problem the strain measures are not commonly written in this form.

The above conclusion as well as the derivation of the pure strain measure $\boldsymbol{\epsilon}$ is consistent with [18]; however, the development of a pure large displacement-small strain measure for the geometrically exact TWB theory requires to find expressions for the Green strain in terms of the director field.

In order to do that we proceed recalling Eq. (11) and writing the Green strain tensor as

$$\boldsymbol{E} = \frac{1}{2}[(\boldsymbol{\Lambda} + \boldsymbol{\epsilon} \otimes \boldsymbol{E}_1)^T (\boldsymbol{\Lambda} + \boldsymbol{\epsilon} \otimes \boldsymbol{E}_1) + \boldsymbol{I}]. \tag{25}$$

This expression can be expanded to give:

$$\boldsymbol{E} = \frac{1}{2}[\boldsymbol{\Lambda}^T \boldsymbol{\Lambda} + \boldsymbol{\Lambda}^T (\boldsymbol{\epsilon} \otimes \boldsymbol{E}_1) + (\boldsymbol{E}_1 \otimes \boldsymbol{\epsilon})\, \boldsymbol{\Lambda} + (\boldsymbol{E}_1 \otimes \boldsymbol{\epsilon})(\boldsymbol{\epsilon} \otimes \boldsymbol{E}_1) + \boldsymbol{I}], \tag{26}$$

where we have used the property $(\boldsymbol{a} \otimes \boldsymbol{b})^T = (\boldsymbol{b} \otimes \boldsymbol{a})$**.** Exploiting the facts that $\boldsymbol{A}(\boldsymbol{a} \otimes \boldsymbol{b}) = (\boldsymbol{A}\boldsymbol{a}) \otimes \boldsymbol{b}$ and $(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{A} = \boldsymbol{a} \otimes (\boldsymbol{A}^T \boldsymbol{b})$ then we can simplify the above expression as

$$\begin{aligned} \boldsymbol{E} &= \frac{1}{2}[\boldsymbol{\Lambda}^T \boldsymbol{\epsilon} \otimes \boldsymbol{E}_1 + \boldsymbol{E}_1 \otimes \boldsymbol{\Lambda}^T \boldsymbol{\epsilon} + (\boldsymbol{E}_1 \otimes \boldsymbol{\epsilon})(\boldsymbol{\epsilon} \otimes \boldsymbol{E}_1)] \\ &= \frac{1}{2}\left[\boldsymbol{\Lambda}^T \boldsymbol{\epsilon} \otimes \boldsymbol{E}_1 + \boldsymbol{E}_1 \otimes \boldsymbol{\Lambda}^T \boldsymbol{\epsilon} + \left((\boldsymbol{E}_1 \otimes \boldsymbol{\epsilon})\boldsymbol{\epsilon}\right) \otimes \boldsymbol{E}_1\right] \end{aligned} \tag{27}$$

The last term can be rearranged if we consider that $(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{c} = (\boldsymbol{c} \cdot \boldsymbol{b})\boldsymbol{a}$ and then we obtain

$$\boldsymbol{E} = \frac{1}{2}[\boldsymbol{\Lambda}^T \boldsymbol{\epsilon} \otimes \boldsymbol{E}_1 + \boldsymbol{E}_1 \otimes \boldsymbol{\Lambda}^T \boldsymbol{\epsilon} + (\boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}) \boldsymbol{E}_1 \otimes \boldsymbol{E}_1] \tag{28}$$

Being $\boldsymbol{\Lambda} = \boldsymbol{e}_j \otimes \boldsymbol{E}_j$, we can see that

$$\boldsymbol{E} = \frac{1}{2}\left[\left(\left(\boldsymbol{e}_j \otimes \boldsymbol{E}_j\right)^T \boldsymbol{\epsilon}\right) \otimes \boldsymbol{E}_1 + \boldsymbol{E}_1 \otimes \left(\left(\boldsymbol{e}_j \otimes \boldsymbol{E}_j\right)^T \boldsymbol{\epsilon}\right) + (\boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}) \boldsymbol{E}_1 \otimes \boldsymbol{E}_1\right] \tag{29}$$

Again, using $(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{c} = (\boldsymbol{c} \cdot \boldsymbol{b})\boldsymbol{a}$ on the first and second terms:

$$\boldsymbol{E} = \frac{1}{2}\left[(\boldsymbol{\epsilon} \cdot \boldsymbol{e}_j)\boldsymbol{E}_j \otimes \boldsymbol{E}_1 + \boldsymbol{E}_1 \otimes \boldsymbol{E}_j (\boldsymbol{\epsilon} \cdot \boldsymbol{e}_j) + (\boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}) \boldsymbol{E}_1 \otimes \boldsymbol{E}_1\right] \tag{30}$$

From the above equation we see that the last term is a pure nonlinear strain measure; so if it is desired to develop a large deformation-small strain formulation this term can be dropped. Thus, the matrix form of the small Green strain tensor is given by

$$\boldsymbol{E}_s = \begin{bmatrix} \boldsymbol{\epsilon}\cdot\boldsymbol{e}_1 & \frac{1}{2}\boldsymbol{\epsilon}\cdot\boldsymbol{e}_2 & \frac{1}{2}\boldsymbol{\epsilon}\cdot\boldsymbol{e}_3 \\ \frac{1}{2}\boldsymbol{\epsilon}\cdot\boldsymbol{e}_2 & 0 & 0 \\ \frac{1}{2}\boldsymbol{\epsilon}\cdot\boldsymbol{e}_3 & 0 & 0 \end{bmatrix}. \tag{31}$$

Recalling Eq. (23) we can write the explicit vector form of the strain measure as

$$\overline{\boldsymbol{E}}_s = \begin{bmatrix} \boldsymbol{x}_0'\cdot\boldsymbol{e}_1 - 1 + X_\alpha\boldsymbol{e}_\alpha'\cdot\boldsymbol{e}_1 \\ \boldsymbol{x}_0'\cdot\boldsymbol{e}_2 + X_3\boldsymbol{e}_3'\cdot\boldsymbol{e}_2 \\ \boldsymbol{x}_0'\cdot\boldsymbol{e}_3 + X_2\boldsymbol{e}_2'\cdot\boldsymbol{e}_3 \end{bmatrix}. \tag{32}$$

3.4 Generalized Strains

The generalized strain is a strain measure that does not contain geometric information of the beam cross section. In order to obtain its expression we proceed splitting the small Green strain vector in Eq. (32) so we can write

$$\overline{\boldsymbol{E}}_s = \boldsymbol{D}\,\boldsymbol{\varepsilon}, \tag{33}$$

where the cross sectional matrix is given by

$$\boldsymbol{D} = \begin{bmatrix} 1 & X_3 & X_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -X_3 \\ 0 & 0 & 0 & 0 & 1 & X_2 \end{bmatrix}. \tag{34}$$

and then the generalized small strain vector is given by

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \epsilon \\ \kappa_2 \\ \kappa_3 \\ \gamma_2 \\ \gamma_3 \\ \kappa_1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}_0'\cdot\boldsymbol{e}_1 - 1 \\ \boldsymbol{e}_1\cdot\boldsymbol{e}_3' \\ \boldsymbol{e}_1\cdot\boldsymbol{e}_2' \\ \boldsymbol{x}_0'\cdot\boldsymbol{e}_2 \\ \boldsymbol{x}_0'\cdot\boldsymbol{e}_3 \\ \boldsymbol{e}_2'\cdot\boldsymbol{e}_3 \end{bmatrix}. \tag{35}$$

As it can be seen, the generalized small strain vector only has six components, which is of value for the finite element computational umplementation. Recalling the nonlinear counterpart of the generalized small strain vector has nine components from [11, 12], we have

$$
\boldsymbol{\varepsilon}_L = \begin{bmatrix} \epsilon \\ \kappa_2 \\ \kappa_3 \\ \gamma_2 \\ \gamma_3 \\ \kappa_1 \\ \chi_2 \\ \chi_3 \\ \chi_{23} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(\boldsymbol{x}_0' \cdot \boldsymbol{x}_0' - 1) \\ \boldsymbol{x}_0' \cdot \boldsymbol{e}_3' \\ \boldsymbol{x}_0' \cdot \boldsymbol{e}_2' \\ \boldsymbol{x}_0' \cdot \boldsymbol{e}_2 \\ \boldsymbol{x}_0' \cdot \boldsymbol{e}_3 \\ \boldsymbol{e}_2' \cdot \boldsymbol{e}_3 \\ \boldsymbol{e}_2' \cdot \boldsymbol{e}_2' \\ \boldsymbol{e}_3' \cdot \boldsymbol{e}_3' \\ \boldsymbol{e}_2' \cdot \boldsymbol{e}_3' \end{bmatrix}. \tag{36}
$$

It can be seen that not only the number of generalized strain is reduced from nine to six, but also the expressions for the axial strain and the curvatures is different; as the shear strains are linear even in the nonlinear description of strains, they remain unchanged.

3.5 Objectivity of the small strain tensor

In previous sections is has been demonstrated that the discrete form of the small strain tensor is a linear pure strain measure. A very important property that this vector must possess is the frame invariance, i.e. it must be an objective measure of the strain state of the beam. Several works have been devoted to demonstrate the preservation of the objectivity of the discrete strain measures of geometrically exact beam formulations [21-28].

In this context, the important contributions of Crisfield and Jelenic [23, 26] shown that geometrically exact beam finite element formulations parametrized with iterative spins, incremental rotation vectors and total rotation vector fail to satisfy the objectivity of its discrete strain measures. Recently, Mäkinen [29] showed that their conclusions regarding the objectivity of the discrete strain measures of formulations parametrized with the total and the incremental rotation vector are incorrect. The misleading conclusions in [23, 26] about the Total and Updated Lagrangian formulations arise from the fact that linear interpolation does not preserve an observer transformation, which in the cited work was assumed.

Pointing toward the development of a geometrically exact beam formulation where the discrete strain measures are objective, interesting works presented alternatives that gained this property by avoiding the interpolation of rotation variables [21, 22, 27]. This was aided by parametrization the equation of motion in terms of nodal triads, with obtention the discrete forms via interpolation of directors. Although the discrete strain measures derived in this works preserve the objectivity property, the linearization of the spins was not consistent and the tangent stiffness matrix results to be non-symmetrical (implying the loss of the quadratic convergence property). A consistent derivation of objective discrete strain measure for geometrically exact thin walled composite beams was presented by Saravia et al. [19].

In order to check the objectivity of the generalized small strain measures we start by superposing a rigid body motion onto the configuration and then we test the invariance of the discrete version of the strains.

A rigid body motion modifies the current configuration as

$$\boldsymbol{x}_0^* = \boldsymbol{c} + \boldsymbol{Q}\,\boldsymbol{x}_0 \qquad \boldsymbol{e}_i^* = \boldsymbol{Q}\boldsymbol{e}_i \tag{37}$$

where $\boldsymbol{c} \in \mathbb{R}^3$and $\boldsymbol{Q} \in \mathrm{SO}(3)$.

Now we first assume, for simplicity, zero initial strain and then impose the rigid body motion to the generalized small strains in Eq. (35). If, for example, we take the axial strain, i.e. $\bar{\epsilon}$, it could be demonstrated that its discrete version is given by

$$\bar{\epsilon} = \left(N_j'\boldsymbol{x}_0^j\right)\cdot\left(N_j\hat{\boldsymbol{e}}_3^j\right), \tag{38}$$

where N_j are linear shape functions and $\hat{\boldsymbol{e}}_i^j$are nodal director triads; summation over the nodal index $j = 1{:}\,2$ was implicitly assumed.

After the imposition of the rigid body motion the discrete axial deformation is given by

$$\bar{\epsilon}^* = \left[\boldsymbol{c} + \boldsymbol{Q}\left(N_j\boldsymbol{x}_0^j\right)\right]'\cdot\boldsymbol{Q}\left(N_j\hat{\boldsymbol{e}}_3^j\right) \tag{39}$$

Taking the derivative of the first term we get

$$\bar{\epsilon}^* = \left[\boldsymbol{c}' + \boldsymbol{Q}'\left(N_j\boldsymbol{x}_0^j\right) + \boldsymbol{Q}\left(N_j'\boldsymbol{x}_0^j\right)\right]\cdot\boldsymbol{Q}\left(N_j\hat{\boldsymbol{e}}_3^j\right); \tag{40}$$

and being $\boldsymbol{c}' = \boldsymbol{Q}' = \boldsymbol{0}$ the expression simplifies to

$$\bar{\epsilon}^* = \boldsymbol{Q}\left(N_j'\boldsymbol{x}_0^j\right)\cdot\boldsymbol{Q}\left(N_j\hat{\boldsymbol{e}}_3^j\right). \tag{41}$$

Since the scalar product of vectors is invariant under orthogonal transformations we can see that the above expression can be simplified as

$$\bar{\epsilon}^* = \boldsymbol{Q}\left(N_j'\boldsymbol{x}_0^j\right)\cdot\boldsymbol{Q}\left(N_j\hat{\boldsymbol{e}}_3^j\right) = N_j'\boldsymbol{x}_0^j\cdot N_j\hat{\boldsymbol{e}}_3^j. \tag{42}$$

Finally, comparing the last expression with Eq. (38) we verify that

$$\bar{\epsilon}^* = \bar{\epsilon}, \tag{43}$$

so the discrete version of the axial generalized small strain is an objective measure of strain.

Continuing with the curvatures, for example the flexural curvature in direction 2, the discrete measure of strain takes the form

$$\bar{\kappa}_2 = N_j\hat{\boldsymbol{e}}_1^j\cdot N_j'\hat{\boldsymbol{e}}_3^j. \tag{44}$$

Proceeding as above we can see that after the imposition of the rigid body motion to the configuration the discrete flexural measure is modified as

$$\bar{\kappa}_2^* = \left[\boldsymbol{Q}\left(N_j\hat{\boldsymbol{e}}_1^j\right)\right]\cdot\left[\boldsymbol{Q}\left(N_j\hat{\boldsymbol{e}}_3^j\right)\right]'. \tag{45}$$

After derivation we can operate as

$$\bar{\kappa}_2^* = \left[\boldsymbol{Q}\left(N_j \hat{\boldsymbol{e}}_1^j\right)\right] \cdot \left[\boldsymbol{Q}'\left(N_j \hat{\boldsymbol{e}}_3^j\right) + \boldsymbol{Q}\left(N_j \hat{\boldsymbol{e}}_3^j\right)'\right]; \tag{46}$$

again, being $\boldsymbol{Q}' = \boldsymbol{0}$ and taking the derivative to the interpolated director field, we have

$$\bar{\kappa}_2^* = \boldsymbol{Q}\left(N_j \hat{\boldsymbol{e}}_1^j\right) \cdot \boldsymbol{Q}\left(N_j' \hat{\boldsymbol{e}}_3^j\right). \tag{47}$$

Once again, the invariance of the scalar products under orthogonal transformation gives

$$\bar{\kappa}_2^* = N_j \hat{\boldsymbol{e}}_1^j \cdot N_j' \hat{\boldsymbol{e}}_3^j = \bar{\kappa}_2. \tag{48}$$

And thus we conclude that also the discrete version of the small flexural strain is objective under rigid body motions.

It is important to note that since the shear small strain measures have the same expression as the nonlinear shear strain measures developed in [12], and in this work its invariance was proved, we do not repeat the development here.

Finally, we conclude from the above expressions that the discrete version of the generalized small strain vector is not affected by superimposed rigid body motions; and thus it is an objective measure of strain. It is interesting to note that since linear interpolation of vector fields is invariant under rigid body motion,

$$\boldsymbol{Q}\, N_j' \hat{\boldsymbol{e}}_i^j = N_j'\left(\boldsymbol{Q}\, \hat{\boldsymbol{e}}_i^j\right) \tag{49}$$

and the scalar product is invariant under orthogonal transformations, the above conclusion clearly makes sense. The frame invariance of the remaining generalized strains can be proven in a similar manner. We note that the generalized strains could also be obtained by interpolation of nodal strains as, for example, $\kappa_2 = N_j'\left(\boldsymbol{x}_{0j} \cdot \hat{\boldsymbol{e}}_i^j\right)$ but, although the frame invariance property is maintained, the resulting strain measure is less accurate.

4 VARIATIONAL FORMULATION

The variational formulation relies on virtual work principle of the thin-walled composite beam, which is given by

$$\delta W\left(\boldsymbol{\phi}, \dot{\boldsymbol{\phi}}, \ddot{\boldsymbol{\phi}}, \delta\boldsymbol{\phi}\right) = \delta W_i - \delta W_e - \delta W_a, \tag{50}$$

where $\delta W_i, \delta W_e$ and δW_a represent the internal, external and inertial virtual works. The deformation measure is only present in the internal energy, so if we take as reference the large strain formulation the present approach only modifies the expression of the internal virtual work of the beam, i.e. [30]:

$$\delta W_i = \int_L \delta\boldsymbol{\varepsilon}^T \boldsymbol{S}\, dx, \tag{51}$$

The obtention of the finite element stiffness matrix is based on the derivation of the virtual small strain vector, which is obtained performing the variation of the Eq. (35).

4.1 The virtual strain vector

The variation of the small generalized strain vector gives

$$\delta\boldsymbol{\varepsilon} = \begin{bmatrix} \delta\boldsymbol{x}_0' \cdot \boldsymbol{e}_1 + \boldsymbol{x}_0' \cdot \delta\boldsymbol{e}_1 \\ \delta\boldsymbol{e}_1 \cdot \boldsymbol{e}_3' + \boldsymbol{e}_1 \cdot \delta\boldsymbol{e}_3' \\ \delta\boldsymbol{e}_1 \cdot \boldsymbol{e}_2' + \boldsymbol{e}_1 \cdot \delta\boldsymbol{e}_2' \\ \delta\boldsymbol{x}_0' \cdot \boldsymbol{e}_2 + \boldsymbol{x}_0' \cdot \delta\boldsymbol{e}_2 \\ \delta\boldsymbol{x}_0' \cdot \boldsymbol{e}_3 + \boldsymbol{x}_0' \cdot \delta\boldsymbol{e}_3 \\ \delta\boldsymbol{e}_2' \cdot \boldsymbol{e}_3 + \boldsymbol{e}_2' \cdot \delta\boldsymbol{e}_3 \end{bmatrix}. \tag{52}$$

The last expression can be written in matrix form as

$$\delta\boldsymbol{\varepsilon} = \mathbb{H}\,\delta\boldsymbol{\varphi}, \tag{53}$$

where

$$\mathbb{H} = \begin{bmatrix} \boldsymbol{e}_1^T & \mathbf{0} & \boldsymbol{x}_0' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{e}_3'^{\,T} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{e}_1^T \\ \mathbf{0} & \mathbf{0} & \boldsymbol{e}_2'^{\,T} & \mathbf{0} & \mathbf{0} & \boldsymbol{e}_1^T & \mathbf{0} \\ \boldsymbol{e}_2^T & \mathbf{0} & \mathbf{0} & \boldsymbol{x}_0'^{\,T} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \boldsymbol{e}_3^T & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{x}_0'^{\,T} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{e}_2'^{\,T} & \boldsymbol{e}_3^T & \mathbf{0} \end{bmatrix}, \quad \delta\boldsymbol{\varphi} = \begin{bmatrix} \delta\boldsymbol{u}_0' \\ \delta\boldsymbol{\theta} \\ \delta\boldsymbol{e}_1 \\ \delta\boldsymbol{e}_2 \\ \delta\boldsymbol{e}_3 \\ \delta\boldsymbol{e}_2' \\ \delta\boldsymbol{e}_3' \end{bmatrix}. \tag{54}$$

The dimension of the matrix $\mathbb{H}$ is reduced from 54 vector entries in the nonlinear strain formulation to 42 in the linear counterpart; also, the number of nonzero entries is reduced from 15 to 12. Since the above relationship is key to the obtention of the element stiffness matrix, the reduction of computational cost generated by the large deformation-small strain formulation is evident.

The above expression can be further simplified if we consider that $\delta\boldsymbol{e}_1 = \delta\boldsymbol{e}_2 \times \boldsymbol{e}_3 + \boldsymbol{e}_2 \times \delta\boldsymbol{e}_3$, then we can reduce the dimension of $\mathbb{H}$ and $\delta\boldsymbol{\varphi}$ such that.

$$\mathbb{H} = \begin{bmatrix} \boldsymbol{e}_1^T & \mathbf{0} & (\tilde{\boldsymbol{e}}_3\boldsymbol{x}_0')^T & -(\tilde{\boldsymbol{e}}_2\boldsymbol{x}_0')^T & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & (\tilde{\boldsymbol{e}}_3\boldsymbol{e}_3')^T & -(\tilde{\boldsymbol{e}}_2\boldsymbol{e}_3')^T & \mathbf{0} & \boldsymbol{e}_1^T \\ \mathbf{0} & \mathbf{0} & (\tilde{\boldsymbol{e}}_3\boldsymbol{e}_2')^T & -(\tilde{\boldsymbol{e}}_2\boldsymbol{e}_2')^T & \boldsymbol{e}_1^T & \mathbf{0} \\ \boldsymbol{e}_2^T & \mathbf{0} & \boldsymbol{x}_0'^{\,T} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \boldsymbol{e}_3^T & \mathbf{0} & \mathbf{0} & \boldsymbol{x}_0'^{\,T} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{e}_2'^{\,T} & \boldsymbol{e}_3^T & \mathbf{0} \end{bmatrix}, \quad \delta\boldsymbol{\varphi} = \begin{bmatrix} \delta\boldsymbol{u}_0' \\ \delta\boldsymbol{\theta} \\ \delta\boldsymbol{e}_2 \\ \delta\boldsymbol{e}_3 \\ \delta\boldsymbol{e}_2' \\ \delta\boldsymbol{e}_3' \end{bmatrix}. \tag{55}$$

We have also used the geometric relation $\tilde{\boldsymbol{e}}_2'\boldsymbol{e}_2' = \mathbf{0}$.

4.2 Linearization

We can use the definition of the directional derivative to find the linearized virtual work of the internal forces in the direction $\delta\boldsymbol{\phi}$ as

$$\mathcal{L}[\delta W_i(\boldsymbol{\phi};\delta\boldsymbol{\phi})] = \delta W_i(\boldsymbol{\phi}_0;\delta\boldsymbol{\phi}) + \mathcal{D}_{\boldsymbol{\phi}}\,\delta W_i(\boldsymbol{\phi}_0;\delta\boldsymbol{\phi}) \cdot \Delta\boldsymbol{\phi}, \tag{56}$$

being

$$\mathcal{D}_{\boldsymbol{\phi}}\, \delta W_i(\boldsymbol{\phi}_0; \delta\boldsymbol{\phi}) \cdot \Delta\boldsymbol{\phi} = \int_L (\delta\boldsymbol{\varepsilon}^T \Delta\boldsymbol{S} + \Delta\delta\boldsymbol{\varepsilon}^T \boldsymbol{S}) dx. \tag{57}$$

The incremental beam stress is easy to obtain as $\Delta\boldsymbol{S} = \mathbb{D}\, \Delta\boldsymbol{\varepsilon}$, but the incremental virtual strains require some work. After derivation, the increment of the generalized small strain vector gives

$$\Delta\delta\boldsymbol{\varepsilon} = \begin{bmatrix} \delta\boldsymbol{x}_0' \cdot \Delta\boldsymbol{e}_1 + \Delta\boldsymbol{x}_0' \cdot \delta\boldsymbol{e}_1 + \boldsymbol{x}_0' \cdot \Delta\delta\boldsymbol{e}_1 \\ \Delta\delta\boldsymbol{e}_1 \cdot \boldsymbol{e}_3' + \delta\boldsymbol{e}_1 \cdot \Delta\boldsymbol{e}_3' + \Delta\boldsymbol{e}_1 \cdot \delta\boldsymbol{e}_3' + \boldsymbol{e}_1 \cdot \Delta\delta\boldsymbol{e}_3' \\ \Delta\delta\boldsymbol{e}_1 \cdot \boldsymbol{e}_2' + \delta\boldsymbol{e}_1 \cdot \Delta\boldsymbol{e}_2' + \Delta\boldsymbol{e}_1 \cdot \delta\boldsymbol{e}_2' + \boldsymbol{e}_1 \cdot \Delta\delta\boldsymbol{e}_2' \\ \delta\boldsymbol{x}_0' \cdot \Delta\boldsymbol{e}_2 + \Delta\boldsymbol{x}_0' \cdot \delta\boldsymbol{e}_2 + \boldsymbol{x}_0' \cdot \Delta\delta\boldsymbol{e}_2 \\ \delta\boldsymbol{x}_0' \cdot \Delta\boldsymbol{e}_3 + \Delta\boldsymbol{x}_0' \cdot \delta\boldsymbol{e}_3 + \boldsymbol{x}_0' \cdot \Delta\delta\boldsymbol{e}_3 \\ \Delta\delta\boldsymbol{e}_2' \cdot \boldsymbol{e}_3 + \delta\boldsymbol{e}_2' \cdot \Delta\boldsymbol{e}_3 + \Delta\boldsymbol{e}_2' \cdot \delta\boldsymbol{e}_3 + \boldsymbol{e}_2' \cdot \Delta\delta\boldsymbol{e}_3 \end{bmatrix}. \tag{58}$$

Again, the evaluation of $\Delta\delta\boldsymbol{\varepsilon}$ requires 21 scalar products in place of 25 of the large strain formulation.

It is common to write the geometrical stiffness part of the virtual work considering that

$$\Delta\delta\boldsymbol{\varepsilon}^T \boldsymbol{S} = \delta\boldsymbol{\varphi}^T \mathbb{G}\, \Delta\boldsymbol{\varphi}. \tag{59}$$

where the matrix $\mathbb{G}$ is given by

$$\mathbb{G} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{S}_1 & \boldsymbol{Q}_2 & \boldsymbol{Q}_3 & \boldsymbol{0} & \boldsymbol{0} \\ & \boldsymbol{A} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ & & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{M}_3 & \boldsymbol{M}_2 \\ & & & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ & & & & \boldsymbol{0} & \boldsymbol{M}_1 & \boldsymbol{0} \\ & & Sym & & & \boldsymbol{0} & \boldsymbol{0} \\ & & & & & & \boldsymbol{0} \end{bmatrix}. \tag{60}$$

From the computational point of view, the most expensive element of $\mathbb{G}$ is $\boldsymbol{A}$. This element comes from the terms $\boldsymbol{a} \cdot \Delta\delta\boldsymbol{e}_i$, which involve second variations of the director field. The explicit expression of the element $\boldsymbol{A}$ is

$$\begin{aligned} \boldsymbol{A} = \sum_{j=1}^{2} \Big\{ & N_1 N_j \left[\Xi\big(\tilde{\tilde{\boldsymbol{e}}}_1^j \boldsymbol{x}_0'\big) + \boldsymbol{T}_j \tilde{\boldsymbol{x}}_0' \tilde{\tilde{\boldsymbol{e}}}_1^j \boldsymbol{T}_j^T \right] \\ & + M_2 \left[N_j \left[\Xi\big(\tilde{\tilde{\boldsymbol{e}}}_1^j \boldsymbol{e}_3'\big) + \boldsymbol{T}_j \tilde{\boldsymbol{e}}_3' \tilde{\tilde{\boldsymbol{e}}}_1^j \boldsymbol{T}_j^T \right] + N_j' \left[\Xi\big(\tilde{\tilde{\boldsymbol{e}}}_3^j \boldsymbol{e}_1\big) + \boldsymbol{T}_j \tilde{\boldsymbol{e}}_1 \tilde{\tilde{\boldsymbol{e}}}_3^j \boldsymbol{T}_j^T \right] \right] \\ & + M_3 \left[N_j \left[\Xi\big(\tilde{\tilde{\boldsymbol{e}}}_1^j \boldsymbol{e}_2'\big) + \boldsymbol{T}_j \tilde{\boldsymbol{e}}_2' \tilde{\tilde{\boldsymbol{e}}}_1^j \boldsymbol{T}_j^T \right] + N_j' \left[\Xi\big(\tilde{\tilde{\boldsymbol{e}}}_2^j \boldsymbol{e}_1\big) + \boldsymbol{T}_j \tilde{\boldsymbol{e}}_1 \tilde{\tilde{\boldsymbol{e}}}_2^j \boldsymbol{T}_j^T \right] \right] \\ & + Q_3 N_j \left[\Xi\big(\tilde{\tilde{\boldsymbol{e}}}_2^j \boldsymbol{x}_0'\big) + \boldsymbol{T}_j \tilde{\boldsymbol{x}}_0' \tilde{\tilde{\boldsymbol{e}}}_2^j \boldsymbol{T}_j^T \right] + Q_2 N_j \left[\Xi\big(\tilde{\tilde{\boldsymbol{e}}}_3^j \boldsymbol{x}_0'\big) + \boldsymbol{T}_j \tilde{\boldsymbol{x}}_0' \tilde{\tilde{\boldsymbol{e}}}_3^j \boldsymbol{T}_j^T \right] \\ & + T \left[N_j' \left[\Xi\big(\tilde{\tilde{\boldsymbol{e}}}_2^j \boldsymbol{e}_3\big) + \boldsymbol{T}_j \tilde{\boldsymbol{e}}_3 \tilde{\tilde{\boldsymbol{e}}}_2^j \boldsymbol{T}_j^T \right] + N_j \left[\Xi\big(\tilde{\tilde{\boldsymbol{e}}}_3^j \boldsymbol{e}_2'\big) + \boldsymbol{T}_j \tilde{\boldsymbol{e}}_2' \tilde{\tilde{\boldsymbol{e}}}_3^j \boldsymbol{T}_j^T \right] \right] \Big\} \end{aligned} \tag{61}$$

where Ξ is an operator such that $\boldsymbol{\Xi}_{\boldsymbol{T}}(\tilde{\boldsymbol{e}}_i \boldsymbol{a}) = \Delta[\boldsymbol{T}(\tilde{\boldsymbol{e}}_i \boldsymbol{a})]$.

The relation between the variables $\delta\boldsymbol{\varphi}$ and $\delta\widehat{\boldsymbol{\phi}}_j$ is given by:

$$\delta\boldsymbol{\varphi} \cong \sum_{j=1}^{nn} \mathbb{B}_j \delta\widehat{\boldsymbol{\phi}}_j, \tag{62}$$

being:

$$\mathbb{B}_j = \begin{bmatrix} \boldsymbol{N}_j' & \mathbf{0} \\ \mathbf{0} & \boldsymbol{N}_j \\ \mathbf{0} & \boldsymbol{N}_j \tilde{\boldsymbol{e}}_1^{j^T} \boldsymbol{T}_j^T \\ \mathbf{0} & \boldsymbol{N}_j \tilde{\boldsymbol{e}}_2^{j^T} \boldsymbol{T}_j^T \\ \mathbf{0} & \boldsymbol{N}_j \tilde{\boldsymbol{e}}_3^{j^T} \boldsymbol{T}_j^T \\ \mathbf{0} & \boldsymbol{N}_j' \tilde{\boldsymbol{e}}_2^{j^T} \boldsymbol{T}_j^T \\ \mathbf{0} & \boldsymbol{N}_j' \tilde{\boldsymbol{e}}_3^{j^T} \boldsymbol{T}_j^T \end{bmatrix}, \quad \delta\widehat{\boldsymbol{\Phi}}_j = \begin{bmatrix} \delta\widehat{\boldsymbol{u}}_{0j} \\ \delta\widehat{\boldsymbol{\theta}}_j \end{bmatrix}. \tag{63}$$

After the above derivations it is straightforward to obtain the tangent stiffness matrix for the small strain-large deformation formulation; for the sake of brevity we do not include the full derivation in this work, the reader can refer to [12] if he needs additional details.

5 NUMERICAL TESTS

In order to evaluate the accuracy of the present formulation we perform two numerical tests. The objective of the tests is to show briefly that the present formulation gives correct values of displacements still in large deformation cases.

5.1 Bending of a curved cantilever composite beam

In the first test we analyze the evolution of the displacement field and the generalized strains of a curved cantilever beam subjected to a tip oblique load $\mathrm{P} = \{4.0 \times 10^4, -4.0 \times 10^4, 8.0 \times 10^4\}$; the magnitude of the load is such that the induce displacements are large. The curved beam has a reference configuration defined by a 45º circular segment lying in the *xy* plane with radius $R = 100$, see Figure 2.

Figure 2. 45º Cantilever beam.

The material properties of the composite beam are listed in Table 1; the cross section of the beam is boxed with sides of length $b = h = 1$ and thickness $t = 0.1$. The lamination sequence is $\{45, -45, -45, 45\}$.

E_{11}	E_{22}	G_{12}	G_{23}	ν_{12}
45.0×10^9	12.0×10^9	5.5×10^9	5.5×10^9	0.3

Table 1– Material properties of EFG-Epoxy layers.

The load generates a large deformation behavior, thus the ability of the large deformation-small strain formulation can be addressed by a close comparison with the large deformation-large strain formulation presented in [12]. As it can be seen from the evolution of the displacement field at the beam tip (see Figure 3), the present formulation behaves very well for the large deformation case.

Figure 3. 45º Cantilever Beam – Evolution of displacements at the beam tip

The Figures 4 and 5 show the progression of the generalized small strain components. It can be observed that the present formulation behaves very well still for moderate strain cases.

Figure 4. 45º Cantilever Beam – Evolution of strains

Figure 5. 45º Cantilever Beam – Evolution of strains

5.2 Simple pendulum on tip of cantilever beam

The next test was thought to test the behavior of the formulation in a multibody environment and to examine the behavior of the beam when it is subjected to large rigid body motions. Again, we have chosen to compare the displacement and deformation time histories of the present formulation against data obtained with the large strain approach in [31].

The test consist on an L-shaped mechanism formed by a simple pendulum attached to a cantilever beam, as the Figure 6 shows. The idea of the test is that gravity makes the flexible pendulum fall and, as the beam is very flexible, a very complex dynamic response is induced in the whole mechanism. Not only the displacements and rotations are finite, also the elastic deformations are quite high, probably one order of magnitude higher that what can be found in practical situations.

Both beam and the pendulum are 10m long, the whole model contains 40 beam finite elements and 1 hing joint elements. The beams have a rectangular cross section with $b = h = 0.1$ and thickness $t = 0.01$. The lamination sequence is $\{45, -45, -45, 45\}$ and the material properties of the laminate are given in Table 1.

The multibody algorithm rely on the introduction of Lagrange multipliers for the imposition of the kinematical constraints of the hinge joint. The simulation lasts 10 seconds and is temporally discretized by the generalized alpha method using 1000 time steps.

Figure 6. L-shaped pendulum configuration

The Figure 7 shows the comparison of the evolution of displacements measured in the pendulum tip; all displacement components are showed in the same picture. It can be clearly seen that all components of displacements are calculated with remarkable precision, still when the strains are assumed to be small; an animated picture of the evolution of the dynamic response of the mechanism can be found at www.martinsaravia.com/investigacion.

Figure 7. Evolution of displacements at the pendulum tip.

Figures 8 and 9 show the time history of the six components of the generalized strain vector; it can clearly observed that the present formulation gives excellent results in a multibody problem with moderate strains.

Figure 8. Evolution of axial and flexural strains at the beam root.

Figure 9. Evolution of shear and torsional strains at the beam root.

There are several possible computational implementations of the formulation, the fastest takes approximately 153 seconds for the evaluation of the element tangent matrices; 80040 callback to the computational routine that performs this task are performed and the full simulations lasts 322 seconds. The large strain formulation takes 211 seconds for evaluation of the element tangent matrices, while the whole simulations lasts 390 seconds. This represents a reduction of about 38% in computational time; this improvement in the computational time comes mainly from the simplification of the linearization term in Eq. (61). This term is the driver of the computational efficiency of the formulation, note that the operator $\boldsymbol{\Xi}$ is actually a very complex trigonometric function of the incremental rotation vector, see [19].

CONCLUSIONS

This work has presented a formulation for modelling thin walled composite beam that exhibit small strain-large deformation behavior. The formulation is especially suited for modeling high aspect ratio composite beams that undergo large rigid body motions, such as wind turbine wings, satellite arms, automotive body stiffeners, etc.

The presented formulation assumes that the kinematic behavior of the beam can undergo arbitrary changes while the strains is restricted to be small, thus compatibilizing the strain measure with the constitutive law of composite laminates. The latter is achieved obtaining a pure small strain measure written solely in terms of scalar products of position and director vectors. The discrete version of the generalized small strain results to be objective under rigid body motion.

The formulation has a finite element implementation that is simpler than that of the finite strain theory previously developed by the author. The number of generalized strain measures is reduced from nine to six, and thus the variational formulation is considerably simpler.

In terms of computational cost, numerical experiments showed that the proposed formulation is computationally more efficient than the finite strain form of the theory. The storage of strain variables is reduced in a 30% and the computational time used for the evaluation of the tangent stiffness matrices is reduced 38% compare to the consumption of the large strain formulation.

Several comparisons of displacement and strain histories that the presented formulation gives excellent results still in extreme deformation cases. All displacement variables have shown good accuracy, the maximum error detected in the test was 2%.

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