Exercise 3 solutions - TFY4345 Classical Mechanics

2020

1 Pendulum on spinning a wheel

With the origin of our coordinate system in the center of the rotating rim, the Cartesian components of the mass m become

$$x = a\cos(\omega t) + b\sin(\theta)$$
$$y = a\sin(\omega t) - b\cos(\theta).$$

The velocities are

$$\dot{x} = -a\omega\sin(\omega t) + b\dot{\theta}\cos(\theta)$$
$$\dot{y} = a\omega\cos(\omega t) + b\dot{\theta}\sin(\theta).$$

Taking the time derivative once again gives the acceleration:

$$\ddot{x} = -a\omega^2\cos(\omega t) + b(\ddot{\theta}\cos(\theta) - \dot{\theta}^2\sin(\theta))$$
$$\ddot{y} = -a\omega^2\sin(\omega t) + b(\ddot{\theta}\sin(\theta) + \dot{\theta}^2\cos(\theta)).$$

It should be clear that the single generalize coordinate is θ. The kinetic and potential energies are

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2),\ V = mgy.$$

Inserting what we found earlier, the Lagrangian becomes

$$L = \frac{1}{2}m[a^2\omega^2 + b\dot{\theta}^2 + 2b\theta^2 a\omega\sin(\theta - \omega t)] - mg[a\sin(\omega t) - b\cos(\theta)].$$

The derivatives needed for the equation of motion are

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{\theta}} = mb^2\ddot{\theta} + mba\omega(\dot{\theta} - \omega)\cos(\theta - \omega t),$$
$$\frac{\partial L}{\partial\theta} = mba\dot{\theta}\omega\cos(\theta - \omega t) - mgb\sin(\theta).$$

Inserting this into Euler's equation, and solving for $\ddot{\theta}$ gives

$$\ddot{\theta} = \frac{\omega^2 a}{b}\cos(\theta - \omega t) - \frac{g}{b}\sin(\theta).$$

Notice that, for $\omega = 0$, this reduces to the equation for the simple pendulum.

2 Bead on a ring

The potential energy is given by

$$U = mgh = mgR(1 - \cos(\theta)),$$

while the kinetic energy is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left((r\dot{\varphi})^2 + (R\dot{\theta})\right) = \frac{1}{2}mR^2\left(\sin^2(\theta)\dot{\varphi}^2 + \dot{\theta}^2\right).$$

The Euler equation for φ is given by

$$\frac{\partial L}{\partial \varphi} = 0 \implies \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\varphi}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(m\sin(\theta)R^2\dot{\varphi}\right) = 0$$

The shafts is driven with a constant speed, so $\dot{\varphi} = \omega = \text{const}$. The equation for θ is given by

$$\begin{aligned} &\frac{\partial L}{\partial \theta} = mR^2\cos(\theta)\sin(\theta)\dot{\varphi}^2 - mgR\sin(\theta),\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} = mR^2\ddot{\theta}, \\ &\implies \ddot{\theta} = \cos(\theta)\sin(\theta)\dot{\varphi}^2 - \frac{g}{R}\sin(\theta) \end{aligned}$$

In the equilibrium position, we have that $\ddot{\theta} = 0$. This means

$$\cos(\theta) = \frac{g}{R\dot{\varphi}^2} = \frac{g}{R\omega^2}.$$

3 Atwood's machine

The angular velocity of the pulley is

$$\omega = \frac{\dot{x}_2}{a}.$$

This means the kinetic energy of the system is

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}I\omega^2$$

The length of the string is a constant, so

$$x_1 + x_2 = \ell = \text{const}.$$

Inserting this into the kinetic energy gives

$$T = \frac{1}{2}(m_2 - m_1)\dot{x}_2^2 + \frac{1}{2}I\left(\frac{\dot{x}_2}{a}\right)^2.$$

The potential energy is

$$V = -m_1gx_1 - m_2gx_2 = -m_1g(\ell - x_2) - m_2g(x_2),$$

so the Lagrangian is

$$L = \frac{1}{2}\left(m_1 + m_2 + \frac{I}{a^2}\right)\dot{x}_2^2 + m_1 g(\ell - x_2) + m_2 g x_2.$$

The canonical momentum is

$$p_2 = \frac{\partial L}{\partial \dot{x}_2} = \left(m_1 + m_2 + \frac{I}{a^2}\right)\dot{x}_2$$

The conditions are met so that we can write the Hamiltonian as

$$H = T + V = \frac{1}{2}\frac{p_2^2}{m_1 + m_2 + I/a^2} - m_1 g(\ell - x_2) - m_2 g x_2.$$

Hamilton's equations then become

$$\begin{aligned}
\dot{x}_2 &= \frac{\partial H}{\partial p_2} = \frac{p_2}{m_1 + m_2 + I/a^2} \\
\dot{p}_2 &= -\frac{\partial H}{\partial x_2} = (m_2 - m_1)g.
\end{aligned}$$

Lastly, H is conserved as

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -\frac{\partial L}{\partial t} = 0.$$

4 Particle on a moving wedge

(a) The position of the mass m in a coordinate system moving with the wedge is

$$\mathbf{r}' = r\cos(\theta)\hat{e}_x + r\sin(\theta)\hat{e}_y.$$

In the laboratory coordinates, which does not move with the wedge, the wedge has position $x\hat{e}_x$, so the position of the mass m is

$$\mathbf{r} = (x + r\cos(\theta))\hat{e}_x + \sin(\theta)\hat{e}_y.$$

The velocity is

$$\dot{\mathbf{r}} = (\dot{x} + \dot{r}\sin(\theta) - r\dot{\theta}\sin(\theta))\hat{e}_x + (\dot{r}\sin(\theta) + r\dot{\theta}\cos(\theta))\hat{e}_y,$$

while the square velocity becomes

$$\begin{aligned}
\dot{r}^2 = &\left(\dot{x} + \dot{r}\cos(\theta) - r\dot{\theta}\sin(\theta)\right)^2 + \left(\dot{r}\sin(\theta) + r\dot{\theta}\cos(\theta)\right)^2 = \\
&\dot{x}^2 + 2\dot{x}\left(\dot{r}\cos(\theta) - r\dot{\theta}\sin(\theta)\right) + \left(\dot{r}\cos(\theta) - r\dot{\theta}\sin(\theta)\right)^2 + (\dot{r}\cos(\theta))^2 + 2r\dot{r}\dot{\theta}\cos(\theta)\sin(\theta) + (r\dot{\theta}\sin(\theta))^2 \\
&= \dot{x}^2 + \underline{(\dot{r}\cos(\theta))^2} + \underline{(r\dot{\theta}\sin(\theta))^2} + 2\dot{x}\dot{r}\cos(\theta) - 2\dot{x}r\dot{\theta}\sin(\theta) - \cancel{2r\dot{r}\dot{\theta}\cos(\theta)\sin(\theta)} \\
&+ \underline{(\dot{r}\sin(\theta))^2} + \cancel{2r\dot{r}\dot{\theta}\cos(\theta)\sin(\theta)} + \underline{(r\dot{\theta}\cos(\theta))^2} \\
&= \dot{x}^2 + \dot{r}^2 + (r\dot{\theta})^2 + 2\dot{x}\dot{r}\cos(\theta) - 2\dot{x}r\dot{\theta}\sin(\theta).
\end{aligned}$$

The potential energy is given by

$$V = -mgr\sin(\theta).$$

The restriction of the little mass to stay on the wedge is given by $r - R = 0$, so the total Lagrangian, including the undetermined multiplier becomes

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{r}^2 + (r\dot{\theta})^2 + 2\dot{x}\dot{r}\cos(\theta) - 2\dot{x}r\dot{\theta}\sin(\theta)\right) + \frac{1}{2}M\dot{x}^2 + mgr\sin(\theta) + \lambda(r - R).$$

The derivatives with respect to x are:

$$\begin{aligned}
&\frac{\partial L}{\partial x} = 0,\ \frac{\partial L}{\partial \dot{x}} = m(\dot{x} + \dot{r}\cos(\theta) - r\dot{\theta}\sin(\theta)) + M\dot{x}, \\
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} = m(\ddot{x} + \ddot{r}\cos(\theta) - \sin(\theta)(\dot{r}\dot{\theta} + r\ddot{\theta}) - r\dot{\theta}^2\cos(\theta)) + M\ddot{x}
\end{aligned}$$

With respect to θ:

$$\begin{aligned}
&\frac{\partial L}{\partial \theta} = -m\dot{x}\dot{r}\sin(\theta) - m\dot{x}r\dot{\theta}\cos(\theta) + mgr\cos(\theta), \quad \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} - \dot{x}r\sin(\theta), \\
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} - m\sin(\theta)(\ddot{x}r + \dot{x}\dot{r}) - m\cos(\theta)\dot{\theta}\dot{x}r,
\end{aligned}$$

And finally r:

$$\begin{aligned}
&\frac{\partial L}{\partial r} = m(r\dot{\theta}^2 - \dot{x}\dot{\theta}\sin(\theta)) + mg\sin(\theta) + \lambda, \quad \frac{\partial L}{\partial \dot{r}} = m\dot{r} + m\dot{x}\cos(\theta) \\
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{r}} = m\ddot{r} + m\ddot{x}\cos(\theta) - m\dot{x}\dot{\theta}\sin(\theta).
\end{aligned}$$

We can then use the restriction $r = R$, which implies $\dot{r} = \ddot{r} = 0$, so we get the equations of motion

$$\begin{aligned}
&\ddot{x}(m + M) - mR\ddot{\theta}\sin(\theta) - mR\dot{\theta}^2\cos(\theta) = 0 \\
&mR^2\ddot{\theta} - mR(\ddot{x}\sin(\theta) + g\cos(\theta)) = 0 \\
&m\ddot{x}\cos(\theta) - mR\dot{\theta}^2 - mg\sin(\theta) - \lambda = 0.
\end{aligned}$$

Cleaning up gives

$$\begin{aligned}
\ddot{x} &= \frac{mR}{m + M}\left(\ddot{\theta}\sin(\theta) + \dot{\theta}^2\cos(\theta)\right), \\
\ddot{\theta} &= \frac{\ddot{x}\sin(\theta) + g\cos(\theta)}{R}, \\
\lambda &= m\left(\ddot{x}\cos(\theta) - R\dot{\theta}^2 - g\sin(\theta)\right).
\end{aligned}$$

(b) Exploiting a common factor,

$$mR\dot{\theta}^2 = \frac{1}{\cos(\theta)}\left[(m + M)\ddot{x} - mR\ddot{\theta}\sin(\theta)\right],$$

we can insert this into the expression for λ:

$$
\begin{aligned}
\lambda &= m\ddot{x}\cos(\theta) - gm\sin(\theta) - \frac{1}{\cos(\theta)}\Big[(m+M)\ddot{x} - mR\ddot{\theta}\sin(\theta)\Big] \\
&= \ddot{x}\left[m\cos(\theta) - \frac{m+M}{\cos(\theta)}\right] - gm\sin\theta - mR\ddot{\theta}\frac{\sin(\theta)}{\cos(\theta)} \\
&= \frac{mR}{m+M}\left(\ddot{\theta}\sin(\theta) + \dot{\theta}^2\cos(\theta)\right)\left[m\cos(\theta) - \frac{m+M}{\cos(\theta)}\right] - gm\sin\theta - mR\ddot{\theta}\frac{\sin(\theta)}{\cos(\theta)}
\end{aligned}
$$

This is the the force of constraint working on m, and thus is equal to the to the normal force of M on m.