Exercise 2 solutions - TFY4345 Classical Mechanics

2020

1 Damped oscillator

(a) The frictional force is

$$F_f = -\frac{\partial \mathcal{F}}{\partial \dot{x}}.$$

The work done by friction is force times distance, so the work per unit time is

$$\dot{W}_f = -F_f v = \frac{\partial \mathcal{F}}{\partial \dot{x}}\dot{x} \implies \mathcal{F} = C\dot{x}^2.$$

(As $\mathcal{F}$ is a (velocity) potential, we can dismiss any constants, just as with regular potentials.) This means

$$\dot{W}_f = 2C\dot{x}^2 = 2\mathcal{F}.$$

(b) The Lagrangian with a velocity-dependent potential is

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} + \frac{\partial \mathcal{F}}{\partial \dot{x}} = 0.$$

Inserting the Lagrangian for a harmonic oscillator,

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2,$$

and the given velocity potential $\mathcal{F} = 3\pi\mu a\dot{x}^2$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} = m\ddot{x},\ \frac{\partial L}{\partial x} = -kx,\ \frac{\partial \mathcal{F}}{\partial \dot{x}} = 6\pi\mu a\dot{x},$$
$$\implies m\ddot{x} + 6\pi\mu a + kx = 0,$$

or

$$\ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = 0, \quad \lambda = \frac{3\pi\mu a}{m},\ \omega_0 = \sqrt{\frac{k}{m}}.$$

(c) If we assume the solution to be of the from

$$x(t) = Ae^{\omega_a t} + Be^{\omega_b t},$$

we get

$$A(\omega_0^2 + 2\lambda\omega_a + \omega^2)e^{\omega_a t} + B(\omega_0^2 + 2\lambda\omega_b + \omega_b^2)e^{\omega_b t} = 0,$$

so

$$\omega_{a/b} = -\lambda \pm \sqrt{\lambda^2 - \omega_0^2}$$

this gives us

$$x(t) = e^{-\lambda t}\left(A\exp\left[\omega t\sqrt{(\lambda/\omega)^2 - 1}\right] + B\exp\left[-\omega t\sqrt{(\lambda/\omega)^2 - 1}\right]\right).$$

Now, as $\lambda/\omega \ll 1$, we get that $\sqrt{(\lambda/\omega) - 1} \approx i$. This means that, after applying the initial conditions, we get the solution

$$x(t) = x_0 e^{-\lambda t}\cos(\omega t).$$

The instantaneous energy dissipation is therefore

$$\dot{W}_f = F_f \dot{x} = 2\lambda m \dot{x}^2 = 2\lambda m x_0^2 e^{-\lambda t}\cos^2(\omega t).$$

If we then take the average over a period $2\pi/\omega_0$, we can assume the exponential is more or less constant (as $\lambda << \omega_0$), so the time averaged dissipation is

$$\overline{\dot{W}_f} = m\lambda(\omega_0 \dot{x})^2 e^{-\lambda t}.$$

2 Operator identities

The Einstein summation convention is used throughout this exercise, so repeated indeces are summed over. The i'th component of the curl of the curl of $\mathbf{A}$ can be written

3 Shortest path in polar coordinates

4 Forces of constraint