Programmikeelte semantika 2012/2014
Koduülesanded 2

Mart Karu

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1 Ülesanne 1

Osaliselt järjestatud hulk $(P_{fin}(X), \subseteq)$ ei ole ccpo, sest ccpo eelduseks on ahelatel vähima ülemise raja olemasolu, mida ei ole võimalik leida moodustatud hulgas. Kuigi moodustatud hulga elementideks on lõplikud hulgad, on nende hulkade hulk lõpmatu hulga X lõpmatuse tõttu.

2 Ülesanne 2

Ccpo olemasolu tuvastamiseks tuleb esmalt kontrollida hulga $(D_0 \times D_1, \sqsubseteq)$ osaliselt järjestatuks olemist, kontrollides refleksiivsust, transitiivsust ja antisümmeetrilisust hulga elementidel ning leida ahelates vähim ülemine raja:

Osalise järjestatuse kontroll:

1.refleksiivus: $(d_0, d_1) \sqsubseteq (d_0, d_1)$ kehtib (tehtes sisaldub võrdsus), sest $d_0 = d_0$ ja $d_1 \sqsubseteq_0 d_1$

2.transitiivsus: kui $(d_0, d_1) \sqsubseteq (e_0, e_1)$ ja $(e_0, e_1) \sqsubseteq (f_0, f_1)$, siis kehtib ka $(d_0, d_1) \sqsubseteq (f_0, f_1)$

3.antisümmeetrilisus: kui $(d_0, d_1) \sqsubseteq (e_0, e_1)$ ja $(e_0, e_1) \sqsubseteq (d_0, d_1)$, siis $(d_0, d_1) = (e_0, e_1)$

Vähima ülemise raja tuvastamine:

Kui $(D_0, \sqsubseteq_0)$ ja $(D_1, \sqsubseteq_1)$ on ccpo-d ja tehe $\times$ on mõlema operandi suhtes monotoonne (ehk $f_0 : D_0 \to D_0 \times D_1$ on monotoonne ja $f_1 : D_1 \to D_0 \times D_1$ on monotoonne), siis raamatu lemma 4.30 järgi on ka $(D_0 \times D_1, \sqsubseteq)$ ccpo ja sellel eksisteerib vähim ülemine raja.

3 Ülesanne 3

4 Ülesanne 4

4.1 Denotatsioonisemantikas

$S_{ds}[\![\text{while } x \neq y \text{ do (if } x \leq y \text{ then } y := y - x \text{ else } x := x - y)]\!]s = FIX\ F\ s$,
kus $F\ g = cond(B[\![x \neq y]\!], g \circ S_{ds}[\![\text{if } x \leq y \text{ then } y := y - x \text{ else } x := x - y]\!], id)$,

millest tuletub

$$F\ g\ s = \begin{cases} g(S_{ds}[\![\text{if } x \leq y \text{ then } y := y - x \text{ else } x := x - y]\!]s) & \text{kui } sx \neq sy \\ s & \text{kui } sx = sy \end{cases}$$

ja mille abil saab alt-üles leida vähima ülemise raja:

$$(F_0(\bot))s = undef$$

$$(F_1(\bot))s = \begin{cases} F_0(S_{ds}\llbracket \text{if } x \le y \text{ then } y := y - x \text{ else } x := x - y\rrbracket s) & \text{kui } sx \ne sy \\ s & \text{kui } sx = sy \end{cases}$$

$$= \begin{cases} undef & \text{kui } sx \ne sy \\ s & \text{kui } sx = sy \end{cases}$$

$$s' = \begin{cases} s[y \mapsto sy - sx] & \text{kui } sx \le sy \\ s[x \mapsto sx - sy] & \text{kui } sx > sy \end{cases}$$

$$= \begin{cases} s'x = sx \wedge s'y = sy - sx & \text{kui } sx \le sy \\ s'x = sx - sy \wedge s'y = sy & \text{kui } sx > sy \end{cases}$$

$$(F_2(\bot))s = \begin{cases} F_1(S_{ds}\llbracket \text{if } x \le y \text{ then } y := y - x \text{ else } x := x - y\rrbracket s = s') & \text{kui } sx \ne sy \\ s & \text{kui } sx = sy \end{cases}$$

$$= \begin{cases} \left\{\begin{array}{ll} undef & \text{kui } s'x \ne s'y \\ s' & \text{kui } s'x = s'y \end{array}\right\} & \text{kui } sx \ne sy \\ \\ s & \text{kui } sx = sy \end{cases}$$

$$= \begin{cases} \left\{\begin{array}{ll} \left\{\begin{array}{ll} undef & \text{kui } s'x \ne s'y \\ s[y \mapsto sy - sx] & \text{kui } s'x = s'y \end{array}\right\} & \text{kui } sx \le sy \\ \left\{\begin{array}{ll} undef & \text{kui } s'x \ne s'y \\ s[x \mapsto sx - sy] & \text{kui } s'x = s'y \end{array}\right\} & \text{kui } sx > sy \end{array}\right\} & \text{kui } sx \ne sy \\ \\ s & \text{kui } sx = sy \end{cases}$$

$$= \begin{cases} \left\{\begin{array}{ll} \left\{\begin{array}{ll} undef & \text{kui } sx \ne (sy - sx) \\ s[y \mapsto sy - sx] & \text{kui } sx = (sy - sx) \end{array}\right\} & \text{kui } sx \le sy \\ \left\{\begin{array}{ll} undef & \text{kui } (sx - sy) \ne sy \\ s[x \mapsto sx - sy] & \text{kui } (sx - sy) = sy \end{array}\right\} & \text{kui } sx > sy \end{array}\right\} & \text{kui } sx \ne sy \\ \\ s & \text{kui } sx = sy \end{cases}$$

$$= \begin{cases} \left\{\begin{array}{ll} \left\{\begin{array}{ll} undef & \text{kui } 2sx \ne sy \\ s[y \mapsto sy - sx] & \text{kui } 2sx = sy \end{array}\right\} & \text{kui } sx \le sy \\ \left\{\begin{array}{ll} undef & \text{kui } sx \ne 2sy \\ s[x \mapsto sx - sy] & \text{kui } sx = 2sy \end{array}\right\} & \text{kui } sx > sy \end{array}\right\} & \text{kui } sx \ne sy \\ \\ s & \text{kui } sx = sy \end{cases}$$

$$= \begin{cases} \left\{\begin{array}{ll} undef & \text{kui } 2sx \ne sy \\ s[y \mapsto sy - sx] & \text{kui } 2sx = sy \end{array}\right\} & \text{kui } sx < sy \\ \left\{\begin{array}{ll} undef & \text{kui } sx \ne 2sy \\ s[x \mapsto sx - sy] & \text{kui } sx = 2sy \end{array}\right\} & \text{kui } sx > sy \\ s & \text{kui } sx = sy \end{cases}$$

$$= \begin{cases} undef & \text{kui } (sx \ne sy) \wedge (2sx \ne sy) \wedge (sx \ne 2sy) \\ s[y \mapsto sy - sx] & \text{kui } 2sx = sy \\ s[x \mapsto sx - sy] & \text{kui } sx = 2sy \\ s & \text{kui } sx = sy \end{cases}$$

$$= \begin{cases} undef & \text{kui } (sx \ne sy) \wedge (2sx \ne sy) \wedge (sx \ne 2sy) \\ s[y \mapsto \frac{sy}{2}] & \text{kui } 2sx = sy \\ s[x \mapsto \frac{sx}{2}] & \text{kui } sx = 2sy \\ s & \text{kui } sx = sy \end{cases}$$

$$(F_3(\bot))s = \begin{cases} F_2(S_{ds}\llbracket \text{if } x \le y \text{ then } y := y - x \text{ else } x := x - y\rrbracket s = s') & \text{kui } sx \ne sy \\ s & \text{kui } sx = sy \end{cases}$$

$$= \begin{cases} \left\{\begin{array}{ll} undef & \text{kui } (s'x \ne s'y) \wedge (2s'x \ne s'y) \wedge (s'x \ne 2s'y) \\ s'[y \mapsto \frac{s'y}{2}] & \text{kui } 2s'x = s'y \\ s'[x \mapsto \frac{s'x}{2}] & \text{kui } s'x = 2s'y \\ s' & \text{kui } s'x = s'y \end{array}\right\} & \text{kui } sx \ne sy \\ s & \text{kui } sx = sy \end{cases}$$

$$
= \begin{cases}
\left\{\begin{array}{ll}
\left.\begin{cases}
undef & \text{kui } (3sx \neq sy) \wedge (3sx \neq 2sy) \wedge (2sx \neq sy) \\
s[y \mapsto \frac{sy}{3}][y \mapsto \frac{sy}{3}] & \text{kui } 3sx = sy \\
s[y \mapsto \frac{sy}{3}][x \mapsto \frac{sx}{2}] & \text{kui } 3sx = 2sy \\
s[y \mapsto \frac{sy}{2}] & \text{kui } 2sx = sy
\end{cases}\right\} & \text{kui } sx \leq sy \\
\left.\begin{cases}
undef & \text{kui } (2sx \neq 3sy) \wedge (sx \neq 3sy) \wedge (sx \neq 2sy) \\
s[x \mapsto \frac{sx}{3}][y \mapsto \frac{sy}{2}] & \text{kui } 2sx = 3sy \\
s[x \mapsto \frac{sx}{3}][x \mapsto \frac{sx}{3}] & \text{kui } sx = 3sy \\
s[x \mapsto \frac{sx}{2}] & \text{kui } sx = 2sy
\end{cases}\right\} & \text{kui } sx > sy
\end{array}\right\} & \text{kui } sx \neq sy \\
s & \text{kui } sx = sy
\end{cases}
$$

$$
= \begin{cases}
undef & \text{kui } (3sx \neq sy) \wedge (3sx \neq 2sy) \wedge (2sx \neq sy) \wedge \\
 & \wedge (2sx \neq 3sy) \wedge (3sx \neq 2sy) \wedge (sx \neq 2sy) \wedge (sx \neq sy) \\
s[y \mapsto \frac{sy}{3}] & \text{kui } 3sx = sy \\
s[y \mapsto \frac{sy}{3}][x \mapsto \frac{sx}{2}] & \text{kui } 3sx = 2sy \\
s[y \mapsto \frac{sy}{2}] & \text{kui } 2sx = sy \\
s[x \mapsto \frac{sx}{3}][y \mapsto \frac{sy}{2}] & \text{kui } 2sx = 3sy \\
s[x \mapsto \frac{sx}{3}] & \text{kui } sx = 3sy \\
s[x \mapsto \frac{sx}{2}] & \text{kui } sx = 2sy \\
s & \text{kui } sx = sy
\end{cases}
$$

$$
= \begin{cases}
undef & \text{kui } (3sx \neq sy) \wedge (3sx \neq 2sy) \wedge (2sx \neq sy) \wedge \\
 & \wedge (2sx \neq 3sy) \wedge (3sx \neq 2sy) \wedge (sx \neq 2sy) \wedge (sx \neq sy) \\
s[y \mapsto \frac{sy}{3}] & \text{kui } 3sx = sy \\
s[x \mapsto \frac{sx}{3}] & \text{kui } sx = 3sy \\
s[y \mapsto \frac{sy}{3}][x \mapsto \frac{sx}{2}] & \text{kui } 3sx = 2sy \\
s[x \mapsto \frac{sx}{3}][y \mapsto \frac{sy}{2}] & \text{kui } 2sx = 3sy \\
s[y \mapsto \frac{sy}{2}] & \text{kui } 2sx = sy \\
s[x \mapsto \frac{sx}{2}] & \text{kui } sx = 2sy \\
s & \text{kui } sx = sy
\end{cases}
$$

$$
(F_n(\perp))s = \begin{cases}
undef & \text{kui } (i * sx \neq j * sy) \wedge (1 \leq i \geq n) \wedge (1 \leq j \geq n) \\
s[y \mapsto \frac{sy}{i}][x \mapsto \frac{sx}{j}] & \text{kui } (i * sx = j * sy) \wedge (1 \leq i \geq n) \wedge (1 \leq j \geq n) \wedge (i \neq j) \\
s & \text{kui } sx = sy
\end{cases}
$$

$$
(FIX\ F)s = \begin{cases}
s[y \mapsto \frac{sy}{i}][x \mapsto \frac{sx}{j}] & \text{kui } (i * sx = j * sy) \wedge (i \neq j) \\
s & \text{kui } sx = sy
\end{cases}
$$

4.2 Jätkuedastussemantikas

$S'_{gs}[\![\text{while } x \neq y \text{ do (if } x \leq y \text{ then } y := y - x \text{ else } x := x - y)]\!] = FIX\ G$,
kus $G(g)c = cond(B[\![x \neq y]\!], S'_{cs}[\![\text{if } x \leq y \text{ then } y := y - x \text{ else } x := x - y]\!](gc), c)$

Reegli $S'_{cs}[\![S]\!]c = c \circ S_{ds}[\![S]\!]$ (raamatus punkt 4.74) järgi võib tuletada:

$(FIX\ G)c\ s = (c \circ (FIX\ F))(s) = c(FIX\ F\ s)$

$$
= \begin{cases}
c(s[y \mapsto \frac{sy}{i}][x \mapsto \frac{sx}{j}]) & \text{kui } (i * sx = j * sy) \wedge (i \neq j) \\
cs & \text{kui } sx = sy
\end{cases}
$$

5 Ülesanne 5