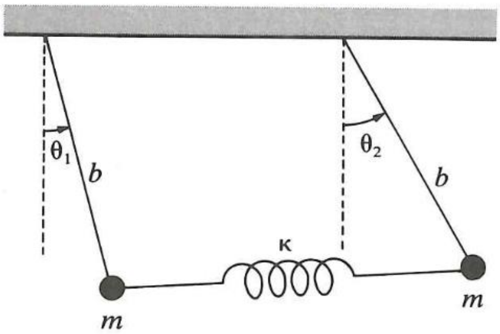
Classical Mechanics TFY 4345 – Solution set 6 (lectures 31-36)

1. Coupled pendula.



Solution: We choose θ_1 and θ_2 (Figure 12-5) as the generalized coordinates. The potential energy is chosen to be zero in the equilibrium position. The kinetic and potential energies of the system are, for small angles,

$$T = \tfrac{1}{2}m(b\dot{\theta}_1)^2 + \tfrac{1}{2}m(b\dot{\theta}_2)^2 \quad \textbf{(12.81)}$$

$$U = mgb(1 - \cos\theta_1) + mgb(1 - \cos\theta_2)$$
$$+\tfrac{1}{2}\kappa(b\sin\theta_1 - b\sin\theta_2)^2 \quad \textbf{(12.82)}$$

Using the small oscillation assumption $\sin\theta \approx \theta$ and $\cos\theta \approx 1 - \theta^2/2$, we can write

$$U = \frac{mgb}{2}(\theta_1^2 + \theta_2^2) + \frac{\kappa b^2}{2}(\theta_1 - \theta_2)^2 \quad \textbf{(12.83)}$$

The components of $\{\mathbf{A}\}$ and $\{\mathbf{m}\}$ are

$$\{\mathbf{m}\} = \begin{Bmatrix} mb^2 & 0 \\ 0 & mb^2 \end{Bmatrix} \quad \textbf{(12.84)}$$

$$\{\mathbf{A}\} = \begin{Bmatrix} mgb^2 + \kappa b^2 & -\kappa b^2 \\ -\kappa b^2 & mgb + \kappa b^2 \end{Bmatrix} \quad \textbf{(12.85)}$$

The determinant needed to find the eigenfrequencies ω is

$$\begin{vmatrix} mgb + \kappa b^2 - \omega^2 mb^2 & -\kappa b^2 \\ -\kappa b^2 & mgb + \kappa b^2 - \omega^2 mb^2 \end{vmatrix} = 0 \quad \textbf{(12.86)}$$

which gives the characteristic equation

$$b^2(mg + \kappa b - \omega^2 mb)^2 - (\kappa b^2)^2 = 0$$
$$(mg + \kappa b - \omega^2 mb)^2 = (\kappa b)^2$$

or

$$mg + \kappa b - \omega^2 mb = \pm\kappa b \quad \textbf{(12.87)}$$

Taking the plus sign, $\omega = \omega_1$,

$$mg + \kappa b - \omega_1^2 mb = \kappa b$$

$$\omega_1^2 = \frac{g}{b} \quad \textbf{(12.88)}$$

Taking the minus sign in Equation 12.87, $\omega = \omega_2$,

$$mg + \kappa b - \omega_2^2 mb = -\kappa b$$

$$\omega_2^2 = \frac{g}{b} + \frac{2\kappa}{m} \quad \textbf{(12.89)}$$

Putting the values of ω_1 and ω_2 into Equation 12.40 gives, for $k = 1$,

$$(mgb + \kappa b^2 - \omega_r^2 mb^2)a_{1r} - \kappa b^2 a_{2r} = 0 \quad \textbf{(12.90)}$$

If $r = 1$, then

$$\left(mgb + \kappa b^2 - \frac{g}{b}mb^2\right)a_{11} - \kappa b^2 a_{21} = 0$$

and

$$a_{11} = a_{21} \quad \textbf{(12.91)}$$

If $r = 2$, then

$$\left(mgb + \kappa b^2 - \frac{g}{b}mb^2 - \frac{2\kappa}{m}mb^2\right)a_{12} - \kappa b^2 a_{22} = 0$$

and

$$a_{12} = -a_{22} \quad \textbf{(12.92)}$$

We write the coordinates θ_1 and θ_2 in terms of the normal coordinates by

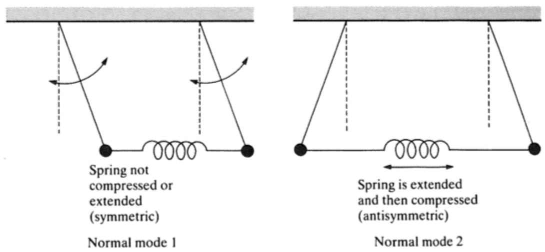
$$\left.\begin{aligned} \theta_1 &= a_{11}\eta_1 + a_{12}\eta_2 \\ \theta_2 &= a_{21}\eta_1 + a_{22}\eta_2 \end{aligned}\right\} \quad \textbf{(12.93)}$$

Using Equations 12.91 and 12.92, Equations 12.93 become

$$\left.\begin{aligned} \theta_1 &= a_{11}\eta_1 - a_{22}\eta_2 \\ \theta_2 &= a_{11}\eta_1 + a_{22}\eta_2 \end{aligned}\right\} \quad \textbf{(12.94)}$$

The normal modes are easily determined, by adding and subtracting θ_1 and θ_2, to be

$$\left.\begin{aligned} \eta_1 &= \frac{1}{2a_{11}}(\theta_1 + \theta_2) \\ \eta_2 &= \frac{1}{2a_{22}}(\theta_2 - \theta_1) \end{aligned}\right\} \quad \textbf{(12.95)}$$



2. Two coupled oscillators.

$$T = \frac{1}{2} m \dot{x}_1^{\,2} + \frac{1}{2} m \dot{x}_2^{\,2}$$

$$V = \frac{1}{2} k x_1^2 + \frac{1}{2} k (x_2 - x_1)^2$$

$$\Longrightarrow \quad \bar{\bar{m}} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$$

$$V = \frac{1}{2} k x_1^2 + \frac{1}{2} k \left(x_2^2 - 2x_1 x_2 + x_1^2 \right)$$

$$= 2 \cdot \frac{1}{2} k x_1^2 + \frac{1}{2} k x_2^2 - \frac{1}{2} k x_1 x_2 - \frac{1}{2} k x_2 x_1$$

$$\equiv \frac{1}{2} \sum_{j,k} A_{jk} q_j q_k$$

$$\Longrightarrow \quad \bar{\bar{A}} = \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}$$

Eigenfrequencies

$$\begin{vmatrix} 2k - \omega^2 m & -k \\ -k & k - \omega^2 m \end{vmatrix} = 0$$

$$\Longrightarrow \quad (2k - \omega^2 m)(k - \omega^2 m) - k^2 = 0$$

$$\Longrightarrow \quad \dots \quad \Longrightarrow \quad \omega^2 = \frac{k}{2m} \left(3 \pm \sqrt{5} \right) \qquad (>0)$$

Eigenfrequencies:

1° $\quad (A_{11} - \omega_1^2 m_{11}) a_{11} + (A_{21} - \omega_1^2 M_{12}) a_{12} = 0$

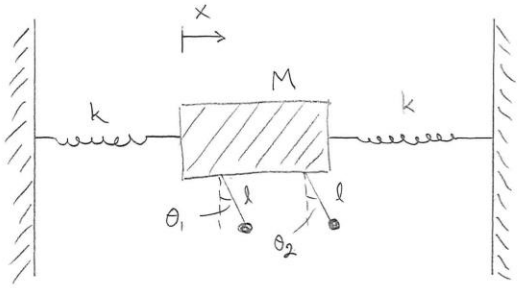
$$\Longrightarrow \quad a_{11} = \frac{2}{1 + \sqrt{5}} a_{12} \qquad \bar{a}_1 = c_1 \cdot \begin{bmatrix} 1 \\ \frac{2}{1+\sqrt{5}} \end{bmatrix}$$

2° $\quad (A_{21} - \omega_2^2 m_{21}) a_{21} + (A_{22} - \omega_2^2 m_{22}) a_{22} = 0$

$$\Longrightarrow \quad a_{22} = \frac{2}{1 + \sqrt{5}} a_{21} \qquad \bar{a}_2 = c_2 \begin{bmatrix} \frac{2}{1+\sqrt{5}} \\ 1 \end{bmatrix}$$

3. Oscillating body with two attached pendula.

3.



Equilibrium: $x=0,\ \theta_1=\theta_2=0$

$V_1 = 2\cdot\frac{1}{2}kx^2 = kx^2$

$V_2 = Mg\ell(1-\cos\theta) \approx \frac{1}{2}Mg\ell\theta_1^2$

$V_3 = \frac{1}{2}Mg\ell\theta_2^2$

$$V = kx^2 + \frac{1}{2}mg\ell(\theta_1^2+\theta_2^2) \equiv \frac{1}{2}\sum_{j,k}^{3} A_{jk}\,q_j q_k$$

$$\Rightarrow \quad \bar{\bar{A}} = \begin{bmatrix} 2k & 0 & 0 \\ 0 & mg\ell & 0 \\ 0 & 0 & mg\ell \end{bmatrix}$$

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}+\ell\dot{\theta}_1)^2 + \frac{1}{2}m(\dot{x}+\ell\dot{\theta}_2)^2 \equiv \frac{1}{2}\sum m_{jk}\dot{q}_j\dot{q}_k$$

$$\Rightarrow \quad \bar{\bar{m}} = \begin{bmatrix} M+2m & m\ell & m\ell \\ m\ell & m\ell^2 & 0 \\ m\ell & 0 & m\ell^2 \end{bmatrix} \qquad \text{(non-diagonal)}$$

Eigenfrequencies:

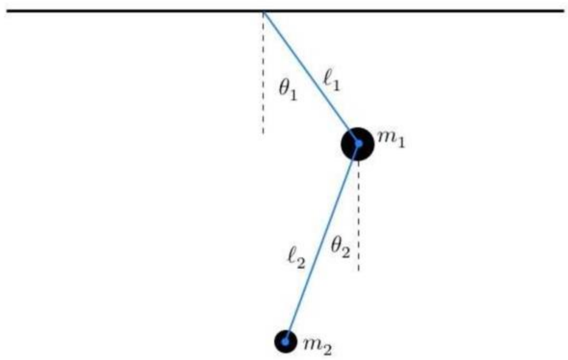
$$\begin{vmatrix} 2k-\omega_r^2(M+2m & -\omega_r^2 m\ell & -\omega_r^2 m\ell \\ -\omega_r^2 m\ell & mg\ell-\omega_r^2 m\ell & 0 \\ -\omega_r^2 m\ell & 0 & mg\ell-\omega_r^2 m\ell \end{vmatrix} = 0$$

$$\Rightarrow \ \dots \ \Rightarrow \quad \omega_r^2 = \frac{g}{\ell} \qquad \text{(three roots)}$$

$$\text{or} \quad \omega_r^2 = \frac{g}{2\ell}\left(1+\frac{2m}{M}\right)+\frac{k}{M} \pm \sqrt{\frac{g}{2\ell}\left[\left(1+\frac{2m}{m}\right)+\frac{k}{M}\right]^2 - 2\frac{k}{m}g\ell}$$

4. Double pendulum. (Here: Compendium notation where A tensor = "V" and m tensor = "T")

Note: Approximate cosine at small angles



As a second example, consider the double pendulum, with $m_1 = m_2 = m$ and $\ell_1 = \ell_2 = \ell$. The kinetic and potential energies are

$$T = m\ell^2\dot{\theta}_1^2 + m\ell^2\cos(\theta_1 - \theta_1)\,\dot{\theta}_1\dot{\theta}_2 + \tfrac{1}{2}m\ell^2\dot{\theta}_2^2 \tag{10.39}$$

$$V = -2mg\ell\cos\theta_1 - mg\ell\cos\theta_2\ , \tag{10.40}$$

leading to

$$\mathrm{T} = \begin{pmatrix} 2m\ell^2 & m\ell^2 \\ m\ell^2 & m\ell^2 \end{pmatrix} \quad , \quad \mathrm{V} = \begin{pmatrix} 2mg\ell & 0 \\ 0 & mg\ell \end{pmatrix} \ . \tag{10.41}$$

Then

$$\omega^2\mathrm{T} - \mathrm{V} = m\ell^2 \begin{pmatrix} 2\omega^2 - 2\omega_0^2 & \omega^2 \\ \omega^2 & \omega^2 - \omega_0^2 \end{pmatrix} \ , \tag{10.42}$$

with $\omega_0 = \sqrt{g/\ell}$. Setting the determinant to zero gives

$$2(\omega^2 - \omega_0^2)^2 - \omega^4 = 0 \quad \Rightarrow \quad \omega^2 = (2 \pm \sqrt{2})\,\omega_0^2\ . \tag{10.43}$$

We find the unnormalized eigenvectors by setting $(\omega_i^2\,\mathrm{T} - V)\,\psi^{(i)} = 0$. This gives

$$\psi^+ = C_+ \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \quad , \quad \psi^- = C_- \begin{pmatrix} 1 \\ +\sqrt{2} \end{pmatrix} \ , \tag{10.44}$$

where $C_\pm$ are constants.