ORIE 5355/INFO 5370 HW 3: Algorithmic Pricing

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- Date: 2021-10-14
- Late days used for this assignment: 0
- Total late days used (counting this assignment): 2
- People with whom you discussed this assignment:

  https://github.com/martinsun0/People-Data-Systems/blob/main/HW3/HW3_assignment.ipynb

After you finish the homework, please complete the following (short, anonymous) post-homework survey: https://forms.gle/1SxoC4YoBhUQiwPP7 and include the survey completion code below.

Question 0 [2 points]

Survey completion code:

We have marked questions in blue . Please put answers in black (do not change colors). You'll want to write text answers in "markdown" mode instead of code. In Jupyter notebook, you can go to Cell > Cell Type > Markdown, from the menu. Please carefully read the late days policy and grading procedure here.

Conceptual component [6 points]

Please complete the following pricing ethics scenario questionaire: https://forms.gle/DK4djayEar2ZX92y6, and include the survey completion code below. **We will discuss these issues in class on either 10/18 or 10/20 (Exact date to be announced). You must complete the questionaire before the day of that class, even if you turn in the rest of the homework later. The questionaire will close the morning of the class that we discuss these issues.**

Survey completion code: hw3_pricingethics_2021

Survey completion code: Based on the first letter of your first name, explain your answers to the following questions, in at most three sentences each.

First letter A-C: 1, 6, 11, 16

First letter D-H: 2, 7, 12, 17

First letter I-M: 3, 8, 13, 18

First letter N-S: 4, 9, 14, 19

First letter T-Z: 5, 10, 15, 20

3: More or less simply supply and demand which happens all the time. When something has much higher demand, offerers will tend to get more value from it. There is nothing unfair about that.

8: This is fair pricing strategy. The places with lots of competition are required to reduce prices to stay afloat, however nothing unethical occurs when this is not done in a non-competitive area (unless it is exploitative of a community).

13: This is unfair solely because the neighbourhood is disadvantaged. The higher prices are due to higher buyer perceived value, but that may be because this community needs the item that much more. Essential items should not be priced higher in disadvantaged communities.

18: The fact is that drivers need to spend more time and money to get to these passengers. It doesn't matter who the passengers are - the drivers should be compensated somewhat for this extra distance. It's also not an essential product but rather more of a convenience/luxury service.

Be prepared to discuss your answers to at least these questions in class (I might randomly call on people), but you should also be willing/able to discuss your answers to other questions.

Programming component

Helper code

In [1]:
```python
import numpy as np
import pandas as pd
import warnings
warnings.filterwarnings('ignore')
import seaborn as sns
sns.set_theme()
import os, sys, math
import matplotlib.pyplot as plt
```

In [2]:
```python
df_train = pd.read_csv('HW3_data_train.csv')
test_demand_curve = pd.read_csv('test_demand.csv')
```

In [3]:
```python
df_train.head()
```

Out[3]:

	Location	Income	Offered price	Purchased
0	Africa	10.38	3.16	False
1	Europe	26.33	3.47	True
2	Europe	24.06	3.78	True
3	Africa	16.18	3.74	False
4	Asia Pacific	13.73	4.75	False

In [4]:
```python
df_train.shape, test_demand_curve.shape
```

Out[4]:
```
((4000, 4), (199, 2))
```

Problem 1: Demand estimation and pricing without covariates

First, we will use the training data to construct estimates of the demand at each price without leveraging the covariates, and then use that estimated function to calculate optimal prices.

1a) Naive method: empirical estimate of demand $d(p)$ at each price

Fill in the below function, that takes in a dataframe and the number of bins into which to separate the historical prices. The function should output a dataframe that has one row for each price bin, with two columns: the bin interval, and the estimated demand $d(p)$ (the fraction of potential customers who purchase at price p) in that bin.

Use the following function to create bins: https://pandas.pydata.org/pandas-docs/stable/reference/api/pandas.qcut.html

In [5]:
```python
# Example with 10 bins:
# df_train['bin_with_10_bins'] = pd.qcut(df_train['Offered price'], 10)
# df_train.head()
# grouped_df = df_train.groupby(['bin_with_10_bins'])['Purchased'].sum().reset_index(name ='count')
# grouped_df['count'].sum()
# df_train.groupby(['bin_with_10_bins'])['Purchased'].count()
```

For example, with 2 bins and passing in df_train to the function, you should see the following output:

	Price_bin	Demand_at_price
0	(0.499, 3.54]	0.583417
1	(3.54, 6.5]	0.050050

In [6]:
```python
def create_empirical_estimate_demand_df(df, number_of_pricing_bins):
    df['bin'] = pd.qcut(df['Offered price'], number_of_pricing_bins)
    grouped = df.groupby(['bin'])['Purchased'].sum().reset_index(name ='count')
    total_per_bin = df.groupby(['bin'])['Purchased'].count()
    grouped['demand'] = grouped['count'].divide(total_per_bin.values)
    return grouped.drop('count', axis=1)
```

In [7]:
```python
demand_df = create_empirical_estimate_demand_df(df_train, 10)
```

Fill in the below function, that takes in a single price and your empirical df from the above function and outputs the prediction for the demand $d(p)$ at that price.

In [8]:
```python
def get_prediction_empirical(empirical_df, price):
    min = empirical_df['bin'].iloc[0].left
    max = empirical_df['bin'].iloc[-1].right
    if price < min:
        price = min
    elif price > max:
        price = max
    demand = empirical_df.loc[empirical_df['bin'].apply(lambda a: a.left <= price <= a.right)]['demand'].iloc[(
    return demand
```

In [9]:
```python
prices_to_predict = np.linspace(min(df_train['Offered price']), max(df_train['Offered price']), 200)
```

Plot in a single figure the outputs of your function as a line plot -- where the X axis corresponds to prices in `prices_to_predict` and the Y axis the predicted Demand at that price -- for the following three inputs to the function:

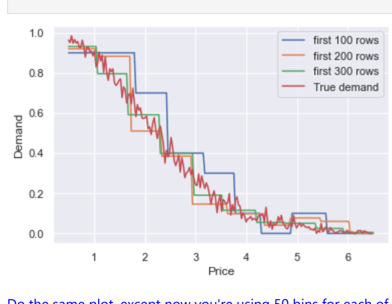
```
1. the dataframe is the first 100 rows of df_train, with 10 bins.

2. the dataframe is the first 500 rows of df_train, with 10 bins.

3. the dataframe is all the rows of df_train, with 10 bins.
```

In the same figure, include the "true" test-time demand curve, `test_demand_curve`.

In [10]:
```python
n_bins = 10
df1 = create_empirical_estimate_demand_df(df_train.iloc[:100], n_bins)
df2 = create_empirical_estimate_demand_df(df_train.iloc[:500], n_bins)
df3 = create_empirical_estimate_demand_df(df_train, n_bins)
```

In [11]:
```python
pred_demand1 = [get_prediction_empirical(df1, price) for price in prices_to_predict]
pred_demand2 = [get_prediction_empirical(df2, price) for price in prices_to_predict]
pred_demand3 = [get_prediction_empirical(df3, price) for price in prices_to_predict]
```

In [12]:
```python
sns.lineplot(prices_to_predict,pred_demand1)
sns.lineplot(prices_to_predict,pred_demand2)
sns.lineplot(prices_to_predict,pred_demand3)
sns.lineplot(test_demand_curve.iloc[:,0],test_demand_curve.iloc[:,1])
plt.ylabel('Demand')
plt.legend(['first 100 rows','first 200 rows','first 300 rows','True demand']);
```

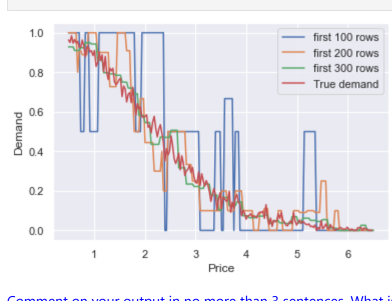


Do the same plot, except now you're using 50 bins for each of the three data frames.

In [13]:
```python
n_bins = 50
df1 = create_empirical_estimate_demand_df(df_train.iloc[:100], n_bins)
df2 = create_empirical_estimate_demand_df(df_train.iloc[:500], n_bins)
df3 = create_empirical_estimate_demand_df(df_train, n_bins)
```

In [14]:
```python
pred_demand_50_1 = [get_prediction_empirical(df1, price) for price in prices_to_predict]
pred_demand_50_2 = [get_prediction_empirical(df2, price) for price in prices_to_predict]
pred_demand_50_3 = [get_prediction_empirical(df3, price) for price in prices_to_predict]
```

In [15]:
```python
sns.lineplot(prices_to_predict,pred_demand_50_1)
sns.lineplot(prices_to_predict,pred_demand_50_2)
sns.lineplot(prices_to_predict,pred_demand_50_3)
sns.lineplot(test_demand_curve.iloc[:,0],test_demand_curve.iloc[:,1])
plt.ylabel('Demand')
plt.legend(['first 100 rows','first 200 rows','first 300 rows','True demand']);
```



Comment on your output in no more than 3 sentences. What is the effect of using more data and more bins?

The estimated demand follows the "true" demand curve reasonably well. Of course, the more rows of data we use, the more accurate the data appears to be. Additionally, the more bins we have, the less data we have for each bin, which appears to dramatically reduce the prediction precision when we have a limited number of rows, because there is so little data for each bin.

1b) Demand estimation using logistic regression

First, Fill in the below function that fits a logistic regression to predict the probability of purchase at a price ($d(p)$). The logistic regression should just have two coefficients: one for the intercept, and one for the price. The function takes in a dataframe that you will use as your training data for your model, and should return your fitted model.

In [16]:
```python
from sklearn.linear_model import LogisticRegression
```

In [17]:
```python
# encoded_df = pd.get_dummies(df_train.drop('bin', axis=1), drop_first=True)
# X = encoded_df.drop('Purchased', axis=1)
# Y = encoded_df['Purchased']
# encoded_df
```

In [18]:
```python
def fit_logistic_regression_demand_just_on_price(df):
    X = df['Offered price']
    Y = df['Purchased']
    model = LogisticRegression()
    model.fit(X.values.reshape(-1,1), Y)
    return model
```

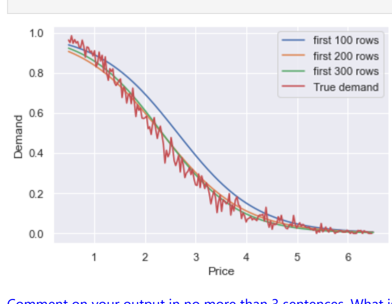
Fill in the below function, that takes in a single price and your trained model and outputs the prediction for the demand $d(p)$ at that price.

In [19]:
```python
def get_prediction_logistic(fitted_model, price):
    return fitted_model.predict_proba(np.array([price]).reshape(-1,1))[0,1]
```

For each of the three training dataframes as in part A, fit a model and get the predictions for each of the prices in `prices_to_predict` using your above function. Generate the same lineplot as above. Also include the "true" test-time demand curve, `test_demand_curve`.

In [20]:
```python
def get_logistic_predictions(df_subset, prices_to_predict):
    lr_model = fit_logistic_regression_demand_just_on_price(df_subset)
    pred_demand = [get_prediction_logistic(lr_model, price) for price in prices_to_predict]
    return pred_demand
```

In [21]:
```python
lr_pred1 = get_logistic_predictions(df_train.iloc[:100], prices_to_predict)
lr_pred2 = get_logistic_predictions(df_train.iloc[:500], prices_to_predict)
lr_pred3 = get_logistic_predictions(df_train, prices_to_predict)

sns.lineplot(prices_to_predict,lr_pred1)
sns.lineplot(prices_to_predict,lr_pred2)
sns.lineplot(prices_to_predict,lr_pred3)
sns.lineplot(test_demand_curve.iloc[:,0],test_demand_curve.iloc[:,1])
plt.ylabel('Demand')
plt.legend(['first 100 rows','first 200 rows','first 300 rows','True demand']);
```



Comment on your output in no more than 3 sentences. What is the effect of using logistic regression instead of the empirical distribution?

Fitting the probability based on a continuous sigmoid function as in logistic regression gives us a very smooth predicted demand, as we expect - the precision is high. With empirical binned estimates, we cannot get the same continuous estimate, and can only estimate based on demand per bin, which is discrete and choppy. The logistic model appears very accurate aside from the 100 row subset (lack of data), meaning that true demand is rather continuous and non-anomalous for this case.

1c) Optimal pricing using your demand estimates

Fill in the following function that takes in two lists: a list of prices, and a list of predicted demand d(p) at that price. The function outputs the revenue maximizing price given the data and the corresponding revenue. You may use a "brute force" technique, that loops through all the possible prices and calculates the revenue using that price.

In [22]:
```python
def get_revenue_maximizing_price_and_revenue(price_options, demand_predictions):
    revenues = []
    for option in range(len(price_options)):
        revenue = price_options[option]*demand_predictions[option]
        revenues.append(revenue)
    best = np.argmax(revenues)
    return price_options[best], revenues[best]
```

Print out the optimal price and the predicted optimal revenue from the predictions for your naive and logistic models, using 100 rows and all the data, each.

In [23]:
```python
print('Price, Revenue for Naive models:')
print([get_revenue_maximizing_price_and_revenue(prices_to_predict,pred) for pred in [pred_demand1,pred_demand2,
print('Price, Revenue for Logistic models:')
print([get_revenue_maximizing_price_and_revenue(prices_to_predict,pred) for pred in [lr_pred1,lr_pred2,lr_pred3
```

```
Price, Revenue for Naive models:
[(2.42964824120603, 1.700753768844221), (1.7060301507537687, 1.5053207212533253), (2.278894472361809, 1.3481578
68550674)]
Price, Revenue for Logistic models:
[(2.1884422110552766, 1.4070878495510661), (2.007537688442211, 1.2038956372592826), (1.9773869346733668, 1.2295
480454251977)]
```

Now, we're going to use the "true" test-time demand curve, `test_demand_curve`. For each of the above predicted optimal prices, calculate the revenue resulting from that price used on the true demand curve. Also print out the true optimal price and corresponding revenue for that curve.

In [24]:
```python
def get_matched_demand_vector(corrected_df, price_vector):
    demand_vector = [corrected_df.loc[(corrected_df['Price_bin_left'] <= a) & (corrected_df['Price_bin_right']
    return demand_vector
```

In [25]:
```python
test_demand_curve_corrected = pd.read_csv('test_demand2.csv')
true_demand_matched = get_matched_demand_vector(test_demand_curve_corrected, prices_to_predict)

print('Price, Revenue for true data from "test_demand":')
print(get_revenue_maximizing_price_and_revenue(prices_to_predict,true_demand_matched))
```

```
Price, Revenue for true data from "test_demand":
(2.0979899497487438, 1.2008231949219783)
```

How do your estimates compare to the actual revenue? Discuss in no more than 3 sentences.

The estimates are quite good - our empirical Naive estimate gave us an optimal price of 2.28 at 1.35 estimated revenue while the logistic model gave us an optimal price of 1.98 at a 1.23 estimated revenue. The true value output an optimal price of 2.10 at 1.2 revenue, which is close to estimations (although estimations appear to overestimate demand slightly). Because our demand estimations aren't perfect, our optimal pricing is also imperfect.

Problem 2: Demand estimation and pricing with covariates

Now, we are going to ask you to do personalized pricing, based on just a two binarized covariates.

First, take `df_train` and create a new column for "low" and "high" wealth, based on if the income level is above or below the median income level. Second, create a new column for Location: `1` if the location is either America, and `0` if the location is anything else.

For this section, we will use all the df_train data, as opposed to just the first few rows.

In [26]:
```python
median_income = df_train['Income'].median()
df_train['High Wealth'] = 0
df_train['High Wealth'].loc[df_train['Income'] >= median_income] = 1
df_train_2 = pd.get_dummies(df_train, columns=['Location']).drop(['Income','Location_Africa', 'Location_Asia Pa
```

2a) Demand estimation

First, Fill in the below function that fits a logistic regression to predict the probability of purchase at a price ($d(p)$). The logistic regression should now have more coefficients than before: 1 for each covariate, and any interactions (including interactions between price and covariates) that you wish to add. If you add more interactions, you may wish to add regularization.

In [27]:
```python
def fit_logistic_regression_demand_with_covariates(df):
    X = df.drop('Purchased', axis=1)
    Y = df['Purchased']
    model = LogisticRegression()
    model.fit(X,Y)
    return model

cov_model = fit_logistic_regression_demand_with_covariates(df_train_2)
```

Fill in the below function, that takes in a single price, covariates, and your trained model, and outputs the prediction for the demand $d(p)$ at that price. For example, one of the covariate inputs to the function can be `['NotAmerica', 'LowWealth']`.

In [28]:
```python
def get_prediction_logistic_covariates(fitted_model, price, covariates):
    input = np.zeros(3)
    input[0] = price

    if 'America' in covariates:
        input[2] = 1
    if 'HighWealth' in covariates:
        input[1] = 1

    return fitted_model.predict_proba(input.reshape(1,-1))[0,1]
```

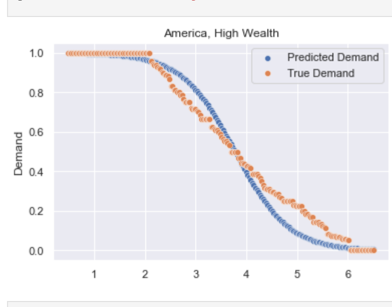
In [29]:
```python
test_demand_curve_America_HighWealth = pd.read_csv('test_demand_America_HighWealth2.csv')
test_demand_curve_NotAmerica_HighWealth = pd.read_csv('test_demand_NotAmerica_HighWealth2.csv')
test_demand_curve_America_LowWealth = pd.read_csv('test_demand_America_LowWealth2.csv')
test_demand_curve_NotAmerica_LowWealth = pd.read_csv('test_demand_NotAmerica_LowWealth2.csv')

true_America_HighWealth = get_matched_demand_vector(test_demand_curve_America_HighWealth, prices_to_predict)
true_NotAmerica_HighWealth = get_matched_demand_vector(test_demand_curve_NotAmerica_HighWealth, prices_to_predi
true_America_LowWealth = get_matched_demand_vector(test_demand_curve_America_LowWealth, prices_to_predict)
true_NotAmerica_LowWealth = get_matched_demand_vector(test_demand_curve_NotAmerica_LowWealth, prices_to_predict
```

Fit a model and get the predictions for each of the prices in `prices_to_predict` using your above function and each unique covariate combination.

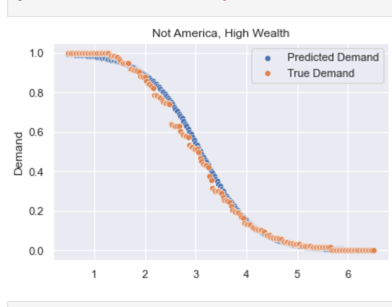
For each covariate combination, generate the same lineplot as in 1a and 1b (separately for each covariate combination). Also include the "true" test-time demand curve for the appropriate covariate combination (For example, `test_demand_NotAmerica_HighWealth`).

In [30]:
```python
pred_America_HighWealth = [get_prediction_logistic_covariates(cov_model, a, ['America','HighWealth']) for a in
pred_NotAmerica_HighWealth = [get_prediction_logistic_covariates(cov_model, a, ['NotAmerica','HighWealth']) for
pred_America_LowWealth = [get_prediction_logistic_covariates(cov_model, a, ['America','LowWealth']) for a in pr
pred_NotAmerica_LowWealth = [get_prediction_logistic_covariates(cov_model, a, ['NotAmerica','LowWealth']) for a
```

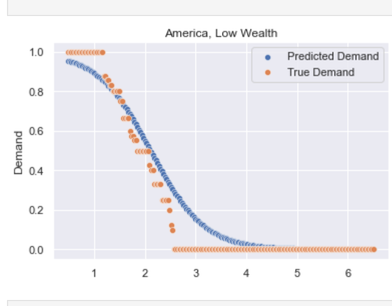
In [31]:
```python
sns.scatterplot(prices_to_predict,pred_America_HighWealth)
sns.scatterplot(prices_to_predict,true_America_HighWealth)
plt.ylabel('Demand')
plt.legend(['Predicted Demand','True Demand'])
plt.title('America, High Wealth');
```



In [32]:
```python
sns.scatterplot(prices_to_predict,pred_NotAmerica_HighWealth)
sns.scatterplot(prices_to_predict,true_NotAmerica_HighWealth)
plt.ylabel('Demand')
plt.legend(['Predicted Demand','True Demand'])
plt.title('Not America, High Wealth');
```



In [33]:
```python
sns.scatterplot(prices_to_predict,pred_America_LowWealth)
sns.scatterplot(prices_to_predict,true_America_LowWealth)
plt.ylabel('Demand')
plt.legend(['Predicted Demand','True Demand'])
plt.title('America, Low Wealth');
```



In [34]:
```python
sns.scatterplot(prices_to_predict,pred_NotAmerica_LowWealth)
sns.scatterplot(prices_to_predict,true_NotAmerica_LowWealth)
plt.ylabel('Demand')
plt.legend(['Predicted Demand','True Demand'])
plt.title('Not America, Low Wealth');
```

2b) Pricing

Now, use your code from 1c to output predicted optimal prices, predicted revenue, and and actual revenue using the test data curve, for each covariate combination.

In [67]:
```python
prices = np.zeros([2,4])
revenues = np.zeros([3,4])

prices[0,0], revenues[0,0] = get_revenue_maximizing_price_and_revenue(prices_to_predict, pred_America_HighWealt
prices[1,0], revenues[1,0] = get_revenue_maximizing_price_and_revenue(prices_to_predict, true_America_HighWealt
actual_revenue1 = true_America_HighWealth[[ind for ind,x in enumerate(prices_to_predict) if x == prices[0,0]][(
print('America, High Wealth, Predicted over True: Optimal Price, Revenue')
print(get_revenue_maximizing_price_and_revenue(prices_to_predict, pred_America_HighWealth))
print(get_revenue_maximizing_price_and_revenue(prices_to_predict, true_America_HighWealth))
print(actual_revenue1)

prices[0,1], revenues[0,1] = get_revenue_maximizing_price_and_revenue(prices_to_predict, pred_NotAmerica_HighWe
prices[1,1], revenues[1,1] = get_revenue_maximizing_price_and_revenue(prices_to_predict, true_NotAmerica_HighWe
actual_revenue2 = true_America_HighWealth[[ind for ind,x in enumerate(prices_to_predict) if x == prices[0,1]][(
print('Not America, High Wealth, Predicted over True: Optimal Price, Revenue')
print(get_revenue_maximizing_price_and_revenue(prices_to_predict, pred_NotAmerica_HighWealth))
print(get_revenue_maximizing_price_and_revenue(prices_to_predict, true_NotAmerica_HighWealth))
print(actual_revenue2)

prices[0,2], revenues[0,2] = get_revenue_maximizing_price_and_revenue(prices_to_predict, pred_America_LowWealth
prices[1,2], revenues[1,2] = get_revenue_maximizing_price_and_revenue(prices_to_predict, true_America_LowWealth
actual_revenue3 = true_America_HighWealth[[ind for ind,x in enumerate(prices_to_predict) if x == prices[0,2]][(
print('America, Low Wealth, Predicted over True: Optimal Price, Revenue')
print(get_revenue_maximizing_price_and_revenue(prices_to_predict, pred_America_LowWealth))
print(get_revenue_maximizing_price_and_revenue(prices_to_predict, true_America_LowWealth))
print(actual_revenue3)

prices[0,3], revenues[0,3] = get_revenue_maximizing_price_and_revenue(prices_to_predict, pred_NotAmerica_LowWea
prices[1,3], revenues[1,3] = get_revenue_maximizing_price_and_revenue(prices_to_predict, true_NotAmerica_LowWea
actual_revenue4 = true_America_HighWealth[[ind for ind,x in enumerate(prices_to_predict) if x == prices[0,3]][(
print('Not America, Low Wealth, Predicted over True: Optimal Price, Revenue')
print(get_revenue_maximizing_price_and_revenue(prices_to_predict, pred_NotAmerica_LowWealth))
print(get_revenue_maximizing_price_and_revenue(prices_to_predict, true_NotAmerica_LowWealth))
print(actual_revenue4)
```

```
America, High Wealth, Predicted over True: Optimal Price, Revenue
(2.972361809045226, 2.4424682577206087)
(3.2738693467336684, 2.182579564489112)
2.1231155778894473
Not America, High Wealth, Predicted over True: Optimal Price, Revenue
(2.42964824120603, 1.8972428666515946)
(2.4899497487437188, 1.8485990558854881)
2.159687325516471
America, Low Wealth, Predicted over True: Optimal Price, Revenue
(1.7060301507537687, 1.1685625179462906)
(1.4949748743718594, 1.1959798994974875)
1.7060301507537687
Not America, Low Wealth, Predicted over True: Optimal Price, Revenue
(1.2537688442211055, 0.7346870013657666)
(1.1331658291457285, 0.8173655156407035)
1.2537688442211055
```

Suppose each of the 4 covariate combinations make up an equal part of the population. What would be the resulting revenue achieved at test time using your above prices?

In [68]:
```python
print(np.mean(revenues[0,:]))
print(np.mean(revenues[1,:]))
print(np.mean([actual_revenue1,actual_revenue2,actual_revenue3,actual_revenue4]))
```

```
1.5607401609210652
1.5111310088781977
1.8106504745951981
```

The revenue we calculated is in proportion of the population. Thus, the resulting revenue is simply the average of the revenues multiplied by the total population because the populations are the same. These are: 1.56 predicted and 1.51 true (from predicting optimal price and revenue from true demand). The revenue from the actual revenue prediction would be 1.81.

Comment on your outputs in no more than 3 sentences. What is the effect of using different prices for differerent covariate groups?

Different covariate groups exhibit different purchasing decisions. Setting a reasonably low price for one group could mean a prohibitively high price for another, resulting in near-zero demand. Thus, when possible, pricing should be customized and optimized for each covariate group.

Problem 3: Pricing under capacity constraints

Now, we are going to build up to implementing the Bellman equation approach discussed in class, to price a single copy of an item to be sold over T time periods. For simplicity, we will use "test_demand_curve" as $d(p)$.

In [38]:
```python
price_options = list(test_demand_curve.Price)
demand_predictions = list(test_demand_curve.Demand_at_price)
```

3a) Implementing one step of the Bellman equation

Recall the "Bellman equation" taught in class. Suppose we have 1 copy of the item at time t. Then, my expected revenue given I price the item at p_t is:

$$V_t = d(p_t)p_t + (1 - d(p_t))V_{t+1}$$

Implement the following function that returns optimal price p_t and the resulting value V_t, given the demand curve and V_{t+1}.

In [39]:
```python
def get_single_step_revenue_maximizing_price_and_revenue(Vtplus1, price_options, demand_predictions):
    expected_revenues = []

    for option in range(len(price_options)):
        expected_revenue = demand_predictions[option]*price_options[option] + (1 - demand_predictions[option])
        expected_revenues.append(expected_revenue)

    best_option = np.argmax(expected_revenues)
    return price_options[best_option], expected_revenues[best_option]
```

3b) Calculating prices over time

Implement the following function that returns a list (of length T) of optimal prices for each time period, and a expected revenue number for those prices.

Hint: your function should loop through each time step, *starting* at time $t = T - 1$ (the last time period, since the first time period is time $t = 0$). Each iteration of the loop should call the function from part 3a. Recall that we can define $V_T = 0$, since even if the item is unsold at time T, we have finished trying to sell it.

In [40]:
```python
def get_prices_over_time_and_expected_revenue(prices, demand_predictions, T):
    optimal_prices = []
    optimal_revenues = []

    Vt = 0

    for t in range(T-1,-1, -1):
        best_price, best_revenue = get_single_step_revenue_maximizing_price_and_revenue(Vt, prices, demand_pred
        optimal_prices.append(best_price)
        optimal_revenues.append(best_revenue)
        Vt = best_revenue

    return optimal_prices, optimal_revenues
```

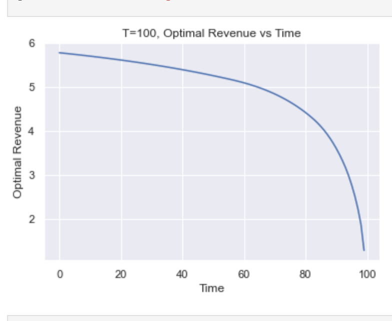
Plot a line plot for your optimal prices over time when $T = 100$ and $T = 10$. Also print out the expected revenue using these prices and for each T.

In [41]:
```python
optimal_prices_t100, optimal_revenues_t100 = get_prices_over_time_and_expected_revenue(price_options, demand_pr
optimal_prices_t10, optimal_revenues_t10 = get_prices_over_time_and_expected_revenue(price_options, demand_pred
time_100 = np.arange(99,-1,-1)
time_10 = np.arange(9,-1,-1)
```

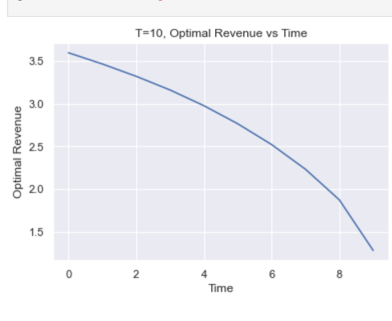
In [42]:
```python
sns.lineplot(time_100, optimal_prices_t100)
plt.ylabel('Optimal Price')
plt.xlabel('Time')
plt.title('T=100, Optimal Price vs Time');
```

In [43]:
```python
sns.lineplot(time_10, optimal_prices_t10)
plt.ylabel('Optimal Price')
plt.xlabel('Time')
plt.title('T=10, Optimal Price vs Time');
```

In [44]:
```python
sns.lineplot(time_100, optimal_revenues_t100)
plt.ylabel('Optimal Revenue')
plt.xlabel('Time')
plt.title('T=100, Optimal Revenue vs Time');
```



In [45]:
```python
sns.lineplot(time_10, optimal_revenues_t10)
plt.ylabel('Optimal Revenue')
plt.xlabel('Time')
plt.title('T=10, Optimal Revenue vs Time');
```



3c) [Bonus, 3 points] Prices over time with multiple copies

Now, suppose that you have K copies of the item, that you must sell over a time period T. Implement the two-dimensional dynamic program as discussed in class. Plot a line plot where the X axis is time as in 3b, but now you have K lines where each line indicates the price at time T if you have K items left.

Hint: As in 3a and 3b, you may find it useful to first optimize the price $p_{t,k}$ given the values $V_{t+1,k}$, $V_{t+1,k-1}$. Then, have a 2nd function that loops through t, k in an appropriate order.

In [46]:
```python
def capacity_get_single_step_revenue_maximizing_price_and_revenue(Vtplus1k, Vtplus1kless1, price_options, demar
    expected_revenues = []

    for option in range(len(price_options)):
        expected_revenue = demand_predictions[option]*(price_options[option] + Vtplus1kless1) + (1 - demand_pre
        expected_revenues.append(expected_revenue)

    best_option = np.argmax(expected_revenues)
    return price_options[best_option], expected_revenues[best_option]
```

In [47]:
```python
def capacity_get_prices_over_time_and_expected_revenue(prices, demand_predictions, T, K):
    optimal_prices = np.zeros([K+1, T+1])
    optimal_revenues = np.zeros([K+1, T+1])

    for t in range(T-1,-1, -1):
        for k in range(K,-1,-1):

            if k == 1:
                optimal_revenues[k-1,t] = 0

            best_price, best_revenue = capacity_get_single_step_revenue_maximizing_price_and_revenue(optimal_re
            optimal_revenues[k,t] = best_revenue
            optimal_prices[k,t] = best_price

    return optimal_prices, optimal_revenues
```

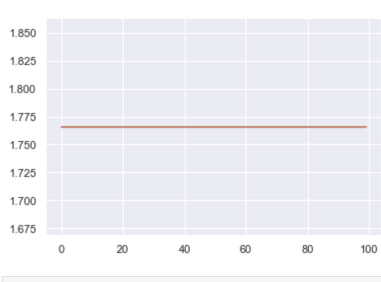
In [48]:
```python
capacity_get_single_step_revenue_maximizing_price_and_revenue(10, 20, price_options, demand_predictions)
```

Out[48]:
```
(0.56, 20.40238805970149)
```

In [49]:
```python
price_matrix, revenue_matrix = capacity_get_prices_over_time_and_expected_revenue(price_options, demand_predict
```

In [50]:
```python
sns.lineplot(time_100, price_matrix[8,0:100])
sns.lineplot(time_100, price_matrix[4,0:100])
```

Out[50]: <AxesSubplot:>



In [ ]: