Exercise 12 - TFY4345 Classical Mechanics

2020

1 Generating function F_4

The generating function F is given by

$$F = q_i p_i - Q_i P_i + F_4(p, P, t).$$

This means the time derivative can be written as

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \dot{p}_i q_i + p_i \dot{q}_i - \dot{P}_i Q_i - P_i \dot{Q}_i + \frac{\mathrm{d}F(p, P, t)}{\mathrm{d}t}.$$

Inserting this into the relation between the original Hamiltonian H and the new one, K

$$p_i \dot{q}_i - H(q, p, t) = P_i \dot{Q}_i - K(Q, P, t) + \frac{\mathrm{d}F}{\mathrm{d}t}$$

gives

$$\begin{aligned} \cancel{p_i \dot{q}_i} - H(q, p, t) = \cancel{P_i \dot{Q}_i} - K(Q, P, t) + \dot{p}_i q_i + \cancel{p_i \dot{q}_i} - \dot{P}_i Q_i - \cancel{P_i \dot{Q}_i} + \frac{\mathrm{d}F(p, P, t)}{\mathrm{d}t} \\ \dot{p}_i q_i + H(q, p, t) = \dot{P}_i Q_i + K(Q, P, t) - \frac{\mathrm{d}F_4(p, P, t)}{\mathrm{d}t} \end{aligned}$$

We can expand

$$\frac{\mathrm{d}F_4(p, P, t)}{\mathrm{d}t} = \frac{\partial F_4}{\partial t} + \frac{\partial F_4}{\partial p_i} \dot{p}_i + \frac{\partial F_4}{\partial P_i} \dot{P}_i,$$

which gives

$$\dot{p}_i q_i + H(q, p, t) = \dot{P}_i Q_i + K(Q, P, t) - \left(\frac{\partial F_4}{\partial t} + \frac{\partial F_4}{\partial p_i} \dot{p}_i + \frac{\partial F_4}{\partial P_i} \dot{P}_i \right).$$

This only holds if

$$K = H + \frac{\partial F_4}{\partial t}, \quad q_i = -\frac{\partial F_4}{\partial p_i}, \quad Q_i = \frac{\partial F_4}{\partial P_i},$$

which is the equations we were looking

2 The Poisson bracket

The Hamiltonian for the harmonic oscillator is

$$H = \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right),$$

and we have the canonical transformations

$$q = \sqrt{\frac{2P}{m\omega}}\sin(Q), \quad p = \sqrt{2Pm\omega}\cos(Q), \quad H = \omega P.$$

The Poisson bracket in the original is

$$[q,H]_{q,p} = \underbrace{\frac{\partial q}{\partial q}}_{=1}\frac{\partial H}{\partial p} - \underbrace{\frac{\partial q}{\partial q}}_{=0}\frac{\partial H}{\partial p} = \frac{\partial H}{\partial p} = \frac{p}{m}.$$

In the new coordinates the bracket is

$$[q,H]_{Q,P} = \frac{\partial q}{\partial Q}\frac{\partial H}{\partial P} - \frac{\partial q}{\partial P}\underbrace{\frac{\partial H}{\partial Q}}_{=0} = \frac{\partial q}{\partial Q}\frac{\partial H}{\partial P},$$

where

$$\begin{aligned}
&\frac{\partial q}{\partial Q} = \frac{\partial}{\partial Q}\left(\sqrt{\frac{2P}{m\omega}}\sin(Q)\right) = \sqrt{\frac{2P}{m\omega}}\cos(Q)\cdot\frac{\sin(Q)}{\sin(Q)} = q\cot(Q),\\
&\frac{\partial H}{\partial P} = \omega, \quad \cot(Q) = \frac{\cos(Q)}{\sin(Q)} = m\omega\frac{p}{q}.
\end{aligned}$$

This gives

$$[q,H]_{Q,P} = \omega q\cot Q = \frac{p}{m} = [q,H]_{q,p}.$$

The fact that $[q,H] \neq 0$ means that q is not a constant of motion.

3 The symplectic condition

4 Free particle