Exercise 2 solutions - TFY4345 Classical Mechanics

2020

1 Damped oscillator

(a) The frictional force is

$$F_f = -\frac{\partial \mathcal{F}}{\partial \dot{x}}.$$

The work done by friction is force times distance, so the work per unit time is

$$\dot{W}_f = -F_f \dot{x} = \frac{\partial \mathcal{F}}{\partial \dot{x}}\dot{x} \implies \mathcal{F} = C\dot{x}^2.$$

(As $\mathcal{F}$ is a (velocity) potential, we can dismiss any constants, just as with regular potentials.) This means that

$$\dot{W}_f = 2C\dot{x}^2 = 2\mathcal{F}.$$

(b) The Lagrangian with a velocity-dependent potential is

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} + \frac{\partial \mathcal{F}}{\partial \dot{x}} = 0.$$

Inserting the Lagrangian for a harmonic oscillator,

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2,$$

and the given velocity potential $\mathcal{F} = 3\pi\mu a\dot{x}^2$, we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} = m\ddot{x},\ \frac{\partial L}{\partial x} = -kx,\ \frac{\partial \mathcal{F}}{\partial \dot{x}} = 6\pi\mu a\dot{x},\\
&\implies m\ddot{x} + 6\pi\mu a + kx = 0,
\end{aligned}$$

or

$$\ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = 0, \quad \lambda = \frac{3\pi\mu a}{m},\ \omega_0 = \sqrt{\frac{k}{m}}.$$

(c) If we assume the solution to be of the form

$$x(t) = Ae^{\omega_a t} + Be^{\omega_b t},$$

we get

$$A(\omega_0^2 + 2\lambda\omega_a + \omega^2)e^{\omega_a t} + B(\omega_0^2 + 2\lambda\omega_b + \omega_b^2)e^{\omega_b t} = 0,$$

so

$$\omega_{a/b} = -\lambda \pm \sqrt{\lambda^2 - \omega_0^2}.$$

This gives us

$$x(t) = e^{-\lambda t} \left(A \exp\left[\omega_0 t \sqrt{(\lambda/\omega_0)^2 - 1}\right] + B \exp\left[-\omega_0 t \sqrt{(\lambda/\omega_0)^2 - 1}\right] \right).$$

Now, as $\lambda/\omega_0 \ll 1$, we get that $\sqrt{(\lambda/\omega_0) - 1} \approx i$. This means that, after applying the initial conditions, we get the solution

$$x(t) = x_0 e^{-\lambda t} \cos(\omega_0 t).$$

The instantaneous energy dissipation is therefore

$$\dot{W}_f = F_f \dot{x} = 2\lambda m \dot{x}^2 = 2\lambda m (\omega_0 x_0)^2 e^{-2\lambda t} \cos^2(\omega_0 t).$$

If we then take the average over a period $2\pi/\omega_0$, we can assume the exponential is more or less constant (as $\lambda \ll \omega_0$), so the time averaged dissipation is

$$\overline{\dot{W}_f} = \frac{\omega_0}{2\pi} \int_0^{2\pi} \mathrm{d}t \, 2\lambda m (\omega_0 x_0)^2 e^{-2\lambda t} \cos^2(\omega_0 t) \approx \frac{\lambda m (\omega_0 x_0)^2}{\pi} e^{-2\lambda t} \int_0^{2\pi} \mathrm{d}x \, \cos^2(x) = m\lambda(\omega_0 x_0)^2 e^{-2\lambda t}.$$

2 Operator identities

The Einstein summation convention is used throughout this exercise, so repeated indices are summed over. The i'th component of the curl of the curl of $\mathbf{A}$ can be written

$$[\nabla \times (\nabla \times \mathbf{A})]_i = \varepsilon_{ijk} \partial_j [\nabla \times \mathbf{A}]_k = \varepsilon_{ijk} \partial_j \varepsilon_{klm} \partial_l A_m = \varepsilon_{ijk} \varepsilon_{klm} \partial_j \partial_l A_m.$$

Using the fact that $\varepsilon_{ijk} = \varepsilon_{kij}$, and the identity $\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$, this becomes

$$[\nabla \times (\nabla \times \mathbf{A})]_i = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \partial_j \partial_l A_m = \partial_j \partial_i A_j - \partial_j \partial_j A_i = \nabla \cdot (\nabla_i \mathbf{A}) - (\nabla \cdot \nabla) A_i,$$

or

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla \cdot (\nabla \mathbf{A}) - \nabla^2 \mathbf{A}.$$

3 Shortest path in polar coordinates

A distance element in polar coordinates are given by

$$\mathrm{d}s = \sqrt{\mathrm{d}r^2 + r^2 \mathrm{d}\varphi^2} = \sqrt{1 + r^2 \varphi'^2} \mathrm{d}r.$$

This means the distance along a curve $\mathcal{C}$ is

$$s = \int_{\mathcal{C}} \mathrm{d}s = \int_{\mathcal{C}} \sqrt{1 + r^2 \varphi'^2} \mathrm{d}r.$$

With r as our parameter, the Euler equation reads

$$\frac{\mathrm{d}}{\mathrm{d}r} \frac{\partial f}{\partial \varphi'} - \frac{\partial f}{\partial \varphi} = 0,$$

where $f(\varphi', r) = \sqrt{1 + r^2\varphi'^2}$ is what we are trying to minimize. We see that

$$\frac{\partial f}{\partial \varphi} = 0 \implies \frac{\mathrm{d}}{\mathrm{d}r}\frac{\partial f}{\partial \varphi'} = 0,$$

so

$$\begin{aligned}
\frac{\partial}{\partial \varphi'}\sqrt{1 + r^2\varphi'^2} &= \frac{-r^2\varphi'}{\sqrt{1 + r^2\varphi'^2}} = a \\
\implies \frac{\varphi^2 r^2}{a} &= \sqrt{1 + r^2\varphi'^2} \\
\implies \frac{\varphi^4 r^4}{a^2} &= 1 + r^2\varphi'^2 \\
\implies \left(\frac{r^2}{a^2} - 1\right) r^2\varphi'^2 &= 1.
\end{aligned}$$

Defining $b^2 = 1/a^2$, we get

$$\begin{aligned}
\varphi'^2 &= \frac{1}{r^2(b^2r^2 - 1)} \\
\implies \varphi' &= \pm\frac{1}{r\sqrt{b^2r^2 - 1}} \\
\implies \varphi(r) &= \pm\int_{r_0}^{r} \frac{\mathrm{d}r}{r\sqrt{b^2r^2 - 1}}.
\end{aligned}$$

This integral can be found in a table, so we get

$$\varphi(r) = \arcsin\left(\frac{-2}{r\sqrt{4b^2}}\right) + c.$$

Setting $c = 0$, we get

$$\sin(\varphi) = \pm\frac{1}{br} \implies r = \frac{a}{\sin(\varphi)},\ r \geq 0.$$

4 Forces of constraint

When using the method of undetermined multiplier, we do not assume $\dot{r} = 0$, but rather enforce this by a constraint. As we have found before, the kinetic energy is

$$T = \frac{1}{2}m\left(\dot{r}^2 + (r\dot{\beta})^2\right),$$

and the potential energy is

$$V = -mgr\cos(\beta).$$

The constraint needed for a pendulum of length ℓ is $\ell - r = 0$, so the Lagrangian becomes

$$L = \frac{1}{2}m\left(\dot{r}^2 + (r\dot{\beta})^2\right) + mgr\cos(\beta) + \lambda\left(\ell - r\right).$$

This gives the equations of motion

$$m\ddot{r} - mr\dot{\beta}^2 - mg\cos(\beta) + \lambda = 0, \tag{1}$$

$$mr^2\dot{\beta} + mgr\sin(\beta) = 0, \tag{2}$$

as well as the constraint $\ell - r = 0$. This implies $\dot{r} = \ddot{r} = 0$. The second equation gives, as before,

$$\dot{\beta} + \frac{g}{\ell}\sin(\beta) = 0,$$

while the first gives

$$\lambda = m(\ell\dot{\beta}^2 + g\cos(\beta)).$$

The tension in a pendulum string is the sum of the component of the gravitational force parallel to the string ($mg\cos(\beta)$), and the centripetal force acing on the mass to keep it going in a circle ($m\ell\dot{\beta}^2$). Thus, we recognize λ as the tension in the pendulum string.