

Appendix A:

Fourier transform:

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

1D Wavelet functions:

$$\Psi_{S_x, T_x}(x) = \Psi\left(\frac{x - T_x}{S_x}\right) \quad \hat{\Psi}_{S_x, T_x}(\omega) = S_x e^{-iT_x \omega} \hat{\Psi}(S_x \omega)$$

where S is the scale parameter, T is the translation parameter

Morlet wavelet:

$$\begin{aligned} M_a(x) &= \pi^{-1/4} e^{iax} e^{-x^2/2} & M_{-a}(x) &= M_a^*(x) \\ \hat{M}_a(\omega) &= \pi^{-1/4} e^{-(\omega-a)^2/2} & \hat{M}_{-a}(-\omega) &= \hat{M}_a(\omega) \end{aligned} \quad (A1)$$

$a > 0$ must be sufficiently large to ensure $\hat{M}_a(0) \approx 0$ (admissible condition)

1D continuous wavelet transform (CWT):

$$W_{\Psi_{S_x, T_x}} f := \frac{\int_{-\infty}^{\infty} f(x) \Psi_{S_x, T_x}^*(x) dx}{\delta x^{1/2} S_x^{1/2}} = \frac{S_x^{1/2} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{iT_x \omega} \hat{\Psi}^*(S_x \omega) d\omega}{\delta x^{1/2}}$$

δx is the interval of the discrete signal f defined at $x = 0, \delta x, 2\delta x, \dots, (N-1)\delta x$

In the discretized version, we assume f is a periodic function given by

$$f(x) = \sum_{k=0}^{N-1} \hat{f}_k e^{i\omega_k x}$$

where

$$\begin{aligned} \hat{f}_k &= \frac{1}{N} \sum_{n=0}^{N-1} f_n e^{-in\omega_k \delta x} \\ \omega_k &= \begin{cases} \frac{2\pi k}{N\delta x}, & k \leq N/2 \\ -\frac{2\pi(N-k)}{N\delta x}, & k > N/2 \end{cases} \end{aligned} \quad (A2)$$

$$\begin{aligned}
W_{\Psi_{S_x T_x}} f &:= \frac{\int_{-\infty}^{\infty} \sum_{k=0}^{N-1} \hat{f}_k e^{i\omega_k x} \Psi_{S_x T_x}^*(x) dx}{\delta x^{1/2} S_x^{1/2}} = \frac{\sum_{k=0}^{N-1} \hat{f}_k \left[\int_{-\infty}^{\infty} e^{-i\omega_k x} \Psi_{S_x T_x}(x) dx \right]^*}{\delta x^{1/2} S_x^{1/2}} \\
&= \frac{\sqrt{2\pi} S_x^{1/2} \sum_{k=0}^{N-1} \hat{f}_k \left[e^{-iT_x \omega_k} \hat{\Psi}(S_x \omega_k) \right]^*}{\delta x^{1/2}} = \frac{\sqrt{2\pi} S_x^{1/2} \sum_{k=0}^{N-1} \hat{f}_k e^{iT_x \omega_k} \hat{\Psi}^*(S_x \omega_k)}{\delta x^{1/2}}
\end{aligned}$$

Consider $\Psi_1 = M_a$ and $\Psi_2 = M_{-a}(x)$:

$$\begin{aligned}
W_{\Psi_1 S_x T_x} f &= \frac{\sqrt{2\pi} S_x^{1/2}}{\delta x^{1/2}} \sum_{k=0}^{N-1} \hat{f}_k e^{iT_x \omega_k} \hat{M}_a(S_x \omega_k) \\
(W_{\Psi_2 S_x T_x} f)^* &= \frac{\sqrt{2\pi} S_x^{1/2}}{\delta x^{1/2}} \sum_{k=0}^{N-1} \hat{f}_k^* e^{-iT_x \omega_k} \hat{M}_{-a}(S_x \omega_k)
\end{aligned}$$

Note that $\hat{M}_a(\omega) \approx 0$ if $\omega \leq 0$, and $\hat{M}_{-a}(\omega) \approx 0$ if $\omega \geq 0$. If N is odd and f is real, it

can be shown with (A2) that $(W_{\Psi_2 S_x T_x} f)^* = W_{\Psi_1 S_x T_x} f$. This is not strictly true when N is

even, but as demonstrated in section xx the deviations are expected to be small.

Therefore,

$$W_{\Psi_1 S_x T_x} f + W_{\Psi_2 S_x T_x} f \approx 2 \operatorname{Re} \{ W_{\Psi_1 S_x T_x} f \} \quad (\text{A3})$$

Reconstruction:

$$\begin{aligned} \int_0^\infty \frac{W_{\Psi 1_{S_x, x}} f + W_{\Psi 2_{S_x, x}} f}{S_x^{3/2}} dS_x &= \frac{\sqrt{2\pi}}{\delta x^{1/2}} \int_0^\infty \frac{\sum_{k=0}^{N-1} \hat{f}_k e^{i\omega_k x} \hat{M}_a^*(S_x \omega_k) + \sum_{k=0}^{N-1} \hat{f}_k e^{i\omega_k x} \hat{M}_{-a}^*(S_x \omega_k)}{S_x} dS_x \\ &\approx \frac{\sqrt{2\pi}}{\delta x^{1/2}} \left[\int_0^\infty \frac{\hat{M}_a^*(u)}{u} du \right] \sum_{k=1}^{N-1} \hat{f}_k e^{i\omega_k x} = \frac{\sqrt{2\pi}}{\delta x^{1/2}} \left[\int_0^\infty \frac{\hat{M}_a^*(u)}{u} du \right] [f(x) - \bar{f}] \end{aligned}$$

For real f , with discretization $S(J) = S_0 2^{J\delta J}$, $\frac{dS}{dJ} = \delta J (\ln 2) S$

$$f(x) - \bar{f} \approx \frac{2\delta x^{1/2}}{\sqrt{2\pi} \left[\int_0^\infty \frac{\hat{M}_a^*(u)}{u} du \right]} \operatorname{Re} \int_0^\infty \frac{W_{\Psi 1_{S_x, x}} f}{S_x^{3/2}} dS_x = \frac{2\delta x^{1/2} \delta J (\ln 2)}{\sqrt{2\pi} \left[\int_0^\infty \frac{\hat{M}_a^*(u)}{u} du \right]} \operatorname{Re} \int_0^\infty \frac{W_{\Psi 1_{S_x, x}} f}{S_x^{1/2}} dJ$$

Covariance:

$$\begin{aligned} &\int_0^{N\delta x} \int_0^\infty \frac{(W_{\Psi 1_{S_x, T_x}} f)(W_{\Psi 1_{S_x, T_x}} g)^* + (W_{\Psi 2_{S_x, T_x}} f)(W_{\Psi 2_{S_x, T_x}} g)^*}{S_x^2} dS_x dT_x \\ &= \frac{2\pi}{\delta x} \int_0^{N\delta x} \int_0^\infty \frac{\sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \hat{f}_m \hat{g}_n^* e^{iT_x(\omega_m - \omega_n)} \hat{M}_a^*(S_x \omega_m) \hat{M}_a(S_x \omega_n) + \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \hat{f}_m \hat{g}_n^* e^{iT_x(\omega_m - \omega_n)} \hat{M}_{-a}^*(S_x \omega_m) \hat{M}_{-a}(S_x \omega_n)}{S_x} dS_x dT_x \\ &\approx 2\pi N \int_0^\infty \frac{\sum_{n=1}^{N-1} \hat{f}_n \hat{g}_n^* \hat{M}_a^*(S_x \omega_n) \hat{M}_a(S_x \omega_n) + \sum_{n=1}^{N-1} \hat{f}_n \hat{g}_n^* \hat{M}_{-a}^*(S_x \omega_n) \hat{M}_{-a}(S_x \omega_n)}{S_x} dS_x \\ &= 2\pi N \left[\int_0^\infty \frac{\hat{M}_a^*(u) \hat{M}_a(u)}{u} du \right] \sum_{n=1}^{N-1} \hat{f}_n \hat{g}_n^* \\ &= 2\pi N \left[\int_0^\infty \frac{\hat{M}_a^*(u) \hat{M}_a(u)}{u} du \right] \left(\frac{1}{N} \sum_{n=0}^{N-1} f_n g_n^* - \bar{f} \bar{g}^* \right) \end{aligned}$$

For real f and g , and discretize with $S(J) = S_0 2^{J\delta J}$, $\frac{dS}{dJ} = \delta J (\ln 2) S$:

$$\begin{aligned} \frac{1}{N} \sum_{n=0}^{N-1} f_n g_n - \bar{f} \bar{g} &\approx \frac{1}{\pi N \left[\int_0^\infty \frac{\hat{M}_a(u) \hat{M}_a(u)}{u} du \right]} \operatorname{Re} \int_0^{N\delta x} \int_0^\infty \frac{(W_{\Psi 1_{S_x, T_x}} f)(W_{\Psi 1_{S_x, T_x}} g)^*}{S_x^2} dS_x dT_x \\ &\approx \frac{\delta x \delta J (\ln 2)}{\pi N \left[\int_0^\infty \frac{\hat{M}_a(u) \hat{M}_a(u)}{u} du \right]} \operatorname{Re} \sum_{x=0}^{N-1} \sum_J \frac{(W_{\Psi 1_{S_J, x}} f)(W_{\Psi 1_{S_J, x}} g)^*}{S_J} \end{aligned}$$

Appendix B: 2D discretized CWT

Assume that the wavelet function is separable

$$\Psi_{S_x, T_x, S_y, T_y}(x, y) = \Psi\left(\frac{x - T_x}{S_x}, \frac{y - T_y}{S_y}\right) = X\left(\frac{x - T_x}{S_x}\right)Y\left(\frac{y - T_y}{S_y}\right)$$

$$\hat{\Psi}_{S_x, T_x, S_y, T_y}(\omega, \Omega) = S_x S_y e^{-iT_x \omega} e^{-iT_y \Omega} \hat{X}(S_x \omega) \hat{Y}(S_y \Omega)$$

For the discretized version we assume f is doubly periodic

$$f(x, y) = \sum_{k=0}^{N-1} \sum_{l=0}^{M-1} \hat{f}_{k,l} e^{i\omega_k x} e^{i\Omega_l y}$$

$$\hat{f}_{k,l} = \frac{1}{NM} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} f_{n,m} e^{-in\omega_k \delta x} e^{-im\Omega_l \delta y} \quad f_{n,m} = \sum_{k=0}^{N-1} \sum_{l=0}^{M-1} \hat{f}_{k,l} e^{in\omega_k \delta x} e^{im\Omega_l \delta y}$$

$$\Omega_l = \begin{cases} \frac{2\pi l}{M\delta y}, l \leq M/2 \\ -\frac{2\pi(M-l)}{M\delta y}, l > M/2 \end{cases} \quad (\text{similar to A2})$$

$$W_{\Psi_{S_x, T_x, S_y, T_y}} f = \frac{2\pi S_x^{1/2} S_y^{1/2} \sum_{k=0}^{N-1} \sum_{l=0}^{M-1} \hat{f}_{k,l} e^{iT_x \omega_k} e^{iT_y \Omega_l} \hat{X}^*(S_x \omega_k) \hat{Y}^*(S_y \Omega_l)}{\delta x^{1/2} \delta y^{1/2}}$$

Consider 4 wavelet functions Ψ with

- 1: $X = M_a, Y = M_a$
- 2: $X = M_a, Y = M_{-a}$
- 3: $X = M_{-a}, Y = M_a$
- 4: $X = M_{-a}, Y = M_{-a}$

Similar to 1D case (A3), if f is real we have

$$(W_{\Psi_{4S_x, T_x, S_y, T_y}} f)^* \approx W_{\Psi_{1S_x, T_x, S_y, T_y}} f \quad (W_{\Psi_{3S_x, T_x, S_y, T_y}} f)^* \approx W_{\Psi_{2S_x, T_x, S_y, T_y}} f$$

Reconstruction:

$$\begin{aligned}
& \int_0^\infty \int_0^\infty \frac{W_{\Psi 1_{S_x, x, S_y, y}} f + W_{\Psi 2_{S_x, x, S_y, y}} f + W_{\Psi 3_{S_x, x, S_y, y}} f + W_{\Psi 4_{S_x, x, S_y, y}} f}{S_x^{3/2} S_y^{3/2}} dS_x dS_y \\
& \approx \frac{2\pi}{\delta x^{1/2} \delta y^{1/2}} \left[\int_0^\infty \frac{\hat{M}_a^*(u)}{u} du \right]^2 \sum_{k=1}^{N-1} \sum_{l=1}^{M-1} \hat{f}_{k,l} e^{i\omega_k x} e^{i\Omega_l y} \\
& = \frac{2\pi}{\delta x^{1/2} \delta y^{1/2}} \left[\int_0^\infty \frac{\hat{M}_a^*(u)}{u} du \right]^2 \left\{ f_{n,m} - \frac{1}{N} \sum_{p=0}^{N-1} f_{p,m} - \frac{1}{M} \sum_{q=0}^{M-1} f_{n,q} + \bar{f} \right\}
\end{aligned}$$

For real f and $S(J) = S_0 2^{J\delta j}$, $\frac{dS}{dJ} = \delta j (\ln 2) S$:

$$\begin{aligned}
& f_{n,m} - \frac{1}{N} \sum_{p=0}^{N-1} f_{p,m} - \frac{1}{M} \sum_{q=0}^{M-1} f_{n,q} + \bar{f} \\
& \approx \frac{\delta x^{1/2} \delta y^{1/2}}{\pi \left[\int_0^\infty \frac{\hat{M}_a^*(u)}{u} du \right]^2} \text{Re} \int_0^\infty \int_0^\infty \frac{W_{\Psi 1_{S_x, x, S_y, y}} f + W_{\Psi 2_{S_x, x, S_y, y}} f}{S_x^{3/2} S_y^{3/2}} dS_x dS_y \quad (\text{B1}) \\
& \approx \frac{\delta x^{1/2} \delta y^{1/2} \delta j^2 (\ln 2)^2}{\pi \left[\int_0^\infty \frac{\hat{M}_a^*(u)}{u} du \right]^2} \text{Re} \sum_J \sum_K \frac{W_{\Psi 1_{S_J, x, S_K, y}} f + W_{\Psi 2_{S_J, x, S_K, y}} f}{S_J^{1/2} S_K^{1/2}}
\end{aligned}$$

Note that $\frac{1}{N} \sum_{p=0}^{N-1} f_{p,m}$ is the mean in the x direction, $\frac{1}{M} \sum_{q=0}^{M-1} f_{n,q}$ is the mean in the y direction, \bar{f} is the “area” mean.

Covariance

$$\begin{aligned}
& \int_0^{M\delta y} \int_0^{N\delta x} \int_0^\infty \int_0^\infty \frac{\left(W_{\Psi 1_{S_x, T_x, S_y, T_y}} f \right) \left(W_{\Psi 1_{S_x, T_x, S_y, T_y}} g \right)^* + \left(W_{\Psi 4_{S_x, T_x, S_y, T_y}} f \right) \left(W_{\Psi 4_{S_x, T_x, S_y, T_y}} g \right)^*}{S_x^2 S_y^2} dS_x dS_y dT_x dT_y \\
& + \int_0^{M\delta y} \int_0^{N\delta x} \int_0^\infty \int_0^\infty \frac{\left(W_{\Psi 2_{S_x, T_x, S_y, T_y}} f \right) \left(W_{\Psi 2_{S_x, T_x, S_y, T_y}} g \right)^* + \left(W_{\Psi 3_{S_x, T_x, S_y, T_y}} f \right) \left(W_{\Psi 3_{S_x, T_x, S_y, T_y}} g \right)^*}{S_x^2 S_y^2} dS_x dS_y dT_x dT_y \\
& = 4\pi^2 MN \left[\int_0^\infty \frac{\hat{M}_a(u) \hat{M}_a(u)}{u} du \right]^2 \sum_{n=1}^{N-1} \sum_{m=1}^{M-1} \hat{f}_{n,m} \hat{g}_{n,m}^* \\
& = 4\pi^2 MN \left[\int_0^\infty \frac{\hat{M}_a(u) \hat{M}_a(u)}{u} du \right]^2 \left\{ \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} \hat{f}_{n,m} \hat{g}_{n,m}^* - \sum_{m=0}^{M-1} \hat{f}_{0,m} \hat{g}_{0,m}^* - \sum_{n=0}^{N-1} \hat{f}_{n,0} \hat{g}_{n,0}^* + \bar{f} \bar{g}^* \right\} \\
& = 4\pi^2 MN \left[\int_0^\infty \frac{\hat{M}_a(u) \hat{M}_a(u)}{u} du \right]^2 \left\{ \frac{1}{NM} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} f_{n,m} g_{n,m}^* - \frac{1}{M} \sum_{l=0}^{M-1} \left(\frac{1}{N^2} \sum_{p=0}^{N-1} \sum_{k=0}^{N-1} f_{p,l} g_{k,l}^* \right) - \frac{1}{N} \sum_{k=0}^{N-1} \left(\frac{1}{M^2} \sum_{q=0}^{M-1} \sum_{l=0}^{M-1} f_{k,q} g_{k,l}^* \right) + \bar{f} \bar{g}^* \right\} \\
& = 4\pi^2 MN \left[\int_0^\infty \frac{\hat{M}_a(u) \hat{M}_a(u)}{u} du \right]^2 \left\{ \frac{1}{NM} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} f_{n,m} g_{n,m}^* - \frac{1}{NM} \sum_{l=0}^{M-1} \sum_{p=0}^{N-1} f_{p,l} \left(\frac{1}{N} \sum_{k=0}^{N-1} g_{k,l}^* \right) - \frac{1}{NM} \sum_{k=0}^{N-1} \sum_{l=0}^{M-1} g_{k,l}^* \left(\frac{1}{M} \sum_{q=0}^{M-1} f_{k,q} \right) + \bar{f} \bar{g}^* \right\}
\end{aligned}$$

For real signals f and g ,

$$\begin{aligned}
& \text{Re} \int_0^{M\delta y} \int_0^{N\delta x} \int_0^\infty \int_0^\infty \frac{\left(W_{\Psi 1_{S_x, T_x, S_y, T_y}} f \right) \left(W_{\Psi 1_{S_x, T_x, S_y, T_y}} g \right)^* + \left(W_{\Psi 2_{S_x, T_x, S_y, T_y}} f \right) \left(W_{\Psi 2_{S_x, T_x, S_y, T_y}} g \right)^*}{S_x^2 S_y^2} dS_x dS_y dT_x dT_y \\
& \approx 2\pi^2 MN \left[\int_0^\infty \frac{\hat{M}_a(u) \hat{M}_a(u)}{u} du \right]^2 \left\{ \frac{1}{NM} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} f_{n,m} g_{n,m} - \frac{1}{NM} \sum_{l=0}^{M-1} \sum_{p=0}^{N-1} f_{p,l} \left(\frac{1}{N} \sum_{k=0}^{N-1} g_{k,l} \right) - \frac{1}{NM} \sum_{k=0}^{N-1} \sum_{l=0}^{M-1} g_{k,l} \left(\frac{1}{M} \sum_{q=0}^{M-1} f_{k,q} \right) + \bar{f} \bar{g} \right\}
\end{aligned}$$

Discretization:

$$\begin{aligned}
& \frac{1}{NM} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} f_{n,m} g_{n,m} - \frac{1}{M} \sum_{l=0}^{M-1} \left(\frac{1}{N^2} \sum_{p=0}^{N-1} \sum_{k=0}^{N-1} f_{p,l} g_{k,l} \right) - \frac{1}{N} \sum_{k=0}^{N-1} \left(\frac{1}{M^2} \sum_{q=0}^{M-1} \sum_{l=0}^{M-1} f_{k,q} g_{k,l} \right) + \bar{f} \bar{g} \\
& \approx \frac{\text{Re} \int_0^{M\delta y} \int_0^{N\delta x} \int_0^\infty \int_0^\infty \frac{\left(W_{\Psi 1_{S_x, T_x, S_y, T_y}} f \right) \left(W_{\Psi 1_{S_x, T_x, S_y, T_y}} g \right)^* + \left(W_{\Psi 2_{S_x, T_x, S_y, T_y}} f \right) \left(W_{\Psi 2_{S_x, T_x, S_y, T_y}} g \right)^*}{S_x^2 S_y^2} dS_x dS_y dT_x dT_y}{2\pi^2 MN \left[\int_0^\infty \frac{\hat{M}_a(u) \hat{M}_a(u)}{u} du \right]^2} \\
& \approx \frac{\delta x \delta y \sum_{x=0}^{N-1} \sum_{y=0}^{M-1} \text{Re} \int_0^\infty \int_0^\infty \frac{\left(W_{\Psi 1_{S_x, T_x, S_y, T_y}} f \right) \left(W_{\Psi 1_{S_x, T_x, S_y, T_y}} g \right)^* + \left(W_{\Psi 2_{S_x, T_x, S_y, T_y}} f \right) \left(W_{\Psi 2_{S_x, T_x, S_y, T_y}} g \right)^*}{S_x^2 S_y^2} dS_x dS_y}{2\pi^2 MN \left[\int_0^\infty \frac{\hat{M}_a(u) \hat{M}_a(u)}{u} du \right]^2} \\
& \approx \frac{\delta x \delta y \delta^2 (\ln 2)^2 \sum_{x=0}^{N-1} \sum_{y=0}^{M-1} \sum_J \sum_K \text{Re} \frac{\left(W_{\Psi 1_{S_J, T_J, S_K, T_K}} f \right) \left(W_{\Psi 1_{S_J, T_J, S_K, T_K}} g \right)^* + \left(W_{\Psi 2_{S_J, T_J, S_K, T_K}} f \right) \left(W_{\Psi 2_{S_J, T_J, S_K, T_K}} g \right)^*}{S_J S_K}}{2\pi^2 MN \left[\int_0^\infty \frac{\hat{M}_a(u) \hat{M}_a(u)}{u} du \right]^2} \quad (\text{B2})
\end{aligned}$$