Educational Innovation Project UCM-UPM
Quantum Computing
Exercise 1: QUIRK Visual Interface

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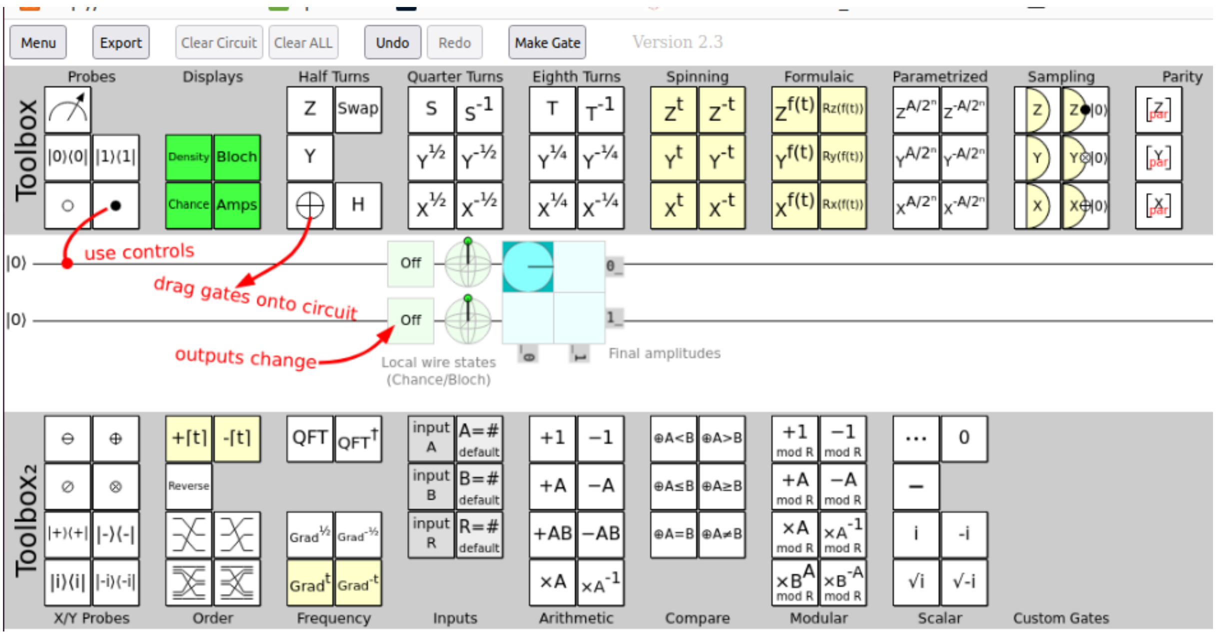
February 2024

1 Loading QUIRK

1. Do not shut down your laboratory computer when you finish
2. Type the following URL on your browser: `https://algassert.com/quirk`
3. Click on "Edit Circuit". You should see the following interface:



You should be able to identify the quantum circuit band. Each black horizontal line corresponds to a single qubit, whose evolution can be followed from left to right. The menus include different quantum gates with one or several qubits. They should be dragged and dropped (Drag'n'Drop) with your mouse to the required place on your quantum circuit.

2 1-qubit quantum gates

1. Hover the mouse pointer over the Bloch spheres of the two qubit horizontal lines that are shown by default. Check that both qubits are set to the state $|0\rangle$.
2. Drop a Pauli-X (=`NOT`) gate over the first qubit line and annotate the change on the corresponding Bloch sphere.
3. Drop a Hadamard-H gate over the second qubit line and annotate the change.

2.1 Exercises

1. What are the following quantum gate products equivalent to?

   (a) $H \cdot H$

   (b) $X \cdot X$

   (c) $Y \cdot Y$

   (d) $Z \cdot Z$

   (e) $X^{1/2} \cdot X^{1/2}$

   (f) $X^{1/4} \cdot X^{1/4}$

   (g) $H \cdot Y \cdot H$

2. Remember that a *global* phase change does not alter a qubit. Build a quantum circuit that, from a single qubit initialized to $|0\rangle$, generates the following 1-qubit states:

   (a) $|1\rangle$

   (b) $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

   (c) $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

   (d) $|R\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle)$

   (e) $|L\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i\,|1\rangle)$.

3 Unitary transformations

Quantum computing has two main fundamental limitations that distinguish it from classical computing: (a) only unitary transformations are typically applied (unless measurements are introduced, which end the coherent computation) and (b) cloning (copying) arbitrary quantum states is not possible. The second feature will be discussed later. Let us focus on the first one in the next lines.
Recall that unitary transformations U preserve the complex scalar product among state vectors. Denoting $|\psi_i'\rangle \equiv U\,|\psi_i\rangle$,

$$\langle\psi_1|\psi_2\rangle = \langle\psi_1'|\psi_2'\rangle \Leftrightarrow \langle\psi_1|\psi_2\rangle = \langle\psi_1|\,U^\dagger U\,|\psi_2\rangle \Leftrightarrow U^\dagger U = I, \tag{1}$$

Hence, U is always invertible (note that qubits are finite dimensional systems) and

$$U^{-1} = U^\dagger. \tag{2}$$

This equation is useful to check for the unitarity of a transformation; use it to solve the following exercises.

Exercise 3.1 (conceptual, not on QUIRK) Which of the following operations is/are NOT unitary, if any?

1. Sorting a list of elements (e.g. by alphabetic order $|p\rangle|l\rangle|a\rangle|n\rangle|c\rangle|k\rangle \to |a\rangle|c\rangle|k\rangle|l\rangle|n\rangle|p\rangle$).

2. Applying a certain permutation π ($|\psi_1\rangle|\psi_2\rangle\ldots|\psi_N\rangle \to |\psi_{\pi(1)}\rangle|\psi_{\pi(2)}\rangle\ldots|\psi_{\pi(N)}\rangle$).

3. Adding two elements $a, b \in \{0, 1\ldots, d-1\}$ mod d, keeping none of them ($|a\rangle|b\rangle \to |a \overset{d}{\oplus} b\rangle|0\rangle$).

4. Adding two elements $a, b \in \{0, 1\ldots, d-1\}$ mod d, keeping one and only one of them ($|a\rangle|b\rangle \to |a \overset{d}{\oplus} b\rangle|b\rangle$).

5. Symmetryzing two states. That is, if we have $|\psi\rangle\,|\phi\rangle$, generating $\frac{1}{\sqrt{2}}\,(|\psi\rangle\,|\phi\rangle + |\phi\rangle\,|\psi\rangle)$.

6. Given a list of elements, generating a new list of elements that verify certain properties without keeping a copy of the original list. For instance, if the list is numerical, keeping only numbers that are greater that some given number.

Exercise 3.2 Check on QUIRK that the Pauli-X, Y and Z operations, represented on the computational basis $\{|0\rangle, |1\rangle\}$ by the Pauli matrices σ_x, σ_y and σ_z, are unitary. Compute their inverses and check that they are equal to their transpose conjugates.

Unitarity imposes serious limitations on what is possible. Hence, on quantum computing it is usual to add auxiliary or *ancillary* qubits. They keep the necessary information to reverse the quantum operation. Using a measurement to collapse (project) the quantum state to a certain subspace is also usual.

4 Multi-qubit quantum gates

4.1 Control gates: `CNOT` and others

1. Hover the mouse pointer over the two default qubit lines and check that both qubits are on the $|0\rangle$ state.
2. Drop a `NOT` gate over the second line and annotate the change on the corresponding Bloch sphere.
3. Drop a `control` gate (small black circle) over the first line and on the same horizontal position that the first Pauli-X (`NOT`) gate. A new vertical segment joining the `control` and the `NOT` gate should appear. Annotate any change on any of the Bloch spheres.
4. Change the state of the first qubit to $|1\rangle$ by clicking on the $|0\rangle$ symbol on the left of the corresponding qubit line. Annotate any changes on the Bloch spheres.
5. Repeat the experiment using an `anti-control` (small white circle) instead of the `control`.
6. Repeat the experiment using a Hadamard gate instead of the `NOT` one.

4.2 Building entangled Bell states

Using a Hadamard gate and a `CNOT`[1] gate whose control is applied to the output of the Hadamard gate, build a quantum circuit that turns a 2-qubit initial state $|0\rangle \otimes |0\rangle$ to the following Bell states:

1. $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)$
2. $|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle)$
3. $|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle)$
4. $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle)$

4.3 No qubit cloning, even by the CNOT gate

Because of the No-Cloning Theorem, a process whose effect is copying the whole quantum state of a system into another identical system is not physically possible. Now, let $\texttt{CNOT}\,|xy\rangle$ represent a `CNOT` gate whose first qubit x is the control qubit. Since

$$\texttt{CNOT}\,|00\rangle = |00\rangle \quad \texttt{CNOT}\,|10\rangle = |11\rangle$$

and, because of linearity,

$$\texttt{CNOT}(\alpha\,|00\rangle + \beta\,|10\rangle) = \alpha\,|00\rangle + \beta\,|11\rangle\,,$$

the `CNOT` gate might appear to allow the cloning of the first qubit onto the second originally set to 0. This would be a violation of the no-cloning theorem. In order to prove that the `CNOT` gate does NOT provide a way of violating the no-cloning theorem, study the following exercises.

[1] `NOT` gate with control.

1. The 1-qubit state $|-\rangle$ is defined as above in 2b by

$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

   Using the base $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ to describe a 2-qubit quantum state, answer the following questions:

   (a) What is the 2-qubit state if each qubit has been initialized to $|-\rangle$?

   (b) The first qubit is initialized to $|-\rangle$ and the second, to $|0\rangle$. What is the 2-qubit state if a `CNOT` gate has been applied targeting the second qubit, the first one being the control?

   (c) Can you use a `CNOT` gate for breaking the no-clone theorem? Please, justify your answer on the previous questions of this exercise.

2. Using Hadamard gates, prepare 2 quantum registers, each one composed of 2 qubits. The states of this registers will be:

   - Register A: $|-\rangle \otimes |-\rangle = (HX|0\rangle) \otimes (HX|0\rangle)$
   - Register B: the first qubit is initialized to $|-\rangle = HX|0\rangle$. The second one is initialized by applying a `CNOT` gate to the quantum state $|0\rangle$. The control of this `CNOT` gate is the first qubit, previously initialized to $|-\rangle$.

   Operator X is the Pauli-X operator (`NOT` gate). Answer the following questions:

   (a) Compute the probability of measuring a $|1\rangle$ state in each of the four qubits.

   (b) Is there any difference among the four Bloch spheres? Anything strange on the first? Go on nonetheless.

   (c) Apply an additional Hadamard gate to each qubit before the measurement. Taking into account that $H^2 = I$, do you see anything strange? Can a `CNOT` gate alter the *control* qubit?

   (d) You can understand this behavior later on a piece of paper (or by a matricial language like Octave or Matlab) to see how entanglement makes the supposed `CNOT` gate act differently from the classical case. Use the Kronecker (tensor) product and the matrix formalism, to obtain the effect of 2 Hadamard gates, $H \otimes H$, on a Bell state, $|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle)$. Can you explain the QUIRK results?

4.4 Quantum entanglement of distant qubits via SWAP gates

On the memory of current state-of-the-art quantum computers the qubits are coupled only to their near neighbours. In order to entangle distant qubits, the 2-qubit swapping gate (`SWAP`) is used. Such a gate is implemented on QUIRK. Please, use it to solve the following exercise:

1. Build a 4-qubit quantum circuit. It should generate 2 Bell pairs, $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and $|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$. These pairs are formed by qubits (1,3) for $|\Phi^+\rangle$ and by (2,4) for $|\Phi^-\rangle$.

   As a restriction, consider that the `CNOT` gates can only be applied to nearest-neighbour qubits. That is, they can entangle qubits (1,2), (2,3) or (3,4). You should use a single `SWAP` gate, as it would be done on an actual quantum computer.

2. Check that qubits (1,3) and (2,4) are entangled. To this end, rewrite the previous circuit without using the `SWAP` gate, and the apply it in reversed order to the output of the previous circuit. You should recover the initial state $|0000\rangle$. Use the fact that $UU^\dagger = I$, with $U = \texttt{CNOT} \cdot (H \otimes I)$ and $U^\dagger = (H \otimes I) \cdot \texttt{CNOT}$, since $H^\dagger = H^{-1} = H$ and $\texttt{CNOT}^\dagger = \texttt{CNOT}^{-1} = \texttt{CNOT}$.

3. Build a gate that swaps the values of 2 qubits with 3 `CNOT` gates. Compare its action over the 4 possible input states with the `SWAP` gate.

4. Can you implement the permutation of 5 qubits $(1, 2, 3, 4, 5) \rightarrow (2, 4, 1, 3, 5)$ in Quirk?