Exercise 9 solutions - TFY4345 Classical Mechanics

2020

Notational Note: There is some different notation for the matrices in the lagrangian for small coupled oscillations. The compendium uses $\hat{V}$, $\hat{T}$, while the lecture notes uses $\bar{\bar{A}}$, $\bar{\bar{m}}$, I stick to the latter here. The components of the matrices are denoted A_{ij}, m_{ij}.

1 Coupled pendula

The kinetic energy of the masses are

$$T = \frac{1}{2}m\left[(b\dot{\theta}_1)^2 + (b\dot{\theta}_2)^2\right].$$

The displacement of the masses in the vertical direction is given by $b(1-\cos(\theta))$. As we are considering small oscillations, we only care about the stretching of the spring due to the horizontal movement of the pendula. This is given by $b(\sin(\theta_1) - \sin(\theta_2))$. Thus, the potential energy of the system is

$$V = mgb\left[(1-\cos(\theta_1)) + (1-\cos(\theta_2))\right] + \frac{1}{2}kb^2\left[\sin(\theta_1) - \sin(\theta_2)\right]^2.$$

Using the small angle approximation, we get $\sin(\theta) \approx \theta$, $1-\cos(\theta) \approx \frac{1}{2}\theta^2$, so the potential energy becomes

$$\frac{1}{2}mgb(\theta_1^2 + \theta_2^2) - \frac{1}{2}kb^2(\theta_1 - \theta_2) = \frac{1}{2}\left((mgb + kb^2)(\theta_1^2 + \theta_2^2) - 2kb^2\theta_1\theta_2\right)$$

The lagrangian is then

$$L = \frac{1}{2}m\left[(b\dot{\theta}_1)^2 + (b\dot{\theta}_2)^2\right] + \frac{1}{2}(mgb + kb^2)(\theta_1^2 + \theta_2^2) - \frac{1}{2}kb^2(\theta_1 - \theta_2) = \frac{1}{2}\left(m_{ij}\theta_i\theta_i + A_{ij}\theta_i\theta_j\right),$$

where

$$\bar{\bar{m}} = \begin{pmatrix} mb^2 & 0 \\ 0 & mb^2 \end{pmatrix}, \quad \bar{\bar{A}} = \begin{pmatrix} mgb + kb^2 & -kb^2 \\ -kb^2 & mgb + kb^2 \end{pmatrix}.$$

The eigenfrequencies of the system are then given by the equation

$$\det\left(\bar{\bar{A}} - \omega^2\bar{\bar{m}}\right) = 0.$$

Writing out the determinant, we get

$$\begin{vmatrix} mgb + kb^2 - \omega^2 mb^2 & -kb^2 \\ kb^2 & mgb + kb^2 - \omega^2 mb^2 \end{vmatrix} = (mgb + kb^2 - \omega^2 mb^2)^2 - (mb^2)^2 = 0,$$

or

$$mgb + kb^2 - \omega^2 mb^2 = \pm mb^2 \implies \omega^2 = \frac{g}{b} + (1 \mp 1)\frac{k}{m}.$$

This leaves us with the eigenfrequencies

$$\omega_1^2 = \frac{g}{b}, \quad \omega_2^2 = \frac{g}{b} + 2\frac{k}{m}$$

The equation for the eigenfrequencies $\mathbf{a_i} = (a_{1i}, a_{2i})$, corresponding to ω_i^2, is

$$(\bar{\bar{A}} - \omega_r^2 \bar{\bar{m}})\mathbf{a_r} = (mgb + kb^2 - \omega_r^2 mb^2)a_{1r} - (kb^2)a_{2r} = 0$$

Inserting ω_1^2, this gives

$$\left(mgb + kb^2 - \frac{g}{b}mb^2\right)a_{11} - (kb^2)a_{21} = (kb^2)a_{11} - (kb^2)a_{21} = 0 \implies a_{11} = a_{21}.$$

ω_2^2 gives

$$\left(mgb + kb^2 - \left(\frac{g}{b} + 2\frac{k}{m}\right)mb^2\right)a_{12} - (kb^2)a_{22} = -(kb^2)a_{12} - (kb^2)a_{22} = 0 \implies a_{12} = -a_{22}.$$

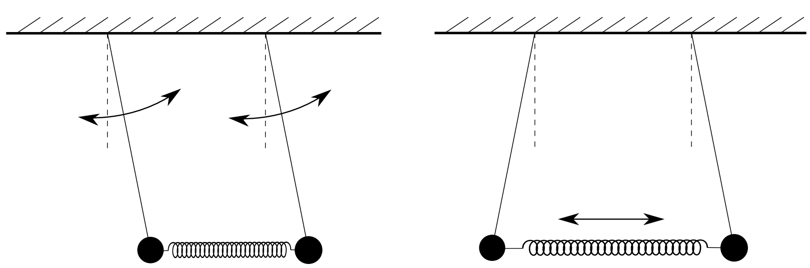
The solution, in our original coordinates θ_i, in terms of the normal coordinates η_r, is then

$$\begin{cases} \theta_1 = a_{11}\eta_1 + a_{12}\eta_2 = a_{11}\eta_1 + a_{22}\eta_2 \\ \theta_2 = a_{21}\eta_1 + a_{22}\eta_2 = a_{11}\eta_1 - a_{22}\eta_2) \end{cases}$$

To physically describe the normal coordinates, we need to express them in terms of our original coordinates. Adding the equations together, we get

$$\begin{cases} \eta_1 = \dfrac{1}{2a_{11}}(\theta_1 + \theta_2) \\ \eta_2 = \dfrac{1}{2a_{22}}(\theta_1 - \theta_2). \end{cases}$$

η_1 corresponds to the pendula oscillating in sync, while η_2 corresponds to them oscillating in opposite direction. This explains why the frequency ω_2 is higher than ω_1. The second mode stretches the spring, while the first mode leaves the distance between the pendula constant. This means the restoring force will be higher for the second mode, leading to a higher frequency.



2 Two coupled oscillators

Let x_1 and x_2 denote the distance of the two blocks from A. The kinetic and potential energy of the system is then

$$T = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2), \quad V = \frac{1}{2}k(x_1^2 + (x_1 - x_2)^2).$$

Writing out the lagrangian in matrix form gives

$$L = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) + \frac{1}{2}k(2x_1^2 - 2x_1x_2 + x_2^2) = \frac{1}{2}\left(m_{ij}\dot{x}_i\dot{x}_j + A_{ij}x_ix_j\right),$$

where

$$\bar{\bar{m}} = m\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \quad \bar{\bar{A}} = k\begin{pmatrix}2 & -1\\ -1 & 1\end{pmatrix}$$

If we introduce $\omega_0^2 = k/m$. This means the equation for the eigenfrequencies is

$$\det\left(A - \omega^2 m\right) = \begin{vmatrix}2\omega_0^2 - \omega & -\omega_0^2\\ -\omega_0^2 & \omega_0^2 - \omega^2\end{vmatrix} = (2\omega_0^2 - \omega^2)(\omega_0^2 - \omega^2) - \omega_0^4 = 2\omega_0^4 - 3\omega_0^2\omega^2 + \omega^4 - \omega_0^4$$

$$= (\omega^2)^2 - 3\omega_0^2(\omega^2) + \omega_0^4 = 0 \implies \omega^2 = \frac{1}{2}\left(3\omega_0^2 \pm \sqrt{(3\omega_0^2)^2 - 4\omega_0^4}\right) = \frac{3 \pm \sqrt{5}}{2}\omega_0^2.$$

The equations for the normal coordinates are

$$\begin{cases} a_{11} = \left(1 - \dfrac{3-\sqrt{5}}{2}\right)a_{21} = -\left(\dfrac{1+\sqrt{5}}{2}\right)a_{21} \\ a_{12} = \left(1 - \dfrac{3+\sqrt{5}}{2}\right)a_{22} = -\left(\dfrac{1+\sqrt{5}}{1+\sqrt{5}}\right)\left(\dfrac{1-\sqrt{5}}{2}\right)a_{22} = \left(\dfrac{2}{1+\sqrt{5}}\right)a_{22}. \end{cases}$$

The golden ratio! Here, we picked out one of two possible equations. But as they are linearly dependent (we have inserted ω^2 such that the determinant is zero), we could have chosen either one. In vector form,

$$\mathbf{a}_1 = C_1\begin{pmatrix}-1\\ \frac{2}{1+\sqrt{5}}\end{pmatrix} \quad \mathbf{a}_2 = C_2\begin{pmatrix}\frac{2}{1+\sqrt{5}}\\ 1\end{pmatrix}$$

These modes corresponds to vibration in the same and opposite direction. However, this system is not as symmetrical as the last one, so the amplitudes of the oscillations have different absolute values.

3 Oscillating body with two attached pendula

The potential energy of the block is $2\frac{1}{2}kx^2$, and using the same small oscillation approximation as in the first exercise the potential energy of both pendula is $\frac{1}{2}mg\ell\theta$. The total potential energy of the system is thus

$$V = kx^2 + \frac{1}{2}\ell(\theta_1^2 + \theta_2^2) = \frac{1}{2}\sum_{ij} A_{ij}q_iq_j,$$

where $(q_1, q_2, q_3) = (x, \theta_1, \theta_2)$ and

$$\bar{\bar{A}} = \begin{pmatrix} 2k & 0 & 0 \\ 0 & mgl & 0 \\ 0 & 0 & mgl \end{pmatrix}$$

When we find the kinetic energy, we must remember that the velocities of the pendula are not only due to the change in θ, but also dependent on how the box moves. The kinetic energy is

$$\begin{aligned} T &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x} + \ell\dot{\theta}_1)^2 + \frac{1}{2}m(\dot{x} + \ell\dot{\theta}_2)^2 \\ &= \frac{1}{2}(M + 2m)\dot{x}^2 + \frac{1}{2}m\ell\dot{x}\theta_1 + \frac{1}{2}m\ell\dot{x}\theta_2 + \frac{1}{2}m\ell^2\dot{\theta}_1 + \frac{1}{2}m\ell^2\dot{\theta}_2 = \frac{1}{2}\sum_{ij} m_{ij}q_iq_j \end{aligned}$$

in matrix form,

$$\bar{\bar{m}} = \begin{pmatrix} M + 2m & m\ell & m\ell \\ m\ell & m\ell^2 & 0 \\ m\ell & 0 & m\ell^2 \end{pmatrix}$$

The equation for the eigenfrequencies are

$$\begin{aligned} &\det\left(\bar{\bar{A}} - \omega^2\bar{\bar{m}}\right) = \begin{vmatrix} 2k - \omega^2(M + 2m) & -\omega^2 m\ell & -\omega^2 m\ell \\ -\omega^2 m\ell & mg\ell - \omega^2 m\ell^2 & 0 \\ -\omega^2 m\ell & 0 & mg\ell - \omega^2 m\ell^2 \end{vmatrix} \\ &= (mg\ell - \omega^2 m\ell^2)\Big[(mg\ell - \omega^2 m\ell^2)(2k + \omega^2(M + 2m)) - (\omega^2 m\ell)^2\Big] - (\omega^2 m\ell)^2\left(mg\ell - \omega^2 m\ell^2\right) \\ &= (mg\ell - \omega^2 m\ell^2)\left(2mg\ell k - \omega^2[2m\ell^2 k - (M + 2m)mg\ell] + \omega^4 Mm\ell^2\right) = 0. \end{aligned}$$

Thus, either $\omega^2 = \omega_1^2 = g/\ell$, or (after dividing by $Mm\ell^2$)

$$\begin{aligned} &\omega^4 - \left(\frac{2k}{M} + \frac{g}{\ell}\left[1 + 2\frac{m}{M}\right]\right)\omega^2 + 2\left(\frac{gk}{\ell M}\right) = 0 \\ \Longrightarrow\ &\omega^2 = \frac{k}{M} + \frac{1}{2}\frac{g}{\ell}\left[1 + 2\frac{m}{M}\right] \pm \sqrt{\left(\frac{k}{M} + \frac{1}{2}\frac{g}{\ell}\left[1 + 2\frac{m}{M}\right]\right)^2 - 2\frac{gk}{\ell M}}. \end{aligned}$$

4 Double pendulum

We have found the kinetic and potential energy of the double pendulum before,

$$T = m\ell^2\dot{\theta}_1^2 + m\ell^2\cos(\theta_1 - \theta_2)\dot{\theta}_1^2\dot{\theta}_2^2 + \frac{1}{2}m\ell^2\dot{\theta}_2^2, \quad V = -mg\ell\cos(\theta_1) - mg\ell\cos(\theta_2).$$

With the small angle approximation this becomes

$$T = \frac{1}{2}m\ell^2\left(2\dot{\theta}_1^2 + 2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2\right), \quad V = \frac{1}{2}mg\ell\left(2\theta_1^2 + \theta_2^2\right) + \text{const}.$$

In matrix form,

$$\bar{\bar{m}} = \begin{pmatrix} 2m\ell^2 & m\ell^2 \\ m\ell^2 & m\ell^2 \end{pmatrix} \quad \bar{\bar{A}} = \begin{pmatrix} 2mg\ell & 0 \\ 0 & mg\ell \end{pmatrix}$$

Defining $\omega_0 = \sqrt{g/\ell}$, the eigenfrequencies are given by

$$\det\left(\bar{\bar{A}} - \omega^2 \bar{\bar{m}}\right) = m\ell \begin{pmatrix} 2(\omega_0^2 - \omega^2) & -\omega^2 \\ -\omega^2 & \omega_0^2 - \omega^2 \end{pmatrix} = (2[\omega^2 - \omega_0^2]^2 - \omega^4) = 0$$

$$\implies \omega^4 - 4\omega_0^2\omega^2 + 2\omega_0^4 = 0 \implies \omega^2 = 2\omega^2 \pm \sqrt{4\omega_0^4 - 2\omega_0^2} = \left(2 \pm \sqrt{2}\right)\omega_0^2.$$

The eigenvectors are (choosing the equation from the lower row)

$$(-1 \mp \sqrt{2})a_{1i} - (2 \pm \sqrt{2})a_{2i} = 0 \implies a_{1i} = -\frac{1 \pm \sqrt{2}}{2 \pm \sqrt{2}} a_{2i} = -\frac{1}{\sqrt{2}} \frac{1 \pm \sqrt{2}}{\sqrt{2} \pm 1} a_{2i} = \mp \frac{1}{\sqrt{2}} a_{2i}$$

In vector form, they are

$$\mathbf{a}_1 = c_1 \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}, \quad \mathbf{a}_2 = c_2 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$$