Exercise 8 solutions - TFY4345 Classical Mechanics

2020

1 Principal moments of inertia of a triangular slab

(a) Since the mas has uniform density, we can write the mass area density as $M = 1/2ab\rho$. Let x_{CM} denote the x-component of the center of mass. Using the definition of CM, we find

$$x_{CM} = \frac{1}{M}\int_0^a \mathrm{d}x \int_0^{b(1-x/a)} \mathrm{d}y \rho x = \frac{\rho b}{M}\int_0^a \mathrm{d}x \left(1 - \frac{x}{a}\right) = \frac{a^2 b\rho}{M}\int_0^1 \mathrm{d}u(1-u)u = \frac{\rho a^2 b}{6M} = \frac{a}{3}.$$

We used the substitution $u = 1 - x/a$ which implies a $\mathrm{d}x = -a\mathrm{d}u$. Because of the symmetry in the problem (the slab is a triangle), the calculation of y_{CM} is the same, only exchanging $a \leftrightarrow b$, so the result is $y_{CM} = \mathrm{b}/3$.

(b) The slab is two dimensional, and laying in the xy-plane. If we look at the definition of the off-diagonal entries in moment of inertia tensor,

$$I_{ij} = -\int_V \mathrm{d}V x_i x_j,$$

$I_{zx} = I_{xz} = I_{zy} = I_{yz} = 0$, as z = 0. This also implies that $I_{xx} + I_{yy} = I_{zz}$, so all we need to calculate is I_{xx}, I_{yy} and I_{xy}.

$$\begin{aligned}
I_{xy} &= -\rho\int_0^a \mathrm{d}x \int_0^{v(1-x/a)} \mathrm{d}y yx = -\frac{\rho b^2}{2}\int_0^a \mathrm{d}x x\left(1 - \frac{x}{a}\right)^2 = -\frac{\rho b^2}{2}\int_0^a \mathrm{d}x \left(x - \frac{2}{a}x^2 + \frac{1}{a^2}x^3\right) \\
&= -\frac{\rho b^2 a^2}{2}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = \frac{Mab}{12} \\
I_{xy} &= -\rho\int_0^a \mathrm{d}x \int_0^{v(1-x/a)} \mathrm{d}y y^2 = \frac{\rho b^3}{3}\left(1 - \frac{x}{a}\right)^3 = \frac{\rho a b^3}{3}\int_0^1 \mathrm{d}u u^3 = \frac{Mb^2}{6}.
\end{aligned}$$

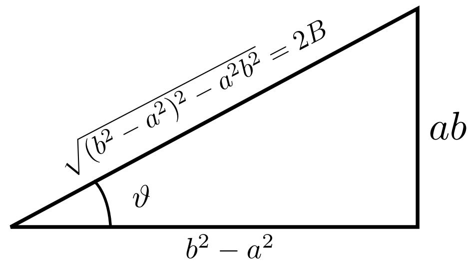
Lastly, I_{yy} can a gain be found just by the exchange $a \leftrightarrow b$. In matrix form,

$$I = \frac{M}{6}\begin{pmatrix} b^2 & -\frac{1}{2}ab & 0 \\ -\frac{1}{2}ab & a^2 & 0 \\ 0 & 0 & a^2 + b^2 \end{pmatrix}$$

(c) We can remove the common factor $M/6$, so insert our values into the new variables, we get

$$A = \frac{1}{2}(a^2 + b^2), \quad B = \frac{1}{2}\sqrt{(b^2 - a^2) + a^2 b^2}, \quad \vartheta = \tan^{-1}\left(\frac{ab}{b^2 - a^2}\right).$$

The last equation describes a triangle with side lengths b^2-a^2, ab and $\sqrt{(b^2-a^2)+a^2b^2}=2B$, and an angle ϑ opposite the side of length ab. This gives us the relations $ab = 2B\sin(\vartheta)$ and $b^2-a^2 = 2B\cos(\vartheta)$. It follows that



$$
\begin{aligned}
a^2 &= \frac{1}{2}(b^2+a^2) - \frac{1}{2}(b^2-a^2) = A - B\cos(\vartheta)\\
b^2 &= \frac{1}{2}(b^2+a^2) + \frac{1}{2}(b^2-a^2) = A + B\cos(\vartheta)
\end{aligned}
$$

Putting all this together, we get

$$
I = \frac{M}{18}\begin{pmatrix} A+B\cos(\vartheta) & B\sin(\vartheta) & 0\\ B\sin(\vartheta) & A-B\cos(\vartheta) & 0\\ 0 & 0 & 2A\end{pmatrix}
$$

To find the principal moments of inertia, we must find solve the characteristic equation for the principal moments of inertia ω

$$
\begin{aligned}
\det(I-\omega)=0 \implies &\begin{vmatrix} A+B\cos(\vartheta)-\omega & B\sin(\vartheta) & 0\\ B\sin(\vartheta) & A-B\cos(\vartheta)-\omega & 0\\ 0 & 0 & 2A-\omega\end{vmatrix}\\
&= (2A-\omega)\big[(A+B\cos(\vartheta)-\omega)(A-B\cos(\vartheta)-\omega)-B\sin(\vartheta)\big]\\
&= (2A-\omega)[A^2-B^2+\omega^2-2\omega A]\\
&= (2A-\omega)[(A-\omega)^2-B^2]=0,
\end{aligned}
$$

which has the solutions $\omega_1=2A$, $\omega_2=A+B$ and $\omega_3=A-B$. By inspection, the first eigenvector is $\mathbf{v}=(0,0,1)$. We can then only look at the relevant part of the matrix to find the others. Inserting $\omega=A+B$,

$$
0 = (I-\omega\mathbb{1})\mathbf{v} = B\begin{pmatrix}\cos(\vartheta)-1 & \sin(\vartheta)\\ \sin(\vartheta) & -\cos(\vartheta)-1\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}
$$

Setting $v_2=1$, we get $v_1=\sin(\vartheta)/(1-\cos(\vartheta))$. The normalized eigenvector then becomes

$$
\begin{aligned}
\mathbf{v} &= \frac{1}{\sqrt{1+\frac{\sin^2(\vartheta)}{(1-\cos(\vartheta))^2}}}\left(\frac{\sin(\vartheta)}{1-\cos(\vartheta)},1,0\right) = \frac{1-\cos(\vartheta)}{\sqrt{1-2\cos(\theta)+\cos^2(\vartheta)+\sin^2(\vartheta)}}\left(\frac{\sin(\vartheta)}{1-\cos(\vartheta)},1,0\right)\\
&= \frac{1-\cos(\vartheta)}{\sqrt{1-\cos(\vartheta)}}\left(\frac{\sin(\vartheta)}{1-\cos(\vartheta)},1,0\right) = \frac{1}{\sqrt{2}\sin(\vartheta/2)}\left(\sin(\vartheta),2\sin^2(\vartheta/2),0\right)
\end{aligned}
$$

Using $2\sin(\vartheta/2)\cos(\vartheta/2)=\sin(\vartheta)$, this gives (HVOR KOMMER $\sqrt{2}$ FRA????!?!?!)

$$
\mathbf{v} = \left(\sqrt{2}\cos(\vartheta/2), -\sqrt{2}\sin(\vartheta/2), 0\right)
$$

The last eigenvector is then found in a similar manner by setting $\omega=A-B$, yielding

$$
\mathbf{v} = (-\sin(\vartheta/2),\cos(\vartheta/2),0)
$$

2 Precession of a frisbee

(a) The Euler equation for the motion of a spinning free body (no torque) is

$$\left(\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}\right)_b + \boldsymbol{\omega} \times \mathbf{L} = 0$$

Writing this out in component form gives

$$\begin{aligned}
I_1\dot{\omega}_{x'} + \omega_{y'}\omega_{z'}(I_3 - I_2) = 0,\\
I_2\dot{\omega}_{y'} + \omega_{z'}\omega_{x'}(I_1 - I_3) = 0,\\
I_3\dot{\omega}_{z'} + \omega_{x'}\omega_{y'}(I_2 - I_1) = 0.
\end{aligned}$$

As shown in the compendium (5.G), the components of the angular velocity in the body frame is

$$\begin{aligned}
\omega_{x'} &= \dot{\phi}\sin(\theta)\sin(\psi) + \dot{\theta}\cos(\psi)\\
\omega_{y'} &= \dot{\phi}\sin(\theta)\cos(\psi) - \dot{\theta}\sin(\psi)\\
\omega_{z'} &= \dot{\phi}\cos(\theta) + \dot{\psi}.
\end{aligned}$$

(b) From the component form of the equations of motion, we see that

$$I_1 = I_2 \implies I_3\dot{\omega}_{z'} = 0 \implies \omega_{z'} = \text{const.}$$

From the figure in the exercise, we can see that $L_{z'} = L\cos(\theta)$. The body axes are the principal axes of the frisbee, so $L_{z'} = I_3\omega_{z'} = \text{const.} \implies \theta = \text{const.}$, i.e. $\dot{\theta} = 0$. Using the Euler equation, we then get

$$\dot{\omega}_{x'} = -\Omega\omega_{y'}, \quad \dot{\omega}_{y'} = \Omega\omega_y, \quad \Omega = \frac{I_3 - I_1}{I_1}\omega_{z'}.$$

This is the equation of two sinusoidal functions, 90° out of phase. (An example of a solution is $\omega_{x'} = \cos(\Omega), \omega_{y'} = \sin(\Omega t)$). This implies[1]

$$\omega_{x'}^2 + \omega_{y'}^2 = \left(\dot{\phi}\sin(\theta)\sin(\psi)\right)^2 + \left(\dot{\phi}\sin(\theta)\cos(\psi)\right)^2 = \dot{\phi}^2\sin(\theta) = \text{const.},$$

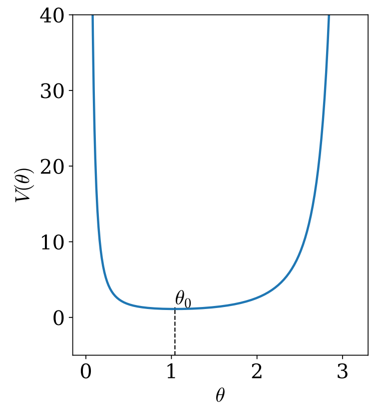
i.e. that $\phi = \text{const.}$.

3 Precession of a heavy spinning top

[1]A more general proof for this is that $\frac{\mathrm{d}}{\mathrm{d}t}\left(\omega_{x'}^2 + \omega_{y'}^2\right) = 2\omega_{x'}\dot{\omega}_{x'} + 2\omega_{y'}\dot{\omega}_{y'} = -2\Omega\dot{\omega}_{y'}\dot{\omega}_{x'} + 2\Omega\dot{\omega}_{y'}\dot{\omega}_{x'} = 0$

The effective potential of the spinning top is

$$V(\theta) = \frac{(p_\phi - p_\psi \cos(\theta))^2}{2I_1 \sin(\theta)^2}.$$



The shape is shown in the plot. This case is similar to that of orbiting planets. In that case, the centrifugal force creates an effective potential as a function of r, with a minimum. If the planet has an energy corresponding to that minimum, it is in a circular orbit, with constant radius. In this case, the effective potential is a function of the angle. Thus, the stable configuration with constant $\theta = \theta_0$ corresponds to the minimum of the potential. This is found by differentiating $V(\theta)$ and setting it equal to zero:

$$\left.\frac{\partial V}{\partial \theta}\right|_{\theta=\theta_0} = \frac{-\cos(\theta_0)(p_\phi - p_\psi \cos(\theta_0)^2 + p_psi \sin^2(\theta_0)(p_\phi - p_\psi \cos(\theta_0)))}{I_1 \sin^3(\theta_0)} - Mgh \sin(\theta_0) = 0.$$

Defining $\beta = p_\phi - p_\psi \cos(\theta_0)$, we can simplify this to

$$0 = \cos(\theta_0)\beta^2 - p_\psi \sin^2(\theta_0)\beta + MghI_1 \sin^4(\theta_0) = 0,$$

so we can solve for β:

$$\beta_\pm = \frac{p_\psi \sin^2(\theta_0)}{2\cos(\theta_0)} \left(1 \pm \sqrt{1 - \frac{4MgHI_1 \cos(\theta_0)}{p_\psi^2}}\right).$$

We can see that β must be a real quantity from its definition. Thus, if $\theta_0 < \pi/2$, so $\cos(\theta_0) > 0$, we get the restriction

$$p_\psi^2 \geq 4MghI_1 \cos(\theta_0)$$

on physical configurations of the system. Inserting $p_\psi = I_3\omega_3$, we get

$$\omega_3 \geq \frac{2}{I_3}\sqrt{MghI_1 \cos(\theta_0)}.$$

This is a lower bound for the angular momentum needed by the spinning to be able to precess at an constant angle θ.

The rate of precession is then given by

$$\dot{\phi}_{0(\pm)} = \frac{\beta_\pm}{I_1 \sin^2(\theta_0)}.$$

This means we have to different configurations with stable precession, given by the two roots $\beta_\pm$. $\dot{\phi}_{0(+)}$ gives fast precession, while $\dot{\phi}_{0(+)}$ gives slow precession. if $\omega_3 \gg \frac{2}{I_3}\sqrt{MghI_1 \cos(\theta_0)}$, then $p_\psi^2 \gg 4MghI_1 \cos(\theta_0)$. We can us $\sqrt{1+x} = 1 - \frac{1}{2}x + \mathcal{O}(x^2)$ then expand the root in equation for

β as

$$\sqrt{1-\frac{4MghI_1\cos(\theta_0)}{p_\psi^2}} \approx 1+\frac{2MghI_1\cos(\theta_0)}{p_\psi^2}, \quad p_\psi = I_3\omega_3$$
$$\implies \beta_\pm \approx \frac{I_3\omega_3\sin^2(\theta_0)}{2\cos(\theta_0)}\left(\frac{(I_3\omega_3)^2(1\pm 1)+2MghI_1\cos(\theta_0)}{(I_3\omega_3)^2}\right)$$
$$\implies \dot{\phi}_{0(\pm)} \approx \frac{(I_3\omega_3)^2(1\pm 1)+2MghI_1\cos(\theta_0)}{2I_1I_3\omega_3\cos(\theta_0)} = \frac{I_3\omega_3}{2I_1\cos(\theta_0)}(1\pm 1)+\frac{Mgh}{I_3\omega_3}.$$

The two stable configurations thus precess with the angular velocities

$$\dot{\phi}_{0(+)} = \frac{I_3\omega_3}{I_1\cos(\theta_0)}, \quad \dot{\phi}_{0(-)} = \frac{Mgh}{I_3\omega_3}$$