# Solutions 8: TFY 4345 (lectures 41-46)

1. Generating function $F_4$

$$F = q_i p_i - Q_i P_i + F_4(p, P, t) \qquad (4^{th} \text{ option with } p, P)$$

$$\cancel{p_i \dot{q}_i} - H = \cancel{P_i \dot{Q}_i} - K + \cancel{p_i \dot{q}_i} + \dot{p}_i q_i - \cancel{P_i \dot{Q}_i} - \dot{P}_i Q_i + \frac{d}{dt} F_4(p, P, t)$$

$$\Rightarrow -\dot{p}_i q_i - H = -Q_i \dot{P}_i - K + \frac{\partial F_4}{\partial t} + \frac{\partial F_4}{\partial p_i} \dot{p}_i + \frac{\partial F_4}{\partial P_i} \dot{P}_i$$

$$\Rightarrow \boxed{q_i = -\frac{\partial F_4}{\partial p_i}; \quad Q_i = \frac{\partial F_4}{\partial P_i}; \quad K = H + \frac{\partial F_4}{\partial t}}$$

2. Poisson brackets, canonical transformation

$$H = \frac{1}{2m}(p^2 + m^2 \omega^2 q^2) \qquad \text{harmonic oscillator}$$

$$q = \sqrt{\frac{2P}{m\omega}} \sin Q, \quad p = m\omega q \cot Q, \quad P = \frac{E}{\omega}, \quad H = \omega P$$

$$[q, H]_{q,p} = \underbrace{\frac{\partial q}{\partial q}}_{=1} \frac{\partial H}{\partial p} - \underbrace{\frac{\partial q}{\partial p}}_{=0} \frac{\partial H}{\partial q} = \frac{\partial H}{\partial p} = \frac{p}{m} = \omega q \cot Q$$

$$[q, H]_{Q,P} = \frac{\partial q}{\partial Q} \frac{\partial H}{\partial P} - \frac{\partial q}{\partial P} \underbrace{\frac{\partial H}{\partial Q}}_{=0} = \frac{\partial q}{\partial Q} \frac{\partial H}{\partial P}$$

$$\frac{\partial q}{\partial Q} = \sqrt{\frac{2P}{m\omega}} \cos Q = \underbrace{\sqrt{\frac{2P}{m\omega}} \sin Q}_{q} \cdot \frac{\cos Q}{\sin Q} = q \cot Q$$

$$\frac{\partial H}{\partial P} = \omega \quad \Rightarrow [q, H]_{Q,P} = \omega q \cot Q =: \frac{p}{m}$$

$$[q, H] \neq 0 \quad \Longrightarrow \quad q \text{ is not a constant of motion}$$

3. Canonical transformation (Goldstein 9-6)

$$\begin{cases} Q = \log(1+\sqrt{q}\cos p) \\ P = 2(1+\sqrt{q}\cos p)\sqrt{q}\sin p \end{cases}$$

a) Symplectic condition $\bar{\bar{M}}^T \bar{\bar{J}} \bar{\bar{M}} = \bar{\bar{J}}$

Antisymmetric matrix $\bar{\bar{J}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Jacobian matrix $\bar{\bar{M}} = \begin{bmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{bmatrix}$ $\left[ M_{ij} = \frac{\partial \zeta_i}{\partial \eta_j} \right]$

transformed

original

$$\Rightarrow \bar{\bar{M}} = \begin{bmatrix} \frac{1}{2}\left(\frac{1}{q+\sqrt{q}\sec(p)}\right) & \frac{-\sqrt{q}\sin p}{1+\sqrt{q}\cos p} \\ \left(\frac{1}{\sqrt{q}} + 2\cos p\right)\sin p & 2(\sqrt{q}\cos p + q\cos(2p)) \end{bmatrix}$$

Now: Check $\bar{\bar{M}}^T \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \bar{\bar{M}}$ (for example, Mathematica)

$$\Rightarrow \bar{\bar{M}}^T \bar{\bar{J}} \bar{\bar{M}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \bar{\bar{J}} \quad \checkmark$$

As long as $q, p$ canonical, $Q$ and $P$ are too.

b) Type 3 generating functions

$$q = -\frac{\partial F_3(p, Q, t)}{\partial p}, \quad P = -\frac{\partial F_3(p, Q, t)}{\partial Q}$$

$$F_3 = (p, Q, t) = -(e^Q - 1)^2 \tan p$$

$$\Rightarrow \begin{cases} q = -\frac{\partial F_3}{\partial p} = (e^Q - 1)^2 \sec^2(p) \\ P = -\frac{\partial F_3}{\partial Q} = 2e^Q(e^Q - 1)\tan p \end{cases} \Rightarrow \underline{Q = \log(1+\sqrt{q}\cos p)}$$

few steps

$$\Rightarrow \underline{P} = 2e^{\log(1+\sqrt{q}\cos p)}\left(e^{\log(1+\sqrt{q}\cos p)} - 1\right)\tan p = \ldots = \underline{2(1+\sqrt{q}\cos p)\sqrt{q}\sin p}$$

4. Hamilton-Jacobi theory, free particle

Hamiltonian $H = \frac{p^2}{2m}$, independent of $x$ $(=: q)$

$$H\left(q, \frac{\partial S}{\partial q}; t\right) + \frac{\partial S}{\partial t} = 0$$

$$\Rightarrow H(p) = H\left(\frac{\partial S}{\partial q}\right) = \frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2$$

$$\Rightarrow \frac{\partial S}{\partial t} = -\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2$$

independent of time

$$S = W(x) - Et$$

$$\Rightarrow -E = -\frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 \Rightarrow \frac{\partial S}{\partial x} = \sqrt{2mE} = \alpha$$

$$\Rightarrow W(x) = \sqrt{2mE}\, x + C$$

Now: $S(x,t) = \sqrt{2mE}\, x + C - Et = \alpha x + C - \frac{\alpha^2}{2m} t$

Transformed Hamiltonian:

$$K = H + \frac{\partial S}{\partial t} = 0 \Rightarrow \frac{p^2}{2m} - \frac{\alpha^2}{2m} = 0$$

This means that $p = \alpha =$ constant, set it equal to $p_0$

$$\Rightarrow S(x, p_0, t) = p_0 x - \frac{p_0^2}{2m} t \qquad (C \text{ arbitrary} \to 0)$$

New variables

$$\begin{cases} P = \frac{\partial S}{\partial x} = p_0 \\ Q = \frac{\partial S}{\partial p_0} = x - \frac{p_0}{m} t =: x_0 \end{cases}$$

How does this look in the phase space? Think!

Solution of the original problem:

$$\begin{cases} x = Q + \frac{p_0}{m} t \\ p = p_0 \end{cases}$$

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The new canonical variables are just the initial position $x_0$ and the initial momentum $p_0$.