Gravgård

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Alt som ikke kom med...

0.1 Tree level pion star

We introduce the new dimensionless variable $1 + x^2 = \mu_I^2/\bar{m}^2$. This is reminicent of the dimensionless Fermi momentum $x_f = p_f/m$ in {section: cold fermi star}. By an argument using a right triangle, we can vertify that $\cos a = b$ implies $\sin a = \sqrt{1-b^2}$. Substituting the dimensionless variable into the free energy density, we get

$$\mathcal{F} = -\frac{u_0}{2}\left(1 + x^2 + \frac{1}{1+x^2}\right). \tag{1}$$

We have introduced the characteristic energy density $u_0 = \bar{m}^2 f^2$. As we found in {section: cold fermi star}, pressure is given by negative the free energy density, normalized to $\mu_I = \bar{m}$, or $x = 0$. We choose $p_0 = u_0$, so the dimensionless pressure can be written

$$\tilde{p} = -\frac{1}{u_0}(\mathcal{F} - \mathcal{F}_{x=0}) = \frac{1}{2}\frac{x^4}{1+x^2}. \tag{2}$$

The charge density corresponding to a chemical potential is given by minus the derivative of the free energy with respect to that chemical potential. We must, however, not assume any dependence of α on μ_I. The isospin density therefore is

$$n_I = -\frac{\partial \mathcal{F}}{\partial \mu_I} = f^2 \mu_I \sin^2\alpha = \frac{u_0}{\mu_I}\frac{2x^2 + x^4}{1+x^2}., \tag{3}$$

With this, the dimensionless energy density

$$\tilde{u} = -\tilde{p} + \frac{1}{u_0} n_I \mu_I = \frac{1}{2}\frac{4x^2 + x^4}{1+x^4} \tag{4}$$

0.2 Feil analyse av three-flavor vakum

$$\frac{1}{4}\mathrm{Tr}\left\{\nabla_\mu \Sigma_\alpha \nabla^\mu \Sigma_\alpha^\dagger\right\} = -\frac{1}{2}\sin^2\alpha\left\{\mu_I^2\left[n_1^2 + n_2^2 + \frac{1}{4}(n_6^2 + n_7^2)\right] + \frac{1}{4}[\mu_I^2 + 3\mu_8^2][n_4^2 + n_5^2]\right\} \tag{5}$$

$$\frac{1}{4}\mathrm{Tr}\left\{\chi\Sigma^\dagger + \Sigma\chi^\dagger\right\} = M_1^2\cos\alpha \tag{6}$$

$$\mathcal{H} = -\frac{1}{8}\sin^2\alpha\left[\mu_I^2\left(4a^2 + b^2 + c^2\right) + 3\mu_8^2(b^2 + c^2) + 2\sqrt{3}\mu_8\mu_I(b^2 - c^2)\right] + M_1^2\cos\alpha \tag{7}$$

$$= -\frac{1}{8}\sin^2\alpha\left[4\mu_I^2 a^2 + (\mu_I + \sqrt{3}\mu_8)^2 b^2 + (\mu_I - \sqrt{3}\mu_8)^2 c^2\right] \tag{8}$$

Choose, without loss of generality, $n_1 = n_4 = 0$, which leaves $n_2 = \cos\beta$, $n_5 = \sin\beta$, and thus

$$\Sigma = \exp\left\{i\alpha(\cos\beta\lambda_2 + \sin\beta\lambda_5)\right\} = \cos\alpha + i(\lambda_2\cos\beta + \lambda_5\sin\beta)\sin\alpha. \tag{9}$$

0.3 Feil three-flavor chpt beregning

We need to parametrize the ground state, as we did in subsection: parametrization, and define Let

$$\Sigma_\alpha = \exp\left\{i\alpha n_a\lambda_a\right\} = \cos\alpha + i n_a\lambda_a\sin\alpha, \quad \alpha = \frac{1}{f}\sqrt{\pi_a^0\pi_a^0}, \quad n_a = \frac{\pi_a^0}{\sqrt{\pi_b^0\pi_b^0}}. \tag{10}$$

The relevant terms are then

$$\frac{1}{4}\mathrm{Tr}\left\{\nabla_\mu\Sigma_\alpha\nabla^\mu\Sigma_\alpha^\dagger\right\} = \frac{1}{2}\sin^2\alpha\left[\mu_I^2(n_1^2+n_2^2)+\frac{1}{4}(\mu_I+2\mu_s)^2(n_4^2+n_5^2)+\frac{1}{4}(\mu_I-2\mu_s)^2(n_6^2+n_7^2)\right] \tag{11}$$

$$\frac{1}{4}\mathrm{Tr}\left\{\chi\Sigma^\dagger+\Sigma\chi^\dagger\right\} = M_1^2\cos\alpha \tag{12}$$

We notice that both terms are independent of μ_B. With this, the static Hamiltonian is

$$\mathcal{H}_0 = -\frac{1}{2}f^2\sin^2\alpha\left[\mu_I^2a^2+\mu_{K^\pm}^2b^2+\mu_{K^0}^2c^2\right]-f^2M_1^2\cos\alpha \tag{13}$$

We have defined the chemical potentials $\mu_{K^\pm}=\frac{1}{2}(\mu_I+2\mu_S)=\mu_u-\mu_s$ and $\mu_{K^\pm}=\frac{1}{2}(\mu_I-2\mu_S)=-\mu_d+\mu_s$, and

$$a^2=n_1^2+n_2^2,\quad b^2=n_4^2+n_5^2,\quad c^2=n_6^2+n_7^2,\quad a^2+b^2+c^2=1-n_3^2-n_8^2. \tag{14}$$

All the terms with a square chemical potential factors are positive definite, which means that the Hamiltonian will always be minimized by $n_3=n_8=0$. Furhtermore, we can without loss of generality chose $n_1=n_4=n_6=0$. This corresponds to changing basis of $\mathfrak{su}(3)$. Depending on the signs of μ_I and μ_S, we must have either $b=0$ or $c=0$ If $\mathrm{sgn}(\mu_I)=\mathrm{sgn}(\mu_S)$, then $\mu_{K^\pm}>\mu_{K^0}$, and $c=0$. Likewise, if $\mathrm{sgn}(\mu_I)=-\mathrm{sgn}(\mu_S)$, then $\mu_{K^\pm}<\mu_{K^0}$, and $b=0$. To begin with, we assume the former. Define $a^2=\cos^2\beta$, which implies $b^2=\sin^2\beta$. The Hamiltonian density is then

$$\mathcal{H}_0 = -\frac{1}{2}f^2\left[\mu_I^2\cos^2\beta+\mu_{K^\pm}^2\sin^2\beta\right]\sin^2\alpha-f^2M_1^2\cos\alpha. \tag{15}$$

The β parameter is set, as α, by minimizing $\mathcal{H}$. We have

$$\frac{\partial\mathcal{H}}{\partial\beta}=\frac{1}{2}(\mu_I^2-\mu_{K^\pm}^2)f^2\sin^2\alpha\cos2\beta,\quad \frac{\partial^2\mathcal{H}}{\partial\beta^2}=f^2(\mu_I^2-\mu_{K^\pm}^2)\sin^2\alpha\sin2\beta. \tag{16}$$

We see that, if we are in the pion condensate phase where $\alpha\neq0$, the stationary points for β are 0 and $\pi/2$. However, which one these that is a minimum depends on the sign of $\mu_I^2-\mu_{K^\pm}^2$, as this determines the sign of the second derivative. For $\mu_I^2>\mu_{K^\pm}^2$, $\beta=0$, while for $\mu_I^2<\mu_{K^\pm}^2$ we have $\beta=\pi/2$. The analysis for $\mathrm{sgn}(\mu_I)=-\mathrm{sgn}(\mu_S)$ is the same, only with $\mu_{K^\pm}$ changed to μ_{K^0}. The different ground states are then

$$\Sigma_0=\mathbb{1},\quad \Sigma_\pi=\exp\{i\alpha\lambda_2\},\quad \Sigma_{K^\pm}=\exp\{i\alpha\lambda_5\},\quad \Sigma_{K^0}=\exp\{i\alpha\lambda_7\}. \tag{17}$$

As we found for two flavors, this corresponds to a transformation of the vacuum to a new ground state by, $\Sigma_0\to A_\alpha\Sigma_0A_\alpha$. We must therefore transform the excitations around ground state in the same way. However, now the transformation depend on which phase we are in. We therefore parametrize the fields as

$$\Sigma=A_\alpha^i[U(x)\Sigma_0U(x)]A_\alpha^i,\quad U(x)=\exp\left\{i\frac{\pi_a\lambda_a}{2f}\right\},\quad A_\alpha^i=\exp\{i\alpha\lambda_i\}. \tag{18}$$

Here, there is no sum over i. Rather, $i=2$, 5, or 7, dependent on if we are in the pion condensate, the charged kaon condensate or neutral kaon condensate.

0.3.1 failed Leading order

We work in the pion condensate, with $e=0$. The relevant terms are then

$$\begin{aligned}\frac{f^2}{8B_0}\mathrm{Tr}\left\{\chi\Sigma^\dagger+\Sigma\chi^\dagger\right\} = &-\frac{1}{4}(m_u+m_d)\cos\alpha(\pi_1^2+\pi_2^2+\pi_3^2)-\frac{1}{4}\left[(m_u+m_s)\cos^2\frac{\alpha}{2}-m_d\sin^2\frac{\alpha}{2}\right](\pi_4^2+\pi_5^2)\\ &-\left[(m_d+m_s)\cos^2\frac{\alpha}{2}-m_u\sin^2\frac{\alpha}{2}\right](\pi_6^2+\pi_7^2)+\frac{1}{12}\left[(m_u+m_d+2m_s)\cos\alpha+2m_s\right]\pi_8^2\\ &-\frac{1}{2\sqrt{3}}(m_u-m_d)\pi_3\pi_8-\frac{1}{2}(m_u+m_d)\sin\alpha\pi_2+\frac{1}{2}(m_u+m_d)\cos\alpha+\frac{1}{4}m_s(\cos\alpha+1)\end{aligned} \tag{19}$$

0.4 Two-flavor electromagnetic effecets

When including contribution from a dynamical photon field, the leading order Lagrangian is [**eckerRoleResonancesChiral19** **urechVirtualPhotonsChiral1995**]

$$\mathcal{L}_2^{\mathrm{EM}} = \frac{1}{4} f^2 \mathrm{Tr}\left\{\nabla_\mu \Sigma \nabla^\mu \Sigma^\dagger\right\} + \frac{1}{4} f^2 \mathrm{Tr}\left\{\chi \Sigma^\dagger + \Sigma \chi^\dagger\right\} + e^2 C \mathrm{Tr}\left\{Q \Sigma Q \Sigma^\dagger\right\} \tag{20}$$

Q is the quark charge matrix, which for $N_f = 2$ is

$$Q = \frac{1}{3}\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{2}\mathbb{1} + \frac{1}{6}\tau_3. \tag{21}$$

C and dimensionfull constant, and $\chi = 2B_0 m$, where m is the quark mass matrix **??**. To find the electromagnetic effect on the pion mass, we assume $\mu_I = 0$. We use the parametrization $\Sigma = \exp\{i\pi_a \tau_a / f\}$, and the covariant derivative is in this case

$$\nabla_\mu \Sigma = \partial_\mu \Sigma - ie\mathcal{A}_\mu [Q, \Sigma]. \tag{22}$$

We expand to second order in π_a/f, which gives

$$\frac{1}{4} f^2 \mathrm{Tr}\left\{\nabla_\mu \Sigma \nabla^\mu \Sigma\right\} = \frac{1}{2}\partial_\mu \pi_a \partial^\mu \pi_a + e\mathcal{A}^\mu (\pi_1 \partial_\mu \pi_2 - \pi_2 \partial_\mu \pi_1) + e^2 \mathcal{A}^2 (\pi_1^2 + \pi_2^2), \tag{23}$$

$$\frac{1}{4} f^2 \mathrm{Tr}\left\{\chi \Sigma^\dagger + \Sigma \chi^\dagger\right\} = \bar{m}^2 \left(f^2 - \frac{1}{2}\pi_a \pi_a\right), \tag{24}$$

$$\mathrm{Tr}\left\{Q \Sigma Q \Sigma^\dagger\right\} = \frac{5}{9} - \frac{\pi_1^2 + \pi_2^2}{f^2}. \tag{25}$$

Inserting this into Eq. (20), we get

$$\mathcal{L}_2^{\mathrm{EM}} = \bar{m}^2 f^2 + \frac{5}{9} e^2 C + \frac{1}{2}\partial_\mu \pi_a \partial^\mu \pi_a - \frac{1}{2}\bar{m}_\pm^2 (\pi_1^2 + \pi_2^2) - \frac{1}{2}\bar{m}^2 \pi_3^2 + e\mathcal{A}^\mu (\pi_1 \partial_\mu \pi_2 - \pi_2 \partial_\mu \pi_1) + e^2 \mathcal{A}^2 (\pi_1^2 + \pi_2^2). \tag{26}$$

where

$$\bar{m}_\pm^2 = \bar{m}^2 + 2\frac{e^2}{f^2} C. \tag{27}$$

This is the leading order electromagnetic contribution to the mass. It only affects the π_1, π_2 pions, as they are linear combinations of the charged pions $\pi^\pm$, while $\pi_3 = \pi^0$, the neutral pion. To leading order, $\bar{m} = m_\pi$, the neutral pion mass, and $\bar{m}_\pm = m_{\pi^\pm}$ From the values listed in **??**, we find

$$\Delta m_\pm := \frac{e}{f}\sqrt{2C} = \sqrt{m_{\pi_\pm}^2 - m_\pi^2} = 35.50\,\mathrm{MeV}. \tag{28}$$

Compare/use Urec's results: C = 61.1 × 10 −6 (GeV)^4

This corresponds to $C = 0.3771\, u_0 = 5.824 \cdot 10^{-5}\,\mathrm{GeV}^4$. We now no longer assume $\mu_I = 0$. The zeroth-order expansion in π/f is

$$\Sigma = e^{i\alpha\tau_1} = \sin\alpha + i\tau_1 \cos\alpha. \tag{29}$$

This gives the contributions

$$\mathrm{Tr}\left\{\nabla_\mu \Sigma \nabla^\mu \Sigma^\dagger\right\} = 2\sin^2\alpha \left(\mu_I^2 + 2e\mu\mathcal{A}_0 + e^2\mathcal{A}^2\right), \tag{30}$$

$$\mathrm{Tr}\left\{\chi \Sigma^\dagger + \Sigma \chi^\dagger\right\} = 4\bar{m}^2 \cos\alpha, \tag{31}$$

$$\mathrm{Tr}\left\{Q \Sigma Q \Sigma^\dagger\right\} = \cos^2\alpha - \frac{4}{9}. \tag{32}$$

We are interested in the static Lagrangian, the Lagrangian for $\pi_a = \mathcal{A}_\mu = 0$. Inserting these terms into Eq. (20), we get

$$\mathcal{L}_2^{\mathrm{EM},0} = f^2 \left[\frac{1}{2}\mu_I^2 \sin^2\alpha + \bar{m}^2 \cos\alpha + \frac{1}{2}\Delta m_{\pi_\pm}^2 \left(\cos^2\alpha - \frac{4}{9}\right)\right]. \tag{33}$$

0.5 possible terms

These are our building blocks for constructing a general, G-invariant effective Lagrangian. The trace of a product of d_μ's are invariant under G,

$$\mathrm{Tr}\left\{d_\mu d_\nu \ldots d_\rho\right\} \to \mathrm{Tr}\left\{h d_\mu h^{-1} h d_\nu h^{-1} h \ldots d_\rho h^{-1}\right\} = \mathrm{Tr}\left\{d_\mu d_\nu \ldots d_\rho\right\}, \tag{34}$$

where we have used the cyclic property of trace. However, the terms must also obey the other symmetries of the Lagrangian, such as C or P-parity and Lorentz invariance. The last criterion excludes any terms with free space-time indices. In **??**, we will construct an effective Lagrangian in powers of d. The lowest order terms are therefore

$$\mathrm{Tr}\left\{d_\mu\right\} \mathrm{Tr}\left\{d^\mu\right\}, \quad \mathrm{Tr}\left\{d_\mu d^\mu\right\}. \tag{35}$$