Classical Mechanics TFY 4345 – Solution set 8

1. Principal moments of inertia of a triangular slab.

(a)
Since the slab is uniform its mass M is $M = A \times \rho = \frac{1}{2}ab\rho$. Let x_{CM} denote the x-component of the center of mass (CM). Using the definition of CM we find,

$$x_{cm} = \frac{1}{M}\int_0^a dx \int_0^{b(1-\frac{x}{a})} dy \rho x = \frac{\rho b}{M}\int_0^a dx(1-\frac{x}{a})$$
$$= \frac{a^2 b\rho}{M}\int_0^1 du(1-u)u = \frac{\rho a^2 b}{6M} = \frac{a}{3}, \tag{26}$$

where we used the substitution $u = 1 - \frac{x}{a}$ which implies $dx = -adu$. Because of the geometry in the problem (the slab has a triangular shape) the calculation of y_{CM} is completely analogous, and the result is $y_{CM} = \frac{b}{3}$.

(b)
The slab is two dimensional and is laying in the xy-plane ($z = 0$). This implies that $I_{zx} = I_{xz} = I_{zy} = I_{yz} = 0$ and $I_{zz} = I_{xx} + I_{yy}$. All we need to calculate is then reduced to I_{xx},I_{yy} and $I_{xy} = I_{yx}$:

$$I_{xx} = \rho\int_0^a dx \int_0^{b(1-\frac{x}{a})} y^2 dy = \frac{\rho b^3}{3}\int_0^a dx(1-\frac{x}{3})^3 = \frac{\rho a b^3}{3}\int_0^1 u^3 du = \frac{M}{6}b^2. \tag{27}$$

The computation of I_{yy} is completely analogous and the result is $I_{yy} = \frac{M}{6}a^2$. Finally we compute I_{xy},

$$I_{xy} = -\rho\int_0^a dx \int_0^{b(1-\frac{x}{a})} yx dy = -\frac{\rho b^2}{2}\int_0^a x(1-\frac{x}{a})^2 = -\frac{\rho a^2 b^2}{24} = -\frac{M}{12}ab. \tag{28}$$

Putting it all together we can write the inertia tensor on matrix form,

$$I = \frac{M}{6}\begin{pmatrix} b^2 & -\frac{1}{2}ab & 0 \\ -\frac{1}{2}ab & a^2 & 0 \\ 0 & 0 & a^2+b^2 \end{pmatrix} \tag{29}$$

(c)
By comparing the two matrices in the problem text we see that we can write the new variables as,

$$A = \frac{1}{2}(a^2 + b^2), B = \frac{1}{2}\sqrt{(b^2 - a^2)^2 + a^2b^2}, \vartheta = \tan^{-1}\left(\frac{ab}{b^2 - a^2}\right). \tag{30}$$

The last equation describes a right triangle with side lengths $b^2 - a^2$, ab and $\sqrt{b^2 - a^2)^2 + a^2b^2} = 2B$, where the angle ϑ is opposite to the side whose length is ab. From this observation we deduce that $ab = 2B\cos\vartheta$ and $b^2 - a^2 = 2B\cos\vartheta$. It follows that

$$a^2 = \frac{1}{2}(b^2 + a^2) - \frac{1}{2}(b^2 - a^2) = A - B\cos\vartheta, \tag{31}$$

$$b^2 = \frac{1}{2}(b^2 + a^2) + \frac{1}{2}(b^2 - a^2) = A + B\cos\vartheta, \tag{32}$$

and putting it all together we get,

$$I = \frac{M}{18}\begin{pmatrix} A + B\cos\vartheta & B\sin\vartheta & 0 \\ B\sin\vartheta & A - B\cos\vartheta & 0 \\ 0 & 0 & 2A \end{pmatrix} \tag{33}$$

2. Precession of a frisbee.

The Euler equation free body (no torque):

$$\left(\frac{d\vec{L}}{dt}\right)_{body} + \vec{\omega} \times \vec{L} = 0 \tag{29}$$

From Eq. (29) we find the Euler equation on component form:

$$\begin{aligned} I_1\dot{\omega}_{x'} + \omega_{y'}\omega_{y'}(I_3 - I_2) &= 0 & (30)\\ I_2\dot{\omega}_{y'} + \omega_{x'}\omega_{z'}(I_1 - I_3) &= 0 & (31)\\ I_2\dot{\omega}_{z'} + \omega_{x'}\omega_{y'}(I_2 - I_1) &= 0 & (32) \end{aligned}$$

Angular velocities in body frame:

$$\begin{aligned} \omega_{x'} &= \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi & (33)\\ \omega_{y'} &= \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi & (34)\\ \omega_{z'} &= \dot{\varphi}\cos\theta + \dot{\psi} & (35) \end{aligned}$$

3b) When $I_1 = I_2$ Eq. (32) implies that $\omega_{z'} = \text{constant}$. This again implies that the angular momentum around z' axis body frame is constant, i.e. $L_z' = L\cos\theta = \text{constant}$, which implies

$$\theta = \text{constant} \tag{36}$$

, i.e. $\dot{\theta} = 0$. Eq. (30) and (31) can now be expressed as:

$$\begin{aligned} \dot{\omega}_{x'} &= -\Omega\omega_{y'} & (37)\\ \dot{\omega}_{y'} &= \Omega\omega_{x'} & (38) \end{aligned}$$

with $\Omega = \frac{I_3 - I_1}{I_1}\omega_{z'} = \text{constant}$. This implies [1] that $\omega_{x'}^2 + \omega_{y'}^2 = \text{constant}$. Which in turn implies:

$$\omega_{x'}^2 + \omega_{y'}^2 = (\dot{\varphi}\sin\theta\sin\psi)^2 + (\dot{\varphi}\sin\theta\cos\psi)^2 = \dot{\varphi}^2\sin\theta = \text{constant} \tag{39}$$

We thus find that:

$$\dot{\varphi} = c_1 \tag{40}$$

where c_1 is a constant. Inserting Eq. (33) and (34) into Eq. (37) and (38) gives

$$\begin{aligned} \dot{\varphi}\dot{\psi}\sin\theta\cos\psi &= -\Omega\dot{\varphi}\sin\theta\cos\psi & (41)\\ -\dot{\varphi}\dot{\psi}\sin\theta\sin\psi &= \Omega\dot{\varphi}\sin\theta\sin\psi & (42) \end{aligned}$$

This implies that

$$\dot{\psi} = -\Omega = -\frac{(I_3 - I_1)}{I_1}\omega_z' = \left(\frac{1}{I_3} - \frac{1}{I_1}\right) L\cos\theta \tag{43}$$

[1]Using Eq. (37) and (38) we find: $\frac{\mathrm{d}}{\mathrm{d}t}\left[\omega_{x'}^2 + \omega_{y'}^2\right] = 2\omega_{x'}\dot{\omega}_{x'} + 2\omega_{y'}\dot{\omega}_{y'} = -2\Omega\omega_{x'}\omega_{y'} + 2\Omega\omega_{y'}\omega_{x'} = 0$ which shows that $\omega_{x'}^2 + \omega_{y'}^2$ does not change with time.

Inserting Eq. (43) into Eq. (35) gives

$$\dot{\phi} = \frac{L}{I_1} \tag{44}$$

3c)

$$\frac{\dot{\varphi}}{\omega_z'} = \frac{\dot{\varphi}}{L\cos(\theta)/I_3} = \frac{L/I_1}{L\cos(\theta)/I_3} = \frac{I_3}{I_1\cos\theta} \approx \frac{I_3}{I_1} = 2 \tag{45}$$

3. Precession of a heavy spinning top.

The shifted total energy and potential

$$E' = \frac{1}{2} I_1 \dot{\theta}^2 + V(\theta)\,; \qquad V(\theta) = \frac{(p_\phi - p_\psi \cos\theta)^2}{2 I_1 \sin^2\theta} + Mgh\cos\theta.$$

Potential shown below. Note the analogy with the central force problem!

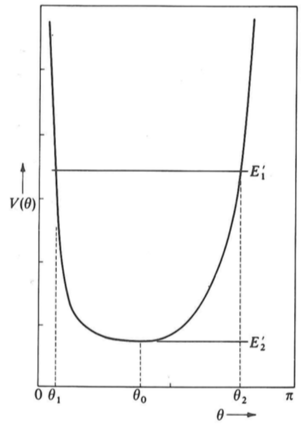

FIGURE 11-14

The value of θ_0 can be obtained by setting the derivative of $V(\theta)$ equal to zero. Thus,

$$\left.\frac{\partial V}{\partial \theta}\right|_{\theta=\theta_0} = \frac{-\cos\theta_0 (p_\phi - p_\psi \cos\theta_0)^2 + p_\psi \sin^2\theta_0 (p_\phi - p_\psi\cos\theta_0)}{I_1 \sin^3\theta_0} - Mgh\sin\theta_0 = 0 \tag{11.165}$$

If we define

$$\beta \equiv p_\phi - p_\psi \cos\theta_0 \tag{11.166}$$

then Equation 11.165 becomes

$$(\cos\theta_0)\,\beta^2 - (p_\psi \sin^2\theta_0)\,\beta + (MghI_1 \sin^4\theta_0) = 0 \tag{11.167}$$

This is a quadratic in β and can be solved with the result

$$\beta = \frac{p_\psi \sin^2\theta_0}{2\cos\theta_0}\left(1 \pm \sqrt{1 - \frac{4MghI_1\cos\theta_0}{p_\psi^2}}\right) \tag{11.168}$$

Because β must be a real quantity, the radicand in Equation 11.168 must be positive. If $\theta_0 < \pi/2$, we have

$$p_\psi^2 \geq 4MghI_1 \cos\theta_0 \tag{11.169}$$

But from Equation 11.159a, $p_\psi = I_3\omega_3$; thus,

$$\omega_3 \geq \frac{2}{I_3}\sqrt{MghI_1\cos\theta_0} \tag{11.170}$$

We therefore conclude that a steady precession can occur at the fixed angle of inclination θ_0 only if the angular velocity of spin is larger than the limiting value given by Equation 11.170.

From Equation 11.156, we note that we can write (for $\theta = \theta_0$)

$$\dot{\phi}_0 = \frac{\beta}{I_1 \sin^2\theta_0} \tag{11.171}$$

We therefore have two possible values of the precessional angular velocity $\dot{\phi}_0$, one for each of the values of β given by Equation 11.168:

$$\dot{\phi}_{0(+)} \to \text{Fast precession}$$

and

$$\dot{\phi}_{0(-)} \to \text{Slow precession}$$

If ω_3 (or p_ψ) is large (a fast top), then the second term in the radicand of Equation 11.168 is small, and we may expand the radical. Retaining only the first nonvanishing term in each case, we find

$$\left.\begin{aligned} \dot{\phi}_{0(+)} &\cong \frac{I_3\omega_3}{I_1\cos\theta_0} \\ \dot{\phi}_{0(-)} &\cong \frac{Mgh}{I_3\omega_3} \end{aligned}\right\} \tag{11.172}$$

It is the slower of the two possible precessional angular velocities, $\dot{\phi}_{0(-)}$, that is usually observed.