

Exercise 3 solutions - TFY4345 Classical Mechanics

2020

1 Pendulum on spinning a wheel

With the origin of our coordinate system in the center of the rotating rim, the Cartesian components of the mass m become

$$\begin{aligned}x &= a \cos(\omega t) + b \sin(\theta) \\y &= a \sin(\omega t) - b \cos(\theta).\end{aligned}$$

The velocities are

$$\begin{aligned}\dot{x} &= -a\omega \sin(\omega t) + b\dot{\theta} \cos(\theta) \\ \dot{y} &= a\omega \cos(\omega t) + b\dot{\theta} \sin(\theta).\end{aligned}$$

Taking the time derivative once again gives the acceleration:

$$\begin{aligned}\ddot{x} &= -a\omega^2 \cos(\omega t) + b(\ddot{\theta} \cos(\theta) - \dot{\theta}^2 \sin(\theta)) \\ \ddot{y} &= -a\omega^2 \sin(\omega t) + b(\ddot{\theta} \sin(\theta) + \dot{\theta}^2 \cos(\theta)).\end{aligned}$$

It should be clear that the single generalize coordinate is θ . The kinetic and potential energies are

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2), \quad V = mgy.$$

Inserting what we found earlier, the Lagrangian becomes

$$L = \frac{1}{2}m[a^2\omega^2 + b\dot{\theta}^2 + 2b\dot{\theta}a\omega \sin(\theta - \omega t)] - mg[a \sin(\omega t) - b \cos(\theta)].$$

The derivatives needed for the equation of motion are

$$\begin{aligned}\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} &= mb^2\ddot{\theta} + mbaw(\dot{\theta} - \omega) \cos(\theta - \omega t), \\ \frac{\partial L}{\partial \theta} &= mba\dot{\theta}\omega \cos(\theta - \omega t) - mgb \sin(\theta).\end{aligned}$$

Inserting this into Euler's equation, and solving for $\ddot{\theta}$ gives

$$\ddot{\theta} = \frac{\omega^2 a}{b} \cos(\theta - \omega t) - \frac{g}{b} \sin(\theta).$$

Notice that, for $\omega = 0$, this reduces to the equation for the simple pendulum.

2 Bead on a ring

The potential energy is given by

$$U = mgh = mgR(1 - \cos(\theta)),$$

while the kinetic energy is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m \left((r\dot{\varphi})^2 + (R\dot{\theta})^2 \right) = \frac{1}{2}mR^2 \left(\sin^2(\theta)\dot{\varphi}^2 + \dot{\theta}^2 \right).$$

The Euler equation for φ is given by

$$\frac{\partial L}{\partial \varphi} = 0 \implies \frac{d}{dt} \frac{\partial L}{\partial \dot{\varphi}} = \frac{d}{dt} (m \sin(\theta) R^2 \dot{\varphi}) = 0$$

The shafts is driven with a constant speed, so $\dot{\varphi} = \omega = \text{const.}$ The equation for θ is given by

$$\begin{aligned} \frac{\partial L}{\partial \theta} &= mR^2 \cos(\theta) \sin(\theta) \dot{\varphi}^2 - mgR \sin(\theta), \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = mR^2 \ddot{\theta}, \\ \implies \ddot{\theta} &= \cos(\theta) \sin(\theta) \dot{\varphi}^2 - \frac{g}{R} \sin(\theta) \end{aligned}$$

In the equilibrium position, we have that $\ddot{\theta} = 0$. This means

$$\cos(\theta) = \frac{g}{R\dot{\varphi}^2} = \frac{g}{R\omega^2}.$$

3 Atwood's machine

The angular velocity of the pulley is

$$\omega = \frac{\dot{x}_2}{a}.$$

This means the kinetic energy of the system is

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}I\omega^2$$

The length of the string is a constant, so

$$x_1 + x_2 = \ell = \text{const.}$$

Inserting this into the kinetic energy gives

$$T = \frac{1}{2}(m_2 - m_1)\dot{x}_2^2 + \frac{1}{2}I \left(\frac{\dot{x}_2}{a} \right)^2.$$

The potential energy is

$$V = -m_1gx_1 - m_2gx_2 = -m_1g(\ell - x_2) - m_2g(x_2),$$

so the Lagrangian is

$$L = \frac{1}{2} \left(m_1 + m_2 + \frac{I}{a^2} \right) \dot{x}_2^2 + m_1 g(\ell - x_2) + m_2 g x_2.$$

The canonical momentum is

$$p_2 = \frac{\partial L}{\partial \dot{x}_2} = \left(m_1 + m_2 + \frac{I}{a^2} \right) \dot{x}_2$$

The conditions are met so that we can write the Hamiltonian as

$$H = T + V = \frac{1}{2} \frac{p_2^2}{m_1 + m_2 + I/a^2} - m_1 g(\ell - x_2) - m_2 g x_2.$$

Hamilton's equations then become

$$\begin{aligned} \dot{x}_2 &= \frac{\partial H}{\partial p_2} = \frac{p_2}{m_1 + m_2 + I/a^2} \\ \dot{p}_2 &= -\frac{\partial H}{\partial x_2} = (m_2 - m_1)g. \end{aligned}$$

Lastly, H is conserved as

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t} = 0.$$

4 Particle on a moving wedge

(a) The position of the mass m in a coordinate system moving with the wedge is

$$\mathbf{r}' = r \cos(\theta) \hat{e}_x + r \sin(\theta) \hat{e}_y.$$

In the laboratory coordinates, which does not move with the wedge, the wedge has position $x \hat{e}_x$, so the position of the mass m is

$$\mathbf{r} = (x + r \cos(\theta)) \hat{e}_x + r \sin(\theta) \hat{e}_y.$$

The velocity is

$$\dot{\mathbf{r}} = (\dot{x} + \dot{r} \cos(\theta) - r \dot{\theta} \sin(\theta)) \hat{e}_x + (\dot{r} \sin(\theta) + r \dot{\theta} \cos(\theta)) \hat{e}_y,$$

while the square velocity becomes

$$\begin{aligned} \dot{r}^2 &= \left(\dot{x} + \dot{r} \cos(\theta) - r \dot{\theta} \sin(\theta) \right)^2 + \left(\dot{r} \sin(\theta) + r \dot{\theta} \cos(\theta) \right)^2 = \\ &= \dot{x}^2 + 2\dot{x} \left(\dot{r} \cos(\theta) - r \dot{\theta} \sin(\theta) \right) + \left(\dot{r} \cos(\theta) - r \dot{\theta} \sin(\theta) \right)^2 + (\dot{r} \cos(\theta))^2 + 2r\dot{r}\dot{\theta} \cos(\theta) \sin(\theta) + (r\dot{\theta} \sin(\theta))^2 \\ &= \dot{x}^2 + \underline{(\dot{r} \cos(\theta))^2} + \underline{(r\dot{\theta} \sin(\theta))^2} + 2\dot{x}\dot{r} \cos(\theta) - 2\dot{x}r\dot{\theta} \sin(\theta) - \underline{2r\dot{r}\dot{\theta} \cos(\theta) \sin(\theta)} \\ &\quad + \underline{(\dot{r} \sin(\theta))^2} + \underline{2r\dot{r}\dot{\theta} \cos(\theta) \sin(\theta)} + \underline{(r\dot{\theta} \cos(\theta))^2} \\ &= \dot{x}^2 + \dot{r}^2 + (r\dot{\theta})^2 + 2\dot{x}\dot{r} \cos(\theta) - 2\dot{x}r\dot{\theta} \sin(\theta). \end{aligned}$$

The potential energy is given by

$$V = -mgr \sin(\theta).$$

The restriction of the little mass to stay on the wedge is given by $r - R = 0$, so the total Lagrangian, including the undetermined multiplier becomes

$$L = \frac{1}{2}m \left(\dot{x}^2 + \dot{r}^2 + (r\dot{\theta})^2 + 2\dot{x}\dot{r}\cos(\theta) - 2\dot{x}r\dot{\theta}\sin(\theta) \right) + \frac{1}{2}M\dot{x}^2 + mgr \sin(\theta) + \lambda(r - R).$$

The derivatives with respect to x are:

$$\begin{aligned} \frac{\partial L}{\partial x} &= 0, \quad \frac{\partial L}{\partial \dot{x}} = m(\dot{x} + \dot{r}\cos(\theta) - r\dot{\theta}\sin(\theta)) + M\dot{x}, \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} &= m(\ddot{x} + \ddot{r}\cos(\theta) - \sin(\theta)(\dot{r}\dot{\theta} + r\ddot{\theta}) - r\dot{\theta}^2\cos(\theta)) + M\ddot{x} \end{aligned}$$

With respect to θ :

$$\begin{aligned} \frac{\partial L}{\partial \theta} &= -m\dot{x}\dot{r}\sin(\theta) - m\dot{x}r\dot{\theta}\cos(\theta) + mgr\cos(\theta), \quad \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} - \dot{x}r\sin(\theta), \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} &= mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} - m\sin(\theta)(\ddot{x}r + \dot{x}\dot{r}) - m\cos(\theta)\dot{\theta}\dot{x}r, \end{aligned}$$

And finally r :

$$\begin{aligned} \frac{\partial L}{\partial r} &= m(r\dot{\theta}^2 - \dot{x}\dot{\theta}\sin(\theta)) + mg\sin(\theta) + \lambda, \quad \frac{\partial L}{\partial \dot{r}} = m\dot{r} + m\dot{x}\cos(\theta) \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{r}} &= m\ddot{r} + m\ddot{x}\cos(\theta) - m\dot{x}\dot{\theta}\sin(\theta). \end{aligned}$$

We can then use the restriction $r = R$, which implies $\dot{r} = \ddot{r} = 0$, so we get the equations of motion

$$\begin{aligned} \ddot{x}(m + M) - mR\ddot{\theta}\sin(\theta) - mR\dot{\theta}^2\cos(\theta) &= 0 \\ mR^2\ddot{\theta} - mR(\ddot{x}\sin(\theta) + g\cos(\theta)) &= 0 \\ m\ddot{x}\cos(\theta) - mR\dot{\theta}^2 - mg\sin(\theta) - \lambda &= 0. \end{aligned}$$

Cleaning up gives

$$\begin{aligned} \ddot{x} &= \frac{mR}{m + M} \left(\ddot{\theta}\sin(\theta) + \dot{\theta}^2\cos(\theta) \right), \\ \ddot{\theta} &= \frac{\ddot{x}\sin(\theta) + g\cos(\theta)}{R}, \\ \lambda &= m \left(\ddot{x}\cos(\theta) - R\dot{\theta}^2 - g\sin(\theta) \right). \end{aligned}$$

(b) Exploiting a common factor,

$$mR\dot{\theta}^2 = \frac{1}{\cos(\theta)} \left[(m + M)\ddot{x} - mR\ddot{\theta}\sin(\theta) \right],$$

we can insert this into the expression for λ :

$$\begin{aligned}
\lambda &= m\ddot{x} \cos(\theta) - gm \sin(\theta) - \frac{1}{\cos(\theta)} \left[(m+M)\ddot{x} - mR\ddot{\theta} \sin(\theta) \right] \\
&= \ddot{x} \left[m \cos(\theta) - \frac{m+M}{\cos(\theta)} \right] - gm \sin \theta - mR\ddot{\theta} \frac{\sin(\theta)}{\cos(\theta)} \\
&= \frac{mR}{m+M} \left(\ddot{\theta} \sin(\theta) + \dot{\theta}^2 \cos(\theta) \right) \left[m \cos(\theta) - \frac{m+M}{\cos(\theta)} \right] - gm \sin \theta - mR\ddot{\theta} \frac{\sin(\theta)}{\cos(\theta)}
\end{aligned}$$

This is the the force of constraint working on m , and thus is equal to the to the normal force of M on m .