Exercise 6 solutions - TFY4345 Classical Mechanics

2020

1 Elastic scattering in laboratory coordinates

The relations we start with are

$$\cos(\vartheta) = \frac{\cos(\Theta) + \rho}{\sqrt{1 + 2\rho\cos(\Theta) + \rho^2}}, \quad \sigma'(\vartheta) = \sigma(\Theta)\frac{(1 + 2\rho\cos(\Theta) + \rho^2)^{3/2}}{1 + \rho\cos(\Theta)}, \quad \rho = m_1/m_2.$$

By the assumption of equal masses, we get $\rho = 1$, and thus

$$\cos(\vartheta) = \frac{1 + \cos(\Theta)}{\sqrt{2}\sqrt{1 + \cos(\Theta)}} = \sqrt{\frac{1 + \cos(\Theta)}{2}}.$$

Using the provided trigonometric identity,

$$\cos(\vartheta) = \cos(\Theta/2) \implies \vartheta = \Theta/2$$

This means scattering angels of above 90° is not possible in the lab system.

The relation for the cross sections becomes

$$\sigma'(\vartheta) = \sigma(\Theta)\frac{2^{3/2}(1 + \cos(\Theta))^{3/2}}{1 + \cos(\Theta)} = 4\sqrt{\frac{1 + \cos(\Theta)}{2}}\sigma(\Theta) = 4\cos(\Theta/2)\sigma(\Theta), \quad \vartheta \leq \pi/2$$

This means that for isotropic scattering (when the cross section in the CM frame $\sigma(\Theta)$ is constant), the cross section in the lab frame goes as the cosine of the scattering angle, also in the lab frame.

The velocity of of particle i after the collision is denoted by v_i in the lab frame, and v_i' in the CM frame, while the velocity of the CM frame in the lab frame is denote by V. Assume $i = 1$ was the incident particle. We then have the relation (see lecture notes, chapter 4)

$$v_1^2 = v_1'^2 + V^2 + 2v_1'V\cos(\Theta), \quad V = \frac{\mu}{m_2}v_0, \quad \rho = \frac{\mu}{m_2}\frac{v_0}{v_1'}$$

where $\mu = m_1m_2/(m_1 + m_2)$ is the reduced mass, while v_0 is the initial velocity of the incident particle. Inserting $m_1 = m_2$, we get

$$V = \frac{1}{2}v_0, \quad v_1' = \frac{1}{2}v_0$$

This then gives

$$v_1^2 = 2(\frac{1}{2}v_0)^2 + 2(\frac{1}{2}v_0)^2\cos(\Theta) = \frac{1}{2}v_0(1+\cos(\Theta)).$$

The relation for the kinetic energy before and after the collision is therefore,

$$\frac{E_1}{E_0} = \frac{v_1^2}{v_o^2} = \frac{1+\cos(\Theta)}{2} = \cos^2(\vartheta)$$

2 Rotating system in cylindrical coordinates.

Cylindrical coordinates are given by the relations

$$x = r\cos(\theta), \quad y = r\sin(\theta), z = z,$$

giving

$$\begin{cases} \dot{x}^2 = \dot{r}\cos(\theta) - r\dot{\theta}\sin(\theta) \\ \dot{y}^2 = \dot{r}\sin(\theta) + r\dot{\theta}\sin(\theta) \\ \dot{z}^2 = \dot{z}^2 \end{cases}$$

The kinetic energy of a particle of mass m is then

$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) = \frac{1}{2}m\Big[\dot{r}^2\big(\cos^2(\theta)+\sin^2(\theta)\big) + 2\dot{r}r\dot{\theta}\big(\cos(\theta)\sin(\theta)-\cos(\theta)\sin(\theta)\big) \tag{1}$$

$$+ r^2\dot{\theta}\big(\cos^2(\theta)+\sin^2(\theta)\big) + \dot{z}^2\Big] = \frac{1}{2}(\dot{r}^2 + (r\dot{\theta})^2 + \dot{z}^2), \tag{2}$$

giving the lagrangian

$$L = T - V = \frac{1}{2}m[\dot{r}^2 + (r\dot{\theta})^2 + \dot{z}^2] - V(r,\theta,z).$$

The needed derivatives are

$$\begin{cases} r: \quad \dfrac{\partial}{\partial r}L = mr\dot{\theta}^2 - \dfrac{\partial V}{\partial r}, \quad \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial}{\partial \dot{r}}L = \dfrac{\mathrm{d}}{\mathrm{d}t}m\dot{r} = m\ddot{r} \\ \theta: \quad \dfrac{\partial}{\partial \theta}L = -\dfrac{\partial V}{\partial \theta}, \quad \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial}{\partial \dot{\theta}}L = \dfrac{\mathrm{d}}{\mathrm{d}t}\Big(mr^2\dot{\theta}\Big) = 2m\dot{r}r\dot{\theta} + mr^2\ddot{\theta} \\ z: \quad \dfrac{\partial L}{\partial z} = -\dfrac{\partial V}{\partial z}, \quad \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial}{\partial \dot{z}}L = \dfrac{\mathrm{d}}{\mathrm{d}t}m\dot{z} = m\ddot{z}, \end{cases}$$

which gives the equations of motion

$$\begin{cases} m\ddot{r} = mr\dot{\theta}^2 - \dfrac{\partial V}{\partial r} \\ 2m\dot{r}r\dot{\theta} + mr^2\ddot{\theta} = -\dfrac{\partial V}{\partial \theta} \\ m\ddot{z} = -\dfrac{\partial V}{\partial z}. \end{cases}$$

The canonical momenta can be read of the earlier derivatives, and are

$$\begin{cases} p_r = m\dot{r} \\ p_\theta = mr^2\dot{\theta} \\ p_z = m\dot{z}. \end{cases}$$

This means that we can rewrite the kinetic energy term as

$$T = \frac{1}{2}\left[\frac{p_r^2}{m} + \frac{p_\theta^2}{r^2 m} + \frac{p_z^2}{m}\right]$$

As the assumptions of a time independent Lagrangian, a conservative force and a kinetic energy which is quadratic in the velocity terms (see Goldstein p. 338-339), we can write

$$H = T + V = \frac{1}{2}\left[\frac{p_r^2}{m} + \frac{p_\theta^2}{r^2 m} + \frac{p_z^2}{m}\right] + V(r, \theta, z)$$

Hamiltons equations of motion are then

$$\begin{cases} \dot{q}_r = \dfrac{\partial H}{\partial p_r} = \dfrac{p_r}{m}, \quad -\dot{p}_r = \dfrac{\partial H}{\partial r} = \dfrac{\partial V}{\partial r} - \dfrac{p_\theta^2}{r^3 m} \\ \dot{q}_\theta = \dfrac{\partial H}{\partial p_\theta} = \dfrac{p_\theta}{r^2 m}, \quad -\dot{p}_\theta = \dfrac{\partial H}{\partial \theta} = \dfrac{\partial V}{\partial \theta} \\ \dot{q}_z = \dfrac{\partial H}{\partial p_z} = \dfrac{p_z}{m}, \quad -\dot{p}_z = \dfrac{\partial H}{\partial z} = \dfrac{\partial V}{\partial z} \end{cases}$$

3 Centrifugal force and gravitation

In the rotating coordinate system, the centrifugal force acting on an element of length $\mathrm{d}r$, a distance r from the center, is

$$\mathrm{d}F_c = r\omega^2\lambda\,\mathrm{d}r.$$

The gravitational pull on the same piece of rod is given by

$$\mathrm{d}F_g = \frac{Gm\lambda\mathrm{d}r}{r^2} = \frac{gR^2\lambda\mathrm{d}r}{r^2},$$

where g is the gravitational acceleration on the surface of the earth. Balancing the centrifugal and gravitational force then amounts to demanding that

$$F_c = \int_R^{R+L} r\omega^2\lambda\,\mathrm{d}r = \omega^2\rho\lambda(2RL + L^2) \tag{3}$$

$$= F_g = \int_R^{R+L} \frac{gR^2\lambda\mathrm{d}r}{r^2} = 2g_0R^2\lambda\frac{L}{(R+L)R} \tag{4}$$

$$\implies L^2 + 3RL + \left(2R^2 - \frac{2g_0R}{\omega^2}\right) = 0 \implies L = -\frac{3R}{2} + \frac{1}{2}\sqrt{R^2 + \frac{8gR}{\omega^2}}. \tag{5}$$

Setting values for the earth, $\omega = 2\pi/(1\text{day})$, $R = 6.4\cdot10^3\text{km}$, $g = 9.8\text{m}\,\text{s}^{-1}$, we get $L = 1.4\cdot10^5\text{km}$, or $1/e$ of the way to the moon.

4 Coriolis effect on a falling particle

We do not need to take into account the centrifugal force, as it is proportional to ω^2 and thus negligible. The acceleration seen in the coordinate system rotating with the earth is then (see chapter 5.H in the compendium, chapter 4.10 in Goldstein, p. 174 - 180)

$$\mathbf{a_s} = \mathbf{a_r} + 2\omega \times \mathbf{v_r}$$

When taking the cross product, the coriolis force is negligible compared to the acceleration due to the gravity of the earth ($\omega|\mathbf{v_r}| \ll g$), and it is thus a very good first order approximation that the motion of the particle is

$$\dot{x} = 0,\ \dot{y} = 0,\ \dot{z} = -gt.$$

We then have

$$\omega \times \mathbf{v}_r = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_x & \mathbf{e}_x \\ -\omega\cos(\alpha) & 0 & \omega\sin(\alpha) \\ 0 & 0 & -gt \end{vmatrix} = -\omega gt\cos(\alpha)\mathbf{e}_y$$

The acceleration in the non-rotating frame of reference is

$$\mathbf{a_s} = -g\mathbf{e_z}$$

The equations of motion in the rotating frame of reference thus become

$$(\mathbf{a}_r)_x = \ddot{x} = 0, \quad (\mathbf{a}_r)_y = \ddot{y} = 2\omega gt\cos(\alpha), \quad (\mathbf{a_r})_z = \ddot{z} = -g$$

Integrating twice, and setting $y(0) = \dot{y}(0) = \dot{z}(0) = 0$ then gives the motion due to the coriolis effect,

$$y(t) = \frac{1}{3}\omega g\cos(\alpha)t^3, \quad z(t) = z_0 - \frac{1}{2}gt^2$$

The fall time from the height $h = z_0$ is $t = \sqrt{2h/g}$, so the total eastward deflection is

$$d = \frac{1}{3}\omega\cos(\alpha)\sqrt{\frac{8h^3}{g}}.$$

An object dropped from 100m, at latitude 45° north is deflected approximately 1.55cm.