Imprecision Attenuates Updating*

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Abstract

Agents often base decisions on noisy signals, attenuating Bayesian updating toward the prior expectation—a well-established phenomenon in the normal-normal signal-extraction model. We show this attenuation effect extends to all symmetric, log-concave distributions. By introducing a notion of precision based on likelihood-ratio dominance, we prove that when both the prior and noise are symmetric and log-concave, the posterior mean moves closer to the prior mean as the signal becomes less precise. We discuss applications to cognitive imprecision, prior precision, and overconfidence.

Keywords: signal extraction, Bayes's rule, reversion to the mean, cognitive imprecision
JEL Classifications: C11, D83, D84

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1 Introduction

In many economic contexts, agents often do not respond optimally to fundamental variables: numerical estimates are biased toward default values (Tversky and Kahneman, 1974), firms and households adjust only partially to changes in macroeconomic conditions (Sims, 2003), and consumers underreact to non-salient taxes (Chetty et al., 2009). A recent body of literature suggests that these and similar behavioral phenomena can be explained by *cognitive imprecision* (Gabaix, 2019; Woodford, 2020; Enke and Graeber, 2023; Enke et al., 2024): agents base their decisions on noisy internal signals of the true variables of interest.

Models of cognitive imprecision account for such behavior by showing noise in cognition attenuates the Bayesian updating process, thereby compressing behavior towards some default action. This effect is typically formalized through the *normal-normal model*: the agent observes a noisy signal $S = X + \varepsilon$, where the state $X \in \mathbb{R}$ is normally distributed, and ε is independent, normally distributed noise. The agent's posterior mean is then compressed toward the prior mean, more so when the signal is less precise: the posterior mean lies between the signal and the prior mean and the posterior mean is closer to the prior mean the larger the variance of ε.[1] We refer to this effect as *imprecision attenuates updating*. When the agent's action is determined by their posterior mean, then attenuation of updating translates to attenuation of behavior. One important piece of evidence for cognitive imprecision relies on this effect: subjects who report higher cognitive uncertainty tend to exhibit more attenuated behavioral responses (Enke and Graeber, 2023; Enke et al., 2024).

Despite the empirical relevance of this attenuation effect, it remained unknown how far it extends beyond normal distributions, which rely on strong parametric assumptions. Although normal distributions are justified in certain contexts — such as through the central limit theorem or under rational inattention with a normal prior and quadratic loss — these justifications are limited to specific situations. Therefore, identifying a non-parametric class of signal structures under which imprecision attenuates updating is desirable to provide a more robust theoretical foundation for interpreting empirical observations as implications of cognitive imprecision.

To address this gap, we show imprecision attenuates updating holds for all additive noise models with symmetric, log-concave distributions: when the state X and noise ε have (possibly different) symmetric and log-concave densities, the posterior mean moves closer to the prior mean as the signal becomes less precise, for any signal realization s (Theorem 1).

To formulate this result, we introduce an appropriate notion of precision, which we call the *precision order*. It requires the noise distribution of a less precise signal to be further away from zero in the sense of likelihood-ratio domination: $\tilde{S} = X + \tilde{\varepsilon}$ is less precise than $S = X + \varepsilon$ if the likelihood ratio $f_{\tilde{\varepsilon}}(x)/f_{\varepsilon}(x)$ is nondecreasing as x moves away from 0. More concretely, we show that if we scale up a log-concave noise term, the resulting signal is less precise in our sense, and thus, our precision

[1] Formally, if $X \sim \mathcal{N}(\mu, \sigma_X^2)$ and $\varepsilon \sim \mathcal{N}(0, \sigma_\varepsilon^2)$, then

$$\mathbb{E}[X|S = s] = \frac{\sigma_X^2}{\sigma_X^2 + \sigma_\varepsilon^2} s + \left(1 - \frac{\sigma_X^2}{\sigma_X^2 + \sigma_\varepsilon^2}\right) \mu.$$

The larger σ_ε^2, the closer the posterior mean is to the prior mean.

order applies to log-concave location-scale experiments.

Symmetric, log-concave distributions include many common probability distributions beyond the normal, such as logistic, extreme value, and double-exponential distributions. Our focus on this class is motivated by Chambers and Healy (2012), who demonstrate symmetry and quasi-concavity (unimodality) are necessary conditions for the posterior mean to robustly lie between the prior mean and the signal realization. More precisely, they show that if the prior is not symmetric and quasi-concave, one can find a symmetric and quasi-concave noise density (and vice versa) under which the posterior mean does not lie between the prior mean and the signal realization. We show one need only strengthen quasi-concavity to log-concavity to obtain our result.

Although our main result extends the attenuation effect to a broader class of distributions, some caution is warranted. As Chambers and Healy (2012) demonstrate, without symmetry and quasi-concavity, one can easily construct examples where the agent overreacts to signals; that is, the posterior mean is more extreme than the signal. We add to this caution by showing that the Blackwell order Moreover, we show that a more informative signal structure in the Blackwell order can lead to a posterior mean closer to the prior mean for some signal realizations (section 3).

Finally, we leverage Theorem 1 to establish implications for signal extraction problems, which we believe are of broader economic relevance:

- *Prior Precision*: Our main theorem implies a converse comparative-statics result regarding prior precision (Corollaries 2 and 3): increasing the precision of the prior brings the posterior mean closer to the prior mean for any given signal realization.

- *Average Posterior Means*: We extend our analysis to *average* posterior means conditional on the true state, which are relevant when agents acquire conditionally independent signals about a common variable. We show that, conditional on the true state, the average posterior mean lies between the state and the prior mean (Proposition 1).

- *Comparative Statics for Average Posterior Mean*: Greater (over)confidence in the signal brings the average posterior mean closer to the true state (Proposition 2), whereas greater prior precision brings it closer to the prior mean (Proposition 3).

Related Literature This paper introduces a new information order and provides comparative-statics results for signal structures with additive independent noise, commonly referred to as location experiments.

Prior work has explored orderings on location experiments related to the value of information. Boll (1955) shows that one location experiment Blackwell-dominates another if and only if the latter's noise term can be derived by adding an independent noise term to the former's. This condition is very restrictive, making location experiments comparable only in limited contexts. In response, Lehmann (1988) introduced the Lehmann order to rank location experiments for *monotone* decision problems. Our proposed precision order is distinct from the Blackwell order, neither implying nor being implied by it, while strengthening the Lehmann order.

Other comparative statics results have been established for location experiments with log-concave noise distributions, which satisfy the monotone likelihood-ratio property. Milgrom (1981) shows that the strict monotone likelihood-ratio property ensures higher signals constitute "good news," in the sense of producing first-order stochastically dominant posteriors, for any prior.[2] Conversely, Dawid (1973) examines violations of this property under heavy-tailed noise distributions, showing that if the noise is sufficiently heavy-tailed relative to the prior, extreme observations are "rejected," yielding posteriors that revert to the prior (for further discussion, see O'Hagan and Pericchi, 2012). Kartik et al. (2021) provides a comparative statics result with economic applications for agents with heterogeneous priors, termed "information validates the prior," in which a more informative experiment brings another agent's posterior mean closer to one's prior mean.

2 Model

We restrict attention to signal structures, where the signal equals the one-dimensional state plus some independent noise, which are called location experiments. Formally, the random signal S is a location experiment if it is the sum of the random state X and random noise ε that is independent of X, which we write as $\varepsilon \perp\!\!\!\perp X$. We assume the noise ε has mean 0; that is, the signal is unbiased. If not, one could easily derive an unbiased signal by subtracting the mean of ε from S.

Assumption 1 (Location Experiment)**.**

$$S = X + \varepsilon \tag{1}$$

$$\varepsilon \perp\!\!\!\perp X \tag{2}$$

$$\mathbb{E}[\varepsilon] = 0 \tag{3}$$

Further, we make the technical assumption that the state X and noise ε admit positive, continuously differentiable densities f_X and f_ε with finite absolute first moments. This guarantees finite conditional expectation $\mathbb{E}[X|S = s]$. We call f_X the *prior density* and f_ε the *noise density*.

Assumption 2. *The random state X and noise ε admit positive, continuously differentiable densities f_X, f_ε and have finite absolute first moments.*

Following Chambers and Healy (2012), we assume symmetric and quasi-concave prior and noise densities.[3] They show that without these assumptions, the posterior mean does not necessarily lie between the prior mean and the signal realization. We believe that our comparative statics result is only interesting when the posterior mean is attenuated in the first place.

Assumption 3. *The prior and noise densities f_X and f_ε are symmetric and quasi-concave.*

[2] See also Chambers and Healy (2011) for a strengthening of Milgrom's result and Heinsalu (2020) for an extension and economic application of Chambers and Healy (2011).

[3] Quasi-concave densities are also called unimodal densities.

In particular, the prior density f_X is symmetric around $\mathbb{E}[X]$ and the noise density f_ε is symmetric around zero by (3).

Chambers and Healy (2012) show the following result, which we include for later reference.

Fact 1 (Chambers and Healy, 2012, Proposition 3)**.** *Under Assumptions 1 to 3, for any signal realization s, the posterior mean $\mathbb{E}[X|S=s]$ lies weakly between the prior mean $\mathbb{E}[X]$ and the signal s. Formally,*

$$\begin{aligned} \forall s \leq \mathbb{E}[X]: \qquad & s \leq \mathbb{E}[X|S=s] \leq \mathbb{E}[X], \\ \forall s \geq \mathbb{E}[X]: \qquad & s \geq \mathbb{E}[X|S=s] \geq \mathbb{E}[X]. \end{aligned}$$

Throughout the paper, we maintain Assumptions 1 to 3. In particular, all considered prior and noise densities are assumed to be positive, symmetric, and quasi-concave, unless otherwise specified.

3 Precision Order

One challenge to formalizing the idea that imprecision attenuates updating is to find the right order on signal structures. We show that the following new order, which we call the *precision order*, is characteristic of our desired result. Below, we discuss the relation between the precision order and well-known information orders like the Blackwell and Lehmann orders.

We define the precision order for the class of location experiments that satisfy Assumptions 1 to 3. We first define this order on noise terms (or, more generally, on random variables) and then extend it to the associated location experiments.

Definition 1 (Precision Order)**.** *Let $\tilde{\varepsilon}$ and ε be random variables with positive, symmetric-around-0, and quasi-concave densities $f_{\tilde{\varepsilon}}$ and f_ε, respectively. We say $\tilde{\varepsilon}$ is less precise than ε if the likelihood ratio*

$$\frac{f_{\tilde{\varepsilon}}(x)}{f_\varepsilon(x)}$$

is nondecreasing in x for $x > 0$. Further, we say the location experiment $\tilde{S} = X + \tilde{\varepsilon}$ is less precise than location experiment $S = X + \varepsilon$ if $\tilde{\varepsilon}$ is less precise than ε.

Note that by the symmetry of densities, the definition implies the likelihood ratio $f_{\tilde{\varepsilon}}(x)/f_\varepsilon(x)$ is weakly *de*creasing in x for $x < 0$. Thus, the precision order requires that, for positive values, $\tilde{\varepsilon}$ likelihood-ratio dominates ε and, for negative values, $\tilde{\varepsilon}$ is likelihood-ratio dominated by ε. In other words, the less precise location experiment has a noise term further away from 0 in the sense of likelihood-ratio domination. Figure 1 gives an example of two noise distributions ranked by the precision order. The more precise distribution has a higher density at zero, which falls faster as we move away from zero.

A natural question is whether the precision order holds under natural operations on the noise densities. In section 4.1, we show that when we *scale up* a symmetric, log-concave density by a constant $k > 1$, the resulting density is less precise.

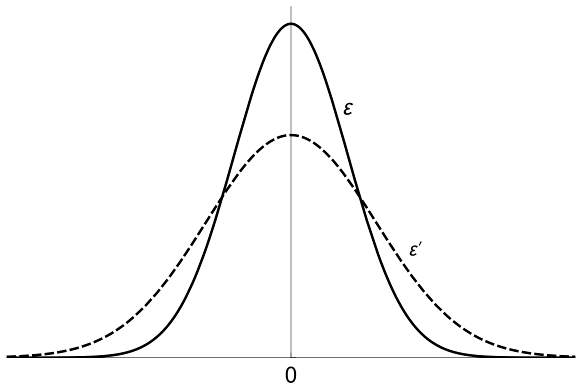

Figure 1: The densities of $\varepsilon \sim \mathcal{N}(0,1)$ and $\tilde{\varepsilon} \sim \mathcal{N}(0,1.5^2)$: the former distribution is more precise.

How does the precision order relate to existing information orders? The precision order is neither implied by nor implies the Blackwell order. To see why the precision order is not implied by the Blackwell order, consider the latter's restrictive nature for location experiments. As Boll (1955) shows, two location experiments are Blackwell ordered if only if the noise term of one experiment can be obtained from the other's noise term by adding independent noise. Cramér's decomposition theorem further implies that a normal location experiment (i.e., $S = X + \varepsilon$, $\varepsilon \sim \mathcal{N}(\mu, \sigma^2)$) Blackwell dominates another location experiment only if the latter is also normal (Cramér, 1936). By contrast, the precision order allows for normal location experiments to be less precise than experiments with non-normal noise. On the other hand, the precision order does not imply the Blackwell order, as illustrated by the noise term $\tilde{\varepsilon} = \varepsilon + \varepsilon'$, that is the independent sum of a standard Cauchy ε and $\varepsilon' \sim U[-1,1]$. Computing the density of $\tilde{\varepsilon}$ shows that it is not less precise than ε.

The precision order lies at the intersection of two important orders: the convex order (also called mean-preserving spread order) on noise distributions, and the Lehmann order on location experiments. One can show that if the location experiment $S = X + \varepsilon'$ is less precise than $S = X + \varepsilon$, then noise ε' is a mean-preserving spread of noise ε.[4] The Lehmann order characterizes what experiments are more valuable in all *monotone* decision problems (Lehmann, 1988; Quah and Strulovici, 2009). This order is defined only for location experiments with log-concave noise, because only these location experiments satisfy the required monotone likelihood-ratio property. One can show for location experiments with log-concave noise, the precision order implies the Lehmann order.[5]

[4]If S' is less precise than S, the CDFs of ε' and ε cross once. Together with ε' and ε having equal means, this implies one distribution majorizes the other; see also Diamond and Stiglitz (1974).

[5]On such location experiments, the Lehmann order coincides with the dispersive order on the noise distributions, which requires that any two quantiles are weakly further apart under the more dispersed noise distribution. Formally, let ε and ε' have CDFs F and G, respectively, and let F^{-1} and G^{-1} denote here the right-continuous inverses. ε is smaller in the dispersive order than ε' if $\forall\, 0 < \alpha \leq \beta < 1 : F^{-1}(\beta) - F^{-1}(\alpha) \leq G^{-1}(\beta) - G^{-1}(\alpha)$. When we truncate the noise to positive values, the precision order implies the likelihood-ratio order. As is known, the likelihood-ratio order implies the hazard-rate order (e.g., Shaked and Shanthikumar, 2007, Theorem 1.C.1). The hazard-rate order together with densities being log-concave implies the dispersive order (Bagai and Kochar, 1986, Theorem 1). One can easily see that if the dispersive order holds for all $1/2 \leq \alpha \leq \beta < 1$, then it holds for all $0 < \alpha \leq \beta < 1$.

4 Results

Our main result establishes that *imprecision attenuates updating* if and only if location experiments are ranked by the precision order, under the additional assumption of a log-concave prior.

Theorem 1 (Imprecision Attenuates Updating)**.** *Let the prior density be log-concave. Consider two location experiments $S = X + \varepsilon$ and $\tilde{S} = X + \tilde{\varepsilon}$. For every signal realization, the posterior mean under $\tilde{S}$ is closer to the prior mean than is the posterior mean under S, if and only if $\tilde{S}$ is less precise than S. Formally:*

$$\begin{aligned} \forall s \leq \mathbb{E}[X]: &\qquad \mathbb{E}[X|S=s] \leq \mathbb{E}[X|\tilde{S}=s] \leq \mathbb{E}[X] \\ \forall s \geq \mathbb{E}[X]: &\qquad \mathbb{E}[X|S=s] \geq \mathbb{E}[X|\tilde{S}=s] \geq \mathbb{E}[X] \end{aligned} \tag{4}$$

if and only if $\tilde{S}$ is less precise than S.

An immediate implication of this result is as follows. Consider two agents who observe the same signal, such as an economic forecast, but differ in their assessment of the signal precision. The agent who assumes greater precision will have a posterior mean closer to the signal, for any signal realization.

The proof, relegated to the Appendix, demonstrates necessity and sufficiency of the precision order. Necessity is the easier direction and shown by constructing a counterexample when two location experiments are not precision-ordered. Sufficiency is more involved. We can suppose without loss that the signal realization s is larger than the prior mean. To the right of s, the more precise signal results in a likelihood-ratio dominated posterior, lowering decreases the posterior mean. To the left of the s, the more precise signal results in a likelihood-ratio dominant posterior, raising the posterior mean. The proof establishes that the former effect dominates, using a sequence of inequalities and bounds, leveraging symmetry, quasi-, and log-concavity assumptions.

For this as well as the following results, an analogous *strict* version of the result holds. We can require that the noise term is replaced by a *strictly* less precise noise in the sense that the likelihood ratio in Definition 1 is strictly decreasing. Then, for signal realizations distinct from the prior mean, the posterior mean is strictly closer to s.

Next, we show that our precision order arises endogenously in *location-scale experiments* with log-concave, and many non-log-concave, noise densities.

4.1 Location-Scale Experiments

We introduce to our location experiment a *scale parameter* $\sigma \in \mathbb{R}_{\geq 0}$ that scales the noise term, such that

$$S_\sigma = X + \sigma\varepsilon. \tag{5}$$

We show comparative statics on the scale parameter σ.

Lemma 1. *If ε is symmetric around 0 and $\log f_\varepsilon(e^x)$ is concave, then $\forall \sigma' > \sigma > 0$, $\sigma'\varepsilon$ is less precise than $\sigma\varepsilon$. In particular, $\log f_\varepsilon(e^x)$ is concave if the density $f_\varepsilon(x)$ is log-concave.*[6]

In the Appendix, we show that Lemma 1 follows from a known result. Many commonly used distributions are symmetric and log-concave, such as the normal, logistic, extreme value, and double-exponential distributions. Further, we give examples of symmetric and non-log-concave distributions for which $\log f_\varepsilon(e^x)$ is nevertheless concave, such as the Student-t, Cauchy, and the "double" Pareto distribution.

Together, Theorem 1 and Lemma 1 imply the following important result. By Lemma 1, the result still holds if we weaken the assumption that the noise density f_ε is log-concave to $\log f_\varepsilon(e^x)$ being concave.

Corollary 1. *Let the prior and noise densities be log-concave. The posterior mean is weakly closer to the prior mean under a larger scale parameter, for any signal realization s. Formally, if $\tilde{\sigma} > \sigma > 0$, then*

$$\begin{aligned} \forall s \leq \mathbb{E}[X]: \qquad & \mathbb{E}[X|S_\sigma = s] \leq \mathbb{E}[X|S_{\tilde{\sigma}} = s] \leq \mathbb{E}[X], \\ \forall s \geq \mathbb{E}[X]: \qquad & \mathbb{E}[X|S_\sigma = s] \geq \mathbb{E}[X|S_{\tilde{\sigma}} = s] \geq \mathbb{E}[X]. \end{aligned}$$

Location-scale experiments with log-concave noise density further satisfy the monotone likelihood-ratio property, which implies that the posterior mean is non-decreasing in the signal, as in the normal-normal model. This suggests that location-scale experiments with symmetric and log-concave prior and noise density are a useful class of models that maintains key properties of the normal-normal model.

The next section shows comparative statics results for changing the prior instead of the noise, exploiting a symmetry in location experiments.

4.2 Comparative Statics on the Prior

For location experiments, the posterior density is symmetric in the prior and in the noise density. This follows from the more general property of Bayesian updating that the posterior is proportional to the product of the prior and the likelihood, $p(x|s) \propto p(x,s) = p(x)p(s|x)$. In the case of location experiments, this implies the density of the posterior conditional on $S = s$ is proportional to $f_X(x)f_\varepsilon(s-x)$. Using this insight, Theorem 1 immediately implies a dual result for making the prior more precise.

Corollary 2. *Let the noise density be log-concave. The posterior mean is weakly closer to the prior mean under a more precise prior, for any signal realization s. Formally, if $\mathbb{E}[X] = \mathbb{E}[\tilde{X}]$ and $\tilde{X}$ is*

[6] This can be seen easily if the log-density is differentiable. Define $\phi = \log f$ and $\psi = \exp$. Then, ϕ is concave and decreasing for positive values and ψ is convex, increasing, and obtains positive values only, which implies $(\phi \circ \psi)''(x) = \psi''(x)(\phi' \circ \psi)(x) + (\psi'(x))^2(\phi'' \circ \psi)(x) \leq 0$.

more precise than X, then

$$\forall s \leq \mathbb{E}[X]: \qquad \mathbb{E}[X|X+\varepsilon = s] \leq \mathbb{E}[\tilde{X}|\tilde{X}+\varepsilon = s] \leq \mathbb{E}[X],$$
$$\forall s \geq \mathbb{E}[X]: \qquad \mathbb{E}[X|X+\varepsilon = s] \geq \mathbb{E}[\tilde{X}|\tilde{X}+\varepsilon = s] \geq \mathbb{E}[X].$$

To illustrate Corollary 2, suppose that two agents observe the same signal about the state but one agent has a more precise prior (but with the same mean). Corollary 2 implies that the agent with the more precise prior has a posterior mean that is closer to the prior mean, for any signal realization.

By the same argument, an analogous dual result to Corollary 1 holds for scaling the prior instead of the noise density. If X has density $f_X(x)$, then kX has density $1/kf_X(x/k)$.

Corollary 3. *Let the prior and noise densities be log-concave. For any signal realization s, the posterior mean is weakly closer to the prior mean if we scale down the prior. Formally, if $0 < \tilde{k} < k$ and we normalize the prior mean to zero, $\mathbb{E}[X] = 0$, then*

$$\forall s \leq \mathbb{E}[X]: \qquad \mathbb{E}[kX|kX+\varepsilon = s] \leq \mathbb{E}[\tilde{k}X|\tilde{k}X+\varepsilon = s] \leq \mathbb{E}[X],$$
$$\forall s \geq \mathbb{E}[X]: \qquad \mathbb{E}[kX|kX+\varepsilon = s] \geq \mathbb{E}[\tilde{k}X|\tilde{k}X+\varepsilon = s] \geq \mathbb{E}[X].$$

4.3 Average Posterior Means

Our previous comparative statics results hold for any signal realization and thus speaks to situations where agents observe the same signal. However, in many situations, agents observe *distinct* signals from the same signal structure, that is their signal realization are independent conditional on the state. Further, often we do not observe agent's signal realizations but only the *average* posterior mean, such as when we observe only aggregate behavior from a population of individuals. What can be said about comparative statics with respect to the average posterior means given some true state X?

Before we prove comparative statics results, we show that the *average* posterior mean necessarily lies between the state X and the prior mean $\mathbb{E}[X]$, extending Fact 1.

Proposition 1. *For any state x, the conditional average posterior mean $\mathbb{E}[\mathbb{E}[X|S]|X = x]$ lies weakly between the state x and the prior mean $\mathbb{E}[X]$. Formally,*

$$\forall x \leq \mathbb{E}[X]: \qquad x \leq \mathbb{E}[\mathbb{E}[X|S]|X = x] \leq \mathbb{E}[X],$$
$$\forall x \geq \mathbb{E}[X]: \qquad x \geq \mathbb{E}[\mathbb{E}[X|S]|X = x] \geq \mathbb{E}[X].$$

The proof is illustrated using Figure 2. Let $X > \mathbb{E}[X] = 0$. Conditional on state X, the distribution of the signal S is is symmetric around X so its expectation equals X. Taking the conditional expectation $\mathbb{E}[X|S]$ moves the distribution closer to $\mathbb{E}[X]$ as indicated by the arrows. Given our assumptions, the conditional expectation is antisymmetric and $s > 0$ has a higher likelihood than $-s < 0$. Thus, the overall effect on the average posterior mean is negative and $\mathbb{E}[\mathbb{E}[X|S]|X] <$

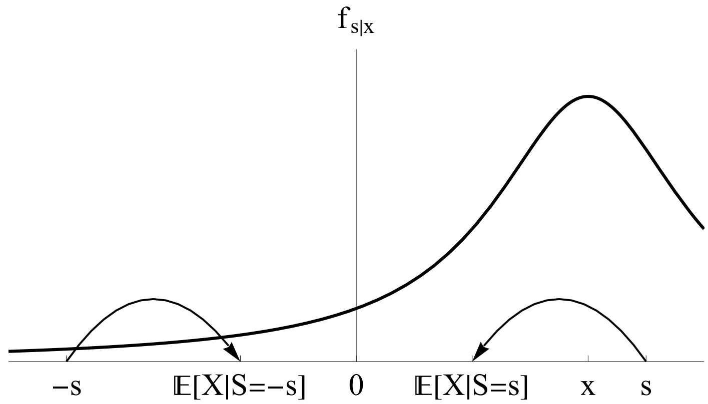

Figure 2: Illustrating the proof of Proposition 1.

$\mathbb{E}[S|X] = X$. Further, because the density of s is larger than the density of $-s$, integrating over all s leads to a positive expectation, so $\mathbb{E}[X] < \mathbb{E}[\mathbb{E}[X|S]|X]$.

Overconfidence First, we prove a comparative statics result for overconfidence in the signal. We consider two agents, A and B, that face the same objective signal structure $S = X + \varepsilon$ but update differently because they have different confidence in the signal. That is, agent $i \in \{A, B\}$ forms their conditional expectation $\mathbb{E}_i[X|S = s]$ as if $S = X + \varepsilon_i$, $\mathbb{E}_i[X|S = s] := \mathbb{E}[X|X + \varepsilon_i = s]$. Especially empirically relevant is the case *over*confidence in the signal, also called overprecision, which is pervasive (Moore and Healy, 2008). We define a relative notion of overconfidence by generalizing the definition in Ortoleva and Snowberg (2015), which is based on the normal-normal model, using our Definition 1.

Definition 2. *A is more confident than B in the signal if ε_A is more precise than ε_B.*

Using this definition, we prove the following comparative statics result.

Proposition 2. *Let the prior density be log-concave and A be more confident than B. Conditional on any state x, the average posterior mean of A is weakly closer to the state than the average posterior mean of B. Formally,*

$$\forall x \leq \mathbb{E}[X]: \qquad x \leq \mathbb{E}[\mathbb{E}_A[X|S]|X = x] \leq \mathbb{E}[\mathbb{E}_B[X|S]|X = x],$$
$$\forall x \geq \mathbb{E}[X]: \qquad x \geq \mathbb{E}[\mathbb{E}_A[X|S]|X = x] \geq \mathbb{E}[\mathbb{E}_B[X|S]|X = x].$$

This shows that more overconfident agents have, on average, posterior means closer to the state X and further away from their prior mean $\mathbb{E}[X]$.

The proof builds on the proof of Proposition 1 and is illustrated using Figure 3. Because A is more overconfident in the signal, their posterior mean is closer to the signal for any signal realization s as well as $-s$. For positive X, because $s > 0$ is more likely than $-s$, the overall effect on the average posterior mean is positive. Then, the result follows from Proposition 1.

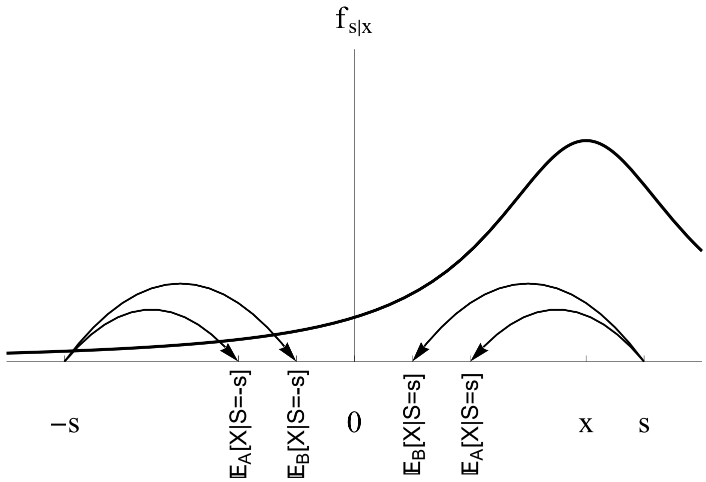

Figure 3: Illustrating the proof of Proposition 2

Prior Precision Second, we show comparative statics with respect to the prior precision. Given some state X, consider two agents, A and B, with symmetric and quasi-concave priors that have the same mean, where agent A's prior is more precise than B's. Then, we have from the argument in Proposition 2 and the Corollary 2, we immediately obtain the following result.

Proposition 3. *Let the noise density be log-concave and A's prior be less precise than B's. Conditional on any state x, the average posterior mean of A is weakly closer to the state than the average posterior mean of B. Formally,*

$$\forall x \leq \mathbb{E}[X]: \qquad x \leq \mathbb{E}[\mathbb{E}_A[X|S]|X=x] \leq \mathbb{E}[\mathbb{E}_B[X|S]|X=x],$$
$$\forall x \geq \mathbb{E}[X]: \qquad x \geq \mathbb{E}[\mathbb{E}_A[X|S]|X=x] \geq \mathbb{E}[\mathbb{E}_B[X|S]|X=x].$$

5 Conclusion

In this paper, we have extended the attenuation effect of cognitive imprecision beyond the normal-normal model to encompass all symmetric, log-concave distributions. By introducing a new order of precision, based on likelihood-ratio dominance, we demonstrated that imprecision attenuates Bayesian updating toward prior beliefs across a broad class of distributions commonly used in economic modeling. This generalization provides a more robust theoretical foundation for interpreting empirical observations of attenuated behavior as resulting from cognitive imprecision.

Our findings also have broader implications for signal extraction problems. We established comparative statics results regarding prior precision, showing that increased prior precision brings the posterior mean closer to the prior mean for any given signal realization. Additionally, we analyzed average posterior means, demonstrating how overconfidence and prior precision affect the average posterior mean. Perhaps surprisingly, our results show that the posterior means of overconfident agents are on average closer to the truth while those of agents with more precise priors are further away from it.

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6 Appendix

6.1 Proof of Theorem 1

Proof. We, first, show necessity and, second, sufficiency of the precision order for the ordering requirement (4).

Lemma 2 (Necessity)**.** *If $\tilde{\varepsilon}$ is* not *less precise than ε, then there exist a symmetric, log-concave prior and a signal realization $s \in \mathbb{R}$ such that the ordering requirement (4) of Theorem 1 is violated.*

Proof. We first find a uniform prior and a signal realization s such that (4 is violated. Then, we approximate the uniform prior with a symmetric, log-concave prior density that is positive everywhere

First, if $\tilde{\varepsilon}$ is not less precise than ε, then the likelihood-ratio $f_{\tilde{\varepsilon}}(x)/f_{\varepsilon}(x)$ has a strictly negative derivative for some $x = s > 0$.[7] By continuous differentiability, $f_{\tilde{\varepsilon}}(x)/f_{\varepsilon}(x)$ has a strictly negative derivative on some interval $[s-\delta, s+\delta]$ around s.

Consider the uniform prior on $[-\delta, \delta]$, $f_X^U(x) = 1/(2\delta)$. Let f be the posterior density upon observing $X + \tilde{\varepsilon} = s$ and g be the posterior density upon observing $X + \varepsilon = s$. The posterior density $f(x)$ is proportional to the product $f_X^U(x) f_{\tilde{\varepsilon}}(s-x)$ and the posterior density $g(x)$ is proportional to the product $f_X^U(x) f_{\varepsilon}(s-x)$, so the ratio $f(x)/g(x)$ of the posterior densities is proportional to

$$\frac{f_{\tilde{\varepsilon}}(s-x)}{f_{\varepsilon}(s-x)}.$$

By the above, this likelihood ratio is strictly increasing in x on the support $[-\delta, \delta]$ of the prior, so the posterior under signal $\tilde{S} = s$ strictly likelihood-ratio dominates the posterior under signal $S = s$. Hence, the posterior mean is strictly greater under the less precise signal, $\mathbb{E}[X|X + \tilde{\varepsilon} = s] > \mathbb{E}[X|X + \varepsilon = s]$, despite $s > 0 = \mathbb{E}[X]$, violating (4).

Second, we approximate the uniform prior $[-\delta, \delta]$ with $\delta > 0$ with a symmetric, log-concave prior, such that the posterior mean converges (pointwise) for any signal realization s. For any $d > 0$, define

[7] If both noise densities f_{ε} and $f_{\tilde{\varepsilon}}$ are continuously differentiable and positive, then the likelihood ratio $f_{\tilde{\varepsilon}}/f_{\varepsilon}$ is continuously differentiable.

the prior density $f_X^d(x)$ via

$$\log f_X^d(x) = \begin{cases} c, & \text{if } x \in [-\delta, \delta] \\ c - d(x-\delta)^2, & \text{if } x > \delta \\ c - d(x-(-\delta))^2, & \text{if } x < -\delta. \end{cases}$$

where c is the unique constant such that f_X integrates to 1. The density f_X is symmetric around 0, log-concave, and integrable. As we let d go to $+\infty$, the density f_X^d pointwise converges to the uniform density f_X^U. Use Lebesgue dominated convergence theorem we show that as we let d go to $+\infty$, the posterior mean under signal realization s converges:

$$\frac{\int x f_X^d(x) f_\varepsilon(s-x)dx}{\int f_X^d(x) f_\varepsilon(s-x)dx} \xrightarrow[d\to\infty]{} \frac{\int x f_X^U(x) f_\varepsilon(s-x)dx}{\int f_X^U(x) f_\varepsilon(s-x)dx}.$$

To show that the numerator converges, note that by $f_X^d(x) < 1/(2\delta)$, the integrand is bounded by $x f_\varepsilon(s-x)/(2\delta)$, which is integrable by ε having a finite first absolute moment. Thus, by Lebesgue dominated convergence theorem, the integral converges to $\int x f_X^U(x) f_\varepsilon(s-x)dx$.

To show that the denominator converges, note that for any $d > 0$, $f_X^d(x) < 1/(2\delta)$. Thus, the integrand is bounded by $f_\varepsilon(s-x)/(2\delta)$, which is integrable. By the Lebesgue dominated convergence theorem, the integral converges to $\int f_X^U(x) f_\varepsilon(s-x)dx$. Because the denominator is strictly positive, the ratio converges.

By $\mathbb{E}[X|X+\tilde{\varepsilon}=s] > \mathbb{E}[X|X+\varepsilon=s]$ under uniform prior f_X^U, we can choose a d large enough such that the same ordering requirement holds under prior f_X^d. □

The rest of the proof concerns the harder direction, that is, showing that the precision order is sufficient for (4). Without loss of generality, the signal realization is zero, $s=0$. If $\mathbb{E}[X]=0$, then the posterior mean is zero by Fact 1 and we are done. Assume $\mathbb{E}[X] < 0$ (the case $\mathbb{E}[X] > 0$ is analogous). We prove that for strictly more precise noise, the posterior mean becomes strictly closer to 0. Let $\tilde{\varepsilon}$ be less precise than ε, and $f_{\tilde{\varepsilon}}$ and f_ε denote their respective densities. Further, let f denote the posterior density under noise $\tilde{\varepsilon}$ after observing signal $s=0$ and g analogously under noise ε.

The posterior mean is

$$\mathbb{E}[X|X+\varepsilon=0] = \int_{-\infty}^{0} x g(x)dx + \int_{0}^{\infty} x g(x)dx = \int_{0}^{\infty} -x(g(-x)-g(x))dx,$$

and analogously with density f instead of g for the signal with greater noise. The proof revolves around showing that the following integral is positive:

$$\mathbb{E}[X|X+\varepsilon=0] - \mathbb{E}[X|X+\tilde{\varepsilon}=0] = \int_0^\infty -x\big[(g(-x)-g(x)) - (f(-x)-f(x))\big]dx \tag{6}$$

First, we prove the following result regarding the integrand of (6), which uses that $\frac{f_\varepsilon(x)}{f_{\tilde{\varepsilon}}(x)}$ is strictly decreasing for $x > 0$.

Lemma 3. *There is some $c > 0$ such that the integrand of (6) is strictly negative for $x \in [0, c)$ and strictly positive for $x \in (-c, \infty)$.*

Proof. Again, the density $f(x)$ is proportional to $f_X(x) f_{\tilde{\varepsilon}}(x)$ and $g(x)$ to $f_X(x) f_{\varepsilon}(x)$. Thus, there is some factor $C > 0$ such that

$$g(-x) - g(x) = C\frac{f_{\varepsilon}(-x)}{f_{\tilde{\varepsilon}}(-x)} f(-x) + C\frac{f_{\varepsilon}(x)}{f_{\tilde{\varepsilon}}(x)} f(x) = C\frac{f_{\varepsilon}(x)}{f_{\tilde{\varepsilon}}(x)}(f(-x) - f(x)),$$

where we have used the symmetry of the noise densities. Thus, the integrand of (6) is negative if, and only if, $C\frac{f_{\varepsilon}(x)}{f_{\tilde{\varepsilon}}(x)} > 1$.

The ratio $\frac{f_{\varepsilon}(x)}{f_{\tilde{\varepsilon}}(x)}$ is strictly decreasing for $x > 0$ by assumption. As both are densities that integrate to 1, the ratio must cross $1/C$ and by strictly decreasing ratio, the crossing point must be unique up to sign. Let c be the unique positive x at which $\frac{g(x)}{f(x)} = 1/C$. Then, for $x \in [0, c)$ we have $C\frac{f_{\varepsilon}(x)}{f_{\tilde{\varepsilon}}(x)} > 1$ and for $x \in (c, \infty)$ we have $C\frac{f_{\varepsilon}(x)}{f_{\tilde{\varepsilon}}(x)} < 1$. □

Without loss, we can rescale the space, so that $c = 1$. Using Lemma 3, we have that

$$\begin{gathered}\int_0^{\infty} -x\big[(g(-x) - g(x)) - (f(-x) - f(x))\big]dx \\ > \\ \int_0^1 -1 \cdot \big[(g(-x) - g(x)) - (f(-x) - f(x))\big]dx + \int_1^{\infty} -1 \cdot \big[(g(-x) - g(x)) - (f(-x) - f(x))\big]dx \\ = \\ (-G_2 + G_3 - G_1 + G_4) - (-F_2 + F_3 - F_1 + F_4) \end{gathered} \tag{7}$$

where F_1 through F_4 are the probabilities of according to f on four mutually exclusive and exhaustive intervals

$$\begin{aligned} F_1 &:= \int_1^{\infty} f(-x)dx = \int_{-\infty}^{-1} f(x)dx \\ F_2 &:= \int_0^1 f(-x)dx = \int_{-1}^{0} f(x)dx \\ F_3 &:= \int_0^1 f(x)dx \\ F_4 &:= \int_1^{\infty} f(x)dx \end{aligned}$$

so $F_1, F_2, F_3, F_4 > 0$ and $F_1 + F_2 + F_3 + F_4 = 1$. G_1 through G_4 are defined analogously. We thus need to show that

$$-G_2 + G_3 - G_1 + G_4 > -F_2 + F_3 - F_1 + F_4. \tag{8}$$

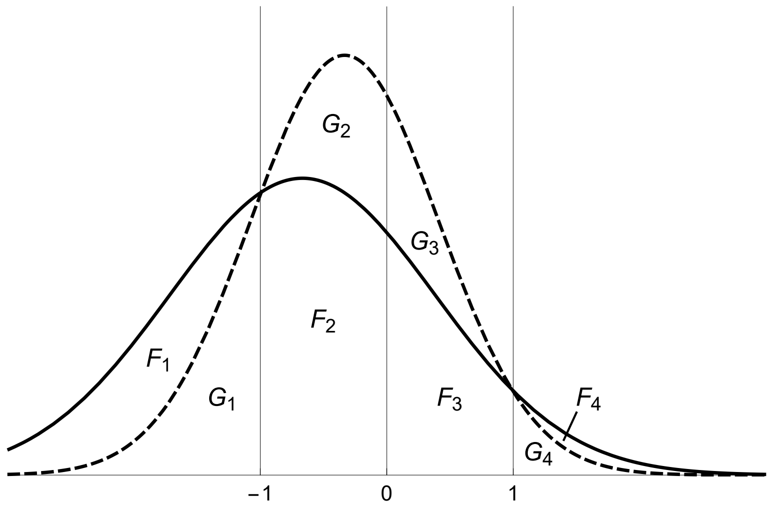

Figure 4: The posteriors f (bold) and g (dashed) after observing $X + \tilde{\varepsilon} = 0$ and $X + \varepsilon = 0$, respectively.

The proof proceeds as follows. As the Figure 4 depicts, G_2 and G_3 are larger than F_2 and F_3, respectively, and G_1 and G_4 are smaller than F_1 and F_4, respectively. We show in Lemma 7 that (8) would hold if G_2 and G_3 were larger than F_2 and F_3, each, by the same factor and similarly for G_1 and G_4. Lemma 4 to 6 argue that in fact these ratios are not the same and thus $-G_2 + G_3 - G_1 + G_4$ is even larger, proving (8).

Before that, we prove the following lemma, which uses the log-concavity and symmetry of the prior density as well as the symmetry of the noise density.

Lemma 4. *The posterior probability-ratio* $\frac{f(-x)}{f(x)}$ *is strictly increasing in* x.

Proof. By Bayes' law, f is proportional to the product $f_X(x)f_{\tilde{\varepsilon}}(s - x) = f_X(x)f_{\tilde{\varepsilon}}(x)$. The ratio is strictly increasing if its logarithm, which is as follows, is strictly increasing in x. Using the symmetry of $f_{\tilde{\varepsilon}}$ and f_X we obtain

$$\begin{aligned}\log\left(\frac{f(-x)}{f(x)}\right) &= \log\left(\frac{f_X(-x)f_{\tilde{\varepsilon}}(-x)}{f_X(x)f_{\tilde{\varepsilon}}(x)}\right) = \log\left(\frac{f_X(-x)}{f_X(x)}\right) \\ &= \log f_X(-x) - \log f_X(x) = \log f_X(x + 2\mathbb{E}[X]) - \log f_X(x),\end{aligned}$$

where we have used that $f_X(x)$ is symmetric around $\mathbb{E}[X] < 0$, so

$$\log f_X(-x) = \log f_X\big(\mathbb{E}[X] + (-x - \mathbb{E}[X])\big) = \log f_X\big(\mathbb{E}[X] - (-x - \mathbb{E}[X])\big) = \log f_X(x + 2\mathbb{E}[X]).$$

By strict concavity of $\log f_X$ and $\mathbb{E}[X] < 0$, the difference $\log f_X(x + 2\mathbb{E}[X]) - \log f_X(x)$ is strictly increasing. □

Using Lemma 4, we prove two lemmas regarding ratios of the terms in (7).

Lemma 5. $\frac{F_1}{F_4} \geq \frac{F_2}{F_3}$.

Proof. We have that

$$\frac{F_1}{F_4} = \frac{\int_1^\infty f(-x)dx}{\int_1^\infty f(x)dx} = \frac{\int_1^\infty \frac{f(-x)}{f(x)} f(x)dx}{\int_1^\infty f(x)dx} \geq \frac{\int_0^1 \frac{f(-x)}{f(x)} f(x)dx}{\int_0^1 f(x)dx} = \frac{\int_0^1 f(-x)dx}{\int_0^1 f(x)dx} = \frac{F_2}{F_3}.$$

The inequality holds because by Lemma 4. The two inner terms are the expectation of $\frac{f(-x)}{f(x)}$ with respect to the posterior distribution f conditional on the domain $[1, \infty)$ and $[0, 1]$, respectively. The former distribution first-order stochastically dominates the latter, thus the inequality follows from $\frac{f(-x)}{f(x)}$ being strictly increasing in x for $x > 0$. $\square$

The following lemma uses that $\frac{f_\varepsilon(x)}{f_{\bar{\varepsilon}}(x)}$ is decreasing for $x > 0$.

Lemma 6. *$1 < \frac{G_2}{F_2} < \frac{G_3}{F_3}$ and $\frac{G_1}{F_1} < \frac{G_4}{F_4} < 1$.*

Proof. Using $g(x) = \frac{f_\varepsilon(x)}{f_{\bar{\varepsilon}}(x)} f(x)C$, where C is the ratio of the integration constants, and $\frac{f_\varepsilon(-x)}{f_{\bar{\varepsilon}}(-x)} = \frac{f_\varepsilon(x)}{f_{\bar{\varepsilon}}(x)}$ (by symmetry), we have

$$\frac{G_2}{F_2} = \frac{\int_0^1 \frac{g(-x)}{f(-x)} f(-x)Cdx}{\int_0^1 f(-x)dx} = \frac{\int_0^1 \frac{f_\varepsilon(x)}{f_{\bar{\varepsilon}}(x)} Cf(-x)dx}{\int_0^1 f(-x)dx}$$
$$\frac{G_3}{F_3} = \frac{\int_0^1 \frac{g(x)}{f(x)} f(x)dx}{\int_0^1 f(x)dx} = \frac{\int_0^1 \frac{f_\varepsilon(x)}{f_{\bar{\varepsilon}}(x)} Cf(x)dx}{\int_0^1 f(x)dx}.$$

Thus, both ratios are expectations of the function $\frac{f_\varepsilon(x)}{f_{\bar{\varepsilon}}(x)}$ over the interval $[0, 1]$ multiplied by C but with densities $f(-x)/(\int_0^1 f(-x)dx)$ and $f(x)/(\int_0^1 f(x)dx)$, respectively. Because $\frac{f(-x)}{f(x)}$ is increasing in $x > 0$ by Lemma 4, the former density likelihood-ratio dominates the latter. This implies first-order stochastic dominance, which in turn implies a strictly smaller expectation since $\frac{f_\varepsilon(x)}{f_{\bar{\varepsilon}}(x)}$ is a strictly decreasing function by assumption. Thus, $\frac{G_2}{F_2} < \frac{G_3}{F_3}$.

Moreover, by Lemma 3 and having normalized $c = 1$, $\frac{f_\varepsilon(x)}{f_{\bar{\varepsilon}}(x)} C > 1$ in the interval $[0, 1)$. Hence, $\frac{G_2}{F_2}$ and $\frac{G_3}{F_3}$, which are expectations of this ratio, are strictly greater than 1.

The proof of $\frac{G_1}{F_1} < \frac{G_4}{F_4} < 1$ is analogous, but with expectation over the domain $[1, \infty)$. $\square$

Define $\tilde{G}_2 = kF_2$ and $\tilde{G}_3 = kF_3$ with $k > 1$, and $\tilde{G}_1 = lF_1$ and $\tilde{G}_4 = lF_4$ with $l < 1$, such that $\tilde{G}_2 + \tilde{G}_3 = G_2 + G_3$ and $\tilde{G}_1 + \tilde{G}_4 = G_1 + G_4$. By Lemma 6, $\frac{G_3}{F_3} > \frac{\tilde{G}_3}{F_3} = \frac{G_2+G_3}{F_2+F_3} = \frac{\tilde{G}_2}{F_2} > \frac{G_2}{F_2}$, so $Q_3 > \tilde{Q}_3$ and $\tilde{Q}_2 > Q_2$, implying $-G_2 + G_3 > -\tilde{G}_2 + \tilde{G}_3$. Analogously, $-G_1 + G_4 > -\tilde{G}_1 + \tilde{G}_4$.

$$-G_2 + G_3 - G_1 + G_4 > -\tilde{G}_2 + \tilde{G}_3 - \tilde{G}_1 + \tilde{G}_4. \tag{9}$$

Finally, the following lemma concludes the proof.

Lemma 7. *$-\tilde{G}_2 + \tilde{G}_3 - \tilde{G}_1 + \tilde{G}_4 > -F_2 + F_3 - F_1 + F_4$.*

Proof. Define $r := \frac{F_2}{F_3} > 0$ and $R := \frac{F_1}{F_4} > 0$ where $R > r$ by Lemma 5, and $P := F_2 + F_3$ and $p := F_1 + F_4$.

We have $\tilde{G}_1+\tilde{G}_2+\tilde{G}_3+\tilde{G}_4 = G_1+G_2+G_3+G_4 = F_1+F_2+F_3+F_4 = 1$. Thus, we can define $\Delta := (\tilde{G}_2+\tilde{G}_3)-(F_2+F_3) = (F_1-F_4)-(\tilde{G}_1+\tilde{G}_4)$ with $\Delta > 0$ as well as $kP = P+\Delta$ and $lp = p-\Delta$. From $P = F_2+F_3$ and $r = F_2/F_3$, it follows that $-F_2+F_3 = P(\frac{1}{1+r}-\frac{r}{1+r}) = -P\frac{r-1}{r+1}$ and analogously $-F_1+F_4 = -p\frac{R-1}{R+1}$. So, $(-\tilde{G}_2+\tilde{G}_3)-(-F_2+F_3) = -\Delta\frac{r-1}{r+1}$ and $(-\tilde{G}_1+\tilde{G}_4)-(-F_1+F_4) = \Delta\frac{R-1}{R+1}$. Note that $\frac{d}{dx}\frac{x-1}{x+1} = \frac{2}{(x+1)^2} > 0$ for $x > 0$. Then, by $R > r$, $\frac{R-1}{R+1} > \frac{r-1}{r+1}$, so $-\tilde{G}_2+\tilde{G}_3-\tilde{G}_1+\tilde{G}_4 > -F_2+F_3-F_1+F_4$. $\square$

By (9) and Lemma 7, we obtain $-G_2+G_3-G_1+G_4 > -F_2+F_3-F_1+F_4$. $\square$

6.2 Proof of Lemma 1

Proof. By our definition, the symmetric around 0 random variable ε is more precise than the symmetric around 0 random variable ε if $[\varepsilon|\varepsilon > 0]$ is smaller in the likelihood ratio order than $[\varepsilon|\varepsilon > 0]$. It is not hard to show that for a non-negative continuous random variables X, aX is smaller in the likelihood ratio order than X for all $0 < a < 1$ if, and only if, $\log f_\varepsilon(e^x)$ is concave for $x > 0$ (e.g. Hu et al., 2004). Applying this result to $X = [\varepsilon|\varepsilon > 0]$ yields the result.

The function $\log f_\varepsilon(e^x)$ is concave for $x > 0$ in particular if f_ε is log-concave and symmetric around 0. Note that

$$\frac{d^2}{dx^2}\log f_\varepsilon(e^x) = \frac{d}{dx}e^x(\log f_\varepsilon)'(e^x) = e^x(\log f_\varepsilon)'(e^x) + e^{2x}(\log f_\varepsilon)''(e^x).$$

The latter term is negative because f_ε is log-concave and the former term is negative because f_ε is also symmetric around 0. $\square$

The main text gives several examples of commonly encountered symmetric, log-concave distributions. Below, we prove for symmetric distributions that are not log-concave that $\log f_\varepsilon(e^x)$ is nevertheless concave.

Non-log-concave examples The *Student-t* distribution with parameter $\nu > 0$, which includes as a special case the Cauchy distribution, gives

$$\begin{aligned} f(x) &\propto \left(1+\frac{x^2}{\nu}\right)^{-(\nu+1)/2} \\ \Rightarrow \log f(e^x) &= C - \frac{\nu+1}{2}\log(1+\frac{1}{v}e^{2x}) \\ \Rightarrow \frac{d^2}{dx^2}\log f(e^x) &= \frac{d}{dx} - \frac{\nu+1}{2}\frac{\frac{2}{\nu}e^{2x}}{1+\frac{1}{\nu}e^{2x}} = -\frac{\nu+1}{2}\frac{\frac{4}{\nu}e^{2x}}{(1+\frac{1}{\nu}e^{2x})^2} < 0, \end{aligned}$$

and hence has log-concave $f(e^x)$.

Creating a symmetric distribution from the *Pareto* distribution, analogous to the double-exponential distribution, with $\alpha > 0$ gives

$$f(x) \propto x^{-\alpha-1} \Rightarrow \log f(e^x) = C - (\alpha+1)x,$$

with log-linear and hence log-concave $f(e^x)$.

6.3 Proof of Proposition 1

Proof. By symmetry and translation invariance of location experiments, it is without loss to suppose that $X \geq \mathbb{E}[X] = 0$.

First, we show the inequality $\mathbb{E}[X] = 0 \leq \mathbb{E}[\mathbb{E}[X|S]|X]$. We have

$$\begin{aligned}\mathbb{E}[\mathbb{E}[X|S]|X] &= \int_{-\infty}^{\infty} \mathbb{E}[X|S=s] f_\varepsilon(s-X)ds \\ &= \int_{0}^{\infty} \big(\mathbb{E}[X|S=s] f_\varepsilon(s-X) + \mathbb{E}[X|S=-s] f_\varepsilon(-s-X)\big)ds. \qquad (10)\end{aligned}$$

By symmetry of the prior and noise densities, $\mathbb{E}[X|S=s] = -\mathbb{E}[X|S=-s]$ and by Fact 1, $\mathbb{E}[X|S=s] > 0$. By symmetry around 0 and quasi-concavity of the noise density, $f_\varepsilon(-s-X) = f_\varepsilon(s+X) < f_\varepsilon(s-X)$ and hence, the integrand of (10) is positive for every $s \in [0,\infty)$.

Second, we show $\mathbb{E}[\mathbb{E}[X|S]|X] \leq X$. We have by symmetry of f_ε around 0,

$$\begin{aligned}X &= \int_{-\infty}^{\infty} s f_\varepsilon(s-X)ds \\ &= \int_{0}^{\infty} \big(s f_\varepsilon(s-X) + (-s) f_\varepsilon(-s-X)\big)ds. \qquad (11)\end{aligned}$$

We need to show that (10) is less or equal to (11). By Fact 1, $s \geq \mathbb{E}[X|S=s]$ but $-s \leq \mathbb{E}[X|S=-s]$, preventing a direct conclusion of the result. However, by symmetry, $s - \mathbb{E}[X|S=s] = \mathbb{E}[X|S=-s] - (-s)$ and as before $f_\varepsilon(s-X) > f_\varepsilon(-s-X)$. Thus, for every $s \in [0,\infty)$, the integrand of (10) is less or equal to the integrand of (11). □

6.4 Proof of Proposition 3

We prove Proposition 3. Proposition 2 follows by analogous arguments.

Proof. For the first statement, we can again exploit the symmetry of the conditional expectations $\mathbb{E}_A[X|S]$ and $\mathbb{E}_B[X|S]$ and that the density of S is larger for $x > 0$ than for $x < 0$.

Suppose, without loss, that $X > \mathbb{E}[X] = 0$. By Proposition 1, we have $\mathbb{E}[X] \leq \mathbb{E}[\mathbb{E}_A[X|S]|X]$ and $\mathbb{E}[X] \leq \mathbb{E}[\mathbb{E}_B[X|S]|X]$. It remains to show that $\mathbb{E}[\mathbb{E}_A[X|S]|X] \leq \mathbb{E}[\mathbb{E}_B[X|S]|X]$.

$$\mathbb{E}[\mathbb{E}_A[X|S]|X] = \int_{-\infty}^{\infty} \mathbb{E}_A[X|S=s] f_\varepsilon(s-X)ds = \int_{0}^{\infty} \big(\mathbb{E}_A[X|S=s] f_\varepsilon(s-X) - \mathbb{E}_A[X|S=-s] f_\varepsilon(s-X)\big)ds$$
$$\mathbb{E}[\mathbb{E}_B[X|S]|X] = \int_{-\infty}^{\infty} \mathbb{E}_B[X|S=s] f_\varepsilon(s-X)ds = \int_{0}^{\infty} \big(\mathbb{E}_B[X|S=s] f_\varepsilon(s-X) - \mathbb{E}_B[X|S=-s] f_\varepsilon(s-X)\big)ds$$

By Theorem 1, we have $\mathbb{E}_A[X|S=s] \leq \mathbb{E}_B[X|S=s]$ and $\mathbb{E}_A[X|S=-s] \geq \mathbb{E}_B[X|S=-s]$ for $s > 0$, preventing a direct conclusion of the result. However, by symmetry of the prior and the noise density, we know that $\mathbb{E}_B[X|S=s] - \mathbb{E}_B[X|S=s] = \mathbb{E}_A[X|S=-s] - \mathbb{E}_B[X|S=-s]$ and by symmetry and quasi-concavity of the noise density and $X > 0$, we have that $f_\varepsilon(s - X) > f_\varepsilon(s + X) = f_\varepsilon(-s - X)$ for $s > 0$. Thus, the integrand of the first equation is smaller than the integrand of the second for any $s > 0$.

For the second statement, conditional on X, the distribution of S is the same for A and B. For any realization S, $\mathbb{E}_A[X|S]$ is closer to $\mathbb{E}[X]$ than $\mathbb{E}_B[X|S]$. So, the conditional distribution of the absolute distance of $\mathbb{E}_A[X|S]$ to $\mathbb{E}[X]$ is smaller in first-order stochastic dominance than the one of $\mathbb{E}_B[X|S]$. $\square$