The free particle, and gaussian wave-packets

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October 17, 2019

The Schrödinger equation for free space is

$$i\hbar \frac{\partial}{\partial t'} \Psi(x, t') = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \Psi(x, t)$$

Introducing the dimensionless $q = \sqrt{2m/\hbar\omega_0} x, t = t'\omega_0$, where ω_0 is characetristic quantity with dimensions s^{-1}, this becomes

$$\frac{\partial}{\partial t} \Psi(q, t) = i \frac{\partial^2}{\partial q^2} \Psi(q, t) \tag{1}$$

This is the diffusion equation, with $D = i$, and can be solved with a fourier trasform (which has a physical interpetation). Given a normalized initial condition,

$$\Psi(q, t) = f(q), \quad |\Psi(q, 0)|^2 = 1,$$

Taking the inverse fourier transfor w.r.t q, to the variable p, we get

$$\begin{aligned}
&\frac{\partial}{\partial t} \Phi(p, t) = -ip^2 \Phi(p, t), \\
&\implies \Phi(p, t) = C(p) e^{ip^2 t}, \quad \Phi(p, 0) = C(p) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(q) e^{iqp} \mathrm{d}q \\
&\implies \Psi(x, t) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} f(q) e^{iqp} \mathrm{d}q \, e^{-i(pq + p^2 t)} \mathrm{d}p
\end{aligned}$$

Of particular interest is the gaussian wave packet with expected position $q = q_0$, and expected momentum $p = p_0$,

$$\Psi(q, 0) = \frac{1}{\sqrt{4\pi\sigma^2}} \exp\left[\left(\frac{q - q_0}{2\sigma}\right)^2 - iqp_0\right] \tag{2}$$

giving us the integral

$$\Psi(x, t) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi\sigma^2}} \exp\left[\left(\frac{q - q_0}{2\sigma}\right)^2 - iqp_0\right] e^{iqp} \mathrm{d}q \, e^{-i(pq + p^2 t)} \mathrm{d}p$$

Taking the inner integral first, we have

$$\begin{aligned}
\int_{\mathbb{R}} \exp\left[\left(\frac{q-q_0}{2\sigma}\right)^2 - iqp_0\right] e^{iqp} \mathrm{d}q &= \int_{\mathbb{R}} \exp\left[\frac{q^2}{4\sigma^2} + \frac{q_0^2}{4\sigma^2} - q(q_0/2\sigma^2 + ip_0)\right] e^{iqp} \mathrm{d}q \\
&= \exp\left(\frac{q_0^2}{4\sigma^2} - (q_0/2 + \sigma i p_0)\right) \int_{\mathbb{R}} \exp\left[\left(\frac{q}{\sqrt{4\sigma^2}} - (q_0/2 + \sigma i p_0)\right)^2\right] e^{iqp} \mathrm{d}q \\
&= C \int_{\mathbb{R}} \exp\left[\left(\frac{q-A}{\sqrt{2\sigma}}\right)^2\right] e^{iqp}
\end{aligned}$$

.... or

$$\begin{aligned}
C(p) = \int_{\mathbb{R}} \exp\left[\left(\frac{q-q_0}{2\sigma}\right)^2 - iqp_0\right] e^{iqp} \mathrm{d}q &= \int_{\mathbb{R}} \exp\left[\left(\frac{q-q_0}{2\sigma}\right)^2\right] e^{iq(p-p_0)} \mathrm{d}q \\
\implies C(p+p_0) e^{iq_0 p} = \int_{\mathbb{R}} \exp\left[\left(\frac{q}{2\sigma}\right)^2\right] e^{iqp} \mathrm{d}q &= 2\sigma \int_{\mathbb{R}} [q^2] e^{iq(p)} \mathrm{d}q'
\end{aligned}$$