Rigidity and Reconstruction of Convex Polytopes

– an application of Wachspress Geometry –

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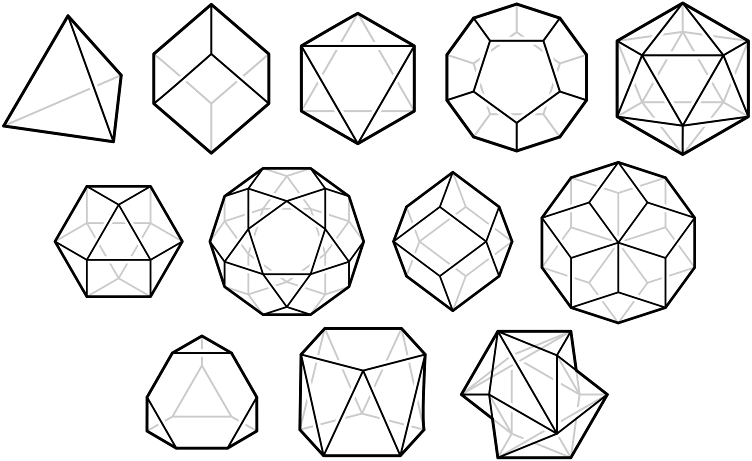
Polyhedra

= polygons glued edge to edge to form a closed surface.



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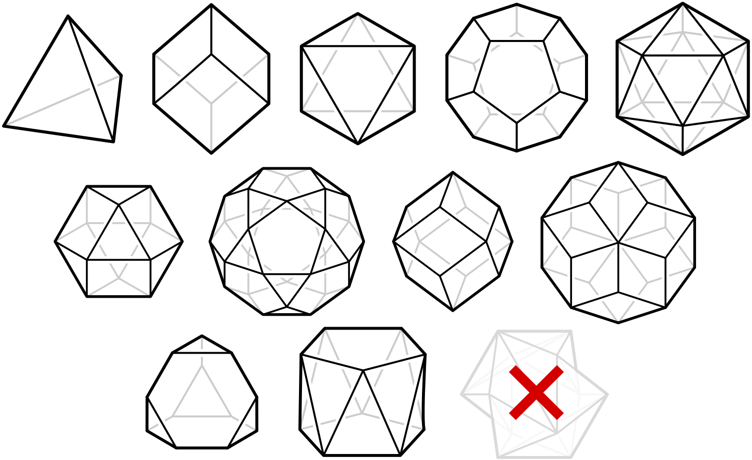
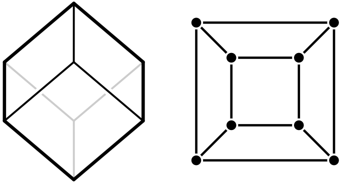
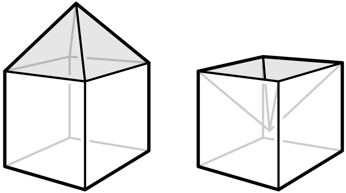


Polyhedral combinatorics and geometry





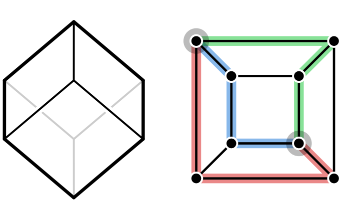
Steinitz' Theorem: (1922)

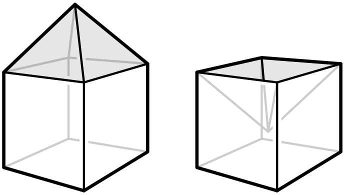
G is the edge graph of a polyhedron $\iff$ G is 3-connected and planar

Cauchy's rigidity theorem: (1813)

A polyhedron is determined by the shape of its faces.

Polyhedral combinatorics and geometry





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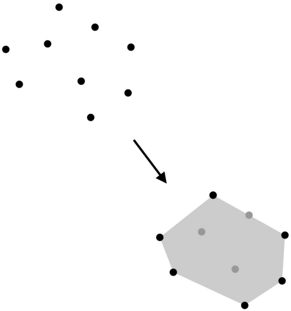
Polytopes

$$\{(\alpha_1, ..., \alpha_n) \in \mathbb{R}^n_{\geq 0} \mid \alpha_1 + \cdots + \alpha_n = 1\}$$

$$P = \mathrm{conv}\{p_1, ..., p_n\} = \Big\{ \sum_i \alpha_i p_i \mid \alpha \in \overset{\downarrow}{\Delta_n} \Big\} \subset \mathbb{R}^d$$



d	
$d = 0$	point
1	line segment
2	polygon
3	polyhedron
4	polychoron
$\vdots$	$\vdots$

polytope

Polytopes

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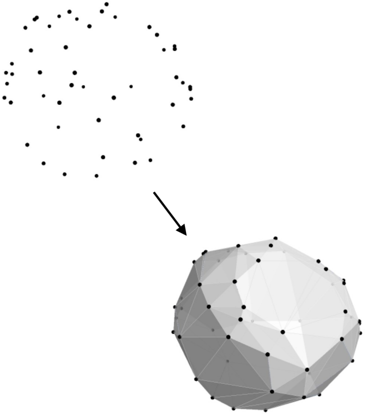
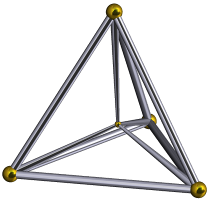
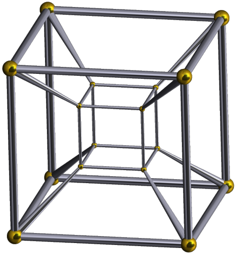
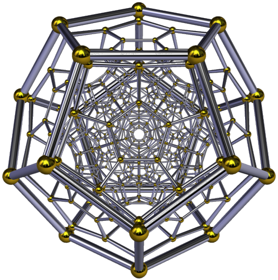


$$\left.\begin{array}{r|l} d = 0 & \text{point} \\ 1 & \text{line segment} \\ 2 & \text{poly}\underline{\text{gon}} \\ 3 & \text{poly}\underline{\text{hedron}} \\ 4 & \text{poly}\underline{\text{choron}} \\ \vdots & \quad \vdots \end{array}\right\} \text{poly}\underline{\text{tope}}$$

Higher dimensional polytopes

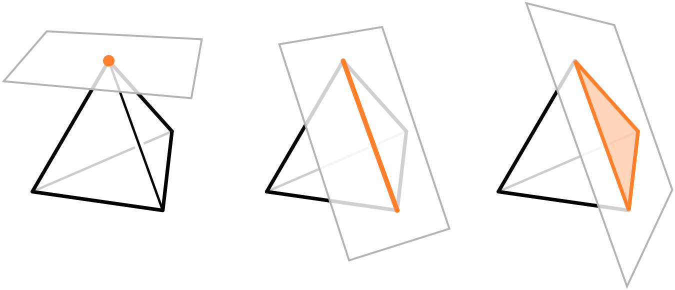






Faces of a polytope

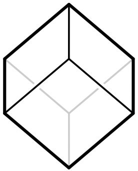
$:=$ intersections with supporting hyperplanes

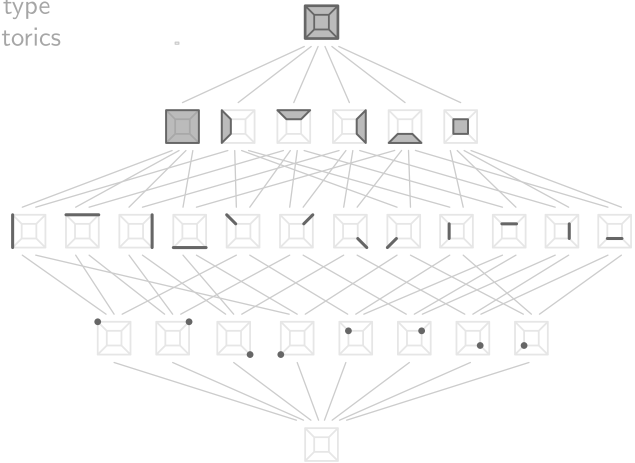


Note: for technical reasons $\varnothing$ and P are also considered as faces.

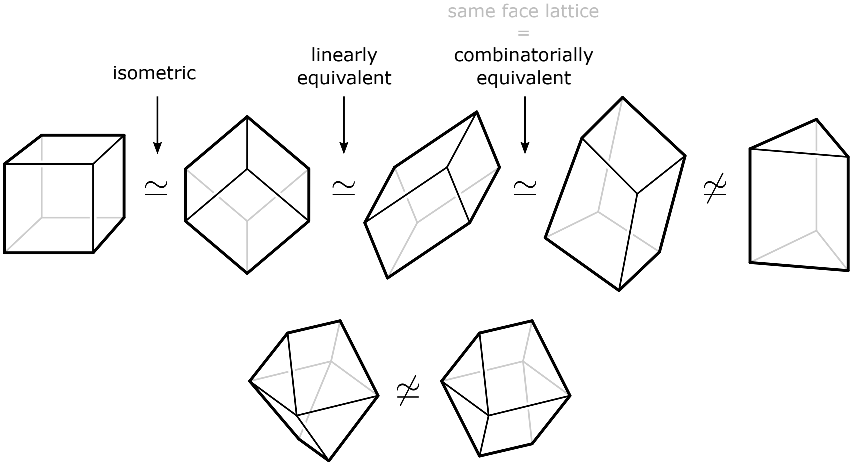
The combinatorics of a polytope

face lattice
$\cong$ combinatorial type
$\cong$ (full) combinatorics

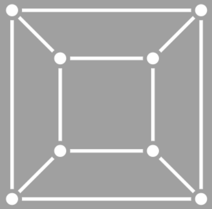


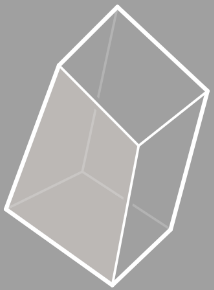


Combinatorial equivalence



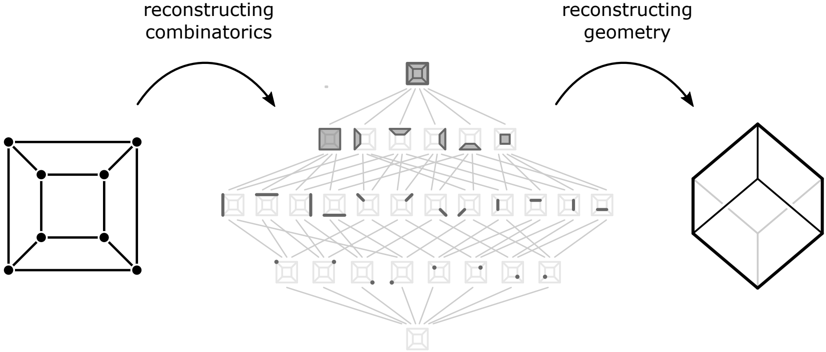
Reconstruction of polytopes



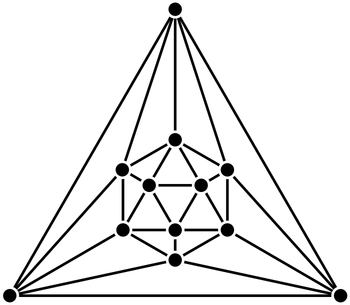


Reconstruction of polytopes

"In how far is a polytope determined by partial combinatorial and geometric data, up to isometry, affine transformation or combinatorial equivalence?"



Reconstructing combinatorics ($d = 3$)



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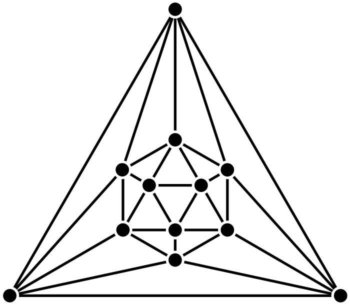

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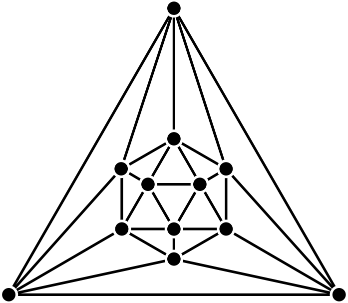

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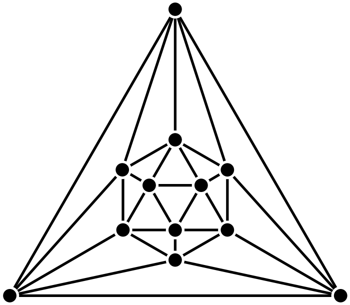

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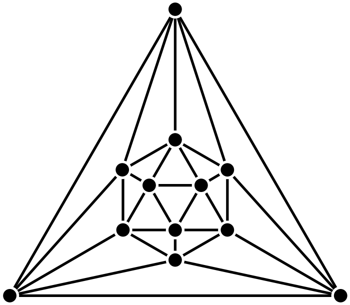

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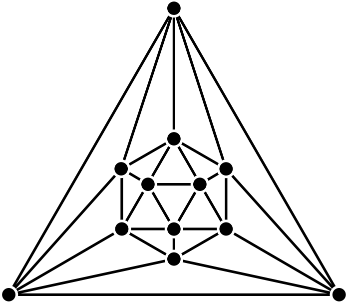

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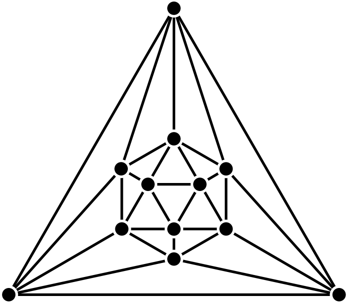

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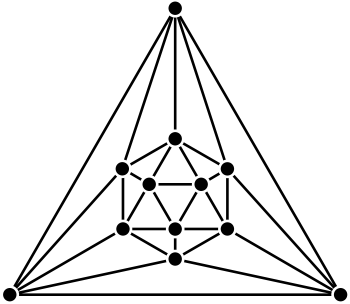

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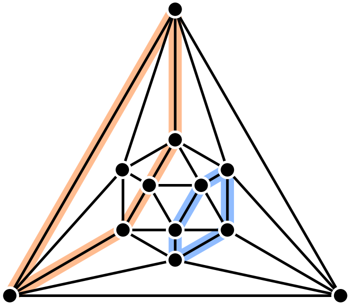

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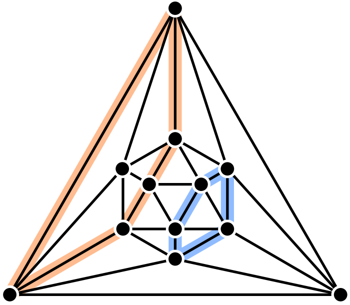

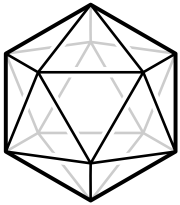

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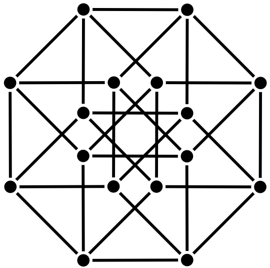

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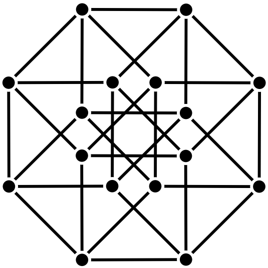

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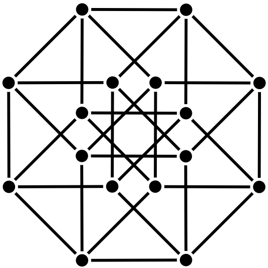

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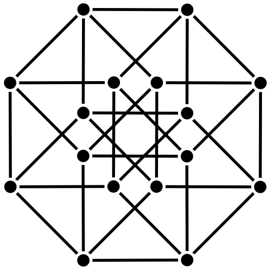

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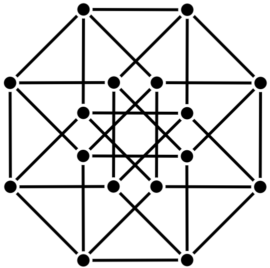

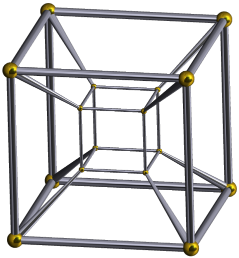

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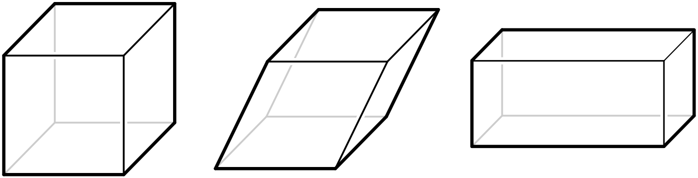
Reconstructing geometry

Given the full combinatorics, can we reconstruct from ...

- edge lengths ✗
- dihedral angles ✗



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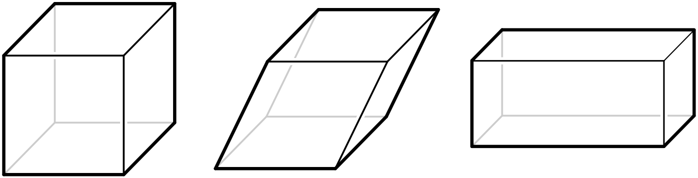
$\Big\}$ edge lengths + dihedral angles ✓ (Stoker)



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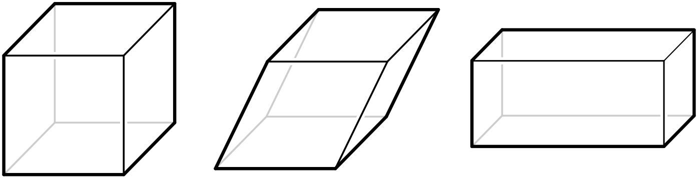
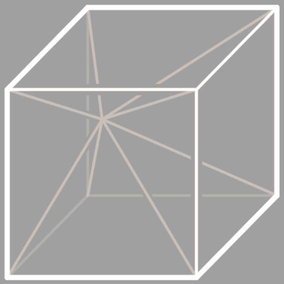


Cauchy's rigidity theorem (Cauchy, 1813)

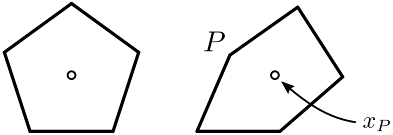
A polytope is uniquely determined (up to isometry) by its combinatorics and the shapes of its 2-faces.

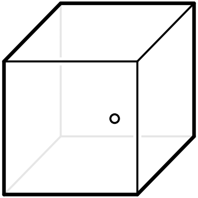
Reconstruction of pointed polytopes



Pointed polytopes

:= polytope $P \subset \mathbb{R}^d$ + point $x_P \in \mathbb{R}^d$





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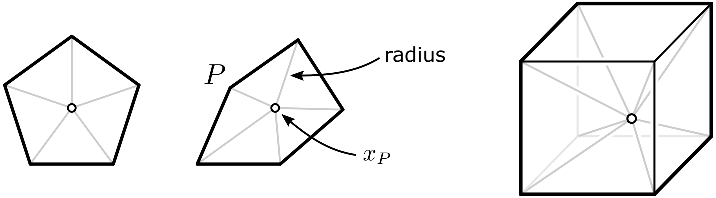

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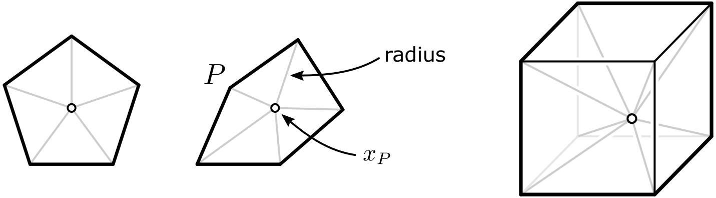

Questions:

- Is a pointed polytope determined by the graph, edge lengths and radii?

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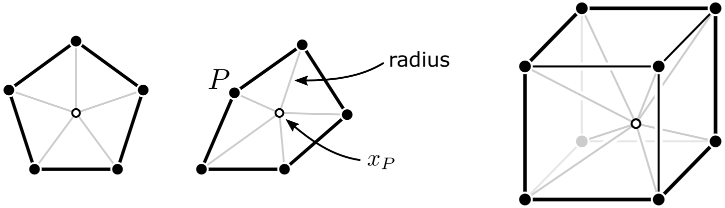

Questions:

- Is a pointed polytope determined by the graph, edge lengths and radii?
- ... also as a framework?

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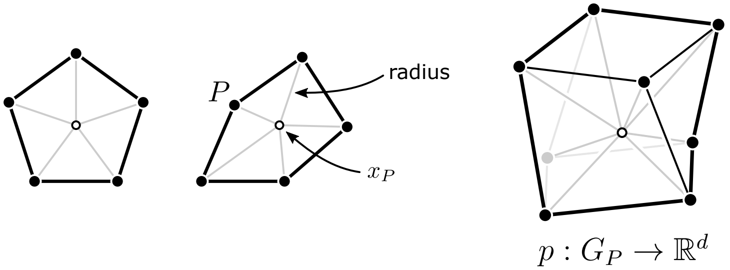

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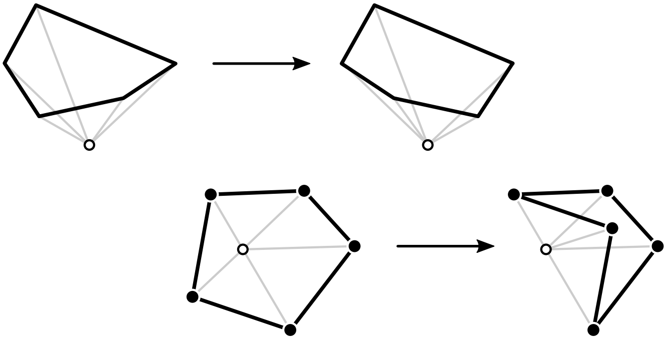
Pointed polytopes and frameworks



Central conjectures

Conjecture. (W., 2023)

A pointed polytope P with $x_P \in \mathrm{int}(P)$ is uniquely determined (up to isometry) by its edge graph, edge lengths and radii.

... across all dimensions and all combinatorial types!

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Conjecture. (W., 2023)

If $P \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^e$ are pointed polytopes with the same edge graph and

(i) $x_Q \in \operatorname{int}(Q)$

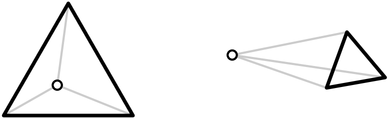
(ii) *edges in Q are <u>at most</u> as long as in P,*

(iii) *radii in Q are <u>at least</u> as large as in P,*

then P and Q are isometric.

"A polytope cannot become larger if all its edges become shorter."

Central conjectures



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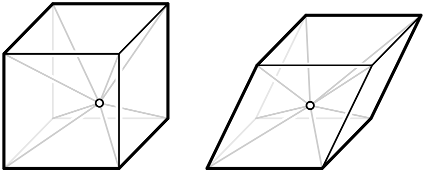
Conjecture holds in special cases (W., 2023)

I. P and Q are centrally symmetric

II. P and Q are "close"

III. P and Q are combinatorially equivalent



$P \subset \mathbb{R}^d$ $\qquad$ $Q \subset \mathbb{R}^e$

Conjecture holds in special cases (W., 2023)

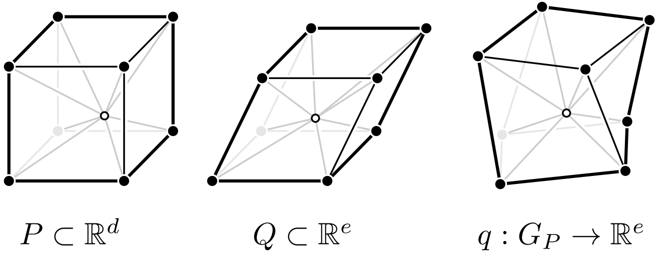
I. P and $q\colon G_P \to \mathbb{R}^e$ are centrally symmetric

- $\cong$ universally rigid as a centrally symmetric framework

II. P and $q\colon G_P \to \mathbb{R}^e$ are "close"

- $\cong$ locally rigid as a framework

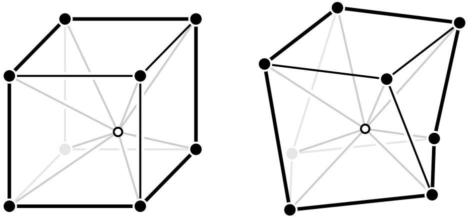
III. P and Q are combinatorially equivalent



Why is this surprising?

II. P and $q\colon G_P \to \mathbb{R}^e$ are "close"

$\cong$ locally rigid as a framework



$$\#\text{DOFs} - \#\text{constraints} = (\overset{V}{8} + 1) \times \overset{d}{3} - (\overset{E}{12} + \overset{V}{8}) = 7 = 6 + 1.$$

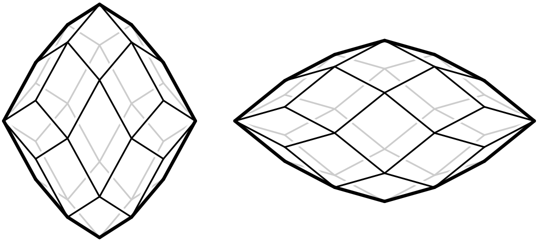
How to prove it ...

Two step plan:

1. prove theorem using “better” coordinates in place of radii.
2. infer the radii version from the “better” coordinates version.

Key theorem (W., 2023)

A pointed polytope is uniquely determined (up to affine transformations) by its edge graph, edge lengths and Wachspress coordinates.



Wachspress coordinates are …

I. … the unique rational GBCs of lowest possible degree (Warren, 2003)

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Theorem. (Warren)

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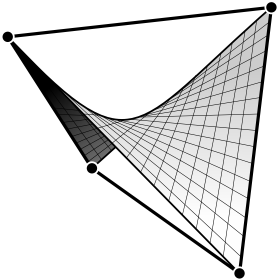

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II. ... a "shadow" of a higher rank objects

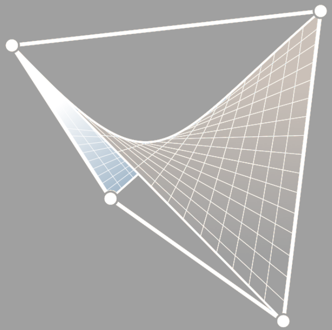
Theorem. (Lovász, $d = 3$; Izmestiev, $d > 3$)

A polytope framework is a spectral embedding of the edge graph using suitable weights on vertices and edges

weight matrix $M \in \mathbb{R}^{n \times n}$ ... **Lovász-Izmestiev matrix** of P

$$\alpha_i(x) = \sum_j M_{ij}(x) \quad \text{(W., 2023)}$$

Thank you.



M. Winter, *"Rigidity, Tensegrity and Reconstruction of Polytopes under Metric Constraints"* (2023)