

Exercise 4 solutions - TFY4345 Classical Mechanics

2020

1 Mathematical pendulum in accelerated motion

The position and velocity of the mass is

$$\begin{aligned}x &= \ell \sin(\theta), & y &= \frac{1}{2}at^2 - \ell \cos(\theta), \\ \dot{x} &= \ell \dot{\theta} \cos(\theta), & \dot{y} &= at + \ell \dot{\theta} \sin(\theta),\end{aligned}$$

so the kinetic energy is given by

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m\left((\ell \dot{\theta})^2 + (at)^2 + 2at\ell \dot{\theta} \sin(\theta)\right),$$

and the potential energy is

$$V = mgy = mg\left(\frac{1}{2}at^2 - \ell \cos(\theta)\right).$$

The Lagrangian is

$$L = \frac{1}{2}m\left((\ell \dot{\theta})^2 + (at)^2 + 2at\ell \dot{\theta} \sin(\theta)\right) - mg\left(\frac{1}{2}at^2 - \ell \cos(\theta)\right),$$

so the canonical momentum is

$$p_{\theta} = \frac{\partial L}{\partial \dot{\theta}} = m\left(\ell^2 \dot{\theta} + at\ell \sin(\theta)\right) \implies \dot{\theta} = \frac{p_{\theta} - mta\ell \sin(\theta)}{m\ell^2}.$$

This gives the Hamiltonian

$$\begin{aligned}H &= \dot{\theta}p_{\theta} - L \\ &= p_{\theta} \frac{p_{\theta} - mta\ell \sin(\theta)}{m\ell^2} - \frac{1}{2}m \left[\ell^2 \left(\frac{p_{\theta} - mta\ell \sin(\theta)}{m\ell^2} \right)^2 + (\ell at)^2 + 2at\ell \sin(\theta) \left(\frac{p_{\theta} - mta\ell \sin(\theta)}{m\ell^2} \right) \right] \\ &\quad + mg\left(\frac{1}{2}at^2 - \ell \cos(\theta)\right)\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{m\ell^2} (p_\theta^2 - p_\theta m t a \ell \sin(\theta)) - \\
&\frac{1}{2m\ell^2} [p_\theta^2 - 2p_\theta m t a \ell \sin(\theta) + (m t a \ell \sin(\theta))^2 + (m \ell t a)^2 + 2p_\theta m a t \ell \sin(\theta) - 2(m t a \ell \sin(\theta))^2] \\
&+ m g \left(\frac{1}{2} a t^2 - \ell \cos(\theta) \right) \\
&= \frac{1}{2m\ell^2} (p_\theta - m t a \ell \sin(\theta))^2 - \frac{1}{2} m a^2 t^2 + \frac{1}{2} m a g t^2 - m g \ell \cos(\theta).
\end{aligned}$$

The Hamiltonian equations of motion

$$\begin{aligned}
\dot{\theta} &= \frac{\partial H}{\partial p_\theta} = \frac{p_\theta - m t a \ell \sin(\theta)}{m\ell^2} \\
\dot{p}_\theta &= -\frac{\partial H}{\partial \theta} = \frac{a t \cos(\theta)}{\ell} [p_\theta - m a t \ell \sin(\theta)] - m g \ell \sin(\theta).
\end{aligned}$$

Furthermore, we see that $H \neq T + V$, so the Hamiltonian function is not the total energy of the system. Furthermore,

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t} \implies \frac{dH}{dt} \neq 0,$$

as the Lagrangian has an explicit time dependence. The pendulum is in an accelerating motion with the respect to the inertial frame of reference. This mean that H will not be conserved.

2 Spherically symmetrical potential

Spherical coordinates defined by

$$x = r \sin(\theta) \cos(\varphi), \quad y = r \sin(\theta) \sin(\varphi), \quad z = r \cos(\theta)$$

This mean that the square velocity is (take a deep breath)

$$\begin{aligned}
v^2 &= \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \\
&= (\dot{r} \sin(\theta) \cos(\varphi) + r \dot{\theta} \cos(\theta) \cos(\varphi) - r \dot{\varphi} \sin(\theta) \sin(\varphi))^2 \\
&+ (\dot{r} \sin(\theta) \sin(\varphi) + r \dot{\theta} \cos(\theta) \sin(\varphi) + r \dot{\varphi} \sin(\theta) \cos(\varphi))^2 + (\dot{r} \cos(\theta) - r \dot{\theta} \sin(\theta))^2 \\
&= \underline{\dot{r}^2 \sin^2(\theta) \cos^2(\varphi)} + \underline{\dot{r}^2 \dot{\theta}^2 \cos^2(\theta) \cos^2(\varphi)} + \underline{\dot{r}^2 \dot{\varphi}^2 \sin^2(\theta) \sin^2(\varphi)} + \underline{2\dot{r}r\dot{\theta} \sin(\theta) \cos(\theta) \cos^2(\varphi)} \\
&- \underline{2\dot{r}r\dot{\varphi} \sin^2(\theta) \cos(\varphi) \sin(\varphi)} - \underline{2r^2\dot{\theta}\dot{\varphi} \sin(\theta) \cos(\theta) \sin(\varphi) \cos(\varphi)} + \underline{\dot{r}^2 \sin^2(\theta) \sin^2(\varphi)} \\
&+ \underline{\dot{r}^2 \dot{\theta}^2 \cos^2(\theta) \sin^2(\varphi)} + \underline{\dot{r}^2 \dot{\varphi}^2 \sin^2(\theta) \cos^2(\varphi)} + \underline{2\dot{r}r\dot{\theta} \sin(\theta) \cos(\theta) \sin^2(\varphi)} \\
&+ \underline{2\dot{r}r\dot{\varphi} \sin^2(\theta) \sin(\varphi) \cos(\varphi)} + \underline{2r^2\dot{\theta}\dot{\varphi} \sin(\theta) \cos(\theta) \sin(\varphi) \cos(\varphi)} + \underline{\dot{r}^2 \cos^2(\theta)} + \underline{r^2 \dot{\theta}^2 \sin^2(\theta)} \\
&- \underline{2\dot{r}r\dot{\theta} \sin(\theta) \cos(\theta)} \\
&= \dot{r}^2 \sin^2(\theta) + r^2 \dot{\theta}^2 \cos^2(\theta) + r^2 \dot{\varphi}^2 \sin^2(\theta) + \underline{2\dot{r}r\dot{\theta} \sin(\theta) \cos(\theta)} \\
&+ \dot{r}^2 \cos^2(\theta) + r^2 \dot{\theta}^2 \sin^2(\theta) - \underline{2\dot{r}r\dot{\theta} \sin(\theta) \cos(\theta)} \\
&= \dot{r}^2 + (r\dot{\theta})^2 + (r\dot{\varphi} \sin(\theta))^2
\end{aligned}$$

The Lagrangian is

$$L = T - V = \frac{1}{2}m \left[\dot{r}^2 + (r\dot{\theta})^2 + (r\dot{\varphi} \sin(\theta))^2 \right] - \frac{k}{r},$$

so the canonical momenta are

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}, \quad p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = mr^2 \sin^2(\theta)\dot{\varphi}.$$

This means we can rewrite the kinetic energy in terms of the momenta:

$$T = \frac{1}{2m} \left[p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2 \sin^2(\theta)} \right].$$

The Hamiltonian becomes

$$H = T + V = \frac{1}{2m} \left[p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2 \sin^2(\theta)} \right] - \frac{k}{r}.$$

Hamilton's equation of motion

$$\begin{aligned} \dot{r} &= \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \\ \dot{\theta} &= \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \\ \dot{\varphi} &= \frac{\partial H}{\partial p_\varphi} = \frac{p_\varphi}{mr^2 \sin^2(\theta)} \\ \dot{p}_r &= -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} + \frac{p_\varphi^2}{mr^3 \sin^2(\theta)} + \frac{k}{r^2} \\ \dot{p}_\theta &= -\frac{\partial H}{\partial \theta} = \frac{p_\varphi^2 \cos(\theta)}{mr^2 \sin^3(\theta)} \\ \dot{p}_\varphi &= -\frac{\partial H}{\partial \varphi} = 0. \end{aligned}$$

3 Earth's orbit

(See compendium, chapter 4)

The eccentricity of a circle is 1, so using the formula for the eccentricity of an orbit we get that

$$\varepsilon = 0 = \sqrt{1 + \frac{2E\ell^2}{mk^2}} \implies 1 + \frac{2E\ell^2}{mk^2} = 0 \implies E = -\frac{mk^2}{2\ell^2}.$$

On the other hand, for a circular orbit $E = V_{\min}$. Using the effective 1D potential

$$E(r) = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{\ell^2}{mr^2} - \frac{k}{r} = V_{\min}.$$

If the mass of the sun is halved, then as the constant in the potential $k = GMm$ is proportional to the mass of the sun, it is halved, $k \rightarrow k/2$. This means the energy is change to

$$E \rightarrow E' = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{\ell^2}{mr^2} - \frac{k}{2r} = V_{min} + \frac{k}{2r} = -\frac{mk^2}{2\ell^2} + \frac{k}{2r}.$$

The original radius of the original orbit is given by the minimum of the effective potential,

$$V(r) = \frac{1}{2}\frac{\ell^2}{mr^2} - \frac{k}{r},$$

so

$$V'(r) = -\frac{\ell^2}{mr^3} + \frac{k}{r^2} = 0 \implies r = \frac{\ell^2}{km}.$$

This means the new energy is

$$E' = -\frac{mk^2}{2\ell^2} + \frac{mk^2}{2\ell^2} = 0,$$

The new eccentricity is therefore

$$\varepsilon = \sqrt{1+0} = 1,$$

which means the new orbit is a parabola, and thus unbounded. The earth just escapes to infinity.

4 Einsteins correction

As the central force is given by

$$f(r) = -\frac{k}{r^2} + \frac{\beta}{r^3},$$

the potential is (up to a constant)

$$V(r) = -\frac{k}{r} + \frac{\beta}{2r^2}.$$

In the compendium, we can find that the angle of an object in a central potential is

$$\theta(r) = \int_{r_0}^r \frac{1/r^2 dr}{\sqrt{\frac{2mE}{\ell^2} - \frac{2mV(r)}{\ell^2} - \frac{1}{r^2}}}.$$

Inserting our potential, setting $u = 1/r$, and using $\gamma^2 = 1 + \beta m/\ell^2$, this becomes

$$\theta(r) = - \int_{u_0}^u \frac{du}{\sqrt{\frac{2mE}{\ell^2} - \frac{2mu}{\ell^2} - \gamma^2 u^2}}.$$

By introducing the constants

$$a = \frac{2mE}{\ell^2} \quad b = \frac{2mK}{\ell^2} \quad c^2 = -\gamma^2,$$

We get the integral on a known form which can be found in tables:

$$\theta(r) = - \int_{u_0}^u \frac{du}{\sqrt{a + bu + cu^2}} = -\frac{1}{\sqrt{-c}} \arccos \left(-\frac{b + 2cu}{\sqrt{b^2 - 4ac}} \right).$$

Now,

$$-\frac{b+2cu}{\sqrt{b^2-4ac}} = \frac{2\gamma^2 u - 2mk/\ell^2}{\sqrt{(2mk/\ell^2)^2 + 4(2mE/\ell^2)\gamma^2}} = \frac{\frac{\ell^2\gamma^2}{mk}u - 1}{\sqrt{1 + \frac{2E\gamma^2\ell^2}{mk}}} = \frac{p/r - 1}{\varepsilon},$$

where

$$p = \frac{\ell^2\gamma^2}{mk}, \quad \varepsilon = \sqrt{1 + \frac{2E\gamma^2\ell^2}{mk}}.$$

This means the angle of the object is given by

$$\theta(r) = \frac{-1}{\gamma} \arccos\left(\frac{p/r - 1}{\varepsilon}\right).$$

Turning this around,

$$\frac{p}{r} = 1 + \varepsilon \cos(\gamma\theta), \quad \text{where } \gamma = \sqrt{1 + \frac{m\beta}{\ell^2}} \approx 1 + \frac{m\beta}{2\ell^2}, \quad \frac{m\beta}{\ell} \ll 1.$$

If $E < 0$, then this is an ellipse with slow precession. The semi-major axis for $\gamma = 1$ is

$$a = \frac{p}{1 - \varepsilon^2} = \frac{\gamma^2\ell^2/mk}{1 - (1 + 2E\gamma^2\ell^2/mk^2)} = \frac{k}{2|E|}.$$

This, then, is a perturbation to this, with the smallness parameter $\eta = \beta/ka$, so $\gamma = 1 + m\eta ka/(2\ell^2)$. For Mercury, $\eta = 1.42 \cdot 10^{-7}$, which is the perihelion precession of $43''$ per century.