Exercise 4 solutions - TFY4345 Classical Mechanics

2020

1 Mathematical pendulum in accelerated motion

The position and velocity of the mass is

$$\begin{aligned} x &= \ell\sin(\theta), \quad y = \frac{1}{2}at^2 - \ell\cos(\theta), \\ \dot{x} &= \ell\dot{\theta}\cos(\theta), \quad y = at + \ell\dot{\theta}\sin(\theta), \end{aligned}$$

so the kinetic energy is given by

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2}m\left((\ell\dot{\theta})^2 + (at)^2 + 2at\ell\dot{\theta}\sin(\theta)\right),$$

and the potential energy is

$$V = mgy = mg\left(\frac{1}{2}at^2 - \ell\cos(\theta)\right).$$

The Lagrangian is

$$L = \frac{1}{2}m\left((\ell\dot{\theta})^2 + (at)^2 + 2at\ell\dot{\theta}\sin(\theta)\right) - mg\left(\frac{1}{2}at^2 - \ell\cos(\theta)\right),$$

so the canonical momentum is

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = m\left(\ell^2\dot{\theta} + at\ell\sin(\theta)\right) \implies \dot{\theta} = \frac{p_\theta - mta\ell\sin(\theta)}{m\ell^2}.$$

This gives the Hamiltonian

$$\begin{aligned} H &= \dot{\theta}p_\theta - L \\ &= p_\theta\frac{p_\theta - mta\ell\sin(\theta)}{m\ell^2} - \frac{1}{2}m\left[\ell^2\left(\frac{p_\theta - mta\ell\sin(\theta)}{m\ell^2}\right)^2 + (\ell at)^2 + 2at\ell\sin(\theta)\left(\frac{p_\theta - mta\ell\sin(\theta)}{m\ell^2}\right)\right] \\ &+ mg\left(\frac{1}{2}at^2 - \ell\cos(\theta)\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{m\ell^2}\left(p_\theta^2 - p_\theta mta\ell\sin(\theta)\right) - \\
&\frac{1}{2m\ell^2}\left[p_\theta^2 - 2p_\theta mta\ell\sin(\theta) + (mta\ell\sin(\theta))^2 + (m\ell ta)^2 + 2p_\theta mat\ell\sin(\theta) - 2(mta\ell\sin(\theta))^2\right] \\
&+ mg\left(\frac{1}{2}at^2 - \ell\cos(\theta)\right) \\
&= \frac{1}{2m\ell^2}\left(p_\theta - mta\ell\sin(\theta)\right)^2 - \frac{1}{2}ma^2t^2 + \frac{1}{2}magt^2 - mg\ell\cos(\theta).
\end{aligned}
$$

The Hamiltonian equations of motion

$$
\begin{aligned}
\dot\theta &= \frac{\partial H}{\partial p_\theta} = \frac{p_\theta - mta\ell\sin(\theta)}{m\ell^2} \\
\dot p_\theta &= -\frac{\partial H}{\partial \theta} = \frac{at\cos(\theta)}{\ell}\left[p_\theta - mat\ell\sin(\theta)\right] - mg\ell\sin(\theta).
\end{aligned}
$$

Furthermore, we see that $H \neq T + V$, so the Hamiltonian function is not the total energy of the system. Furthermore,

$$
\frac{\mathrm{d}H}{\mathrm{d}t} = -\frac{\partial L}{\partial t} \implies \frac{\mathrm{d}H}{\mathrm{d}t} \neq 0,
$$

as the Lagrangian has an explicit time dependence. The pendulum is in an accelerating motion with the respect to the inertial frame of reference. This mean that H will not be conserved.

2 Spherically symmetrical potential

Spherical coordinates defined by

$$
x = r\sin(\theta)\cos(\varphi), \quad y = r\sin(\theta)\cos(\varphi), \quad z = r\cos(\theta)
$$

This mean that the square velocity is (take a deep breath)

$$
\begin{aligned}
v^2 &= \dot x^2 + \dot y^2 + \dot z^2 \\
&= \left(\dot r\sin(\theta)\cos(\varphi) + r\dot\theta\cos(\theta)\cos(\varphi) - r\dot\varphi\sin(\theta)\sin(\varphi)\right)^2 \\
&+ \left(\dot r\sin(\theta)\sin(\varphi) + r\dot\theta\cos(\theta)\sin(\varphi) + r\dot\varphi\sin(\theta)\cos(\varphi)\right)^2 + \left(\dot r\cos(\theta) - r\dot\theta\sin(\theta)\right)^2 \\
&= \underline{\dot r^2\sin^2(\theta)\cos^2(\varphi)} + \underline{r^2\dot\theta^2\cos^2(\theta)\cos^2(\varphi)} + \underline{r^2\dot\varphi^2\sin^2(\theta)\sin^2(\varphi)} + \underline{\underline{2\dot r r\dot\theta\sin(\theta)\cos(\theta)\cos^2(\varphi)}} \\
&- \cancel{2\dot r r\dot\varphi\sin^2(\theta)\cos(\varphi)}\sin(\varphi) - \cancel{2r^2\dot\theta\dot\varphi\sin(\theta)\cos(\theta)}\sin(\varphi)\cos(\varphi) + \underline{\dot r^2\sin^2(\theta)\sin^2(\varphi)} \\
&+ \underline{r^2\dot\theta^2\cos^2(\theta)\sin^2(\varphi)} + \underline{r^2\dot\varphi^2\sin^2(\theta)\cos^2(\varphi)} + \underline{\underline{2\dot r r\dot\theta\sin(\theta)\cos(\theta)\sin^2(\varphi)}} \\
&+ \cancel{2\dot r r\dot\varphi\sin^2(\theta)\sin(\varphi)\cos(\varphi)} + \cancel{2r^2\dot\theta\dot\varphi\sin(\theta)\cos(\theta)\sin(\varphi)\cos(\varphi)} + \dot r^2\cos^2(\theta) + r^2\dot\theta^2\sin^2(\theta) \\
&- 2\dot r r\dot\theta\sin(\theta)\cos(\theta) \\
&= \dot r^2\sin^2(\theta) + r^2\dot\theta^2\cos^2(\theta) + r^2\dot\varphi^2\sin^2(\theta) + \cancel{2\dot r r\dot\theta\sin(\theta)\cos(\theta)} \\
&+ \dot r^2\cos^2(\theta) + r^2\dot\theta^2\sin^2(\theta) - \cancel{2\dot r r\dot\theta\sin(\theta)\cos(\theta)} \\
&= \dot r^2 + (r\dot\theta)^2 + (r\dot\varphi\sin(\theta))^2
\end{aligned}
$$

The Lagrangian is

$$L = T - V = \frac{1}{2}m\left[\dot{r}^2 + (r\dot{\theta})^2 + (r\dot{\varphi}\sin(\theta))^2\right] - \frac{k}{r},$$

so the canonical momenta are

$$p_r = \frac{\partial L}{\partial r} = m\dot{r}, \quad p_\theta = \frac{\partial L}{\partial \theta} = mr^2\dot{\theta}, \quad p_\varphi = \frac{\partial L}{\partial \varphi} = mr^2\sin^2(\varphi)\dot{\varphi}.$$

This means we can rewrite the kinetic energy in terms of the momenta:

$$T = \frac{1}{2m}\left[p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2\sin^2(\theta)}\right].$$

The Hamiltonian becomes

$$H = T + V = \frac{1}{2m}\left[p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2\sin^2(\theta)}\right] - \frac{k}{r}.$$

Hamilton's equation of motion

$$\begin{aligned}
\dot{r} &= \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \\
\dot{\theta} &= \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \\
\dot{\varphi} &= \frac{\partial H}{\partial p_\varphi} = \frac{p_\varphi}{mr^2\sin^2(\theta)} \\
\dot{p}_r &= -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} + \frac{p_\varphi^2}{mr^3\sin^2(\theta)} + \frac{k}{r^2} \\
\dot{p}_\theta &= -\frac{\partial H}{\partial \theta} = \frac{p_\varphi^2\cos(\theta)}{mr^2\sin^3(\theta)} \\
\dot{p}_\varphi &= -\frac{\partial H}{\partial \varphi} = 0.
\end{aligned}$$

3 Earth's orbit

(Se compendium, chapter 4)
The eccentricity of a circle is 1, so using the formula for the for the eccentricity of an orbit we get that

$$\varepsilon = 0 = \sqrt{1 + \frac{2E\ell^2}{mk^2}} \implies 1 + \frac{2E\ell^2}{mk^2} = 0 \implies E = -\frac{mk^2}{2\ell^2}.$$

On the other hand, for a circular orbit $E = V_{min}$. Using the effective 1D potential

$$E(r) = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{\ell^2}{mr^2} - \frac{k}{r} = V_{\min}.$$

If the mass of the sun is halved, then as the constant in the potential $k = GMm$ is proportional to the mass of the sun, it is halved, $k \to k/2$. This means the energy is change to

$$E \to E' = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{\ell^2}{mr^2} - \frac{k}{2r} = V_{min} + \frac{k}{2r} = -\frac{mk^2}{2\ell^2} + \frac{k}{2r}.$$

The original radius of the original orbit is given by the minimum of the effective potential,

$$V(r) = \frac{1}{2}\frac{\ell^2}{mr^2} - \frac{k}{r},$$

so

$$V'(r) = -\frac{\ell^2}{mr^3} + \frac{k}{r^2} = 0 \implies r = \frac{\ell^2}{km}.$$

This means the new energy is

$$E' = -\frac{mk^2}{2\ell^2} + \frac{mk^2}{2\ell^2} = 0,$$

The new eccentricity is therefore

$$\varepsilon = \sqrt{1+0} = 1,$$

which means the new orbit is a parabola, and thus unbounded. The earth just escapes to infinity.

4 Einsteins correction

As the central force is given by

$$f(r) = -\frac{k}{r^2} + \frac{\beta}{r^3},$$

the potential is (up to a constant)

$$V(r) = -\frac{k}{r} + \frac{\beta}{2r^2}.$$

In the compendium, we can find that the angle of an object in a central potential is

$$\theta(r) = \int_{r_0}^{r} \frac{1/r^2 \,\mathrm{d}r}{\sqrt{\frac{2mE}{\ell^2} - \frac{2mV(r)}{\ell^2} - \frac{1}{r^2}}}.$$

Inserting our potential, setting $u = 1/r$, and using $\gamma = 1 + \beta m/\ell^2$, this becomes

$$\theta(r) = -\int_{u_0}^{u} \frac{\mathrm{d}u}{\sqrt{\frac{2mE}{\ell^2} - \frac{2mu}{\ell^2} - \gamma^2 u^2}}.$$

By introducing the constants

$$a = \frac{2mE}{\ell^2} \quad b = \frac{2mK}{\ell^2} \quad c^2 = -\gamma^2,$$

We get the integral on a known form which can be found in tables:

$$\theta(r) = -\int_{u_0}^{u} \frac{\mathrm{d}u}{\sqrt{a + bu + cu^2}} = -\frac{1}{\sqrt{-c}} \arccos\left(-\frac{b + 2cu}{\sqrt{b^2 - 4ac}}\right).$$

Now,

$$-\frac{b+2cu}{\sqrt{b^2-4ac}} = \frac{2\gamma^2 u - 2mk/\ell^2}{\sqrt{\left(2mk/\ell^2\right)^2 + 4\left(2mE/\ell^2\right)\gamma^2}} = \frac{\frac{\ell^2\gamma^2}{mk}u - 1}{\sqrt{1+\frac{2E\gamma^2\ell^2}{mk}}} = \frac{p/r-1}{\varepsilon},$$

where

$$p = \frac{\ell^2\gamma^2}{mk}, \quad \varepsilon = \sqrt{1+\frac{2E\gamma^2\ell^2}{mk}}.$$

This means the angle of the object is given by

$$\theta(r) = \frac{-1}{\gamma}\arccos\left(\frac{p/r-1}{\varepsilon}\right).$$

Turning this around,

$$\frac{p}{r} = 1+\varepsilon\cos(\gamma\theta), \quad \text{where } \gamma = \sqrt{1+\frac{m\beta}{\ell^2}} \approx 1+\frac{m\beta}{2\ell^2}, \ \frac{m\beta}{\ell} \ll 1.$$

If $E<0$, then this is an ellipse with slow precession. The semi-major axis for $\gamma=1$ is

$$a = \frac{p}{1-\varepsilon^2} = \frac{\gamma^2\ell^2/mk}{1-(1+2E\gamma^2\ell^2/mk^2)} = \frac{k}{2|E|}.$$

This, then, is a perturbation to this, with the smallness parameter $\eta = \beta/ka$, so $\gamma = 1+m\eta ka/(2\ell^2)$. For Mercury, $\eta = 1.42\cdot 10^{-7}$, which is the perihelion precession of $43''$ per century.