Exercise 3 solutions - TFY4345 Classical Mechanics

2020

1 Pendulum on spinning a wheel

With the origin of our coordinate system in the center of the rotating rim, the Cartesian components of the mass m become

$$x = a\cos(\omega t) + b\sin(\theta)$$
$$y = a\sin(\omega t) - b\cos(\theta).$$

The velocities are

$$\dot{x} = -a\omega\sin(\omega t) + b\dot{\theta}\cos(\theta)$$
$$\dot{y} = a\omega\cos(\omega t) + b\dot{\theta}\sin(\theta).$$

Taking the time derivative once again gives the acceleration:

$$\ddot{x} = -a\omega^2\cos(\omega t) + b(\ddot{\theta}\cos(\theta) - \dot{\theta}^2\sin(\theta))$$
$$\ddot{y} = -a\omega^2\sin(\omega t) + b(\ddot{\theta}\sin(\theta) + \dot{\theta}^2\cos(\theta)).$$

It should be clear that the single generalize coordinate is θ. The kinetic and potential energies are

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2),\, V = mgy.$$

Inserting what we found earlier, the Lagrangian becomes

$$L = \frac{1}{2}m[a^2\omega^2 + b\dot{\theta}^2 + 2b\theta^2 a\omega\sin(\theta - \omega t)] - mg[a\sin(\omega t) - b\cos(\theta)].$$

The derivatives needed for the equation of motion are

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{\theta}} = mb^2\ddot{\theta} + mba\omega(\dot{\theta} - \omega)\cos(\theta - \omega t),$$
$$\frac{\partial L}{\partial\theta} = mba\dot{\theta}\omega\cos(\theta - \omega t) - mgb\sin(\theta).$$

Inserting this into Euler's equation, and solving for $\ddot{\theta}$ gives

$$\ddot{\theta} = \frac{\omega^2 a}{b}\cos(\theta - \omega t) - \frac{g}{b}\sin(\theta).$$

Notice that, for $\omega = 0$, this reduces to the equation for the simple pendulum.

2 Bead on a ring

(FIGUR)

The potential energy is given by

$$U = mgh = mgR(1-\cos(\theta)),$$

while the kinteic energy is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left((r\dot{\varphi})^2 + (R\dot{\theta})\right) = \frac{1}{2}mR^2\left(\sin^2(\theta)\dot{\varphi}^2 + \dot{\theta}^2\right).$$

The Euler equation for φ is given by

$$\frac{\partial L}{\partial \varphi} = 0 \implies \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\varphi}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(m\sin(\theta)R^2\dot{\varphi}\right) = 0 \implies \dot{\varphi} = \omega = \text{const.}$$

The equation for θ is given by

$$\frac{\partial L}{\partial \theta} = mR^2\cos(\theta)\sin(\theta)\dot{\varphi}^2 - mgR\sin(\theta),\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} = mR^2\ddot{\theta},$$
$$\implies \ddot{\theta} = R\cos(\theta)\sin(\theta)\dot{\varphi}^2 - g\sin(\theta)$$

In the equilibrium position, we have that $\ddot{\theta} = 0$. This means

$$\cos(\theta) = \frac{g}{R\dot{\varphi}^2} = \frac{g}{R\omega^2}.$$

Note: we could have set $\dot{\varphi} = 0$ right at the beginning, and treat θ as the sole generalized variable.

3 Atwood's machine

The angular velocity of the pulley is

$$\omega = \frac{\dot{x}_2}{a}.$$

This means the kinetic energy of the system is

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}I\omega^2$$

The length of the string is a constant, so

$$x_1 + x_2 = \ell = \text{const.}$$

Inserting this into the kinetic energy gives

$$T = \frac{1}{2}(m_2 - m_1)\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}I\left(\frac{\dot{x}}{a}\right)^2.$$

The potential energy is

$$U = -m_1gx_1 - m_2gx_2 = -m_1g(\ell - x_2) - m_2g(x_2).$$

4 Particle on a moving wedge