Exercise 8 solutions - TFY4345 Classical Mechanics

2020

1 Principal moments of inertia of a triangular slab

(a) Since the mas has uniform density, we can write the mass area density as $M = 1/2ab\rho$. Let x_{CM} denote the x-component of the center of mass. Using the definition of CM, we find (EXPLAIN UPPER LIMIT?)

$$x_{CM} = \frac{1}{M}\int_0^a \mathrm{d}x \int_0^{b(1-x/a)} \mathrm{d}y\rho x = \frac{\rho b}{M}\int_0^a \mathrm{d}x\left(1-\frac{x}{a}\right) = \frac{a^2 b\rho}{M}\int_0^1 \mathrm{d}u(1-u)u = \frac{\rho a^2 b}{6M} = \frac{a}{3}.$$

We used the substitution $u = 1 - x/a$ which implies a $\mathrm{d}x = -a\mathrm{d}u$. Because of the symmetry in the problem (the slab is a triangle), the calculation of y_{CM} is the same, only exchanging $a \leftrightarrow b$, so the result is y_{CM}.

(b) The slab is two dimensional, and laying in the xy-plane. If we look at the definition of the off-diagonal entries in moment of inertia tensor,

$$I_{ij} = -\int_V \mathrm{d}V x_i x_j,$$

$I_{zx} = I_{xz} = I_{zy} = I_{yz} = 0$, as z = 0. This also implies that $I_{xx} + I_{yy} = I_{zz}$, so all we need to calculate is I_{xx}, I_{yy} and $I_{xy})I_{yx}$.

$$\begin{aligned}
I_{xy} &= -\rho\int_0^a \mathrm{d}x\int_0^{v(1-x/a)} \mathrm{d}y yx = -\frac{\rho b^2}{2}\int_0^a \mathrm{d}x x\left(1-\frac{x}{a}\right)^2 = -\frac{\rho b^2}{2}\int_0^a \mathrm{d}x\left(x - \frac{2}{a}x^2 + \frac{1}{a^2}x^3\right) \\
&= -\frac{\rho b^2 a^2}{2}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = \frac{Mab}{12} \\
I_{xy} &= -\rho\int_0^a \mathrm{d}x\int_0^{v(1-x/a)} \mathrm{d}y y^2 = \frac{\rho b^3}{3}\left(1-\frac{x}{a}\right)^3 = \frac{\rho a b^3}{3}\int_0^1 \mathrm{d}u u^3 = \frac{Mb^2}{6}.
\end{aligned}$$

Lastly, I_{yy} can a gain be found just by the exchange $a \leftrightarrow b$. In matrix form,

$$I = \frac{M}{6} = \begin{pmatrix} b^2 & -\frac{1}{2}ab & 0 \\ -\frac{1}{2}ab & a^2 & 0 \\ 0 & 0 & a^2+b^2 \end{pmatrix}$$

(c) We can remove the common factor $M/6$, so insert our values into the new variables, we get

$$A = \frac{1}{2}(a^2 + b^2), \quad B = \frac{1}{2}\sqrt{(b^2 - a^2) + a^2 b^2}, \quad \vartheta = \tan^{-1}\left(\frac{ab}{b^2 - a^2}\right).$$

(FIGUR) The last equation describes a traingle with side lengths $b^2 - a^2$, ab and $\sqrt{(b^2 - a^2) + a^2 b^2}$, and an angle ϑ opposite the side of length ab.

2 Precession of a frisbee

3 Precession of a heavy spinning top