Semester assigment ElMag - Simulation of aurora

Martin Johsnrud

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Auroras come from charged particles launch into outer space by sun storms beeing guided by the magnetic field of the earth down in the atmosphere. This project simulates how the path of these charged particles, spesifically protons, are affected by magnetic field using a Runge-Kutta method.

Theory

Parametres

This problem has many parametres, $q, M, R_\odot, m, \mu_0$, respectivley the proton charge and mass, the earths mass and magnetic dipole moment and the magnetic peremability of space. They span many orders of magnitude, which is inconvinient in numerical computing. By reducing them to 4 characteristic sizes, length, magnetic flux density, time and energy, they can be used as convenien units fitting the problem. They are

$$x_0 = R_\odot, \quad B_0 = \frac{\mu_0 m}{x_0^3}, \quad t_0 = \frac{M}{q}\frac{5\cdot 10^5}{B_0}, \quad E_0 = M\frac{x_0^2}{t_0^2}.$$

A numeric factor has been added to the time, so that x_0/t_0 is about the speed of a typical coronal ejection. [1] As some of them do not have permanet and definit values they will not give very precise results. Their values are roughly [2]

$$x_0 = 6.47\cdot 10^6\mathrm{m}, \quad B_0 = \frac{1.26\cdot 10^{-6}\, 8\cdot 10^{22}}{x_0^3} = 3.7\cdot 10^{-4}\,\frac{\mathrm{kg}}{\mathrm{s^2\,A}}$$
$$t_0 = \frac{1.7\cdot 10^{-27}}{1.6\cdot 10^{-19}}\frac{5\cdot 10^5}{B_0} = 14\mathrm{s}, \quad E_0 = M\frac{x_0^2}{t_0^2} = 3.6\cdot 10^{-16}\,\frac{\mathrm{kg\,m^2}}{\mathrm{s^2}}$$

These are used to define the dimensionless varibales

$$x_i^* = x_i/x_0, \quad B_i^* = B_i/B_0, \quad t_i^* = t_i/t_0, \quad E^* = E/E_0,$$

which are used thourgout the report, but the asterisk will be dropped for convenience.

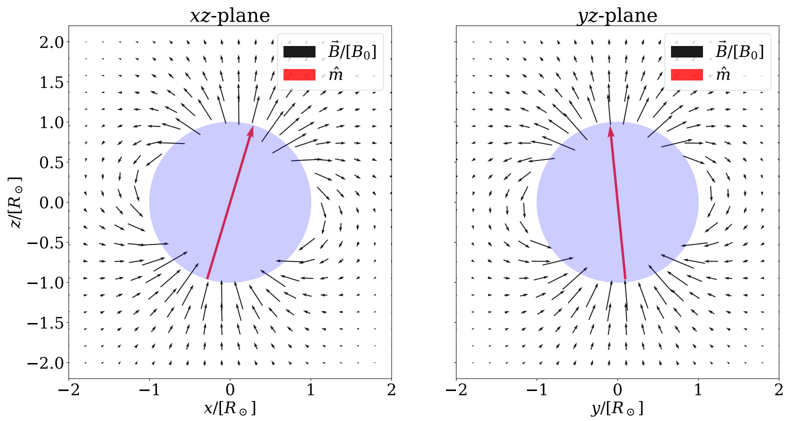

Figure 1: The earths magnetic field, in the xz- and yz-plane.

Magnetic field

We can model the earths magnetic field as a diploe, with dipole moment $\vec{m}$. Let the x-axis point from the centre of the earth towards the sun, the y-axis roughly paralel to the earths orbit, and then the z-axis northwards. The magnetic field of the earth is then given by

$$B_j = \frac{1}{4\pi} \frac{3\hat{m}_i \hat{x}_i \hat{x}_j - \hat{m}_j}{r^3}, \tag{1}$$

where

$$r = \sqrt{x_i x_i}, \quad \hat{x}_i = x_i/r, \quad \hat{m}_i = m_i/m$$

This is the only force being considered in this simulation.

Charged particles

The force from the magnetic field of the earth on a particle with charge q, position $\hat{e}_i x_i(t)$ and veloxity $\hat{e}_i \dot{x}_i(t)$ is

$$F_i = \epsilon_{ijk} \dot{x}_j B_k. \tag{2}$$

Newtons second law gives the equation of motion for the particle,

$$\ddot{x}_i = \epsilon_{ijk} \dot{x}_j B_k \tag{3}$$

Rewriting to a first order set of equations yields

$$\frac{\mathrm{d}}{\mathrm{d}t} y = \frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x_i \\ \dot{x}_i \end{pmatrix} = \begin{pmatrix} \dot{x}_i \\ \epsilon_{ijk} \dot{x}_j B_k \end{pmatrix} = f(y).$$

This can then be solved numerically by a Runge-Kutta method. The simulation done here uses the Runge-Kutta 4 method.

Accuracy

The fact that magnetic forces never do any work can be use to test the accuracy of the numerical scheme. The kinetic energy of the particle should be conserved, as the simultation does not take into account friction, nor electromagnetic radiation. Kinetic energy is given by

$$E = \frac{1}{2}\dot{x}^2.$$

Relative error,

$$\Delta E(t) = \frac{|E(t) - E(0)|}{E(0)},$$

is thus a usefull quantity for evaluating the precission of the simulations.

Results and discussion

Figure 1 shows the magnetic field as seen from the behind the erath in its orbit, and on the other side than the sun. The results are shown in figure 2, with a closer look at the most intricate paths in figure 3.

Several particles with different starting positions are simulated as they approach the earth. They are then deflected by the magnetic field of the earth, to varying degrees. Particles approaching the equator are deflected northwards, and then flung away. The spiraling paths are consistent with the fact that magnetic forces act orthogonally to the velocity.

Particles comming towards the earth further north comes closer to the surface of the earth, while some even reach it and goes though the surface, as there is nothing in the simulation taking into account the earth. This shows that the simultaion captures both the ability of the earths magnetic field to shield the earth, and why auroras only are seen closer to the poles.

The relative error in energy for all paths, as a function of time, is show in figure 4. This shows that the step length used, $\Delta t = 0.005$, is more than enough for this purpose.

References

[1] Wikipedia contributers, *Coronal mass ejection,* `https://en.wikipedia.org/wiki/Coronal_mass_ejection#physical_properties` [Online; accessed 21-02-2020]

[2] Wolfram alpha LLC, *Magnetic dipole moment of the earth*, `https://www.wolframalpha.com/input/?i=magnetic+dipole+moment+of+the+earth` [Online; accessed 21-02-2020]

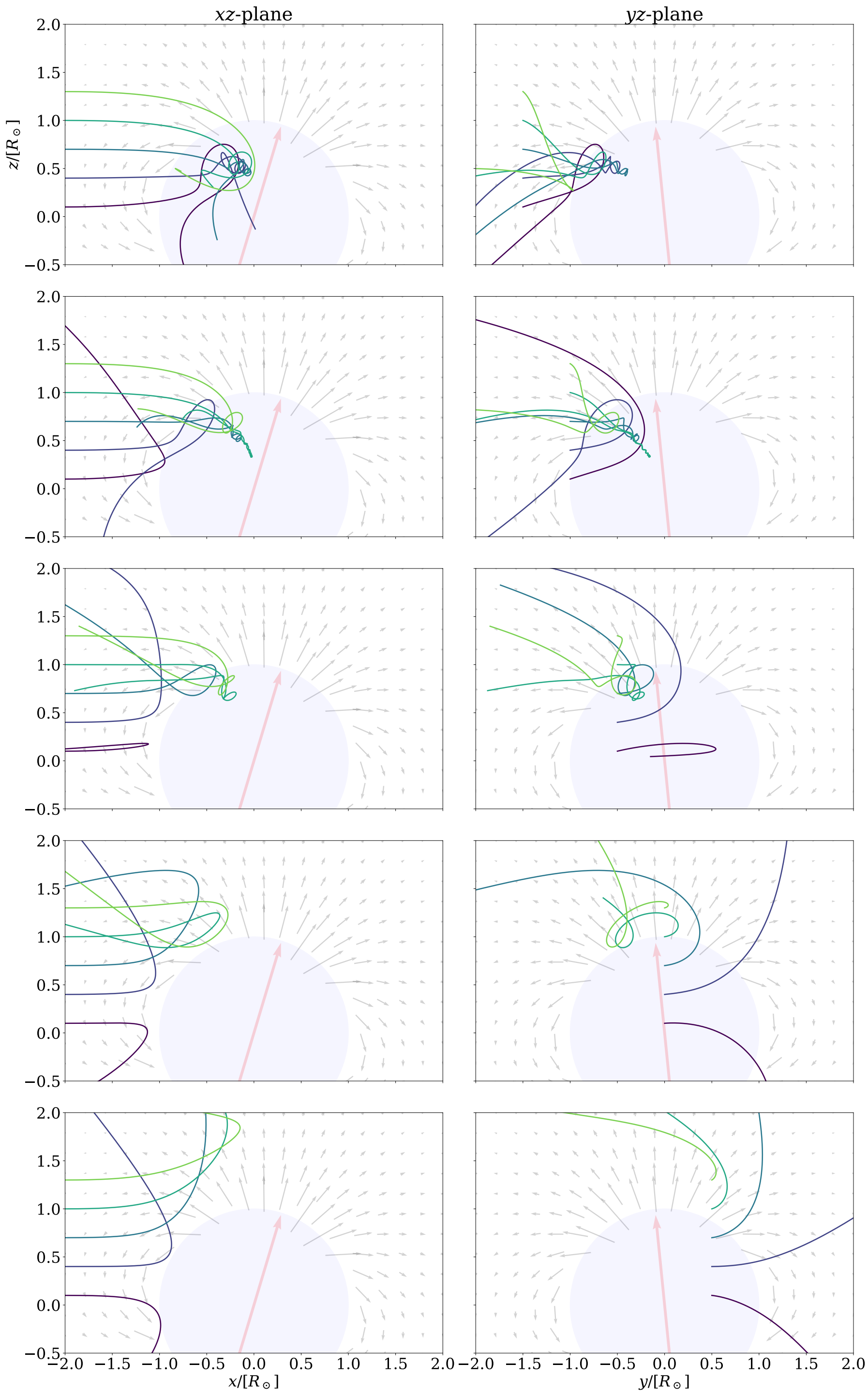

Figure 2: Charged particles approach earth from different starting points. Evne though the lines may overlap with the earth, it does not mean that they have hit the ground, as the graph only contains two dimensions. The magnetic dipole moment is not to scale, and only inteded to indicate the direction.

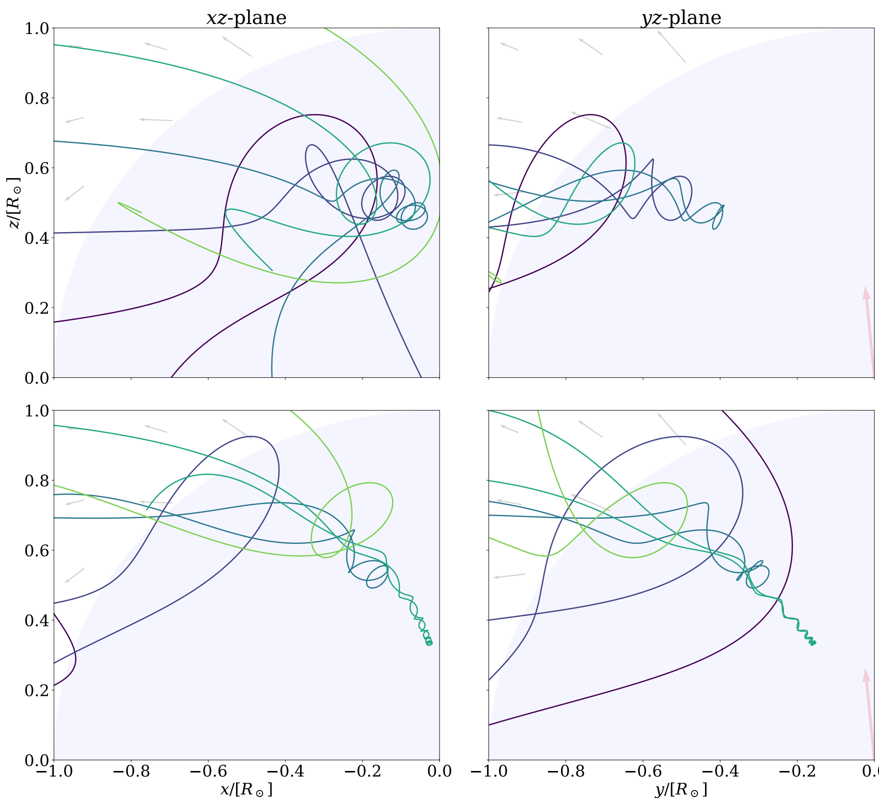

Figure 3: A closer look at the two first series. The magnetic force leads to a change in direction of the particles, while the speed is maintaine, and the particles are sent back into space.

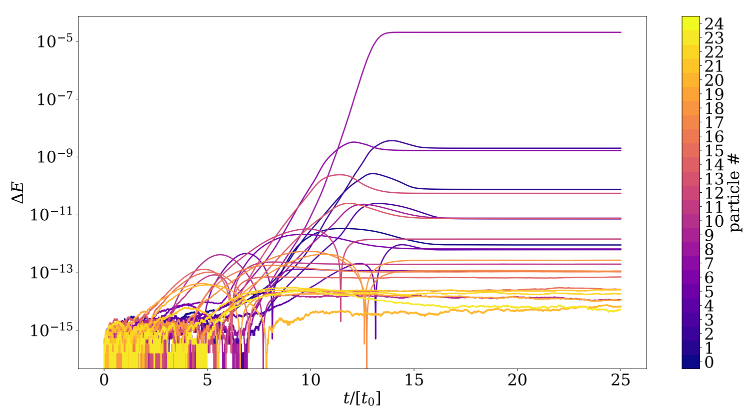

Figure 4: The relative shifts in energy, in comparison with the starting values. Particle 8 has the largest value, $2.036 \cdot 10^{-5}$. This corresponds to the particle with the tightest turns in its path.