Introduction to Caustics

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Abstract

Caustics are the envelopes of light rays that have been reflected or refracted by a curved surface, or the envelopes of projections of those light rays. We see them as bright fringes where light "bunches up", for instance, at the bottom of a swimming pool, on the inside of a mug as it reflects incident light, or projected by a glass as light shines through it. They are due to the focusing of light, that is, many rays intersecting along the caustic — this notion allows for a powerful mathematical description. This project covers the caustics of plane reflections and refractions, projections of perturbed plane wavefronts, and dispersion in prisms.

1. Objectives

The purpose of this project is to study caustics from an intuitive point of view. It will explore some basic problems in geometrical optics from which caustics arise, in particular those that are close to our daily experience or those that can be studied without specialised equipment. This report consists of:

1. A mathematical breakdown of caustics.
2. Analytical solutions to a few sample problems.
3. Experiments performed by me.
4. Computer simulations as a visual aid.

Though caustics show up in many unexpected places such as quantum mechanics, this project is concerned exclusively with geometrical optics. Furthermore, caustic curves will be explored here in 2D only, mostly due to ease of visualisation and ease of experimentation (although it should be noted that in this report I have intentionally provided methods for calculating caustics that generalise to higher dimensions). This project will not analyse the shapes of caustics in anisotropic media, for which the eikonal equation and a generalised version of Snell's law (Holm, 2012, p. 131) are required. Derivations will employ some interesting mathematics, but will mostly neglect the more advanced theory used in caustics research and will forgo quite a bit of mathematical rigour.

The aim of this report is to compile introductory information on caustics in such a way that a reader uninformed about the topic may develop a clear sense of what caustics are and how to calculate them. Its contents are intended to serve as a stepping stone for understanding harder problems related to caustics, and to provide mathematical tools which may be used in other contexts.

2. Introduction

2.1. Envelope

The envelope of a family of curves is the set of limits of the curves' points of intersection (Bruce and Giblin (1981)). There are intuitive ways to calculate it in $\mathbb{R}^2$, but there is a notion that applies to more complex problems. Consider a set of curves in parametric form, $\vec{u}(\vec{x})$, where $\vec{x}$ and $\vec{u}$ each have n coordinates. By definition, along the envelope the curves intersect each other and $\vec{u}(\vec{x})$ is non-invertible. The inverse function theorem then implies the Jacobian of $\vec{u}$ is singular, so $\vec{x}$ is subject to some constraint. Furthermore, we are interested only in envelopes of light rays, which are normal to a set of wavefronts. There thus exists a scalar $f(\vec{x})$ such that $\vec{u}(\vec{x}) = \nabla f(\vec{x})$. Envelopes of light rays are also degenerate singularities (locally "flat", vanishing gradient and Hessian) with respect to $\vec{x}$ of $F(\vec{x},\, \vec{u}) = \vec{u} \cdot \vec{x} - f(\vec{x})$ (Auer (1980), Berry (1976)).

2.2. Equation of a ray deflected by a surface

Let $\vec{i}$ be the direction of incident rays, $\vec{d}$ that of rays deflected by a surface. If the surface is parametrised as $\vec{c}(t) = (u(t), v(t))$, we obtain its normal, $\vec{n}$, and the equation of deflected rays, $\vec{r}$, as follows:

$$\vec{n} = \pm \frac{(-\dot{v}(t), \dot{u}(t))}{\sqrt{\dot{u}(t)^2 + \dot{v}(t)^2}} \qquad (x, y) \equiv \vec{r}(\lambda, t) = \vec{c}(t) + \lambda \vec{d}(t) \tag{1}$$

2.2.1. Reflection

To reflect $\vec{i}$ along $\vec{n}$ one can consider the parallelogram defined by $\vec{i}$ and $\vec{d}$. Due to symmetry with respect to $\vec{n}$ we find the direction of reflected rays is:

$$\vec{d} = \vec{i} - 2(\vec{i} \cdot \vec{n})\vec{n} \tag{2}$$

2.2.2. Refraction

In order to find the direction of refracted rays, let us apply Snell's law:

$$n_i \sin\theta_i = n_d \sin\theta_d \tag{3}$$

We can decompose the direction of refracted rays as $\vec{d} = a\vec{i} + b\vec{n}$. To obtain the values of coefficients a and b, for $\vec{d}$ a unit vector that continues travelling forward ($a > 0$), we may solve the following system:

$$\begin{cases} a^2 + b^2 + 2ab(\vec{i} \cdot \vec{n}) = 1 \\ (a\vec{i} + b\vec{n}) \cdot \vec{n} = -\cos\theta_d \end{cases} \tag{4}$$

It yields:

$$\vec{d} = \frac{n_i}{n_d}\vec{i} - \left(\frac{n_i}{n_d}\vec{i} \cdot \vec{n} + \sqrt{1 - \left(\frac{n_i}{n_d}\right)^2 \left(1 - (\vec{i} \cdot \vec{n})^2\right)} \right) \vec{n} \tag{5}$$

2.3. Caustics

Based on Section 2.1 and Eq. (1), we know solving the following equation for $\lambda(t)$ provides the envelope of rays $\vec{r}(\lambda(t),\, t)$, i.e. their caustic [1]:

$$\det(\mathbf{J}_{\vec{r}}) = \left|\frac{\partial \vec{r}}{\partial(\lambda, t)}\right| = \begin{vmatrix} \frac{\partial x}{\partial \lambda} & \frac{\partial x}{\partial t} \\ \\ \frac{\partial y}{\partial \lambda} & \frac{\partial y}{\partial t} \end{vmatrix} = 0 \tag{6}$$

The trajectory density of a mapping $\vec{u}(\vec{x})$, where $\vec{x}$ determines wavefronts and $\vec{u}$ determines ray positions, is $I(\vec{u}) = \sum_i \left|\frac{d\vec{u}}{d\vec{x}_i(\vec{u})}\right|^{-1}$, adding over each $\vec{x}_i$ corresponding to a given $\vec{u}$ — this is effectively the rays' scattering cross section, proportional to the scattered intensity. Caustics are thus regions invariant to first order in $\vec{x}$ of infinite intensity. We call reflection caustics *catacaustics* and refraction caustics *diacaustics*.

2.4. Far Field of a Perturbed Plane Wave

Consider a plane wave propagating along the Z axis in an isotropic medium, where $\vec{R} \equiv (x, y)$ at $z = 0$ is deformed into $z = f(\vec{R})$. Let us assume $f \gg \lambda$ so we can ignore interference and $|\nabla f| \ll 1$ for algebraic simplicity. According to Berry (1976) this happens at an interface, between air and a material of refractive index n, with profile $f(\vec{R})n/(n-1)$. The direction of rays on the XY plane at "infinity" ($z \gg 2\lambda$) is thus $\vec{\Omega}(\vec{R}) = -\nabla f(\vec{R})$. It follows from Section 2.1 that the caustics at infinity are the zeros of $\left|\frac{d\vec{\Omega}}{d\vec{R}}\right|$. These caustics take many shapes, but the fact that they are degenerate singularities allows us to classify them.

2.5. Dispersion Caustics

Berry (2022) explains how the dispersion of white light by a prism produces a virtual caustic of colours, though Newton himself believed the virtual rays converged at a point (understandable, given their caustic is extremely narrow). Berry parametrises the refractive index n of a prism in terms of an abstract colour parameter μ, defined by the crystal's refractive index for green light n_g:

$$n = n_g(1 + \mu), \qquad \mu \in [-\mu_{\text{max}},\, \mu_{\text{max}}] \tag{7}$$

This parametrisation yields an expression for the refracted rays, $y(x, \mu)$. In the limit of weak dispersion ($\mu_{\text{max}} \ll 1$) one can expand $y(x, \mu)$ to second order in μ (maintaining the non-injective relation between y and μ), which results in a good approximation of the caustic:

$$y_c(x) = A - Bx - \frac{(C - Dx)^2}{4(E - Fx)}, \qquad x \in [x(-\mu_{\text{max}}),\, x(\mu_{\text{max}})] \tag{8}$$

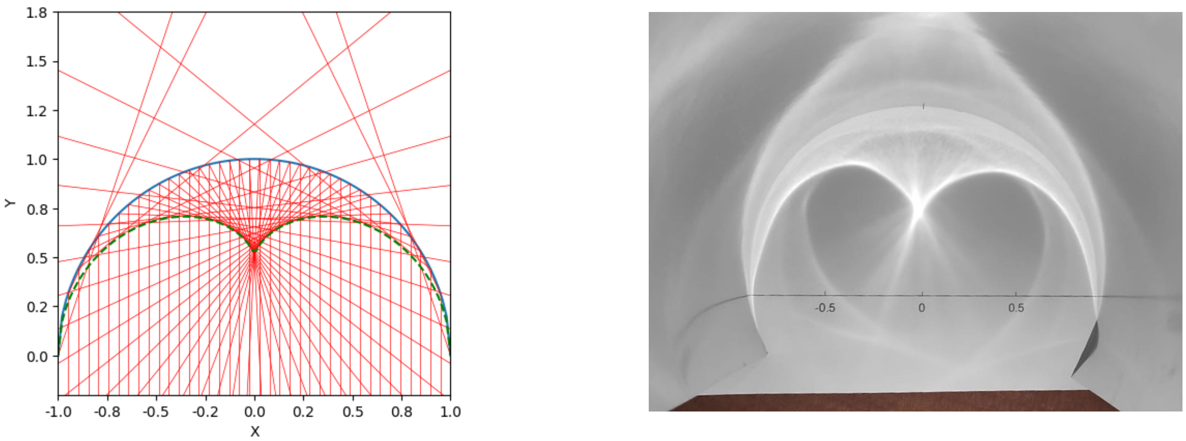
Berry provides the above coefficients for a "standard" prism, as well as formulas for $\mu(x, y)$ and indications on how to convert μ to RGB, enabling one to render the coloured rays.

[1] The envelope of a family of implicit plane curves $F(x, y, t) = 0$ is typically obtained by solving $F(x, y, t) = \partial_t F(x, y, t) = 0$. However, the procedure I explain immediately generalises to 3D and has a deeper geometric significance, in my opinion.

3. Discussion

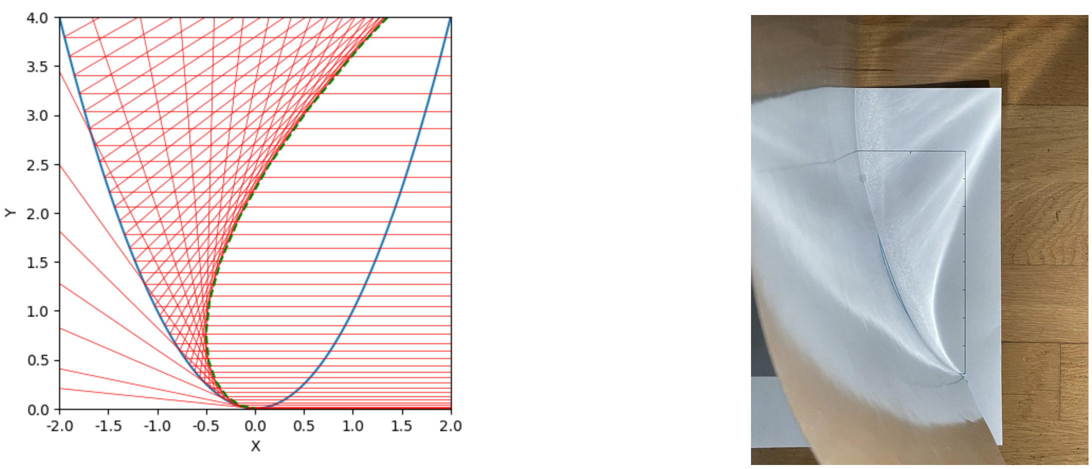
3.1. Catacaustics

Using the methods outlined in Section 2 we can calculate the mathematical formulas of reflection caustics generated by mirrors of various shapes, and try to verify them with a side by side comparison to the real caustics. To produce these I used a metallic cardboard sheet that I folded over the graph of different functions, as well as my mobile phone's flashlight. For the sake of mathematical simplicity light rays were assumed to arrive parallel, from "infinity", and accordingly I placed my flashlight far away from the mirror. The derivation of the formulas can be found in Appendix A.1. For more examples of catacaustics see Weisstein (2023).



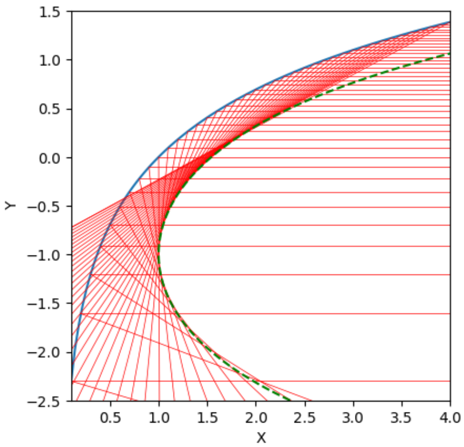
(a) $y = \sqrt{1-|x|^{2/3}}\left(1/2+|x|^{2/3}\right)$ **(b)** Caustic photograph.

Figure 1: Catacaustic of a semicircle.

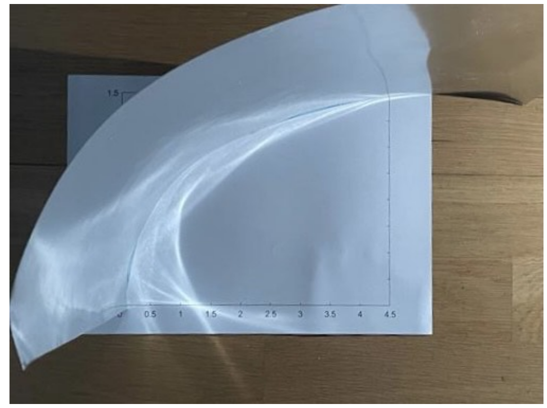


(a) $x = 3/2 \cdot t - 2t^3,\ y = 3t^2$ **(b)** Caustic photograph.

Figure 2: Catacaustic of a parabola.



(a) $x = \cosh(y+1)$

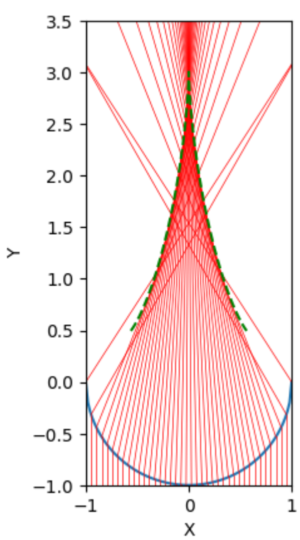


(b) Caustic photograph.

Figure 3: Catacaustic of the natural logarithm.

3.2. Diacaustics

We now use the tools in Section 2 to calculate refraction caustics. Fig. 4 shows the theoretical caustic of rays refracted once at a circular interface in order to compare it qualitatively to the caustic generated by a water glass (the full problem involves more than one refraction and is thus more complicated). In fact, the formulas involved for a single refraction are already rather unwieldy, which is why the explicit expression of $y_c(t)$ has been omitted from the caption of Fig. 4a. Luckily one can perform a Taylor expansion of y_c in powers of $\frac{1}{n}$ (where n is the refractive index of water) — unfortunately, water is not very refractive, this approach makes more sense for materials with $n \gtrsim 2$. Regardless, for problems more complicated than this one it is likely most sensible to provide a numerical solution. Water's refractive index was taken to be $n = 1.3325$ (Polyanskiy (2023)). The analytical expression of the caustic is in Appendix A.2.



(a) $x = 1/n^2 \cdot \cos^3 t,\ y = y_c(t)$

(b) Caustic photograph.

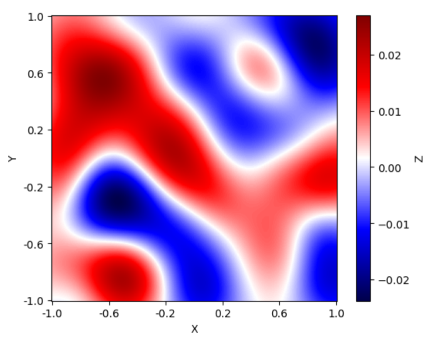
Figure 4: Diacaustic of a water glass.

3.3. Caustics in Swimming Pools and Droplets

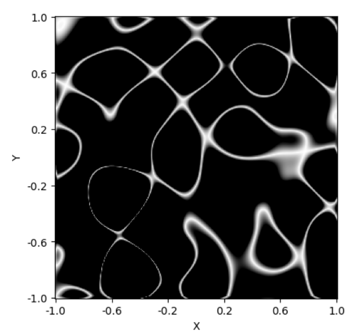
The irregular surface of a swimming pool creates caustics, which we can calculate using the formulas in Section 2.4. To understand what happens at the bottom of swimming pools, we may simulate a random wavefront and its far field projection, using a programme inspired by Nolte (2021) (code by Ferguson (2023)). Figure 5a shows such a wavefront, while Figure 5b plots the singularities of its Hessian. The far field intensity is shown in Figure 5c — knowing the height of the wavefront at (x, y), the direction of rays $\vec{\Omega}$ and the distance d to a screen, we can calculate the coordinates (x', y') of the intersection of rays with said screen:

$$(x', y') = (x, y) + (d - f(x, y))\vec{\Omega}(x, y) \tag{9}$$

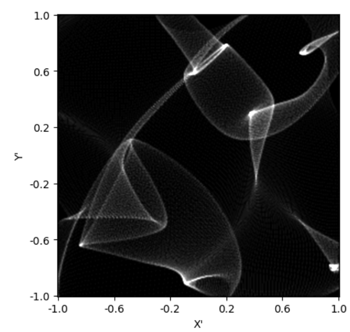
Intensity is then calculated as $I(x', y') = \sum_i \left| \frac{d(x', y')}{d(x, y)_i} \right|^{-1}$. As a numerical approximation we can add the number of rays crossing a given (x', y') coming from any $(x, y)_i$ points, on a discrete grid. [2]



(a) Original wavefront.



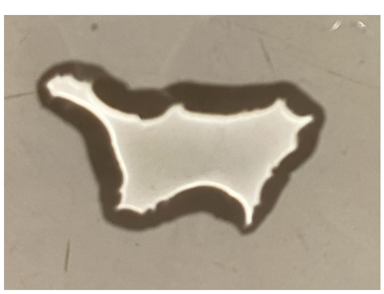
(b) Singularities of Jacobian of $\vec{\Omega}$.



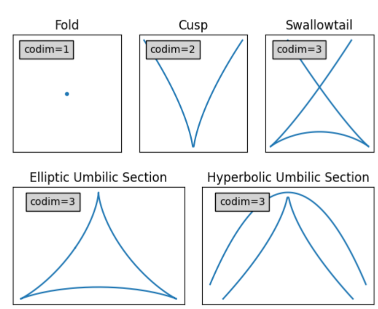
(c) Intensity at $z = 5$.

Figure 5: Caustics of a perturbed plane wavefront.

Caustics like these also appear, for instance, when we shine light through water droplets on flat glass surfaces. Fig. 6a was obtained by shining my kitchen lamp's light through a wet pan lid. Fig. 6b shows an important result from catastrophe theory: degenerate singularities of some scalar function, which is indeed what caustics are, may be classified into five elementary types (considering rays in 3D, algebraically speaking codimension ≤ 3), Auer (1980). It is easy to make out some of these shapes in Fig. 6a, in particular cusps.



(a) Far field of fluorescent lamp through droplet.



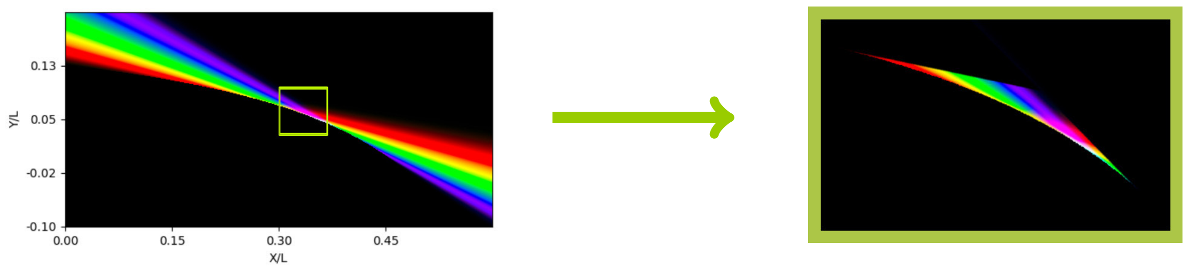
(b) Elementary catastrophes of codimension 1-3.

Figure 6: Caustics of a water droplet.

[2]This is equivalent to computing a Fraunhofer diffraction integral (Berry (1976)).

3.4. Caustics in Newton's Prism

As shown by Berry (2022), the width of the caustic of colours generated by a typical prism is on the order of micrometers, and therefore unfortunately invisible to the naked eye. I thus will merely limit myself to rendering the caustic for the "standard" prism studied by Berry: apex angle $\alpha = \pi/3$, refractive index $n_g = 1.5$ and apex-incidence distance L. The prism will have an unnaturally large dispersion $\mu_{\text{max}} = 0.1$ in order to make the caustic clearly visible. (Code by Ferguson (2023)). [3]



(a) Simulated rays.

(b) Ray overlap region in detail.

Figure 7: Virtual caustic of colours in a prism.

Fig. 7 clearly shows how the virtual rays do not converge at a point, but rather fold over themselves and create a curved caustic. This caustic is in fact a smooth line formed by a series of fold catastrophes.

4. Conclusion

Though many of the mathematical tools used to describe envelopes, caustics and catastrophes may seem somewhat abstract, I hope this report has made it clear that they directly explain easily observable (or at the very least intuitive) phenomena. Indeed, all experiments presented here are easily reproducible and modifiable. An obvious way to improve them would be to use a better reflector. The metallic carboard I used had a grainy surface and was difficult to fold smoothly, so it was hard to produce the desired catacaustics.

Given the problems I have presented are very visual, it is relatively easy to gain a qualitative understanding of the behaviour of caustics. This understanding can be applied to other subjects, for instance classical mechanics, where one could calculate the envelope of all possible trajectories in a given potential — this may be useful for understanding problems in which a statistical approach is necessary (for semiclassical and quantum considerations see Child (1996)). Caustics can also be used to describe, in a broader sense, focusing phenomena, such as the focusing of tsunamis due to underwater islands acting as "lenses" (Berry (2007)).

From a pure geometrical optics perspective, the study of caustics has direct applications in fields such as jewelry design, computer graphics, optical lens design or telecommunications. For instance, Abd Rahman et al. (2015) consider the caustics of a parabolic reflector to design a multi-beam satellite antenna for coverage of Peninsular Malaysia. Needless to say, understanding caustics is crucial in many technical environments.

[3] As a technical sidenote, Berry calculates colours in the CIE RGB space, but I have calculated them in the sRGB space.

A. Formulas

A.1. Catacaustics

A.1.1. Semicircle

Let us parametrise the reflecting surface as:

$$\vec{c} = (\cos t,\, \sin t), \quad \vec{n} = -(\cos t,\, \sin t), \quad t \in [0, \pi] \tag{10}$$

Using Eq. (2) to find the reflection of rays incident from $(0,\, 1)$, we obtain:

$$\vec{d} = (0,\, 1) - 2\sin t(\cos t,\, \sin t) = (-\sin(2t),\, \cos(2t)) \quad \Rightarrow \quad \vec{r} = \vec{c} + \lambda\vec{d} \tag{11}$$

Now calculate the Jacobian determinant:

$$\det(\mathbf{J}_{\vec{r}}) = \begin{vmatrix} -\sin(2t) & -\sin t - 2\lambda\cos(2t) \\ \cos(2t) & \cos t - 2\lambda\sin(2t) \end{vmatrix} = 2\lambda - \sin t = 0 \quad \Rightarrow \quad \lambda = \frac{1}{2}\sin t \tag{12}$$

And we obtain an equation for the caustic:

$$\vec{r} = \left(\cos^3 t,\, \sin t\left(\frac{1}{2} + \cos^2 t\right)\right) \quad \Longleftrightarrow \quad y = \sqrt{1 - |x|^{2/3}}\left(\frac{1}{2} + |x|^{2/3}\right) \tag{13}$$

A.1.2. Parabola

Let us parametrise the reflecting surface as:

$$\vec{c} = (t,\, t^2), \quad \vec{n} = \frac{(-2t,\, 1)}{\sqrt{1+4t^2}}, \quad t \in (-\infty,\, 0] \tag{14}$$

Using Eq. (2) to find the reflection of rays incident from $(-1,\, 0)$, we obtain:

$$\vec{d} = (-1,\, 0) - \frac{4t}{1+4t^2}(-2t,\, 1) = \frac{(4t^2-1,\, -4t)}{1+4t^2} \quad \Rightarrow \quad \vec{r} = \vec{c} + \lambda\vec{d} \tag{15}$$

Now calculate the Jacobian determinant:

$$\det(\mathbf{J}_{\vec{r}}) = \begin{vmatrix} \dfrac{4t^2-1}{4t^2+1} & 1 + \lambda\dfrac{16t}{(4t^2+1)^2} \\ \dfrac{-4t}{4t^2+1} & 2t + \lambda\dfrac{4(4t^2-1)}{(4t^2+1)^2} \end{vmatrix} = \frac{2(2\lambda + 4t^3 + t)}{4t^2+1} = 0 \quad \Rightarrow \quad \lambda = -\left(\frac{t}{2} + 2t^3\right) \tag{16}$$

And we obtain an equation for the caustic:

$$\vec{r} = \left(\frac{3}{2}t - 2t^3,\, 3t^2\right) \tag{17}$$

A.1.3. Natural Log

Let us parametrise the reflecting surface as:

$$\vec{c} = (e^t,\, t), \quad \vec{n} = \frac{(1,\, -e^t)}{\sqrt{1+e^{2t}}}, \quad t \in \mathbb{R} \tag{18}$$

Using Eq. (2) to find the reflection of rays incident from $(-1,\, 0)$, we obtain:

$$\vec{d} = (-1,\, 0) + 2\frac{(1,\, -e^t)}{1+e^{2t}} = \frac{(1-e^{2t},\, -2e^t)}{1+e^{2t}} \quad \Rightarrow \quad \vec{r} = \vec{c} + \lambda \vec{d} \tag{19}$$

Now calculate the Jacobian determinant:

$$\det(\mathbf{J}_{\vec{r}}) = \begin{vmatrix} -\tanh t & e^t - \lambda \operatorname{sech}^2 t \\ -\operatorname{sech} t & 1 + \lambda \operatorname{sech} t \tanh t \end{vmatrix} = 1 - \lambda \operatorname{sech} t = 0 \quad \Rightarrow \quad \lambda = \cosh t \tag{20}$$

And we obtain an equation for the caustic:

$$\vec{r} = (\cosh t,\, t-1) \quad \Longleftrightarrow \quad x = \cosh(y+1) \tag{21}$$

A.2. Diacaustics

A.2.1. Semicircle

Let us parametrise the refracting surface as:

$$\vec{c} = (\cos t,\, \sin t), \quad \vec{n} = (\cos t,\, \sin t), \quad t \in [\pi, 2\pi] \tag{22}$$

Using Eq. (5) with rays incident from $(0,\, 1)$ and defining the reciprocal refractive index of water $\mu := \frac{1}{n}$, we obtain:

$$\vec{d} = \mu \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \left(\mu \sin t + \sqrt{1-\mu^2\cos^2 t}\right) \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} = \begin{pmatrix} -\mu \sin t \cos t - \cos t\sqrt{1-\mu^2\cos^2 t} \\ \mu \cos^2 t - \sin t\sqrt{1-\mu^2\cos^2 t} \end{pmatrix} \tag{23}$$

Now calculate the Jacobian determinant of ray trajectories:

$$\det(\mathbf{J}_{\vec{r}}) = \lambda \left(1 + \frac{\mu \sin t}{\sqrt{1-\mu^2\cos^2 t}}\right) - \sqrt{1-\mu^2\cos^2 t} = 0 \quad \Rightarrow \quad \lambda = \frac{1-\mu^2\cos^2 t}{\mu \sin t + \sqrt{1-\mu^2\cos^2 t}} \tag{24}$$

And after quite a lot of algebra we finally reach a parametric equation for the caustic:

$$\begin{cases} x = \mu^2 \cos^3 t \\ y = \mu^2 \cos^2 t \sin t + \dfrac{1-\mu^2\cos^2 t}{\sin t + \frac{1}{\mu}\sqrt{1-\mu^2\cos^2 t}} = \mu - \mu^2 \sin^3 t + \mu^3 \left(1 - \dfrac{3\cos^2 t}{2}\right) + \mathcal{O}(\mu^4) \end{cases} \tag{25}$$

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