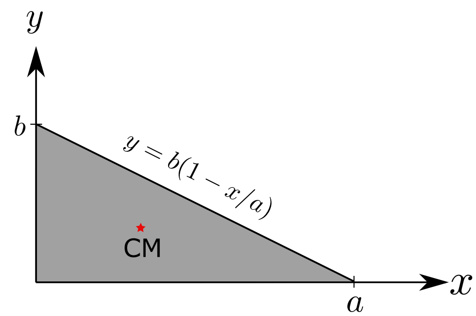
Exercise 8 - TFY4345 Classical Mechanics

2020

1 Principal moments of inertia of a triangular slab

(Exam Aug. 2019)



(a) Compute the center-of-mass (CM) for the planar triangle in the figure, assuming it to be of uniform two-dimensional mass density ρ.

(b) Compute the inertia tensor *with respect to the origin* for the same triangle.

(c) (Optional) If the origin is shifted to the CM, the inertia tensor becomes (this can be show by using the Steiner's parallel axis theorem)

$$I_{CM} = \begin{pmatrix} I_{11} & I_{12} & 0 \\ I_{21} & I_{22} & 0 \\ 0 & 0 & I_{33} \end{pmatrix} = \frac{M}{18} \begin{pmatrix} a^2 & \frac{1}{2}ab & 0 \\ \frac{1}{2}ab & b^2 & 0 \\ 0 & 0 & a^2 + b^2 \end{pmatrix}$$

where $I_{xy} = I_{yx}$ and $I_{xx} + I_{yy} = I_{zz}$ in the general form show first. Define next

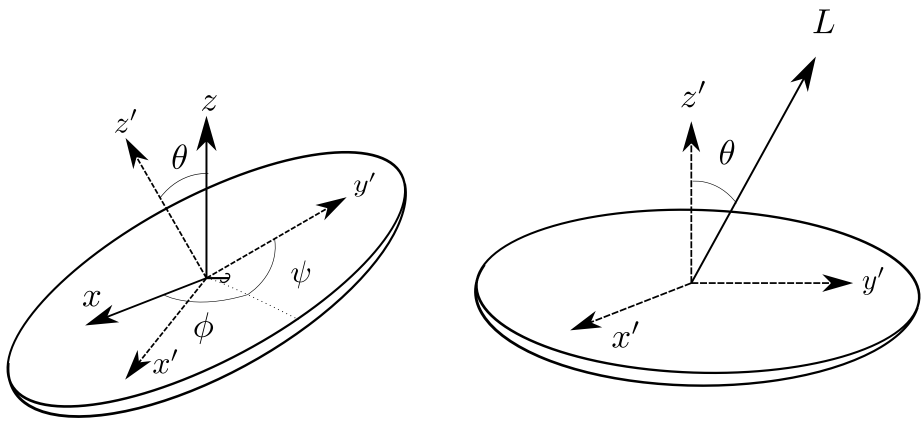
$$A = \frac{1}{2}(I_{xx} + I_{yy}), \quad B = \sqrt{\frac{1}{2}(I_{xx} - I_{yy})^2 + I_{xy}^2}, \quad \vartheta = \tan^{-1}\left(\frac{2I_{xy}}{I_{xx} - I_{yy}}\right).$$

Derive the principal moments of inertia and the principal axes by using the general form of the inertia tensor, and these new variables. [Hint] The last equations comprises a relationship that can be described by a right triangle.

2 Precession of a frisbee

(Exam Aug. 2016)

Consider an axial-symmetric body with the principal moments of inertia $I_1 = I_2 \neq I_3$, rotating with angular momentum $\mathbf{L} = L\mathbf{e}_z$ in the laboratory frame. (The unmarked coordinate system.)



(a) Derive the equations of motion for the body, using the Euler equations and the angles θ, ψ and ϕ. Find the components of $\boldsymbol{\omega}$ in the body system. (See lecture notes, we derived this already!)

(b) Find the expression for the Euler angles $\dot{\theta}, \dot{\psi}, \dot{\phi}$ as a function of I_i, L, θ.

(b) Assume $I_3 = 2I_1$. The precession (wobble) of the frisbee is given by $\dot{\phi}$. Show that the precession is twice as fast as the rotation frequency of the frisbee, assuming that θ is small (i.e. that $\cos(\theta) \approx 1$).

3 Precession of a heavy spinning top

(Based on example p. 208-223 in Goldstein 3rd. ed., p. 70-74 in the compendium)
In the example we define the shifted energy as

$$E' = \frac{1}{2} I_1 \dot{\theta}^2 + V(\theta), \quad V(\theta) = \frac{(p_\phi - p_\psi \cos(\theta))^2}{2I_1 \sin^2(\theta)} + Mgh\cos(\theta),$$

which is a constant of motion. We also found the constants of motion

$$p_\psi = I_3(\dot{\phi}\cos(\theta) + \dot{\psi}) = I_3\omega_3, \quad p_\phi = (I_1 \sin^2(\theta) + I_3\cos^2(\theta))\dot{\phi} + I_3\dot{\psi}\cos(\theta).$$

θ_0 was defined to be the constant angle of inclination of spinning top with regular precession. This means that the symmetry axis rotates around the z at a fixed angle θ_0. Consider the shape of the effective potential $V(\theta)$ at θ_0. *What is the condition for the spinning top to stay at a constant angle θ_0?* Think back to our treatment of orbits. The following change of variables will come in handy for the result:

$$\beta = p_\phi - p_\psi \cos(\theta_0) = I_1 \sin^2(\theta)\dot{\phi}_0$$

You will encounter a quadratic equation for β. Show that for the equilibrium precession inclination angle θ_0, the following must hold true:

$$\omega_3 \geq \frac{2}{I_3}\sqrt{MghI_1\cos(\theta_0)}.$$

What can you say about the corresponding precession angular velocity $\dot{\phi}_0$? Express $\dot{\phi}_0$ when $\omega_3 \gg \frac{2}{I_3}\sqrt{MghI_1\cos(\theta_0)}$