Universidad Complutense de Madrid

Proyecto de Innovación Educativa en Computación Cuántica

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Friday group 1

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Session 1: Visual introduction to quantum circuits with Quirk

2. Exercises

2.1. What are the following quantum gate products equivalent to?

(a) $H \cdot H = I$

(b) $X \cdot X = I$

(c) $Y \cdot Y = I$

(d) $Z \cdot Z = I$

(e) $X^{1/2} \cdot X^{1/2} = X$

(f) $X^{1/4} \cdot X^{1/4} = X^{1/2}$

$$X^{1/2} = \pm\frac{1}{2}\begin{pmatrix} 1 \pm i & 1 \mp i \\ 1 \mp i & 1 \pm i \end{pmatrix}$$

However, in QUIRK's implementation

$$X^{1/2} = \frac{1}{2}\begin{pmatrix} 1 + i & 1 - i \\ 1 - i & 1 + i \end{pmatrix}$$

(g) $H \cdot Y \cdot H = -Y$

2.2. Remember that a *global* phase change does not alter a qubit. Build a quantum circuit that, from a single qubit initialized to $|0\rangle$, generates the following 1-qubit states:

(a) $|1\rangle = X\,|0\rangle$

(b) $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = H\,|0\rangle$

(c) $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = H \cdot X\,|0\rangle$

(d) $|R\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle) = H \cdot X^{1/2}\,|0\rangle$

(e) $|L\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i\,|1\rangle) = H \cdot X^{1/2} \cdot X\,|0\rangle$

3. Unitary transformations

3.1. (conceptual, not on QUIRK) Which of the following operations is/are NOT unitary, if any? Not unitary: a), c), d), e), f)

(a) Sorting a list of elements.

A sorted list could have been given in any initial configuration, so this operation is not invertible and thus not unitary.

(b) Applying a certain permutation.

Say we have a quantum state $|\psi\rangle = |\psi\rangle_1\,|\psi\rangle_2 \ldots |\psi\rangle_n$ and a permutation operator which acts as $P\,|\psi\rangle = |\psi\rangle_{P_1}\,|\psi\rangle_{P_2} \ldots |\psi\rangle_{P_n}$, $\{P_i\}_{i=1}^n \in S_n$. It is clear that it preserves the inner product, since

$$\langle\phi|\,P^\dagger P\,|\psi\rangle = \langle\phi|_{P_1}\,|\psi\rangle_{P_1}\,\langle\phi|_{P_2}\,|\psi\rangle_{P_2} \ldots \langle\phi|_{P_n}\,|\psi\rangle_{P_n} = \langle\phi|_1\,|\psi\rangle_1\,\langle\phi|_2\,|\psi\rangle_2 \ldots \langle\phi|_n\,|\psi\rangle_n = \langle\phi|\psi\rangle$$

Therefore the permutation operator P is unitary.

(c) Adding two elements (a and b) keeping none of them.

Any two numbers in a field can add to another number in said field, therefore the operation is not invertible and thus not unitary.

* This question was later changed to: *Adding two elements* $a, b \in \{0, 1, \ldots, d-1\} \mod d$*, keeping none of them* ($|a\rangle\,|b\rangle \to |a \overset{d}{\oplus} b\rangle\,|0\rangle$)*.*

For the same reason as in the original question, the operation is not invertible. Even if a, b belong to some finite list, there are multiple possible combinations which result in the same sum.

(d) Adding two elements (a and b) keeping one and only one of them

This operation may be invertible, however in matrix form:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a+b \\ b \end{pmatrix}$$

The matrix inverse of this operation is not its adjoint, in fact it is

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

* This question was later changed to: *Adding two elements* $a, b \in \{0, 1, \ldots, d-1\} \mod d$*, keeping one and only one of them* ($|a\rangle\,|b\rangle \to |a \overset{d}{\oplus} b\rangle\,|b\rangle$)*.*

In this case the operation *would* be unitary. If $a, b \in \{0, 1, \ldots, d-1\}$, clearly also $a \overset{d}{\oplus} b \in \{0, 1, \ldots, d-1\}$. We can then build a single element Hilbert space $\mathcal{H} = \text{span}\{|i\rangle\}_{i=0}^{d-1}$ and a tensor product space $\mathcal{H}^2 = \mathcal{H} \otimes \mathcal{H}$. In these spaces both $|a\rangle\,|b\rangle$ and $|a \overset{d}{\oplus} b\rangle\,|b\rangle$ would be basis kets — the operation would then be norm preserving only if it permuted the basis kets, i.e. if it didn't map multiple kets to one. This is certainly the case, since $a \overset{d}{\oplus} b \neq a' \overset{d}{\oplus} b \;\; \forall \;\; a \neq a'$.

(e) Symmetryzing two states. That is, if we have $|i\rangle\,|j\rangle$, generating $\frac{1}{\sqrt{2}}(|i\rangle\,|j\rangle + |j\rangle\,|i\rangle)$.

The symmetrisation operator $P_S = \frac{1}{2} \sum_{\pi \in S_2} \pi$ is a projector to the subspace of symmetric states and thus not invertible. Originally $|i\rangle\,|j\rangle$ could have had an unknown antisymmetric component which symmetrisation would eliminate.

(f) Given a list of elements, generating a new list of elements that verify certain properties without keeping a copy of the original list. For instance, if the list is numerical, keeping only numbers that are greater that some given number.

In the very example provided, the original list could have had any number below the cutoff, so the operation is not in general invertible.

3.2. Check on QUIRK that the Pauli-X, Y and Z operations, represented on the computational basis $\{|0\rangle, |1\rangle\}$ by the Pauli matrices σ_x, σ_y and σ_z, are unitary. Compute their inverses and check that they are equal to their transpose conjugates.

We can easily check by hand that

$$\sigma_x = \sigma_x^{-1} = \sigma_x^\dagger = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_y = \sigma_y^{-1} = \sigma_y^\dagger = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_z = \sigma_z^{-1} = \sigma_z^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

4. Multi-qubit quantum gates

4.1. Control gates: `CNOT` and others

(a) Hover the mouse pointer over the two default qubit lines and check that both qubits are on the $|0\rangle$ state.

(b) Drop a `NOT` gate over the second line and annotate the change on the corresponding Bloch sphere.

$\texttt{NOT}|0\rangle = |1\rangle$

(c) Drop a `control` gate (small black circle) over the first line and on the same horizontal position that the first Pauli-X (`NOT`) gate. A new vertical segment joining the `control` and the `NOT` gate should appear. Annotate any change on any of the Bloch spheres.

$\texttt{CNOT}|00\rangle = |00\rangle$

A controlled gate only takes effect if the control qubit has a $|1\rangle$ component.

(d) Change the state of the first qubit to $|1\rangle$ by clicking on the $|0\rangle$ symbol on the left of the corresponding qubit line. Annotate any changes on the Bloch spheres.

$\texttt{CNOT}|01\rangle = |11\rangle$

The controlled gate produces a bit flip.

(e) Repeat the experiment using an `anti-control` (small white circle) instead of the `control`.

The `anti-control` does the opposite, it takes effect if the control qubit has a $|0\rangle$ component.

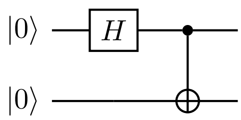
(f) Repeat the experiment using a Hadamard gate instead of the `NOT` one.

The control logic is the same as before, the only difference is $H|0\rangle = |+\rangle$.

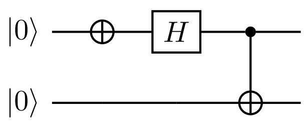
4.2. Building entangled Bell states

Using a Hadamard gate and a `CNOT` gate whose control is applied to the output of the Hadamard gate, build a quantum circuit that turns a 2-qubit initial state $|0\rangle \otimes |0\rangle$ to the following Bell states:

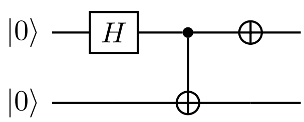
(a) $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)$



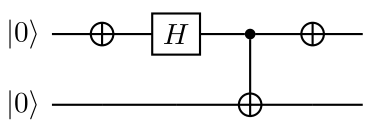
(b) $|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle)$



(c) $|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle)$



(d) $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle)$



4.3. No qubit cloning, even by the CNOT gate

4.3.1. The 1-qubit state $|-\rangle$ is defined as above in 2b by

$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

Using the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ to describe a 2-qubit quantum state, answer the following questions:

(a) What is the 2-qubit state if each qubit has been initialized to $|-\rangle$?

The state is $|-\rangle \otimes |-\rangle = \frac{1}{2}(|00\rangle - |01\rangle - |10\rangle + |11\rangle)$.

(b) The first qubit is initialized to $|-\rangle$ and the second, to $|0\rangle$. What is the 2-qubit state if a `CNOT` gate has been applied targeting the second qubit, the first one being the control?

The state is $\texttt{CNOT}(|0\rangle \otimes |-\rangle) = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$.

(c) Can you use a `CNOT` gate for breaking the no-clone theorem? Please, justify your answer on the previous questions of this exercise.

At the very least, as this counterexample shows, we cannot use the `CNOT` gate to clone states such that $\texttt{CNOT}(|0\rangle \otimes |\psi\rangle) = |\psi\rangle \otimes |\psi\rangle$.

4.3.2. Using Hadamard gates, prepare 2 quantum registers, each one composed of 2 qubits. The states of this registers will be:

- Register A: $|-\rangle \otimes |-\rangle = (HX\,|0\rangle) \otimes (HX\,|0\rangle)$
- Register B: the first qubit is initialized to $|-\rangle = HX\,|0\rangle$. The second one is initialized by applying a `CNOT` gate to the quantum state $|0\rangle$. The control of this `CNOT` gate is the first qubit, previously initialized to $|-\rangle$.

Operator X is the Pauli-X operator (`NOT` gate). Answer the following questions:

(a) Compute the probability of measuring a $|1\rangle$ state in each of the four qubits.

The probability to measure a $|1\rangle$ in each qubit is 50%.

(b) Is there any difference among the four Bloch spheres?

The Bloch spheres for each qubit are very different, we can only obtain states of the form $|00\text{xx}\rangle$ or $|11\text{xx}\rangle$, meaning the qubits in register B can only be measured in the same state. The first two are in a superposition of all possible states.

(c) Include an additional Hadamard gate in each qubit before the measurement. Taking into account that $H^2 = I$, do you see anything strange? Can a `CNOT` gate alter the *control* qubit? Does the concept of individual qubit make any sense in the presence of quantum entanglement effects?

Given that $H^2 = I$, the first two qubits now become $X \otimes X\,|00\rangle = |11\rangle$, whereas the second two become the Bell state $|\Psi^+\rangle$ (as in exercise 4.2). A `CNOT` gate *does* in general alter the control qubit — in fact, it only doesn't when the control qubit is either in the $|0\rangle$ or in the $|1\rangle$ state. The notion of individual qubit breaks down when logic gates apply operations to all qubits and they become entangled (their measurement outcomes become correlated).

(d) Using the Kronecker (tensor) product and the matrix formalism, compute by hand (or by a matricial language like Octave or Matlab) the effect of 2 Hadamard gates, $H \otimes H$, on a Bell state, $\left|\Phi^-\right\rangle = \dfrac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle)$. Can you explain the QUIRK results?

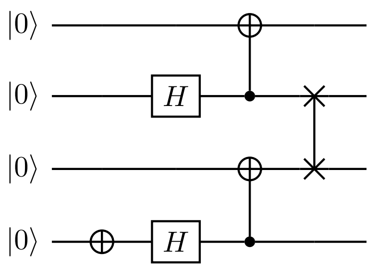
The result is $|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$. This explains the previous exercise (c), where register B is initialised to $|\Phi^-\rangle$. Now we see explicitly that $H^{\otimes 2}\,|\Phi^-\rangle = |\Psi^+\rangle$.

4.4. Quantum entanglement of distant qubits via SWAP gates

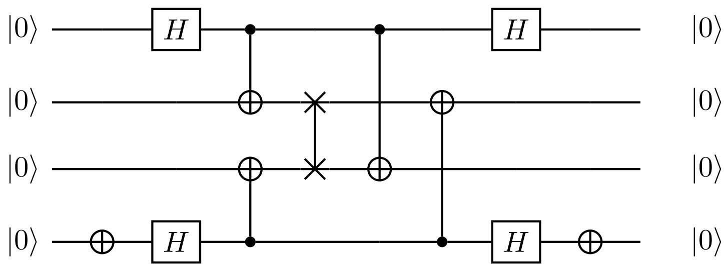
On the memory of current state-of-the-art quantum computers the qubits are coupled only to their near neighbours. In order to entangle distant qubits, the 2-qubit swapping gate

(SWAP) is used. Such a gate is implemented on QUIRK. Please, use it to solve the following exercise:

(a) Build a 4-qubit quantum circuit. It should generate 2 Bell pairs, $|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$. These pairs are formed by qubits $(1,3)$ for $|\Psi^+\rangle$ and by $(2,4)$ for $|\Psi^-\rangle$. As a restriction, consider that the `CNOT` gates can only be applied to nearest-neighbour qubits. That is, they can entangle qubits $(1,2)$, $(2,3)$ or $(3,4)$. You should use a single `SWAP` gate, as it would be done on an actual quantum computer.



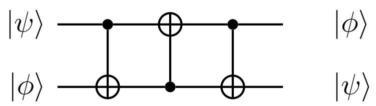
(b) Check that qubits $(1,3)$ and $(2,4)$ are entangled, so that they form Bell pairs. In order to prove this, use the fact that $UU^\dagger = I$. Here, $U =$`CNOT`$\cdot(H\otimes I)$ and $U^\dagger = (H\otimes I)\cdot$ `CNOT`, since $H^\dagger = H^{-1} = H$ and $\texttt{CNOT}^\dagger = \texttt{CNOT}^{-1} = \texttt{CNOT}$. Hence, if you reverse the quantum circuit that generated the Bell pairs $|\Psi^-\rangle\,|\Psi^+\rangle$, a $|0000\rangle$ state should be recovered. Check this with the pairs $(1,3)$ and $(2,4)$. Repeat the process with qubits $(1,2)$ and $(3,4)$, that should not be entangled. Repeat the process, changing the initial state of each of the 4 qubits.



By applying the circuit which creates the Bell pairs in reversed order (swapping the corresponding qubits) we recover $|0000\rangle$ as expected.

(c) Build a gate that swaps the values of 2 qubits with 3 `CNOT` gates. Compare its action over the 4 possible input values with the `SWAP` gate.

The `SWAP` gate may be decomposed as follows:



This composite gate has the same effect as a `SWAP` gate. In the $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ basis, the matrices of these gates read

$$\texttt{CNOT}_{1\to 2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\texttt{CNOT}_{2\to1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\texttt{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The matrix product $\texttt{CNOT}_{1\to2}\texttt{CNOT}_{2\to1}\texttt{CNOT}_{1\to2}$ indeed yields a $\texttt{SWAP}$ gate. Its action on the basis elements is given by its column vectors.

(d) Can you implement the permutation of 5 qubits $(1,2,3,4,5) \to (2,4,1,3,5)$ in Quirk?