Exercise 4 solutions - TFY4345 Classical Mechanics

2020

1 Mathematical pendulum

The position of the mass is

$$
\begin{aligned}
x = \ell \sin(\theta), \quad y &= \frac{1}{2}at^2 - \ell \cos(\theta) \\
\dot{x} = \ell\dot{\theta}\cos(\theta) \quad y &= at + \ell\dot{\theta}\sin(\theta),
\end{aligned}
$$

so the kinetic energy is given by

$$
T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2}m\left((\ell\dot{\theta})^2 + (at)^2 + 2at\ell\dot{\theta}\sin(\theta)\right),
$$

and the potential energy is

$$
V = mgy = mg\left(\frac{1}{2}at^2 - \ell\cos(\theta)\right).
$$

The Lagrangian is

$$
L = \frac{1}{2}m\left((\ell\dot{\theta})^2 + (at)^2 + 2at\ell\dot{\theta}\sin(\theta)\right) - mg\left(\frac{1}{2}at^2 - \ell\cos(\theta)\right),
$$

so the canonical momentum is

$$
p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m\left(\ell^2\dot{\theta} + at\ell\sin(\theta)\right) \implies \dot{\theta} = \frac{p_\theta - mta\ell\sin(\theta)}{m\ell^2}.
$$

This gives the Hamiltonian

$$
\begin{aligned}
H &= \dot{\theta}p_\theta - L \\
&= p_\theta\frac{p_\theta - mta\ell\sin(\theta)}{m\ell^2} - \frac{1}{2}m\left[\ell^2\left(\frac{p_\theta - mta\ell\sin(\theta)}{m\ell^2}\right)^2 + (\ell at)^2 + 2at\ell\sin(\theta)\left(\frac{p_\theta - mta\ell\sin(\theta)}{m\ell^2}\right)\right] \\
&+ mg\left(\frac{1}{2}at^2 - \ell\cos(\theta)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{m\ell^2}\left(p_\theta^2 - p_\theta mta\ell\sin(\theta)\right) - \\
&\frac{1}{2m\ell^2}\left[p_\theta^2 - 2p_\theta mta\ell\sin(\theta) + (mta\ell\sin(\theta))^2 + (m\ell ta)^2 + 2p_\theta mat\ell\sin(\theta) - 2(mta\ell\sin(\theta))^2\right] \\
&+ mg\left(\frac{1}{2}at^2 - \ell\cos(\theta)\right) \\
&= \frac{1}{2m\ell^2}\left(p_\theta - mta\ell\sin(\theta)\right)^2 - \frac{1}{2}ma^2t^2 + \frac{1}{2}magt^2 - mg\ell\cos(\theta).
\end{aligned}
$$

The Hamiltonian equations of motion

$$
\begin{aligned}
\dot{\theta} &= \frac{\partial H}{\partial p_\theta} = \frac{p_\theta - mta\ell\sin(\theta)}{m\ell^2} \\
\dot{p}_\theta &= -\frac{\partial H}{\partial \theta} = \frac{at\cos(\theta)}{\ell}\left[p_\theta - mat\ell\sin(\theta)\right] - mg\ell\sin(\theta).
\end{aligned}
$$

Furthermore, we see that $H \neq T + V$, so the Hamiltonian function is not the total energy of the system. Furthermore,

$$
\frac{\mathrm{d}H}{\mathrm{d}t} = -\frac{\partial L}{\partial t} \implies \frac{\mathrm{d}H}{\mathrm{d}t} \neq 0,
$$

as the Lagrangian has an explicit time dependence. The pendulum is in an accelerating motion with the respect to the inertial frame of reference. This mean that H will not be conserved.

2 Spherically symmetrical potential

Spherical coordinates defined by

$$
x = r\sin(\theta)\cos(\varphi), \quad y = r\sin(\theta)\cos(\varphi), \quad z = r\cos(\theta)
$$

This mean that the square velocity is

$$
\begin{aligned}
v^2 &= \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \\
&= (\dot{\sin}(\theta)\cos(\vartheta) + r\dot{\theta}\cos(\theta)\cos(\vartheta) - r\dot{\varphi}\sin(\theta)\sin(\varphi))^2 \\
&+ (\dot{r}\sin(\theta)\sin(\varphi) + r\dot{\theta}\cos(\theta)\sin(\varphi) + r\dot{\varphi}\sin(\theta)\cos(\varphi))^2 + (r\cos(\theta) - r\dot{\theta}\sin(\theta))^2 \\
&= \underline{\dot{r}^2\sin^2(\theta)\cos^2(\varphi)} + r^2\dot{\theta}^2\cos(\theta)\cos^2(\varphi) + r^2\dot{\varphi}^2\sin^2(\theta)\sin^2(\varphi) + \underline{\underline{2\dot{r}r\dot{\theta}\sin(\theta)\cos(\theta)\cos^2(\varphi)}} \\
&- \cancel{2\dot{r}r\dot{\varphi}\sin^2(\theta)\cos(\varphi)\sin(\varphi)} - \cancel{2r^2\dot{\theta}\dot{\varphi}\sin(\theta)\cos(\theta)\sin(\varphi)\cos(\varphi)} + \underline{\dot{r}^2\sin^2(\theta)\sin^2(\varphi)} \\
&+ r^2\dot{\theta}^2\cos^2(\theta)\sin^2(\varphi) + r^2\dot{\varphi}^2\sin^2(\theta)\cos^2(\varphi) + \underline{\underline{2\dot{r}r\dot{\theta}\sin(\theta)\cos(\theta)\sin^2(\varphi)}} \\
&+ \cancel{2\dot{r}r\dot{\varphi}\sin^2(\theta)\sin(\varphi)\cos(\varphi)} + \cancel{2r^2\dot{\theta}\dot{\varphi}\sin(\theta)\cos(\theta)\sin(\varphi)\cos(\varphi)} + \dot{r}^2\cos^2(\theta) + r^2\dot{\theta}^2\sin^2(\theta) \\
&- 2\dot{r}r\dot{\theta}\sin(\theta)\cos(\theta) \\
&= \dot{r}^2\sin^2(\theta) + r^2\dot{\theta}^2\cos^2(\theta) + r^2\dot{\varphi}^2\sin^2(\theta) + \cancel{2\dot{r}r\dot{\theta}\sin(\theta)\cos(\theta)} \\
&+ \dot{r}^2\cos^2(\theta) + r^2\dot{\theta}^2\sin^2(\theta) - \cancel{2\dot{r}r\dot{\theta}\sin(\theta)\cos(\theta)} \\
&= \dot{r}^2 + (r\dot{\theta})^2 + (r\dot{\varphi}\sin(\theta))^2
\end{aligned}
$$

The Lagrangian is

$$L = T - V = \frac{1}{2}m\left[\dot{r}^2 + (r\dot{\theta})^2 + (r\dot{\varphi}\sin(\theta))^2\right] - \frac{k}{r},$$

so the canonical momenta are

$$p_r = \frac{\partial L}{\partial r} = m\dot{r}, \quad p_\theta = \frac{\partial L}{\partial \theta} = mr^2\dot{\theta}, \quad p_\vartheta = \frac{\partial L}{\partial \varphi} = mr^2\sin^2(\varphi)\dot{\varphi}.$$

This means we can rewrite the kinetic energy in terms of the momenta:

$$T = \frac{1}{2m}\left[p_r^2 + \frac{p_\theta}{r^2} + \frac{p_\varphi}{r^2\sin^2(\theta)}\right].$$

The Hamiltonian becomes

$$H = T + V = \frac{1}{2m}\left[p_r^2 + \frac{p_\theta}{r^2} + \frac{p_\varphi}{r^2\sin^2(\theta)}\right] - \frac{k}{r}.$$

Hamilton's equation of motion

$$\begin{aligned}
\dot{r} &= \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \\
\dot{\theta} &= \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \\
\dot{\varphi} &= \frac{\partial H}{\partial p_\varphi} = \frac{p_\varphi}{mr^2\sin^2(\theta)} \\
\dot{p}_r &= -\frac{\partial H}{\partial r} = \frac{p_\theta}{mr^3} + \frac{p_\varphi}{mr^3\sin^2(\theta)} + \frac{k}{r^2} \\
\dot{p}_\theta &= -\frac{\partial H}{\partial \theta} = \frac{p_\varphi\cos(\theta)}{mr^2\sin^3(\theta)} \\
\dot{\varphi} &= -\frac{\partial H}{\partial \varphi} = 0.
\end{aligned}$$

3 Earth's orbit

4 Einsteins correction