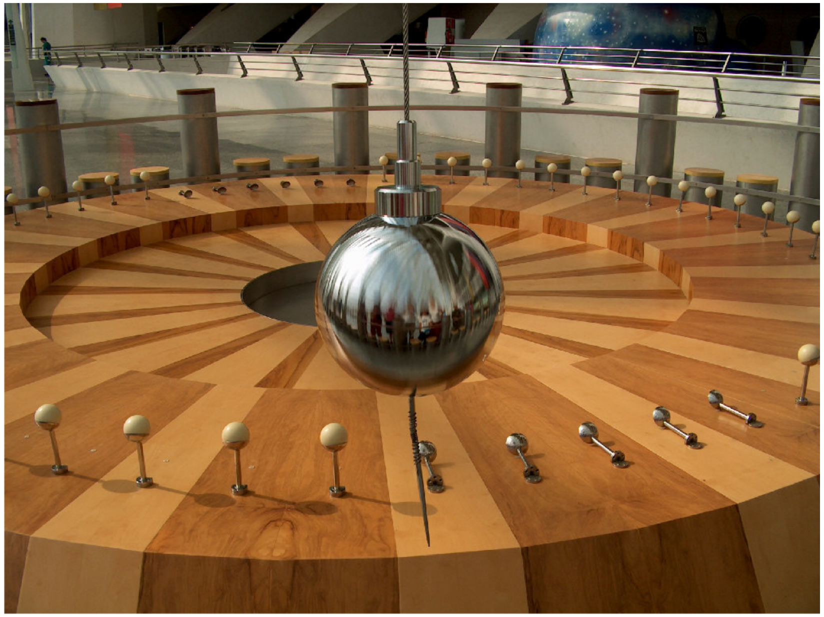
Pendulum

Oscillatory motion and and chaos



Background

- Oscillatory phenomena occur in many areas of physics
- Planetary orbits, motion of electrons in atoms, currents and voltages in electronic circuits, etc.
- The simplest mechanical example: **pendulum** (harmonic case, small angles)
- Elementary description does not usually include complicating factors: friction, external driving force, large angles (nonlinearities)
- **Chaotic behavior** arises as we consider such ***real effects***

Simple harmonic motion

- Let us assume only two forces: gravity and tension of the rod (or string)
- The force perpendicular to the rod is

$$F_\theta = -mg sin\theta$$

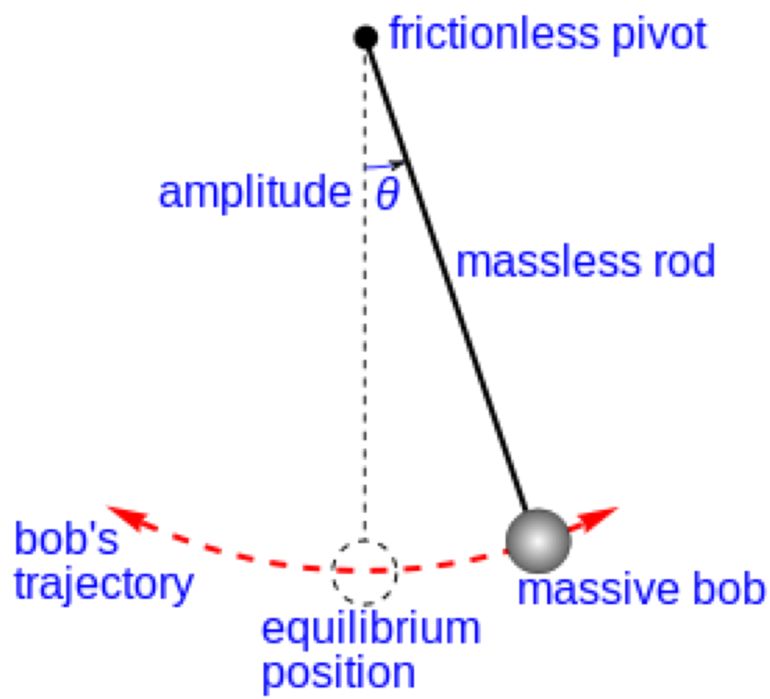
- Newton's second law and small angles lead to (check)

$$\boxed{\frac{d^2\theta}{dt^2} = -\frac{g}{\ell}\theta}$$

- Simple harmonic motion, where the general solutions is

$$\theta = \theta_0 sin(\Omega t + \phi), \qquad \Omega = \sqrt{g/\ell}$$

- ***Sinusoidal oscillations, no decay***
- Angular frequency *Ω* is independent of the mass and amplitude



Analytical solution

- Let us next consider the **numerical approach**
- The equation of motion (previous slide) is a second-order differential equation (DE); let us present it as two first-order DEs

$$\frac{d\omega}{dt} = -\frac{g}{\ell}\theta$$

$$\frac{d\theta}{dt} = \omega$$

- Convert these to difference equations, for Euler method

$$\boxed{\begin{aligned} \omega_{i+1} &= \omega_i - \frac{g}{\ell}\theta_i \Delta t \\ \theta_{i+1} &= \theta_i + \omega_i \Delta t \end{aligned}}$$

Numerical solution

- The trajectory is now defined by angle θ, angular velocity ω, and time (saved as arrays in the program)

- The Euler method is inherently unstable for this problem
- The total energy is given by

$$E = \frac{1}{2}m\ell^2\omega^2 + mg\ell(1 - cos\theta)$$

- The first term is the kinetic energy, second term is the gravitational potential energy
- For small angle, this reduces to (Taylor expansion, check)

$$E = \frac{1}{2}m\ell^2(\omega^2 + \frac{g}{\ell}\theta^2)$$

- Numerical solution by substitution (Euler method)

$$E_{i+1} = E_i + \frac{1}{2}m\ell^2(\omega_i^2 + \frac{g}{\ell}\theta_i^2)(\Delta t)^2$$

- The second term is always positive → *the energy of the Euler solution increases always with time*
- **The Euler method is not a good choice**

- Other numerical methods (Runge-Kutta, Verlet, etc.) work better for this problem
- A simple modification of the Euler method yields a suitable algorithm

$$\omega_{i+1} = \omega_i - \frac{g}{\ell}\theta_i \Delta t \qquad \text{(forward derivative)}$$

$$\theta_{i+1} = \theta_i + \omega_{i+1}\Delta t \qquad \text{(backward derivative)}$$

- Use the **old values** for computing the angular velocity ω_{i+1}, **and** use the **updated** ω_{i+1} for computing the angle θ_{i+1}
- This is known as the **Euler-Cromer method**
- The total energy is now conserved for each complete period
- While Euler method is based on a one-sided evaluation of the derivative (forward), the Euler-Cromer method corrects this feature with a ***more symmetrical description*** ← note that we have now two equations of motion, two derivatives (angle and angular velocity)