# Rigidity Theory
## of Frameworks and Polytopes


- Martin Winter (TU Berlin)
  winter@math.tu-berlin.de

- Tuesday 10:00 – 12:00
  October 15th 2024 – February 11th 2025

15/10/2022

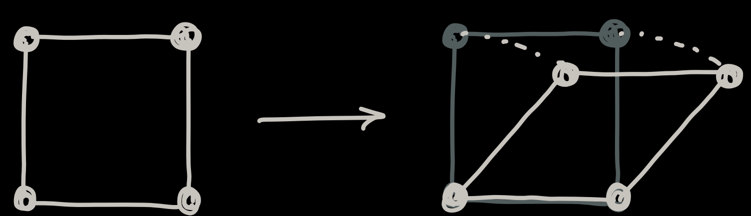
# §0. Introduction

- Rigidity is a very broad concept in mathematics (search "Rigidity Theorems" on Wikipedia)
- Inspired by physics/structural engineering and with applications in architecture, chemistry etc
- ... but also deeply mathematical
- Rigidity is about degrees of freedom and rigidity theory is mostly interested in their absence.

  Absence of rigidity $\Longrightarrow$ there is more than one way to do things
- Rigidity can be studied for many objects (Riemannien manifolds, algebraic varieties, ...)

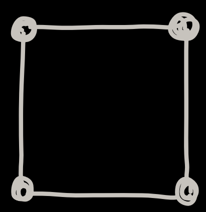
BUT

- This lecture focuses on Geometric + Combinatorial Rigidity Theory
- central objects: frameworks (dt. "Fachwerk")

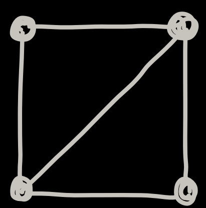


flexible
(edges stay of constant length)

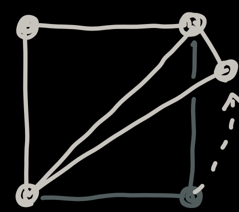
- think of it physically
  - edges are rigid metal bars of fixed length
  - vertices are universal joints (i.e. the bars can move freely but must stay attached)



flexible (we can deform while keeping edge lengths)

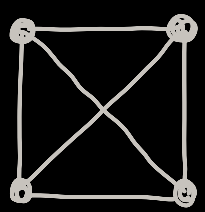


rigid in 2D
but flexible in 3D

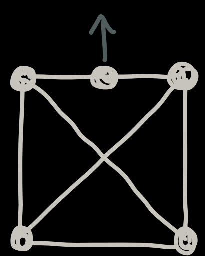


into 3D

<u>NOTE</u>: we can still move vertices and preserve edge lengths (translate + rotate)
= trivial



rigid in <u>all</u> dimensions (universally rigid)

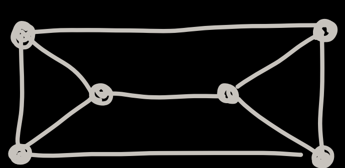


rigid ... but not in reality

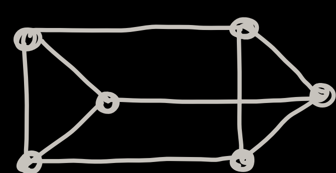
we say: not infinitesimally rigid
(first-order theory will occupy us for the first few weeks)

Q: So can we see rigidity from the graph structure?

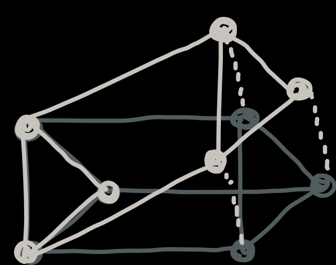
---

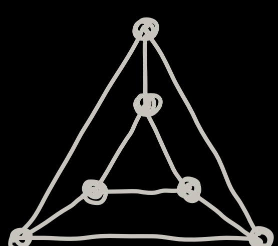


rigid

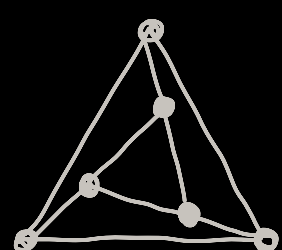


flexible





rigid but not inf. rigid



rigid

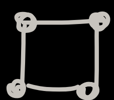
- Rigidity is <u>not</u> a combinatorial property

BUT

- the triangular prism graph is flexible only in very special cases (in 2D)

  ⟶ we say it is "generically rigid"

  this is a combinatorial property

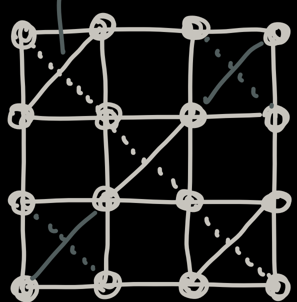
 is "generically flexible" (  is rigid )

- In reality we will only encounter generic configurations except if deliberately crafted.

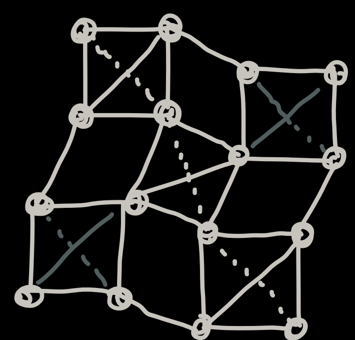
BUT... non-generic frameworks are interesting as well

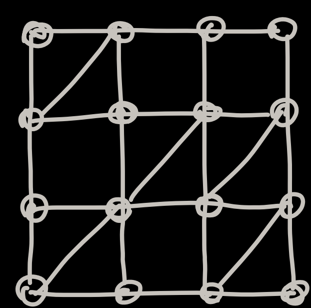
---

braces



cross-braced grid



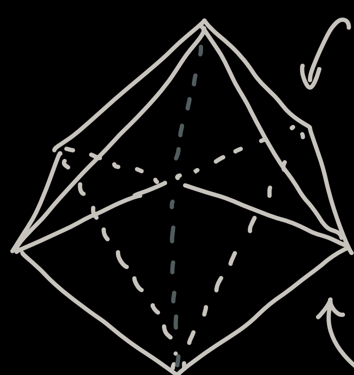


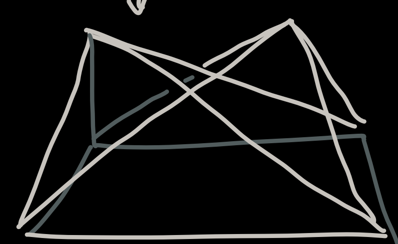
rigid (even though fewer bars than the left framework)

---

rigid

flexible



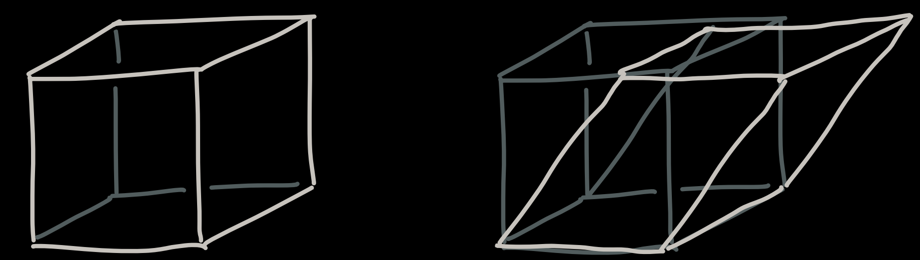


Bricard octahedra

Thm (Cauchy's rigidity theorem) (1813)

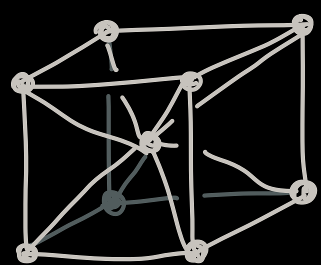
- All convex triangulated surfaces are rigid

BUT: what about other surfaces, e.g. triangulated tori?



flexible

BUT ... rigid if we fix the faces to be squares
(also Cauchy's theorem) ← triangulate the faces



coned polytope framework

→ rigid (deep result)

BUT ... not first-order rigid

Conj: second-order rigid

## Rigidity Theory

- Frameworks
- we want everything rigid
- typical questions:
  - how can we ensure rigidity?
  - what sort of rigidity are we dealing with?
  - how much rigid or how far away from rigid are we?
- stress, ...

we study this

## Mobility Theory

- Linkages
- we want things to move
- typical questions
  - in how many ways does something move (DOFs)
  - topological properties of the realization space
  - how can I make things move in a precisely determined way?
- kinematic chains, forces, ...