Exercise 5 - TFY4345 Classical Mechanics

2020

1 Effective potential and scattering center

(FIGURE)
A particle with mass m moves in an attractive potential $V(r)$. Show, on the basis of energy conservation, how the problem can be looked upon as a one-dimensional problem with the effective potential $V'(r)$ What is the condition for the particle to reach the scattering centre, $r = 0$?

(Hint: Once setting the condition between E and V', consider the limit where $r \to 0$.)

2 Scattering from a spherical obstacle

(a) FInd the differential cross section $\sigma(\Theta)$ for scattering against a hard sphere of radius a. Notice when the impact para meter is $s = 0$ the scattering angle is $\Theta = \pi$, and when $s > a$ the scattering angle is $\Theta = 0$ (no scattering).

(Hint: Use simple geometrical considerations to find a relation between the impact parameter s and the scattering angle Θ.)

(b) Calculate the total cross section σ. Why is the final expression for σ reasonable?

3 Scattering by an attractive hard sphere

A hard sphere has radius a. For $r > a$, the sphere yields a Kepler potential $V(r) = -k/r$, where $k > 0$. Particles coming in from infinity have mass m and original velocity v_0. The part of the particles having impact parameter $s \geq s_{\max}$ will hit the sphere's surface. Find $s_{\max}$, and the corresponding "effective" scattering cross section $\sigma_{\text{eff}} = \pi s_{\max}^2$.

(Hint: Consider the conservation of energy and angular momentum.)

4 Average energies in the Kepler problem

(a) The solution to the Kepler problem (particle in a central fore potential $V = -k/r$) is given by the following expression (in polar coordinates):

$$r = \frac{p}{1 + \varepsilon \cos(\theta)}.$$

Here, p and ε are constants (whose definitions can be found in the compendium, chapter 4). Use the expression defining p and ε to show that the total energy is

$$E = -\frac{k}{2p}(1 - \varepsilon^2),$$

Use thte viral theorem and find the average kinetic energy $\langle T \rangle$, and the average potential energy $\langle V \rangle$.

(b) The average potential energy an be calculated as the average over one orbit period:

$$\langle V \rangle = \frac{1}{t_p} \int_0^{t_p} \mathrm{d}t V,$$

where t_p is the total orbital period (time for one complete orbit). Find $\langle V \rangle$ by direct calculation of this integral (not using the viral theorem).

Hint 1: change variable of integration from t to θ
Hint 2: the residue method (E. Kreyzig, 9th ed., chapter 16.4) gives

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{1 + \varepsilon \cos(\theta)} = \frac{2\pi}{\sqrt{1 - \varepsilon^2}}.$$

Hint 3:

$$\int_0^{2\pi} \frac{\mathrm{d}\theta \cos(\theta)}{(1 + \varepsilon \cos(\theta))^2} = -\frac{\mathrm{d}}{\mathrm{d}\varepsilon} \int_0^{2\pi} \frac{\mathrm{d}\theta}{1 + \varepsilon \cos(\theta)}$$

(c) The average kinetic energy is:

$$\langle T \rangle = \frac{1}{t_p} \int_0^{t_p} \mathrm{d}t \, T = \frac{1}{t_p} \int_0^{t_p} \mathrm{d}t \, \frac{1}{2} m \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \right)^2.$$

Find $\langle T \rangle$ by direct calculation of this integral.
Hint: Do the integral by parts, and use Newton's equation

$$m \frac{\mathrm{d}^2 \mathbf{r}}{\mathrm{d}t^2} = -\nabla V = -\frac{k}{r^3} \mathbf{r}.$$

Note also that the integration (by parts) is take over a full period where the particle has returned back to its starting position.