Classical Mechanics TFY 4345 – Solution set 10

1. Velocity addition rule and Lorentz transformation matrices.

$$x'_\mu = L_{\mu\nu} x_\mu \,, \qquad x''_\mu = L'_{\mu\nu} x'_\mu$$

collinear motion in x_1-direction

$\Rightarrow$ L' and L have the same form

$$L = \begin{bmatrix} \gamma_1 & 0 & 0 & i\beta\gamma_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma_1 & 0 & 0 & \gamma_1 \end{bmatrix}$$

Prescription:

$$L_{jk} = \delta_{jk} + (\gamma - 1)\frac{\beta_j \beta_k}{\beta^2}$$

$$L_{j4} = i\gamma\beta_j \,, \quad L_{4k} = -i\beta\gamma_k$$

$$L_{44} = \gamma$$

$$L' = \begin{bmatrix} \gamma_2 & 0 & 0 & i\beta\gamma_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma_2 & 0 & 0 & \gamma_2 \end{bmatrix}$$

L, L' orthogonal

$\hookrightarrow$ $L^{+} = L^{-1}$

Now: There has to be $L'' = L' \cdot L$ such that

$$x''_\mu = L''_{\mu\nu} x_\mu$$

$$\Rightarrow \quad L'' = L' \cdot L = \begin{bmatrix} \gamma_1\gamma_2(1+\beta_1\beta_2) & 0 & 0 & i\gamma_1\gamma_2(\beta_1+\beta_2) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\gamma_1\gamma_2(\beta_1+\beta_2) & 0 & 0 & \gamma_1\gamma_2(1+\beta_1\beta_2) \end{bmatrix}$$

For L'' we can see

$$\begin{cases} \gamma = \gamma_1\gamma_2(1+\beta_1\beta_2) \\ \beta\gamma = \gamma_1\gamma_2(\beta_1+\beta_2) \end{cases} \Rightarrow \quad \beta = \frac{\cancel{\gamma_1\gamma_2}(\beta_1+\beta_2)}{\cancel{\gamma_1\gamma_2}(1+\beta_1\beta_2)}$$

Multiply with c

$$\Rightarrow \quad \boxed{v = \frac{v_1 + v_2}{1 + (v_1 v_2 / c^2)}}$$

if $v_1 < c$, $v_2 < c$

$\rightarrow$ $v < c$ also

2. Light from a fluorescent tube. [Exam 2016]

In the S frame the tube lights up at the point z at time t. Seen from coordinate system S' :

$$
\begin{aligned}
z' &= \gamma(z - vt) && (45)\\
t' &= \gamma(t - \frac{vz}{c^2}) && (46)
\end{aligned}
$$

In the S frame the tube lights up at the point $z + \Delta z$ at time t. Seen from coordinate system S' :

$$
\begin{aligned}
z' + \Delta z' &= \gamma(z + \Delta z - vt) && (47)\\
t' + \Delta t' &= \gamma(t - \frac{v(z + \Delta z)}{c^2}) && (48)
\end{aligned}
$$

$$
\begin{aligned}
\Delta z' &= \gamma \Delta z && (49)\\
\Delta t' &= -\frac{\gamma v \Delta z}{c^2} && (50)
\end{aligned}
$$

Seen from S' the fluroscent tube does not light up instantaneously everwhere (like in S), the lighting up propagates with the velocity:

$$
u = \frac{\Delta z'}{\Delta t'} = \frac{\gamma \Delta z}{-\frac{\gamma v \Delta z}{c^2}} = -\frac{c^2}{v} \tag{51}
$$

3. Relativistic Doppler effect.

$L = c\Delta t - v\Delta t$ ← wave train

$\lambda = \frac{c\Delta t - v\Delta t}{n}$ (observer in K)

$\nu = \frac{c}{\lambda} = \frac{cn}{c\Delta t - v\Delta t}$

IF

According to source: (K′)

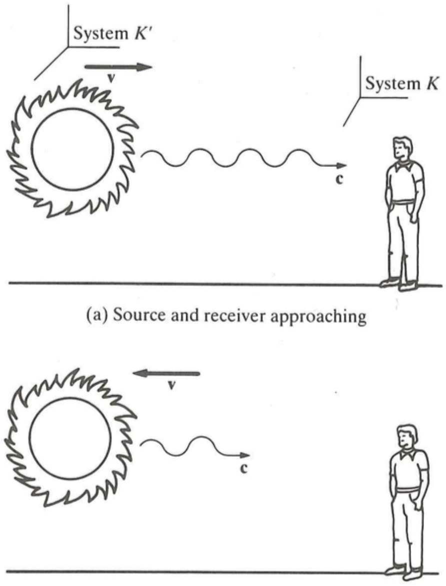
$n = \nu_0 \Delta t'$

$\Delta t' = \frac{\Delta t}{\gamma}$ ← proper time "eigentime"

$$\Rightarrow \nu = \frac{c \cdot \nu_0 \Delta t'}{(c-v)\Delta t} = \frac{c \cdot \cancel{\Delta t}}{\gamma (c-v) \cancel{\Delta t}} \nu_0$$

$$= \frac{\sqrt{1-v^2/c^2}}{1-v/c} \nu_0 = \frac{\sqrt{1-v/c}\sqrt{1+v/c}}{(\sqrt{1-v/c})^2} \nu_0$$

$$= \frac{\sqrt{1+\beta}}{\sqrt{1-\beta}} \nu_0 \quad \text{(approaching)}$$



(a) Source and receiver approaching

(b) Source and receiver receding

Receding:

$L = c\Delta t + v\Delta t$

$$\Rightarrow \nu = \ldots = \frac{\sqrt{1-v^2/c^2}}{1+v/c} \nu_0 = \frac{\sqrt{1-\beta}}{\sqrt{1+\beta}} \nu_0$$

This is the source of "red shift" in astronomy ⟹ "big bang", expanding universe

(Note added: This is not the sole reason. The expansion of space has an important contribution, the underlying theory is beyond this course)