Deriving the laplacian in spherical cordinates

Martin Johnsrud

August 4, 2019

The definition of spherical coordinates is

$$
\begin{aligned}
x &= r\cos(\phi)\sin(\theta) & r &= \sqrt{x^2+y^2+z^2}, \quad r \in [0, \infty) \\
y &= r\sin(\phi)\sin(\theta) & \phi &= \arctan\left(\frac{y}{x}\right), \quad \phi \in [0, 2\pi) \\
z &= r\cos(\theta) & \theta &= \arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right), \quad \theta \in [0, \pi)
\end{aligned}
$$

To see the inverse relationship, we just substitute

$$
\begin{aligned}
r =& \sqrt{(r\cos(\phi)\sin(\theta))^2 + (r\sin(\phi)\sin(\theta))^2 + (r\cos(\theta))^2} \\
=& r\sqrt{(\cos^2\phi + \sin^2\phi)\sin^2\theta + \cos^2\theta} = r \\
\phi =& \arctan\left(\frac{r\sin(\phi)\sin(\theta)}{r\cos(\phi)\sin(\theta)}\right) = \arctan(\tan\phi) = \phi \\
\theta =& \arctan\left(\frac{\sqrt{(r\cos(\phi)\sin(\theta))^2 + (r\sin(\phi)\sin(\theta))^2}}{r\cos(\theta)}\right) \\
=& \arctan\left(\frac{\sqrt{\sin^2(\theta)}}{\cos(\theta)}\right) = \theta.
\end{aligned}
$$

The differential of a function can be written as

$$
\mathrm{d}f = \nabla f \cdot \mathrm{d}\vec{r},
$$

where $\mathrm{d}\vec{r}$ is a differential line element. In cartesian cooridnates, this simply becomes

$$
\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y + \frac{\partial f}{\partial z}\mathrm{d}z
$$

In spherical coordinates, a differential line element is

$$
\mathrm{d}\vec{r} = \hat{r}\mathrm{d}r + \hat{\phi}r\sin(\theta)\mathrm{d}\phi + \hat{\theta}\mathrm{d}\theta,
$$

To find nabla represented in spherical coordinates, we simply compare the terms in

$$\begin{aligned}\nabla f \cdot (\hat{r}\mathrm{d}r + \hat{\phi}r\sin(\theta)\mathrm{d}\phi + \hat{\theta}\mathrm{d}\theta) =& \frac{\partial f}{\partial r}\mathrm{d}r + \frac{\partial f}{\partial \phi}\mathrm{d}\phi + \frac{\partial f}{\partial \theta}\mathrm{d}\theta \\ \implies \nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}&\frac{1}{r\sin(\theta)}\frac{\partial}{\partial \phi} + \hat{\phi}\frac{1}{r}\frac{\partial}{\partial \phi}\end{aligned}$$

We need the unit vector to get further. The definition of a unit-vector is

$$\hat{n} = \frac{\mathrm{d}\boldsymbol{r}/\mathrm{d}n}{|\mathrm{d}\boldsymbol{r}/\mathrm{d}n|},$$

where the $\boldsymbol{r}$ is the position vector, given by

$$\begin{aligned}\boldsymbol{r} &= x\hat{x} + y\hat{y} + z\hat{z} \\ \boldsymbol{r} &= r\cos(\phi)\sin(\theta)\hat{x} + r\sin(\phi)\sin(\theta)\hat{y} + r\cos(\theta)\hat{z}\end{aligned}$$

To find the unit vectors in the spherical coordinats system, we need

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}r} &= \cos(\phi)\sin(\theta)\hat{x} + \sin(\phi)\sin(\theta)\hat{y} + \cos(\theta)\hat{z} \\ \left|\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}r}\right| &= \sqrt{\cos^2(\phi)\sin^2(\theta) + \sin(\phi)\sin^2(\theta) + \cos^2(\theta)} = 1 \\ \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}\phi} &= -r\sin(\phi)\sin(\theta)\hat{x} + r\cos(\phi)\sin(\theta)\hat{y} \\ \left|\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}\phi}\right| &= r\sqrt{\sin^2(\phi)\sin^2(\theta) + \cos^2(\phi)\sin^2(\theta)} = r\sin(\theta) \\ \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}\theta} &= r\cos(\phi)\cos(\theta)\hat{x} + r\sin(\phi)\cos(\theta)\hat{y} - r\sin(\theta)\hat{z} \\ \left|\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}\theta}\right| &= r\sqrt{\cos^2(\phi)\cos^2(\theta) + \sin^2(\phi)\cos^2(\theta) + \sin^2(\theta)} = r\end{aligned}$$

This gives us the unit vectors

$$\begin{aligned}\hat{r} &= \cos(\phi)\sin(\theta)\hat{x} + \sin(\phi)\sin(\theta)\hat{y} + \cos(\theta)\hat{z} \\ \hat{\phi} &= -\sin(\phi)\hat{x} + \cos(\phi)\hat{y} \\ \hat{\theta} &= \cos(\phi)\cos(\theta)\hat{x} + \sin(\phi)\cos(\theta)\hat{y} - \sin(\theta)\hat{z}\end{aligned}$$

To find the laplace operator, we just calculate the effect of applying ∇ twice

$$\left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r\sin(\theta)}\frac{\partial}{\partial \phi} + \hat{\phi}\frac{1}{r}\frac{\partial}{\partial \theta}\right)\left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r\sin(\theta)}\frac{\partial}{\partial \phi} + \hat{\phi}\frac{1}{r}\frac{\partial}{\partial \theta}\right)f =$$

$$\left(\frac{\partial^2}{\partial r^2} + \hat{\theta}\frac{1}{r\sin(\theta)}\frac{\partial \hat{r}}{\partial \phi}\frac{\partial}{\partial r} + \frac{1}{r^2\sin^2(\theta)}\left[\frac{\partial^2}{\partial \phi^2} + \frac{\partial \hat{\theta}}{\partial \phi}\frac{\partial}{\partial \phi}\right] + \frac{1}{r^2\sin(\theta)}\frac{\partial \hat{\phi}}{\partial \phi}\frac{\partial}{\partial \phi} + \right.$$
$$\left.\hat{\phi}\frac{1}{r}\frac{\partial \hat{r}}{\partial \theta}\frac{\partial}{\partial r} + \frac{1}{r^2\sin(\theta)}\frac{\partial \hat{\theta}}{\partial \theta}\frac{\partial}{\partial \phi} + \frac{cos(\theta)}{r^2\sin^2(\theta)}\frac{\partial}{\partial \phi} + \frac{1}{r^2}\frac{\partial \hat{\phi}}{\partial \theta}\right)f$$

The partial derivatives needed are

$$\frac{\partial\hat{\theta}}{\partial\phi} = -\sin(\phi)\cos(\theta)\hat{x} + \cos(\phi)\cos(\theta)\hat{y} = \cos(\theta)\hat{\phi}$$
$$\frac{\partial\hat{\phi}}{\partial\phi} = -\cos{(\phi)}\hat{x} - \sin{(\phi)}\hat{y} + (\sin(\theta)\hat{z} - \sin(\theta)\hat{z}c)$$