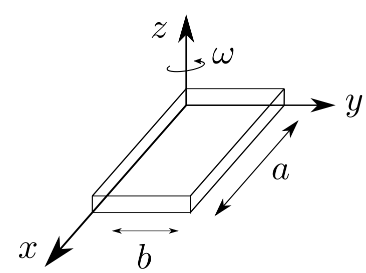
Exercise 7 - TFY4345 Classical Mechanics

2020

1 Inertia tensor

A very thin uniform rectangular slab of mass M has been placed in the xyz-coordinates system, such that the origin is in one of the slab corners, and the sides are along the x- and y-axes. The corresponding side lengths are a and b. Since the slab is very thin, we can assume that $z = 0$ throughout.

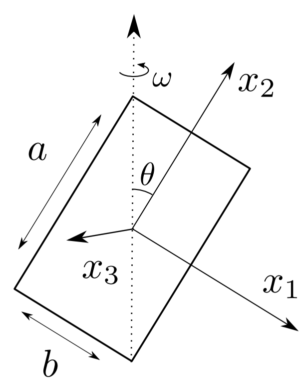


(a) Evaluate the individual elements of the inertia tensor.

(b) Set $a = b$, and solve for the principal moments of inertia, and the corresponding principal axes.

2 Rotated tilted slab

(Exam Aug. 2018)
Consider the same thin slab. However, now it rotates around an axis parallel to its diagonal, with an angular velocity vector $\boldsymbol{\omega}$. As the axis of rotation has change, so has its principal axis. The principal axes x_1 and x_2 are parallel to the slab edges, as indicated in the figure, while the axis x_3 is perpendicular to the slab, and goes through its center. The side lengths of the slab are a and b, with the principal moments of inertia



$$I = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} = M \begin{pmatrix} \frac{1}{12}a^2 & 0 & 0 \\ 0 & \frac{1}{12}b^2 & 0 \\ 0 & 0 & \frac{1}{12}(a^2+b^2) \end{pmatrix}$$

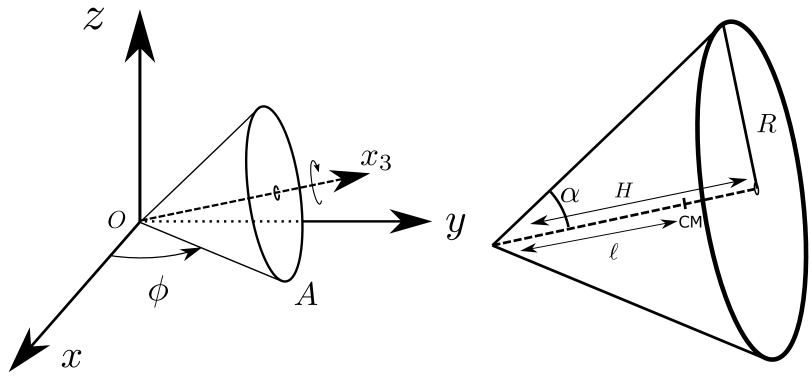
(a) Derive the angular momentum vector $\mathbf{L}$ of the slab, in terms of a, b, M and ω.

(b) What is the angle α between $\mathbf{L}$ and $\boldsymbol{\omega}$?

(c) What is the rotational kinetic energy T?

3 Cone rolling on a plane

(Exam Dec. 2016)
We shall consider the motion of a solid cone that is rolling on the surface (xy-plane), without slipping. The center of mass of the cone is situated on the symmetry axis x_3, which goes through the center of the bottom of the cone. It is at a distance ℓ from the origin O. The height of the cone is H, and the radius of the bottom of the cone is R, and the cone half angle is α. These are related by $R\cos(\alpha) = \sin(\alpha)H$. The momentary line of contact between the cone and the xy-plane is the line OA, which is at an angle ϕ relative to the x-axis.



(a) Calculate the velocity of the center of mass, V_{CM} as a function of ℓ, α and $\frac{\mathrm{d}}{\mathrm{d}t}\phi = \dot{\phi}$.

(b) Explain why the angular velocity vector $\boldsymbol{\omega}$ of the rolling cone is directed along the line OA, the line of contact between the cone and the xy-plane. Show that

$$\omega = |\boldsymbol{\omega}| = \frac{1}{\tan(\alpha)}\dot{\phi}$$

(c) x_3 is one of the principal axis of rotations, what are the other? Remember that the coordinate transformation that takes xyz into the principal axis $x_1x_2x_3$ is a rotation. Due to the symmetry of the cone, we have some freedom in choosing the other two axis. Choose x_1 to be in the plane spanned by $\boldsymbol{\omega}$ and x_3, and find the components of $\boldsymbol{\omega}$ along the principal axes.

(d) The principal moment of inertia of the cone, for rotation around the point O is

$$I = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} = \frac{3}{20}M \begin{pmatrix} R^2 + 4H^2 & 0 & 0 \\ 0 & R^2 + 4H^2 & 0 \\ 0 & 0 & 2R^2 \end{pmatrix},$$

where M is the mass of the cone. Calculate the kinetic energy of the cone as a function of M, H, α and $\dot{\phi}$.