Classical Mechanics TFY 4345 – Solution set 5

1. Effective potential and scattering centre.

(1) Energy-conservation combined with conservation of angular momentum gives us:

$$m\dot{r}^2/2 + (V + \frac{l^2}{2mr^2}) = E. \quad (1)$$

This is an effective 1D problem with $V' = V + \frac{l^2}{2mr^2}$. In order for the particle to reach the center it needs to have sufficiently high energy to overcome the potential barrier, i.e. $E > V'(r \to 0)$. This can be written as

$$Er^2 > r^2V + l^2/(2m). \quad (2)$$

For $r \to 0$, the l.h.s. goes to zero, so that the condition on the potential becomes:

$$(r^2V)|_{r\to 0} < -l^2/(2m). \quad (3)$$

This can be fulfilled either with $V(r) = -k/r^2$ where $k > l^2/2m$ or if $V(r) = -A/r^n$ with $n > 2$ where A is a positive constant.

2. Scattering from a spherical obstacle.

(1) *See figure in the Norwegian version of the solution*. The scattering angle Θ satisfies $2\Psi + \Theta = \pi$. From the figure, we see that the impact parameter is given by $s = a\cos\Theta/2$, so that

$$|ds/d\Theta| = (a/2)\sin(\Theta/2). \tag{1}$$

. Therefore, the differential scattering cross section is

$$\sigma(\Theta) = s|ds/d\Theta|/\sin\Theta = a^2/4. \tag{2}$$

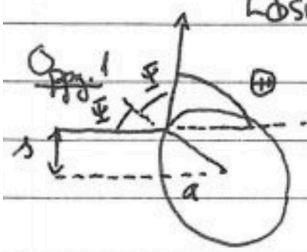
The total cross section is obtained by integration over Θ, so that

$$\sigma = 2\pi\int_0^{\pi}\sigma(\Theta)\sin\Theta d\Theta = \pi a^2. \tag{3}$$

This is physically sensible since it is the actual cross-sectional area of the sphere.

Løsning Øving 7

Oppg. 1



Spredningsvinkel Θ oppfyller $2\Psi + \Theta = \pi$

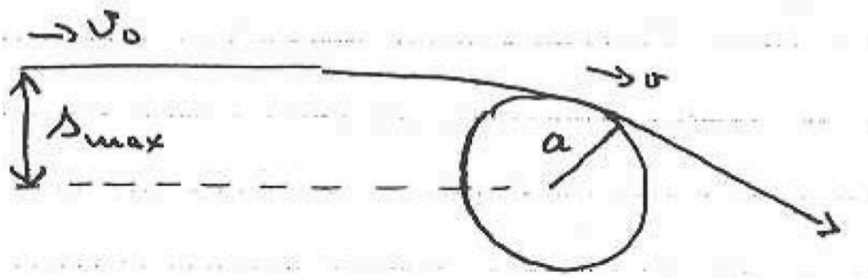
Av figuren er støtparameteren $s = a\sin\left(\frac{\pi}{2} - \frac{\Theta}{2}\right) = a\cos\frac{\Theta}{2}$

⊃: $|ds/d\Theta| = (a/2)\sin\Theta/2$.

$\underline{\sigma(\Theta)} = \frac{s}{\sin\Theta}\left|\frac{ds}{d\Theta}\right| = \frac{a\cos\Theta/2}{\sin\Theta}\cdot\frac{a}{2}\sin\frac{\Theta}{2} = \underline{\frac{a^2}{4}}$, uavhengig av Θ.

Totalt tverrsnitt $\underline{\sigma} = 2\pi\int_0^{\pi}\sigma(\Theta)\sin\Theta\, d\Theta = \underline{\pi a^2}$, rimelig svar.

3. Scattering by an attractive hard sphere.



(2) *See figure in the Norwegian version of the solution.* When the particle touches the surface at $r = a$, conservation of the total energy dictates that

$$E = mv_0^2/2 = mv^2/2 - k/a. \tag{4}$$

Also, conservation of angular momentum provides us with l infinitely far away being equal to l when the particle touches the surface, i.e.

$$l = mv_0 s_{\text{max}} = mva \tag{5}$$

Combining these two equations allow us to identify s_{max}:

$$s_{\text{max}} = \sqrt{a^2 + 2ka/(mv_0^2)}. \tag{6}$$

All particles with impact parameter $s < s_{\text{max}}$ will hit the surface, so that $\sigma_{\text{eff}} = \pi s_{\text{max}}^2$.

4. Average energies in the Kepler problem.

$$\begin{aligned} p &= \frac{\ell^2}{mk} && (1) \\ \epsilon^2 &= 1 + \frac{2E\ell^2}{mk^2} && (2) \end{aligned}$$

Eliminating ℓ gives:

$$E = -\frac{k}{2p}(1-\epsilon^2) \tag{3}$$

The total energy is constant, this means that the average total energy is also constant:

$$\langle T\rangle + \langle V\rangle = \langle E\rangle = E \tag{4}$$

The virial theorem for a gravitational potential (Eq. 4.14 Brevik compendium) gives

$$\langle T\rangle = -\frac{1}{2}\langle V\rangle \tag{5}$$

Combining eq. (4) and (5) gives the average kinetic and potenial energy as a function of p and ϵ:

$$\begin{aligned} \langle T\rangle &= \frac{k}{2p}\left(1-\epsilon^2\right) && (6) \\ \langle V\rangle &= -\frac{k}{p}\left(1-\epsilon^2\right) && (7) \end{aligned}$$

(b) The solution to the Kepler problem in polar coordinates:

$$r = \frac{p}{1+\epsilon\cos(\theta)} \tag{8}$$

Average potential energy over one period:

$$\langle V\rangle = \frac{1}{t_p}\int_0^{t_p} dt\, V = -\frac{1}{t_p}\int_0^{t_p} dt\,\frac{k}{r} \tag{9}$$

where t_p is the orbital period. Resulting average potential energy:

$$\begin{aligned} \langle V\rangle &= -\frac{1}{t_p}\int_0^{t_p} dt \frac{k}{p}(1+\epsilon\cos(\theta)) && (10) \\ &= -\frac{k}{p} - \frac{k\epsilon}{p}\frac{1}{t_p}\int_0^{t_p} dt\cos(\theta) && (11) \\ &= -\frac{k}{p} - \frac{k\epsilon}{p}\langle\cos\theta\rangle && (12) \end{aligned}$$

The integral in Eq.(11) can be transformed by using the expression for the angular momentum: $\ell = mr^2\frac{d\theta}{dt}$:

$$\begin{aligned} \langle\cos(\theta)\rangle &= \frac{1}{t_p}\int_0^{t_p} dt\cos(\theta) && (13) \\ &= \frac{1}{t_p}\int_0^{2\pi} d\theta\frac{mr^2}{\ell}\cos\theta && (14) \\ &= \frac{mp^2}{\ell t_p}\int_0^{2\pi}\frac{d\theta\cos(\theta)}{[1+\epsilon\cos(\theta)]^2} && (15) \end{aligned}$$

The integral in Eq. (15) can be expressed as the derivative of a simpler integral:

$$\int_0^{2\pi} \frac{d\theta \cos(\theta)}{[1+\epsilon\cos(\theta)]^2} = -\frac{d}{d\epsilon}\int_0^{2\pi} \frac{d\theta}{[1+\epsilon\cos(\theta)]} \tag{16}$$

The integral on the right side of Eq. (16) can be calculated using the Residue method (E. Kreyzig, 9th edition, Chapter 16.4):

$$\int_0^{2\pi} \frac{d\theta}{[1+\epsilon\cos(\theta)]} = \frac{2\pi}{\sqrt{1-\epsilon^2}} \tag{17}$$

(see appendix for detailed calculation). Combining Equations (12-17) gives the time average of $\cos(\theta)$

$$\langle\cos(\theta)\rangle = -\frac{mp^2}{\ell t_p}\frac{d}{d\epsilon}\frac{2\pi}{\sqrt{1-\epsilon^2}} = -\frac{mp^2}{\ell t_p}\frac{2\pi\epsilon}{(1-\epsilon^2)^{3/2}} \tag{18}$$

The orbit period is given in Section 4.5 in the Brevik compendium:

$$t_p = \frac{2\pi m}{\ell}ab = \frac{2\pi m}{\ell}\frac{p}{1-\epsilon^2}\frac{p}{\sqrt{1-\epsilon^2}} = \frac{2\pi mp^2}{\ell}\frac{1}{(1-\epsilon^2)^{3/2}} \tag{19}$$

where a and b are the major and minor-axis of the ellipse.
Eliminating t_p from Eq. (19) gives:

$$\langle\cos(\theta)\rangle = -\epsilon \tag{20}$$

Inserting Eq. (20) into Eq. (12) gives the average potential energy:

$$\langle V\rangle = -\frac{k}{p}\left(1-\epsilon^2\right) \tag{21}$$

(c) The average kinetic energy is:

$$\langle T\rangle = \frac{1}{t_p}\int_0^{t_p} dt\, T = \frac{1}{t_p}\int_0^{t_p} dt\,\frac{1}{2}m\left(\frac{d\vec{r}}{dt}\right)^2 \tag{22}$$

Integration by parts gives:

$$\begin{aligned}
\langle T\rangle &= \left[\frac{1}{t_p}\frac{1}{2}m\vec{r}\cdot\frac{d\vec{r}}{dt}\right]_0^{t_p} - \frac{1}{t_p}\int_0^{t_p} dt\,\frac{1}{2}m\vec{r}\cdot\frac{d^2\vec{r}}{dt^2} && (23)\\
&= -\frac{1}{t_p}\int_0^{t_p} dt\,\frac{1}{2}m\vec{r}\cdot\frac{d^2\vec{r}}{dt^2} && (24)\\
&= \frac{1}{t_p}\int_0^{t_p} dt\,\frac{1}{2}\vec{r}\cdot\nabla(-\frac{k}{r}) && (25)\\
&= \frac{1}{t_p}\int_0^{t_p} dt\frac{k}{2r} && (26)\\
&= \langle\frac{k}{2r}\rangle && (27)\\
&= -\frac{1}{2}\langle V\rangle && (28)\\
&= \frac{k}{2p}\left(1-\epsilon^2\right) && (29)
\end{aligned}$$

The result agrees with the virial theorem for a gravitational field: $\langle T\rangle = -\frac{1}{2}\langle V\rangle$