Optimization Models and Applications

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ISE 316, Fall 2014, Lecture 1

September 3, 2014

Outline

- Course Information
- Example of an Optimization Problem
- Terminology and Formal Formulation
- General Optimization Problem
- Solving of an Optimization Problem in Excel

Part I.
Course Information

What is it about?

Theory

- Linear Programming (LP)
- Integer Programming (IP)
- Nonlinear Programming (NP)
- Robust Optimization, Stochastic Programming and Multi-criteria Programming

Practice

- Modeling
- Solving problems using Excel Solver and AMPL

1st rule

1st rule

There are no stupid questions!

Evaluation

Homework	25%
Quiz #1	20%
Quiz #2	25%
Case study	20%
Class participation	10%

Homework

- Several homework
- **All must be completed to receive a grade for the course**
- Require modeling and solving simple problems
- Homework will be penalized for each day they are late
- After solutions are released, they will not be accepted
- **No exceptions! Also, no exception to the no-exception rule!**

Case study

- Is a hands-on experience on real Optimization problems
- **It must be completed to receive a grade**
- Groups of three-four people study an Optimization problem, propose a model and solve it using a tool of their choice
- The result is a **short report on the whole experience**
- There will be an informal discussion after the beginning of the study and another after the report is due

Which skills I will gain?

- Report writing
- Group work
- Presentation
- Consultancy

"This is something I can talk about in job interviews!"

Textbooks

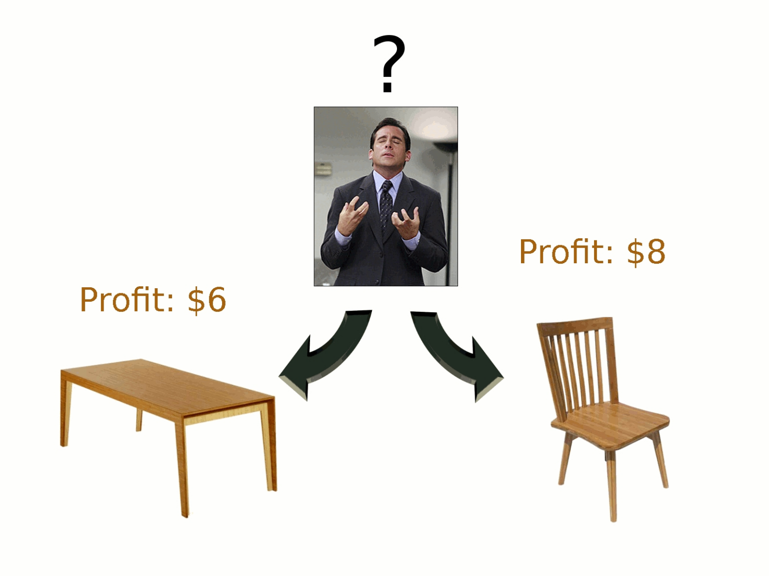
- Select chapters of *Introduction to Operations Research* by F.S. Hiller and G.J. Lieberman
- Select chapters of *Introduction to Mathematical Programming: Applications and Algorithms* by W.L. Winston and M. Venkataramanan
- Select chapters of *Operations Research: Applications and Algorithms* by Wayne L. Winston

Software

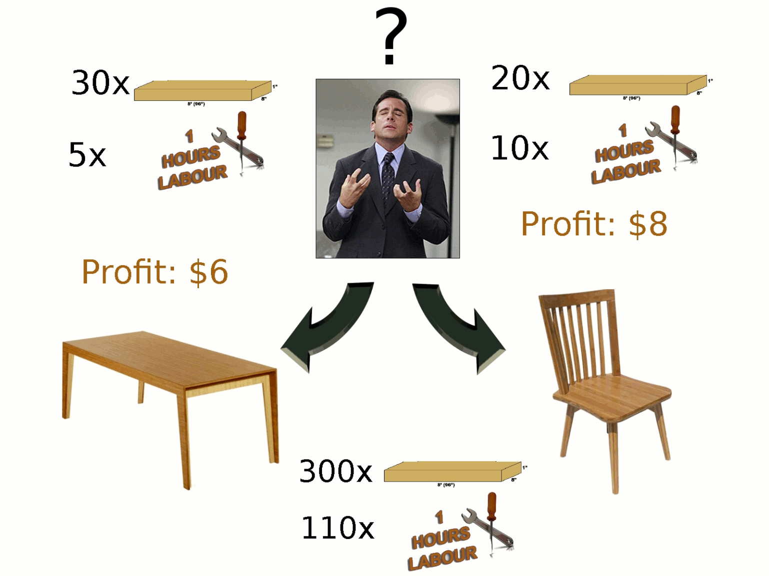
- I will use **Excel Solver** and **AMPL**
- You are allowed to use whatever language you prefer
- Alternatives are GAMS and Mosel
- Online option http://neos.mcs.anl.gov/neos/

Part II.
Motivation and Example

What would you produce?



And now?


?
30x
5x
1 HOURS LABOUR
Profit: $6
20x
10x
1 HOURS LABOUR
Profit: $8
300x
110x
1 HOURS LABOUR

Part III. Terminology and Formal Formulation

Variables

- x_1 - how many chairs we are going to produce
- x_2 - how many tables we are going to produce

Objective function

One chair gives us profit $8 and one table will gives us $6.
Our profit will be

$$6x_1 + 8x_2$$

Material constraint

for one chair we need 20 and for one table we need 30 pieces of wood and our budget is 300 pieces of wood

$$20x_1 + 30x_2 \leq 300$$

Labour constraint

for one chair we need 10 and for one table we need 5 hours of work. Budget is 110 hours of work

$$10x_1 + 5x_2 \leq 110$$

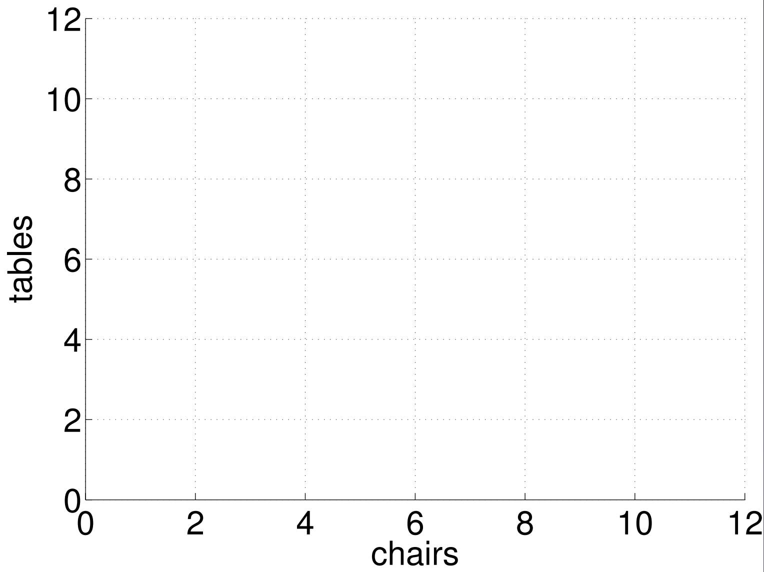
Final optimization formulation

Optimization Problem

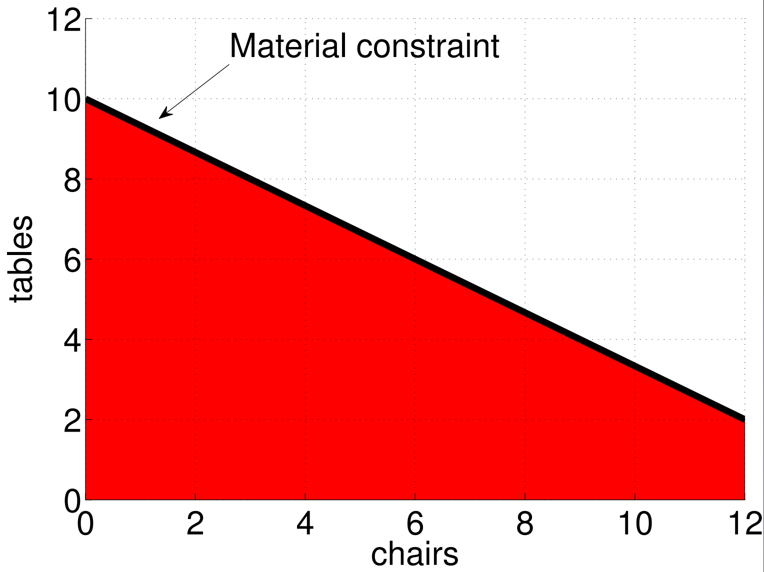
$$\begin{array}{rl} \text{maximize} & 6\ x_1 + \ 8x_2 \\ \text{subject to} & 20x_1 + 30x_2 \le 300 \\ & 10x_1 + \ 5x_2 \le 110 \\ & x_1 \ge 0 \\ & x_2 \ge 0 \end{array}$$

- Objective function is linear (convex)
- All constraints are linear (convex)
- $\Rightarrow$ this optimization problem is linear (**Linear Program**)

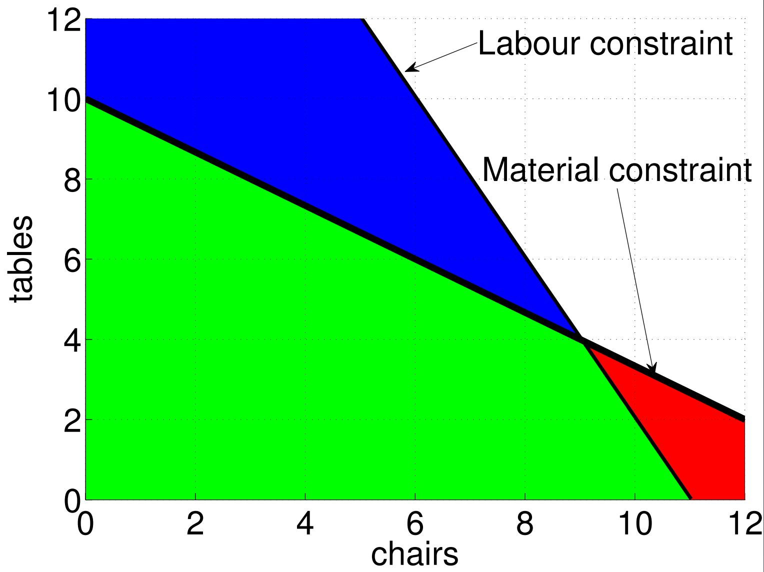
Solution



Solution

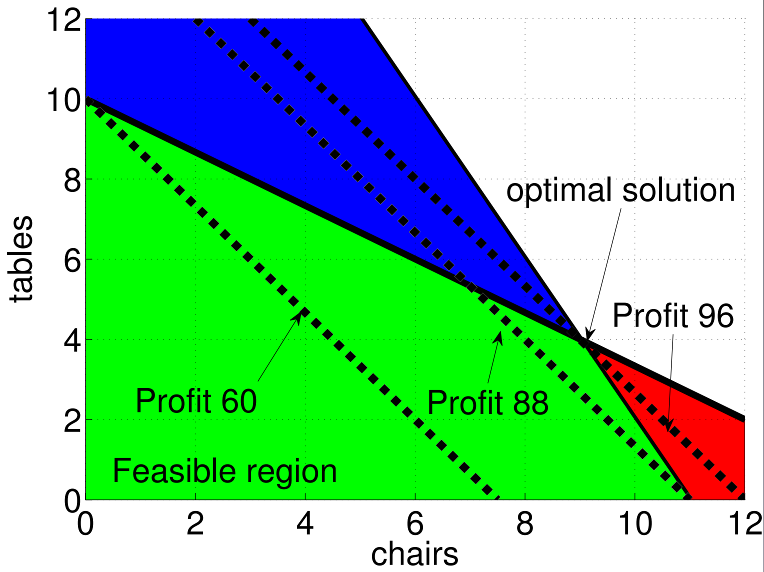


Solution



Labour constraint
Material constraint
tables
chairs
0
2
4
6
8
10
12

Solution



12
10
8
6
4
2
0
0
2
4
6
8
10
12
tables
chairs
optimal solution
Profit 96
Profit 60
Profit 88
Feasible region

Part IV.
General Optimization Problem, Convexity

Consider the vector $x \in \mathbf{R}^n$ of optimization variables

Optimization Problem

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{subject to} \quad & f_1(x) \le b_1 \\ & f_2(x) \le b_2 \\ & \vdots \\ & f_m(x) \le b_m \end{aligned}$$

We denote by Ω **feasible region**:

$$\Omega = \{x \in \mathbf{R}^n \ : \ f_1(x) \le b_1, \dots, f_m(x) \le b_m\}$$

Remember!

If all functions $f_i(x)$ ($i = 0, 1, \dots, m$) are convex, then the optimization problem is (usually) easy to solve (by a computer).

Convex set

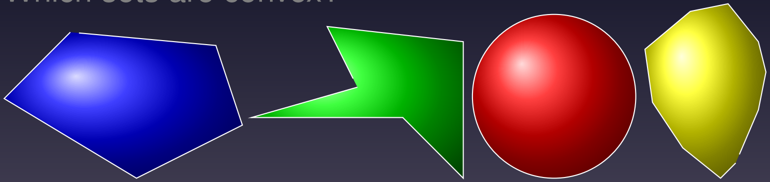
Definition

Set $S \subset \mathbf{R}^n$ is convex, if

$$\forall x_1, x_2 \in S, \forall \lambda \in [0,1] \Rightarrow (1-\lambda)x_1 + \lambda x_2 \in S$$

(if you take any two points from S then the whole line segment has to be in S)

Which sets are convex?

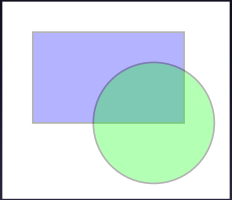


Set of real numbers $\mathbf{R}$ Set of Integers

Set operation

Preserving convexity

Intersection of convex sets is convex set!



Adding a new constraint in optimization problem is actually intersection of original feasible set with a new set.

Convex function

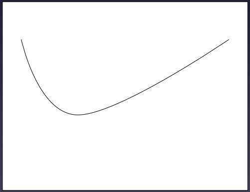
Definition

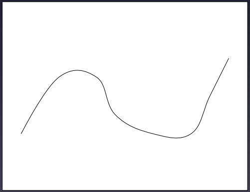
Function $f : \mathbf{R}^n \to \mathbf{R}$ is convex iff $\forall x, y \in dom(f), \forall \lambda \in [0, 1]$

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y)$$

(if you connect any two points on f then the whole line segment is above the function)

Convex vs. nonconvex





Operations preserving convexity

Multiplying by a positive constant

Let $a > 0$ and f be a convex function. Then $a \cdot f$ is also convex

Summing functions

If functions $\{f_i(x)\}_{i=1}^m$ are convex then also

$$\sum_{i=1}^{m} f_i(x)$$

is also convex

Linear function

A linear function $f(x) = a_1 x_1 + \cdots + a_n x_n$ is convex for any $a_i \in \mathbf{R}$

Part V.

Solving of an Optimization Problem in Excel

Variables

x1	9	number of chairs produced
x2	4	number of tables produced

Optimization problem

objective function

6 * x1 + 8 * x2	=	86			
constraints					
20 * x1 + 30 * x2	=	300	<=	300	Material constraint
10 * x1 + 5 * x2	=	110	<=	110	Labour constraint
x1	=	9	>=	0	Non-negative constraint
x2	=	4	>=	0	Non-negative constraint

The diet model

Task

Find the cheapest diet when restricted to the following food options?

food	potato	eggs	milk	chicken
cost (cents per 100g)	15	30	17	92

Nutritions and the daily requirement

	Yield (per 100g)				Daily requirement
	potato	eggs	milk	chicken	
Energy (kcal)	390	210	60	198	2000
Protein (g)	13	19	3	18	56
Sugar (g)	60	5	10	0	17

Final optimization formulation

Variables:

x_1 - amount of potato, x_2 - amount of eggs, x_3 - amount of milk and x_4 - amount of chicken

Optimization Problem

$$
\begin{array}{ll}
\text{minimize} & 15x_1 + 30x_2 + 17x_3 + 92x_4 \\
\text{subject to} & 390x_1 + 210x_2 + 60x_3 + 198x_4 \geq 2000 \\
& 13x_1 + 19x_2 + 3x_3 + 18x_4 \geq 56 \\
& 60x_1 + 5x_2 + 10x_3 + 0x_4 \geq 17 \\
& x_1 \geq 0 \\
& x_2 \geq 0 \\
& x_3 \geq 0 \\
& x_4 \geq 0
\end{array}
$$

Variables		
x1	**5.1282051282**	amount of potato
x2	**0**	amount of eggs
x3	**0**	amount of milk
x4	**0**	amount of chicken

Optimization problem

objective function

15 x1+30 x2+17 x3+92 x4	=	76.9231			
constraints					
390 x1+210 x2+60 x3+ 198 x4	=	2000	>=	2000	Energy
13 x1+19 x2+3 x3+18 x4	=	66.6667	>=	56	Protein
60 x1+5 x2+10 x3+0 x4	=	307.692	>=	17	Sugar
x1	=	5.12821	>=	0	Non-negative constraint
x2	=	0	>=	0	Non-negative constraint
x3	=	0	>=	0	Non-negative constraint
x4	=	0	>=	0	Non-negative constraint

Post optimization modification

We do not want to get Diabetes, let us add limit on sugar!

Variables

x1	1.0328638498	amount of potato
x2	7.6056338028	amount of eggs
x3	0	amount of milk
x4	0	amount of chicken

Optimization problem

objective function

15 x1+30 x2+17 x3+92 x4	=	243.662			
constraints					
390 x1+210 x2+60 x3+ 198 x4	=	2000	>=	2000	Energy
13 x1+19 x2+3 x3+18 x4	=	157.934	>=	56	Protein
60 x1+5 x2+10 x3+0 x4	=	100	>=	17	Sugar
x1	=	1.03286	>=	0	Non-negative constraint
x2	=	7.60563	>=	0	Non-negative constraint
x3	=	0	>=	0	Non-negative constraint
x4	=	0	>=	0	Non-negative constraint
60 x1+5 x2+10 x3+0 x4	=	100	<=	100	Sugar

Attention!

There is a homework due on 9th of September at 9.20am!