

# Exercise 2 solutions - TFY4345 Classical Mechanics

2020

## 1 Damped oscillator

(a) The frictional force is

$$F_f = -\frac{\partial \mathcal{F}}{\partial \dot{x}}.$$

The work done by friction is force times distance, so the work per unit time is

$$\dot{W}_f = -F_f \dot{x} = \frac{\partial \mathcal{F}}{\partial \dot{x}} \dot{x} \implies \mathcal{F} = C\dot{x}^2.$$

(As  $\mathcal{F}$  is a (velocity) potential, we can dismiss any constants, just as with regular potentials.) This means

$$\dot{W}_f = 2C\dot{x}^2 = 2\mathcal{F}.$$

(b) The Lagrangian with a velocity-dependent potential is

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} + \frac{\partial \mathcal{F}}{\partial \dot{x}} = 0.$$

Inserting the Lagrangian for a harmonic oscillator,

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2,$$

and the given velocity potential  $\mathcal{F} = 3\pi\mu a\dot{x}^2$ , we get

$$\begin{aligned} \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} &= m\ddot{x}, \quad \frac{\partial L}{\partial x} = -kx, \quad \frac{\partial \mathcal{F}}{\partial \dot{x}} = 6\pi\mu a\dot{x}, \\ \implies m\ddot{x} + 6\pi\mu a + kx &= 0, \end{aligned}$$

or

$$\ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = 0, \quad \lambda = \frac{3\pi\mu a}{m}, \quad \omega_0 = \sqrt{\frac{k}{m}}.$$

(c) If we assume the solution to be of the form

$$x(t) = Ae^{\omega_a t} + Be^{\omega_b t},$$

we get

$$A(\omega_0^2 + 2\lambda\omega_a + \omega_a^2)e^{\omega_a t} + B(\omega_0^2 + 2\lambda\omega_b + \omega_b^2)e^{\omega_b t} = 0,$$

so

$$\omega_{a/b} = -\lambda \pm \sqrt{\lambda^2 - \omega_0^2}.$$

This gives us

$$x(t) = e^{-\lambda t} \left( A \exp \left[ \omega_0 t \sqrt{(\lambda/\omega_0)^2 - 1} \right] + B \exp \left[ -\omega_0 t \sqrt{(\lambda/\omega_0)^2 - 1} \right] \right).$$

Now, as  $\lambda/\omega_0 \ll 1$ , we get that  $\sqrt{(\lambda/\omega_0)^2 - 1} \approx i$ . This means that, after applying the initial conditions, we get the solution

$$x(t) = x_0 e^{-\lambda t} \cos(\omega_0 t).$$

The instantaneous energy dissipation is therefore

$$\dot{W}_f = F_f \dot{x} = 2\lambda m \dot{x}^2 = 2\lambda m (\omega_0 x_0)^2 e^{-2\lambda t} \cos^2(\omega_0 t).$$

If we then take the average over a period  $2\pi/\omega_0$ , we can assume the exponential is more or less constant (as  $\lambda \ll \omega_0$ ), so the time averaged dissipation is

$$\overline{\dot{W}_f} = \frac{\omega_0}{2\pi} \int_0^{2\pi} dt \, 2\lambda m (\omega_0 x_0)^2 e^{-2\lambda t} \cos^2(\omega_0 t) \approx \frac{\lambda m (\omega_0 x_0)^2}{\pi} e^{-2\lambda t} \int_0^{2\pi} dx \cos^2(x) = m\lambda (\omega_0 x_0)^2 e^{-2\lambda t}.$$

## 2 Operator identities

The Einstein summation convention is used throughout this exercise, so repeated indices are summed over. The  $i$ 'th component of the curl of the curl of  $\mathbf{A}$  can be written

$$[\nabla \times (\nabla \times \mathbf{A})]_i = \varepsilon_{ijk} \partial_j [\nabla \times \mathbf{A}]_k = \varepsilon_{ijk} \partial_j \varepsilon_{klm} \partial_l A_m = \varepsilon_{ijk} \varepsilon_{klm} \partial_j \partial_l A_m.$$

Using the fact that  $\varepsilon_{ijk} = \varepsilon_{kij}$ , and the identity  $\varepsilon_{ijk} \varepsilon_{imn} = \delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km}$ , this becomes

$$[\nabla \times (\nabla \times \mathbf{A})]_i = (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \partial_j \partial_l A_m = \partial_j \partial_i A_j - \partial_j \partial_j A_i = \nabla \cdot (\nabla_i \mathbf{A}) - (\nabla \cdot \nabla) A_i,$$

or

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla \cdot (\nabla \mathbf{A}) - \nabla^2 \mathbf{A}.$$

## 3 Shortest path in polar coordinates

A distance element in polar coordinates are given by

$$ds = \sqrt{dr^2 + r^2 d\varphi^2} = \sqrt{1 + r^2 \varphi'^2} dr.$$

This means the distance along a curve  $\mathcal{C}$  is

$$s = \int_{\mathcal{C}} ds = \int_{\mathcal{C}} \sqrt{1 + r^2 \varphi'^2} dr.$$

With  $r$  as our parameter, the Euler equation reads

$$\frac{d}{dr} \frac{\partial f}{\partial \varphi'} - \frac{\partial f}{\partial \varphi} = 0,$$

where  $f(\varphi', r) = \sqrt{1 + r^2 \varphi'^2}$  is what we are trying to minimize. We see that

$$\frac{\partial f}{\partial \varphi} = 0 \implies \frac{d}{dr} \frac{\partial f}{\partial \varphi'} = 0,$$

so

$$\begin{aligned} \frac{\partial}{\partial \varphi'} \sqrt{1 + r^2 \varphi'^2} &= \frac{-r^2 \varphi'}{\sqrt{1 + r^2 \varphi'^2}} = a \\ \implies \frac{\varphi'^2 r^2}{a} &= \sqrt{1 + r^2 \varphi'^2} \\ \implies \frac{\varphi'^4 r^4}{a^2} &= 1 + r^2 \varphi'^2 \\ \implies \left( \frac{r^2}{a^2} - 1 \right) r^2 \varphi'^2 &= 1. \end{aligned}$$

Defining  $b^2 = 1/a^2$ , we get

$$\begin{aligned} \varphi'^2 &= \frac{1}{r^2(b^2 r^2 - 1)} \\ \implies \varphi' &= \pm \frac{1}{r \sqrt{b^2 r^2 - 1}} \\ \implies \varphi(r) &= \pm \int_{r_0}^r \frac{dr}{r \sqrt{b^2 r^2 - 1}}. \end{aligned}$$

This integral can be found in a table, so we get

$$\varphi(r) = \arcsin \left( \frac{-2}{r \sqrt{4b^2}} \right) + c.$$

Setting  $c = 0$ , we get

$$\sin(\varphi) = \pm \frac{1}{br} \implies r = \frac{a}{\sin(\varphi)}, \quad r \geq 0.$$

## 4 Forces of constraint

When using the method of undetermined multiplier, we do not assume  $\dot{r} = 0$ , but rather enforce this by a constraint. As we have found before, the kinetic energy is

$$T = \frac{1}{2} m \left( \dot{r}^2 + (r \dot{\beta})^2 \right),$$

and the potential energy is

$$V = -mgr \cos(\beta).$$

The constraint needed for a pendulum of length  $\ell$  is  $\ell - r = 0$ , so the Lagrangian becomes

$$L = \frac{1}{2} m \left( \dot{r}^2 + (r \dot{\beta})^2 \right) + mgr \cos(\beta) + \lambda (\ell - r).$$

This gives the equations of motion

$$m\ddot{r} - mr\dot{\beta}^2 - mg \cos(\beta) + \lambda = 0, \quad (1)$$

$$mr^2\dot{\beta} + mgr \sin(\beta) = 0, \quad (2)$$

as well as the constraint  $\ell - r = 0$ . This implies  $\dot{r} = \ddot{r} = 0$ . The second equation gives, as before,

$$\dot{\beta} + \frac{g}{\ell} \sin(\beta) = 0,$$

while the first gives

$$\lambda = m(\ell\dot{\beta} + g \cos(\beta)).$$

The tension in a pendulum string is the sum of the component of the gravitational force parallel to the string ( $mg \cos(\beta)$ ), and the centripetal force acting on the mass to keep it going in a circle ( $m\ell\dot{\beta}^2$ ). Thus, we recognize  $\lambda$  as the tension in the pendulum string.