Classical Mechanics TFY4345 – Solution set 2

1. Friction (from previous years → additional numbering below)

(1c) Frictional force $F_f = -\partial\mathcal{F}/\partial v$. The work performed by the system against friction, per unit time, is $\dot{W} = -F_f v$ which with $\mathcal{F} = Cv^2$ becomes $\dot{W} = 2\mathcal{F}$. The Lagrange equations read:

$$\frac{d}{dt}\frac{\partial L}{\partial v} - \frac{\partial L}{\partial x} + \frac{\partial \mathcal{F}}{\partial v} = 0. \tag{5}$$

Insert the Lagrangian $L = T - V = mv^2/2 - kx^2/2$ and $\mathcal{F} = 3\pi\mu a v^2$ to obtain:

$$\ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = 0 \tag{6}$$

where $\lambda = 3\pi\mu a/m$ and $\omega_0 = \sqrt{k/m}$. Assuming $\lambda/\omega_0 \ll 1$, the solution is:

$$x(t) = x_0 \mathrm{e}^{-\lambda t}\cos\omega_0 t \tag{7}$$

and the average energy dissipation $\bar{\dot{W}}$ over a period $2\pi/\omega_0$ can be computed by treating $\mathrm{e}^{-\lambda t}$ as a constant since it remains virtually unchanged over a time-interval $2\pi/\omega$:

$$\bar{\dot{W}} \simeq m\lambda(\omega_0 x_0)^2 \mathrm{e}^{-2\lambda t}. \tag{8}$$

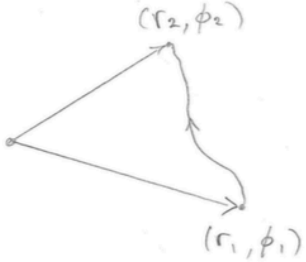
2. Levi-Civita tensor

(1b) We have that:

$$\begin{aligned}
[\nabla\times(\nabla\times\mathbf{A})]_i &= \varepsilon_{ijk}\partial_j(\nabla\times\mathbf{A})_k \\
&= \varepsilon_{ijk}\partial_j\varepsilon_{klm}\partial_l A_m \\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\partial_j\partial_l A_m \\
&= \partial_i(\nabla\cdot\mathbf{A}) - \nabla^2 A_i.
\end{aligned}$$

We have then shown that $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$.

3. Variational calculus



$$ds = \sqrt{dr^2 + r^2 d\phi^2} = \sqrt{1 + r^2 \phi'^2}\, dr$$

$$S = \int_1^2 ds = \int_{r_1}^{r_2} \sqrt{1 + r^2\phi'^2}\, dr$$

Euler equation: $\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0 \implies \frac{\partial f}{\partial \phi} - \frac{d}{dr}\frac{\partial f}{\partial \phi'} = 0$

$$\frac{\partial f}{\partial \phi} = 0, \quad \frac{\partial f}{\partial \phi'} = \frac{\partial}{\partial \phi'}\sqrt{1 + r^2\phi'^2} = \frac{\cancel{2}\,\phi' r^2}{\cancel{2}\sqrt{1 + r^2\phi'^2}}$$

$$\frac{d}{dr}\frac{\partial f}{\partial \phi'} = 0 \implies \frac{\partial f}{\partial \phi'} = \text{constant} = a$$

$$\frac{\phi' r^2}{\sqrt{1 + r^2\phi'^2}} = a \implies \frac{\phi' r^2}{a} = \sqrt{1 + r^2\phi'^2}$$

$$\implies \frac{\phi'^2 r^4}{a^2} = \cancel{\pm}\left(1 + r^2\phi'^2\right)$$

$$\implies \left(\frac{r^2}{a^2} - 1\right) r^2 \phi'^2 = 1$$

New constant $b^2 = \frac{1}{a^2}$

$$\implies (b^2r^2 - 1)\, r^2 \phi'^2 = 1 \implies \phi'^2 = \frac{1}{r^2(b^2r^2 - 1)}$$

$$\implies \phi' = \pm \frac{1}{r\sqrt{b^2r^2 - 1}}$$

Table!

$$\implies \phi = \pm \int \frac{dr}{r\sqrt{b^2r^2 - 1}} \overset{\phi}{=} \pm \arcsin\left(\frac{-2}{r\sqrt{4b^2}}\right) + (c)$$

$$\implies \sin\phi = \pm \frac{1}{br} \implies \underbrace{r = \pm \frac{a}{\sin\phi}, \quad r \geq 0}$$

$b = \frac{1}{a}$ $\qquad$ $a = \text{constant}$ $\qquad$ Must be!

(choice of $b \leftrightarrow a$, does not matter)

4. Simple pendulum revisited

Kinetic energy:

$$T = \frac{1}{2}m\left(\frac{d\vec{r}}{dt}\right)^2 = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\beta}^2\right) \tag{1}$$

Potential energy:

$$V = mgh = -mgr\cos\beta \tag{2}$$

Lagrangian:

$$L = T - V = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\beta}^2\right) + mgr\cos\beta \tag{3}$$

Lagrangian with Lagrange multiplier to fix the length of the string ($\ell - r = 0$):

$$\tilde{L} = T - V + \lambda(\ell - r) = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\beta}^2\right) + mgr\cos\beta + \lambda(\ell - r) \tag{4}$$

Lagrange equation for r coordinate:

$$\frac{d}{dt}\frac{\partial\tilde{L}}{\partial\dot{r}} - \frac{\partial\tilde{L}}{\partial r} = 0 \tag{5}$$

giving

$$m\ddot{r} - mr\dot{\beta}^2 - mg\cos\beta + \lambda = 0 \tag{6}$$

Lagrange equation for β coordinate:

$$\frac{d}{dt}\frac{\partial\tilde{L}}{\partial\dot{\beta}} - \frac{\partial\tilde{L}}{\partial\beta} = 0 \tag{7}$$

giving:

$$mr^2\ddot{\beta} + mgr\sin\beta = 0 \tag{8}$$

The constraint on the position of the particle implies:

$$\ell - r = 0 \;\Rightarrow\; \dot{r} = 0 \;\text{ and }\; \ddot{r} = 0 \tag{9}$$

Final result:

$$\ddot{\beta} + \frac{g}{\ell}\sin\beta = 0 \tag{10}$$

$$\lambda = mr\dot{\beta}^2 + mg\cos\beta \tag{11}$$

The tension T in a pendulum string is the sum of the component of gravitational force parallel to the string ($mg\cos\beta$), and the centripetal force acting on the mass ($mr\dot{\beta}^2$), i.e. $T = mr\dot{\beta}^2 + mg\cos\beta$, which is the same as Eq. (11). This means that λ is the tension in the pendulum string.