Optimization Models and Applications

Martin Takáč

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Outline

- Linear Programming - General (Matrix) Form
- Duality
- Interpretation of Dual Variables and Sensitivity Analysis
- Problems that can be Written as Linear Programs
- Solving Easy LP Problems in AMPL
- Convexity and More
- Relaxations and Lower Bounds

Part I.
Linear Programming - General (Matrix) Form

Linear Programming (LP)

$$
\begin{aligned}
\text{minimize} \quad & f_0(x) \\
\text{subject to} \quad & f_1(x) \le b_1 \\
& f_2(x) \le b_2 \\
& \vdots \\
& f_m(x) \le b_m
\end{aligned}
$$

Linear Programming (LP)

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& \vdots \\
& f_m(x) \le b_m
\end{aligned}
$$

For LP all functions are linear. We define the LP in standard form

$$
\begin{aligned}
\text{maximize} \quad & c_1 \; x_1 + \cdots + c_n \; x_n \\
\text{subject to} \quad & a_{1,1} \; x_1 + \cdots + a_{1,n} \; x_n \le b_1 \\
& a_{2,1} \; x_1 + \cdots + a_{2,n} \; x_n \le b_2 \\
& \vdots \\
& a_{m,1} x_1 + \cdots + a_{m,n} x_n \le b_m \\
& x \ge 0
\end{aligned}
$$

LP in Matrix Form

We can write LP using $\sum$ notation

$$
\begin{aligned}
\text{maximize} \quad & \sum_{i=1}^{n} c_i x_i \\
\text{subject to} \quad & \sum_{i=1}^{n} a_{j,i} x_i \leq b_j \qquad j = 1, 2, \dots, m \\
& x \geq 0
\end{aligned}
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& x \geq 0
\end{aligned}
$$

or in a matrix form

$$
\begin{aligned}
\text{maximize} \quad & \mathbf{c} \cdot \mathbf{x} \\
\text{subject to} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\
& \mathbf{x} \geq 0
\end{aligned}
$$

LP in Matrix Form - Example

$$
\begin{array}{ll}
\text{minimize} & 15\ x_1 + 30x_2 + 17x_3 + 92x_4 \\
\text{subject to} & 390x_1 + 210x_2 + 60x_3 + 198x_4 \geq 2000 \\
& 13\ x_1 + 19x_2 + 3\ x_3 + 18x_4 \geq 56 \\
& 60\ x_1 + 5x_2 + 10x_3 + 0x_4 \geq 17 \\
& x_i \geq 0
\end{array}
$$

In a matrix form

$$
\begin{array}{lrcl}
\color{red}{\text{maximize}} & \begin{pmatrix} -15 & -30 & -17 & -92 \end{pmatrix} & \mathbf{x} & \\
\text{subject to} & -\begin{pmatrix} 390 & 210 & 60 & 198 \\ 13 & 19 & 3 & 18 \\ 60 & 5 & 10 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} & \leq - & \begin{pmatrix} 2000 \\ 56 \\ 17 \end{pmatrix} \\
& & \mathbf{x} \geq & 0
\end{array}
$$

Part II.
Duality

Duality

For each LP there is a dual LP

Primal LP (P)		Dual LP (D)	
maximize	$\mathbf{c} \cdot \mathbf{x}$	minimize	$\mathbf{b} \cdot \mathbf{y}$
subject to	$\mathbf{A}\mathbf{x} \le \mathbf{b}$	subject to	$\mathbf{y}^T \mathbf{A} \ge \mathbf{c}^T$
	$\mathbf{x} \ge 0$		$\mathbf{y} \ge 0$

or equivalently

minimize	$\mathbf{b} \cdot \mathbf{y}$
subject to	$\mathbf{A}^T \mathbf{y} \ge \mathbf{c}$
	$\mathbf{y} \ge 0$

Primal-Dual Properties

Let

- (P) and (D) are feasible
- $\tilde{\mathbf{x}}$ is any feasible and $\mathbf{x}^*$ is optimal to (P)
- $\tilde{\mathbf{y}}$ and $\mathbf{y}^*$ is optimal to (D)

Then

- strong duality $c^T x^* = b^T y^*$
- weak duality $c^T \tilde{x} \leq b^T \tilde{y}$ *proof on black board*
- complementarity conditions

$$(Ax^* - b)^T y^* = 0$$
$$(A^T y^* - c)^T x^* = 0$$

Primal-Dual Properties

- if (P) is not feasible then (D) is unbounded
- if (D) is not feasible then (P) is unbounded

Example

(P)

$$
\begin{array}{ll}
\text{maximize} & x_1 \\
\text{subject to} & x_1 \leq -1 \\
 & x_1 \geq \ \ 0
\end{array}
$$

Primal-Dual Properties

- if (P) is not feasible then (D) is unbounded
- if (D) is not feasible then (P) is unbounded

Example

(P)		(D)	
maximize	x_1	minimize	$-y_1$
subject to	$x_1 \leq -1$	subject to	$y_1 \geq 1$
	$x_1 \geq 0$		$y_1 \geq 0$

Part III.
Interpretation of Dual Variables and Sensitivity Analysis

Production Problem

- The WYNDOR GLASS CO. produces high-quality glass products, including windows and glass doors
- it has three plants
  1. Aluminum frames and hardware are made in Plant 1
  2. frames are made in Plant 2
  3. Plant 3 produces the glass and assembles the products

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  Because of declining earnings, top management has decided to revamp the company's product line. Unprofitable products are being discontinued, releasing production capacity to launch two new products having large sales potential:
  1. Product 1: An 8-foot glass door with aluminum framing

     Product 1 requires some of the production capacity in Plants 1 and 3, but none in Plant 2.
  2. Product 2: A 4×6 foot double-hung wood-framed window

     Product 2 needs only Plants 2 and 3.

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2. Product 2: A 4×6 foot double-hung wood-framed window

   Product 2 needs only Plants 2 and 3.

The marketing division has concluded that the company could sell as much of either product as could be produced by these plants.

Production Problem

Task for OR consultant

Determine what the production rates should be for the two products in order to **maximize their total profit**, subject to the **restrictions imposed by the limited production capacities**.

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Input:

Product 1 requires 1 hour of work in Plant 1 and 3 hours of work in Plant 3

Product 2 requires 2 hours of work in Plant 2 and 2 hours of work in Plant 3

We have available Plant 1 for 4 hours, Plan 2 for 12 hours and Plant 3 for 18 hours

The profit from Product 1 is $3,000 and from Product 2 is $5,000

Solution

x_1 - Amount of Product 1
x_2 - Amount of Product 2

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x_1 - Amount of Product 1

x_2 - Amount of Product 2

$$
\begin{array}{ll}
\text{maximize} & 3x_1 + 5x_2 \\
\text{subject to} & x_1 \leq 4 \\
& 2x_2 \leq 12 \\
& 3x_1 + 2x_2 \leq 18 \\
& x_i \geq 0
\end{array}
$$

Solution

x_1 - Amount of Product 1

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\end{array}
$$

$$
\begin{array}{ll}
\text{minimize} & 4y_1 + 12y_2 + 18y_3 \\
\text{subject to} & y_1 + 3y_3 \geq 3 \\
& 2y_2 + 2y_3 \geq 5 \\
& y_i \geq 0
\end{array}
$$

Solution

x_1 - Amount of Product 1

x_2 - Amount of Product 2

$$\begin{array}{ll} \text{maximize} & 3x_1+5x_2 \\ \text{subject to} & x_1 \le 4 \\ & 2x_2 \le 12 \\ & 3x_1+2x_2 \le 18 \\ & x_i \ge 0 \end{array}$$

$$\begin{array}{ll} \text{minimize} & 4y_1+12y_2+18y_3 \\ \text{subject to} & y_1 + 3y_3 \ge 3 \\ & 2y_2 + 2y_3 \ge 5 \\ & y_i \ge 0 \end{array}$$

The optimal solution to (P) and (D) are

$$x^* = (2 \quad 6)^T$$

and

$$y^* = (0 \quad \tfrac{3}{2} \quad 1)^T$$

Interpretation of Dual Variables

- The dual variable y_i is interpreted as the contribution to profit per unit of resource i.
- In other words, the y_i values are just the shadow prices.

Definition

The **shadow price** of a constraint is the amount of change in the optimal objective as the RHS of that constraint is increased by 1 unit, and the rest of the data is kept constant.

Part IV.
Problems that can be Written as Linear Programs

Linear Regression

Given a data set $\{y_i,\, t_{i1}, \ldots, t_{in}\}_{i=1}^{m}$ of m statistical units, a linear regression model assumes that the relationship between the dependent variable y_i and the n-vector of regressors t_i is linear.
To find the relationship we can solve

$$\min_{x_1,\ldots,x_n} \sum_{i=1}^{m} (y_i - x_1 t_{i1} - \cdots - x_n t_{in})^2$$

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$$\min_{x_1,\dots,x_n} \sum_{i=1}^{m} (y_i - x_1 t_{i1} - \dots - x_n t_{in})^2$$

Two common modification:

$$\min_{x_1,\dots,x_n} \sum_{i=1}^{m} |y_i - x_1 t_{i1} - \dots - x_n t_{in}|$$

and

$$\min_{x_1,\dots,x_n} \max_{i} |y_i - x_1 t_{i1} - \dots - x_n t_{in}|$$

Rewriting ℓ_1 as LP

$$\min_{x_1,\dots,x_n} \sum_{i=1}^{m} |y_i - x_1 t_{i1} - \cdots - x_n t_{in}|$$

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We introduce auxiliary variables $z_1, \dots, z_m$

Rewriting ℓ_1 as LP

$$\min_{x_1,\dots,x_n} \sum_{i=1}^{m} |y_i - x_1 t_{i1} - \cdots - x_n t_{in}|$$

We introduce auxiliary variables $z_1, \dots, z_m$ The final LP is

$$\begin{array}{rl} \text{minimize in } x \in \mathbf{R}^n \text{ and } z \in \mathbf{R}^m & \displaystyle\sum_{i=1}^{m} z_i \\ \text{subject to} & y_i - (x_1 t_{i1} + \cdots + x_n t_{in}) \le z_i \\ & y_i - (x_1 t_{i1} + \cdots + x_n t_{in}) \ge -z_i \end{array}$$

Rewriting ℓ_∞ as LP

$$\min_{x_1,\ldots,x_n} \max_i |y_i - x_1 t_{i1} - \cdots - x_n t_{in}|$$

Rewriting ℓ_∞ as LP

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Rewriting ℓ_∞ as LP

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$$\begin{array}{rl} \text{minimize in } x \in \mathbf{R}^n \text{ and } z \in \mathbf{R} & z \\ \text{subject to} & y_i - (x_1 t_{i1} + \cdots + x_n t_{in}) \le z \\ & y_i - (x_1 t_{i1} + \cdots + x_n t_{in}) \ge -z \end{array}$$

Part V.
Solving Easy LP Problems in AMPL

$$
\begin{array}{ll}
\text{maximize} & 3x_1+5x_2 \\
\text{subject to} & x_1 \leq 4 \\
& 2x_2 \leq 12 \\
& 3x_1+2x_2 \leq 18 \\
& x_i \geq 0
\end{array}
$$

AMPL Model

```
var X1 >=0;
var X2 >=0;
maximize Profit: 3*X1+5*X2;
subject to Plant1: X1 <= 4;
subject to Plant2: 2*X2 <= 12;
subject to Plant3: 3*X1+2*X2 <= 18;
```

The diet model

Task

Find the cheapest diet when restricted to the following food options?

food	potato	eggs	milk	chicken
cost (cents per 100g)	15	30	17	92

Nutritions and the daily requirement

	Yield (per 100g)				Daily requirement
	potato	eggs	milk	chicken	
Energy (kcal)	390	210	60	198	2000
Protein (g)	13	19	3	18	56
Sugar (g)	60	5	10	0	17

Final optimization formulation

Variables:

x_1 - amount of potato, x_2 - amount of eggs, x_3 - amount of milk and x_4 - amount of chicken

Optimization Problem

$$
\begin{array}{rl}
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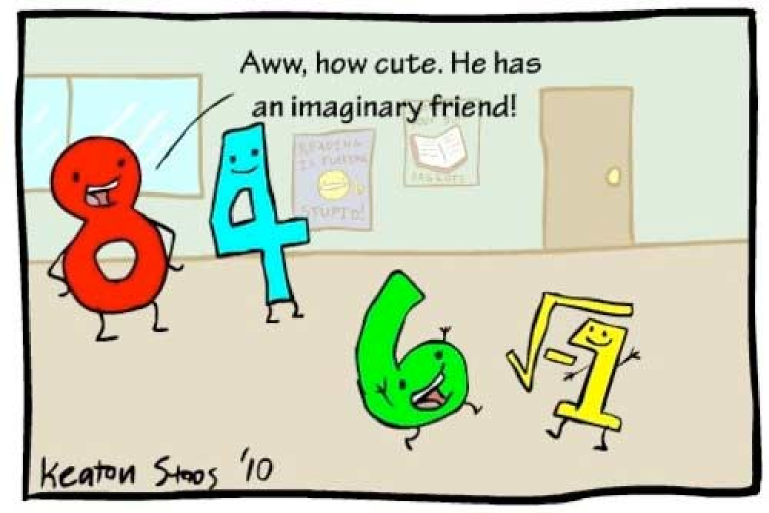
& x_1 \geq 0 \\
& x_2 \geq 0 \\
& x_3 \geq 0 \\
& x_4 \geq 0
\end{array}
$$

AMPL Model

```
var P >=0;
var E >=0;
var M >=0;
var C >=0;

minimize Cost: 15*P+30*E+17*M+92*C;

subject to Energy: 390*P+210*E+60*M+198*C >= 2000;
subject to Protein: 13*P+ 19*E+ 3*M+ 18*C >=   56;
subject to Sugar:   60*P+  5*E+10*M       >=   17;
```


Aww, how cute. He has an imaginary friend!
Keaton Stoos '10

Part VI.
Convexity and More

Local and Global Optima

Consider an optimization problem

$$\min_{x \in \Omega} f(x) \tag{1}$$

Point $\tilde{x}$ is **local optimum** if there exists $r > 0$ such that

$$\forall y \in \Omega \cap B_r(\tilde{x}) : f(y) \geq f(\tilde{x})$$

PS:

$$B_r(\tilde{x}) = \{x \in \mathbf{R}^n \ : \ \|x - \tilde{x}\| \leq r\}$$

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Point x^* is **global optimum** if

$$\forall y \in \Omega : f(y) \geq f(x^*)$$

Convex Problem

Theorem

If Ω is convex and $f(x)$ is convex then **every** *local optimum is a global optimum!*

Proof.

on black board $\square$

Convex Problem

Theorem

If Ω is convex and $f(x)$ is convex then **every** *local optimum is a global optimum!*

Proof.

on black board $\square$

Remember (again:)

Convex problems ARE EASY!

Sufficient Condition for Convexity

Let function $f \in C^2$

Theorem

if $f''(x) \succeq 0$ for all $x \in \Omega$ then f is convex on Ω!

What does $\succeq$ mean?

Matrix $A \in \mathbf{R}^{n \times n} \succeq 0$ (is positive semi-definite) if for all $x \in \mathbf{R}^n$ it holds

$$x^T A x \geq 0$$

How to find out if $A \succeq 0$?

PS:

$$f''(x) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}$$

Which Function if Convex?

$$f(x_1, x_2) = x_1^2 + 3x_2$$
$$f(x_1, x_2) = 3x_1^2 + 9x_2^2$$
$$f(x_1, x_2) = x_1^2 - 10x_1x_2 + x_2^2$$
$$f(x_1, x_2) = x_1x_2$$
$$f(x_1, x_2) = x_1^2 - x_2^2$$
$$f(x_1) = \sin^2(x_1) + \cos^2(x_1)$$

When is the Feasible Region Convex?

Feasible region:

$$\Omega = \{x \in \mathbf{R}^n \ : \ f_1(x) \leq b_1, \ldots, f_m(x) \leq b_m\}$$

When is the Feasible Region Convex?

Feasible region:

$$\Omega = \{x \in \mathbf{R}^n \ : \ f_1(x) \le b_1, \dots, f_m(x) \le b_m\}$$

Answer:

$$\Omega = \Omega_1 \cap \Omega_2 \cap \cdots \cap \Omega_m$$

where

$$\Omega_j = \{x \in \mathbf{R}^n \ : \ f_j(x) \le b_j\}$$

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Feasible region:

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Answer:

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On which requirement on $f_j(x)$ will be the set Ω_j convex?

Part VII. **Relaxations and Lower Bounds**

Consider sets $\Omega \subseteq \Omega' \subseteq \mathbf{R}^n$ and function $f : \Omega' \to \mathbf{R}$

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$$\min_{x \in \Omega} f(x) \tag{OPT}$$

or

$$\min_{x \in \Omega'} f(x) \tag{OPT'}$$

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Terminology

(OPT') is RELAXATION of (OPT) *typically, Ω' arises if you ignore some constraints*

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For **MINIMIZATION** problems, the RELAXATION gives us LOWER-BOUND.

- is $\{\min f(x), x \in [0,1]\}$ relaxation of $\{\min f(x), x \in [-2,1]\}$?
- is $\{\min f(x), x \in [-3,1]\}$ relaxation of $\{\min f(x), x \in [-1,0]\}$?
- is $\{\min f(x), g(x) \leq b\}$ relaxation of $\{\min f(x), g(x) \leq b-1\}$, if $g(x)$ is convex?
- is $\{\min f(x), g(x) \leq b\}$ relaxation of $\{\min f(x), g(x) \leq b+1\}$, if $g(x)$ is convex?

- is $\{\min f(x), x \in [0,1]\}$ relaxation of $\{\min f(x), x \in [-2,1]\}$? **NO**
- is $\{\min f(x), x \in [-3,1]\}$ relaxation of $\{\min f(x), x \in [-1,0]\}$?
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PS: Relaxation can be even if you replace objective function f by $\tilde{f}$ if

$$\tilde{f}(x) \leq f(x), \forall x \in \Omega$$

Example - Knapsack Problem

At a flea market in Rome, you spot n objects (old pictures, a vessel, rusty medals. . .) that you could re-sell in your antique shop for about double the price.

- You want these objects to pay for your flight ticket to Rome, which cost C.
- You don't want a heavy backpack (limitation by Airline company to 23kg), so you want to buy the objects that will not exceed the limit.

Question: Formulate as optimization problem.

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Question: Formulate as optimization problem.
Question: Without being able to solve Integer programming, under which condition we can say that we do not want to buy anything?

Answer

Let item i has price p_i and weight w_i. Then we define variables $x_i \in \{0, 1\}$ (0= we do not buy that item)

$$\begin{array}{rll} \text{maximize} & \sum_{i=1}^n x_i p_i & \\ \text{subject to} & \sum_{i=1}^n w_i \quad x_i \leq & 23 \\ & \qquad\qquad x_i \in \{0, 1\} & \end{array}$$

Answer

Let item i has price p_i and weight w_i. Then we define variables $x_i \in \{0, 1\}$ (0= we do not buy that item)

$$\begin{array}{rrl} \text{maximize} & \sum_{i=1}^{n} x_i p_i & \\ \text{subject to} & \sum_{i=1}^{n} w_i \quad x_i \leq & 23 \\ & x_i \in & \{0, 1\} \end{array}$$

We relax the problem

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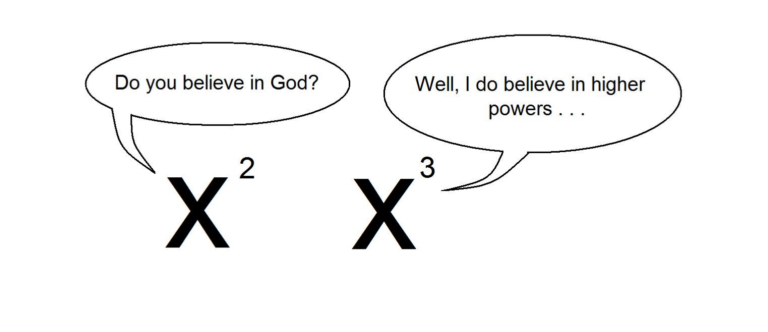
if the relaxed problem has optimal objective value $\leq C$ then we know for sure that the original optimization problem will have also optimal objective value $\leq C$ and hence we will not buy stuff!


Do you believe in God?
Well, I do believe in higher powers . . .
X^2
X^3