Exercise 3 solutions - TFY4345 Classical Mechanics

2020

1 Pendulum on spinning a wheel

With the origin of our coordinate system in the center of the rotating rim, the Cartesian components of the mass m become

$$\begin{aligned} x &= a\cos(\omega t) + b\sin(\theta) \\ y &= a\sin(\omega t) - b\cos(\theta). \end{aligned}$$

The velocities are

$$\begin{aligned} \dot{x} &= -a\omega\sin(\omega t) + b\dot{\theta}\cos(\theta) \\ \dot{y} &= a\omega\cos(\omega t) + b\dot{\theta}\sin(\theta). \end{aligned}$$

Taking the time derivative once again gives the acceleration:

$$\begin{aligned} \ddot{x} &= -a\omega^2\cos(\omega t) + b(\ddot{\theta}\cos(\theta) - \dot{\theta}^2\sin(\theta)) \\ \ddot{y} &= -a\omega^2\sin(\omega t) + b(\ddot{\theta}\sin(\theta) + \dot{\theta}^2\cos(\theta)). \end{aligned}$$

It should be clear that the single generalize coordinate is θ. The kinetic and potential energies are

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2),\ V = mgy.$$

Inserting what we found earlier, the Lagrangian becomes

$$L = \frac{1}{2}m[a^2\omega^2 + b\dot{\theta}^2 + 2b\theta^2 a\omega\sin(\theta - \omega t)] - mg[a\sin(\omega t) - b\cos(\theta)].$$

The derivatives needed for the equation of motion are

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{\theta}} &= mb^2\ddot{\theta} + mba\omega(\dot{\theta} - \omega)\cos(\theta - \omega t), \\ \frac{\partial L}{\partial\theta} &= mba\dot{\theta}\omega\cos(\theta - \omega t) - mgb\sin(\theta). \end{aligned}$$

Inserting this into Euler's equation, and solving for $\ddot{\theta}$ gives

$$\ddot{\theta} = \frac{\omega^2 a}{b}\cos(\theta - \omega t) - \frac{g}{b}\sin(\theta).$$

Notice that, for $\omega = 0$, this reduces to the equation for the simple pendulum.

2 Bead on a ring

(FIGUR)

The potential energy is given by

$$U = mgh = mgR(1 - \cos(\theta)),$$

while the kinteic energy is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left((r\dot{\varphi})^2 + (R\dot{\theta})\right) = \frac{1}{2}mR^2\left(\sin^2(\theta)\dot{\varphi}^2 + \dot{\theta}^2\right).$$

The Euler equation for φ is given by

$$\frac{\partial L}{\partial \varphi} = 0 \implies \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\varphi}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(m\sin(\theta)R^2\dot{\varphi}\right) = 0 \implies \dot{\varphi} = \omega = \text{const.}$$

The equation for θ is given by

$$\begin{aligned}
&\frac{\partial L}{\partial \theta} = mR^2\cos(\theta)\sin(\theta)\dot{\varphi}^2 - mgR\sin(\theta),\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} = mR^2\ddot{\theta}, \\
&\implies \ddot{\theta} = R\cos(\theta)\sin(\theta)\dot{\varphi}^2 - g\sin(\theta)
\end{aligned}$$

In the equilibrium position, we have that $\ddot{\theta} = 0$. This means

$$\cos(\theta) = \frac{g}{R\dot{\varphi}^2} = \frac{g}{R\omega^2}.$$

Note: we could have set $\dot{\varphi} = 0$ right at the beginning, and treat θ as the sole generalized variable.

3 Atwood's machine

The angular velocity of the pulley is

$$\omega = \frac{\dot{x}_2}{a}.$$

This means the kinetic energy of the system is

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}I\omega^2$$

The length of the string is a constant, so

$$x_1 + x_2 = \ell = \text{const.}$$

Inserting this into the kinetic energy gives

$$T = \frac{1}{2}(m_2 - m_1)\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}I\left(\frac{\dot{x}}{a}\right)^2.$$

The potential energy is

$$V = -m_1 g x_1 - m_2 g x_2 = -m_1 g(\ell - x_2) - m_2 g(x_2),$$

so the Lagrangian is

$$L = \frac{1}{2}\left(m_1 + m_2 + \frac{I}{a^2}\right)\dot{x}_2^2 + m_1 g(\ell - x_2) + m_2 g x_2.$$

The canonical momentum is

$$p_2 = \frac{\partial L}{\partial \dot{x}^2} = \left(m_1 + m_2 + \frac{I}{a^2}\right)\dot{x}_2$$

The conditions are met so that we can write the Hamiltonian as

$$H = T + v = \frac{1}{2}\frac{p_2^2}{m_1 + m_2 + I/a^2} - m_1 g(\ell - x_2) - m_2 g x_2.$$

Hamilton's equations then become

$$\dot{x}_2 = \frac{\partial H}{\partial p_2} = \frac{p_2}{m_1 + m_2 + I/a^2}$$
$$\dot{p}_2 = -\frac{\partial H}{\partial x_2} = (m_2 - m_1)g.$$

Lastly, H is conserved as

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -\frac{\partial L}{\partial t} = 0.$$

4 Particle on a moving wedge

(FIGUR)

The position of the mass m in a coordinate system moving with the wedge is

$$\mathbf{r}' = r\cos(\theta)\hat{e}_x + r\sin(\theta)\hat{e}_y.$$

In the laboratory coordinates, which does not move with the wedge, the wedge has position $x\hat{e}_x$, so the position of the mass m is

$$\mathbf{r} = (x + r\cos(\theta))\hat{e}_x + \sin(\theta)\hat{e}_y.$$

The velocity is

$$\dot{\mathbf{r}} = (\dot{x} + \dot{r}\sin(\theta) - r\dot{\theta}\sin(\theta))\hat{e}_x + (\dot{r}\sin(\theta) + r\dot{\theta}\cos(\theta))\hat{e}_y,$$

while the square velocity becomes

$$\begin{aligned}
\dot{r}^2 = &\left(\dot{x} + \dot{r}\cos(\theta) - r\dot{\theta}\sin(\theta)\right)^2 + \left(\dot{r}\sin(\theta) + r\dot{\theta}\cos(\theta)\right)^2 = \\
&\dot{x}^2 + 2\dot{x}\left(\dot{r}\cos(\theta) - r\dot{\theta}\sin(\theta)\right) + \left(\dot{r}\cos(\theta) - r\dot{\theta}\sin(\theta)\right)^2 + (\dot{r}\cos(\theta))^2 + 2r\dot{r}\dot{\theta}\cos(\theta)\sin(\theta) + (r\dot{\theta}\sin(\theta))^2 \\
&= \dot{x}^2 + \underline{(\dot{r}\cos(\theta))^2} + \underline{(r\dot{\theta}\sin(\theta))^2} + 2\dot{x}\dot{r}\cos(\theta) - 2\dot{x}r\dot{\theta}\sin(\theta) - \cancel{2r\dot{r}\dot{\theta}\cos(\theta)\sin(\theta)} \\
&+ \underline{(\dot{r}\sin(\theta))^2} + \cancel{2r\dot{r}\dot{\theta}\cos(\theta)\sin(\theta)} + \underline{(r\dot{\theta}\cos(\theta))^2} \\
&= \dot{x}^2 + \dot{r}^2 + (r\dot{\theta})^2 + 2\dot{x}\dot{r}\cos(\theta) - 2\dot{x}r\dot{\theta}\sin(\theta).
\end{aligned}$$

The potential energy is given by

$$V = -mgr\sin(\theta).$$

The restriction of the little mass to stay on the wedge is given by $r - R = 0$, so the total Lagrangian, including the undetermined multiplier becomes

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{r}^2 + (r\dot{\theta})^2 + 2\dot{x}\dot{r}\cos(\theta) - 2\dot{x}r\dot{\theta}\sin(\theta)\right) + \frac{1}{2}M\dot{x}^2 + mgr^2\sin(\theta) + \lambda(r - R).$$