Appendix A:

Fourier transform:

$$\hat{f}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ix\omega}dx$$

1D Wavelet functions:

$$\Psi_{S_x,T_x}(x)=\Psi\left(\frac{x-T_x}{S_x}\right) \qquad \hat{\Psi}_{S_x,T_x}(\omega)=S_x e^{-iT_x\omega}\hat{\Psi}(S_x\omega)$$

where *S* is the scale parameter, *T* is the translation parameter

Morlet wavelet:

$$\mathrm{M}_a(x)=\pi^{-1/4}e^{iax}e^{-x^2/2} \qquad \mathrm{M}_{-a}(x)=\mathrm{M}_a{}^*(x)$$

$$\hat{\mathrm{M}}_a(\omega)=\pi^{-1/4}e^{-(\omega-a)^2/2} \qquad \hat{\mathrm{M}}_{-a}(-\omega)=\hat{\mathrm{M}}_a(\omega) \tag{A1}$$

$a>0$ must be sufficiently large to ensure $\hat{\mathrm{M}}_a(0)\approx 0$ (admissible condition)

1D continuous wavelet transform (CWT):

$$W_{\Psi_{S_x,T_x}}f:=\frac{\int_{-\infty}^{\infty} f(x)\Psi_{S_x,T_x}{}^*(x)dx}{\delta x^{1/2}S_x^{1/2}}=\frac{S_x^{1/2}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{iT_x\omega}\hat{\Psi}^*(S_x\omega)d\omega}{\delta x^{1/2}}$$

δx is the interval of the discrete signal *f* defined at $x=0,\delta x,2\delta x,\cdots,(N-1)\delta x$
This rescaled definition is consistent with Torrence and Compo's software.

In the discretized version, we assume *f* is a periodic function given by

$$f(x)=\sum_{k=0}^{N-1}\hat{f}_k e^{i\omega_k x}$$

where

$$\hat{f}_k=\frac{1}{N}\sum_{n=0}^{N-1} f_n e^{-in\omega_k\delta x}$$

$$\omega_k=\begin{cases}\dfrac{2\pi k}{N\delta x}, k\le N/2\\ -\dfrac{2\pi(N-k)}{N\delta x}, k>N/2\end{cases} \tag{A2}$$

$$W_{\Psi_{S_x,T_x}} f := \frac{\int_{-\infty}^{\infty} \sum_{k=0}^{N-1} \hat{f}_k e^{i\omega_k x} \Psi_{S_x,T_x}{}^*(x)dx}{\delta x^{1/2} S_x^{1/2}} = \frac{\sum_{k=0}^{N-1} \hat{f}_k \left[\int_{-\infty}^{\infty} e^{-i\omega_k x} \Psi_{S_x,T_x}(x)dx\right]^*}{\delta x^{1/2} S_x^{1/2}}$$

$$= \frac{\sqrt{2\pi} S_x^{1/2} \sum_{k=0}^{N-1} \hat{f}_k \left[e^{-iT_x\omega_k} \hat{\Psi}(S_x\omega_k)\right]^*}{\delta x^{1/2}} = \frac{\sqrt{2\pi} S_x^{1/2} \sum_{k=0}^{N-1} \hat{f}_k e^{iT_x\omega_k} \hat{\Psi}^*(S_x\omega_k)}{\delta x^{1/2}}$$

(Compare Torrence and Compo 1998 paper, equation 4)

Consider $\Psi 1 = \mathrm{M}_a$ and $\Psi 2 = \mathrm{M}_{-a}(x)$:

$$W_{\Psi 1_{S_x,T_x}} f = \frac{\sqrt{2\pi} S_x^{1/2}}{\delta x^{1/2}} \sum_{k=0}^{N-1} \hat{f}_k e^{iT_x\omega_k} \hat{\mathrm{M}}_a(S_x\omega_k)$$

$$\left(W_{\Psi 2_{S_x,T_x}} f\right)^* = \frac{\sqrt{2\pi} S_x^{1/2}}{\delta x^{1/2}} \sum_{k=0}^{N-1} \hat{f}_k{}^* e^{-iT_x\omega_k} \hat{\mathrm{M}}_{-a}(S_x\omega_k)$$

Note that $\hat{\mathrm{M}}_a(\omega) \approx 0$ if $\omega \le 0$, and $\hat{\mathrm{M}}_{-a}(\omega) \approx 0$ if $\omega \ge 0$. If *N* is odd and *f* is real, it can be shown with (A2) that $\left(W_{\Psi 2_{S_x,T_x}} f\right)^* = W_{\Psi 1_{S_x,T_x}} f$. This is not strictly true when *N* is even, but as demonstrated in section xx the deviations are expected to be small. Therefore,

$$W_{\Psi 1_{S_x,T_x}} f + W_{\Psi 2_{S_x,T_x}} f \approx 2\,\mathrm{Re}\left\{W_{\Psi 1_{S_x,T_x}} f\right\} \qquad \text{(A3)}$$

Reconstruction:

$$\int_0^\infty \frac{W_{\Psi 1_{S_x,x}} f + W_{\Psi 2_{S_x,x}} f}{S_x^{3/2}} dS_x = \frac{\sqrt{2\pi}}{\delta x^{1/2}} \int_0^\infty \frac{\sum_{k=0}^{N-1} \hat{f}_k e^{i\omega_k x} \hat{\mathrm{M}}_a^*(S_x\omega_k) + \sum_{k=0}^{N-1} \hat{f}_k e^{i\omega_k x} \hat{\mathrm{M}}_{-a}^*(S_x\omega_k)}{S_x} dS_x$$

$$\approx \frac{\sqrt{2\pi}}{\delta x^{1/2}} \left[\int_0^\infty \frac{\hat{\mathrm{M}}_a^*(u)}{u} du\right] \sum_{k=1}^{N-1} \hat{f}_k e^{i\omega_k x} = \frac{\sqrt{2\pi}}{\delta x^{1/2}} \left[\int_0^\infty \frac{\hat{\mathrm{M}}_a^*(u)}{u} du\right] \left[f(x) - \bar{f}\right]$$

For real *f*, with discretization $S(J) = S_0 2^{J\delta j}$, $\frac{dS}{dJ} = \delta j(\ln 2)S$

$$f(x) - \bar{f} \approx \frac{2\delta x^{1/2}}{\sqrt{2\pi}\left[\int_0^\infty \frac{\hat{\mathrm{M}}_a^*(u)}{u} du\right]} \mathrm{Re} \int_0^\infty \frac{W_{\Psi 1_{S_x,x}} f}{S_x^{3/2}} dS_x = \frac{2\delta x^{1/2} \delta j(\ln 2)}{\sqrt{2\pi}\left[\int_0^\infty \frac{\hat{\mathrm{M}}_a^*(u)}{u} du\right]} \mathrm{Re} \int_0^\infty \frac{W_{\Psi 1_{S_x,x}} f}{S_x^{1/2}} dJ$$

Covariance:

$$\int_0^{N\delta x} \int_0^\infty \frac{\left(W_{\Psi 1_{S_x,T_x}} f\right)\left(W_{\Psi 1_{S_x,T_x}} g\right)^* + \left(W_{\Psi 2_{S_x,T_x}} f\right)\left(W_{\Psi 2_{S_x,T_x}} g\right)^*}{S_x^2} dS_x dT_x$$

$$= \frac{2\pi}{\delta x} \int_0^{N\delta x} \int_0^\infty \frac{\sum_{m=0}^{N-1}\sum_{n=0}^{N-1} \hat{f}_m \hat{g}_n^* e^{iT_x(\omega_m - \omega_n)} \hat{\mathrm{M}}_a^*(S_x\omega_m) \hat{\mathrm{M}}_a(S_x\omega_n) + \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} \hat{f}_m \hat{g}_n^* e^{iT_x(\omega_m - \omega_n)} \hat{\mathrm{M}}_{-a}^*(S_x\omega_m) \hat{\mathrm{M}}_{-a}(S_x\omega_n)}{S_x} dS_x dT_x$$

$$\approx 2\pi N \int_0^\infty \frac{\sum_{n=1}^{N-1} \hat{f}_n \hat{g}_n^* \hat{\mathrm{M}}_a^*(S_x\omega_n) \hat{\mathrm{M}}_a(S_x\omega_n) + \sum_{n=1}^{N-1} \hat{f}_n \hat{g}_n^* \hat{\mathrm{M}}_{-a}^*(S_x\omega_n) \hat{\mathrm{M}}_{-a}(S_x\omega_n)}{S_x} dS_x$$

$$= 2\pi N \left[\int_0^\infty \frac{\hat{\mathrm{M}}_a^*(u) \hat{\mathrm{M}}_a(u)}{u} du\right] \sum_{n=1}^{N-1} \hat{f}_n \hat{g}_n^*$$

$$= 2\pi N \left[\int_0^\infty \frac{\hat{\mathrm{M}}_a^*(u) \hat{\mathrm{M}}_a(u)}{u} du\right] \left(\frac{1}{N} \sum_{n=0}^{N-1} f_n g_n^* - \bar{f}\bar{g}^*\right)$$

For real *f* and *g*, and discretize with $S(J) = S_0 2^{J\delta j}$, $\frac{dS}{dJ} = \delta j(\ln 2)S$:

$$\frac{1}{N} \sum_{n=0}^{N-1} f_n g_n - \bar{f}\bar{g} \approx \frac{1}{\pi N\left[\int_0^\infty \frac{\hat{\mathrm{M}}_a(u) \hat{\mathrm{M}}_a(u)}{u} du\right]} \mathrm{Re} \int_0^{N\delta x} \int_0^\infty \frac{\left(W_{\Psi 1_{S_x,T_x}} f\right)\left(W_{\Psi 1_{S_x,T_x}} g\right)^*}{S_x^2} dS_x dT_x$$

$$\approx \frac{\delta x \delta j(\ln 2)}{\pi N\left[\int_0^\infty \frac{\hat{\mathrm{M}}_a(u) \hat{\mathrm{M}}_a(u)}{u} du\right]} \mathrm{Re} \sum_{x=0}^{N-1} \sum_J \frac{\left(W_{\Psi 1_{S_J,x}} f\right)\left(W_{\Psi 1_{S_J,x}} g\right)^*}{S_J}$$

Appendix B: 2D discretized CWT

Assume that the wavelet function is separable

$$\Psi_{S_x,T_x,S_y,T_y}(x,y)=\Psi\left(\frac{x-T_x}{S_x},\frac{y-T_y}{S_y}\right)=\mathrm{X}\left(\frac{x-T_x}{S_x}\right)\mathrm{Y}\left(\frac{y-T_y}{S_y}\right)$$

$$\hat{\Psi}_{S_x,T_x,S_y,T_y}(\omega,\Omega)=S_xS_ye^{-iT_x\omega}e^{-iT_y\Omega}\hat{\mathrm{X}}(S_x\omega)\hat{\mathrm{Y}}(S_y\Omega)$$

For the discretized version we assume *f* is doubly periodic

$$f(x,y)=\sum_{k=0}^{N-1}\sum_{l=0}^{M-1}\hat{f}_{k,l}e^{i\omega_k x}e^{i\Omega_l y}$$

$$\hat{f}_{k,l}=\frac{1}{NM}\sum_{n=0}^{N-1}\sum_{m=0}^{M-1}f_{n,m}e^{-in\omega_k\delta x}e^{-im\Omega_l\delta y}\qquad f_{n,m}=\sum_{k=0}^{N-1}\sum_{l=0}^{M-1}\hat{f}_{k,l}e^{in\omega_k\delta x}e^{im\Omega_l\delta y}$$

$$\Omega_l=\begin{cases}\dfrac{2\pi l}{M\delta y}, l\le M/2\\ -\dfrac{2\pi(M-l)}{M\delta y}, l>M/2\end{cases}$$ (similar to A2)

$$W_{\Psi_{S_x,T_x,S_y,T_y}}f=\frac{2\pi S_x^{1/2}S_y^{1/2}\sum_{k=0}^{N-1}\sum_{l=0}^{M-1}\hat{f}_{k,l}e^{iT_x\omega_k}e^{iT_y\Omega_l}\hat{\mathrm{X}}^*(S_x\omega_k)\hat{\mathrm{Y}}^*(S_y\Omega_l)}{\delta x^{1/2}\delta y^{1/2}}$$

Consider 4 wavelet functions Ψ with

1: $\mathrm{X}=\mathrm{M}_a$, $\mathrm{Y}=\mathrm{M}_a$

2: $\mathrm{X}=\mathrm{M}_a$, $\mathrm{Y}=\mathrm{M}_{-a}$

3: $\mathrm{X}=\mathrm{M}_{-a}$, $\mathrm{Y}=\mathrm{M}_a$

4: $\mathrm{X}=\mathrm{M}_{-a}$, $\mathrm{Y}=\mathrm{M}_{-a}$

Similar to 1D case (A3), if *f* is real we have

$$\left(W_{\Psi 4_{S_x,T_x,S_y,T_y}}f\right)^*\approx W_{\Psi 1_{S_x,T_x,S_y,T_y}}f\qquad \left(W_{\Psi 3_{S_x,T_x,S_y,T_y}}f\right)^*\approx W_{\Psi 2_{S_x,T_x,S_y,T_y}}f$$

Reconstruction:

$$\int_0^\infty \int_0^\infty \frac{W_{\Psi 1_{S_x,x,S_y,y}} f + W_{\Psi 2_{S_x,x,S_y,y}} f + W_{\Psi 3_{S_x,x,S_y,y}} f + W_{\Psi 4_{S_x,x,S_y,y}} f}{S_x^{3/2} S_y^{3/2}} dS_x dS_y$$

$$\approx \frac{2\pi}{\delta x^{1/2} \delta y^{1/2}} \left[\int_0^\infty \frac{\hat{\mathrm{M}}_a^{\;*}(u)}{u} du \right]^2 \sum_{k=1}^{N-1} \sum_{l=1}^{M-1} \hat{f}_{k,l} e^{i\omega_k x} e^{i\Omega_l y}$$

$$= \frac{2\pi}{\delta x^{1/2} \delta y^{1/2}} \left[\int_0^\infty \frac{\hat{\mathrm{M}}_a^{\;*}(u)}{u} du \right]^2 \left\{ f_{n,m} - \frac{1}{N} \sum_{p=0}^{N-1} f_{p,m} - \frac{1}{M} \sum_{q=0}^{M-1} f_{n,q} + \bar{f} \right\}$$

For real *f* and $S(J) = S_0 2^{J\delta j}$, $\frac{dS}{dJ} = \delta j (\ln 2) S$:

$$f_{n,m} - \frac{1}{N} \sum_{p=0}^{N-1} f_{p,m} - \frac{1}{M} \sum_{q=0}^{M-1} f_{n,q} + \bar{f}$$

$$\approx \frac{\delta x^{1/2} \delta y^{1/2}}{\pi \left[\int_0^\infty \frac{\hat{\mathrm{M}}_a^{\;*}(u)}{u} du \right]^2} \mathrm{Re} \int_0^\infty \int_0^\infty \frac{W_{\Psi 1_{S_x,x,S_y,y}} f + W_{\Psi 2_{S_x,x,S_y,y}} f}{S_x^{3/2} S_y^{3/2}} dS_x dS_y \qquad \text{(B1)}$$

$$\approx \frac{\delta x^{1/2} \delta y^{1/2} \delta j^2 (\ln 2)^2}{\pi \left[\int_0^\infty \frac{\hat{\mathrm{M}}_a^{\;*}(u)}{u} du \right]^2} \mathrm{Re} \sum_J \sum_K \frac{W_{\Psi 1_{S_J,x,S_K,y}} f + W_{\Psi 2_{S_J,x,S_K,y}} f}{S_J^{1/2} S_K^{1/2}}$$

Note that $\frac{1}{N} \sum_{p=0}^{N-1} f_{p,m}$ is the mean in the *x* direction, $\frac{1}{M} \sum_{q=0}^{M-1} f_{n,q}$ is the mean in the *y* direction, $\bar{f}$ is the “area” mean.

Covariance

$$\int_0^{M\delta y}\int_0^{N\delta x}\int_0^{\infty}\int_0^{\infty}\frac{\left(W_{\Psi 1_{S_x,T_x,S_y,T_y}}f\right)\left(W_{\Psi 1_{S_x,T_x,S_y,T_y}}g\right)^*+\left(W_{\Psi 4_{S_x,T_x,S_y,T_y}}f\right)\left(W_{\Psi 4_{S_x,T_x,S_y,T_y}}g\right)^*}{S_x^2S_y^2}dS_xdS_ydT_xdT_y$$

$$+\int_0^{M\delta y}\int_0^{N\delta x}\int_0^{\infty}\int_0^{\infty}\frac{\left(W_{\Psi 2_{S_x,T_x,S_y,T_y}}f\right)\left(W_{\Psi 2_{S_x,T_x,S_y,T_y}}g\right)^*+\left(W_{\Psi 3_{S_x,T_x,S_y,T_y}}f\right)\left(W_{\Psi 3_{S_x,T_x,S_y,T_y}}g\right)^*}{S_x^2S_y^2}dS_xdS_ydT_xdT_y$$

$$=4\pi^2MN\left[\int_0^{\infty}\frac{\hat{\mathrm{M}}_a(u)\hat{\mathrm{M}}_a(u)}{u}du\right]^2\sum_{n=1}^{N-1}\sum_{m=1}^{M-1}\hat{f}_{n,m}\hat{g}_{n,m}{}^*$$

$$=4\pi^2MN\left[\int_0^{\infty}\frac{\hat{\mathrm{M}}_a(u)\hat{\mathrm{M}}_a(u)}{u}du\right]^2\left\{\sum_{n=0}^{N-1}\sum_{m=0}^{M-1}\hat{f}_{n,m}\hat{g}_{n,m}{}^*-\sum_{m=0}^{M-1}\hat{f}_{0,m}\hat{g}_{0,m}{}^*-\sum_{n=0}^{N-1}\hat{f}_{n,0}\hat{g}_{n,0}{}^*+\bar{f}\bar{g}^*\right\}$$

$$=4\pi^2MN\left[\int_0^{\infty}\frac{\hat{\mathrm{M}}_a(u)\hat{\mathrm{M}}_a(u)}{u}du\right]^2\left\{\frac{1}{NM}\sum_{n=0}^{N-1}\sum_{m=0}^{M-1}f_{n,m}g_{n,m}{}^*-\frac{1}{M}\sum_{l=0}^{M-1}\left(\frac{1}{N^2}\sum_{p=0}^{N-1}\sum_{k=0}^{N-1}f_{p,l}g_{k,l}{}^*\right)-\frac{1}{N}\sum_{k=0}^{N-1}\left(\frac{1}{M^2}\sum_{q=0}^{M-1}\sum_{l=0}^{M-1}f_{k,q}g_{k,l}{}^*\right)+\bar{f}\bar{g}^*\right\}$$

$$=4\pi^2MN\left[\int_0^{\infty}\frac{\hat{\mathrm{M}}_a(u)\hat{\mathrm{M}}_a(u)}{u}du\right]^2\left\{\frac{1}{NM}\sum_{n=0}^{N-1}\sum_{m=0}^{M-1}f_{n,m}g_{n,m}{}^*-\frac{1}{NM}\sum_{l=0}^{M-1}\sum_{p=0}^{N-1}f_{p,l}\left(\frac{1}{N}\sum_{k=0}^{N-1}g_{k,l}{}^*\right)-\frac{1}{NM}\sum_{k=0}^{N-1}\sum_{l=0}^{M-1}g_{k,l}{}^*\left(\frac{1}{M}\sum_{q=0}^{M-1}f_{k,q}\right)+\bar{f}\bar{g}^*\right\}$$

For real signals *f* and *g*,

$$\mathrm{Re}\int_0^{M\delta y}\int_0^{N\delta x}\int_0^{\infty}\int_0^{\infty}\frac{\left(W_{\Psi 1_{S_x,T_x,S_y,T_y}}f\right)\left(W_{\Psi 1_{S_x,T_x,S_y,T_y}}g\right)^*+\left(W_{\Psi 2_{S_x,T_x,S_y,T_y}}f\right)\left(W_{\Psi 2_{S_x,T_x,S_y,T_y}}g\right)^*}{S_x^2S_y^2}dS_xdS_ydT_xdT_y$$

$$\approx 2\pi^2MN\left[\int_0^{\infty}\frac{\hat{\mathrm{M}}_a(u)\hat{\mathrm{M}}_a(u)}{u}du\right]^2\left\{\frac{1}{NM}\sum_{n=0}^{N-1}\sum_{m=0}^{M-1}f_{n,m}g_{n,m}-\frac{1}{NM}\sum_{l=0}^{M-1}\sum_{p=0}^{N-1}f_{p,l}\left(\frac{1}{N}\sum_{k=0}^{N-1}g_{k,l}\right)-\frac{1}{NM}\sum_{k=0}^{N-1}\sum_{l=0}^{M-1}g_{k,l}\left(\frac{1}{M}\sum_{q=0}^{M-1}f_{k,q}\right)+\bar{f}\bar{g}\right\}$$

Discretization:

$$\frac{1}{NM}\sum_{n=0}^{N-1}\sum_{m=0}^{M-1}f_{n,m}g_{n,m}-\frac{1}{M}\sum_{l=0}^{M-1}\left(\frac{1}{N^2}\sum_{p=0}^{N-1}\sum_{k=0}^{N-1}f_{p,l}g_{k,l}\right)-\frac{1}{N}\sum_{k=0}^{N-1}\left(\frac{1}{M^2}\sum_{q=0}^{M-1}\sum_{l=0}^{M-1}f_{k,q}g_{k,l}\right)+\bar{f}\bar{g}$$

$$\approx\frac{\mathrm{Re}\int_0^{M\delta y}\int_0^{N\delta x}\int_0^{\infty}\int_0^{\infty}\frac{\left(W_{\Psi 1_{S_x,T_x,S_y,T_y}}f\right)\left(W_{\Psi 1_{S_x,T_x,S_y,T_y}}g\right)^*+\left(W_{\Psi 2_{S_x,T_x,S_y,T_y}}f\right)\left(W_{\Psi 2_{S_x,T_x,S_y,T_y}}g\right)^*}{S_x^2S_y^2}dS_xdS_ydT_xdT_y}{2\pi^2MN\left[\int_0^{\infty}\frac{\hat{\mathrm{M}}_a(u)\hat{\mathrm{M}}_a(u)}{u}du\right]^2}$$

$$\approx\frac{\delta x\delta y\sum_{x=0}^{N-1}\sum_{y=0}^{M-1}\mathrm{Re}\int_0^{\infty}\int_0^{\infty}\frac{\left(W_{\Psi 1_{S_x,x,S_y,y}}f\right)\left(W_{\Psi 1_{S_x,x,S_y,y}}g\right)^*+\left(W_{\Psi 2_{S_x,x,S_y,y}}f\right)\left(W_{\Psi 2_{S_x,x,S_y,y}}g\right)^*}{S_x^2S_y^2}dS_xdS_y}{2\pi^2MN\left[\int_0^{\infty}\frac{\hat{\mathrm{M}}_a(u)\hat{\mathrm{M}}_a(u)}{u}du\right]^2}\qquad\text{(B2)}$$

$$\approx\frac{\delta x\delta y\delta j^2(\ln 2)^2\sum_{x=0}^{N-1}\sum_{y=0}^{M-1}\sum_{J}\sum_{K}\mathrm{Re}\frac{\left(W_{\Psi 1_{S_J,x,S_K,y}}f\right)\left(W_{\Psi 1_{S_J,x,S_K,y}}g\right)^*+\left(W_{\Psi 2_{S_J,x,S_K,y}}f\right)\left(W_{\Psi 2_{S_J,x,S_K,y}}g\right)^*}{S_JS_K}}{2\pi^2MN\left[\int_0^{\infty}\frac{\hat{\mathrm{M}}_a(u)\hat{\mathrm{M}}_a(u)}{u}du\right]^2}$$