# Legendres genereal differential equation

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The general legendre differential equation is given by

$$\left[\frac{\mathrm{d}}{\mathrm{d}x}\left((1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}\right)+\lambda-\frac{m^2}{1-x^2}\right]P(x)=0, \tag{1}$$

and is part of the laplace equation i polar coordinates. We start by solving for $m=0$. Assuming $P(x)\in C^\infty$, we can write

$$P(x)=\sum_{n=0}^{\infty}a_nx^n.$$

Substituting this into the equation gives the terms

$$\begin{aligned}(1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2}P(x)&=(1-x^2)\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}\\&=\sum_{n=0}^{\infty}(a_{n+2}(n+1)(n+2)-a_nn(n-1))x^n\\2x\frac{\mathrm{d}}{\mathrm{d}x}P(x)&=2x\sum_{n=1}^{\infty}na_nx^{n-1}=\sum_{n=0}^{\infty}2na_nx^n.\end{aligned}$$

This lets us rewrite (1) as

$$\sum_{n=0}^{\infty}\big((a_{n+2}(n+1)(n+2)-a_nn(n-1)-2na_n+\lambda a_n\big)x^n=0.$$

Seeing that this has to be true for all $x$, it also has to be true for all $n$. This gives an recursive formula for the coefficients,

$$a_{n+2}=\frac{n(n+1)-\lambda}{(n+1)(n+2)}a_n.$$

This gives us three free parameters, $\lambda, a_0, a_1$. If we look at the asymptotic behaviror of the coefficients, we see that

$$a_{2n}\sim a_n,$$

meaning the series does not converge. The only way to force convergence, is to set

$$\lambda = l(l+1), l \in \mathbb{N}^+,$$

and $a_0 = 0$ if $l$ is even, or $a_1 = 0$ if $l$ is odd. This sets the coefficents equal to 0 after finitly many terms, and gives us a soultion for $m = 0$. The first few instances of $P^l(x)$, up to a multiplicative constant, are

$$\begin{aligned} P^0(x) &= 1, \\ P^1(x) &= x, \\ P^2(x) &= 1 - \frac{1}{3}x^2, \\ P^3(x) &= x - \frac{5}{3}x^3. \end{aligned}$$

To find an explicit formula, we choose

$$a_l = \frac{(2l)!}{2^l l!^2}.$$

We then rearange the recursive relationsship to go backwards,

$$a_{n+2}\frac{(n+1)(n+2)}{(n-l)(n+l+1)} = a_n.$$

To see the pattern that emerges, we find the next to last coefficent.

$$a_{l-2} = \frac{(2l)!}{2^l l!^2}\frac{(l-1)(l)}{(-2)(2l-1)} = \frac{(2l-2)!}{2^l(l-2)!(l-1)!}\frac{(-1)}{1!}$$

generalizing this, we obtain

$$a_{l-2m} = \frac{(-1)^m(2(l-m))!}{2^l(l-2m)!(l-m)!m!} = \frac{(-1)^m}{2^l}\binom{l}{m}\binom{2(l-m)}{l}, \quad m \in [0, ..., \lfloor l/2 \rfloor]$$

This results in an explicit form of the legendre polynomial,

$$P^l(x) = \sum_{n=0}^{\lfloor l/2 \rfloor} \frac{(-1)^m}{2^l}\binom{l}{m}\binom{2(l-m)}{l} x^{l-2m}$$

With some really creative rewriting, we can get a more compact form.

$$P^l(x) = \sum_{n=0}^{\lfloor l/2 \rfloor} \frac{(-1)^m}{2^l}\binom{l}{m}\binom{2(l-m)}{l}\left[\frac{(l-2m)!}{(2(l-m))!}\frac{\mathrm{d}^l}{\mathrm{d}x^l}x^{2(l-m)}\right] =$$
$$\sum_{n=0}^{\lfloor l/2 \rfloor} \frac{1}{2^l l!}\binom{l}{m}\frac{\mathrm{d}^l}{\mathrm{d}x^l}(x^2)^{(l-m)} = \frac{1}{2^l l!}\frac{\mathrm{d}^l}{\mathrm{d}x^l}\sum_{n=0}^{\lfloor l/2 \rfloor}\binom{l}{m}(-1)^m(x^2)^{(l-m)}$$

Recognizing that, for $n > \lfloor l/2 \rfloor$, the term in the sum will be of degree less than $l$, and therefore the derivation oparator will set it to zero, we can change the index, and recognize it as a binomial expansion.

$$\begin{aligned}
P^l(x) &= \frac{1}{2^l l!}\frac{\mathrm{d}^l}{\mathrm{d}x^l}\sum_{n=0}^{l}\binom{l}{m}(-1)^m (x^2)^{(l-m)} \\
&= \frac{1}{2^l l!}\frac{\mathrm{d}^l}{\mathrm{d}x^l}\sum_{n=0}^{l}\binom{l}{m}(-1)^m (x^2)^{(l-m)} = \frac{1}{2^l l!}\frac{\mathrm{d}^l}{\mathrm{d}x^l}(1-x^2)^l
\end{aligned}$$

This is the Rodrigues formula.

To find the solution for $m \neq 0$, we start by plugging our solution in to the equation for $m = 0$. This equation can of course be derivated with respect to x as many times we want, and still remain zero.

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}}{\mathrm{d}x}\left((1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}\right) + l(l+1)\right]P^l(x) = 0 \\
=&\frac{\mathrm{d}}{\mathrm{d}x}\left[\left((1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}}{\mathrm{d}x}\right) + l(l+1)\right]P^l(x) \\
=&\left[\left((1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2} - 4x\frac{\mathrm{d}}{\mathrm{d}x} - 2\right) + l(l+1)\right]\frac{\mathrm{d}}{\mathrm{d}x}P^l(x) \\
=&\left[(1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2} - 4x\frac{\mathrm{d}}{\mathrm{d}x} + l(l+1) - 2\right]\frac{\mathrm{d}}{\mathrm{d}x}P^l(x)
\end{aligned}$$

Generealizing, we get

$$\begin{aligned}
&\frac{\mathrm{d}^m}{\mathrm{d}x^m}\left[\left((1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}}{\mathrm{d}x}\right) + l(l+1)\right]P^l(x) \\
=&\frac{\mathrm{d}^{m-1}}{\mathrm{d}x^{m-1}}\left[(1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2} - 4x\frac{\mathrm{d}}{\mathrm{d}x} + l(l+1) - 2\right]\frac{\mathrm{d}}{\mathrm{d}x}P^l(x) \\
=&\frac{\mathrm{d}^{m-2}}{\mathrm{d}x^{m-2}}\left[(1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2} - 6x\frac{\mathrm{d}}{\mathrm{d}x} + l(l+1) - 6\right]\frac{\mathrm{d}^2}{\mathrm{d}x^2}P^l(x) \\
=&\frac{\mathrm{d}^{m-3}}{\mathrm{d}x^{m-3}}\left[(1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2} - 8x\frac{\mathrm{d}}{\mathrm{d}x} + l(l+1) - 12\right]\frac{\mathrm{d}^3}{\mathrm{d}x^3}P^l(x) \\
&\vdots \\
=&\left[(1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2} - 2(m+1)x\frac{\mathrm{d}}{\mathrm{d}x} + l(l+1) - m(m+1)\right]\frac{\mathrm{d}^m}{\mathrm{d}x^m}P^l(x) \\
=&\left[\frac{\mathrm{d}}{\mathrm{d}x}(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x} + l(l+1) - m(m+1) - 2mx\frac{\mathrm{d}}{\mathrm{d}x}\right]\frac{\mathrm{d}^m}{\mathrm{d}x^m}P^l(x)
\end{aligned}$$

Being hit with devine inspiration, we multiply by $(1-x^2)^{m/2}/(1-x^2)^{m/2}$, and pull the lower part through the operator

$$\left[\frac{\mathrm{d}}{\mathrm{d}x}(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}(1-x^2)^{-m/2}+\frac{l(l+1)}{(1-x^2)^{m/2}}-\frac{(m^2+m)}{(1-x^2)^{m/2}}-2mx\frac{\mathrm{d}}{\mathrm{d}x}(1-x^2)^{-m/2}\right](1-x^2)^{m/2}\frac{\mathrm{d}^m}{\mathrm{d}x^m}P^l(x)$$

$$\begin{aligned}
=&\Bigg[\frac{\mathrm{d}}{\mathrm{d}x}\frac{(1-x^2)}{(1-x^2)^{m/2}}\frac{\mathrm{d}}{\mathrm{d}x}+\frac{\mathrm{d}}{\mathrm{d}x}mx(1-x^2)^{-m/2}\\
&+\frac{l(l+1)}{(1-x^2)^{m/2}}-\frac{(m^2+m)}{(1-x^2)^{m/2}}-2mx\frac{\mathrm{d}}{\mathrm{d}x}(1-x^2)^{-m/2}\Bigg]P^l_m(x)
\end{aligned}$$

$$\begin{aligned}
=&\Bigg[\frac{mx(1-x^2)}{(1-x^2)^{m/2+1}}\frac{\mathrm{d}}{\mathrm{d}x}+(1-x^2)^{-m/2}\frac{\mathrm{d}}{\mathrm{d}x}(1-x^2)^1\frac{\mathrm{d}}{\mathrm{d}x}+\\
&\frac{m}{(1-x^2)^{m/2}}+\frac{m^2x^2}{(1-x^2)^{m/2+1}}+\frac{mx}{(1-x^2)^{m/2}}\frac{\mathrm{d}}{\mathrm{d}x}\\
&+\frac{l(l+1)}{(1-x^2)^{m/2}}-\frac{(m^2+m)}{(1-x^2)^{m/2}}-\frac{2m^2x^2}{(1-x^2)^{m/2+1}}-\frac{2mx}{(1-x^2)^{-m/2}}\frac{\mathrm{d}}{\mathrm{d}x}\Bigg]P^l_m(x)
\end{aligned}$$

$$\begin{aligned}
=&(1-x^2)^{-m/2}\Bigg[\frac{\mathrm{d}}{\mathrm{d}x}(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}+mx\frac{\mathrm{d}}{\mathrm{d}x}+m+\frac{m^2x^2}{(1-x^2)}+mx\frac{\mathrm{d}}{\mathrm{d}x}\\
&+l(l+1)-(m^2+m)-\frac{2m^2x^2}{(1-x^2)}-2mx\frac{\mathrm{d}}{\mathrm{d}x}\Bigg]P^l_m(x)
\end{aligned}$$

$$=(1-x^2)^{-m/2}\left[\frac{\mathrm{d}}{\mathrm{d}x}(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}+l(l+1)-m^2\left(1+\frac{x^2}{(1-x^2)}\right)\right]P^l_m(x)$$

$$=(1-x^2)^{-m/2}\left[\frac{\mathrm{d}}{\mathrm{d}x}(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}+l(l+1)-\frac{m^2}{(1-x^2)}\right]P^l_m(x)=0$$

This proves that the associated legendre function,

$$P(x)^l_m=(1-x^2)^{m/2}\frac{\mathrm{d}^m}{\mathrm{d}x^m}P^l(x)=\frac{1}{l^2l!}(1-x^2)^{2/m}\frac{\mathrm{d}^{l+m}}{\mathrm{d}x^{l+m}}(1-x^2)^l,$$

solves the general legendre equation (1), that is

$$\left[\frac{\mathrm{d}}{\mathrm{d}x}\left((1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}\right)+l(l+1)-\frac{m^2}{1-x^2}\right]P^l_m(x)=0. \qquad \blacksquare$$