Classical Mechanics TFY 4345 – Solution set 6

1. Elastic scattering in laboratory coordinates.

Relations: $\underset{\uparrow \text{ LAB angle}}{\cos\vartheta} = \dfrac{\cos\Theta + \rho}{\sqrt{1+2\rho\cos\Theta+\rho^2}}$, $\rho = \dfrac{m_1}{m_2}$ (elastic)

Θ ← CM angle

$$\sigma'(\vartheta) = \sigma(\Theta)\,\frac{(1+2\rho\cos\Theta+\rho^2)^{3/2}}{1+\rho\cos\Theta}$$

Set masses equal: $\underline{m_1 = m_2}$ (this was missing...) $\Rightarrow \rho = 1$

$$\Rightarrow \cos\vartheta = \frac{\cos\Theta+1}{\sqrt{2+2\cos\Theta}} = \frac{1+\cos\Theta}{\sqrt{2}\cdot\sqrt{1+\cos\Theta}} = \sqrt{\frac{1+\cos\Theta}{2}}$$

$$\Rightarrow \cos\vartheta = \cos\frac{\Theta}{2} \quad\Rightarrow\quad \underline{\vartheta = \frac{\Theta}{2} \quad (\rho = 1)}$$

$\therefore$ Scattering angles $> 90°$ cannot occur in the lab system

Now:

$$\underline{\sigma'(\vartheta)} = \sigma(\Theta)\cdot\frac{(2+2\cos\Theta)^{3/2}}{1+\cos\Theta} = \sigma(\Theta)\cdot 2(1+\cos\Theta)^{1/2}$$

$$= \sigma(\Theta)\cdot 4\cos\frac{\Theta}{2} = \underline{4\cos\vartheta\cdot\sigma(\Theta)}, \quad \vartheta \le \frac{\pi}{2}$$

$\therefore$ For isotropic scattering in Θ ($\sigma(\Theta)$ constant), cross section in ϑ varies as the cosine of the angle

Elastic collision slows down the incident particle:

cosine law (lectures) $\Rightarrow \underset{\text{LAB}}{v_1^2} = \underset{\text{CM}}{v_1'^2} + \underset{\text{CM}}{V^2} + 2v_1'V\cos\Theta$

$$V = \frac{\mu}{m_2}v_0 = \frac{m_1 m_2}{m_1+m_2}\cdot\frac{1}{m_2}\cdot v_0 = \frac{1}{2}v_0 \quad (m_1 = m_2)$$

v_0 ← incident velocity, m_2 at rest

$$\rho = \frac{\mu}{m_2}\frac{v_0}{v_1'} = \frac{1}{2}\frac{v_0}{v_1'}, \quad \rho = 1$$

$$\Rightarrow v_1' = \frac{1}{2}v_0$$

Elastic collision

$$\Rightarrow v_1^2 = \left(\tfrac{1}{2}v_0\right)^2 + \left(\tfrac{1}{2}v_0\right)^2 + 2\cdot\tfrac{1}{2}v_0\cdot\tfrac{1}{2}v_0\cdot\cos\Theta$$

$$= \frac{1}{2}v_0^2 + \frac{1}{2}v_0^2\cos\Theta = \frac{1}{2}v_0^2(1+\cos\Theta)$$

$$\Rightarrow \frac{v_1^2}{v_0^2} = \frac{1+\cos\Theta}{2}; \quad E_1 = \frac{1}{2}m_1v_1^2, \quad E_0 = \frac{1}{2}m_1v_0^2$$

$$\Rightarrow \underline{\frac{E_1}{E_0} = \frac{1+\cos\Theta}{2} = \cos^2\vartheta} \quad \text{(course book: pp. 115-120)}$$

2. Rotating system in cylindrical coordinates.

$$L = T - V = m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2)/2 - V(r,\theta,z) \qquad (7)$$

The Lagrange-equations for the three generalized coordinates then read:

$$m\ddot{r} - mr\dot{\theta}^2 = -\partial_r V,\ d(mr^2\dot{\theta})/dt + \partial_\theta V = 0,\ m\ddot{z} + \partial_z V = 0. \qquad (8)$$

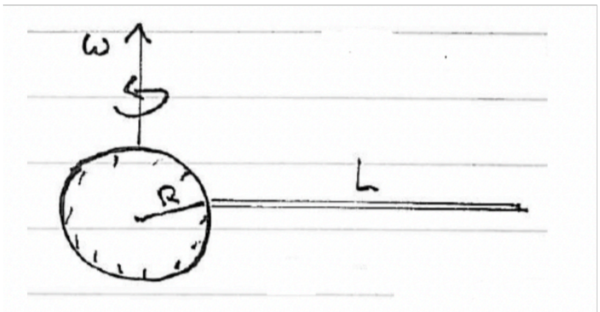
Using Hamilton's equations, we identify the canonical momenta:

$$p_r = m\dot{r},\ p_\theta = mr^2\dot{\theta},\ p_z = m\dot{z}. \qquad (9)$$

These equations can be used to replace the time derivatives of the generalized coordinates in terms of the momenta, so that the final Hamiltonian reads:

$$H = T + V = (p_r^2 + p_\theta^2/r^2 + p_z^2)/(2m) + V(r,\theta,z). \qquad (10)$$

3. Effective potential and scattering centre.



In the rotating coordinate system, the centrifugal force acting on an element of length dr in a distance r from the center equal to $r\omega^2\rho \cdot dr$. Here, ρ is the mass of the rod per unit length. The gravitational pull on the same element dr is $GM\rho \cdot dr/r^2 = g_0 R^2 \rho \cdot dr/r^2$ where g_0 is the gravitational acceleration at the surface of the Earth. Balancing the total graviational and centrifugal force in order to obtain an equilibrium situation, then provides us with:

$$\int_R^{R+L} r\omega^2\rho \cdot dr = g_0 R^2 \rho \int_R^{R+L} dr/r^2. \qquad (1)$$

Performing the integration results in:

$$L^2 + 3RL + (2R^2 - 2g_0 R/\omega^2) = 0 \qquad (2)$$

This is a 2nd order equation for L whose positive solution reads:

$$L = -3R/2 + \sqrt{R^2 + 8g_0 R/\omega^2}/2 \qquad (3)$$

Putting in numbers $R = 6.4$ km, $\omega = 2\pi/(1 \text{ day})$, and $g_0 = 9.8$ m/s^2 gives $L = 1.5 \times 10^5$ km (about halfway to the moon).

4. Coriolis effect on a falling particle.

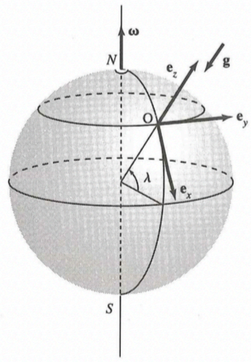


FIGURE 10-9

Although the Coriolis force produces small velocity components in the $\mathbf{e}_y$ and $\mathbf{e}_x$ directions, we can certainly neglect $\dot{x}$ and $\dot{y}$ compared with $\dot{z}$, the vertical velocity. Then, approximately,

$$\dot{x} \cong 0$$
$$\dot{y} \cong 0$$
$$\dot{z} \cong -gt$$

where we obtain $\dot{z}$ by considering a fall from rest. Therefore, we have

$$\boldsymbol{\omega} \times \mathbf{v}_r \cong \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ -\omega\cos\lambda & 0 & \omega\sin\lambda \\ 0 & 0 & -gt \end{vmatrix}$$
$$\cong -(\omega gt\cos\lambda)\mathbf{e}_y$$

The components of $\mathbf{g}$ are

$$g_x = 0$$
$$g_y = 0$$
$$g_z = -g$$

so the equations for the components of $\mathbf{a}_r$ (neglecting terms in ω^2; see Problem 10-13) become

$$(\mathbf{a}_r)_x = \ddot{x} \cong 0$$
$$(\mathbf{a}_r)_y = \ddot{y} \cong 2\omega gt\cos\lambda$$

Thus, the effect of the Coriolis force is to produce an acceleration in the $\mathbf{e}_y$, or easterly, direction. Integrating $\ddot{y}$ twice, we have

$$y(t) \cong \tfrac{1}{3}\omega g t^3 \cos\lambda$$

where $y = 0$ and $\dot{y} = 0$ at $t = 0$. The integration of $\dot{z}$ yields the familiar result for the distance of fall,

$$z(t) \cong z(0) - \tfrac{1}{2}gt^2$$

and the time of fall from a height $h = z(0)$ is given by

$$t \cong \sqrt{2h/g}$$

Hence the result for the eastward deflection d of a particle dropped from rest at a height h and at a northern latitude λ is*

$$d \cong \tfrac{1}{3}\omega\cos\lambda\sqrt{\frac{8h^3}{g}} \qquad \textbf{(10.35)}$$

An object dropped from a height of 100 m at latitude 45° is deflected approximately 1.55 cm (neglecting the effects of air resistance).