

# Exercise 1 solutions - TFY4345 Classical Mechanics

2020

## 1 Halley's comet

a) The gravitational force on the comet is

$$\vec{F} = -\frac{GmM}{r^2}\vec{e}_r,$$

where  $m$  is the mass of the comet, and  $M$  is the mass of the sun. The torque on the comet is therefore

$$\vec{N} = \vec{r} \times \vec{F} = -r\vec{e}_r \times \frac{GmM}{r^2}\vec{e}_r = \frac{GmM}{r}\vec{e}_r \times \vec{e}_r = 0. \quad (1)$$

b) The angular momentum of the comet when the comet is closest to and farthest from the sun is, respectively  $\vec{L}_e = \vec{r}_e \times \vec{p}_e$  and  $\vec{L}_f = \vec{r}_f \times \vec{p}_f$ . Conservation of angular momentum implies that  $\vec{L}_e = \vec{L}_f$ , and in turn that  $|\vec{L}_e| = |\vec{L}_f|$ . At both the point the comet is closest to and farthest from the sun, the position vector  $\vec{r}$  and the momentum vector  $\vec{p}$  are perpendicular. This means that

$$|\vec{L}_e| = |\vec{L}_f| \implies |\vec{r}_e||\vec{p}_e| = |\vec{r}_f||\vec{p}_f| \implies r_e m v_e = r_f m v_f.$$

The velocity of the comet farthest from the sun is therefore

$$v_f = \frac{r_e}{r_f} v_e = \frac{0.6 \text{ AU}}{35 \text{ AU}} 54 \text{ km/s} = 0.9 \text{ km/s} \quad (2)$$

## 2 Simple pendulum

a) By a trigonometric consideration,

$$\vec{R} = \ell \sin(\beta)\vec{e}_x - \ell \cos(\beta)\vec{e}_y$$

b) Potential energy for a mass  $m$  in a uniform gravitational field is  $V = mgh$ , where  $h$  is the height of the mass (in an arbitrary reference frame). In our case,  $\ell \cos(\beta)$ , and hence

$$V = -mg\ell \cos(\beta).$$

A different choice of reference frame will only add a constant to  $V$ , which will not affect the equations of motion.

c) The velocity of the mass  $m$  is given by

$$\vec{v} = \frac{d\vec{R}}{dt} = \ell \cos(\beta) \dot{\beta} \vec{e}_x + \ell \sin(\beta) \dot{\beta} \vec{e}_y.$$

The square of the velocity is then given by

$$v^2 = \vec{v} \cdot \vec{v} = (\ell \dot{\beta})^2 (\cos^2(\beta) \vec{e}_x \cdot \vec{e}_x + \sin^2(\beta) \vec{e}_y \cdot \vec{e}_y) = (\ell \dot{\beta})^2.$$

The kinetic energy is therefore

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m (\ell \dot{\beta})^2$$

d) The Lagrangian of the pendulum is

$$L = T - V = \frac{1}{2} m (\ell \dot{\beta})^2 + m g \ell \cos(\beta).$$

The Lagrange equations are in general

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0.$$

In this case, we have only one variable,  $q_1 = \beta$ . The resulting Lagrange equation is

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\beta}} - \frac{\partial L}{\partial \beta} = 0.$$

Inserting the Lagrangian we found, we get

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\beta}} = \frac{d}{dt} (m \ell^2 \dot{\beta}) = m \ell^2 \ddot{\beta}, \quad \frac{\partial L}{\partial \beta} = -m g \ell \sin(\beta)$$

so the equation of motion is

$$\ddot{\beta} + \frac{g}{\ell} \cos(\beta) = 0$$

### 3 Double pendulum

(a) The position of mass  $m_1$  is, just as in the single pendulum,

$$\vec{R}_1 = \ell_1 \sin(\beta_1) \vec{e}_x - \ell_1 \cos(\beta_1) \vec{e}_y.$$

The position of mass two is then, by the same considerations just using  $\vec{R}_1$  as the point of reference,

$$\begin{aligned} \vec{R}_2 &= \vec{R}_1 + \ell_2 \sin(\beta_2) \vec{e}_x - \ell_2 \cos(\beta_2) \vec{e}_y \\ &= [\ell_1 \sin(\beta_1) + \ell_2 \sin(\beta_2)] \vec{e}_x - [\ell_1 \cos(\beta_1) + \ell_2 \cos(\beta_2)] \vec{e}_y. \end{aligned}$$

The velocity and square velocity of  $m_1$  is also just as in exercise 2:

$$\vec{v}_1 = \frac{d\vec{R}_1}{dt}, \quad v_1^2 = (\ell_1 \dot{\beta}_1)^2.$$

The velocity and velocity square of  $m_2$  is

$$\begin{aligned}\vec{v}_2 &= \frac{d\vec{R}_2}{dt} = \left[ \ell_1 \cos(\beta_1) \dot{\beta}_1 + \ell_2 \cos(\beta_2) \dot{\beta}_2 \right] \vec{e}_x + \left[ \ell_1 \sin(\beta_1) \dot{\beta}_1 + \ell_2 \sin(\beta_2) \dot{\beta}_2 \right] \vec{e}_y, \\ v_2^2 &= \left[ \ell_1 \cos(\beta_1) \dot{\beta}_1 + \ell_2 \cos(\beta_2) \dot{\beta}_2 \right]^2 + \left[ \ell_1 \sin(\beta_1) \dot{\beta}_1 + \ell_2 \sin(\beta_2) \dot{\beta}_2 \right]^2 \\ &= (\ell_1 \dot{\beta}_1)^2 + (\ell_2 \dot{\beta}_2)^2 + 2\ell_1 \ell_2 \cos(\beta_1) \cos(\beta_2) \dot{\beta}_1 \dot{\beta}_2 + 2\ell_1 \ell_2 \sin(\beta_1) \sin(\beta_2) \dot{\beta}_1 \dot{\beta}_2 \\ &= (\ell_1 \dot{\beta}_1)^2 + (\ell_2 \dot{\beta}_2)^2 + 2\ell_1 \ell_2 \dot{\beta}_1 \dot{\beta}_2 \cos(\beta_1 - \beta_2),\end{aligned}$$

where we have used the trigonometric addition law  $\cos(\theta) \cos(\phi) + \sin(\theta) \sin(\phi) = \cos(\theta - \phi)$  in the last line. Total potential energy is

$$V = m_1 g h_1 + m_2 g h_2 = -m_1 g \ell_1 \cos(\beta_1) - m_2 g [\ell_1 \cos(\beta_1) + \ell_2 \cos(\beta_2)].$$

Total kinetic energy is

$$T = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 (\ell_1 \dot{\beta}_1)^2 + \frac{1}{2} m_2 \left[ (\ell_1 \dot{\beta}_1)^2 + (\ell_2 \dot{\beta}_2)^2 + \ell_1 \ell_2 \dot{\beta}_1 \dot{\beta}_2 \cos(\beta_1 - \beta_2) \right],$$

so the Lagrangian is, after gathering some terms,

$$\begin{aligned}L &= T - V \\ &= (m_1 + m_2) \left[ \frac{1}{2} (\ell_1 \dot{\beta}_1)^2 + g \ell_1 \cos(\beta_1) \right] + m_2 \left[ \frac{1}{2} (\ell_2 \dot{\beta}_2)^2 + \ell_1 \ell_2 \dot{\beta}_1 \dot{\beta}_2 \cos(\beta_1 - \beta_2) + g \ell_2 \cos(\beta_2) \right].\end{aligned}$$

b) The Lagrange equations for this problem are

$$\begin{aligned}\frac{d}{dt} \frac{\partial L}{\partial \dot{\beta}_1} - \frac{\partial L}{\partial \beta_1} &= 0, \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{\beta}_2} - \frac{\partial L}{\partial \beta_2} &= 0.\end{aligned}$$

Calculating the quantities needed:

$$\begin{aligned}\frac{\partial L}{\partial \beta_1} &= -(m_1 + m_2) g \ell_1 \sin(\beta_1) - m_2 \ell_1 \ell_2 \dot{\beta}_1 \dot{\beta}_2 \sin(\beta_1 - \beta_2) \\ \frac{\partial L}{\partial \dot{\beta}_1} &= (m_1 + m_2) \ell_1^2 \dot{\beta}_1 + m_2 \ell_1 \ell_2 \dot{\beta}_2 \cos(\beta_1 - \beta_2) \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{\beta}_1} &= (m_1 + m_2) \ell_1^2 \ddot{\beta}_1 + m_2 \ell_1 \ell_2 \left( \ddot{\beta}_2 \cos(\beta_1 - \beta_2) - \dot{\beta}_2 \sin(\beta_1 - \beta_2) (\dot{\beta}_1 - \dot{\beta}_2) \right) \\ \frac{\partial L}{\partial \beta_2} &= m_2 \left[ \ell_1 \ell_2 \dot{\beta}_1 \dot{\beta}_2 \sin(\beta_1 - \beta_2) - g \ell_2 \sin(\beta_2) \right] \\ \frac{\partial L}{\partial \dot{\beta}_2} &= m_2 \left[ \ell_2^2 \dot{\beta}_2 + \ell_1 \ell_2 \dot{\beta}_1 \cos(\beta_1 - \beta_2) \right] \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{\beta}_2} &= m_2 \left[ \ell_2^2 \ddot{\beta}_2 + \ell_1 \ell_2 \left( \ddot{\beta}_1 \cos(\beta_1 - \beta_2) - \dot{\beta}_1 \sin(\beta_1 - \beta_2) (\dot{\beta}_1 - \dot{\beta}_2) \right) \right].\end{aligned}$$

This gives us the equation of motion

$$(m_1 + m_2)\ell_1^2\ddot{\beta}_1 + m_2\ell_1\ell_2 \left( \ddot{\beta}_2 \cos(\beta_1 - \beta_2) - \dot{\beta}_2 \sin(\beta_1 - \beta_2)(\dot{\beta}_1 - \dot{\beta}_2) \right) \\ + (m_1 + m_2)g\ell_1 \sin(\beta_1) + m_2\ell_1\ell_2\dot{\beta}_1\dot{\beta}_2 \sin(\beta_1 - \beta_2) = 0$$

and

$$m_2 \left[ \ell_2^2\ddot{\beta}_2 + \ell_1\ell_2 \left( \ddot{\beta}_1 \cos(\beta_1 - \beta_2) - \dot{\beta}_1 \sin(\beta_1 - \beta_2)(\dot{\beta}_1 - \dot{\beta}_2) \right) \right] \\ - m_2 \left[ \ell_1\ell_2\dot{\beta}_1\dot{\beta}_2 \sin(\beta_1 - \beta_2) - g\ell_2 \sin(\beta_2) \right] = 0$$

Cleaning up some, we get the final result

$$(m_1 + m_2) \left[ \ell_1\ddot{\beta}_1 + g \sin(\beta_1) \right] + m_2\ell_2 \left[ \ddot{\beta}_2 \cos(\beta_1 - \beta_2) + \dot{\beta}_2^2 \sin(\beta_1 - \beta_2) \right] = 0 \\ \ell_2\ddot{\beta}_2 + \ell_1\ddot{\beta}_1 \cos(\beta_1 - \beta_2) - \ell_2\dot{\beta}_1^2 \sin(\beta_1 - \beta_2) + g \sin(\beta_2) = 0.$$