Part I. Frameworks

§1. Frameworks and Rigidity

or "Working towards a notion of rigidity"

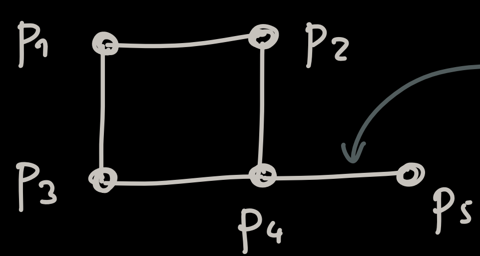
- $G=(V,E)$ will always be a finite simple graph

  V ... vertex set, $V=[n]=\{1,\dots,n\}$

  E ... edge set, we write $ij\in E$ but also $i\sim j$

Def: A (d-dimensional) framework is a pair (G,p) with a graph G and a map $p: V \xrightarrow[\text{realization}]{} \mathbb{R}^d$.

- edges might cross + vert's might intersect



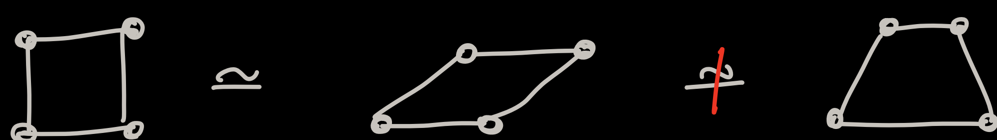
edges are straight lines

- p refers to all points, $p_i\in\mathbb{R}^d$ is one point
- we also write $p\in\mathbb{R}^{dV}=(\mathbb{R}^d)^V=(p_1,\dots,p_n)$

Def:

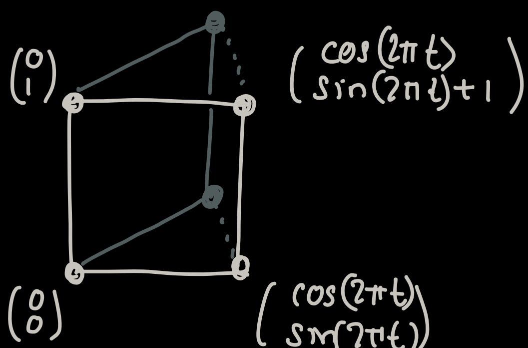
- two frameworks (G,p) and (G,q) are equivalent (we write $(G,p)\simeq(G,q)$) if

$$\|p_i-p_j\|^2=\|q_i-q_j\|^2 \quad \forall ij\in E$$

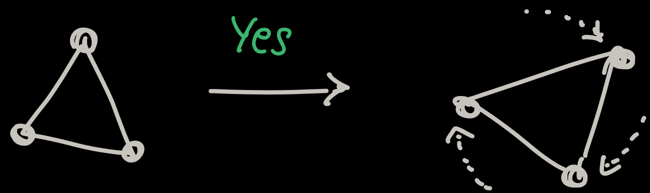


- A **motion** of (G,p) is a continuous function $p(t): [0,1] \times V \longrightarrow \mathbb{R}^d$ so that $p(0) = p$ and $(G, p(t)) \simeq (G,p)$ $\forall t$

<u>Exmpl</u>



Q: Does the triangle have a motion?



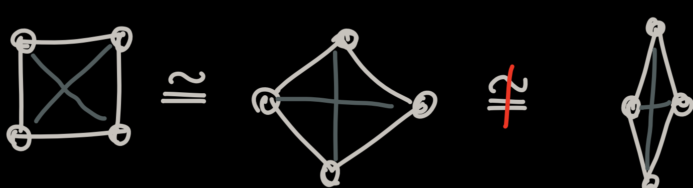
<u>Note</u>: translations and rotations are motions!

... but we don't want to count them.

- Two frameworks (G,p) and (G,q) are **congruent** (and we write $(G,p) \cong (G,q)$) if (will also write $p \cong q$)

$$\|p_j - p_i\| = \|q_j - q_i\| \quad \forall\, i,j \in V$$

Ex: $p \cong q$ iff $p = Tq + v$ for $T \in O(\mathbb{R}^d)$, $v \in \mathbb{R}^d$
called isometries



- a motion $p(t)$ of (G,p) is **trivial** if $(G, p(t)) \cong (G,p)$ $\forall t$. A non-trivial motion is called a **flex**.

Remark: some authors use the terms

flex for motion

non-trivial flex for flex (I might do so as well unintentionally)

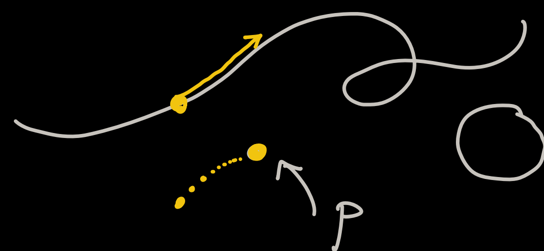
- Even though we have all this in place there are still at least two plausible ways to define rigid / flexible frameworks

Def: A framework (G,p) is

- **continuously rigid** if every motion is trivial
- **locally rigid** if $\exists \varepsilon > 0$ s.t. for all (G,q) with

$$\|p_i - q_i\| < \varepsilon \quad \forall i \in V$$

we have $(G,p) \cong (G,q)$.



Remark: $p(t)$ is a trivial motion for (G,p) iff $p(t)$ is a motion for (K_n, p). $\rightarrow$ each framework of K_n is rigid by definition.

Thm (*): (Asimow - Roth, 1978)

A framework is locally rigid iff it is continuously rigid.

$\Longrightarrow$ we only need one notion of rigidity

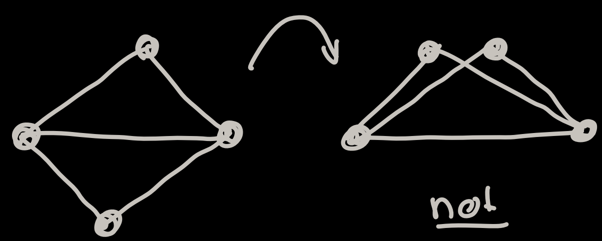
We say that (G,p) is **rigid** if it is either, and **flexible** otherwise.

<u>Note:</u> • there is something much stronger we could ask for:

$$\text{if } (G,p) \simeq (G,q) \text{ then } (G,p) \cong (G,q)$$

• This is known as global rigidity.

"There is a unique framework with these edge lengths."



rigid

<u>not</u> globally rigid

<u>NOTE:</u> complete graphs are rigid by definition

Because for them: l.r. = c.r.

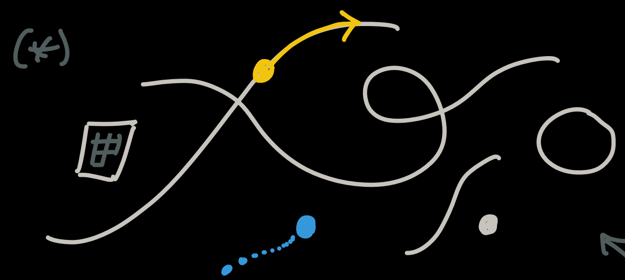
Def: $\mathrm{REAL}(G,p) := \{ q : V \to \mathbb{R}^d \mid (G,p) \simeq (G,q) \}$

$$= \{ q \in \mathbb{R}^{dV} \mid \|p_i - p_j\|^2 = \|q_i - q_j\|^2 \ \forall ij \in E \}$$

... realization space of (G,p)

- REAL is an (quadratic) algebraic variety ← quadratic polynomials — defined by polynomial ident.

(∗)



not cont. rigid

not locally rigid

actually $\mathrm{REAL}^*(G,p)$

- A motion of (G,p) can now be defined as a map

$$p(t) : [0,1] \longrightarrow \mathrm{REAL}(G,p)$$

Q: What is the realization space of

- a point (in 2D) $\longrightarrow \mathbb{R}^2$
- a line (in 2D) $\longrightarrow \mathbb{R}^2 \times SO(\mathbb{R}^2)$ ← no reflections for lines
- a triangle (in 2D) $\longrightarrow \mathbb{R}^2 \times O(\mathbb{R}^2) =: \mathrm{Iso}$
- a square (in 2D) $\longrightarrow S^1 \times \mathrm{Iso}$

⟶ the above picture (∗) is not accurate

- $\mathrm{REAL}(G,p) = \underbrace{\mathrm{REAL}(G,p)/_{\cong}}_{\mathrm{REAL}^*(G,p)}$ "$\times$" Iso

$\mathrm{REAL}^*(G,p)$ ... reduced realization space

Iso ... group of isometries $\subseteq \{ (T,v) \mid T \in O(\mathbb{R}^d),\ v \in \mathbb{R}^d \}$

acts on $\mathrm{REAL}(G,p)$ via $(T,v) \circ p := Tp + v$

- $\mathrm{REAL}^*(G,p)$ is not an alg. variety
  ... BUT still a semi-algebraic set. ← defined by polynomial identities and inequal.

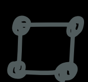
  How can we see this: we have to choose one representative from each isometry class

- Idea: suppose $p_0, \dots, p_d$ are affinely independent (→ or less if $n < d+1$)

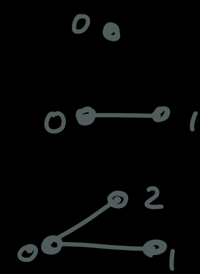
  i.e. $\alpha_0 p_0 + \dots + \alpha_d p_d = 0$ has no non-trivial solutions

  we require

example



$$\begin{cases} p_0 = (0, \dots, 0) & \longrightarrow \text{ kills translations} \\ p_1 = (\underset{\geq 0}{p_1^1}, 0, \dots, 0) & \longrightarrow \text{ kills one rotation} \\ p_2 = (p_2^1, \underset{\geq 0}{p_2^2}, 0, \dots, 0) & \text{ —— '' ——} \\ \vdots & \\ p_d = (p_d^1, \dots, p_d^{d-1}, \underset{\geq 0}{p_d^d}) & \longrightarrow \text{ kills reflections} \end{cases}$$



- all of this clearly expressible via polynomial inequal.
- each framework p has a uniqe congruent representative p^* in REAL^*

  Ex: still continuous?
- each motion $p(t)$ gives a map $p^*(t) : [0,1] \longrightarrow \mathrm{REAL}^*(G,p)$
- a trivial motion $p(t)$ gives a constant map $p^*(t)$

  i.e. $p(t)$ is a flex $\longleftrightarrow$ $p^*(t)$ is not constant locally rigid $\leftrightarrow$ (G,p) is an isolated point in $\mathrm{REAL}^*(G,p)$

continuously rigid $\leftrightarrow$ no non-constant path in $\mathrm{REAL}^*(G,p)$ starts in (G,p)

globally rigid $\leftrightarrow$ (G,p) is the <u>only</u> point in $\mathrm{REAL}^*(G,p)$

<u>Proof</u> (*)

l.r. $\rightarrow$ c.r. : obvious now. A non-constant path cannot start in an isolated point

$\neg$l.r. $\Rightarrow$ $\neg$c.r.

c.r. $\rightarrow$ l.r. : needs some help from real alg. geometry + topology

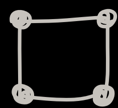
<u>Thm</u>: semi-algebraic sets are locally path connected

i.e. every point has a neighborhood (basis) $U \subseteq \mathrm{REAL}^*(G,p)$ that is path-connected

- if (G,p) is <u>not</u> l.r. then U contains another point $q \neq p$
- Since U is path-connected there is a path $p^*(t)$ from p to q $\longrightarrow$ non-constant $\longrightarrow$ $\neg$c.r.

$\square$

- Studying rigidity of (G,p) is (strictly speaking) studying local properties of $\mathrm{Real}^{+}(G,p)$
- Some simple geometric arguments lead to counting considerations

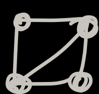
<u>Example</u>: why is  not rigid

- $\dim \mathbb{R}^{2V} = 8$ = DOFs
- each length constraint removes one freedom … there are 4 constraints

$\longrightarrow$ $8-4=4=3+1$ one flex

trivial motions $\binom{d+1}{2}$

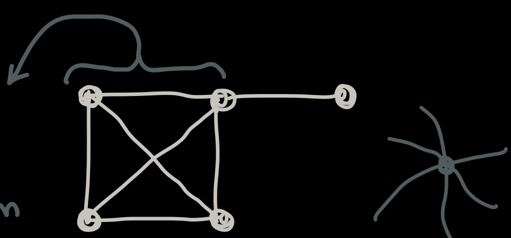
- if we add an edge

 we get $8-5=3+0$

$\rightsquigarrow$ rigid

- Idea good, but we already know this can fail

this part has more edges than necessary for rigidity



$10-7=3+0$ but flexible

$12-8=4=3+1$ but rigid

- counting can never be accurate because realization spaces can be very complicated

Q: So how to figure out whether a framework is rigid?

NOTE: rigidity is decidable

- decidability of first-order theory of real closed fields
- computational algebraic geometry
  ↳ Gröbner bases (computationally infeasible)

Thm: (G,p) is rigid in 1D $\Leftrightarrow$ G is connected

Proof:
- if non-connected: NEVER rigid
- if connected and we fix a vertex, every other vertex is fixed as well. (Ex) $\square$

Thm: (Abbot, 2008) For $d \geq 2$ deciding rigidity is NP hard.

Thm: (Kemper universality theorem)

A suitable linkage can draw your signature

= every algebraic curve is the realization space of a suitable linkage

→ no general solution in sight

we need tricks and tools → §2. First-order Theory