Exercise 9 solutions - TFY4345 Classical Mechanics

2020

1 Coupled pendula

(FIGUR)
(DET SKAL VÆRE ω^2)
The kinetic energy of the masses are

$$T = \frac{1}{2}m\left[(b\dot{\theta}_1)^2 + (b\dot{\theta}_2)^2\right].$$

The displacement of the masses in the vertical direction is given by $b(1 - \cos(\theta))$. As we are considering small oscillations, we only care about the stretching of the spring due to the horizontal movement of the pendula. This is given by $b(\sin(\theta_1) - \sin(\theta_2))$. Thus, the potential energy of the system is

$$U = mgb\left[(1 - \cos(\theta_1)) + (1 - \cos(\theta_2))\right] + \frac{1}{2}kb^2\left[\sin(\theta_1) - \sin(\theta_2)\right]^2.$$

Using the small angle approximation, we get $\sin(\theta) \approx \theta$, $1 - \cos(\theta) \approx \frac{1}{2}\theta^2$, so the potential energy becomes

$$\frac{1}{2}mgb(\theta_1^2 + \theta_2^2) - \frac{1}{2}kb^2(\theta_1 - \theta_2) = \frac{1}{2}\left((mgb + kb^2)(\theta_1^2 + \theta_2^2) - 2kb^2\theta_1\theta_2\right)$$

The lagrangian is then

$$L = \frac{1}{2}m\left[(b\dot{\theta}_1)^2 + (b\dot{\theta}_2)^2\right] + \frac{1}{2}mgb(\theta_1^2 + \theta_2^2) - \frac{1}{2}kb^2(\theta_1 - \theta_2) = \frac{1}{2}\left(m_{ij}\theta_i\theta_i + A_{ij}\theta_i\theta_j\right),$$

where

$$m = \begin{pmatrix} mb^2 & 0 \\ 0 & mb^2 \end{pmatrix}, \quad A = \begin{pmatrix} mgb + kb^2 & -kb^2 \\ kb^2 & mgb + kb^2 \end{pmatrix}.$$

The eigenfrequencies of the system is then given by the equation

$$\det(A - \omega m) = 0.$$

Writing out the determinant, we get

$$\begin{vmatrix} mgb + kb^2 - \omega mb^2 & -kb^2 \\ kb^2 & mgb + kb^2 - \omega mb^2 \end{vmatrix} = (mgb + kb^2 - \omega mb^2)^2 - (mb^2)^2 = 0,$$

or

$$mgb + kb^2 - \omega mb^2 = \pm mb^2 \implies \omega = \frac{g}{b} + (1 \mp 1)\frac{k}{m}.$$

This leaves us with the eigenfrequencies

$$\omega_1 = \frac{g}{b}, \quad \omega_2 = \frac{g}{b} + 2\frac{k}{m}$$

The equation for the eigenfrequencies $\mathbf{a_i} = (a_{1i}, a_{2i})$, corresponding to ω_i, is

$$(m - \omega_r A)\mathbf{a_r} = (mgb + kb^2 - \omega_r mb^2)a_{1r} - (kb^2)a_{2r} = 0$$

Inserting ω_1, this gives

$$\left(mgb + kb^2 - \frac{g}{b}mb^2\right)a_{1q} - (kb^2)a_{21} = (kb^2)a_{11} - (kb^2)a_{21} = 0 \implies a_{11} = a_{21}.$$

ω_2 gives

$$\left(mgb + kb^2 - \left(\frac{g}{b} + 2\frac{k}{m}\right)mb^2\right)a_{12} - (kb^2)a_{22} = -(kb^2)a_{12} - (kb^2)a_{22} = 0 \implies a_{12} = a_{22}.$$

The solution, in our original coordinates θ_i, in terms of the normal coordinates η_r, is then

$$\begin{cases} \theta_1 = a_{11}\eta_1 + a_{12}\eta_2 = a_{11}\eta_1 + a_{22}\eta_2 \\ \theta_2 = a_{21}\eta_1 + a_{22}\eta_2 = a_{11}\eta_1 - a_{22}\eta_2) \end{cases}$$

To physically describe the normal coordinates, we need to express them in terms of our original coordinates. Adding the equations together, we get

$$\begin{cases} \eta_1 = \frac{1}{2a_{11}}(\theta_1 + \theta_2) \\ \eta_2 = \frac{1}{2a_{22}}(\theta_1 - \theta_2). \end{cases}$$

θ_1 corresponds to the pendula oscillating in sync, while θ_2 corresponds to them oscillating in opposite direction. This explains why the frequency ω_2 is higher than ω_1: only this motion distorts the spring, leading to a greater restoring froce. (FIGUR)

2 Two coupled oscillators

(FIGUR)
Let x_1 and x_2 denote the distance of the two block from A. The kinetic and potential energy of the system is then

$$T = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2), \quad U = \frac{1}{2}k(\dot{x}_1^2 + (x_1 - x_2)^2).$$

Writing out the lagrangian in matrix form gives

$$L = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) + \frac{1}{2}k(2\dot{x}_1^2 - 2\dot{x}_1\dot{x}_2 + \dot{x}_2^2) = \frac{1}{2}\left(m_{ij}\dot{x}_i\dot{x}_j + A_{ij}x_ix_j\right),$$

where

$$m_{ij} = m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad A_{ij} = k \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

If we introduce $\omega_0^2 = k/m$ This means the equation for the eigenfrequencies is

$$\det\left(A - \omega^2 m\right) = \begin{vmatrix} 2\omega_0^2 - \omega & -\omega_0^2 \\ -\omega_0^2 & \omega_0^2 - \omega^2 \end{vmatrix} = (2\omega_0^2 - \omega^2)(\omega_0^2 - \omega^2) - \omega_0^4 = 2\omega_0^4 - 3\omega_0^2\omega^2 + \omega^4 - \omega_0^4$$

$$= (\omega^2)^2 - 3\omega_0^2(\omega^2) + \omega_0^4 = 0 \implies \omega^2 = \frac{1}{2}\left(3\omega_0^2 \pm \sqrt{(3\omega_0^2)^2 - 4\omega_0^4}\right) = \frac{3 \pm \sqrt{5}}{2}\omega_0^2 = \omega_0^2 \varphi_{\pm}$$

The equations for the normal coordinates are

$$\begin{cases} a_{11} = \left(1 - \dfrac{3 + \sqrt{5}}{2}\right) a_{21} = \left(\dfrac{1 + \sqrt{5}}{2}\right) a_{21} \\[4mm] a_{12} = \left(1 - \dfrac{3 - \sqrt{5}}{2}\right) a_{22} = \left(\dfrac{1 - \sqrt{5}}{2}\right) a_{22} \end{cases}$$

The golden ratio! Here, we picked out one of two possible equations. But as they are linearly dependent (we have inserted ω^2 such that the determinant is zero), we could have chosen either one. This modes corresponds to vibration in the same and opposite direction. However, this system is not as symmetrical as the las one, so the amplitudes of the oscillations.

3 Oscillating body with two attached pendula

(FIGUR)

4 Double pendulum

(FIGUR)