Exercise 8 solutions - TFY4345 Classical Mechanics

2020

1 Principal moments of inertia of a triangular slab

(a) Since the mas has uniform density, we can write the mass area density as $M = 1/2ab\rho$. Let x_{CM} denote the x-component of the center of mass. Using the definition of CM, we find (EXPLAIN UPPER LIMIT?)

$$x_{CM} = \frac{1}{M}\int_0^a \mathrm{d}x \int_0^{b(1-x/a)} \mathrm{d}y\rho x = \frac{\rho b}{M}\int_0^a \mathrm{d}x\left(1-\frac{x}{a}\right) = \frac{a^2 b\rho}{M}\int_0^1 \mathrm{d}u(1-u)u = \frac{\rho a^2 b}{6M} = \frac{a}{3}.$$

We used the substitution $u = 1 - x/a$ which implies a $\mathrm{d}x = -a\mathrm{d}u$. Because of the symmetry in the problem (the slab is a triangle), the calculation of y_{CM} is the same, only exchanging $a \leftrightarrow b$, so the result is y_{CM}.

(b) The slab is two dimensional, and laying in the xy-plane. If we look at the definition of the off-diagonal entries in moment of inertia tensor,

$$I_{ij} = -\int_V \mathrm{d}V x_i x_j,$$

$I_{zx} = I_{xz} = I_{zy} = I_{yz} = 0$, as z = 0. This also implies that $I_{xx} + I_{yy} = I_{zz}$, so all we need to calculate is I_{xx}, I_{yy} and I_{xy}.

$$\begin{aligned}
I_{xy} &= -\rho\int_0^a \mathrm{d}x\int_0^{v(1-x/a)} \mathrm{d}y yx = -\frac{\rho b^2}{2}\int_0^a \mathrm{d}x x\left(1-\frac{x}{a}\right)^2 = -\frac{\rho b^2}{2}\int_0^a \mathrm{d}x\left(x-\frac{2}{a}x^2+\frac{1}{a^2}x^3\right)\\
&= -\frac{\rho b^2 a^2}{2}\left(\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right) = \frac{Mab}{12}\\
I_{xy} &= -\rho\int_0^a \mathrm{d}x\int_0^{v(1-x/a)} \mathrm{d}y y^2 = \frac{\rho b^3}{3}\left(1-\frac{x}{a}\right)^3 = \frac{\rho a b^3}{3}\int_0^1 \mathrm{d}u u^3 = \frac{Mb^2}{6}.
\end{aligned}$$

Lastly, I_{yy} can a gain be found just by the exchange $a \leftrightarrow b$. In matrix form,

$$I = \frac{M}{6}\begin{pmatrix} b^2 & -\frac{1}{2}ab & 0\\ -\frac{1}{2}ab & a^2 & 0\\ 0 & 0 & a^2+b^2\end{pmatrix}$$

(c) (Exam Aug. 2019)

We can remove the common factor $M/6$, so insert our values into the new variables, we get

$$A = \frac{1}{2}(a^2 + b^2), \quad B = \frac{1}{2}\sqrt{(b^2 - a^2) + a^2 b^2}, \quad \vartheta = \tan^{-1}\left(\frac{ab}{b^2 - a^2}\right).$$

(FIGUR)

The last equation describes a triangle with side lengths $b^2 - a^2$, ab and $\sqrt{(b^2 - a^2) + a^2 b^2} = 2B$, and an angle ϑ opposite the side of length ab. This gives us the relations $ab = 2B\cos(\vartheta)$ and $b^2 - a^2 = 2B\cos(\vartheta)$. (HER ER DET NOE FEIL) It follows that

$$\begin{aligned} a^2 &= \frac{1}{2}(b^2 + a^2) - \frac{1}{2}(b^2 - a^2) = A - B\cos(\vartheta) \\ b^2 &= \frac{1}{2}(b^2 + a^2) + \frac{1}{2}(b^2 - a^2) = A + B\cos(\vartheta) \end{aligned}$$

Putting all this together, we get

$$I = \frac{M}{18}\begin{pmatrix} A + B\cos(\vartheta) & B\sin(\vartheta) & 0 \\ B\sin(\vartheta) & A - B\cos(\vartheta) & 0 \\ 0 & 0 & 2A \end{pmatrix}$$

To find the principal moments of inertia, we must find solve the characteristic equation for the principal moments of inertia ω

$$\begin{aligned} \det(I - \omega) = 0 \implies & \begin{vmatrix} A + B\cos(\vartheta) - \omega & B\sin(\vartheta) & 0 \\ B\sin(\vartheta) & A - B\cos(\vartheta) - \omega & 0 \\ 0 & 0 & 2A - \omega \end{vmatrix} \\ &= (2A - \omega)\big[(A + B\cos(\vartheta) - \omega)(A - B\cos(\vartheta) - \omega) - B\sin(\vartheta)\big] \\ &= (2A - \omega)[A^2 - B^2 + \omega^2 - 2\omega A] \\ &= (2A - \omega)[(A - \omega)^2 - B^2] = 0, \end{aligned}$$

which has the solutions $\omega_1 = 2A$, $\omega_2 = A + B$ and $\omega_3 = A - B$. By inspection, the first eigenvector is $\mathbf{v} = (0, 0, 1)$. We can then only look at the relevant part of the matrix to find the others

$$\begin{aligned} \omega = A + B \implies & \begin{pmatrix} 2A + B(1 + \cos(\vartheta)) & B\sin(\vartheta) \\ B\sin(\vartheta) & 2A + B(1 - \cos(\vartheta)) \end{pmatrix}\mathbf{v} \\ &= \begin{pmatrix} 2[A + B\cos^2(\vartheta/2)] & B\sin(\vartheta) \\ B\sin(\vartheta) & 2[A + B\sin^2(\vartheta/2)] \end{pmatrix}\mathbf{v} = 0 \\ &\implies 0 = \begin{cases} 2[A + B\cos^2(\vartheta/2)]v_1 + B\sin(\vartheta)v_2 \\ B\sin(\vartheta)v_1 + 2[A + B\sin^2(\vartheta/2)]v_2 \end{cases} \end{aligned}$$

(TBD)

$$\mathbf{v}_1 = (\cos(\vartheta/2), \sin(\vartheta/2), 0), \quad \mathbf{v}_1 = (-\sin(\vartheta/2), \cos(\vartheta/2), 0)$$

2 Precession of a frisbee

(a) The Euler equation for the motion of a spinning free body (no torque) is

$$\left(\frac{\mathrm{d}}{\mathrm{d}\mathbf{L}}[t]\right)_b + \boldsymbol{\omega} \times \mathbf{L} = 0$$

Writing this out in component form gives

$$\begin{aligned}
I_1\dot{\omega}_{x'} + \omega_{y'}\omega_{z'}(I_3 - I_2) = 0,\\
I_2\dot{\omega}_{y'} + \omega_{z'}\omega_{x'}(I_1 - I_3) = 0,\\
I_3\dot{\omega}_{z'} + \omega_{x'}\omega_{y'}(I_2 - I_1) = 0.
\end{aligned}$$

As shown in the compendium (5.G), the components of the angular velocity in the body frame is

$$\begin{aligned}
\omega_{x'} &= \dot{\phi}\sin(\theta)\sin(\psi) + \dot{\theta}\cos(\psi)\\
\omega_{y'} &= \dot{\phi}\sin(\theta)\cos(\psi) - \dot{\theta}\sin(\psi)\\
\omega_{z'} &= \dot{\phi}\cos(\theta) + \dot{\psi}.
\end{aligned}$$

(b) From the component form of the equations of motion, we see that

$$I_1 = I_2 \implies I_3\dot{\omega}_{z'} = 0 \implies \omega_{z'} = \text{const.}$$

From the figure in the exercise, we can see that $L_{z'} = L\cos(\theta)$. The body axes are the principal axes of the frisbee, so $L_{z'} = I_3\omega_{z'} = \text{const.} \implies \theta = \text{const.}$, i.e. $\dot{\theta} = 0$. Using the Euler equation, we then get

$$\dot{\omega}_{x'} = -\Omega\omega_{y'}, \quad \dot{\omega}_{y'} = \Omega\omega_y, \quad \Omega = \frac{I_3 - I_1}{I_1}\omega_{z'}.$$

This is the equation of two sinusoidal functions, 90° out of phase. (An example of a solution is $\omega_{x'} = \cos(\Omega), \omega_{y'} = \sin(\Omega t)$). This implies[1]

$$\omega_{x'}^2 + \omega_{y'}^2 = \left(\dot{\phi}\sin(\theta)\sin(\psi)\right)^2 + \left(\dot{\phi}\sin(\theta)\cos(\psi)\right)^2 = \dot{\phi}^2\sin(\theta) = \text{const.},$$

i.e. that $\phi = \text{const.}$.

[1] A more general proof for this is that $\frac{\mathrm{d}}{\mathrm{d}t}\left(\omega_{x'}^2 + \omega_{y'}^2\right) = 2\omega_{x'}\dot{\omega}_{x'} + 2\omega_{y'}\dot{\omega}_{y'} = -2\Omega\dot{\omega}_{y'}\dot{\omega}_{x'} + 2\Omega\dot{\omega}_{y'}\dot{\omega}_{x'} = 0$

3 Precession of a heavy spinning top