Exercise 5 solutions - TFY4345 Classical Mechanics

2020

1 Effective potential and scattering center

The total energy, as given by equation 4.14 in the compendium, is

$$E = \frac{1}{2}m\left(\dot{r}^2 + (r\dot{\theta})^2\right) + V(r).$$

In a central potential, we have that $mr^2\dot{\theta} = \ell$ is a conserved quantity, so we get

$$E = \frac{1}{2}m\dot{r}^2 + \left(\frac{\ell^2}{2mr^2} + V(r)\right) = \frac{1}{2}m\dot{r}^2 + V_{\text{eff}}(r).$$

This is an effective 1D problem, with an effective potential

$$V_{\text{eff}}(r) = V(r) + \frac{\ell^2}{2mr^2}$$

In order for the particle to reach the center, it need to have sufficiently high energy to overcome the potential barrier, i.e. $E > V_{\text{eff}}(r \to 0)$. This can be written as

$$Er^2 > r^2V(r) + \frac{\ell^2}{2m}, \quad r \to 0.$$

The l.h.s. goes to zero, so that the condition becomes

$$(r^2V(r))_{r\to 0} < -\frac{\ell^2}{2m}.$$

This can be fulfilled wither with $-k/r^2$, where $k > \ell^2/2m$, or if $V(r) = -A/r^n$, with $n > 2$ and A a positive constant.

2 Scattering from a spherical obstacle

(FIGUR)

The scattering angle θ satisfies $2\Psi + \theta = \pi$. From the figure, we see that the impact parameter is given by $s = a\sin(\pi/2 - \theta/2) = a\cos(\theta/2)$, so that

$$\left|\frac{\mathrm{d}s}{\mathrm{d}\theta}\right| = \frac{a}{2}\sin\left(\frac{\theta}{2}\right)$$

Using the formula for the differential cross section, as given in equation 4.40 in the compendium, we get

$$\sigma(\theta) = \frac{s}{\sin(\theta)}\left|\frac{\mathrm{d}s}{\mathrm{d}\theta}\right| = \frac{a^2}{2}\frac{\cos(\theta/2)\sin(\theta/2)}{\sin(\theta)} = \frac{a^2}{4}.$$

(b) The total cross section is therefore

$$\sigma = 2\pi\int_0^\pi \sigma(\theta)\sin(\theta)\mathrm{d}\theta = \pi a^2.$$

This is physically sensible, since it is the actual cross-sectinoal area of the sphere.

3 Scattering by an attractive hard sphere

(FIGURE)

The impact parameter s_{max} will send the particle just gracing the surface at $r = a$. Due to conservation of energy, we have that

$$E = \frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 - \frac{k}{a}.$$

Furthermore, conservation of angular momentum means that ℓ infinitely far away is the same as when the particle touches the surface, so

$$\ell = mv_0 s_{\text{max}} = mva.$$

Combining thes two equations, we get

$$s_{\text{max}} = \frac{v}{v_0}a = a\sqrt{1+\frac{2k}{mav_0^2}}.$$

All particles with impact parameter $s < s_{\text{max}}$ will hit the surface, so that $\sigma_{\text{eff}} = \pi s_{\text{max}}^2$.

4 Average energies in the Kepler problem

(a) From the compendium, part 4E, we have

$$p = \frac{\ell^2}{mk} \quad \varepsilon^2 = 1 + \frac{2E\ell^2}{mk^2}.$$

Eliminating ℓ gives us

$$E = -\frac{k}{2p}\left(1-\varepsilon^2\right).$$

The total energy is constant. This means that the average total energy also is constant:

$$\langle T\rangle + \langle V\rangle = \langle E\rangle = E.$$

The viral theorem for a gravitational potential, example 12 in part 4D of the compendium, gives

$$\langle T \rangle = -\frac{1}{2} \langle V \rangle .$$

Combining this gives

$$\begin{aligned}\langle T \rangle &= \frac{k}{2p} \left(1-\varepsilon^2\right)\\ \langle V \rangle &= -\frac{k}{p} \left(1-\varepsilon^2\right)\end{aligned}$$

(b) The solution to the Kepler problem in polar coordinates (found in the compendium) is

$$r = \frac{p}{1+\varepsilon\cos(\theta)}.$$

The average potential energy over one period it

$$\langle V \rangle = \frac{1}{t_p} \int_0^{t_p} \mathrm{d}t V = -\frac{1}{t_p} \int_0^{t_p} \mathrm{d}t \frac{k}{r}.$$

Combining these equations give

$$\begin{aligned}\langle V \rangle &= -\frac{1}{t_p} \int_0^{t_p} \mathrm{d}t \frac{k}{p} \left(1+\varepsilon\cos(\theta)\right) = -\frac{k}{pt_p} \left(\int_0^{t_p} \mathrm{d}t + \varepsilon \int_0^{t_p} \mathrm{d}t \cos(\theta) \right)\\ &= \frac{k}{p} \left(1+\varepsilon \langle \cos(\theta) \rangle\right).\end{aligned}$$

We can find the the last integral by using $\ell = mr^2\dot{\theta}$ and a change of variable

$$\begin{aligned}\langle \cos(\theta) \rangle &= \frac{1}{t_p} \int_0^{t_p} \mathrm{d}t \cos(\theta) = \frac{1}{t_p} \int_0^{2\pi} \mathrm{d}\theta \frac{1}{\dot{\theta}} \cos(\theta) = \frac{m}{\ell t_p} \int_0^{2\pi} \mathrm{d}\theta r(\theta)^2 \cos(\theta)\\ &= \frac{mp^2}{\ell t_p} \int_0^{2\pi} \mathrm{d}\theta \frac{\cos(\theta)}{\left(1+\varepsilon\cos(\theta)\right)^2}.\end{aligned}$$

Using hint 2 and 3 we get

$$\begin{aligned}\langle \cos(\theta) \rangle &= \frac{mp^2}{\ell t_p} \int_0^{2\pi} \mathrm{d}\theta \frac{\cos(\theta)}{\left(1+\varepsilon\cos(\theta)\right)^2} = -\frac{mp^2}{\ell t_p} \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \int_0^{2\pi} \frac{\mathrm{d}\theta}{1+\varepsilon\cos(\theta)}\\ &= -\frac{mp^2}{\ell t_p} \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \frac{2\pi}{\sqrt{1-\varepsilon^2}} = -\frac{2\pi m}{\ell t_p} \frac{p^2 \varepsilon}{(1-\varepsilon^2)^{3/2}}.\end{aligned}$$

Then, using

$$t_p = \frac{2\pi m}{\ell^2} \frac{p^2}{(1-\varepsilon^2)^{3/2}},$$

we get

$$\langle \cos(\theta) \rangle = -\varepsilon,$$

so

$$\langle V \rangle = \frac{k}{p}(1 - \varepsilon^2).$$

(c) Integrating the kinetic energy by parts, with $\dot{\mathbf{r}}^2 = u'v$, gives

$$\langle T \rangle = \frac{m}{2t_p} \int_0^{t_p} \mathrm{d}t \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \right)^2 = \frac{m}{2t_p} \left(\cancel{\mathbf{r} \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \bigg|_0^{t_p}} - \int_0^{t_p} \mathrm{d}t \, \mathbf{r} \cdot \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} \right)$$

(as $\mathbf{r}(0) = \mathbf{r}(t_p)$, and $\dot{\mathbf{r}}(0) = \dot{\mathbf{r}}(t_p)$)

$$= -\frac{1}{2t_p} \int_0^{t_p} \mathrm{d}t \, \mathbf{r} \cdot \left(-\frac{k}{r^3} \mathbf{r} \right) = \frac{1}{2t_p} \int_0^{t_p} \mathrm{d}t \, \frac{k}{r} = -\frac{1}{2} \langle V \rangle = \frac{k}{2p}(1 - \varepsilon^2).$$

This agrees with the result from the viral theorem.