Exercise 5 solutions - TFY4345 Classical Mechanics

2020

1 Effective potential and scattering center

The total energy, as given by equation 4.14 in the compendium, is

$$E = \frac{1}{2}m\left(\dot{r}^2 + (r\dot{\theta})^2\right) + V(r).$$

In a central potential, we have that $mr^2\dot{\theta} = \ell$ is a conserved quantity, so we get

$$E = \frac{1}{2}m\dot{r}^2 + \left(\frac{\ell^2}{2mr^2} + V(r)\right) = \frac{1}{2}m\dot{r}^2 + V_{\text{eff}}(r).$$

This is an effective 1D problem, with an effective potential

$$V_{\text{eff}}(r) = V(r) + \frac{\ell^2}{2mr^2}$$

In order for the particle to reach the center, it need to have sufficiently high energy to overcome the potential barrier, i.e. $E > V_{\text{eff}}(r \to 0)$. This can be written as

$$Er^2 > r^2V(r) + \frac{\ell^2}{2m}, \quad r \leftarrow 0.$$

The l.h.s. goes to zero, so that the condition becomes

$$(r^2V(r))_{r\to 0} < -\frac{\ell^2}{2m}.$$

This can be fulfilled wither with $-k/r^2$, where $k > \ell^2/2m$, or if $V(r) = -A/r^n$, with $n > 2$ and A a positive constant.

2 Scattering from a spherical obstacle

(FIGUR)

The scattering angle θ satisfies $2\Psi + \theta = \pi$. From the figure, we see that the impact parameter is given by $s = a\sin(\pi/2 - \theta/2) = a\cos(\theta/2)$, so that

$$\left|\frac{\mathrm{d}s}{\mathrm{d}\theta}\right| = \frac{a}{2}\sin\left(\frac{\theta}{2}\right)$$

Using the formula for the differential cross section, as given in equation 4.40 in the compendium, we get

$$\sigma(\theta) = \frac{s}{\sin(\theta)} \left| \frac{s}{\theta} \right| = \frac{a^2}{4}.$$

The total cross section is therefore

$$\sigma = 2\pi \int_0^\pi \sin(\theta)\sin(\theta)\mathrm{d}\theta = \pi a^2.$$

This is physically sensible, since it is the actual cross-sectinoal area of the sphere.

3 Scattering by an attractive hard sphere

(FIGURE)

The larges impact parameter s_max will send the particle just gracing the surface at $r = a$. Due to conservation of energy, we have that

$$E = \frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 - \frac{k}{a}.$$

Furthermore, conservation of angular momentum means that ℓ infinitely far away is the same as when the particle touches the surface, so

$$\ell = mv_0 s_{\mathrm{max}} = mva.$$

Combining thes two equations, we get

$$s_{\mathrm{max}} = \frac{v}{v_0}a = a\sqrt{1 + \frac{2k}{mav_0^2}}.$$

All particles with impact parameter $s < s_{\mathrm{max}}$ will hit the surface, so that $\sigma_{\mathrm{eff}} = \pi s_{\mathrm{max}}^2$.

4 Average energies in the Kepler problem