Numerical exerice in classical mechanics

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Introduction

This text explores several different methods for solving differential equations to model a pendulum. The methods are anlyzed for different time steps, to see the advatages of running the different methods.

Theory

Using the small angle approximation for a pendulum, we get the differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\theta(t) = -\frac{g}{l}\theta(t)$$

This can be rewritten in the form

$$\dot{y} = \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\theta \\ \dot{\theta}\end{pmatrix} = f\begin{pmatrix}\theta \\ \dot{\theta}\end{pmatrix} = \begin{pmatrix}\dot{\theta} \\ -\frac{g}{l}\theta,\end{pmatrix}$$

making explicit differential equation solvers straight forward. Eulers method is

$$y_{n+1} = f(y_n)\Delta t,$$

and Runge Kutta 4 is

$$\begin{aligned}
k_1 &= f(y_n), \\
k_2 &= f(y_n + k_1/2)\Delta t, \\
k_3 &= f(y_n + k_2/2)\Delta t, \\
k_4 &= f(y_n + k_3)\Delta t, \\
y_{n+1} &= y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4).
\end{aligned}$$

As we will see later, these methods does not conserve energy. This is achived with the implicit Eurler-Cromer method

$$\dot{\theta}_{n+1} = \dot{\theta}_n - \frac{g}{l}\theta$$

$$\theta_{n+1} = \dot{\theta}_{n+1}\Delta t$$

Results

Eulers method

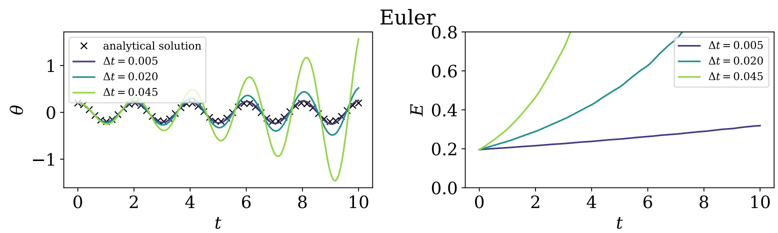

Figure 1: The upper plot shows the oscillations of the pendulum with different time-steps, comparing them to the analytical solution. The lower plot shows how the energy of the different systems increases with time.

Figure 1 shows the resultus of moddeling the system with Eurlers method for using different time steps, $\Delta t \in \{0.005, 0.02, 0.045\}$, over a time period $t \in [0, 10]$. Based on this analysis, we can see that a time step of around $\Delta t = 0.005$ is sufficent for accurate results, however for large enough time periods, all timesteps makes the method unstable.

Comparison with other methods

Figure 2 and 3 compares Eulers method with Euler-Cromer and Runge-Kutta 4 using the same time step ($\Delta t = 0.5$). This comparison shows that Eulers method is much less stabel than the other methods. Runge-Kutta is more computationally expensive as it is a method of a higher order, however Euler-Cromer achives much higher stability without being more expensive.

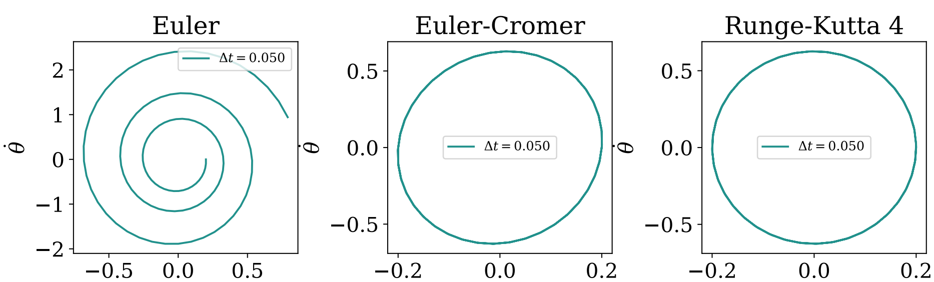

Figure 2: This figure shows motion of the pendulum through phase space, the space of coordinates $(\theta, \dot{\theta})$, parametrized by time t. This is a plot of the same motoion as in figure 3.

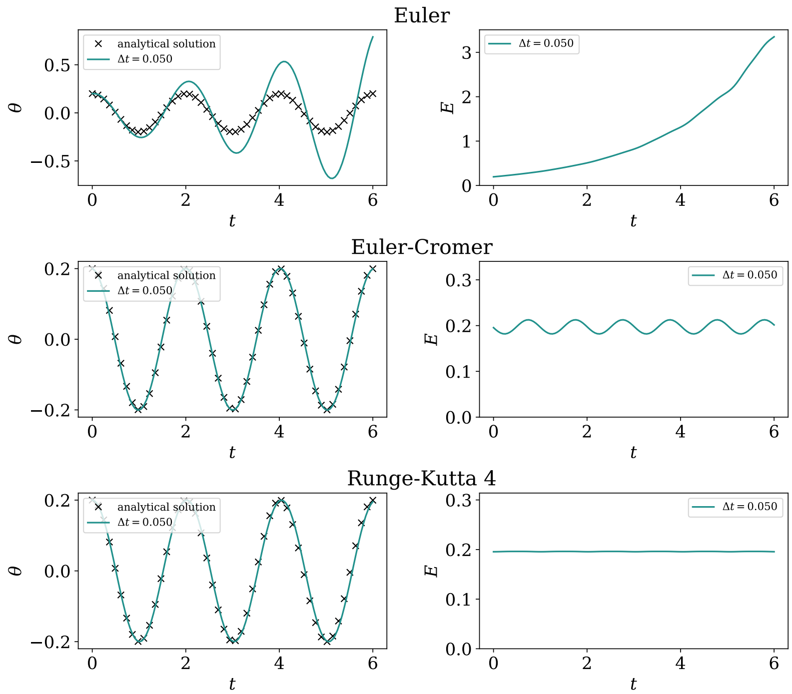

Figure 3: A comparison between the methods discussed in the text. Time is plotted against the angle θ of the pendulum at the left, and the energy E at the right. From top to bottom, the methods are Euler, Euler-Cromer and Runge-Kutta 4.

Euler-Cromer vs. Runge-Kutta 4

Though both Euler-Cromer and Runge-Kutta 4 are more stable methods, they have different advantages. Euler-Cormer conserves energy, so however long the simulation is run for, it will not diverge. The disadvatage of Euler-Cromer can be seen in the phase diagram in figure 4. While the Runge-Kutta path keeps is a (jagged) circle, the path of the Euler-Cromer is elongated when the timestep increases. This means that even though Euler-cromer is more stabel, it deviates more in the start of the simulation. However, as we can see from figure 5, Runge-Kutta looses energy. It will therefore always diverge further from the analytical solution the longer the simulation is run for. For the interval tested here ($t = 10$), Euler-Cromer is fairly accurate with $\Delta t = 0.04$, while Runge-Kutta 4 remains accurate enough up to $\Delta t = 0.16$. For longer simulations, however, Runge-Kutta would need shorter steps.

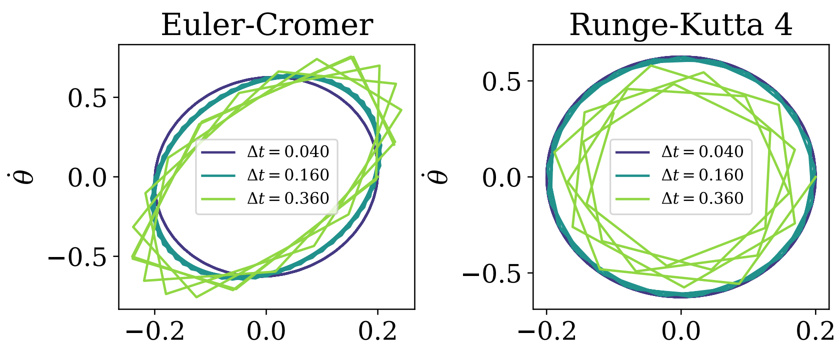

Figure 4: The path in phase space of Euler-Cromer method and Runge-Kutta 4, corresponding to the movements in figure 5

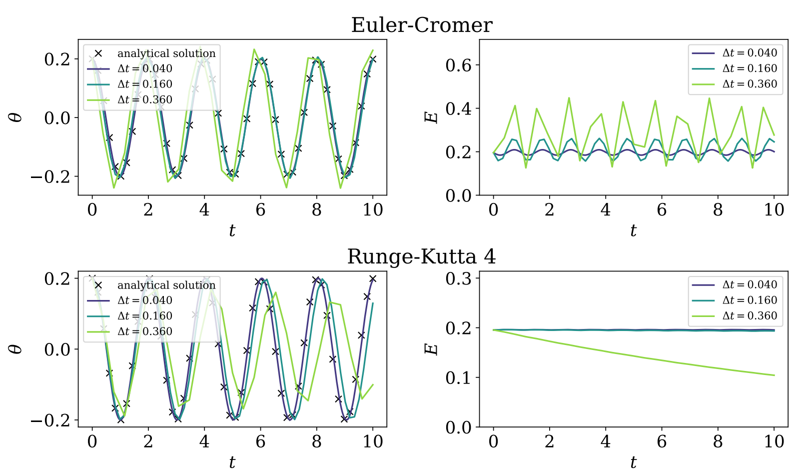

Figure 5: Comparison of the Euler-Cromer method and Runge-Kutta 4.