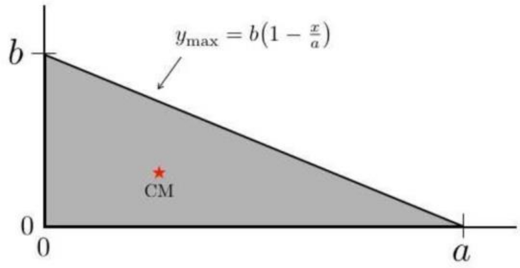
Classical Mechanics TFY 4345 – Exercise 8

1. Principal moments of inertia of a triangular slab. [Exam Aug. 2019]



(a) Compute the center-of-mass (COM) for the planar triangle in the figure right, assuming it to be of uniform two-dimensional mass density ρ.

(b) Compute the inertia tensor *with respect to the origin* for the same triangle.

(c) [Optional] If the origin is shifted in the COM, the inertia tensor becomes (this can be shown by using the Steiner's parallel axis theorem)

$$I^{COM} = \frac{M}{18}\begin{pmatrix} b^2 & \frac{1}{2}ab & 0 \\ \frac{1}{2}ab & a^2 & 0 \\ 0 & 0 & a^2+b^2 \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & 0 \\ I_{yx} & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}$$

where $I_{xy} = I_{yx}$ and $I_{zz} = I_{xx} + I_{yy}$ in the general (latter) form. Define next

$$A = \frac{1}{2}(I_{xx} + I_{yy})$$

$$B = \sqrt{\frac{1}{4}(I_{xx} - I_{yy})^2 + I_{xy}{}^2}$$

$$\vartheta = \tan^{-1}\left(\frac{2I_{xy}}{I_{xx} - I_{yy}}\right)$$

Derive the principal moments of inertia and principal axes of inertia by using the general form of the inertia tensor and these new variables.

Hint: The last equations comprises an inter-relationship that can be described by a right triangle.

2. Precession of a frisbee. [Exam Aug. 2016]

a) Consider an axial-symmetric body with moments of inertia: $I_1 = I_2 \neq I_3$. The angular momentum in the laboratory frame is $\overline{L} = L\hat{e}_z$. Derive the equations of motion for the body, using the Euler equations and the angles θ, ψ, φ. Define here also the components of $\overline{\omega}$. [see lecture notes, we derived this already!]

b) Find the expression for the Euler angles θ, ψ, φ as a function of time.

c) For a Frisbee $I_1 = I_2$ and $I_3 = 2I_1$. The precession (wobble) of the frisbee is given by $\dot{\varphi}$. Show that the precession of the frisbee is twice as fast as the rotation frequency of the frisbee, assuming that the angle θ is small ($i.e.$ $\cos(\theta) \approx 1$).

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3. Precession of a heavy spinning top. [See the lecture notes, course book]

The shifted total energy is a constant of motion and has the expression

$$E' = \frac{1}{2} I_1 \dot{\theta}^2 + V(\theta)$$

where the potential is

$$V(\theta) = \frac{(p_\phi - p_\psi cos\theta)^2}{2I_1 \sin^2\theta} + Mgh\cos\theta.$$

Consider the shape of the effective potential for the equilibrium precession inclination angle θ_0. *What is the condition in this case?* The following change of variables will become handy for the result:

$$\beta = p_\phi - p_\psi cos\theta_0$$

You will encounter a quadratic equation for β. Correspondingly, show that for the equilibrium precession inclination angle θ_0 the following must hold true:

$$\omega_3 \geq \frac{2}{I_3}\sqrt{MghI_1\cos\theta_0}$$

What can you say about the corresponding precession angular velocity $\dot{\phi}_0$?