Exercise 12 - TFY4345 Classical Mechanics

2020

1 Generating function F_4

The generating function F is given by

$$F = q_i p_i - Q_i P_i + F_4(p, P, t).$$

This means the time derivative can be written as

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \dot{p}_i q_i + p_i \dot{q}_i - \dot{P}_i Q_i - P_i \dot{Q}_i + \frac{\mathrm{d}F(p, P, t)}{\mathrm{d}t}.$$

Inserting this into the relation between the original Hamiltonian H and the new one, K

$$p_i \dot{q}_i - H(q, p, t) = P_i \dot{Q}_i - K(Q, P, t) + \frac{\mathrm{d}F}{\mathrm{d}t}$$

gives

$$\begin{aligned} \cancel{p_i \dot{q}_i} - H(q, p, t) = \cancel{P_i \dot{Q}_i} - K(Q, P, t) + \dot{p}_i q_i + \cancel{p_i \dot{q}_i} - \dot{P}_i Q_i - \cancel{P_i \dot{Q}_i} + \frac{\mathrm{d}F(p, P, t)}{\mathrm{d}t} \\ \dot{p}_i q_i + H(q, p, t) = \dot{P}_i Q_i + K(Q, P, t) - \frac{\mathrm{d}F_4(p, P, t)}{\mathrm{d}t} \end{aligned}$$

We can expand

$$\frac{\mathrm{d}F_4(p, P, t)}{\mathrm{d}t} = \frac{\partial F_4}{\partial t} + \frac{\partial F_4}{\partial p_i} \dot{p}_i + \frac{\partial F_4}{\partial P_i} \dot{P}_i,$$

which gives

$$\dot{p}_i q_i + H(q, p, t) = \dot{P}_i Q_i + K(Q, P, t) - \left(\frac{\partial F_4}{\partial t} + \frac{\partial F_4}{\partial p_i} \dot{p}_i + \frac{\partial F_4}{\partial P_i} \dot{P}_i \right).$$

This only holds if

$$K = H + \frac{\partial F_4}{\partial t}, \quad q_i = -\frac{\partial F_4}{\partial p_i}, \quad Q_i = \frac{\partial F_4}{\partial P_i},$$

which is the equations we were looking

2 The Poisson bracket

The Hamiltonian for the harmonic oscillator is

$$H = \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right),$$

and we have the canonical transformations

$$q = \sqrt{\frac{2P}{m\omega}}\sin(Q), \quad p = \sqrt{2Pm\omega}\cos(Q), \quad H = \omega P.$$

The Poisson bracket in the original is

$$[q, H]_{q,p} = \underbrace{\frac{\partial q}{\partial q}}_{=1}\frac{\partial H}{\partial p} - \underbrace{\frac{\partial q}{\partial q}}_{=0}\frac{\partial H}{\partial p} = \frac{\partial H}{\partial p} = \frac{p}{m}.$$

In the new coordinates the bracket is

$$[q, H]_{Q,P} = \frac{\partial q}{\partial Q}\frac{\partial H}{\partial P} - \frac{\partial q}{\partial P}\underbrace{\frac{\partial H}{\partial Q}}_{=0} = \frac{\partial q}{\partial Q}\frac{\partial H}{\partial P},$$

where

$$\frac{\partial q}{\partial Q} = \frac{\partial}{\partial Q}\left(\sqrt{\frac{2P}{m\omega}}\sin(Q)\right) = \sqrt{\frac{2P}{m\omega}}\cos(Q)\cdot\frac{\sin(Q)}{\sin(Q)} = q\cot(Q),$$

$$\frac{\partial H}{\partial P} = \omega, \quad \cot(Q) = \frac{\cos(Q)}{\sin(Q)} = m\omega\frac{p}{q}.$$

This gives

$$[q, H]_{Q,P} = \omega q \cot Q = \frac{p}{m} = [q, H]_{q,p}.$$

The fact that $[q, H] \neq 0$ means that q is not a constant of motion.

3 The symplectic condition

See the appendix at the bottom of the exercise or Goldstein 9.4, 3rd. ed. for explanation of the symplectic condition. The calculations can be done using computer software and will not be shown here. Look at the Jupyter notebook for an example of how this calculation can be done with Python and Sympy.

a) The transformations are

$$\begin{cases} Q = \log\left(1 + \sqrt{q}\cos(p)\right) \\ P = 2(10\sqrt{q}\cos(p)\sqrt{q} + sin(p)). \end{cases}$$

The Jacobian $M_{ij} = \partial \xi_i / \partial \eta_j$

$$M = \begin{pmatrix} \frac{\cos(p)}{2\left(\sqrt{q}+q\cos(p)\right)} & -\frac{\sqrt{q}\sin(p)}{\sqrt{q}\cos(p)+1} \\ \sin(2p) + \frac{\sin(p)}{\sqrt{q}} & 2\sqrt{q}\cos(p) - 4q\sin^2(p) + 2q \end{pmatrix}$$

The symplectic condition is then

$$MJM^T = J,$$

where

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

As shown in the notebook, this condition is met by the transformation in question, which means it is a canonical transformation. If q, p are canonical coordinates, then so are Q, P.

b) Type 3 generating functions obey

$$q = -\frac{\partial F_3(p, Q, t)}{\partial p}, \quad P = -\frac{\partial F_3(p, Q, t)}{\partial Q}.$$

We have

$$F_3(p, Q, t) = -(e^Q - 1)^2 \tan(p).$$

This means

$$\begin{cases} q = -\dfrac{\partial F_3}{\partial p} = (e^Q - 1)^2 \dfrac{1}{\cos^2(p)} \implies Q = \log(1 + \sqrt{q}\cos(p)) \\ P = -\dfrac{\partial F_3}{\partial Q} = 2e^Q(e^Q - 1)\tan(p). \end{cases}$$

Inserting the solution for Q into P gives

$$\begin{aligned} P &= 2\exp\left[\log(1+\sqrt{q}\cos(p))\right]\Big(\exp\left[\log(1+\sqrt{q}\cos(p))\right] - 1\Big)\tan(p) \\ &= 2(1+\sqrt{q}\cos(p))\Big(1+\sqrt{q}\cos(p) - 1\Big)\tan(p) \\ &= 2\sqrt{q}(1+\sqrt{q})\sin(q), \end{aligned}$$

which is the transformation we were looking for.

4 Free particle