Exercise 4 solutions - TFY4345 Classical Mechanics

2020

1 Mathematical pendulum

The position of the mass is

$$x = \ell\sin(\theta), \quad y = \frac{1}{2}at^2 - \ell\cos(\theta)$$
$$\dot{x} = \ell\dot{\theta}\cos(\theta) \quad y = at + \ell\dot{\theta}\sin(\theta),$$

so the kinetic energy is given by

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2}m\left((\ell\dot{\theta})^2 + (at)^2 + 2at\ell\dot{\theta}\sin(\theta)\right),$$

and the potential energy is

$$V = mgy = mg\left(\frac{1}{2}at^2 - \ell\cos(\theta)\right).$$

The Lagrangian is

$$L = \frac{1}{2}m\left((\ell\dot{\theta})^2 + (at)^2 + 2at\ell\dot{\theta}\sin(\theta)\right) - mg\left(\frac{1}{2}at^2 - \ell\cos(\theta)\right),$$

so the canonical momentum is

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = m\left(\ell^2\dot{\theta} + at\ell\sin(\theta)\right) \implies \dot{\theta} = \frac{p_\theta - mta\ell\sin(\theta)}{m\ell^2}.$$

This gives the Hamiltonian

$$\begin{aligned} H &= \dot{\theta}p_\theta - L \\ &= p_\theta\frac{p_\theta - mta\ell\sin(\theta)}{m\ell^2} - \frac{1}{2}m\left[\ell^2\left(\frac{p_\theta - mta\ell\sin(\theta)}{m\ell^2}\right)^2 + (\ell at)^2 + 2at\ell\sin(\theta)\left(\frac{p_\theta - mta\ell\sin(\theta)}{m\ell^2}\right)\right] \\ &+ mg\left(\frac{1}{2}at^2 - \ell\cos(\theta)\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{m\ell^2}\left(p_\theta^2 - p_\theta mta\ell\sin(\theta)\right) - \\
&\frac{1}{2m\ell^2}\left[p_\theta^2 - 2p_\theta mta\ell\sin(\theta) + (mta\ell\sin(\theta))^2 + (m\ell ta)^2 + 2p_\theta mat\ell\sin(\theta) - 2(mta\ell\sin(\theta))^2\right] \\
&+ mg\left(\frac{1}{2}at^2 - \ell\cos(\theta)\right) \\
&= \frac{1}{2m\ell^2}\left(p_\theta - mta\ell\sin(\theta)\right)^2 - \frac{1}{2}ma^2t^2 + \frac{1}{2}magt^2 - mg\ell\cos(\theta).
\end{aligned}
$$

The Hamiltonian equations of motion

$$
\begin{aligned}
\dot{\theta} &= \frac{\partial H}{\partial p_\theta} = \frac{p_\theta - mta\ell\sin(\theta)}{m\ell^2} \\
\dot{p}_\theta &= -\frac{\partial H}{\partial \theta} = \frac{at\cos(\theta)}{\ell}\left[p_\theta - mat\ell\sin(\theta)\right] - mg\ell\sin(\theta).
\end{aligned}
$$

Furthermore, we see that $H \neq T + V$, so the Hamiltonian function is not the total energy of the system. Furthermore,

$$
\frac{\mathrm{d}H}{\mathrm{d}t} = -\frac{\partial L}{\partial t} \implies \frac{\mathrm{d}H}{\mathrm{d}t} \neq 0,
$$

as the Lagrangian has an explicit time dependence. The pendulum is in an accelerating motion with the respect to the inertial frame of reference. This mean that H will not be conserved.

2 Spherically symmetrical potential

3 Earth's orbit

4 Einsteins correction