Classical Mechanics TFY4345 – Solution set 3

1. Pendulum attached on a rotating rim

Solution: We choose the origin of our coordinate system to be at the center of the rotating rim. The Cartesian components of mass m become

$$\left.\begin{aligned} x &= a\cos\omega t + b\sin\theta \\ y &= a\sin\omega t - b\cos\theta \end{aligned}\right\} \qquad \textbf{(7.32)}$$

The velocities are

$$\left.\begin{aligned} \dot{x} &= -a\omega\sin\omega t + b\dot{\theta}\cos\theta \\ \dot{y} &= a\omega\cos\omega t + b\dot{\theta}\sin\theta \end{aligned}\right\} \qquad \textbf{(7.33)}$$

Taking the time derivative once again gives the acceleration:

$$\ddot{x} = -a\omega^2\cos\omega t + b(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)$$

$$\ddot{y} = -a\omega^2\sin\omega t + b(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)$$

It should now be clear that the single generalized coordinate is θ. The kinetic and potential energies are

$$T = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$U = mgy$$

where $U = 0$ at $y = 0$. The Lagrangian is

$$L = T - U = \frac{m}{2}[a^2\omega^2 + b^2\dot{\theta}^2 + 2b\dot{\theta}a\omega\sin(\theta - \omega t)]$$

$$-mg(\mathrm{a}\sin\omega t - b\cos\theta) \qquad \textbf{(7.34)}$$

The derivatives for the Lagrange equation of motion for θ are

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = mb^2\ddot{\theta} + mba\omega(\dot{\theta} - \omega)\cos(\theta - \omega t)$$

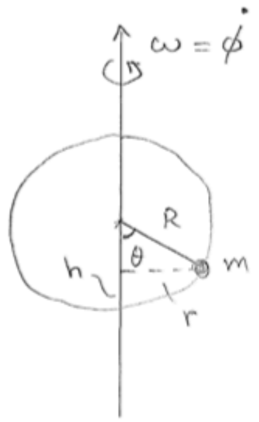
$$\frac{\partial L}{\partial\theta} = mb\dot{\theta}a\omega\cos(\theta - \omega t) - mgb\sin\theta$$

which results in the equation of motion (after solving for $\ddot{\theta}$)

$$\ddot{\theta} = \frac{\omega^2 a}{b}\cos(\theta - \omega t) - \frac{g}{b}\sin\theta \qquad \textbf{(7.35)}$$

Notice that this result reduces to the well-known equation of motion for a simple pendulum if $\omega = 0$.

2. Moving bead attached in a rotating ring



$$U = mgh = mgR(1-\cos\theta)$$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(r\frac{d\phi}{dt}\right)^2 + \frac{1}{2}m\left(R\frac{d\theta}{dt}\right)^2$$

$$= \frac{1}{2}mR^2\sin^2\theta\,(\dot{\phi})^2 + \frac{1}{2}mR^2\dot{\theta}^2$$

contant ω

$$L = T(\dot{\phi},\dot{\theta}) - U(\theta)$$

$$\frac{\partial L}{\partial \phi} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = -\frac{\partial}{\partial t}\left(\frac{\partial}{\partial \dot{\phi}}\left(\frac{1}{2}mR^2\sin^2\theta\,\dot{\phi}^2\right)\right) \equiv 0$$

$$= -mR^2\sin^2\theta\,\ddot{\phi} = 0$$

$$\Longleftrightarrow \ddot{\phi} = 0 \Longleftrightarrow \dot{\phi} = \omega \equiv \text{constant}$$

(initial condition)

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0$$

$$\cancel{m}R^2\sin\theta\cos\theta\,(\dot{\phi})^2 - \cancel{m}gR\sin\theta - \cancel{m}R^2\ddot{\theta} = 0$$

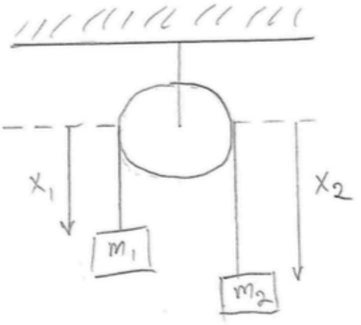
Equilibrium situation: $\theta = \text{constant} \Longleftrightarrow \ddot{\theta} = 0$

$$\Longrightarrow R^{\cancel{2}}\cancel{\sin\theta}\cos\theta\,(\dot{\phi})^2 = g\cancel{R}\,\cancel{\sin\theta}$$

$$\Longrightarrow \cos\theta \equiv \underline{\underline{\cos\theta_0}} = \frac{g}{R(\dot{\phi})^2} = \underline{\underline{\frac{g}{R\omega^2}}}$$

<u>Note:</u> We could have set $\dot{\phi} = \omega$ right in the beginning and treat θ as the sole generalized variable.

3. Moving bead attached in a rotating ring



radius a, moment of inertia I

$$x_1 + x_2 = l = \text{constant}$$

$$\omega = \frac{\dot{x}_2}{a} \quad \left(= -\frac{\dot{x}_1}{a} \right)$$

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} I \omega^2$$

$$= \frac{1}{2} m_1 (-\dot{x}_2)^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} I \left(\frac{\dot{x}_2}{a} \right)^2$$

$$U = -m_1 g x_1 - m_2 g x_2 = -m_1 g (l - x_2) - m_2 g x_2$$

Now: $L = T - U = \frac{1}{2} \left(m_1 + m_2 + \frac{I}{a^2} \right) \dot{x}_2^2 + m_1 g (l - x_2) + m_2 g x_2$

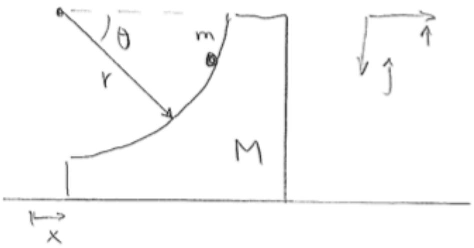
$$p_{x_2} = \frac{\partial L}{\partial \dot{x}_2} = \left(m_1 + m_2 + \frac{I}{a^2} \right) \dot{x}_2$$

$$\Rightarrow \quad H = T + U = \frac{p_{x_2}^2}{2\left(m_1 + m_2 + \frac{I}{a^2} \right)} - m_1 g (l - x_2) - m_2 g x_2$$

Hamiton's equations:

$$\begin{cases} \dot{x}_2 = \dfrac{\partial H}{\partial p_{x_2}} = \dfrac{p_{x_2}}{m_1 + m_2 + \frac{I}{a^2}} \\ \dot{p}_{x_2} = -\dfrac{\partial H}{\partial x_2} = (m_2 - m_1) g \\ \dfrac{dH}{dt} = -\dfrac{\partial L}{\partial t} = 0 \qquad H \text{ conserved} \end{cases}$$

4. Gliding particle on a moving support wedge



Support frame of reference:

$$\bar r' = r\cos\theta\,\hat\imath + r\sin\theta\,\hat\jmath$$

Laboratory coordinates:

$$\bar r = x\hat\imath + r\cos\theta\,\hat\imath + r\sin\theta\,\hat\jmath$$

$$\bar v = \dot x\hat\imath + \dot r\cos\theta\,\hat\imath - r\sin\theta\cdot\dot\theta\,\hat\imath + \dot r\sin\theta\,\hat\jmath + r\cos\theta\cdot\dot\theta\,\hat\jmath$$

$$U = -mgr\sin\theta$$

$$g(x, r, \lambda) = r - R = 0$$

$$L = \tfrac{1}{2}m\left[(\dot x + \dot r\cos\theta - r\dot\theta\sin\theta)^2 + (\dot r\sin\theta + r\dot\theta\cos\theta)^2\right] + \tfrac{1}{2}M\dot x^2 + mgr\sin\theta$$

$$= \tfrac{1}{2}m\left[\dot x^2 + \dot r^2\cos^2\theta + r^2\dot\theta^2\sin^2\theta + 2\dot x\dot r\cos\theta - 2\dot x r\dot\theta\sin\theta - 2\dot r\cos r\dot\theta\sin\theta + \dot r^2\sin^2\theta + r^2\dot\theta^2\cos^2\theta + 2\dot r\sin\theta\, r\dot\theta\cos\theta\right] + \tfrac{1}{2}M\dot x^2 + mgr\sin\theta$$

$$= \tfrac{1}{2}m\left(\dot x^2 + \dot r^2 + r^2\dot\theta^2 + 2\dot x\dot r\cos\theta - 2\dot x r\dot\theta\sin\theta\right) + \tfrac{1}{2}m\dot x^2 + mgr\sin\theta$$

$$\begin{cases} x: & -\frac{d}{dt}\left[m\dot x + m\dot r\cos\theta - mr\dot\theta\sin\theta + M\dot x\right] + \lambda\cdot\overbrace{0}^{\frac{\partial g}{\partial x}} = 0 \\ \theta: & -m\dot x\dot r\sin\theta - m\dot x r\dot\theta\cos\theta + mgr\cos\theta - \frac{d}{dt}\left[mr^2\dot\theta - m\dot x r\sin\theta\right] + \lambda\cdot 0 = 0 \\ r: & mr\dot\theta^2 - m\dot x\dot\theta\sin\theta + mg\sin\theta - \frac{d}{dt}\left[m\dot r + m\dot x\cos\theta\right] + \lambda\cdot 1 = 0 \end{cases}$$

$$\begin{cases} -m\ddot x - m\ddot r\cos\theta + m\dot r\dot\theta\sin\theta + m\dot r\dot\theta\sin\theta + mr\ddot\theta\sin\theta + mr\dot\theta^2\cos\theta - M\ddot x = 0 \\ -m\dot x\dot r\sin\theta - m\dot x r\dot\theta\cos\theta + mgr\cos\theta - 2mr\dot r\dot\theta - mr^2\ddot\theta + m\ddot x r\sin\theta + m\dot x\dot r\sin\theta + m\dot x r\dot\theta\cos\theta = 0 \\ mr\dot\theta^2 - m\cancel{\dot x\dot\theta}\sin\theta - m\ddot r - m\ddot x\cos\theta + m\cancel{\dot x\dot\theta}\sin\theta + \lambda = 0 \end{cases}$$

Now $\dot{r} = \ddot{r} = 0$

$$\begin{cases} -m\ddot{x} + mR\ddot{\theta}\sin\theta + mR\dot{\theta}^2\cos\theta - M\ddot{x} = 0 & (1) \\ \cancel{-m\dot{x}R\dot{\theta}\cos\theta} + \cancel{m}gR\cos\theta - \cancel{m}R^2\ddot{\theta} + \cancel{m}\ddot{x}R\sin\theta + \cancel{m\dot{x}R\dot{\theta}\cos\theta} = 0 & (2) \\ \cancel{m}R\dot{\theta}^2 + \cancel{m}g\sin\theta - \cancel{m}\ddot{x}\cos\theta + \lambda = 0 & (3) \end{cases}$$

$$(1) \Rightarrow \ddot{x}(m+M) = m(R\ddot{\theta}\sin\theta + R\dot{\theta}^2\cos\theta)$$

$$\Rightarrow \ddot{x} = \frac{m}{m+M} R\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right)$$

$$(2) \Rightarrow \ddot{\theta} = \frac{g\cos\theta + \ddot{x}\sin\theta}{R}$$

$$(3) \Rightarrow \lambda = m\ddot{x}\cos\theta - mg\sin\theta - mR\dot{\theta}^2$$

Let us assume: $t=0 \Rightarrow x=0,\ \theta=0$

$$mR\dot{\theta}^2 = \left[(m+M)\ddot{x} - mR\ddot{\theta}\sin\theta\right]\frac{1}{\cos\theta} \qquad (\Leftarrow (1))$$

$$\lambda = m\ddot{x}\cos\theta - mg\sin\theta + \frac{m+M}{\cos\theta}\ddot{x} + \frac{mR\ddot{\theta}\sin\theta}{\cos\theta}$$

$$= \ddot{x}\left[m\cos\theta - \frac{m+M}{\cos\theta}\right] - mg\sin\theta + \frac{mR\ddot{\theta}\sin\theta}{\cos\theta}$$

$$= \frac{m}{m+M}R\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right)\left[m\cos\theta - \frac{m+M}{\cos\theta}\right] - mg\sin\theta + \frac{mR\ddot{\theta}\sin\theta}{\cos\theta}$$

Force of constraint equals to the normal force of M on m.
Note that M moves (accelerates).