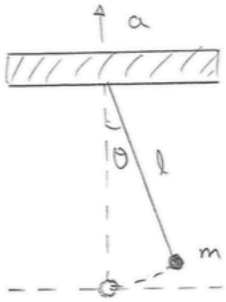
Classical Mechanics TFY4345 – Solution set 4

1. Mathematical pendulum in accelerating motion



$$T = \frac{1}{2} m \bar{v}^2 = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 \right)$$

$$x = l \sin\theta, \quad y = \frac{1}{2} a t^2 - l \cos\theta$$

$$\dot{x} = l \dot{\theta} \cos\theta, \quad \dot{y} = a t + l \dot{\theta} \sin\theta$$

$$\Rightarrow T = \frac{1}{2} m \left[l^2 \dot{\theta}^2 + a^2 t^2 + 2 a t l \dot{\theta} \sin\theta \right]$$

$$U = \left(-l \cos\theta + \frac{1}{2} a t^2 \right) m g$$

$$L = T - U = \frac{1}{2} m \left[l^2 \dot{\theta}^2 + a^2 t^2 + 2 a t l \dot{\theta} \sin\theta \right] + m g l \cos\theta - \frac{1}{2} m g a t^2$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m l^2 \dot{\theta} + m a l t \sin\theta = m \left(l^2 \dot{\theta} + a t l \sin\theta \right)$$

$$\Rightarrow \dot{\theta} = \frac{1}{m l^2} \left(p_\theta - m a t l \sin\theta \right) \qquad (*)$$

$$H = \dot{\theta} p_\theta - L = \frac{1}{2 m l^2} \left[p_\theta - m a t l \sin\theta \right]^2 - \frac{1}{2} m a^2 t^2 - m g l \cos\theta + \frac{1}{2} m g a t^2$$

$$\begin{cases} \dot{\theta} = \dfrac{\partial H}{\partial p_\theta} = \dfrac{1}{m l^2} \left(p_\theta - m a t l \sin\theta \right) \quad \leftarrow (*) \\[2ex] \dot{p}_\theta = -\dfrac{\partial H}{\partial \theta} = \dfrac{1}{l} \left(p_\theta - m a t l \sin\theta \right) \cdot a t \cos\theta - m g l \sin\theta \end{cases}$$

$E = T + U \neq H$ (Hamilton function is not the total energy)

$\dfrac{dH}{dt} = -\dfrac{\partial L}{\partial t} \Rightarrow \dfrac{dH}{dt} \neq 0$ (L has explicit t-dependence)

$\therefore$ The pendulum is in an accelerating motion with respect to the inertial frame of reference $\Rightarrow$ H is not conserved

2. Spherically symmetric potential

$4/3 \qquad U(r) = -k/r \qquad H = T + U \qquad T = \frac{1}{2} m v^2$

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases} \qquad v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 =$$

$$= (\dot{r}\sin\theta\cos\varphi + r\dot{\theta}\cos\theta\cos\varphi - r\dot{\varphi}\sin\theta\sin\varphi)^2$$
$$+ (\dot{r}\sin\theta\sin\varphi + r\dot{\theta}\cos\theta\sin\varphi + r\dot{\varphi}\sin\theta\cos\varphi)^2$$
$$+ (\dot{r}\cos\theta - r\dot{\theta}\sin\theta)^2$$

$$= \dot{r}^2\sin^2\theta\cos^2\varphi + r^2\dot{\theta}^2\cos^2\theta\cos^2\varphi + r^2\dot{\varphi}^2\sin^2\theta\sin^2\varphi$$
$$+ 2\dot{r}r\dot{\theta}\sin\theta\cos\theta\cos^2\varphi - \cancel{2\dot{r}r\dot{\varphi}\sin^2\theta\cos\varphi\sin\varphi} - \cancel{2r^2\dot{\theta}\dot{\varphi}\sin\theta\cos\theta\sin\varphi\cos\varphi}$$
$$+ \dot{r}^2\sin^2\theta\sin^2\varphi + r^2\dot{\theta}^2\cos^2\theta\sin^2\varphi + r^2\dot{\varphi}^2\sin^2\theta\cos^2\varphi$$
$$+ 2\dot{r}r\dot{\theta}\sin\theta\cos\theta\sin^2\varphi + \cancel{2\dot{r}r\dot{\varphi}\sin^2\theta\sin\varphi\cos\varphi} + \cancel{2r^2\dot{\theta}\dot{\varphi}\sin\theta\cos\theta\sin\varphi\cos\varphi}$$
$$+ \dot{r}^2\cos^2\theta + r^2\dot{\theta}^2\sin^2\theta - 2\dot{r}r\dot{\theta}\sin\theta\cos\theta$$

$$= \dot{r}^2\sin^2\theta + \dot{r}^2\cos^2\theta + r^2\dot{\theta}^2\cos^2\theta + r^2\dot{\theta}^2\sin^2\theta + r^2\dot{\varphi}^2\sin^2\theta$$
$$+ 2r\dot{r}\dot{\theta}\left[\sin\theta\cos\theta\cos^2\varphi + \sin\theta\sin\varphi\sin^2\varphi - \sin\theta\cos\theta\right]$$
$$= \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2 \qquad (\text{Remember: } \sin^2\theta + \cos^2\theta = 1)$$

$$\Rightarrow \underline{H = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2) - \frac{k}{r}}$$

Variables: $\dot{r}, \dot{\theta}, \dot{\varphi}, \dot{p}_r, \dot{p}_\theta, \dot{p}_\varphi \Rightarrow H(r, \theta, \varphi, p_r, p_\theta, p_\varphi) = ?$

$$L = T - U \Rightarrow p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$
$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = mr^2\sin^2\theta\,\dot{\varphi}$$

$$\Rightarrow H = \frac{1}{2}m\left[\frac{p_r^2}{m^2} + r^2\frac{p_\theta^2}{m^2 r^4} + r^2\sin^2\theta\frac{p_\varphi^2}{m^2 r^4\sin^4\theta}\right] - \frac{k}{r}$$

$$\underline{H = \frac{1}{2m}\left[p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2\sin^2\theta}\right] - \frac{k}{r}}$$

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \Rightarrow \dot{r} = \frac{p_r}{m}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \Rightarrow \dot{\theta} = \frac{p_\theta}{mr^2}$$

$$\dot{\varphi} = \frac{\partial H}{\partial p_\varphi} = \frac{p_\varphi}{mr^2\sin^2\theta} \Rightarrow \dot{\varphi} = \frac{p_\varphi}{mr^2\sin^2\theta}$$

$$\dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} + \frac{p_\varphi^2}{mr^3\sin^2\theta} + \frac{k}{r^2} = \dot{p}_r$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{p_\varphi^2\cos\theta}{mr^2\sin^3\theta} = \dot{p}_\theta$$

$$\dot{\varphi} = -\frac{\partial H}{\partial \varphi} = 0 \Rightarrow \dot{p}_\varphi = 0$$

3. Earth's orbit

4/4 Earth's orbit, gravitation $F(r) = -\frac{k}{r^2}$

Circular orbit $\Rightarrow E = V_{min}$

$$\varepsilon = 0 = \sqrt{1 + \frac{2El^2}{\mu k^2}} \Rightarrow 1 + \frac{2El^2}{\mu k^2} = 0 \Rightarrow E = -\frac{\mu k^2}{2l^2}$$

reduced mass

(one can use m here as well)

On the other hand: $E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{l^2}{\mu r^2} - \frac{k}{r}$

Mass of the sun $M_\odot \to \frac{1}{2}M_\odot \Rightarrow k \to \frac{k}{2}$ $(k = GM_\odot m)$

$$E \to E' = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{l^2}{\mu r^2} - \frac{k}{2r} = V_{min} + \frac{k}{2r}$$

$$= -\frac{\mu k^2}{2l^2} + \frac{k}{2r}$$

Note: Circular orbit (beginning) $\Rightarrow$ r constant

$r \equiv r_{min}$, $V(r) = -\frac{k}{r} + \frac{l^2}{2\mu r^2}$

$$V'(r) = \frac{k}{r^2} - \frac{l^2}{\mu r^3} = 0 \Rightarrow r = \frac{l^2}{\mu k}$$

$$\Rightarrow E' = -\frac{\mu k^2}{2l^2} + \frac{k}{2}\frac{\mu k}{l^2} = 0$$

New eccentricity: $\varepsilon = \sqrt{1+0} = 1$

$\therefore$ The new orbit is a parable (unbound)

4. Einstein's correction (from previous years, på norsk!)

a) Det sentrale kraftfeltet er gitt ved:

$$f(r) = -\frac{k}{r^2} + \frac{\beta}{r^3} \qquad \Rightarrow \qquad V(r) = -\frac{k}{r} + \frac{\beta}{2r^2}$$

Fra teorien er: $$\theta = \int \frac{\frac{1}{r^2}\,dr}{\sqrt{\frac{2mE}{l^2} - \frac{2mV}{l^2} - \frac{1}{r^2}}} + konst.$$

Innsetting av V og innføring av $u = \frac{1}{r}$ gir når konstanten utelates

$$\theta = -\int \frac{du}{\sqrt{\frac{2mE}{l^2} - \frac{2mku}{l^2} - \gamma^2 u^2}}, \text{ hvor } \gamma^2 = 1 + \frac{\beta m}{l^2}$$

Benytter: $$\int \frac{dx}{\sqrt{a + bx + cx^2}} = \frac{1}{\sqrt{-c}} \arccos\left(-\frac{b + 2cx}{\sqrt{q}}\right), \text{ hvor } q = b^2 - 4ac$$

Her velges $a = \frac{2mE}{l^2}$, $b = \frac{2mk}{l^2}$, $c = -\gamma^2 \Rightarrow q = \left(\frac{2mk}{l^2}\right)^2 \left(1 + \frac{2E\gamma^2 l^2}{mk^2}\right)$,

$$-\frac{b + 2cu}{\sqrt{q}} = \frac{\frac{\gamma^2 l^2 u}{mk} - 1}{\sqrt{1 + \frac{2E\gamma^2 l^2}{mk^2}}}.$$ Definerer $\varepsilon = \sqrt{1 + \frac{2E\gamma^2 l^2}{mk^2}}$, $p = \frac{\gamma^2 l^2}{mk}$

Da blir $\theta = -\frac{1}{\gamma} \arccos \frac{\frac{p}{r} - 1}{\varepsilon}$,

Baneligningen er

$$(1) \quad \frac{p}{r} = 1 + \varepsilon \cos(\gamma\theta), \qquad \text{hvor} \quad \gamma = \sqrt{1 + \frac{m\beta}{\cancel{2}l}} \approx 1 + \frac{m\beta}{2l^2}$$

Antar $E < 0$. Da er ligning (1), ligningen for en ellipse med langsom presesjon.

Store halvakse: $a = \frac{p}{1 - \varepsilon^2}$ (slik som når $\gamma = 1$) $\Rightarrow$

$$a = \frac{\frac{\gamma^2 l^2}{mk}}{1 - \left(1 + \frac{2E\gamma^2 l^2}{mk^2}\right)} = \frac{k}{2|E|}, \qquad \text{som for } \gamma = 1.$$

Vanlig litenhetsparameter er $\eta = \frac{\beta}{ka}$, dvs. $\gamma = 1 + \frac{m\eta ka}{2l^2}$

Verdien $\eta = 1.42 \cdot 10^{-7}$ tilsvarer Merkurs perihelbevegelse, som er $43''$ per hundre år.