

# Exercise 5 solutions - TFY4345 Classical Mechanics

2020

## 1 Effective potential and scattering center

The total energy, as given by equation 4.14 in the compendium, is

$$E = \frac{1}{2}m \left( \dot{r}^2 + (r\dot{\theta})^2 \right) + V(r).$$

In a central potential, we have that  $mr^2\dot{\theta} = \ell$  is a conserved quantity, so we get

$$E = \frac{1}{2}m\dot{r}^2 + \left( \frac{\ell^2}{2mr^2} + V(r) \right) = \frac{1}{2}m\dot{r}^2 + V_{\text{eff}}(r).$$

This is an effective 1D problem, with an effective potential

$$V_{\text{eff}}(r) = V(r) + \frac{\ell^2}{2mr^2}$$

In order for the particle to reach the center, it need to have sufficiently high energy to overcome the potential barrier, i.e.  $E > V_{\text{eff}}(r \rightarrow 0)$ . This can be written as

$$Er^2 > r^2V(r) + \frac{\ell^2}{2m}, \quad r \rightarrow 0.$$

The l.h.s. goes to zero, so that the condition becomes

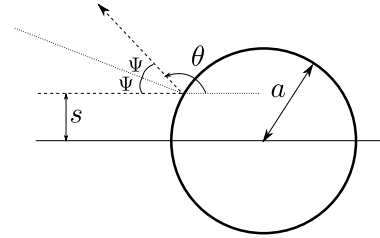
$$(r^2V(r))_{r \rightarrow 0} < -\frac{\ell^2}{2m}.$$

This can be fulfilled wither with  $-k/r^2$ , where  $k > \ell^2/2m$ , or if  $V(r) = -A/r^n$ , with  $n > 2$  and  $A$  a positive constant.

## 2 Scattering from a spherical obstacle

The scattering angle  $\theta$  satisfies  $2\Psi + \theta = \pi$ . From the figure, we see that the impact parameter is given by  $s = a \sin(\pi/2 - \theta/2) = a \cos(\theta/2)$ , so that

$$\left| \frac{ds}{d\theta} \right| = \frac{a}{2} \sin\left(\frac{\theta}{2}\right)$$



Using the formula for the differential cross section, as given in equation 4.40 in the compendium, we get

$$\sigma(\theta) = \frac{s}{\sin(\theta)} \left| \frac{ds}{d\theta} \right| = \frac{a^2 \cos(\theta/2) \sin(\theta/2)}{2 \sin(\theta)} = \frac{a^2}{4}.$$

(b) The total cross section is therefore

$$\sigma = 2\pi \int_0^\pi \sigma(\theta) \sin(\theta) d\theta = \pi a^2.$$

This is physically sensible, since it is the actual cross-sectional area of the sphere.

### 3 Scattering by an attractive hard sphere

The impact parameter  $s_{\max}$  will send the particle just gracing the surface at  $r = a$ . Due to conservation of energy, we have that

$$E = \frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 - \frac{k}{a}.$$

Furthermore, conservation of angular momentum means that  $\ell$  infinitely far away is the same as when the particle touches the surface, so

$$\ell = mv_0 s_{\max} = mva.$$

Combining these two equations, we get

$$s_{\max} = \frac{v}{v_0} a = a \sqrt{1 + \frac{2k}{ma v_0^2}}.$$

All particles with impact parameter  $s < s_{\max}$  will hit the surface, so that  $\sigma_{\text{eff}} = \pi s_{\max}^2$ .

### 4 Average energies in the Kepler problem

(a) From the compendium, part 4E, we have

$$p = \frac{\ell^2}{mk} \quad \varepsilon^2 = 1 + \frac{2E\ell^2}{mk^2}.$$

Eliminating  $\ell$  gives us

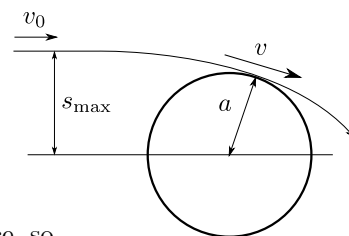
$$E = -\frac{k}{2p} (1 - \varepsilon^2).$$

The total energy is constant. This means that the average total energy also is constant:

$$\langle T \rangle + \langle V \rangle = \langle E \rangle = E.$$

The virial theorem for a gravitational potential, example 12 in part 4D of the compendium, gives

$$\langle T \rangle = -\frac{1}{2} \langle V \rangle.$$



Combining this gives

$$\begin{aligned}\langle T \rangle &= \frac{k}{2p} (1 - \varepsilon^2) \\ \langle V \rangle &= -\frac{k}{p} (1 - \varepsilon^2)\end{aligned}$$

(b) The solution to the Kepler problem in polar coordinates (found in the compendium) is

$$r = \frac{p}{1 + \varepsilon \cos(\theta)}.$$

The average potential energy over one period is

$$\langle V \rangle = \frac{1}{t_p} \int_0^{t_p} dt V = -\frac{1}{t_p} \int_0^{t_p} dt \frac{k}{r}.$$

Combining these equations give

$$\begin{aligned}\langle V \rangle &= -\frac{1}{t_p} \int_0^{t_p} dt \frac{k}{p} (1 + \varepsilon \cos(\theta)) = -\frac{k}{pt_p} \left( \int_0^{t_p} dt + \varepsilon \int_0^{t_p} dt \cos(\theta) \right) \\ &= \frac{k}{p} (1 + \varepsilon \langle \cos(\theta) \rangle).\end{aligned}$$

We can find the last integral by using  $\ell = mr^2\dot{\theta}$  and a change of variable

$$\begin{aligned}\langle \cos(\theta) \rangle &= \frac{1}{t_p} \int_0^{t_p} dt \cos(\theta) = \frac{1}{t_p} \int_0^{2\pi} d\theta \frac{1}{\dot{\theta}} \cos(\theta) = \frac{m}{\ell t_p} \int_0^{2\pi} d\theta r(\theta)^2 \cos(\theta) \\ &= \frac{mp^2}{\ell t_p} \int_0^{2\pi} d\theta \frac{\cos(\theta)}{(1 + \varepsilon \cos(\theta))^2}.\end{aligned}$$

Using hint 2 and 3 we get

$$\begin{aligned}\langle \cos(\theta) \rangle &= \frac{mp^2}{\ell t_p} \int_0^{2\pi} d\theta \frac{\cos(\theta)}{(1 + \varepsilon \cos(\theta))^2} = -\frac{mp^2}{\ell t_p} \frac{d}{d\varepsilon} \int_0^{2\pi} \frac{d\theta}{1 + \varepsilon \cos(\theta)} \\ &= -\frac{mp^2}{\ell t_p} \frac{d}{d\varepsilon} \frac{2\pi}{\sqrt{1 - \varepsilon^2}} = -\frac{2\pi m}{\ell t_p} \frac{p^2 \varepsilon}{(1 - \varepsilon^2)^{3/2}}.\end{aligned}$$

Then, using

$$t_p = \frac{2\pi m}{\ell^2} \frac{p^2}{(1 - \varepsilon^2)^{3/2}},$$

we get

$$\langle \cos(\theta) \rangle = -\varepsilon,$$

so

$$\langle V \rangle = \frac{k}{p} (1 - \varepsilon^2).$$

(c) Integrating the kinetic energy by parts, with  $\dot{\mathbf{r}}^2 = u'v$ , gives

$$\langle T \rangle = \frac{m}{2t_p} \int_0^{t_p} dt \left( \frac{d\mathbf{r}}{dt} \right)^2 = \frac{m}{2t_p} \left( \mathbf{r} \cdot \cancel{\frac{d\mathbf{r}}{dt}} \Big|_0^{t_p} - \int_0^{t_p} dt \mathbf{r} \cdot \frac{d^2\mathbf{r}}{dt^2} \right)$$

(as  $\mathbf{r}(0) = \mathbf{r}(t_p)$ , and  $\dot{\mathbf{r}}(0) = \dot{\mathbf{r}}(t_p)$ )

$$= -\frac{1}{2t_p} \int_0^{t_p} dt \mathbf{r} \cdot \left( -\frac{k}{r^3} \mathbf{r} \right) = \frac{1}{2t_p} \int_0^{t_p} dt \frac{k}{r} = -\frac{1}{2} \langle V \rangle = \frac{k}{2p} (1 - \varepsilon^2).$$

This agrees with the result from the viral theorem.