

Exercise 8 solutions - TFY4345 Classical Mechanics

2020

1 Principal moments of inertia of a triangular slab

(a) Since the mas has uniform density, we can write the mass area density as $M = \rho ab/2$. Let x_{CM} denote the x -component of the center of mass. Using the definition of CM , we find

$$x_{CM} = \frac{1}{M} \int_0^a dx \int_0^{b(1-x/a)} dy \rho x = \frac{\rho b}{M} \int_0^a dx \left(1 - \frac{x}{a}\right) = \frac{a^2 b \rho}{M} \int_0^1 du (1-u)u = \frac{\rho a^2 b}{6M} = \frac{a}{3}.$$

We used the substitution $u = 1 - x/a$ which implies a $dx = -adu$. Because of the symmetry in the problem (the slab is a triangle), the calculation of y_{CM} is the same, only exchanging $a \leftrightarrow b$, so the result is $y_{CM} = b/3$.

(b) The slab is two dimensional, and laying in the xy -plane. If we look at the definition of the off-diagonal entries in moment of inertia tensor,

$$I_{ij} = - \int_V dV x_i x_j,$$

$I_{zx} = I_{xz} = I_{zy} = I_{yz} = 0$, as $z = 0$. This also implies that $I_{xx} + I_{yy} = I_{zz}$, so all we need to calculate is I_{xx} , I_{yy} and I_{xy} .

$$\begin{aligned} I_{xy} &= -\rho \int_0^a dx \int_0^{v(1-x/a)} dy yx = -\frac{\rho b^2}{2} \int_0^a dx x \left(1 - \frac{x}{a}\right)^2 = -\frac{\rho b^2}{2} \int_0^a dx \left(x - \frac{2}{a}x^2 + \frac{1}{a^2}x^3\right) \\ &= -\frac{\rho b^2 a^2}{2} \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = \frac{Mab}{12} \\ I_{xy} &= -\rho \int_0^a dx \int_0^{v(1-x/a)} dy y^2 = \frac{\rho b^3}{3} \left(1 - \frac{x}{a}\right)^3 = \frac{\rho ab^3}{3} \int_0^1 du u^3 = \frac{Mb^2}{6}. \end{aligned}$$

Lastly, I_{yy} can a gain be found just by the exchange $a \leftrightarrow b$. In matrix form,

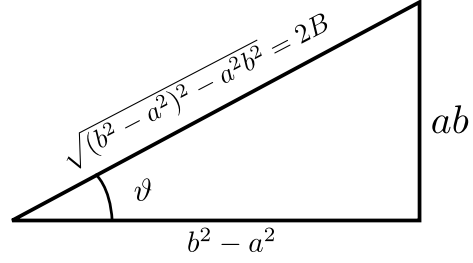
$$I = \frac{M}{6} \begin{pmatrix} b^2 & -\frac{1}{2}ab & 0 \\ -\frac{1}{2}ab & a^2 & 0 \\ 0 & 0 & a^2 + b^2 \end{pmatrix}$$

(c)Insert our values into the new variables (neglecting the common factor), we get

$$A = \frac{1}{2}(a^2 + b^2), \quad B = \frac{1}{2}\sqrt{(b^2 - a^2) + a^2 b^2}, \quad \vartheta = \tan^{-1} \left(\frac{ab}{b^2 - a^2} \right).$$

The last equation describes a triangle with side lengths $b^2 - a^2$, ab and $\sqrt{(b^2 - a^2)^2 + a^2 b^2} = 2B$, and an angle ϑ opposite the side of length ab . This gives us the relations $ab = 2B \sin(\vartheta)$ and $b^2 - a^2 = 2B \cos(\vartheta)$. It follows that

$$\begin{aligned} a^2 &= \frac{1}{2}(b^2 + a^2) - \frac{1}{2}(b^2 - a^2) = A - B \cos(\vartheta) \\ b^2 &= \frac{1}{2}(b^2 + a^2) + \frac{1}{2}(b^2 - a^2) = A + B \cos(\vartheta) \end{aligned}$$



Putting all this together, we get

$$I = \begin{pmatrix} A + B \cos(\vartheta) & B \sin(\vartheta) & 0 \\ B \sin(\vartheta) & A - B \cos(\vartheta) & 0 \\ 0 & 0 & 2A \end{pmatrix}$$

To find the principal moments of inertia, we must find solve the characteristic equation for the principal moments of inertia ω

$$\begin{aligned} \det(I - \omega) = 0 &\implies \begin{vmatrix} A + B \cos(\vartheta) - \omega & B \sin(\vartheta) & 0 \\ B \sin(\vartheta) & A - B \cos(\vartheta) - \omega & 0 \\ 0 & 0 & 2A - \omega \end{vmatrix} \\ &= (2A - \omega)[(A + B \cos(\vartheta) - \omega)(A - B \cos(\vartheta) - \omega) - B^2 \sin^2(\vartheta)] \\ &= (2A - \omega)[A^2 - B^2 + \omega^2 - 2\omega A] \\ &= (2A - \omega)[(A - \omega)^2 - B^2] = 0, \end{aligned}$$

which has the solutions $\omega_1 = 2A$, $\omega_2 = A + B$ and $\omega_3 = A - B$. By inspection, the first eigenvector is $\mathbf{v} = (0, 0, 1)$. We can then only look at the relevant part of the matrix to find the others. Inserting $\omega = A + B$,

$$0 = (I - \omega \mathbf{1})\mathbf{v} = B \begin{pmatrix} \cos(\vartheta) - 1 & \sin(\vartheta) \\ \sin(\vartheta) & -\cos(\vartheta) - 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Setting $v_2 = 1$, we get $v_1 = \sin(\vartheta)/(1 - \cos(\vartheta))$. The normalized eigenvector then becomes

$$\begin{aligned} \mathbf{v} &= \frac{1}{\sqrt{1 + \frac{\sin^2(\vartheta)}{(1 - \cos(\vartheta))^2}}} \begin{pmatrix} \frac{\sin(\vartheta)}{1 - \cos(\vartheta)} \\ 1 \\ 0 \end{pmatrix} = \frac{1 - \cos(\vartheta)}{\sqrt{1 - 2\cos(\vartheta) + \cos^2(\vartheta) + \sin^2(\vartheta)}} \begin{pmatrix} \frac{\sin(\vartheta)}{1 - \cos(\vartheta)} \\ 1 \\ 0 \end{pmatrix} \\ &= \frac{1 - \cos(\vartheta)}{\sqrt{1 - \cos(\vartheta)}} \begin{pmatrix} \frac{\sin(\vartheta)}{1 - \cos(\vartheta)} \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2} \sin(\vartheta/2)} (\sin(\vartheta/2), 2 \sin^2(\vartheta/2), 0) \end{aligned}$$

Using $2 \sin(\vartheta/2) \cos(\vartheta/2) = \sin(\vartheta)$, this gives (HVOR KOMMER $\sqrt{2}$ FRA????!?!?)

$$\mathbf{v} = (\sqrt{2} \cos(\vartheta/2), -\sqrt{2} \sin(\vartheta/2), 0)$$

The last eigenvector is then found in a similar manner by setting $\omega = A - B$, yielding

$$\mathbf{v} = (-\sin(\vartheta/2), \cos(\vartheta/2), 0)$$

2 Precession of a frisbee

(a) The Euler equation for the motion of a spinning free body (no torque) is

$$\left(\frac{d\mathbf{L}}{dt}\right)_b + \boldsymbol{\omega} \times \mathbf{L} = 0$$

Writing this out in component form gives

$$I_1 \dot{\omega}_{x'} + \omega_{y'} \omega_{z'} (I_3 - I_2) = 0,$$

$$I_2 \dot{\omega}_{y'} + \omega_{z'} \omega_{x'} (I_1 - I_3) = 0,$$

$$I_3 \dot{\omega}_{z'} + \omega_{x'} \omega_{y'} (I_2 - I_1) = 0.$$

As shown in the compendium (5.G), the components of the angular velocity in the body frame is

$$\omega_{x'} = \dot{\phi} \sin(\theta) \sin(\psi) + \dot{\theta} \cos(\psi)$$

$$\omega_{y'} = \dot{\phi} \sin(\theta) \cos(\psi) - \dot{\theta} \sin(\psi)$$

$$\omega_{z'} = \dot{\phi} \cos(\theta) + \dot{\psi}.$$

(b) From the component form of the equations of motion, we see that

$$I_1 = I_2 \implies I_3 \dot{\omega}_{z'} = 0 \implies \omega_{z'} = \text{const.}$$

From the figure in the exercise, we can see that $L_{z'} = L \cos(\theta)$. The body axes are the principal axes of the frisbee, so $L_{z'} = I_3 \omega_{z'} = \text{const.} \implies \theta = \text{const.}$, i.e. $\dot{\theta} = 0$. Using the Euler equation, we then get

$$\dot{\omega}_{x'} = -\Omega \omega_{y'}, \quad \dot{\omega}_{y'} = \Omega \omega_{x'}, \quad \Omega = \frac{I_3 - I_1}{I_1} \omega_{z'} = \left(\frac{1}{I_1} - \frac{1}{I_3}\right) L \cos(\theta).$$

This is the equation of two sinusoidal functions, 90° out of phase. (An example of a solution is $\omega_{x'} = \cos(\Omega t)$, $\omega_{y'} = \sin(\Omega t)$). This implies¹

$$\omega_{x'}^2 + \omega_{y'}^2 = \left(\dot{\phi} \sin(\theta) \sin(\psi)\right)^2 + \left(\dot{\phi} \sin(\theta) \cos(\psi)\right)^2 = \dot{\phi}^2 \sin^2(\theta) = \text{const.},$$

i.e. that $\phi = \text{const.}$. Using this to differentiate the expressions of the angular momentum in Euler angles, we get

$$\begin{aligned} \dot{\phi} \dot{\psi} \sin(\theta) \cos(\phi) &= -\Omega \dot{\phi} \sin(\theta) \cos(\psi) \\ -\dot{\phi} \dot{\psi} \sin(\theta) \sin(\phi) &= \Omega \dot{\phi} \sin(\theta) \sin(\psi) \\ \implies \dot{\psi} &= -\Omega = \left(\frac{1}{I_3} - \frac{1}{I_1}\right) L \cos(\theta), \quad \dot{\omega}_{z'}, \end{aligned}$$

¹ A more general proof for this is that $\frac{d}{dt}(\omega_{x'}^2 + \omega_{y'}^2) = 2\omega_{x'} \dot{\omega}_{x'} + 2\omega_{y'} \dot{\omega}_{y'} = -2\Omega \dot{\omega}_{y'} \dot{\omega}_{x'} + 2\Omega \dot{\omega}_{y'} \dot{\omega}_{x'} = 0$

while the last equation of motion gives

$$\dot{\phi} = \frac{L}{I_3} + \left(\frac{1}{I_1} - \frac{1}{I_3} \right) L = \frac{L}{I_1}$$

(c)

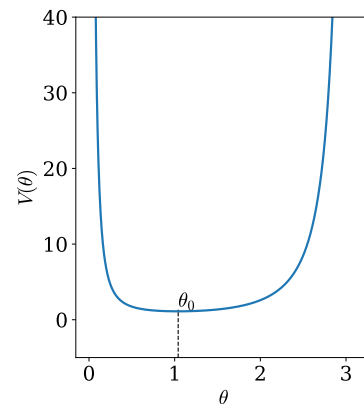
$$\frac{\dot{\phi}}{\omega_{z'}} = \frac{I_3}{I_1 \cos(\theta)} \approx 2.$$

3 Precession of a heavy spinning top

The effective potential of the spinning top is

$$V(\theta) = \frac{(p_\phi - p_\psi \cos(\theta))^2}{2I_1 \sin^2(\theta)} + Mgh \cos(\theta).$$

The shape is shown in the plot. This case is similar to that of orbiting planets. In that case, the centrifugal force creates an effective potential as a function of r , with a minimum. If the planet has an energy corresponding to that minimum, it is in a circular orbit, with constant radius. In this case, the effective potential is a function of the angle. Thus, the stable configuration with constant $\theta = \theta_0$ corresponds to the minimum of the potential. This is found by differentiating $V(\theta)$ and setting it equal to zero:



$$\left. \frac{\partial V}{\partial \theta} \right|_{\theta=\theta_0} = \frac{-\cos(\theta_0)(p_\phi - p_\psi \cos(\theta_0))^2 + p_\psi \sin^2(\theta_0)(p_\phi - p_\psi \cos(\theta_0))}{I_1 \sin^3(\theta_0)} - Mgh \sin(\theta_0) = 0.$$

Defining $\beta = p_\phi - p_\psi \cos(\theta_0)$, we can simplify this to

$$0 = \cos(\theta_0)\beta^2 - p_\psi \sin^2(\theta_0)\beta + MghI_1 \sin^4(\theta_0) = 0,$$

so we can solve for β :

$$\beta_{\pm} = \frac{p_\psi \sin^2(\theta_0)}{2 \cos(\theta_0)} \left(1 \pm \sqrt{1 - \frac{4MgHI_1 \cos(\theta_0)}{p_\psi^2}} \right).$$

We can see that β must be a real quantity from its definition. Thus, if $\theta_0 < \pi/2$, so $\cos(\theta_0) > 0$, we get the restriction

$$p_\psi^2 \geq 4MghI_1 \cos(\theta_0)$$

on physical configurations of the system. Inserting $p_\psi = I_3\omega_3$, we get

$$\omega_3 \geq \frac{2}{I_3} \sqrt{MghI_1 \cos(\theta_0)}.$$

This is a lower bound for the angular momentum needed by the spinning to be able to precess at an constant angle θ .

The rate of precession is then given by

$$\dot{\phi}_{0(\pm)} = \frac{\beta_{\pm}}{I_1 \sin^2(\theta_0)}.$$

This means we have two different configurations with stable precession, given by the two roots β_{\pm} . $\dot{\phi}_{0(+)}$ gives fast precession, while $\dot{\phi}_{0(-)}$ gives slow precession. if $\omega_3 \gg \frac{2}{I_3} \sqrt{MghI_1 \cos(\theta_0)}$, then $p_{\psi}^2 \gg 4MghI_1 \cos(\theta_0)$. We can use $\sqrt{1+x} = 1 - \frac{1}{2}x + \mathcal{O}(x^2)$ then expand the root in equation for β as

$$\begin{aligned} \sqrt{1 - \frac{4MghI_1 \cos(\theta_0)}{p_{\psi}^2}} &\approx 1 + \frac{2MghI_1 \cos(\theta_0)}{p_{\psi}^2}, \quad p_{\psi} = I_3\omega_3 \\ \Rightarrow \beta_{\pm} &\approx \frac{I_3\omega_3 \sin^2(\theta_0)}{2 \cos(\theta_0)} \left(\frac{(I_3\omega_3)^2(1 \pm 1) + 2MghI_1 \cos(\theta_0)}{(I_3\omega_3)^2} \right) \\ \Rightarrow \dot{\phi}_{0(\pm)} &\approx \frac{(I_3\omega_3)^2(1 \pm 1) + 2MghI_1 \cos(\theta_0)}{2I_1 I_3 \omega_3 \cos(\theta_0)} = \frac{I_3\omega_3}{2I_1 \cos(\theta_0)}(1 \pm 1) + \frac{Mgh}{I_3\omega_3}. \end{aligned}$$

The two stable configurations thus precess with the angular velocities

$$\dot{\phi}_{0(+)} = \frac{I_3\omega_3}{I_1 \cos(\theta_0)}, \quad \dot{\phi}_{0(-)} = \frac{Mgh}{I_3\omega_3}.$$