Classical Mechanics TFY 4345 – Exercise 5

1. Effective potential and scattering centre.

A particle with mass *m* moves in an attractive potential $V(r)$. Show, on the basis of energy conservation, how the problem can be looked upon as a one-dimensional problem with effective potential $V'(r)$. What is the condition for the particle to reach the scattering centre, $r = 0$?

[Hint: Once setting the condition between E and V' consider the limit where $r \rightarrow 0$.]

2. Scattering from a spherical obstacle.

a) Find the differential cross section $\sigma(\Theta)$ for scattering against a hard sphere of radius a. Notice that when the impact parameter is $s = 0$ the scattering angle is $\Theta = \pi$, and when $s > a$ the scattering angle is $\Theta = 0$ (no scattering).

Hint: use simple geometrical considerations to find a relation between the impact parameter s and the scattering angle Θ.

b) Calculate the total cross section σ. Why is the final expression for σ reasonable?

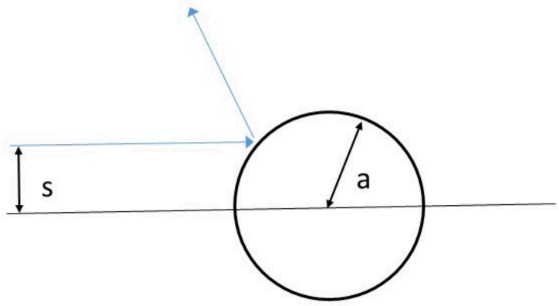

Figure 1: Scattering from a sphere of radius a. Impact parameter: s

3. Scattering by an attractive hard sphere.

A hard sphere has radius a. For $r > a$, the sphere yields a Kepler potential $V = -k/r$, where $k > 0$. Particles coming in from infinity have mass m and original velocity v_0. The part of the particles having impact parameter $s \leq s_{max}$, will hit the sphere's surface. Find s_{max} and the corresponding "effective" scattering cross section $\sigma_{eff} = \pi s^2_{max}$.

[Hint: Consider the conservation of energy and angular momentum.]

[continues →]

4. Average energies in the Kepler problem. [from previous years, see lecture notes Eq. (4.28)]

(a) The solution to the Kepler problem (particle in a central force potential $V = -\frac{k}{r}$) is given by the following expression (polar coordinates):

$$r = \frac{p}{1 + \epsilon \cos(\theta)} \tag{1}$$

where p and ϵ are constant, given in the compendium by I. Brevik (Eq. 4.15). Use the expression defining ϵ and p to show that the total energy is

$$E = -\frac{k}{2p}(1 - \epsilon^2) \tag{2}$$

Use the virial theorem and find the average kinetic energy $\langle T \rangle$, and the average potential energy $\langle V \rangle$.
(b) The average potential energy can be calculated as the average over one orbit period:

$$\langle V \rangle = \frac{1}{t_p} \int_0^{t_p} dt\, V \tag{3}$$

where t_p is the orbital period (time for one complete orbit). Find $\langle V \rangle$ by direct calculation of this integral (not using the virial theorem).

Hint1: change integration variable from t to θ.
Hint2: The Residue method (E. Kreyzig, 9th edition, Chapter 16.4) gives:

$$\int_0^{2\pi} \frac{d\theta}{[1 + \epsilon \cos(\theta)]} = \frac{2\pi}{\sqrt{1 - \epsilon^2}} \tag{4}$$

Hint3:

$$\int_0^{2\pi} \frac{d\theta \cos(\theta)}{[1 + \epsilon \cos(\theta)]^2} = -\frac{d}{d\epsilon} \int_0^{2\pi} \frac{d\theta}{[1 + \epsilon \cos(\theta)]} \tag{5}$$

(c) The average kinetic energy is:

$$\langle T \rangle = \frac{1}{t_p} \int_0^{t_p} dt\, T = \frac{1}{t_p} \int_0^{t_p} dt\, \frac{1}{2} m \left(\frac{d\vec{r}}{dt} \right)^2 \tag{6}$$

Find $\langle T \rangle$ by direct calculation of this integral.
Hint: Integrate Eq. (6) by parts, and use Newton's equation $m\frac{d^2\vec{r}}{dt^2} = -\nabla V = -\frac{k}{r^3}\vec{r}$
Note also that the integration (by parts) is taken over a full period where the particle has returned back to its starting position!