Classical Mechanics TFY4345 - Exercise 1

1. Halley's comet - conservation of angular momentum
(a) Gravitational force on the comet :

$$\vec{F} = -\frac{GmM}{r^2}\vec{e}_r \tag{1}$$

where m is the mass of the coment, and M the mass of the Sun. Torque on the comet:

$$\vec{N} = \vec{r} \times \vec{F} = -r\vec{e}_r \times \frac{GmM}{r^2}\vec{e}_r = -\frac{GmM}{r}\vec{e}_r \times \vec{e}_r = 0 \tag{2}$$

(b)
Angular momentum when the comet is closest to the Sun: $\vec{L}_c = \vec{r}_c \times \vec{p}_c$
Angular momentum when the comet is farthest from to the Sun: $\vec{L}_f = \vec{r}_f \times \vec{p}_f$
Conservation of angular momentum implies $\vec{L}_c = \vec{L}_f$, which in turn implies $|\vec{L}_c| = |\vec{L}_f|$
When the comet is closest to the Sun, the position vector $\vec{r}_c$ is perpendicular to the momentum vector $\vec{p}_c$, the same applies when it the comet is farthest from the Sun.
Therfore: $|\vec{L}_c| = |\vec{L}_f| \Rightarrow |\vec{r}_c||\vec{p}_c| = |\vec{r}_f||\vec{p}_f| \Rightarrow r_c m v_c = r_f m v_f$

Comet velocity farthest from the Sun:

$$v_f = \frac{r_c v_c}{r_f} = \frac{0.6AU}{35AU} 54\text{km/s} = 0.9\text{km/s} \tag{3}$$

2. Simple Pendulum
(a)

$$\vec{R} = l\sin(\beta)\vec{e}_x - l\cos(\beta)\vec{e}_y \tag{4}$$

(b) Potential energy in a uniform gravitational field: $V = mgh$, where h is height (arbitrary reference frame), in our case $h = -l\cos(\beta)$, hence

$$V = -mgl\cos(\beta) \tag{5}$$

(c) The velocity of the mass m is

$$\vec{v} = \frac{d\vec{R}}{dt} = l\cos(\beta)\dot{\beta}\vec{e}_x + l\sin(\beta)\dot{\beta}\vec{e}_y \tag{6}$$

Resulting expression for velocity square :

$$\vec{v}^2 = (l\dot{\beta})^2 \tag{7}$$

Kinetic energy:

$$T = \frac{1}{2}m\vec{v}^2 = \frac{1}{2}m(l\dot{\beta})^2 \tag{8}$$

(d) The Lagrangian of the pendulum is

$$L = T - V = \frac{1}{2}m(l\dot{\beta})^2 + mgl\cos(\beta) \tag{9}$$

Lagrange's equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \tag{10}$$

We have only one varable $q_1 = \beta$. The resulting Lagrange equation is:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\beta}} - \frac{\partial L}{\partial \beta} = 0 \tag{11}$$

which implies:

$$\frac{d}{dt}\left(ml^2\dot{\beta}\right) + mgl\sin\beta = 0 \tag{12}$$

The final result is the well know equation of motion for a simple pendulum:

$$\ddot{\beta} + \frac{g}{l}\sin\beta = 0 \tag{13}$$

3. Double Pendulum

(a)

Position of mass m_1:

$$\vec{R}_1 = l_1\sin(\beta_1)\vec{e}_x - l_1\cos(\beta_1)\vec{e}_y \tag{14}$$

Position of mass m_2:

$$\begin{aligned}\vec{R}_2 &= \vec{R}_1 + l_2\sin(\beta_2)\vec{e}_x - l_2\cos(\beta_2)\vec{e}_y \\ &= \left[l_1\sin(\beta_1) + l_2\sin(\beta_2)\right]\vec{e}_x - \left[l_1\cos(\beta_1) + l_2\cos(\beta_2)\right]\vec{e}_y\end{aligned} \tag{15}$$

Velocity and velocity squared of mass m_1

$$\vec{v}_1 = \frac{d\vec{R}_1}{dt} \tag{16}$$

$$\vec{v}_1^{\,2} = (l_1\dot{\beta}_1)^2 \tag{17}$$

Velocity and velocity squared of mass m_2

$$\vec{v}_2 = \frac{d\vec{R}_2}{dt} = \left[l_1\cos(\beta_1)\dot{\beta}_1 + l_2\cos(\beta_2)\dot{\beta}_2\right]\vec{e}_x + \left[l_1\sin(\beta_1)\dot{\beta}_1 + l_2\sin(\beta_2)\dot{\beta}_2\right]\vec{e}_y \tag{18}$$

$$\begin{aligned}\vec{v}_2^{\,2} &= (l_1\dot{\beta}_1)^2 + (l_2\dot{\beta}_2)^2 + 2l_1l_2\dot{\beta}_1\dot{\beta}_2\cos\beta_1\cos\beta_2 + 2l_1l_2\dot{\beta}_1\dot{\beta}_2\sin\beta_1\sin\beta_2 \\ &= (l_1\dot{\beta}_1)^2 + (l_2\dot{\beta}_2)^2 + 2l_1l_2\dot{\beta}_1\dot{\beta}_2\cos(\beta_1 - \beta_2)\end{aligned} \tag{19}$$

Total kinetic energy:

$$\begin{aligned}T &= \frac{1}{2}m_1\vec{v}_1^{\,2} + \frac{1}{2}m_2\vec{v}_2^{\,2} \\ &= \frac{1}{2}(m_1+m_2)(l_1\dot{\beta}_1)^2 + \frac{1}{2}m_2(l_2\dot{\beta}_2)^2 + m_2l_1l_2\dot{\beta}_1\dot{\beta}_2\cos(\beta_1 - \beta_2)\end{aligned} \tag{20}$$

Total potential energy:

$$V = m_1gh_1 + m_2gh_2 = -m_1gl_1\cos(\beta_1) - m_2g\left[l_1\cos(\beta_1) + l_2\sin(\beta_2)\right] \tag{21}$$

The Lagrangian of the double pendulum:

$$\begin{aligned} L &= T - V \quad (22)\\ &= \frac{1}{2}(m_1+m_2)(l_1\dot{\beta}_1)^2 + \frac{1}{2}m_2(l_2\dot{\beta}_2)^2 + m_2 l_1 l_2 \dot{\beta}_1\dot{\beta}_2\cos(\beta_1-\beta_2) + (m_1+m_2)g l_1\cos(\beta_1) + m_2 g l_2 \cos(\beta_2) \end{aligned}$$

(b) Lagrange's equations for the double pendulum:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\beta}_1} - \frac{\partial L}{\partial \beta_1} = 0 \quad (23)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\beta}_2} - \frac{\partial L}{\partial \beta_2} = 0 \quad (24)$$

Inserting the Lagrangian into these equations gives (after some calculations):

$$(m_1+m_2)l_1\ddot{\beta}_1 + m_2 l_2\ddot{\beta}_2\cos(\beta_1-\beta_2) + m_2 l_2 \dot{\beta}_2^{\,2}\sin(\beta_1-\beta_2) + (m_1+m_2)g\sin\beta_1 = 0 \quad (25)$$

$$l_2\ddot{\beta}_2 + l_1\ddot{\beta}_1\cos(\beta_1-\beta_2) - l_1\dot{\beta}_1^{\,2}\sin(\beta_1-\beta_2) + g\sin(\beta_2) = 0 \quad (26)$$

4. Alternative Lagrangian

(1a) L' and L are equivalent if F satisfies:

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}}\frac{d}{dt}F(q,t) - \frac{\partial}{\partial q}\frac{d}{dt}F(q,t) = 0.$$

Now, we know that

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial q}\dot{q}.$$

Inserting this into the first equation, we obtain:

$$\frac{\partial^2 F}{\partial q \partial t} + \frac{\partial^2 F}{\partial q^2}\dot{q} - \frac{\partial^2 F}{\partial q \partial t} - \frac{\partial^2 F}{\partial q^2}\dot{q} = 0.$$