# Exercise 8 - TFY4345 Classical Mechanics

2020

1 Principal moments of inertia of a triangular slab

(Exam Aug. 2019)

(a) Compute the center-of-mass (CM) for the planar triangle in the figure, assuming it to be of uniform two-dimensional mass density ρ.

(b) Compute the inertia tensor *with respect to the origin,* for the same triangle.

(c) (Optional) If the origin is shifted to the CM, the inertia tensor becomes (this can be show by using the Steiner's parallel axis theorem)
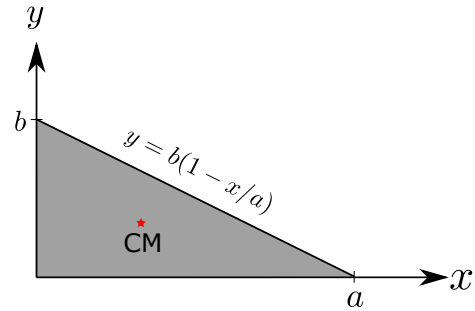
$$I_{CM} = \begin{pmatrix} I_{11} & I_{12} & 0 \\ I_{21} & I_{22} & 0 \\ 0 & 0 & I_{33} \end{pmatrix} = \frac{M}{18} \begin{pmatrix} a^2 & \frac{1}{2}ab & 0 \\ \frac{1}{2}ab & b^2 & 0 \\ 0 & 0 & a^2 + b^2 \end{pmatrix}$$

where $I_{xy} = I_{yx}$ and $I_{xx} + I_{yy} = I_{zz}$ in the general form show first. Define next

$$A = \frac{1}{2}(I_{xx} + I_{yy}), \quad B = \sqrt{\frac{1}{2}(I_{xx} - I_{yy})^2 + I_{xy}^2}, \quad \vartheta = \tan^{-1}\left(\frac{2I_{xy}}{I_{xx} - I_{yy}}\right).$$

Derive the principal moments of inertia and the principal axes by using the general form of the inertia tensor, and these new variables.

[Hint] The last equations comprises a relationship that can be described by a right triangle.

2 Precession of a frisbee

(Exam Aug. 2016)

Consider an axial-symmetric body with the principal moments of inertia $I_1 = I_2 \neq I_3$ for rotation around the axis of symmetry. The angular momentum in the laboratory frame is $\mathbf{L} = L\mathbf{e}_z$, where z is the symmetry axis.
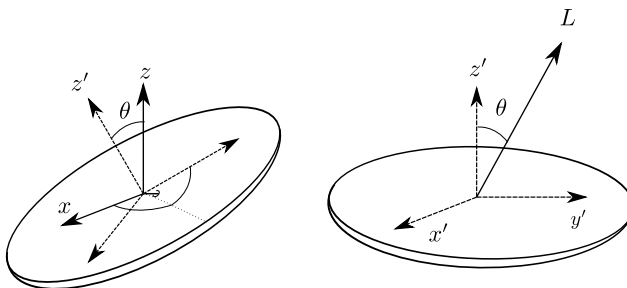
(a) Derive the equations of motion for the body, using the Euler equations and the angles θ, ψ and ϕ. Define also the components of $\boldsymbol{\omega}$. (See lecture notes, we derived this already!)

(b) Find the expression for the Euler angles θ, ψ, ϕ as a function of time.

(c) Assume $I_3 = 2I_1$. The precession (wobble) of the frisbee is given by $\dot{\phi}$. Show that the precession is twice as fast as the rotation frequency of the frisbee, assuming that θ is small (i.e. that $\cos(\theta) \approx 1$).

3 Precession of a heavy spinning top

(Based on example p. 208-223 in Goldstein 3rd. ed., p. 70-74 in the compendium)

In the example we define the shifted energy as

$$E' = \frac{1}{2}I_1\dot{\theta}^2 + V(\theta), \quad V(\theta) = \frac{(p_\phi - p_\psi\cos(\theta)^2)}{2I_1\sin(\theta)^2} + Mgh\cos(\theta),$$

which is a constant of motion. We also defined θ_0 to be the constant angle of inclination of spinning top with regular precession, e.g. that the symmetry axis rotates around the z at a fixed angle. Consider the shape of the effective potential $V(\theta)$ for θ_0. *What is the condition in this case?* The following change of variables will come in handy for the result:

$$\beta = p_\phi - p_\psi\cos(\theta_0).$$

You will encounter a quadratic equation for β. Show that for the equilibrium precession inclination angle θ_0, the following must hold true:

$$\omega_3 \leq \frac{2}{I_3}\sqrt{MghI_1\cos(\theta_0)}.$$

What can you say about the corresponding precession angular velocity $\dot{\phi}_0$?