Active Matter Lectures 1 & 2

* Contents : Introduction ~~to~~ active matter and presentation of some background material (mathematical and conceptual tools).

Explain: lectures every week, 10:15 -> 11:45
Exam : oral exam (format TBD)

* What is active matter?

Commonly accepted definition: AM is composed of elementary units able to locally dissipate energy in a continuous and sustained manner.

-> Dynamics of AM breaks TRS (far from eq.)

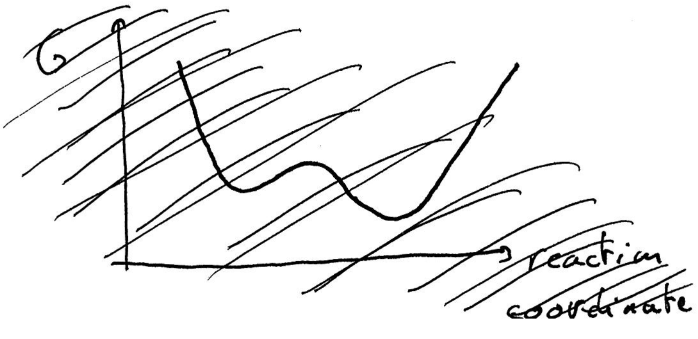
-> In many cases: activity = persistent motion, but can also be local production of forces, chemicals, or growth (reproduction), or several of them. Definition is very broad, includes multiple biological examples:

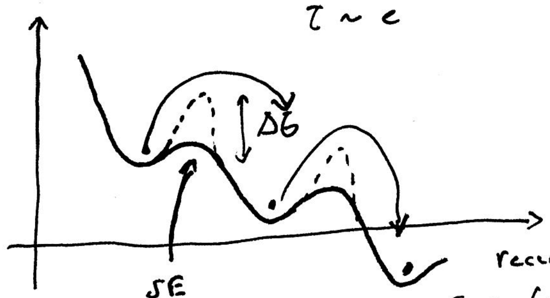
Enzymes, Mol. motors	Bacteria, Cells	Insect swarms	Bird flocks, sheepherds, human crowds.
10^{-9}	10^{-6}	10^{-3}	1 (m)

Enzymes = biological catalysts.

$$S \underset{k_-}{\overset{k_+}{\rightleftharpoons}} P$$

$$\tau \sim e^{\frac{\Delta G}{k_B T}}$$



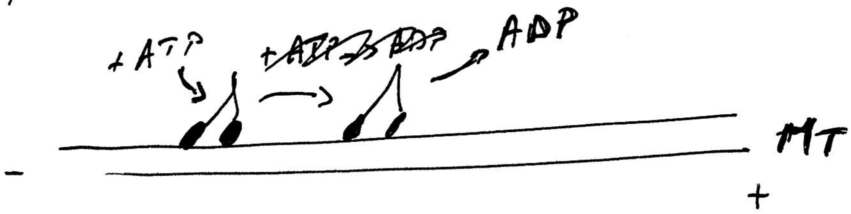


→ E locally turns S into P → locally "produces" chemical currents
(Impose ∇S & ∇P)

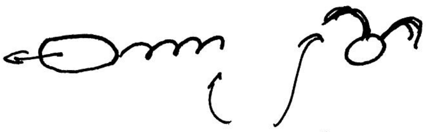
$$E + S \to \underbrace{SE}_{\text{substrate binds to } E.} \to EP$$

(Addressed in lectures 7 & 8)

→ Molecular motors: chemical cycle is coupled to displacement / rotation:

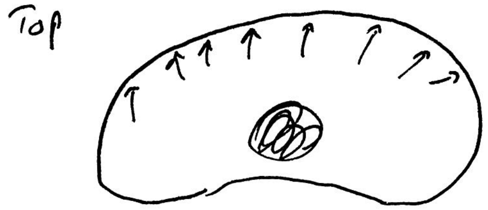


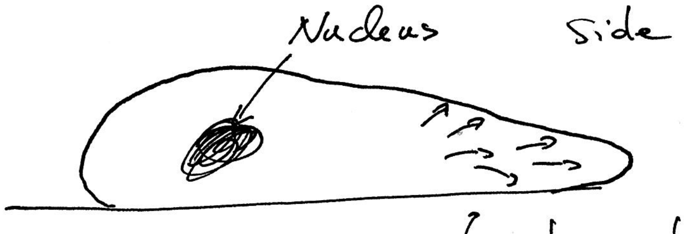
→ Bacteria / algae



Flagella powered by molecular motors.

→ crawling cells (maybe skip this part)





(actin polymerization at the front.

* Why study active Matter ?

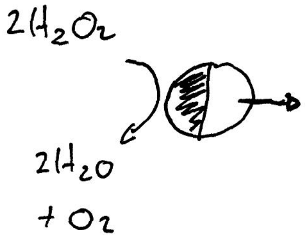
→ Active Matter is driven in bulk (at micro level), unlike driven passive systems (turbulent flows, granulars, etc...)

→ Unique self-organization properties (creation of macro structures, self-assembly)

→ Ideas relevant to many examples in the living world (architecture of cytoskeleton, morphogenesis, collective motion).

→ Application (on the long run): design of bio-inspired Materials. Large part of the field: synthetic active particles:

- self-propelled colloids



(Lecture 7)

- Artificial Microswimmers, Robots...

* Equilibrium properties that AM systems (generally lack):

- Loss of Memory
- TRS (global currents) / Forward / Backward movie)
- ~~Ab~~ No minimizat° principle (dynamics does not converge to a state that minimizes a FE). SS generally unknown.
- Lack of thermodynamic framework: Basic concepts (T, P) not defined in active matter more complex

$\not\exists$ action-react° principle (L 6)
"Exotic Forces"

⇝ Collectives properties of AM generally break intuition based on Eq. 4.

↳ Development of new tools, in particular to construct field theory describing active systems: ~~(Lectures 4~~

- IRR Thermodynamics (L4)
- Coarse-graining, kinetic theory (L5)

* Idea of this course:

→ AM is a (very) large and quickly growing field at the interface of SM, Bio Φ, hydro, Stat Φ. This course = introduction to basic phenomenology of AM and how it is usually apprehended.

⇝ Two important questions (still ~~currently~~ actively studied):

- How is active motion generated and csqs on the dynamics?
- Consequences of activity on the self-organizing properties at macro-scale (Micro → Macro?)

* Hydrodynamics of active motion (Intro to L3)

Many active particles = swimmers.

Fluid obey NS eq:

$$\rho\left(\overset{①}{\partial_t \underline{v}} + \overset{②}{(\underline{v}\cdot\nabla)\underline{v}}\right) = \overset{③}{\eta \nabla^2 \underline{v}} - \overset{\text{pressure}}{\nabla p} + \overset{\text{active forces.}}{\underline{f}}$$

$\nabla \cdot \underline{v} = 0$ Incompressibility.

② + ① = Inertia / convection

③ = dissipation

Dimensional analysis:

$v \to V v$

$x \to L x$ Typical velocity & length scales.

$t \to T t$ with $T = \frac{L}{V}$

$p \to P p$ with $P = \frac{\eta V}{L}$

$f \to F f$ with $F = \frac{\eta V}{L^2}$

NS: $\underbrace{Re}_{\frac{\rho V L}{\eta}} \left(\partial_t \underline{v} + (\underline{v}\cdot\nabla)\underline{v}\right) = \nabla^2 \underline{v} - \nabla p + \underline{f}$

$\frac{\rho V L}{\eta}$ Reynolds number

water: $\begin{cases} \rho = 10^3 \ kg/m^3 \\ \eta = 10^{-3} \ Pa\cdot s \end{cases}$ Bacteria (Ecoli): $V \simeq 10 \ \mu m \cdot s^{-1}$, $L \simeq 1 \ \mu m$

$\Rightarrow \boxed{Re \simeq 10^{-5}}$

=> Microswimmers evolve in the Stokes regime

$$\begin{cases} \nabla^2 \underline{v} - \nabla p + \underline{f} = \underline{0} \\ \underline{\nabla} \cdot \underline{v} = 0 \end{cases}$$

-> Important consequence : S eq. are ~~time~~ kinematically - reversible :

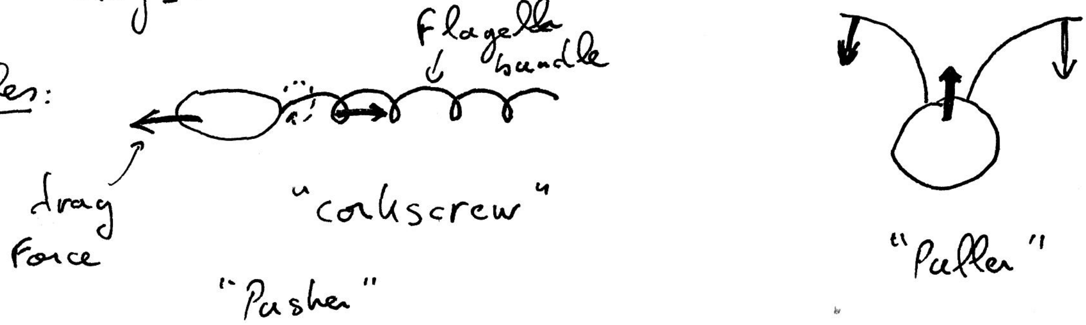
$$\boxed{f \to -f \Rightarrow v \to -v}$$

-> Scallop theorem : sp. cannot be achieved with reciprocal motion :

(No net motion)

=> self-propulsion requires at least 2 independent dofs.

Examples:



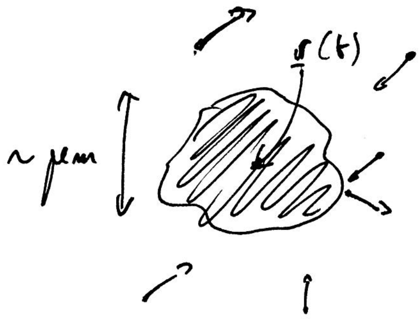
-> swimmers are force and torque - free

* Complementary approach: Minimal description of active motion.

~~How to describe~~

→ How to describe the motion of a particle out-of-equilibrium ?

↳ start by simple example: Brownian motion



Langevin equation:

$$m \frac{d^2 \underline{r}}{dt^2} + \xi \frac{d\underline{r}}{dt} = \underline{f}(t)$$

Inertia — Friction — ⊂ random collisions with solvent molecules.

Def $\tau = \frac{m}{\xi} \to 0$ (in general) → inertia can be neglected. (HW)

$\xi \sim 6\pi\eta \sim 6\pi \cdot 10^3$ Pa.m.s

$m \sim 10^{-15}$ kg

1 pg

$\Rightarrow \boxed{\tau \sim 10^{-17} s}$

$$\Rightarrow \boxed{\frac{d\underline{r}}{dt} = \frac{1}{\xi}\underline{f}(t)}$$ overdamped BM.

(~~Law of large numbers~~) CLT

$\underline{f}$: independent random events → distribut° approaches $\mathcal{N}(0, \sigma)$

$$\Rightarrow \langle \underline{f}_i \rangle = 0 \qquad \langle f_i(t) f_j(t') \rangle = 2D\xi^2 \underline{\delta(t-t')}\, \delta_{ij}$$

No time correlations.

$$\Rightarrow \underline{r}(t) - \underline{r}(0) = \int_0^t d\tau \, \frac{1}{\xi} \underline{f}(\tau)$$

$$\Rightarrow \langle r(t) - r(0) \rangle = 0 \qquad \langle [r(t) - r(0)]^2 \rangle = \int_0^t d\tau_1 d\tau_2 \, 2Dd \, \delta(\tau_1 - \tau_2) = \underline{\underline{2Ddt}}$$

$\begin{cases} \rightarrow \text{No mean motion} \\ \rightarrow \text{MSD is linear in time} \end{cases}$ Diffusion.

Distribution of $\underline{r}$: $P(\underline{x}, t) = \langle \delta(\underline{x} - r(t)) \rangle$

obeys the FP equation : $\partial_t P - D \nabla^2 P = 0$

$$\Rightarrow \quad P(\underline{x}, t) = \frac{e^{-\frac{x^2}{4Dt}}}{(4\pi D t)^{d/2}}$$

assuming that $P(\underline{x}, t=0) = \delta(\underline{x})$

HW: $\int d^d x \; x^2 P(x,t) = 2dDt$

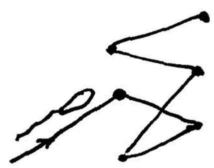
* Active Brownian motion:

-> Langevin description is ~~also~~ not restricted to systems at equilibrium. Adding activity in a minimal way via a self-generated propulsion velocity:

$$\frac{dr}{dt} = \underbrace{\underline{v}_a}_{\text{active velocity}} + \frac{1}{\zeta} f(t)$$

-> Dynamics of $\underline{v}_a$?

Ex: - R & T Motion



- smooth active motion (direct° of $\underline{v}_a$ diffusing in time)

-> In practice, important parameter is the persistence length of the active motion l_p.

→ Dimensional analysis: $\ell_p \simeq v_0 \tau_R$

typical speed (v_0)

τ_R: Timescale associated with tumblings / decorrelation of sp direct

On scales: $\ll \ell_p$: Ballistic motion with speed $\sim v_0$

$\gg \ell_p$: Diffusive motion with effective diffusivity ?

→ Minimal description $\underline{v_a} = v_0 \,\hat{\underline{e}}$ ← unit vector

with $\langle \hat{e}(t) \cdot \hat{e}(t+\tau) \rangle \simeq e^{-\tau/\tau_R}$

Homework: in $d=2$ $\hat{e}(\theta) = (\cos\theta, \sin\theta)$

$$\frac{d\theta}{dt} = \sqrt{2D_R}\,\eta(t) \quad \text{with } \langle \eta(t)\eta(t')\rangle = \delta(t-t')$$
$$\langle \eta(t)\rangle = 0$$

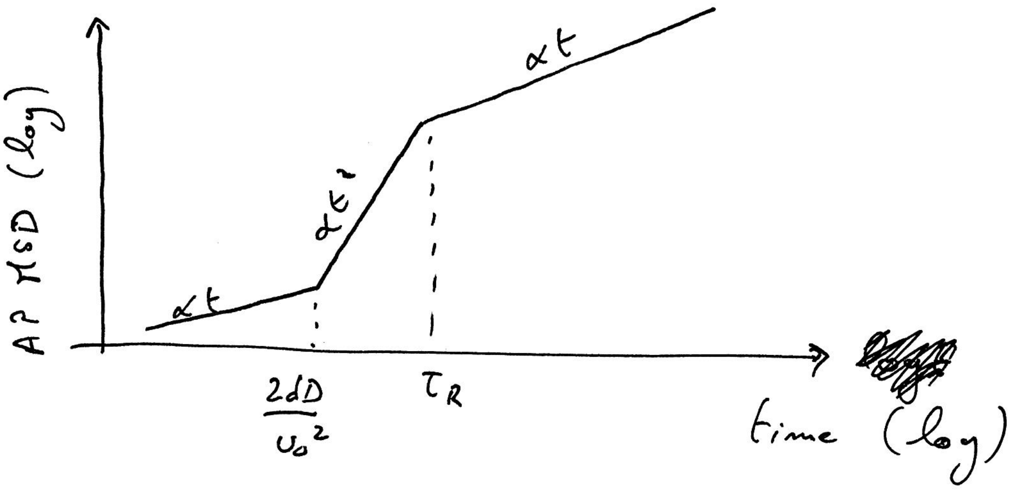
Show that $\langle \hat{e}(\theta(t)) \cdot \hat{e}(\theta(t+\tau)) \rangle = e^{-\tau/\tau_R}$ with $\tau_R = \frac{1}{D_R}$

$$\Rightarrow \quad \Delta r(t) = v_0 \int_0^t \hat{e}(\tau)\, d\tau + \frac{1}{\xi}\int_0^t d\tau\, f(\tau)$$

$$\Rightarrow \langle \Delta r^2(t)\rangle = 2Ddt + v_0^2 \int_0^t d\tau_1 \int_0^t d\tau_2 \; e^{-|\tau_1-\tau_2|/\tau_R}$$

$$\langle \Delta r^2 \rangle = 2Ddt + \underbrace{2 v_0^2 \tau_R^2}_{\ell_p^2} \left(\frac{t}{\tau_R} - 1 + e^{-t/\tau_R} \right)$$

$$\begin{cases} t \ll \tau_R : \langle \Delta r^2 \rangle \simeq 2Ddt + v_0^2 t^2 \\ t \gg \tau_R : \langle \Delta r^2 \rangle \simeq 2 D_{eff}\, dt \qquad D_{eff} = D + \dfrac{v_0^2 \tau_R}{d} \quad (> D) \end{cases}$$



→ Recover same result from FP description?

Focus on 2D: $\hat{e}(\theta) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ $\qquad \frac{d\theta}{dt} = \sqrt{2D_R}$

(Homework)

Define: $P(x, \phi, t) = \langle \delta(x - r(t))\,\delta(\phi - \theta(t)) \rangle$ HW

FP: $\partial_t P + \nabla \cdot \left(v_0 \hat{e}(\theta) P - D \nabla P \right) - D_R \partial^2_{\theta\theta} P = 0$

⇝ No simple analytical solution, but regime $t \gg \underbrace{\frac{1}{D_R}}_{= \tau_R}$

can be described by moment expansion (also very useful for more complex situations)

Define density: $\rho(r,t) = \int_0^{2\pi} d\theta \; P(r, \theta, t)$

$$\Rightarrow \quad \partial_t \rho + v_0 \nabla \cdot (\rho\, p) - D \nabla^2 \rho = 0$$

with $p(r,t) = \frac{1}{\rho} \int_0^{2\pi} d\theta \; \hat{e}(\theta)\, P(r, \theta, t)$ the mean orientation

Equation for $\underline{f}$?

high order in ∇

$$\partial_t \rho \underline{f} = -\frac{v_0}{2}\nabla\rho - D_R \rho \underline{f} - v_0 \nabla\cdot\left(\cancel{\rho \underline{\underline{Q}}}\right) + \cancel{D\nabla^2(\rho \underline{f})}$$

with $\rho \underset{\approx}{Q}_{ij} = \int_0^{2\pi} d\theta \left[\hat{e}_i(\theta)\hat{e}_j(\theta) - \underbrace{\frac{1}{2}\delta_{ij}}\right] P(\underline{r},\theta,t)$

Tensor (Nematic order parameter)

ensure $Tr(Q) = 0$ (why ?) HW

Eq. for Q will depend on $\langle e^3 \rangle$ etc...

↳ Need a closure relation : $\underline{\underline{Q}} = 0$

↳ Express $\underline{f}$ as function of ρ

Write:
$$\partial_t \rho \underline{f} = -D_R \rho \underline{f} + \underline{A}(\underline{r},t)$$

$$\Rightarrow \rho \underline{f} = e^{-tD_R}\left[\rho \underline{f}\big|_0 + \int_0^t e^{t' D_R} A(\underline{r},t')\,dt'\right]$$

$$\underset{t \gg \frac{1}{D_R}}{\simeq} \int_0^t dt'\, e^{-(t-t')D_R} A(\underline{r},t')$$

$$\underset{\tau = t-t'}{\simeq} \int_0^t d\tau\, e^{-\tau D_R} A(\underline{r}, t-\tau)$$

$$\simeq \int_0^\infty d\tau\, e^{-\tau D_R}\left[A(\underline{r},t) - \underbrace{\tau\,\cancel{\partial_t A(\underline{r},t)} + \dots}\right]$$

$$\simeq \frac{1}{D_R} A(\underline{r},t)$$

If A does not vary much on a time scale $1/D_R$

$$\Rightarrow \boxed{\rho \underline{f} \simeq \frac{-v_0}{2D_R}\nabla\rho}$$

$$\Rightarrow \boxed{\partial_t \rho - D_{eff} \nabla^2 \rho = 0}$$

$$D_{eff} = D + \frac{v_0^2}{2 D_R}$$

$\rightsquigarrow$ Equivalent to BM with diffusivity D_{eff}

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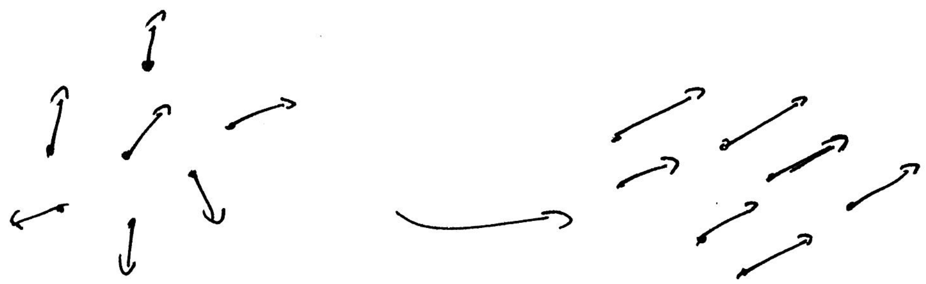
* φ transition is a central concept in AM (Flocking), in L2 review basics of φT and order from field theory perspective.

* φ transitions: central concept in AM studies.

↳ Example:



Flocking ⟺ ordering transition

"like" ferromagnetism

"active ferromagnet".

→ Here, review some basic features of the theories of φ transition at equilibrium, and introduce some tools that we will use to study them in the context of AM

* Important feature: φ transitions are associated with the notion of scale-invariance

Minimal example: the Ising model:

sum over all pairs of n.n.

N spins $s_i = \pm 1$ on a lattice, energy $E = -\frac{J}{2} \sum_{\langle ij \rangle} s_i s_j$

↳ Simple MF analysis: Partition function $Z = \sum_{\{s_i\}} e^{-\beta E}$

Define order parameter $m = \frac{1}{N} \sum_i s_i$ (magnetization)

MF assumption: $s_i \simeq m + \delta s_i$

$$\Rightarrow s_i s_j = m^2 + m(\delta s_i + \delta s_j) + O(\delta s^2) \simeq m(s_i + s_j - m)$$

Rewrite:
$$E_{MF} \simeq -\frac{Jm}{2}\sum_{\langle ij\rangle}(s_i + s_j - m)$$

$$\simeq -\frac{Jm}{2}\left[2z\sum_i s_i - mNz\right]$$

z = # neighbours/site
= coordination number of the lattice

$$\Rightarrow Z_{MF} = \sum_{\{s_i\}} e^{\beta\frac{Jmz}{2}\left[2\sum_i s_i - mN\right]}$$

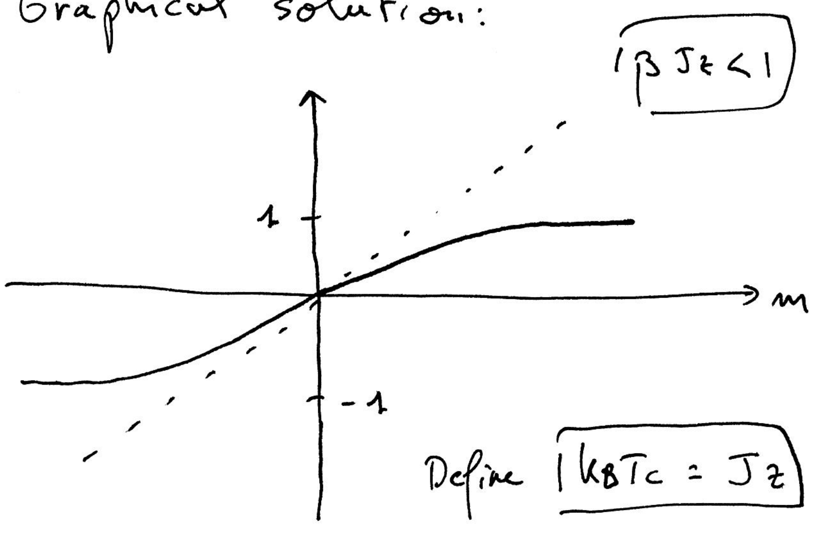
$$= e^{-\beta J\frac{m^2Nz}{2}}\left(\sum_{s_i=\pm 1} e^{\beta Jmz s_i}\right)^N$$

$$\boxed{Z_{MF} = e^{-\beta NJ\frac{m^2 z}{2}}\left(2\cosh(\beta Jmz)\right)^N}$$

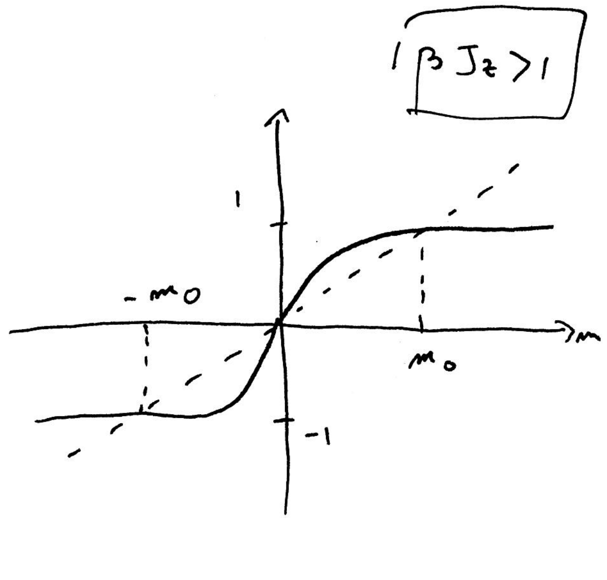
Free energy: $F_{MF} = -k_BT \ln Z_{MF}$
$$= Nk_BT\left(\beta\frac{Jm^2z}{2} - \ln\left(\cosh(\beta Jmz)\right)\right)$$

$\rightsquigarrow$ Minimization: $\dfrac{\partial F_{MF}}{\partial m} = 0 \Leftrightarrow \boxed{m - \tanh(\beta Jmz) = 0}$

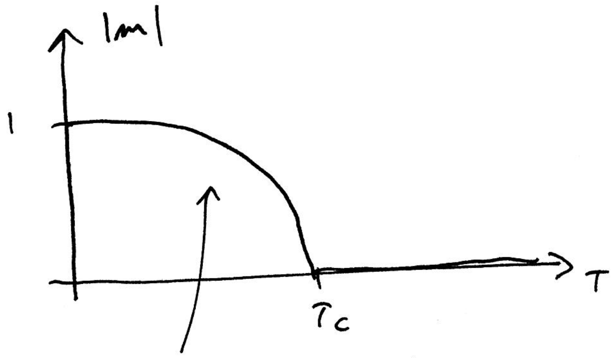
Graphical solution:



Define $\boxed{k_BT_c = Jz}$



⇝ Phase transition at a finite temperature T_c at which the system becomes ordered.



$m = \pm m_0$ (symmetry of the FE is spontaneously broken)

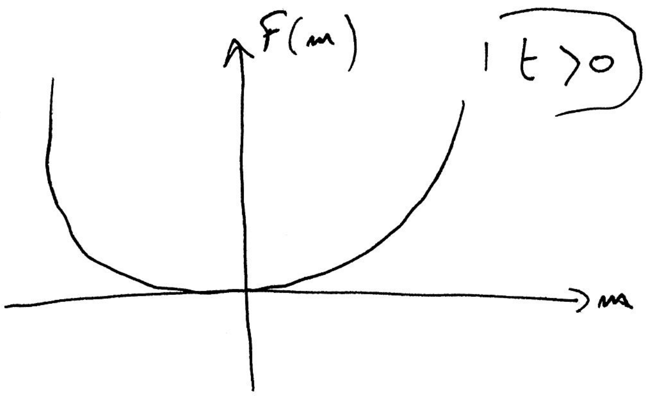
⇝ Expand the free energy at $T \simeq T_c$ (m small):

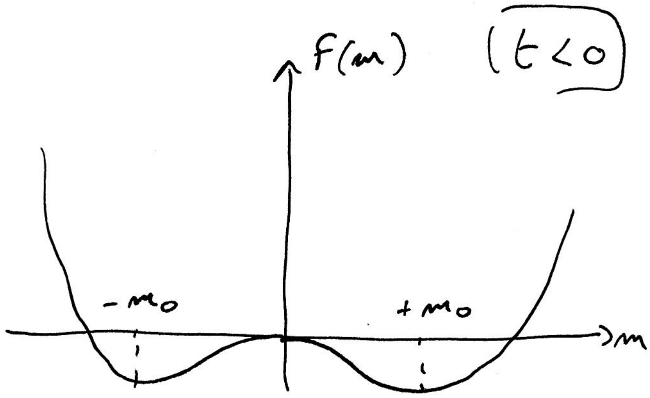
$$F_{MF} \simeq N k_B T \left[\frac{T_c}{T} \frac{m^2}{2} - \ln\left(\cosh\left(m \frac{T_c}{T} \right) \right) \right]$$

$$\boxed{F_{MF} \underset{m \to 0}{\simeq} \frac{N k_B T_c}{2} \left[\left(\frac{T - T_c}{T} \right) m^2 + \left(\frac{T_c}{T} \right)^3 \frac{m^4}{6} + O(m^6) \right]}$$

↳ Landau free energy.

Define $t = \frac{T - T_c}{T_c}$





→ Include spatial dynamics.

$$F[m] = \int d^d\underline{r} \left(f(m) + \frac{K}{2} |\nabla m|^2 \right)$$

Ginzburg-Landau free energy

cost of interfaces

term is invariant by rotations and under $m \to -m$ as allowed by the symmetries of the dynamics.

take $T \simeq T_c$

$$\Rightarrow \quad f(m) = \alpha t \frac{m^2}{2} + \beta \frac{m^4}{4} \qquad \begin{cases} \alpha \simeq N k_B T_c \\ \beta \simeq \frac{N k_B T_c}{3} \simeq \frac{\alpha}{3} \end{cases}$$

"bulk free energy density"

Dynamical equation for m?

$$\frac{\partial m}{\partial t} = -\frac{\delta F}{\delta m} = -(\alpha t + \beta m^2) m + K \nabla^2 m$$

Steady-state homogeneous solution: $(\alpha t + \beta m_0^2) m_0 = 0$

$$\Rightarrow \begin{cases} m_0 = 0 & (t > 0) \\ m_0 = \sqrt{\frac{-\alpha t}{\beta}} & (t < 0) \end{cases}$$

Behavior of m in response to fluctuations?

write $m = m_0 + \delta m$ and study the dynamics of δm at linear ~~order~~ order.

$$\frac{\partial \delta m}{\partial t} = -\upsilon^2 \delta m + K \nabla^2 \delta m + \xi(\underline{x}, t)$$

$$\upsilon^2 = \begin{cases} \alpha t & t > 0 \\ -2\alpha t & t < 0 \end{cases} \quad (> 0)$$

ξ is a GWN satisfying $\langle \xi \rangle = 0$

$$\langle \xi(\underline{x}, t)\, \xi(\underline{x}', t') \rangle = \Xi\, \delta^d(\underline{x} - \underline{x}')\, \delta(t - t')$$

$\rightsquigarrow$ In Fourier space: $\delta \hat{m}_{q,\omega} = \int d^d\underline{r}\, dt\, e^{i\underline{q}\cdot\underline{r} + i\omega t}\, \delta m(\underline{r}, t)$

$$\Rightarrow \quad (i\omega + \upsilon^2 + K q^2)\, \delta \hat{m}_{q,\omega} = \hat{\xi}_{q,\omega}$$

with $\langle \hat{\xi}_{q,\omega}\, \hat{\xi}_{q',\omega'} \rangle = \Xi\, (2\pi)^{d+1}\, \delta(\omega + \omega')\, \delta^d(q + q')$

Fourier transform of the two-point correlation function:

$$\hat{C}_{q,\omega} = \frac{\langle \delta \hat{m}_{q,\omega}\, \delta \hat{m}_{-q,-\omega} \rangle}{(2\pi)^{d+1}\, \delta(\omega + \omega')\, \delta^d(q + q')}$$

$$= \frac{\Xi}{\omega^2 + (\upsilon^2 + K q^2)^2}$$

Equal-time correlation: $\hat{C}_q = \int \frac{d\omega}{2\pi}\, \hat{C}_{q,\omega} = \frac{\Xi}{2(\upsilon^2 + K q^2)}$

$$\Rightarrow \boxed{C(\underline{r}) = \langle \delta m(0)\, \delta m(\underline{r}) \rangle = \frac{\Xi}{2K} \int \frac{d^d q}{(2\pi)^d}\, \frac{e^{-i\underline{q}\cdot\underline{r}}}{\xi^{-2} + q^2}}$$

Correlation length: $\boxed{\xi = \sqrt{\frac{K}{v^2}}}$

$\rightsquigarrow$ Limits of the integral:

$$\begin{cases} C(r) \underset{r \gg \xi}{\sim} \dfrac{e^{-r/\xi}}{r^{(d-1)/2}} \\[2em] C(r) \underset{r \ll \xi}{\sim} \dfrac{1}{r^{d-2}} \end{cases}$$

When ~~Case~~ $\xi \sim |t|^{-1/2}$ is finite $\Rightarrow$ spins are correlated over a typical scale $\sim \xi$.

As $t \to 0$ (CP), $\xi \to +\infty$ and the behavior of the system becomes scale free.

$\rightsquigarrow$ At the CP, the physics is independent of the microscopic details of the dynamics: all the large scale behavior of the system is set by the CL which can be arbitrarily large.

$\hookrightarrow$ Ideas at the root of the powerful concept of universality: physics at a CP only depends on fundamental characteristics like symmetries, conservation laws & dimensionality.

⇝ CP is characterized by a set of critical exponents:

Ex: $|m_0| \sim |t|^{\beta}$, $\xi \sim |t|^{-\nu}$, $C(r) \sim r^{-(d-2+\eta)}$.

↳ BUT: MF predictions are generally incorrect (because we have neglected nonlinearities in the equations). Here, MF breaks down for all $d < d_c = 4$

↳ upper critical dimension

Ising Model:

Beyond MF: use RG

d	2	3	>4
β	1/8	0,3264	1/2
ν	1	0,6300	1/2
η	1/4	0,0363	0
	↑ Exact	↑ Numerics	↳ MF

⇝ Similar ideas hold for systems out-of-equilibrium (when scale-free behavior is present) ⇝ Motivate classification of active systems in terms of their symmetries.

→ Spins with continuous orientation

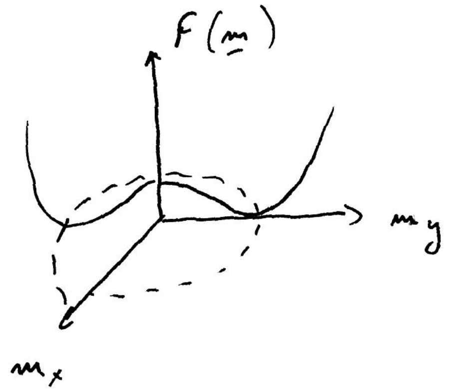
Vector magnetization: $\underline{m} = \frac{1}{N}\langle \sum_i \underline{S}_i \rangle$ $\quad E = -\frac{J}{2} \sum_{\langle ij \rangle} S_i \cdot S_j$

GL free energy: $F = \int d^d r \; \frac{t|\underline{m}|^2}{2} + \frac{|\underline{m}|^4}{4} + \frac{K}{2} \underbrace{|\nabla \underline{m}|^2}_{(\nabla_\alpha m_\beta)(\nabla_\alpha m_\beta)}$

Same procedure:

$$\frac{\partial \underline{m}}{\partial t} = -\left(t + |m|^2\right)\underline{m} + K\nabla^2 \underline{m}$$

SS sol: $|m|^2 = \begin{cases} 0 & t > 0 \\ -t & t < 0 \end{cases} = m_0^2$



$\hookrightarrow \underline{m} = m_0 \hat{u}_{\parallel}$

$\llcorner$ spontaneous symmetry breaking.

Write: $\underline{m} = (m_0 + \delta m_{\parallel})\hat{u}_{\parallel} + \delta \underline{m}_{\perp}$ with $\delta \underline{m}_{\perp} \cdot \hat{u}_{\parallel} = 0$

and focus on the ordered phase: $m_0 > 0$ $(t<0)$

$$\Rightarrow \begin{cases} \dfrac{\partial \delta m_{\parallel}}{\partial t} = +2t\,\delta m_{\parallel} + K\nabla^2 \delta m_{\parallel} + \xi_{\parallel}(\underline{r},t) \\[2ex] \dfrac{\partial \delta \underline{m}_{\perp}}{\partial t} = K\nabla^2 \delta \underline{m}_{\perp} + \underline{\xi}_{\perp}(\underline{r},t) \end{cases}$$

(At linear order)

$\Rightarrow$ $\delta m_{\parallel}$ follows a similar equation as for the Ising model with $\xi = \sqrt{\dfrac{K}{-2t}}$, while $\boxed{\text{for } \delta \underline{m}_{\perp} \;\; \xi = 0}$!!

$\delta \underline{m}_{\perp}$ is a Goldstone mode (massless) $\rightarrow$ results from spontaneous symmetry breaking (No energetic cost associated with global rotation of the system).

Also implies that

$$\hat{C}_{\perp q} = \frac{\eta_\perp \overbrace{(d-1)}^{\#\perp \text{ components}}}{2 K q^2}$$

$$\Rightarrow \underbrace{\langle |\delta m_\perp(r)|^2 \rangle}_{\substack{\text{typical amplitude} \\ \text{of fluctuations}}} = \int \frac{d^d q}{(2\pi)^d} \hat{C}_{\perp,q}$$

$$\propto \int_{1/L \,(\text{system size})}^{1/a \,(\text{lattice size})} dq \; q^{d-3} \sim \begin{cases} L^{2-d} & d < 2 \\ \ln(L/a) & d = 2 \\ a^{2-d} & d > 2 \end{cases}$$

$\Rightarrow$ Fluctuations diverge in all $d \leq 2$ $\rightarrow$ no long-range order is possible. Mermin-Wagner theorem