Øving 1

4. Alternative Lagrangian

(1a) L' and L are equivalent if F satisfies:

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}}\frac{d}{dt}F(q,t) - \frac{\partial}{\partial q}\frac{d}{dt}F(q,t) = 0.$$

Now, we know that

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial q}\dot{q}.$$

Inserting this into the first equation, we obtain:

$$\frac{\partial^2 F}{\partial q \partial t} + \frac{\partial^2 F}{\partial q^2}\dot{q} - \frac{\partial^2 F}{\partial q \partial t} - \frac{\partial^2 F}{\partial q^2}\dot{q} = 0.$$

This needs a better explanation. More detailed.

5. Friction (from previous years → numbering below)

(1c) Frictional force $F_f = -\partial \mathcal{F}/\partial v$. The work performed by the system against friction, per unit time, is $\dot{W} = -F_f v$ which with $\mathcal{F} = Cv^2$ becomes $\dot{W} = 2\mathcal{F}$. The Lagrange equations read:

$$\frac{d}{dt}\frac{\partial L}{\partial v} - \frac{\partial L}{\partial x} + \frac{\partial \mathcal{F}}{\partial v} = 0. \tag{5}$$

Insert the Lagrangian $L = T - V = mv^2/2 - kx^2/2$ and $\mathcal{F} = 3\pi\mu a v^2$ to obtain:

$$\ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = 0 \tag{6}$$

where $\lambda = 3\pi\mu a/m$ and $\omega_0 = \sqrt{k/m}$. Assuming $\lambda/\omega_0 \ll 1$, the solution is:

$$x(t) = x_0 e^{-\lambda t}\cos\omega_0 t \tag{7}$$

and the average energy dissipation $\bar{\dot{W}}$ over a period $2\pi/\omega_0$ can be computed by treating $e^{-\lambda t}$ as a constant since it remains virtually unchanged over a time-interval $2\pi/\omega$:

$$\bar{\dot{W}} \simeq m\lambda(\omega_0 x_0)^2 e^{-2\lambda t}. \tag{8}$$

Øving 3

5. Einstein's correction (from previous years, på norsk!)

a) Det sentrale kraftfeltet er gitt ved:

$$f(r) = -\frac{k}{r^2} + \frac{\beta}{r^3} \qquad \Rightarrow \qquad V(r) = -\frac{k}{r} + \frac{\beta}{2r^2}$$

Fra teorien er: $$\theta = \int \frac{\frac{1}{r^2}dr}{\sqrt{\frac{2mE}{l^2} - \frac{2mV}{l^2} - \frac{1}{r^2}}} + konst.$$

Innsetting av V og innføring av $u = \frac{1}{r}$ gir når konstanten utelates

$$\theta = -\int \frac{du}{\sqrt{\frac{2mE}{l^2} - \frac{2mku}{l^2} - \gamma^2 u^2}}, \text{ hvor } \gamma^2 = 1 + \frac{\beta m}{l^2}$$

Benytter: $$\int \frac{dx}{\sqrt{a + bx + cx^2}} = \frac{1}{\sqrt{-c}} \arccos\left(-\frac{b + 2cx}{\sqrt{q}}\right), \text{ hvor } q = b^2 - 4ac$$

Her velges $a = \frac{2mE}{l^2}$, $b = \frac{2mk}{l^2}$, $c = -\gamma^2 \Rightarrow q = \left(\frac{2mk}{l^2}\right)^2\left(1 + \frac{2E\gamma^2 l^2}{mk^2}\right)$,

$$-\frac{b + 2cu}{\sqrt{q}} = \frac{\frac{\gamma^2 l^2 u}{mk} - 1}{\sqrt{1 + \frac{2E\gamma^2 l^2}{mk^2}}}.$$ Definerer $\varepsilon = \sqrt{1 + \frac{2E\gamma^2 l^2}{mk^2}}$, $p = \frac{\gamma^2 l^2}{mk}$

Da blir $\theta = -\frac{1}{\gamma} \arccos \frac{\frac{p}{r} - 1}{\varepsilon}$,

Baneligningen er

(1) $\frac{p}{r} = 1 + \varepsilon \cos(\gamma\theta)$, hvor $\boxed{\gamma = \sqrt{1 + \frac{m\beta}{2l}} \approx 1 + \frac{m\beta}{2l^2}}$

Mistake $\gamma = \sqrt{1 + \frac{\beta m}{l^2}}$

Used Taylor to get $\gamma \approx 1 + \frac{m\beta}{2l^2}$

Antar $E < 0$. Da er ligning (1), ligningen for en ellipse med langsom presesjon.

Store halvakse: $a = \frac{p}{1 - \varepsilon^2}$ (slik som når $\gamma = 1$) $\Rightarrow$

$$a = \frac{\frac{\gamma^2 l^2}{mk}}{1 - \left(1 + \frac{2E\gamma^2 l^2}{mk^2}\right)} = \frac{k}{2|E|},$$ som for $\gamma = 1$.

Vanlig litenhetsparameter er $\eta = \frac{\beta}{ka}$, dvs. $\gamma = 1 + \frac{m\eta ka}{2l^2}$

Verdien $\eta = 1.42 \cdot 10^{-7}$ tilsvarer Merkurs perihelbevegelse, som er $43''$ per hundre år.

Øving ~~4~~ 4

The integral in Eq. (15) can be expressed as the derivative of a simpler integral:

$$\int_0^{2\pi} \frac{d\theta \cos(\theta)}{[1+\epsilon\cos(\theta)]^2} = -\frac{d}{d\epsilon}\int_0^{2\pi} \frac{d\theta}{[1+\epsilon\cos(\theta)]} \qquad (16)$$

Hint 2

The integral on the right side of Eq. (16) can be calculated using the Residue method (E. Kreyzig, 9th edition, Chapter 16.4):

$$\int_0^{2\pi} \frac{d\theta}{[1+\epsilon\cos(\theta)]} = \frac{2\pi}{\sqrt{1-\epsilon^2}} \qquad (17)$$

(see appendix for detailed calculation). Combining Equations (12-17) gives the time average of $\cos(\theta)$

$$\langle\cos(\theta)\rangle = -\frac{mp^2}{\ell\, t_p}\frac{d}{d\epsilon}\frac{2\pi}{\sqrt{1-\epsilon^2}} = -\frac{mp^2}{\ell\, t_p}\frac{2\pi\epsilon}{(1-\epsilon^2)^{3/2}} \qquad (18)$$

The orbit period is given in Section 4.5 in the Brevik compendium:

$$t_p = \frac{2\pi m}{\ell}ab = \frac{2\pi m}{\ell}\frac{p}{1-\epsilon^2}\frac{p}{\sqrt{1-\epsilon^2}} = \frac{2\pi m p^2}{\ell}\frac{1}{(1-\epsilon^2)^{3/2}} \qquad (19)$$

where a and b are the major and minor-axis of the ellipse.
Eliminating t_p from Eq. (19) gives:

$$\langle\cos(\theta)\rangle = -\epsilon \qquad (20)$$

Inserting Eq. (20) into Eq. (12) gives the average potential energy:

$$\langle V\rangle = -\frac{k}{p}\left(1-\epsilon^2\right) \qquad (21)$$

(c) The average kinetic energy is:

$$\langle T\rangle = \frac{1}{t_p}\int_0^{t_p} dt\, T = \frac{1}{t_p}\int_0^{t_p} dt\,\frac{1}{2}m\left(\frac{d\vec{r}}{dt}\right)^2 \qquad (22)$$

Integration by parts gives:

$$\begin{aligned}
\langle T\rangle &= \overbrace{\left[\frac{1}{t_p}\frac{1}{2}m\vec{r}\cdot\frac{d\vec{r}}{dt}\right]_0^{t_p}}^{=0 \text{ explain why}} - \frac{1}{t_p}\int_0^{t_p} dt\,\frac{1}{2}m\vec{r}\cdot\frac{d^2\vec{r}}{dt^2} && (23)\\
&= -\frac{1}{t_p}\int_0^{t_p} dt\,\frac{1}{2}m\vec{r}\cdot\frac{d^2\vec{r}}{dt^2} && (24)\\
&= \frac{1}{t_p}\int_0^{t_p} dt\,\frac{1}{2}\vec{r}\cdot\nabla(-\frac{k}{r}) && (25)\\
&= \frac{1}{t_p}\int_0^{t_p} dt\,\frac{k}{2r} && (26)\\
&= \langle\frac{k}{2r}\rangle && (27)\\
&= -\frac{1}{\,}\langle V\rangle && (28)
\end{aligned}$$

Øving 4

5. Elastic scattering in laboratory coordinates.

Relations: $\cos\vartheta = \dfrac{\cos\Theta + \rho}{\sqrt{1+2\rho\cos\Theta+\rho^2}}$, $\rho = \dfrac{m_1}{m_2}$ (elastic)

LAB angle (ϑ) — CM angle (Θ)

$$\sigma'(\vartheta) = \sigma(\Theta)\,\frac{(1+2\rho\cos\Theta+\rho^2)^{3/2}}{1+\rho\cos\Theta}$$

Set masses equal: $m_1 = m_2$ (this was missing...) $\Rightarrow \rho = 1$

should be given in exercise text.

$$\Rightarrow \cos\vartheta = \frac{\cos\Theta + 1}{\sqrt{2+2\cos\Theta}} = \frac{1+\cos\Theta}{\sqrt{2}\cdot\sqrt{1+\cos\Theta}} = \sqrt{\frac{1+\cos\Theta}{2}}$$

$$\sqrt{\frac{1+\cos\Theta}{2}} = \cos\left(\frac{\Theta}{2}\right)$$

should be given

$$\Rightarrow \cos\vartheta = \cos\frac{\Theta}{2} \Rightarrow \vartheta = \frac{\Theta}{2} \quad (\rho = 1)$$

$\therefore$ Scattering angles $> 90°$ cannot occur in the lab system

Now:

$$\sigma'(\vartheta) = \sigma(\Theta)\cdot\frac{(2+2\cos\Theta)^{3/2}}{1+\cos\Theta} = \sigma(\Theta)\cdot 2(1+\cos\Theta)^{1/2}$$

$$= \sigma(\Theta)\cdot 4\cos\frac{\Theta}{2} = 4\cos\vartheta\cdot\sigma(\Theta), \quad \vartheta \le \frac{\pi}{2}$$

$\therefore$ For isotropic scattering in Θ ($\sigma(\Theta)$ constant), cross section in ϑ varies as the cosine of the angle

Elastic collision slows down the incident particle:

cosine law (lectures) $\Rightarrow v_1^2 = v_1'^2 + V^2 + 2v_1'V\cos\Theta$

(v_1: LAB; v_1': CM; V: CM)

$$V = \frac{\mu}{m_2}v_0 = \frac{m_1 m_2}{m_1+m_2}\cdot\frac{1}{m_2}\cdot v_0 = \frac{1}{2}v_0 \quad (m_1 = m_2)$$

(v_0: incident velocity, m_2 at rest)

$$\rho = \frac{\mu}{m_2}\frac{v_0}{v_1'} = \frac{1}{2}\frac{v_0}{v_1'}, \quad \rho = 1$$

$$\Rightarrow v_1' = \frac{1}{2}v_0$$

Elastic collision

$$\Rightarrow v_1^2 = \left(\tfrac{1}{2}v_0\right)^2 + \left(\tfrac{1}{2}v_0\right)^2 + 2\cdot\tfrac{1}{2}v_0\cdot\tfrac{1}{2}v_0\cdot\cos\Theta$$

$$= \frac{1}{2}v_0^2 + \frac{1}{2}v_0^2\cos\Theta = \frac{1}{2}v_0^2(1+\cos\Theta)$$

$$\Rightarrow \frac{v_1^2}{v_0^2} = \frac{1+\cos\Theta}{2}; \quad E_1 = \frac{1}{2}m_1v_1^2, \quad E_0 = \frac{1}{2}m_1v_0^2$$

$$\Rightarrow \frac{E_1}{E_0} = \frac{1+\cos\Theta}{2} = \cos^2\vartheta \quad \text{(course book: pp. 115-120)}$$

Øving 5

Classical Mechanics TFY 4345 – Solution set 5 (lectures 25-30)

1. Rotating system in cylindrical coordinates.

$$L = T - V = m(\dot r^2 + r^2\dot\theta^2 + \dot z^2)/2 - V(r,\theta,z) \qquad (7)$$

The Lagrange-equations for the three generalized coordinates then read:

$$m\ddot r - mr\dot\theta^2 = -\partial_r V,\ d(mr^2\dot\theta)/dt + \partial_\theta V = 0,\ m\ddot z + \partial_z V = 0. \qquad (8)$$

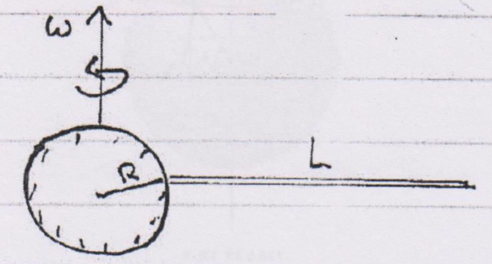
Using Hamilton's equations, we identify the canonical momenta:

$$p_r = m\dot r,\ p_\theta = mr^2\dot\theta,\ p_z = m\dot z. \qquad (9)$$

These equations can be used to replace the time derivatives of the generalized coordinates in terms of the momenta, so that the final Hamiltonian reads:

$$H = T + V = (p_r^2 + p_\theta^2/r^2 + p_z^2)/(2m) + V(r,\theta,z). \qquad (10)$$

2. Effective potential and scattering centre.



In the rotating coordinate system, the centrifugal force acting on an element of length dr in a distance r from the center equal to $r\omega^2\rho\cdot dr$. Here, ρ is the mass of the rod per unit length. The gravitational pull on the same element dr is $GM\rho\cdot dr/r^2 = g_0R^2\rho\cdot dr/r^2$ where g_0 is the gravitational acceleration at the surface of the Earth. Balancing the total graviational and centrifugal force in order to obtain an equilibrium situation, then provides us with:

$$\int_R^{R+L} r\omega^2\rho\cdot dr = g_0R^2\rho\int_R^{R+L} dr/r^2. \qquad (1)$$

Performing the integration results in:

$$L^2 + 3RL + (2R^2 - 2g_0R/\omega^2) = 0 \qquad (2)$$

This is a 2nd order equation for L whose positive solution reads:

$$L = -3R/2 + \sqrt{R^2 + 8g_0R/\omega^2}/2 \qquad (3)$$

Too many steps missing

Putting in numbers $R = 6.4$ km, $\omega = 2\pi/(1\text{ day})$, and $g_0 = 9.8$ m/s^2 gives $L = 1.5\times 10^5$ km (about halfway to the moon).

$\omega = 7{,}29\cdot 10^{-5}$ rad/s should be given

$R = 6400$ km not $6{,}4$ km

Øving 5

5. Precession of a heavy spinning top.

The shifted total energy and potential

$$E' = \frac{1}{2} I_1 \dot{\theta}^2 + V(\theta)\,; \qquad V(\theta) = \frac{(p_\phi - p_\psi \cos\theta)^2}{2 I_1 \sin^2\theta} + Mgh\cos\theta.$$

Potential shown below. Note the analogy with the central force problem!

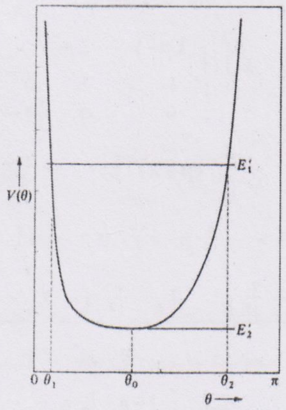


FIGURE 11-14

Explain

The value of θ_0 can be obtained by setting the derivative of $V(\theta)$ equal to zero. Thus,

$$\left.\frac{\partial V}{\partial \theta}\right|_{\theta=\theta_0} = \frac{-\cos\theta_0 (p_\phi - p_\psi \cos\theta_0)^2 + p_\psi \sin^2\theta_0 (p_\phi - p_\psi \cos\theta_0)}{I_1 \sin^3\theta_0} - Mgh\sin\theta_0 = 0 \tag{11.165}$$

If we define

$$\beta \equiv p_\phi - p_\psi \cos\theta_0 \tag{11.166}$$

then Equation 11.165 becomes

$$(\cos\theta_0)\,\beta^2 - (p_\psi \sin^2\theta_0)\beta + (MghI_1 \sin^4\theta_0) = 0 \tag{11.167}$$

This is a quadratic in β and can be solved with the result

$$\beta = \frac{p_\psi \sin^2\theta_0}{2\cos\theta_0}\left(1 \pm \sqrt{1 - \frac{4MghI_1\cos\theta_0}{p_\psi^2}}\right) \tag{11.168}$$

Because β must be a real quantity, the radicand in Equation 11.168 must be positive. If $\theta_0 < \pi/2$, we have

$$p_\psi^2 \geq 4MghI_1 \cos\theta_0 \tag{11.169}$$

Too many steps missing

This question refers to equations that we have no access too. That makes it difficult. EX. 11.159a

Should explain where this is from (forelesning)

But from Equation 11.159a, $p_\psi = I_3\omega_3$; thus,

$$\omega_3 \geq \frac{2}{I_3}\sqrt{MghI_1\cos\theta_0} \tag{11.170}$$

We therefore conclude that a steady precession can occur at the fixed angle of inclination θ_0 only if the angular velocity of spin is larger than the limiting value given by Equation 11.170.

From Equation 11.156, we note that we can write (for $\theta = \theta_0$)

$$\dot{\phi}_0 = \frac{\beta}{I_1 \sin^2\theta_0} \tag{11.171}$$

We therefore have two possible values of the precessional angular velocity $\dot{\phi}_0$, one for each of the values of β given by Equation 11.168:

$$\dot{\phi}_{0(+)} \rightarrow \text{Fast precession}$$

and

$$\dot{\phi}_{0(-)} \rightarrow \text{Slow precession}$$

If ω_3 (or p_ψ) is large (a fast top), then the second term in the radicand of Equation 11.168 is small, and we may expand the radical. Retaining only the first nonvanishing term in each case, we find

$$\left.\begin{aligned} \dot{\phi}_{0(+)} &\cong \frac{I_3\omega_3}{I_1\cos\theta_0} \\ \dot{\phi}_{0(-)} &\cong \frac{Mgh}{I_3\omega_3} \end{aligned}\right\} \tag{11.172}$$

It is the slower of the two possible precessional angular velocities, $\dot{\phi}_{0(-)}$, that is usually observed.

Too many steps missing. Should explain ~~that~~ Taylor is used, when ω_3 is large.

Øving 6

2. Two coupled oscillators.

$$T = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2$$

$$V = \frac{1}{2} k x_1^2 + \frac{1}{2} k (x_2 - x_1)^2$$

$$\Rightarrow \quad \bar{\bar{m}} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$$

$$V = \frac{1}{2} k x_1^2 + \frac{1}{2} k (x_2^2 - 2x_1 x_2 + x_1^2)$$

$$= 2 \cdot \frac{1}{2} k x_1^2 + \frac{1}{2} k x_2^2 - \frac{1}{2} k x_1 x_2 - \frac{1}{2} k x_2 x_1$$

$$\equiv \frac{1}{2} \sum_{j,k} A_{jk} q_j q_k$$

$$\Rightarrow \quad \bar{A} = \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}$$

Eigenfrequencies

$$\begin{vmatrix} 2k - \omega^2 m & -k \\ -k & k - \omega^2 m \end{vmatrix} = 0$$

$$\Rightarrow (2k - \omega^2 m)(k - \omega^2 m) - k^2 = 0$$

$$\Rightarrow \quad \dots \quad \Rightarrow \quad \omega^2 = \frac{k}{2m}(3 \pm \sqrt{5}) \quad (>0)$$

Cannot have same ratio. Should be sym and anti-sym modes.

Eigenfrequencies:

$1°$ $\quad (A_{11} - \omega_1^2 m_{11}) a_{11} + (A_{21} - \omega_1^2 m_{12}) a_{12} = 0$

$$\Rightarrow \quad a_{11} = \frac{2}{1+\sqrt{5}} a_{12} \quad \text{Wrong} \qquad \bar{a}_1 = c_1 \cdot \begin{bmatrix} 1 \\ \frac{2}{1+\sqrt{5}} \end{bmatrix} \quad \text{Wrong}$$

$2°$ $\quad (A_{21} - \omega_2^2 m_{21}) a_{21} + (A_{22} - \omega_2^2 m_{22}) a_{22} = 0$

$$\Rightarrow \quad a_{22} = \frac{2}{1+\sqrt{5}} a_{21} \quad \text{Wrong} \qquad \bar{a}_2 = c_2 \begin{bmatrix} \frac{2}{1+\sqrt{5}} \\ 1 \end{bmatrix} \quad \text{Wrong}$$

$$a_{11} = -\frac{(1+\sqrt{5})}{2} a_{12} \qquad \bar{a}_1 = c_1 \begin{bmatrix} 1 \\ -(1+\sqrt{5})/2 \end{bmatrix}$$

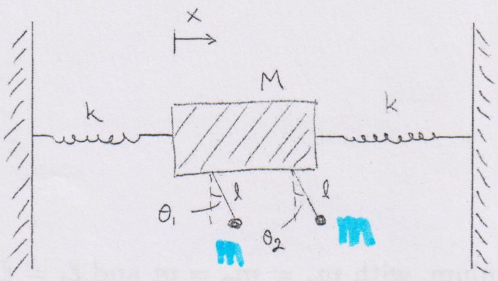
$$a_{22} = \frac{1+\sqrt{5}}{2} a_{21} \qquad \bar{a}_2 = c_2 \begin{bmatrix} (1+\sqrt{5})/2 \\ 1 \end{bmatrix}$$

Øving 6

3. Oscillating body with two attached pendula.

Question does not give the mass of the pendulums. Small m.

3.



Equilibrium: $x=0,\ \theta_1=\theta_2=0$

$V_1 = 2\cdot\frac{1}{2}kx^2 = kx^2$

$V_2 = Mg\ell(1-\cos\theta) \approx \frac{1}{2}Mg\ell\theta_1^2$

$V_3 = \frac{1}{2}Mg\ell\theta_2^2$

Should be small m

$$V = kx^2 + \frac{1}{2}mg\ell(\theta_1^2+\theta_2^2) \equiv \frac{1}{2}\sum_{j,k}^{3} A_{jk}q_jq_k$$

$$\Rightarrow \quad \bar{\bar{A}} = \begin{bmatrix} 2k & 0 & 0 \\ 0 & mg\ell & 0 \\ 0 & 0 & mg\ell \end{bmatrix}$$

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}+\ell\dot{\theta}_1)^2 + \frac{1}{2}m(\dot{x}+\ell\dot{\theta}_2)^2 \equiv \frac{1}{2}\sum m_{jk}\dot{q}_j\dot{q}_k$$

$$\Rightarrow \quad \bar{\bar{m}} = \begin{bmatrix} M+2m & m\ell & m\ell \\ m\ell & m\ell^2 & 0 \\ m\ell & 0 & m\ell^2 \end{bmatrix} \quad \text{(non-diagonal)}$$

Eigenfrequencies:

$$\begin{vmatrix} 2k-\omega_r^2(M+2m) & -\omega_r^2 m\ell & -\omega_r^2 m\ell \\ -\omega_r^2 m\ell & mg\ell-\omega_r^2 m\ell & 0 \\ -\omega_r^2 m\ell & 0 & mg\ell-\omega_r^2 m\ell \end{vmatrix} = 0$$

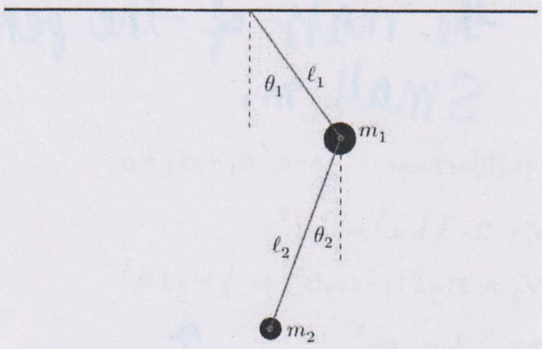
$$\Rightarrow \ \ldots \ \Rightarrow \quad \omega_r^2 = \frac{g}{\ell} \qquad \text{(three roots)}$$

$$\text{or} \quad \omega_r^2 = \frac{g}{2\ell}\left(1+\frac{2m}{M}\right)+\frac{k}{M} \pm \sqrt{\frac{g}{2\ell}\left[\left(1+\frac{2m}{M}\right)+\frac{k}{M}\right]^2 - 2\frac{k}{m}g\ell}$$

This is missing too many steps. Please write a few more steps to help students get there.

Øving 6

4. Double pendulum.



As a second example, consider the double pendulum, with $m_1 = m_2 = m$ and $\ell_1 = \ell_2 = \ell$. The kinetic and potential energies are

$$T = m\ell^2\dot{\theta}_1^2 + m\ell^2\cos(\theta_1 - \theta_1)\,\dot{\theta}_1\dot{\theta}_2 + \tfrac{1}{2}m\ell^2\dot{\theta}_2^2 \tag{10.39}$$

$$V = -2mg\ell\cos\theta_1 - mg\ell\cos\theta_2\,, \tag{10.40}$$

leading to

$$\mathrm{T} = \begin{pmatrix} 2m\ell^2 & m\ell^2 \\ m\ell^2 & m\ell^2 \end{pmatrix} \quad , \quad \mathrm{V} = \begin{pmatrix} 2mg\ell & 0 \\ 0 & mg\ell \end{pmatrix} . \tag{10.41}$$

Then

$$\omega^2\mathrm{T} - \mathrm{V} = m\ell^2 \begin{pmatrix} 2\omega^2 - 2\omega_0^2 & \omega^2 \\ \omega^2 & \omega^2 - \omega_0^2 \end{pmatrix} , \tag{10.42}$$

with $\omega_0 = \sqrt{g/\ell}$. Setting the determinant to zero gives

$$2(\omega^2 - \omega_0^2)^2 - \omega^4 = 0 \quad \Rightarrow \quad \omega^2 = (2 \pm \sqrt{2})\,\omega_0^2 \,. \tag{10.43}$$

We find the unnormalized eigenvectors by setting $(\omega_i^2\,\mathrm{T} - V)\,\psi^{(i)} = 0$. This gives

$$\psi^+ = C_+ \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \quad , \quad \psi^- = C_- \begin{pmatrix} 1 \\ +\sqrt{2} \end{pmatrix} , \tag{10.44}$$

Wrong Wrong

where $C_\pm$ are constants.

Please check

$$\psi^+ = C'_+ \begin{bmatrix} 1 \\ -\frac{\sqrt{2}}{2} \end{bmatrix} \qquad \psi^- = C'_- \begin{bmatrix} 1 \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

$$= C_+ \begin{bmatrix} 1 \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \qquad = C_- \begin{bmatrix} 1 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Øving 7

5. Lorentz transformation of energy and momentum.

Please have explanation of this in the new LF

a) Using Einstein velocity summation formulas (Brevik page 87) gives:

$$
\begin{aligned}
1-\frac{u'^2}{c^2} &= \boxed{1-\frac{u_x^2+u_y^2}{\gamma^2(1-\frac{vu_z}{c^2})^2c^2}-\frac{(u_z-v)^2}{(1-\frac{vu_z}{c^2})^2c^2}} && (1)\\
&= \frac{1}{\gamma^2(1-\frac{vu_z}{c^2})^2}\left[\gamma^2(1-\frac{vu_z}{c^2})^2-\frac{u_x^2+u_y^2}{c^2}-\frac{\gamma^2(u_z-v)^2}{c^2}\right] && (2)\\
&= \frac{1}{\gamma^2(1-\frac{vu_z}{c^2})^2}\left[\gamma^2(1+\boxed{\frac{v^2u_z^2}{c^2}}-\frac{u_z^2}{c^2}-\frac{v^2}{c^2})-\frac{u_x^2+u_y^2}{c^2}\right] && (3)\\
&= \frac{1}{\gamma^2(1-\frac{vu_z}{c^2})^2}\left(1-\frac{u_x^2+u_y^2+u_z^2}{c^2}\right) && (4)\\
&= \frac{1}{\gamma^2(1-\frac{vu_z}{c^2})^2}\left(1-\frac{u^2}{c^2}\right) && (5)
\end{aligned}
$$

Mistake $\frac{V^2 u_z^2}{c^4}$

where we have used that $\gamma^2=\frac{1}{1-\frac{v^2}{c^2}}$

In conclusion we have shown that:

$$\frac{1}{\sqrt{1-u'^2/c^2}}=\gamma\frac{1-\frac{vu_z}{c^2}}{\sqrt{1-u^2/c^2}} \quad (6)$$

b)

Lorentz transformation of energy:

$$
\begin{aligned}
E' &= \frac{mc^2}{\sqrt{1-\frac{u'^2}{c^2}}} && (7)\\
&= \frac{mc^2\gamma(1-\frac{vu_z}{c^2})}{\sqrt{1-\frac{u^2}{c^2}}} && (8)\\
&= \gamma(E-vp_z) && (9)
\end{aligned}
$$

Lots of steps are missing here

Similar calculation for momentum gives:

$$
\begin{aligned}
p'_x &= p_x && (10)\\
p'_y &= p_y && (11)\\
p'_z &= \gamma(p_z-\frac{v}{c^2}E) && (12)
\end{aligned}
$$

3. Canonical transformation (Goldstein 9-6)

$$\begin{cases} Q = \log(1+\sqrt{q}\cos p) \\ P = 2(1+\sqrt{q}\cos p)\sqrt{q}\sin p \end{cases}$$

a) Symplectic condition $\bar{\bar{M}}^T\bar{\bar{J}}\bar{\bar{M}} = \bar{\bar{J}}$ ← Maybe prove this in class?

Antisymmetric matrix $\bar{\bar{J}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Jacobian matrix $\bar{\bar{M}} = \begin{bmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{bmatrix}$ $\left[M_{ij} = \frac{\partial \zeta_i}{\partial \eta_j} \right]$ transformed (ζ_i), original (η_j)

$$\Rightarrow \bar{\bar{M}} = \begin{bmatrix} \frac{1}{2}\left(\frac{1}{q+\sqrt{q}\sec(p)}\right) & \frac{-\sqrt{q}\sin p}{1+\sqrt{q}\cos p} \\ \left(\frac{1}{\sqrt{q}} + 2\cos p\right)\sin p & 2(\sqrt{q}\cos p + q\cos(2p)) \end{bmatrix}$$

Now: Check $\bar{\bar{M}}^T \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \bar{\bar{M}}$ (for example, Mathematica)

$$\Rightarrow \bar{\bar{M}}^T\bar{\bar{J}}\bar{\bar{M}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \bar{\bar{J}}$$ ✓. As long as q, p canonical, Q and P are too.

b) Type 3 generating functions

$$q = -\frac{\partial F_3(p,Q,t)}{\partial p}, \quad P = -\frac{\partial F_3(p,Q,t)}{\partial Q}$$

$$F_3 = (p,Q,t) = -(e^Q-1)^2 \tan p$$

$$\Rightarrow \begin{cases} q = -\frac{\partial F_3}{\partial p} = (e^Q-1)^2\sec^2(p) & \Rightarrow \underline{Q = \log(1+\sqrt{q}\cos p)} \\ P = -\frac{\partial F_3}{\partial Q} = 2e^Q(e^Q-1)\tan p \end{cases}$$

$$\Rightarrow \underline{P} = 2e^{\log(1+\sqrt{q}\cos p)}\left(e^{\log(1+\sqrt{q}\cos p)} - 1\right)\tan p = \underset{\text{few steps}}{\ldots} = \underline{2(1+\sqrt{q}\cos p)\sqrt{q}\sin p}$$