Exercise 8 solutions - TFY4345 Classical Mechanics

2020

1 Principal moments of inertia of a triangular slab

(a) Since the mas has uniform density, we can write the mass area density as $M = \rho\, ab/2$. Let x_{CM} denote the x-component of the center of mass. Using the definition of CM, we find

$$x_{CM} = \frac{1}{M}\int_0^a \mathrm{d}x \int_0^{b(1-x/a)} \mathrm{d}y \rho x = \frac{\rho b}{M}\int_0^a \mathrm{d}x \left(1 - \frac{x}{a}\right) = \frac{a^2 b\rho}{M}\int_0^1 \mathrm{d}u(1-u)u = \frac{\rho a^2 b}{6M} = \frac{a}{3}.$$

We used the substitution $u = 1 - x/a$ which implies $\mathrm{d}x = -a\mathrm{d}u$. Because of the symmetry in the problem (the slab is a triangle), the calculation of y_{CM} is the same, only exchanging $a \leftrightarrow b$, so the result is $y_{CM} = \mathrm{b}/3$.

(b) The slab is two dimensional, and laying in the xy-plane. If we look at the definition of the off-diagonal entries in moment of inertia tensor,

$$I_{ij} = -\int_V \mathrm{d}V x_i x_j,$$

we can see that $I_{zx} = I_{xz} = I_{zy} = I_{yz} = 0$, as z = 0. This also implies that $I_{xx} + I_{yy} = I_{zz}$, so all we need to calculate is I_{xx}, I_{yy} and I_{xy}.

$$\begin{aligned}
I_{xy} &= -\rho\int_0^a \mathrm{d}x \int_0^{v(1-x/a)} \mathrm{d}y\, yx = -\frac{\rho b^2}{2}\int_0^a \mathrm{d}x\, x\left(1-\frac{x}{a}\right)^2 = -\frac{\rho b^2}{2}\int_0^a \mathrm{d}x \left(x - \frac{2}{a}x^2 + \frac{1}{a^2}x^3\right) \\
&= -\frac{\rho b^2 a^2}{2}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = \frac{Mab}{12} \\
I_{xy} &= -\rho\int_0^a \mathrm{d}x \int_0^{v(1-x/a)} \mathrm{d}y\, y^2 = \frac{\rho b^3}{3}\left(1-\frac{x}{a}\right)^3 = \frac{\rho a b^3}{3}\int_0^1 \mathrm{d}u\, u^3 = \frac{Mb^2}{6}.
\end{aligned}$$

Lastly, I_{yy} can a gain be found just by the exchange $a \leftrightarrow b$. In matrix form,

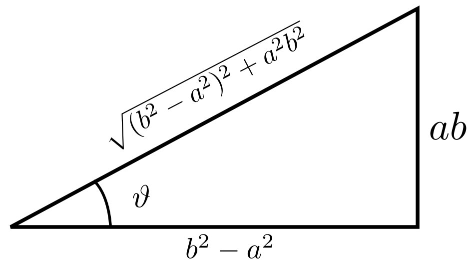
$$I = \frac{M}{6}\begin{pmatrix} b^2 & -\frac{1}{2}ab & 0 \\ -\frac{1}{2}ab & a^2 & 0 \\ 0 & 0 & a^2+b^2 \end{pmatrix}$$

(c) Inserting our values into the new variables (neglecting the common factor), we get

$$A = \frac{1}{2}(a^2+b^2), \quad B = \frac{1}{2}\sqrt{(b^2-a^2)+a^2b^2}, \quad \vartheta = \tan^{-1}\left(\frac{ab}{b^2-a^2}\right).$$

The last equation describes a triangle with side lengths $b^2 - a^2$, ab and $\sqrt{(b^2-a^2)+a^2b^2} = 2B$, and an angle ϑ opposite the side of length ab. This gives us the relations $ab = 2B\sin(\vartheta)$ and $b^2 - a^2 = 2B\cos(\vartheta)$. It follows that

$$a^2 = \frac{1}{2}(b^2+a^2) - \frac{1}{2}(b^2-a^2) = A - B\cos(\vartheta)$$
$$b^2 = \frac{1}{2}(b^2+a^2) + \frac{1}{2}(b^2-a^2) = A + B\cos(\vartheta)$$



Putting all this together, we get

$$I = \begin{pmatrix} A + B\cos(\vartheta) & B\sin(\vartheta) & 0 \\ B\sin(\vartheta) & A - B\cos(\vartheta) & 0 \\ 0 & 0 & 2A \end{pmatrix}$$

To find the principal moments of inertia, we must find solve the characteristic equation for the principal moments of inertia ω

$$\begin{aligned} \det(I-\omega) = 0 \implies & \begin{vmatrix} A + B\cos(\vartheta) - \omega & B\sin(\vartheta) & 0 \\ B\sin(\vartheta) & A - B\cos(\vartheta) - \omega & 0 \\ 0 & 0 & 2A-\omega \end{vmatrix} \\ &= (2A-\omega)\big[(A+B\cos(\vartheta)-\omega)(A-B\cos(\vartheta)-\omega) - B\sin(\vartheta)\big] \\ &= (2A-\omega)[A^2 - B^2 + \omega^2 - 2\omega A] \\ &= (2A-\omega)[(A-\omega)^2 - B^2] = 0, \end{aligned}$$

which has the solutions $\omega_1 = 2A$, $\omega_2 = A+B$ and $\omega_3 = A-B$. By inspection, the first eigenvector is $\mathbf{v} = (0,0,1)$. We can then only look at the relevant part of the matrix to find the others. Inserting $\omega = A+B$,

$$0 = (I - \omega\mathbb{1})\mathbf{v} = B\begin{pmatrix} \cos(\vartheta)-1 & \sin(\vartheta) \\ \sin(\vartheta) & -\cos(\vartheta)-1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Setting $v_2 = 1$, we get $v_1 = \sin(\vartheta)/(1-\cos(\vartheta))$. The normalized eigenvector then becomes

$$\begin{aligned} \mathbf{v} &= \frac{1}{\sqrt{1+\frac{\sin^2(\vartheta)}{(1-\cos(\vartheta))^2}}}\left(\frac{\sin(\vartheta)}{1-\cos(\vartheta)}, 1, 0\right) = \frac{1-\cos(\vartheta)}{\sqrt{1-2\cos(\theta)+\cos^2(\vartheta)+\sin^2(\vartheta)}}\left(\frac{\sin(\vartheta)}{1-\cos(\vartheta)}, 1, 0\right) \\ &= \frac{1-\cos(\vartheta)}{\sqrt{2}\sqrt{1-\cos(\vartheta)}}\left(\frac{\sin(\vartheta)}{1-\cos(\vartheta)}, 1, 0\right) = \frac{1}{2\sin(\vartheta/2)}\left(\sin(\vartheta), 2\sin^2(\vartheta/2), 0\right) \end{aligned}$$

Using $2\sin(\vartheta/2)\cos(\vartheta/2) = \sin(\vartheta)$, this gives

$$\mathbf{v} = (\cos(\vartheta/2), \sin(\vartheta/2), 0)$$

The last eigenvector is then found in a similar manner by setting $\omega = A - B$, yielding

$$\mathbf{v} = (-\sin(\vartheta/2), \cos(\vartheta/2), 0)$$

This is a general solution for any right triangle. Note that the first vector points out of the xy-plane, while the two others rotate as a function of a and b.

2 Precession of a frisbee

(a) The Euler equation for the motion of a spinning free body (no torque) is

$$\left(\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}\right)_{body} + \boldsymbol{\omega} \times \mathbf{L} = 0$$

Writing this out in component form gives

$$\begin{aligned} I_1\dot{\omega}_{x'} + \omega_{y'}\omega_{z'}(I_3 - I_2) &= 0, \\ I_2\dot{\omega}_{y'} + \omega_{z'}\omega_{x'}(I_1 - I_3) &= 0, \\ I_3\dot{\omega}_{z'} + \omega_{x'}\omega_{y'}(I_2 - I_1) &= 0. \end{aligned}$$

As shown in the compendium (5.G), the components of the angular velocity in the body frame is

$$\begin{aligned} \omega_{x'} &= \dot{\phi}\sin(\theta)\sin(\psi) + \dot{\theta}\cos(\psi) \\ \omega_{y'} &= \dot{\phi}\sin(\theta)\cos(\psi) - \dot{\theta}\sin(\psi) \\ \omega_{z'} &= \dot{\phi}\cos(\theta) + \dot{\psi}. \end{aligned}$$

(b) From the component form of the equations of motion, we see that

$$I_1 = I_2 \implies I_3\dot{\omega}_{z'} = 0 \implies \omega_{z'} = \text{const.}$$

From the figure in the exercise, we can see that $L_{z'} = L\cos(\theta)$. The body axes are the principal axes of the frisbee, so $L_{z'} = I_3\omega_{z'} = \text{const.} \implies \theta = \text{const.}$, i.e. $\dot{\theta} = 0$. Using the Euler equation, we then get

$$\dot{\omega}_{x'} = -\Omega\omega_{y'}, \quad \dot{\omega}_{y'} = \Omega\omega_y, \quad \Omega = \frac{I_3 - I_1}{I_1}\omega_{z'} = \left(\frac{1}{I_1} - \frac{1}{I_3}\right)L\cos(\theta).$$

This is the equation of two sinusoidal functions, 90° out of phase. (An example of a solution is $\omega_{x'} = \cos(\Omega t)$, $\omega_{y'} = \sin(\Omega t)$). This implies[1]

$$\omega_{x'}^2 + \omega_{y'}^2 = \left(\dot{\phi}\sin(\theta)\sin(\psi)\right)^2 + \left(\dot{\phi}\sin(\theta)\cos(\psi)\right)^2 = \dot{\phi}^2\sin(\theta) = \text{const.},$$

i.e. that $\phi = \text{const.}$. Using this to differentiate the expressions of the angular momentum in Euler angles, we get

$$\begin{aligned} \dot{\phi}\dot{\psi}\sin(\theta)\cos(\phi) &= -\Omega\dot{\phi}\sin(\theta)\cos(\psi) \\ -\dot{\phi}\dot{\psi}\sin(\theta)\sin(\phi) &= \Omega\dot{\phi}\sin(\theta)\sin(\psi) \\ \implies \dot{\psi} = -\Omega &= \left(\frac{1}{I_3} - \frac{1}{I_1}\right)L\cos(\theta), \end{aligned}$$

while the last equation of motion gives

$$\dot{\phi} = \frac{L}{I_3} + \left(\frac{1}{I_1} - \frac{1}{I_3}\right)L = \frac{L}{I_1}$$

[1] A more general proof for this is that $\frac{\mathrm{d}}{\mathrm{d}t}\left(\omega_{x'}^2 + \omega_{y'}^2\right) = 2\omega_{x'}\dot{\omega}_{x'} + 2\omega_{y'}\dot{\omega}_{y'} = -2\Omega\dot{\omega}_{y'}\dot{\omega}_{x'} + 2\Omega\dot{\omega}_{y'}\dot{\omega}_{x'} = 0$

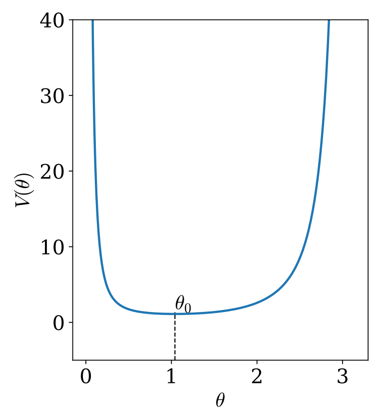
(c)

$$\frac{\dot{\phi}}{\omega_{z'}} = \frac{I_3}{I_1 \cos(\theta)} \approx 2.$$

3 Precession of a heavy spinning top

The effective potential of the spinning top is

$$V(\theta) = \frac{(p_\phi - p_\psi \cos(\theta))^2}{2I_1 \sin^2(\theta)} + Mgh\cos(\theta).$$



The shape is shown in the plot, but it will depend on the parameters. This case is similar to that of orbiting planets. In that case, the centrifugal force creates an effective potential as a function of r, with a minimum. If the planet has an energy corresponding to that minimum, it is in a circular orbit with constant radius. In this case, the effective potential is a function of the angle. Thus, the stable configuration with constant $\theta = \theta_0$ corresponds to the minimum of the potential. This is found by differentiating $V(\theta)$ and setting it equal to zero:

$$\left.\frac{\partial V}{\partial \theta}\right|_{\theta=\theta_0} = \frac{-\cos(\theta_0)(p_\phi - p_\psi \cos(\theta_0))^2 + p_\psi \sin^2(\theta_0)(p_\phi - p_\psi \cos(\theta_0)))}{I_1 \sin^3(\theta_0)} - Mgh\sin(\theta_0) = 0.$$

Defining $\beta = p_\phi - p_\psi \cos(\theta_0)$, we can simplify this to

$$0 = \cos(\theta_0)\beta^2 - p_\psi \sin^2(\theta_0)\beta + MghI_1 \sin^4(\theta_0) = 0,$$

so we can solve for β:

$$\beta_\pm = \frac{p_\psi \sin^2(\theta_0)}{2\cos(\theta_0)} \left(1 \pm \sqrt{1 - \frac{4MgHI_1 \cos(\theta_0)}{p_\psi^2}}\right).$$

We can see that β must be a real quantity from its definition. Thus, if $\theta_0 < \pi/2$, so $\cos(\theta_0) > 0$, we get the restriction

$$p_\psi^2 \geq 4MghI_1 \cos(\theta_0)$$

on physical configurations of the system. Inserting $p_\psi = I_3\omega_3$, we get

$$\omega_3 \geq \frac{2}{I_3}\sqrt{MghI_1 \cos(\theta_0)}.$$

This is a lower bound for the angular momentum needed by the spinning to be able to precess at an constant angle θ.

The rate of precession is then given by

$$\dot{\phi}_{0(\pm)} = \frac{\beta_\pm}{I_1 \sin^2(\theta_0)}.$$

This means we have to different configurations with stable precession, given by the two roots $\beta_\pm$. $\dot{\phi}_{0(+)}$ gives fast precession, while $\dot{\phi}_{0(+)}$ gives slow precession. if $\omega_3 \gg \frac{2}{I_3}\sqrt{MghI_1\cos(\theta_0)}$, then $p_\psi^2 \gg 4MghI_1\cos(\theta_0)$. We can us $\sqrt{1+x} = 1 - \frac{1}{2}x + \mathcal{O}(x^2)$ then expand the root in equation for β as

$$\begin{aligned}
&\sqrt{1 - \frac{4MghI_1\cos(\theta_0)}{p_\psi^2}} \approx 1 + \frac{2MghI_1\cos(\theta_0)}{p_\psi^2}, \quad p_\psi = I_3\omega_3 \\
\implies &\beta_\pm \approx \frac{I_3\omega_3\sin^2(\theta_0)}{2\cos(\theta_0)}\left(\frac{(I_3\omega_3)^2(1\pm 1) + 2MghI_1\cos(\theta_0)}{(I_3\omega_3)^2}\right) \\
\implies &\dot{\phi}_{0(\pm)} \approx \frac{(I_3\omega_3)^2(1\pm 1) + 2MghI_1\cos(\theta_0)}{2I_1I_3\omega_3\cos(\theta_0)} = \frac{I_3\omega_3}{2I_1\cos(\theta_0)}(1\pm 1) + \frac{Mgh}{I_3\omega_3}.
\end{aligned}$$

The two stable configurations thus precess with the angular velocities

$$\dot{\phi}_{0(+)} = \frac{I_3\omega_3}{I_1\cos(\theta_0)}, \quad \dot{\phi}_{0(-)} = \frac{Mgh}{I_3\omega_3}.$$