Exercise 1 solutions - TFY4345 Classical Mechanics

2020

1 Halley's comet

a) The gravitational force on the comet is

$$\vec{F} = -\frac{GmM}{r^2}\vec{e}_r,$$

where m is the mass of the comet, and M is the mass of the sum. The torque on the comet is therefore

$$\vec{N} = \vec{r} \times \vec{F} = -r\vec{e}_r \times \frac{GmM}{r^2}\vec{e}_r = \frac{GmM}{r}\vec{e}_r \times \vec{e}_r = 0.$$

b) The angular momentum of the comet when the comet is closest to and farthest from the sun is, respectively $\vec{L}_e = \vec{r}_e \times \vec{p}_e$ and $\vec{L}_f = \vec{r}_f \times \vec{p}_f$. Conservation of angular momentum implies that $\vec{L}_e = \vec{L}_f$, and in turn that $|\vec{L}_e| = |\vec{L}_f|$. At both the point the comet is closest to and farthest from the sun, the position vector $\vec{r}$ and the momentum vector $\vec{p}$ are perpendicular. This means that

$$|\vec{L}_e| = |\vec{L}_f| \implies |\vec{r}_e||\vec{p}_e| = |\vec{r}_f||\vec{p}_f| \implies r_e m v_e = r_f m v_f.$$

The velocity of the comet farthest from the sun is therefore

$$v_f = \frac{r_e}{r_f}v_e = \frac{0.6\,\mathrm{AU}}{35\,\mathrm{AU}}\,54\,\mathrm{km}\,/\,\mathrm{s} = 0.9\,\mathrm{km}\,/\,\mathrm{s}$$

2 Simple pendulum

a) By a trigonometric consideration,

$$\vec{R} = \ell\sin(\beta)\vec{e}_x - \ell\cos(\beta)\vec{e}_y$$

b) Potential energy for a mass m in a uniform gravitational field is $V = mgh$, where h is the height of the mass (in an arbitrary reference frame). In our case, $\ell\cos(\beta)$, and hence

$$V = -mg\ell\cos(\beta).$$

A different choice of reference frame will only add a constant to V, which will not affect the equations of motion.

c) The velocity of the mass m is given by

$$\vec{v} = \frac{\mathrm{d}\vec{R}}{\mathrm{d}t} = \ell\cos(\beta)\dot{\beta}\vec{e}_x + \ell\sin(\beta)\dot{\beta}\vec{e}_y.$$

The square of the velocity is then given by

$$v^2 = \vec{v}\cdot\vec{v} = (\ell\dot{\beta})^2(\cos^2(\beta)\vec{e}_x\cdot\vec{e}_x + \sin^2(\beta)\vec{e}_y\cdot\vec{e}_y) = (\ell\dot{\beta})^2.$$

The kinetic energy is therefore

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\ell\dot{\beta})^2$$

d) The Lagrangian of the pendulum is

$$L = T - V = \frac{1}{2}m(\ell\dot{\beta})^2 + mg\ell\cos(\beta).$$

The Lagrange equations are in general

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0.$$

In this case, we have only one variable, $q_1 = \beta$. The resulting Lagrange equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\beta}} - \frac{\partial L}{\partial \beta} = 0.$$

Inserting the Lagrangian we found, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\beta}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(m\ell^2\dot{\beta}\right) = m\ell^2\ddot{\beta}, \quad \frac{\partial L}{\partial \beta} = -mg\ell\sin(\beta)$$

so the equation of motion is

$$\ddot{\beta} + \frac{g}{\ell}\cos(\beta) = 0$$

3 Double pendulum

(a) The position of mass m_1 is, just as in the single pendulum,

$$\vec{R}_1 = \ell_1\sin(\beta_1)\vec{e}_x - \ell_1\cos(\beta_1)\vec{e}_y.$$

The position of mass two is then, by the same considerations just using $\vec{R}_1$ as the point of reference,

$$\begin{aligned}\vec{R}_2 &= \vec{R}_1 + \ell_2\sin(\beta_2)\vec{e}_x - \ell_2\cos(\beta_2)\vec{e}_y \\ &= [\ell_1\sin(\beta_1) + \ell_2\sin(\beta_2)]\,\vec{e}_x - [\ell_1\cos(\beta_1) + \ell_2\cos(\beta_2)]\,\vec{e}_y.\end{aligned}$$

The velocity and square velocity of m_1 is also just as in exercise 2:

$$\vec{v}_1 = \frac{\mathrm{d}R_1}{\mathrm{d}t}, \quad v_1^2 = (\ell_1\dot{\beta}_1)^2.$$

The velocity and velocity square of m_2 is

$$\vec{v}_2 = \frac{\mathrm{d}\vec{R}_2}{\mathrm{d}t} = \left[\ell_1\cos(\beta_1)\dot\beta_1 + \ell_2\cos(\beta_2)\dot\beta_2\right]\vec{e}_x + \left[\ell_1\sin(\beta_1)\dot\beta_1 + \ell_2\sin(\beta_2)\dot\beta_2\right]\vec{e}_y,$$

$$\begin{aligned}
v_2^2 &= \left[\ell_1\cos(\beta_1)\dot\beta_1 + \ell_2\cos(\beta_2)\dot\beta_2\right]^2 + \left[\ell_1\sin(\beta_1)\dot\beta_1 + \ell_2\sin(\beta_2)\dot\beta_2\right]^2 \\
&= (\ell_1\dot\beta_1)^2 + (\ell_2\dot\beta_2)^2 + 2\ell_1\ell_2\cos(\beta_1)\cos(\beta_2)\dot\beta_1\dot\beta_2 + 2\ell_1\ell_2\sin(\beta_1)\sin(\beta_2)\dot\beta_1\dot\beta_2 \\
&= (\ell_1\dot\beta_1)^2 + (\ell_2\dot\beta_2)^2 + 2\ell_1\ell_2\dot\beta_1\dot\beta_2\cos(\beta_1-\beta_2),
\end{aligned}$$

where we have used the trigonometric addition law $\cos(\theta)\cos(\phi)+\sin(\theta)\sin(\phi) = \cos(\theta-\phi)$ in the last line. Total potential energy is

$$V = m_1gh_1 + m_2gh_2 = -m_1g\ell_1\cos(\beta_1) - m_2g\left[\ell_1\cos(\beta_1) + \ell_2\cos(\beta_2)\right].$$

Total kinetic energy is

$$T = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1(\ell_1\dot\beta_1)^2 + \frac{1}{2}m_2\left[(\ell_1\dot\beta_1)^2 + (\ell\dot\beta_2)^2 + \ell_1\ell_2\dot\beta_1\dot\beta_2\cos(\beta_1-\beta_2)\right],$$

so the Lagrangian is, after gathering some terms,

$$\begin{aligned}
L &= T - V \\
&= (m_1+m_2)\left[\frac{1}{2}(\ell_1\dot\beta_1)^2 + g\ell_1\cos(\beta_1)\right] + m_2\left[\frac{1}{2}(\ell\dot\beta_2)^2 + \ell_1\ell_2\dot\beta_1\dot\beta_2\cos(\beta_1-\beta_2) + g\ell_2\cos(\beta_2)\right].
\end{aligned}$$

b) The Lagrange equations for this problem are

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot\beta_1} - \frac{\partial L}{\partial\beta_1} &= 0, \\
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot\beta_2} - \frac{\partial L}{\partial\beta_2} &= 0.
\end{aligned}$$

Calculating the quantities needed:

$$\begin{aligned}
\frac{\partial L}{\partial\beta_1} &= -(m_1+m_2)g\ell_1\sin(\beta_1) - m_2\ell_1\ell_2\dot\beta_1\dot\beta_2\sin(\beta_1-\beta_2) \\
\frac{\partial L}{\partial\dot\beta_1} &= (m_1+m_2)\ell_1^2\dot\beta_1 + m_2\ell_1\ell_2\dot\beta_2\cos(\beta_1-\beta_2) \\
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot\beta_1} &= (m_1+m_2)\ell_1^2\ddot\beta_1 + m_2\ell_1\ell_2\left(\ddot\beta_2\cos(\beta_1-\beta_2) - \dot\beta_2\sin(\beta_1-\beta_2)(\dot\beta_1-\dot\beta_2)\right) \\
\frac{\partial L}{\partial\beta_2} &= m_2\left[\ell_1\ell_2\dot\beta_1\dot\beta_2\sin(\beta_1-\beta_2) - g\ell_2\sin(\beta_2)\right] \\
\frac{\partial L}{\partial\dot\beta_2} &= m_2\left[\ell_2^2\dot\beta_2 + \ell_1\ell_2\dot\beta_1\cos(\beta_1-\beta_2)\right] \\
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot\beta_2} &= m_2\left[\ell_2^2\ddot\beta_2 + \ell_1\ell_2\left(\ddot\beta_1\cos(\beta_1-\beta_2) - \dot\beta_1\sin(\beta_1-\beta_2)(\dot\beta_1-\dot\beta_2)\right)\right].
\end{aligned}$$

This gives us the equation of motion

$$
\begin{aligned}
&(m_1+m_2)\ell_1^2\ddot{\beta}_1 + m_2\ell_1\ell_2\left(\ddot{\beta}_2\cos(\beta_1-\beta_2) - \dot{\beta}_2\sin(\beta_1-\beta_2)(\dot{\beta}_1-\dot{\beta}_2)\right) \\
&+(m_1+m_2)g\ell_1\sin(\beta_1) + m_2\ell_1\ell_2\dot{\beta}_1\dot{\beta}_2\sin(\beta_1-\beta_2) = 0
\end{aligned}
$$

and

$$
\begin{aligned}
&m_2\left[\ell_2^2\ddot{\beta}_2 + \ell_1\ell_2\left(\ddot{\beta}_1\cos(\beta_1-\beta_2) - \dot{\beta}_1\sin(\beta_1-\beta_2)(\dot{\beta}_1-\dot{\beta}_2)\right)\right] \\
&-m_2\left[\ell_1\ell_2\dot{\beta}_1\dot{\beta}_2\sin(\beta_1-\beta_2) - g\ell_2\sin(\beta_2)\right] = 0
\end{aligned}
$$

Cleaning up some, we get the final result

$$
\begin{aligned}
(m_1+m_2)\left[\ell_1\ddot{\beta}_1 + g\sin(\beta_1)\right] + m_2\ell_2\left[\ddot{\beta}_2\cos(\beta_1-\beta_2) + \dot{\beta}_2^2\sin(\beta_1-\beta_2)\right] &= 0 \\
\ell_2\ddot{\beta}_2 + \ell_1\ddot{\beta}_1\cos(\beta_1-\beta_2) - \ell_2\dot{\beta}_1^2\sin(\beta_1-\beta_2) + g\sin(\beta_2) &= 0.
\end{aligned}
$$

4 Lagrangian invariance

The original Lagrangian, $L(q,\dot{q},t)$, gives the equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0,$$

while the new Lagrangian, $L'(q,\dot{q},t)$ gives

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}}\left(L(q,\dot{q},t) + \frac{\mathrm{d}F(q,t)}{\mathrm{d}t}\right) - \frac{\partial}{\partial q}\left(L(q,\dot{q},t) + \frac{\mathrm{d}F(q,t)}{\mathrm{d}t}\right) = 0, \\
&\implies \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}}\frac{\mathrm{d}F(q,t)}{\mathrm{d}t} - \frac{\partial}{\partial q}\frac{\mathrm{d}F(q,t)}{\mathrm{d}t} = 0,
\end{aligned}
$$

This means the new Lagrangian will give us the same equation of motion, if

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}}\frac{\mathrm{d}F(q,t)}{\mathrm{d}t} - \frac{\partial}{\partial q}\frac{\mathrm{d}F(q,t)}{\mathrm{d}t} = 0. \tag{1}$$

We can use the chain rule to obtain

$$\frac{\mathrm{d}F(q,t)}{\mathrm{d}t} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial q}\dot{q}.$$

This means that

$$
\begin{aligned}
\frac{\partial}{\partial q}\frac{\mathrm{d}F}{\mathrm{d}t} &= \frac{\partial^2 F}{\partial q\partial t} + \frac{\partial^2 F}{\partial q^2}\dot{q}, \\
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}}\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F}{\partial q}\right) &= \frac{\partial^2 F}{\partial t\partial q} + \frac{\partial^2 F}{\partial q^2}\dot{q}.
\end{aligned}
$$

Inserting this into (1), we get the desired result:

$$\frac{\partial^2 F}{\partial t \partial q} + \frac{\partial^2 F}{\partial q^2}\dot{q} - \left(\frac{\partial^2 F}{\partial q \partial t} + \frac{\partial^2 F}{\partial q^2}\dot{q}\right) = 0.$$