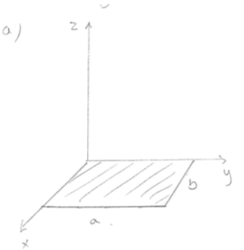
Classical Mechanics TFY 4345 – Solution set 7

1. Inertia tensor.

a)



$$I_{ij} = \int_V \rho(\vec r)\left(\delta_{ij} r^2 - x_i x_j\right) dV$$
$$= \frac{M}{ab}\int_A \left(\delta_{ij} r^2 - x_i x_j\right) dA$$
$$I_{11} = \frac{M}{ab}\int_0^a dx \int_0^b dy\,(y^2+z^2)$$
$$= \frac{M}{ab}\int_0^a dx\left(\frac13 b^3 + \underbrace{z^2}_{=0} b\right) = \frac13 M b^2$$

$$I_{22} = \frac{M}{ab}\int_0^a dx\int_0^b dy\,(x^2+z^2) = \frac13 M a^2$$
$$I_{33} = \frac{M}{ab}\int_0^a dx\int_0^b dy\,(x^2+y^2) = \frac13 M(a^2+b^2)$$
$$I_{12} = \frac{M}{ab}\int_0^a dx\int_0^b dy\,(-xy) = \frac{M}{ab}\int_0^a -\frac12 x b^2 = -\frac14 Mab$$
$$I_{13} = \frac{M}{ab}\int_0^a dx\int_0^b dy\,(-x\underbrace{z}_{=0}) = 0 \qquad \boxed{I_{ij} = I_{ji}}$$
$$I_{23} = \frac{M}{ab}\int_0^a dx\int_0^b dy\,(-y\overbrace{z}^{=0}) = 0$$

$$\bar{\bar{I}} = \begin{bmatrix} \frac13 Mb^2 & -\frac14 Mab & 0 \\ -\frac14 Mab & \frac13 Ma^2 & 0 \\ 0 & 0 & \frac13 M(a^2+b^2) \end{bmatrix}$$

b) Assume $a=b$, denote $\beta = Ma^2$

$$\Rightarrow \bar{\bar{I}} = \begin{bmatrix} \frac13 Ma^2 & -\frac14 Ma^2 & 0 \\ -\frac14 Ma^2 & \frac13 Ma^2 & 0 \\ 0 & 0 & \frac23 Ma^2 \end{bmatrix} = \begin{bmatrix} \frac13\beta & -\frac14\beta & 0 \\ -\frac14\beta & \frac13\beta & 0 \\ 0 & 0 & \frac23\beta \end{bmatrix}$$

$$\det\left(\bar{\bar{I}} - \lambda\cdot\mathbb{1}\right) = 0$$

$$\Rightarrow \begin{vmatrix} \frac13\beta-\lambda & -\frac14\beta & 0 \\ -\frac14\beta & \frac13\beta-\lambda & 0 \\ 0 & 0 & \frac23\beta-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} \frac13\beta-\lambda & -\frac14\beta & 0 \\ \frac1{12}\beta-\lambda & \frac1{12}\beta-\lambda & 0 \\ 0 & 0 & \frac23\beta-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \left(\frac1{12}\beta-\lambda\right)\begin{vmatrix} \frac13\beta-\lambda & -\frac14\beta & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \frac23\beta-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \left(\frac1{12}\beta-\lambda\right)\left[\left(\frac13\beta-\lambda\right)\left(\frac23\beta-\lambda\right)+\frac14\beta\left(\frac23\beta-\lambda\right)\right] = 0$$

$$\Rightarrow \dots \Rightarrow \left(\frac7{12}\beta-\lambda\right)\left(\frac1{12}\beta-\lambda\right)\left(\frac23\beta-\lambda\right) = 0$$

$$\therefore\quad I_1 = \frac7{12}\beta,\quad I_2 = \frac1{12}\beta,\quad I_3 = \frac23\beta$$

Eigenvalue equation: $\bar{\bar{I}}\cdot\bar\omega_{(1)} = I_1\,\omega_{(1)}$ ← vector

$I_1 = \frac7{12}\beta$ $\Rightarrow$
$$\begin{cases} \left(\frac13\beta - \frac7{12}\beta\right)\omega_1 - \frac14\beta\omega_2 = 0 \quad \leftarrow \text{component} \\ -\frac14\beta\omega_1 + \left(\frac13-\frac7{12}\right)\beta\omega_2 = 0 \\ \left(\frac23-\frac7{12}\right)\beta\omega_3 = 0 \end{cases}$$

$\Rightarrow \omega_3 = 0$ and $\omega_1 = -\omega_2$

$$\Rightarrow \bar\omega_{(1)} = \underbrace{\frac1{\sqrt2}}_{\text{norm}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$I_2 = \frac1{12}\beta$ $\Rightarrow$
$$\begin{cases} \left(\frac13\beta - \frac1{12}\beta\right)\omega_1 - \frac14\beta\omega_2 = 0 \\ -\frac14\beta\omega_1 + \left(\frac13-\frac1{12}\right)\beta\omega_2 = 0 \\ \left(\frac23-\frac1{12}\right)\beta\omega_3 = 0 \end{cases}$$

$\Rightarrow \omega_3 = 0$ and $\omega_1 = \omega_2$

$$\Rightarrow \bar\omega_{(2)} = \frac1{\sqrt2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$I_3 = \frac23\beta$ $\Rightarrow$
$$\begin{cases} \left(\frac13-\frac23\right)\beta\omega_1 - \frac14\beta\omega_2 = 0 \\ -\frac14\beta\omega_1 + \left(\frac13-\frac23\right)\beta\omega_2 = 0 \\ \left(\frac23-\frac23\right)\beta\omega_3 = 0 \quad ! \end{cases}$$

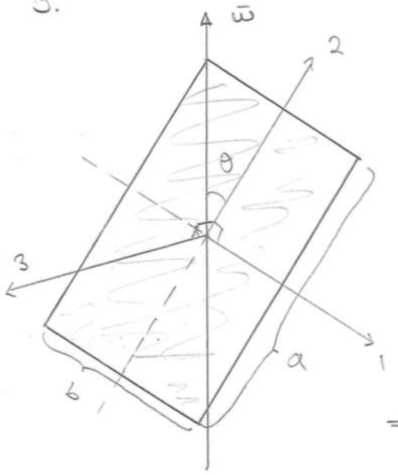
$$\Rightarrow \begin{cases} -\frac13\beta\omega_1 - \frac14\beta\omega_2 = 0 \\ -\frac14\beta\omega_1 - \frac13\beta\omega_2 = 0 \\ 0 = 0 \end{cases}$$

In addition $\bar\omega_i\cdot\bar\omega_j = \delta_{ij}$ Works!

$$\Rightarrow \bar\omega_{(3)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \text{original } \bar\omega_{(3)}$$

2. Rotating tilted slab

3.



$\bar{\omega}$ in 1-2 plane

$I_1 = \frac{1}{12} m a^2$

$I_2 = \frac{1}{12} m b^2$ tensor $\bar{\bar{I}}$, diagonal

$I_3 = \frac{1}{12} m (a^2 + b^2)$

$$\underline{L} = \bar{\bar{I}} \cdot \bar{\omega} = \omega_1 I_1 \hat{e}_1 + \omega_2 I_2 \hat{e}_2 + \omega_3 I_3 \hat{e}_3$$

$$= \underline{\frac{1}{12} m\omega \left[-\sin\theta \cdot a^2 \hat{e}_1 + \cos\theta \cdot b^2 \hat{e}_2 \right]}$$

$$L^2 = \left(\frac{m\omega}{12}\right)^2 \left[\sin^2\theta \cdot a^4 + \cos^2\theta \cdot b^4 \right]$$

$$\sin\theta = \frac{b}{\sqrt{a^2+b^2}}, \quad \cos\theta = \frac{a}{\sqrt{a^2+b^2}}$$

$$\Rightarrow L^2 = \left(\frac{m\omega}{12}\right)^2 \left[\frac{b^2 a^4}{a^2+b^2} + \frac{a^2 b^4}{a^2+b^2} \right]$$

$$= \left(\frac{m\omega}{12}\right)^2 a^2 b^2$$

$$\underline{L = \frac{1}{12} m\omega ab} \quad (*)$$

b) $\bar{\omega} \cdot \bar{L} = \omega L \cos\alpha$ (α unknown)

$$= \omega(-\sin\theta, \cos\theta) \cdot \frac{1}{12} m\omega \left(-\sin\theta \cdot a^2, \cos\theta \cdot b^2\right)$$

$$= \frac{1}{12} m\omega^2 \left(a^2 \sin^2\theta + b^2 \cos^2\theta\right) \quad (*)$$

$$= \cancel{\frac{1}{12} m\omega}^2 \left[\frac{b^2 a^2 + a^2 b^2}{a^2 + b^2} \right] = \cancel{\frac{1}{12} m\omega}\, ab \cdot \cos\alpha$$

$$\Rightarrow \cos\alpha = \frac{2ab}{a^2+b^2} \quad \Rightarrow \underline{\alpha = \arccos \frac{2ab}{a^2+b^2}}$$

E.g. $b = 1$, $a = 2$ $\rightarrow$ $\alpha = 36.7°$, $\theta = 26.6°$

c) $\underline{T_{rot}} = \frac{1}{2} \bar{\omega} \cdot \bar{L} = \underline{\frac{1}{12} m\omega^2 \cdot \frac{a^2 b^2}{a^2 + b^2}}$

3. Cone rolling on a plane

3a) Velocity of centre of mass:

$$V_{\rm cm} = \ell \cos{(\alpha)} \dot{\phi} \tag{38}$$

3b) Cone rotational motion is effectively a rotation around the instantaneous axis OA. The angular velocity around OA is

$$\omega = \frac{V_{\rm cm}}{\ell \sin \phi} = \frac{\cos{(\alpha)}}{\sin{(\alpha)}} \dot{\phi} \tag{39}$$

3c) Let (x_1, x_2, x_3) be the coordinate system aligned with the principal axes of the cone. The projection of $\vec{\omega}$ on these axes is:

$$\vec{\omega} = \omega_1 \vec{e}_{x_1} + \omega_2 \vec{e}_{x_2} + \omega_3 \vec{e}_{x_3} = \omega \sin{(\alpha)} \vec{e}_{x_1} + \omega \cos{(\alpha)} \vec{e}_{x_3} \tag{40}$$

Component of ω along the x_3 axis:

$$\omega_3 = \omega \cos\alpha = \frac{\cos^2{(\alpha)}}{\sin{(\alpha)}} \dot{\phi} \tag{41}$$

3d)

$$\begin{aligned} T &= \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_3 \omega_3^2 && (42) \\ &= \frac{1}{2} I_1 \left[\omega \sin{(\alpha)}\right]^2 + \frac{1}{2} I_3 \left[\omega \cos{(\alpha)}\right]^2 && (43) \\ &= \frac{3}{40} H^2 m \dot{\phi}^2 \left[1 + 5 \cos^2{(\alpha)}\right] && (44) \end{aligned}$$