Leapfrog integration

The variables

The file with the initial conditions contains the following quantities for each of the N particles:

$$\begin{aligned}
&\text{Position: } \mathbf{r}_i, \\
&\text{Velocity: } \mathbf{v}_i, \\
&\text{Energy per mass unit: } u_i, \\
&\text{Mass: } m_i.
\end{aligned}$$

At every timestep the smoothing length, h_i, of each particle is calculated such that the number of particles inside a sphere of radius h_i is constant.

When the positions are known the density of each particle can be calculated:

$$\rho_i = \rho_i(\mathbf{r}).$$

$\rho_i(\mathbf{r})$ is shorthand notation for $\rho_i(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N)$. The pressure and the entropy are given by

$$\begin{aligned}
P_i &= (\gamma - 1)\rho_i u_i, \\
A_i &= \frac{P_i}{\rho_i^\gamma}.
\end{aligned}$$

The acceleration and the entropy derivative of each particle can be written as

$$\mathbf{a}_i = \mathbf{a}_i(\mathbf{r}, \mathbf{v}, \rho, A), \quad \dot{A}_i = \dot{A}_i(\mathbf{r}, \mathbf{v}, \rho, A).$$

Leapfrog integration scheme

We want to determine the time-evolution of the variables, $\mathbf{r}, \mathbf{v}$ and A. At the 0th timestep (i.e. at $t = 0$) the following quantities are known/calculated:

$$\begin{aligned}
&\mathbf{r}_i^0 \text{ is known}, \\
&\mathbf{v}_i^0 \text{ is known}, \\
&A_i^0 \text{ is known}, \\
&\rho_i^0 = \rho_i^0(\mathbf{r}^0), \\
&\mathbf{a}_i^0 = \mathbf{a}_i^0(\mathbf{r}^0, \mathbf{v}^0, \rho^0, A^0), \\
&\dot{A}_i^0 = \dot{A}_i^0(\mathbf{r}^0, \mathbf{v}^0, \rho^0, A^0).
\end{aligned}$$

The superscripts indicate the number of the timestep. First we calculate the velocity and the entropy at $t = \Delta t/2$:

$$\begin{aligned}
\mathbf{v}_i^{1/2} &= \mathbf{v}_i^0 + \mathbf{a}_i^0 \cdot \Delta t/2, \\
A_i^{1/2} &= A_i^0 + \dot{A}_i^0 \cdot \Delta t/2.
\end{aligned}$$

Next we calculate the acceleration and the entropy derivative for each particle at $t = \Delta t$ using the following algorithm:

$$\begin{aligned}
\mathbf{r}_i^1 &= \mathbf{r}_i^0 + \mathbf{v}_i^{1/2}\Delta t, \\
\rho_i^1 &= \rho_i^1(\mathbf{r}^1), \\
\tilde{\mathbf{v}}_i^1 &= \mathbf{v}_i^{1/2} + \mathbf{a}_i^0\Delta t/2, \\
\tilde{A}_i^1 &= A_i^{1/2} + \dot{A}_i^0\Delta t/2, \\
\mathbf{a}_i^1 &= \mathbf{a}_i^1(\mathbf{r}^1, \tilde{\mathbf{v}}^1, \rho^1, \tilde{A}^1), \\
\dot{A}_i^1 &= \dot{A}_i^1(\mathbf{r}^1, \tilde{\mathbf{v}}^1, \rho^1, \tilde{A}^1).
\end{aligned}$$

Then the velocity and the entropy are computed at $t = 3/2\Delta t$:

$$\begin{aligned}
\mathbf{v}_i^{3/2} &= \mathbf{v}_i^{1/2} + \mathbf{a}_i^1\Delta t/2, \\
A_i^{3/2} &= A_i^{1/2} + \dot{A}_i^1\Delta t/2.
\end{aligned}$$

The general algorithm is

$$\begin{aligned}
\mathbf{r}_i^{n+1} &= \mathbf{r}_i^n + \mathbf{v}_i^{n+1/2}\Delta t, \\
\rho_i^{n+1} &= \rho_i^{n+1}(\mathbf{r}^{n+1}), \\
\tilde{\mathbf{v}}_i^{n+1} &= \mathbf{v}_i^{n+1/2} + \mathbf{a}_i^n\Delta t/2, \\
\tilde{A}_i^{n+1} &= A_i^{n+1/2} + \dot{A}_i^n\Delta t/2, \\
\mathbf{a}_i^{n+1} &= \mathbf{a}_i^{n+1}(\mathbf{r}^{n+1}, \tilde{\mathbf{v}}^{n+1}, \rho^{n+1}, \tilde{A}^{n+1}), \\
\dot{A}_i^{n+1} &= \dot{A}_i^{n+1}(\mathbf{r}^{n+1}, \tilde{\mathbf{v}}^{n+1}, \rho^{n+1}, \tilde{A}^{n+1}), \\
\mathbf{v}_i^{n+3/2} &= \mathbf{v}_i^{n+1/2} + \mathbf{a}_i^{n+1}\Delta t/2, \\
A_i^{n+3/2} &= A_i^{n+1/2} + \dot{A}_i^{n+1}\Delta t/2.
\end{aligned}$$