Master

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January 13, 2022

Todo list

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Chapter 1

Introduction

Wirte introduction

Pion stars have recently been proposed [1, 2].

Chapter 2

General Relativity and the TOV-equation

General relativity describes how the fabric of space and time is bent by the presence of matter and energy. It was first written down by Einstein more than a hundred years ago, and is to this day the most accurate model we have for gravitational effects. The theory has made numerous accurate and counterintuitive predictions, which have been borne out by experiments. In this chapter, we will survey the basics of general relativity as well as some mathematical prerequisites. We will then use this to derive the Tolman-Oppenheimer-Volkoff (TOV) equation. This is a differential equation that models massive stellar objects, such as stars. This chapter is based on [3, 4].

2.1 Differential geometry

General relativity is formulated in the language of *differential geometry*, which generalizes multivariable calculus to more general spaces than $\mathbb{R}^n$. Such a space is a differentiable *manifold*, $\mathcal{M}$. A manifold is a set of points that are locally homeomoriphic to $\mathbb{R}^n$. That is, for all points $p \in \mathcal{M}$, there is a neighbourhood U around p and a corresponding set of continous functions,

$$x^\mu : U \subseteq \mathcal{M} \longmapsto V \subseteq \mathbb{R}^n, \tag{2.1}$$

$$p \longmapsto x^\mu(p). \tag{2.2}$$

that has a continous inverse-functions φ_x such that $\varphi_x(x(p)) = p$ for all $p \in U$. $x^\mu = (x^0, ..., x^{n-1})$ are a coordinate function of $\mathcal{M}$. In the case of a differentiable manifold, these must be diffeomorphisms, i.e. infinitly differentiable. Differentiablility of coordinate functions is defined by considering two different coordinate functions, x^μ and x'^μ, with the possibliy overlapping domains U and U'. We can then define a function between subsets of $\mathbb{R}^n$ by mapping via $\mathcal{M}$, the transition map

$$f_{x' \to x} = x^\mu \circ \varphi_{x'} : \mathbb{R}^n \mapsto \mathbb{R}^n. \tag{2.3}$$

It is defined as the function which makes the following diagram commute[1]

$$\begin{tikzcd} \mathbb{R}^n & \mathcal{M} \arrow[l, "x"'] \arrow[d, "x'"] \\ & \mathbb{R}^n \arrow[from=1-1, "f_{x' \to x}"'] \end{tikzcd} \tag{2.4}$$

The map is illustrated in Figure 2.1. A set of functions $\mathcal{A} = \{x^\mu\}$ whose domain cover $\mathcal{M}$ is called an *atlas* of $\mathcal{M}$. If the transition function between *any* two coordinate functions in the atlas is smooth, then we call the

[1] To be rigorous, one has to restrict the domains and image of the coordinate function when combining them. We will leave this implicit here.

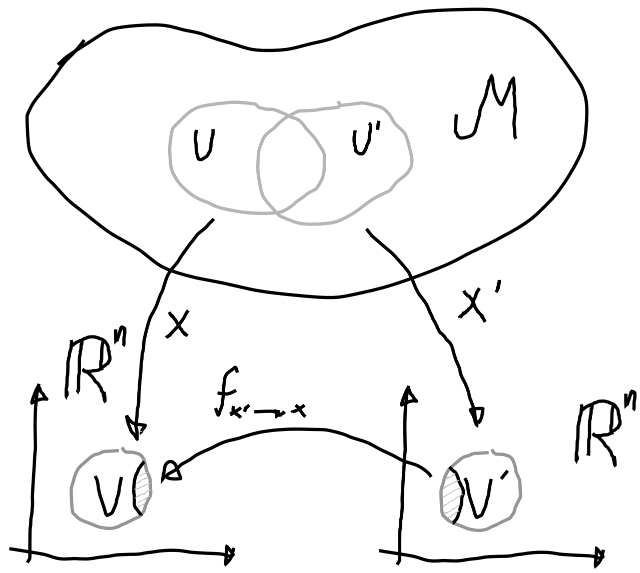

Figure 2.1: Kladd: The transition map between two coordinates

atlas smooth. To uniquily define a differentiable manifold, form smooth transition functions form an *atlas* $\mathcal{A}$. We then define a differentiable, or smooth, manifold as the topological manifold $\mathcal{M}$ together with the *maximal* atlas $\mathcal{A}$. A smooth atlas is maximal if no other coordinate function can be added to the atlas while it retains its smoothness.[2]

Consider two m and n dimensional smooth manifolds $\mathcal{M}$ and $\mathcal{N}$. Let x denote the coordinates on $\mathcal{M}$, while y denotes the coordinates on $\mathcal{N}$. We can define smooth functions between these manifolds in a similar way. Consider the function

$$F : \mathcal{M} \longmapsto \mathcal{N}. \tag{2.5}$$

This is said to be smooth, if for all points $p \in M$, there is a set of local coordinates x around p and y around $F(p)$ so that the map $\tilde{F} = y \circ F \circ x^{-1}$ is smooth. This map is defined by the diagram

$$\begin{tikzcd} \mathcal{M} \arrow[r, "F"] \arrow[d, "x"] & \mathcal{N} \arrow[d, "y"] \\ \mathbb{R}^m \arrow[r, "\tilde{F}"] & \mathbb{R}^n \end{tikzcd} \tag{2.6}$$

We will not be careful with the distinction between F, the funciton between the abstract manifolds, and $\tilde{F}$, the function of theri coordinates, but rather denote both by $F(x)$. We may take the partial derivative of such a function with respect to the coordinates x, $\partial F/\partial x^\mu$. However, this is obviously dependent on our choice of coordinates, as a set of local coordinates can always be scaled. To get a coordinate-independent quantity, we have to introduce the tangent space and the metric.

Vectors and tensors

A curve γ through $\mathcal{M}$ i a function from $\mathbb{R}$ to $\mathcal{M}$,

$$\gamma : \mathbb{R} \longmapsto \mathcal{M} \tag{2.7}$$
$$\lambda \longmapsto \gamma(\lambda). \tag{2.8}$$

Such curves are often denote only by their coordinates and the parameter λ, $x^\mu(\lambda) = (x^\mu \circ \gamma)(\lambda)$. Such a curve defines a directional derivative of a real valued function $f : \mathcal{M} \mapsto \mathbb{R}$. Assume $\gamma(\lambda = 0) = p$. As we

[2]The maximal condition is to ensure that two equivalent atlases correspond to the same differentiable manifold. A single manifold can be combined with different maximal atlases, also called differentiable structures.

are always taking the derivative of functions between $\mathbb{R}^n$, for different n, we can use the chaine rule. The directional derivative of f at p, given by this curve γ, is then

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} f(x(\lambda))\bigg|_p = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\bigg|_{\lambda=0} \frac{\partial}{\partial x^\mu} f(x)\bigg|_p . \tag{2.9}$$

The set of all such directional derivatives at p form a vector space, $T_p\mathcal{M}$, called the *tangent space*. The coordinates x^μ induce a basis of this vectorspace,

$$e_\mu = \frac{\partial}{\partial x^\mu} = \partial_\mu, \tag{2.10}$$

so any element $v \in T_p\mathcal{M}$ can be written

$$v = v^\mu \partial_\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\partial}{\partial x^\mu}. \tag{2.11}$$

Her, λ is the parameter of the curve corresponding to the directional derivative v.[3] It acts on funcitons $f : \mathcal{M} \mapsto \mathbb{R}$ as

$$v(f) = v^\mu \partial_\mu f. \tag{2.12}$$

A function F between two manifolds $\mathcal{M}$ and $\mathcal{N}$ also induces a map between the tangent spaces of these manifolds. This is the *differential* of F at p,

$$\mathrm{d}F_p : T_p\mathcal{M} \longmapsto T_p\mathcal{N}. \tag{2.13}$$

This is a directional derivative on $\mathcal{N}$, and is defined as

$$\mathrm{d}F_p(v)(g) = v(g \circ F), \tag{2.14}$$

for functions $g : \mathcal{N} \mapsto \mathbb{R}$. It thus acts on funcitons on $\mathcal{N}$ by "extending" the derivative v. This is a linear map between vectorspaces, an may be written on component form by considering the differentials of the coordinate functions. Denote the coordinates of $\mathcal{N}$ by y^μ, and $y^\mu \circ F = F^\mu$. Then,

$$\mathrm{d}F_p(\partial_\mu)(g) = \partial_\mu(g \circ F)|_p = \frac{\partial F^\nu}{\partial x^\mu}\bigg|_p \frac{\partial g}{\partial y^\nu}\bigg|_{F(p)} \tag{2.15}$$

The differential is thus a generalization of the Jacobian of a function. In the cas where of a real valued funciton, $f : \mathcal{M} \mapsto \mathbb{R}$, and $g : \mathbb{R} \mapsto \mathbb{R}$, we get

$$\mathrm{d}f_p(v)(g) = v(g \circ f) = (v^\mu \partial_\mu f)|_p \frac{\mathrm{d}g}{\mathrm{d}f}\bigg|_{f(p)}, \tag{2.16}$$

or more simply

$$\mathrm{d}f_p(v) = v^\mu \partial_\mu f|_p \tag{2.17}$$

The differential of a real-valued function is thus a linear map from the vector-space $T_p\mathcal{M}$ to the real numbers. The set of all such maps form the *dual space* of $T_p\mathcal{M}$, denoted $(T_p^*\mathcal{M})$. We can regard each of the coordinate functions as a real valued function,

$$\mathrm{d}x^\mu(\partial_\nu) = \frac{\partial x^\mu}{\partial x_\nu} = \delta^\mu_\nu. \tag{2.18}$$

These form a basis for $T_p^*\mathcal{M}$. We can show this by assuming $v = v^\mu \partial_\mu \in T_p\mathcal{M}$. Then, assuming $\mathrm{d}f_p = \omega_\mu \mathrm{d}x^\mu$, we get

$$\mathrm{d}f(v) = v^\mu \partial_\mu f = v^\mu \omega_\nu \mathrm{d}x^\nu(\partial_\mu) = v^\mu \omega_\mu. \tag{2.19}$$

Thus, we obtain a rigorous just justification for the classical expression

$$\mathrm{d}f = \frac{\partial f}{\partial x^\mu} \mathrm{d}x^\mu, \tag{2.20}$$

[3] There is not only one curve corresponding to any directional derivative, but rather an equivalence class.

however we now interpret it as a covector-field instead of an “infinitesimal displacement”.

This linear map from vectors to real numbers is gerneralized by *tensors*. Given a vector space V, general (n, m) tensor T is a multilinear map, which associates n elements from V and m from its dual V^* to the real numbers, i.e.

$$T : V \times V \times .... \times V^* \times .... \longmapsto \mathbb{R}, \tag{2.21}$$

$$(v, u...; \omega, ...) \longmapsto T(v, u, ...; \omega, ...). \tag{2.22}$$

Multilinear means that T is linear in each argument. The set of all such maps is the tnesor product space $V \otimes V \otimes ... \otimes V^* \otimes ...$, a $\dim(V)^{n+m}$-dimensional vector space. If $\{e_\mu\}$ and $\{e^\mu\}$ are the basis for V and V^*, then we can write the basis of this of the tensor product space as $\{e_\mu \otimes ...e^\mu \otimes ...\}$. The tensor can thus be written

$$T = T^{\mu\nu\cdots}{}_{\rho\ldots}\, e_\mu \otimes e_\nu \otimes \ldots e^\rho \otimes \ldots, \tag{2.23}$$

where

$$T^{\mu\nu\cdots}{}_{\rho\ldots} = T(e^\mu e^\nu, \ldots; e_\rho, \ldots). \tag{2.24}$$

Geometries and the metric

The metric is a symmetric, non-degenerate $(0, 2)$ tensor

$$\mathrm{d}s^2 = g_{\mu\nu}\, \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu. \tag{2.25}$$

It defines the geometry of the manifold $\mathcal{M}$, and is the main object of study in general relativity. As it is invertible, we define $g^{\mu\nu} = (g^{-1})_{\mu\nu}$, which is the components of a $(2, 0)$ tensor. We use this to raise and lower indecies, as is done with the Minkowski metric $\eta_{\mu\nu}$ in special relativity.

Up until now, we have studied the tangent space $T_p\mathcal{M}$ at one point, and the correpsonding dual and tensor product spaces. We are, however, more interested in feilds of vectors, covectors and tensors than. A tensor field T takes the a value $T(p)$ of the tensor product space corresponding to the tangentspace at $p \in \mathcal{M}$, $T_p\mathcal{M}$. We will use a vector felt to illustrate. This vector field can be written as

$$v(p) = v^\mu(p)\partial_\mu|_p. \tag{2.26}$$

We will mostly be working with the compontets v^μ, which are functions of $\mathcal{M}$. For ease of notation, we write the vector as a funciton of the coordinates x, and drop leave the evaluatation at p implicit. The vector field $v(x)$ is unchanged by a coordinate-transformation $x^\mu \to \tilde{x}^\mu$; the coordinate has no effect on it and is only for our convinience. With this, we can deduce the transformation rules of the components under such a transformation:

$$v = v^\mu \partial_\mu = v^\mu \frac{\partial \tilde{x}^\nu}{\partial x^\mu} \tilde{\partial}_\nu = \tilde{v}^\mu \tilde{\partial}_\mu, \tag{2.27}$$

or

$$\tilde{v}^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^\nu} v^\nu. \tag{2.28}$$

For covectors, it is

$$\tilde{\omega}_\mu = \frac{\partial x^\nu}{\partial \tilde{x}^\mu} \omega_\nu \tag{2.29}$$

The gradient of a scalar function f, $\mathrm{d}f = \partial_\mu f \mathrm{d}x^\mu$, is a coordinate-independent derivative, as $\partial_\mu f$ follows the transformation law for covectors. We generalize this by introducing the covariant derivative, ∇, as a map from (n, m) tensor fields to $(n, m + 1)$ tensor fields. When considering a scalar as a $(0, 0)$ tensor, we see that this generalizes the scalar derivative.

We assume

- linearity: $\nabla(T + S) = \nabla T + \nabla S$.
- product rule: $\nabla(T \otimes S) = (\nabla T) \otimes S + T \otimes (\nabla S)$.

- reduces to partial derivative: $\nabla_\mu f = \partial_\mu f$.
- Krönecker delta gives zero: $\nabla_\mu \delta^\rho_\nu = 0$.

With this, we can in general write the covariant derivative as [3]

$$\nabla_\mu v^\nu = \partial_\mu v^\mu + \Gamma^\mu_{\nu\rho} v^\rho, \tag{2.30}$$

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\rho_{\mu\nu} \omega^\rho, \tag{2.31}$$

for vectors and covectors. $\Gamma^\mu_{\nu\rho}$ are called *Christoffel symbols.* The generalization for higher-order tensors is straight forward,

$$\nabla_\mu T^{\nu\dots}{}_{\rho\dots} = \partial_\mu T^{\nu\dots}{}_{\rho\dots} + \Gamma^\mu_{\nu\lambda} T^{\lambda\dots}{}_{\rho\dots} + \dots - \Gamma^\lambda_{\mu\rho} T^{\mu\dots}{}_{\lambda\dots} - \dots \tag{2.32}$$

This is still not enough to to uniquely determin the Christoffel symbols. We will furhtermore assume $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$ and $\nabla_\mu g_{\nu\rho} = 0$. With these, we can find an explicit formula of the Christoffel symbols in terms of the metric,

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\nu\sigma} - \partial_\sigma g_{\mu\nu} + \partial_\nu g_{\sigma\mu}). \tag{2.33}$$

The curvature of the manifold $\mathcal{M}$, with the metic $g_{\mu\nu}$, is encoded in the Riemann tensor. It is defined by

$$[\nabla_\mu, \nabla_\nu] v^\rho = R^\rho{}_{\sigma\mu\nu} v^\sigma, \tag{2.34}$$

which in our case gives the explicit formula

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma} \tag{2.35}$$

Although the Christoffel symbols are not tensors, this quantity is a well-defined tensor due to its definition using covariant derivatives. We can therefore contract some of its indices to get other tensor quantities. We define the Ricci tensor and Ricci scalar as

$$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}, \tag{2.36}$$

$$R = R^\mu{}_\mu = g^{\mu\nu} R_{\mu\nu}. \tag{2.37}$$

These are the quantities we need to start working with general relativity.

Integration on manifolds

The integral of an scalar function on a manifold is not a coordinate independent notion, and we must instead introduce the notion of n-forms. A n-form is a antisymmetric $(0, n)$ tensor. To ease notation, we introduce the symmetrization of

$$T_{(\mu_1 \dots \mu_n)} = \frac{1}{n!} \sum_{\sigma \in S_n} T_{\mu_{\sigma(1)} \dots \mu_{\sigma(n)}}, \tag{2.38}$$

where S_n is the set of all permutations of n objects. The antisymmetrization of a tensor is defined as

$$T_{[\mu_1 \dots \mu_n]} = \frac{1}{n!} \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) T_{\mu_{\sigma(1)]} \dots \mu_{\sigma(n)}}. \tag{2.39}$$

The components of a n form then obey $T_{\mu_1 \dots \mu_n} = T_{(\mu_1 \dots \mu_n)}$. We are interested in the volume one-form, or measure, and therefore define

$$\mathrm{d}^n x = \mathrm{d}x^0 \wedge \dots \wedge \mathrm{d}x^{n-1}. \tag{2.40}$$

Here, $\wedge$ is the wedge product, defined as

$$(A \wedge B)_{\mu_1 \dots \mu_{n+m}} = \frac{(n+m)!}{n!m!} A_{[\mu_1 \dots \mu_n} B_{\mu_{n+1} \dots \mu_{n+m}]}, \tag{2.41}$$

and dx^i is the one-form corresponding to the x^0-coordinate function. Given a different set of coordinates, $\tilde{x}^\mu$, these are related by

$$d^n x = \det\left(\frac{\partial x}{\partial \tilde{x}}\right) d^n \tilde{x}, \tag{2.42}$$

by the properties of the wedge product. This quantity is a tensor-density, rather than a tensor, as it scales as $|\det(\partial x/\partial \tilde{x})|$. To make the measure coordinate independent, we must multiply with a scalar density to conpensate. By the transformation properties of tensors, $\sqrt{|g|} = \sqrt{|\det(g_{\mu\nu})|}$ will do just this. We therefore define the integral of a scalar function f on a manifold $\mathcal{M}$ as

$$I = \int_{\mathcal{M}} d^n x \sqrt{|g|} f(x). \tag{2.43}$$

Stoke's thorem generalizes the fundamental theorem of calculus, as well as the divergence theorem, to manifolds. Let $\mathcal{M}$ be a differential manifold of dimension n, with the boundary $\partial\mathcal{M}$. The boundary is then $n-1$ dimensional, and a metric g on $\mathcal{M}$ will induce a new metric γ on $\partial\mathcal{M}$, and there will be a vector field n^μ of normalized vectors orthogonal to all elements of $T\partial\mathcal{M}$. The generalized divergence theorem then states that, for a vector field V^μ on $\mathcal{M}$,

$$\int_{\mathcal{M}} d^n x \sqrt{|g|}\, \nabla_\mu V^\mu = \int_{\partial\mathcal{M}} d^{n-1} y \sqrt{|\gamma|}\, n_\mu V^\mu. \tag{2.44}$$

2.2 General relativity

General relativity describes how the metric, $g_{\mu\nu}$, behaves in the presence of matter and energy. The matter and energy contents are encoded in the stress-energy tensor $T_{\mu\nu}$, while the Lagrangian should then be a scalar function dependent on $g^{\mu\nu}$. The most obvious and correct, choice is to use the Ricci scalar, which results in the Einstein-Hilbert action,

$$S_{\mathrm{EH}} = \int_{\mathcal{M}} d^n x \sqrt{|g|}\, kR, \tag{2.45}$$

where k is a constant, related to Newton's constant of gravitation by

$$k = \frac{1}{16\pi G}. \tag{2.46}$$

The total action will include contributions from matter field, so that

$$S = S_{\mathrm{EH}} + S_{\mathrm{m}} \tag{2.47}$$

The equations of motion of the dynamical field, which is this case is the metric, is given by Hamiltons principle. Using functional derivatives, as defined in (REF:APPENDIX), this is stated as

Ha appendix på functional derivatives

$$\frac{\delta S}{\delta g^{\mu\nu}} = 0, \tag{2.48}$$

where we have used functional derivatives, We define

$$T_{\mu\nu} = -\frac{2}{\sqrt{|g|}} \frac{\delta S_{\mathrm{m}}}{\delta g^{\mu\nu}}. \tag{2.49}$$

Utlede?

This results in the equations of motion for the metric, the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu}, \tag{2.50}$$

where $\kappa = 8\pi G$. The left hand side of the Einstein field equations is called the Einstein tensor,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}. \tag{2.51}$$

This tensor obeys the identity

$$\nabla^\mu G_{\mu\nu} = 0, \tag{2.52}$$

as a consequence of the more general Bianchi identity.

Spherically symmetric spacetime

As we are going to model a star, we will assume that our metric is spherically symmetric and time independent. In this case, the most general metric can be written, at least locally, as

$$\mathrm{d}s^2 = e^{2\alpha(r)}\mathrm{d}t^2 - e^{2\beta(r)}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2), \tag{2.53}$$

where α and β are general functions of the radial coordinate r, and Ω is the solid angle. In matrix form, this corresponds to

$$g_{\mu\nu} = \begin{pmatrix} e^{2\alpha(r)} & 0 & 0 & 0 \\ 0 & -e^{2\beta(r)} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2(\theta) \end{pmatrix}. \tag{2.54}$$

Using Eq. (2.33), we can now compute the Christoffel symbols in terms of the unknown funcitons. These computations have been done using computer code, which is shown is (REF: KODE). The results are

$$\Gamma^t_{\mu\nu} = \begin{pmatrix} 0 & \frac{d}{dr}\alpha(r) & 0 & 0 \\ \frac{d}{dr}\alpha(r) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.55}$$

$$\Gamma^r_{\mu\nu} = \begin{pmatrix} e^{2\alpha(r)}e^{-2\beta(r)}\frac{d}{dr}\alpha(r) & 0 & 0 & 0 \\ 0 & \frac{d}{dr}\beta(r) & 0 & 0 \\ 0 & 0 & -re^{-2\beta(r)} & 0 \\ 0 & 0 & 0 & -re^{-2\beta(r)}\sin^2(\theta) \end{pmatrix}, \tag{2.56}$$

$$\Gamma^\theta_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\sin(\theta)\cos(\theta) \end{pmatrix}, \tag{2.57}$$

$$\Gamma^\phi_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \frac{\cos(\theta)}{\sin(\theta)} \\ 0 & \frac{1}{r} & \frac{\cos(\theta)}{\sin(\theta)} & 0 \end{pmatrix}. \tag{2.58}$$

With these, we can compute the Riemann tensor, using Eq. (2.35), and then take the trace, as given in Eq. (2.36), resulting in

$$R_{tt} = \left(r\left(\frac{d}{dr}\alpha(r)\right)^2 - r\frac{d}{dr}\alpha(r)\frac{d}{dr}\beta(r) + r\frac{d^2}{dr^2}\alpha(r) + 2\frac{d}{dr}\alpha(r)\right)\frac{e^{2\alpha(r)}e^{-2\beta(r)}}{r}, \tag{2.59}$$

$$R_{rr} = -\frac{1}{r}\left(r\left(\frac{d}{dr}\alpha(r)\right)^2 - r\frac{d}{dr}\alpha(r)\frac{d}{dr}\beta(r) + r\frac{d^2}{dr^2}\alpha(r) - 2\frac{d}{dr}\beta(r)\right), \tag{2.60}$$

$$R_{\theta\theta} = -\left(r\frac{d}{dr}\alpha(r) - r\frac{d}{dr}\beta(r) - e^{2\beta(r)} + 1\right)e^{-2\beta(r)}, \tag{2.61}$$

$$R_{\varphi\varphi} = R_{\theta\theta}\sin^2(\theta). \tag{2.62}$$

All other component are zero. Fianly, the trace of the Ricci tensor gives the Ricci scalar,

$$R = \frac{2e^{-2\beta(r)}}{r^2}\left[r^2\left(\frac{d}{dr}\alpha(r)\right)^2 - r^2\frac{d}{dr}\alpha(r)\frac{d}{dr}\beta(r) + r^2\frac{d^2}{dr^2}\alpha(r) + 2r\frac{d}{dr}\alpha(r) - 2r\frac{d}{dr}\beta(r) - e^{2\beta(r)} + 1\right] \tag{2.63}$$

The Einstein tensor is then

$$G_{tt} = \left(2r\frac{d}{dr}\beta(r) + e^{2\beta(r)} - 1\right)\frac{e^{2\alpha(r)}e^{-2\beta(r)}}{r^2}, \tag{2.64}$$

$$G_{rr} = \frac{2r\frac{d}{dr}\alpha(r) - e^{2\beta(r)} + 1}{r^2}, \tag{2.65}$$

$$G_{\theta\theta} = \left[r\left(\frac{d}{dr}\alpha(r)\right)^2 - r\frac{d}{dr}\alpha(r)\frac{d}{dr}\beta(r) + r\frac{d^2}{dr^2}\alpha(r) + \frac{d}{dr}\alpha(r) - \frac{d}{dr}\beta(r)\right] re^{-2\beta(r)}, \tag{2.66}$$

$$G_{\varphi\varphi} = G_{\theta\theta}\sin^2(\theta). \tag{2.67}$$

The rest of the components vanish. The unknown functions, α and β, are now determined by the matter and energy content of the universe, which is encoded in $T_{\mu\nu}$, as well as the boundary conidtions.

2.3 The TOV equation

We will model a star as being made up of a *perfect fulid*, with energy density ρ and pressure p. The stress-energy tensor of a perfect fluid is

Forklar

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - pg_{\mu\nu}, \tag{2.68}$$

where u_μ is the 4-velocity of the fluid. In the rest frame of the fluid, we may write

$$u_\mu = (u_0, 0, 0, 0)\,. \tag{2.69}$$

This, together with the normalization condition of 4-velocities, $u_\mu u^\mu = 1$, allows us to calcualte that

$$u_\mu u^\mu = g^{\mu\nu}u_\mu u_\nu = g^{00}u_0^2 = 1. \tag{2.70}$$

Using Eq. (2.54), we see that

$$u_0 = e^{\alpha(r)}. \tag{2.71}$$

This gives us the stress-energy tensor of the perfect fluid,

$$T_{\mu\nu} = \begin{pmatrix} \rho(r)e^{2\alpha(r)} & 0 & 0 & 0 \\ 0 & p(r)e^{2\beta(r)} & 0 & 0 \\ 0 & 0 & p(r)r^2 & 0 \\ 0 & 0 & 0 & p(r)r^2\sin^2(\theta) \end{pmatrix}. \tag{2.72}$$

. We will use the tt and rr components of the Einstein field equations, which are

$$8\pi Gr^2\rho(r)e^{2\beta(r)} = 2r\frac{d}{dr}\beta(r) + e^{2\beta(r)} - 1 \tag{2.73}$$

$$8\pi Gr^2 p(r)e^{2\beta(r)} = 2r\frac{d}{dr}\alpha(r) - e^{2\beta(r)} + 1. \tag{2.74}$$

In analogy with the form of the Schwartzhild metmric, we define the function $m(r)$ by

$$e^{2\beta(r)} = \left(1 - \frac{2Gm(r)}{r}\right)^{-1}. \tag{2.75}$$

Substituting this into Eq. (2.73) yields

$$\frac{\mathrm{d}m(r)}{\mathrm{d}r} = 4\pi r^2\,\rho(r). \tag{2.76}$$

The solution is simply

$$m(r) = 4\pi\int_0^r \mathrm{d}r'\,r'^2\rho(r'). \tag{2.77}$$

We see that $m(r)$ is the matter content contained within a radius r. If $\rho = 0$ for $r > R$ and $m(r > R) = M$, then the metric on a constant-time surface, i.e. $\mathrm{d}t = 0$, is

$$\mathrm{d}s^2 = \left(1 - \frac{2GM}{r^2}\right)^{-1} \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2), \tag{2.78}$$

which is the same as the Schwartzhild solution.

Using the Bianchi identity, Eq. (2.52), together with EInstein's equation, we find that

$$\nabla^\mu G_{\mu\nu} = \nabla^\mu T_{\mu\nu} = 0. \tag{2.79}$$

The r-component of this equation is

$$\nabla_\mu T^{\mu r} = \partial_r T^{rr} + \Gamma^\mu_{\mu\nu} T^{\nu r} + \Gamma^r_{\mu\nu} T^{\mu\nu}, \tag{2.80}$$

where we have used the particulare form of $T_{\mu\nu}$ and the Christoffel symbols to eliminate vanishing terms. We calculate

$$\begin{aligned} \nabla_\mu T^{\mu r} &= \partial_r \left(p e^{-2\beta}\right) + (2\Gamma^r_{rr} + \Gamma^t_{tr}) T^{rr} + \Gamma^r_{tt} T^{tt} \\ &= e^{-2\beta} \left(\partial_r p + p \partial_r \alpha + \rho \partial_r \alpha\right) = 0. \end{aligned}$$

This allows us to relate α to p and ρ, via

$$\partial_r \alpha = -\frac{\partial_r p}{p + \rho} \tag{2.81}$$

When we substitute this, together with the definition of $m(r)$, into Eq. (2.74), we obtain

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{G\left(4\pi r^3 p + m\right)(\rho + p)}{r\,(r - 2Gm)}, \tag{2.82}$$

the Tolman-Oppenheimer-Volkoff (TOV) equation.

Appendix A

Code

All code is available at: (LINK)

Link til repo

The code for calculations done in (GR-kapittel) is included below. The code is written in Python in a Jupyter notebook. The `.ipynb` file is available in the repo linked below.

Link til kap-pittel

[17]:
```python
from sympy import MatrixSymbol, Matrix, Array, pprint
from sympy import symbols, diff, exp, log, cos, sin, simplify, Rational
from sympy.core.symbol import Symbol
from sympy import pi

import numpy as np
import sympy as sp
from IPython.display import display, Latex
```

Tensor operations

[18]:
```python
def INDX(i, place, num_indx):
    """
    Acceses an index at 'place' for 'num_indx' order tensor
    T_(a0 ... âp ... an-1) = T[INDX(i, place=p, num_indx=n)] = T[:,...<-p-> ,␣
↪i, :,...<-(n-p-1)->]
    """
    indx = []
    assert place<num_indx
    for j in range(num_indx):
        if place==j: indx.append(i)
        else: indx.append(slice(None))
    return tuple(indx)
```

[19]:
```python
def contract(T, g=None, g_inv=None, num_indx=2, upper=1, indx=(0, 1)):
    """
    contracts indecies indx=(a_p, a_q) on tensor T with 'num_indx',
    'upper' of whom are upper. If upper=0, all indecies are assumed lower.
    With indx=(a_k, a_l), upper=n, num_indx=n+m, this gives
    T^(a_0...a_n-1)_(a_n...a_n+m-1) -> T^(a_0...a_k=a...a_n-1)_(a_n...a_k...
↪a_n+m-1),
    with the necesarry metric. If wrong metric is given, this wil throw error.
    """
    assert indx[0] < indx[1]  # we have to know if the index to the left␣
↪dissapears
    dim = np.shape(T)[0]
    a = (indx[0] < upper) + (indx[1] < upper) # number of upper indecies to be␣
↪contracted
    if a==2: g0 = g # two upper
    elif a==0: g0 = g_inv # two lower
    else: g0 = np.identity(dim, dtype=Rational)

    Tc = Rational(0) * np.ones((T.shape)[:-2], dtype=Rational)
    for i in range(dim):
        for j in range(dim):
            Tc += g0[i, j] * (T[INDX(i, indx[0], num_indx)])[INDX(j, indx[1] -␣
↪1, num_indx - 1)]
```

```python
    return Tc

def raise_indx(T, g_inv, indx, num_indx):
    """
    Raise index 'indx' of a tensor T with 'num_indx' indices.
    """
    dim = np.shape(T)[0]
    Tu = np.zeros_like(T)
    for i in range(dim):
        I = INDX(i, indx, num_indx)
        for j in range(dim):
            J = INDX(j, indx, num_indx)
            Tu[I] += g_inv[i, j] * T[J]
    return Tu

def lower_indx(T, g, indx, num_indx):
    return raise_indx(T, g, indx, num_indx)

def get_g_inv(g):
    return np.array(Matrix(g)**(-1))
```

Calculate Christoffel symbols and Riemann curvature tensor

[20]:
```python
def Christoffel(g, g_inv, var):
    """
    Work out the christoffel symbols, given a metric an its variables
    Γ^i_jk = C[i, j, k]
    """
    dim = len(var)
    C = np.zeros((dim, dim, dim), dtype=Symbol)
    for i in range(dim):
        for j in range(dim):
            for k in range(dim):
                for m in range(dim):
                    C[i, j, k] += Rational(1, 2) * (g_inv)[i, m] * (
                        diff(g[m, k], var[j])
                        + diff(g[m, j], var[k])
                        - diff(g[k, j], var[m])
                    )

    return C
```

[21]:
```python
def Riemann_tensor(C, var):
    """
    Riemann_tensor(Christoffel_symbols, (x_1, ...)) = R[i, j, k, l] = R^i_jkl
    Compute the Riemann tensor from the Christoffel symbols
```

```
    """
    dim = len(var)
    R = np.zeros([dim] * 4, dtype=Symbol)
    indx = [(i, j, k, l)
        for i in range(dim)
        for j in range(dim)
        for k in range(dim)
        for l in range(dim)
    ]

    for (a, b, r, s) in indx:
        R[a, b, r, s] += diff(C[a, b, s], var[r]) - diff(C[a, b, r], var[s])
        for k in range(dim):
            R[a, b, r, s] += C[a, k, r] * C[k, b, s] - C[a, k, s] * C[k, b, r]
    return R
```

Printing functions

[22]:
```
print_latex = False

def print_christoffel(C, var):
    """ A function for dsiplaying christoffels symbols """
    output = []
    for i in range(len(var)):
        txt = "$$"
        txt += "\\Gamma^" + sp.latex(var[i]) + "_{\\mu \\nu} ="
        txt += sp.latex(Matrix(C[i]))
        txt += "$$"
        print(txt) if print_latex else print()
        output.append(display(Latex(txt)))

    return output

def print_matrix(T):
    txt = "$$" + sp.latex(Matrix(T)) +"$$"
    print(txt) if print_latex else print()
    return display(Latex(txt))

def print_scalar(T):
    txt = "$$" + sp.latex(T) +"$$"
    print(txt) if print_latex else print()
    return display(Latex(txt))

def print_eq(eq):
    txt = "$$" + sp.latex(eq) +"=0" + "$$"
    print(txt) if print_latex else print()
    return display(Latex(txt))
```

Metric $g_{\mu\nu}$ for spherically symmetric spacetime

[23]:
```
t, r, th, ph = symbols("t, r, \\theta, \\phi")
x1 = r * cos(ph) * sin(th)
x2 = r * sin(ph) * sin(th)
x3 = r * cos(th)

one = Rational(1)
eta = sp.diag(one, -one, -one, -one)
var = (t, r, th, ph)
J = Matrix([t, x1, x2, x3]).jacobian(var)
g = np.array(simplify(J.T *eta* J))

a = sp.Function("\\alpha", real=True)(r)
b = sp.Function("\\beta", real=True)(r)
g[0, 0] *= exp(2 * a)
g[1, 1] *= exp(2 * b)
g_inv = get_g_inv(g)

print_matrix(g)
print_matrix(g_inv)
```

$$\begin{bmatrix} e^{2\alpha(r)} & 0 & 0 & 0 \\ 0 & -e^{2\beta(r)} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2(\theta) \end{bmatrix}$$

$$\begin{bmatrix} e^{-2\alpha(r)} & 0 & 0 & 0 \\ 0 & -e^{-2\beta(r)} & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{r^2\sin^2(\theta)} \end{bmatrix}$$

[24]:
```
C = Christoffel(g, g_inv, var)
c = print_christoffel(C, var)
```

$$\Gamma^{t}_{\mu\nu} = \begin{bmatrix} 0 & \frac{d}{dr}\alpha(r) & 0 & 0 \\ \frac{d}{dr}\alpha(r) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Gamma^{r}_{\mu\nu} = \begin{bmatrix} e^{2\alpha(r)}e^{-2\beta(r)}\frac{d}{dr}\alpha(r) & 0 & 0 & 0 \\ 0 & \frac{d}{dr}\beta(r) & 0 & 0 \\ 0 & 0 & -re^{-2\beta(r)} & 0 \\ 0 & 0 & 0 & -re^{-2\beta(r)}\sin^{2}(\theta) \end{bmatrix}$$

$$\Gamma^{\theta}_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\sin(\theta)\cos(\theta) \end{bmatrix}$$

$$\Gamma^{\phi}_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \frac{\cos(\theta)}{\sin(\theta)} \\ 0 & \frac{1}{r} & \frac{\cos(\theta)}{\sin(\theta)} & 0 \end{bmatrix}$$

[25]:

```
Rie = Riemann_tensor(C, var)
Ricci = contract(Rie, num_indx=4, upper=1, indx=(0, 2))

for i in range(4):
    print_scalar(Ricci[i, i].factor())
```

$$\frac{\left(r\left(\frac{d}{dr}\alpha(r)\right)^{2} - r\frac{d}{dr}\alpha(r)\frac{d}{dr}\beta(r) + r\frac{d^{2}}{dr^{2}}\alpha(r) + 2\frac{d}{dr}\alpha(r)\right)e^{2\alpha(r)}e^{-2\beta(r)}}{r}$$

$$-\frac{r\left(\frac{d}{dr}\alpha(r)\right)^{2} - r\frac{d}{dr}\alpha(r)\frac{d}{dr}\beta(r) + r\frac{d^{2}}{dr^{2}}\alpha(r) - 2\frac{d}{dr}\beta(r)}{r}$$

$$-\left(r\frac{d}{dr}\alpha(r) - r\frac{d}{dr}\beta(r) - e^{2\beta(r)} + 1\right)e^{-2\beta(r)}$$

$$-\left(r\frac{d}{dr}\alpha(r) - r\frac{d}{dr}\beta(r) - e^{2\beta(r)} + 1\right)e^{-2\beta(r)}\sin^{2}(\theta)$$

[26]:
```
R = contract(Ricci, g_inv=g_inv, upper=0).simplify()
print_scalar(R)
```

$$\frac{2\left(r^{2}\left(\frac{d}{dr}\alpha(r)\right)^{2}-r^{2}\frac{d}{dr}\alpha(r)\frac{d}{dr}\beta(r)+r^{2}\frac{d^{2}}{dr^{2}}\alpha(r)+2r\frac{d}{dr}\alpha(r)-2r\frac{d}{dr}\beta(r)-e^{2\beta(r)}+1\right)e^{-2\beta(r)}}{r^{2}}$$

[27]:
```
G = Ricci - Rational(1, 2) * R * g
for i in range(4):
    G[i, i] = G[i, i].simplify().factor()
    print_scalar(G[i, i])
```

$$\frac{\left(2r\frac{d}{dr}\beta(r)+e^{2\beta(r)}-1\right)e^{2\alpha(r)}e^{-2\beta(r)}}{r^{2}}$$

$$\frac{2r\frac{d}{dr}\alpha(r)-e^{2\beta(r)}+1}{r^{2}}$$

$$r\left(r\left(\frac{d}{dr}\alpha(r)\right)^{2}-r\frac{d}{dr}\alpha(r)\frac{d}{dr}\beta(r)+r\frac{d^{2}}{dr^{2}}\alpha(r)+\frac{d}{dr}\alpha(r)-\frac{d}{dr}\beta(r)\right)e^{-2\beta(r)}$$

$$r\left(r\left(\frac{d}{dr}\alpha(r)\right)^{2}-r\frac{d}{dr}\alpha(r)\frac{d}{dr}\beta(r)+r\frac{d^{2}}{dr^{2}}\alpha(r)+\frac{d}{dr}\alpha(r)-\frac{d}{dr}\beta(r)\right)e^{-2\beta(r)}\sin^{2}(\theta)$$

0.0.1 Stress-energy tensor $T_{\mu\nu}$ for perfect fluid

[28]:
```
p = sp.Function("p")(r)
rho = sp.Function("\\rho")(r)

UU = np.zeros((4, 4), dtype=sp.Rational)
UU[0, 0] = exp(2 * a)

T = (p + rho) * UU - p * g
for i in range(4):
    T[i, i] = T[i, i].simplify()
print_matrix(T)
```

$$\begin{bmatrix} \rho(r)e^{2\alpha(r)} & 0 & 0 & 0 \\ 0 & p(r)e^{2\beta(r)} & 0 & 0 \\ 0 & 0 & r^2p(r) & 0 \\ 0 & 0 & 0 & r^2p(r)\sin^2(\theta) \end{bmatrix}$$

0.0.2 Einstin equation

$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi GT_{\mu\nu}$

[29]:
```
G_newton = sp.Symbol("G")

eq = []
for i in range(len(G)):
    eq.append((G[i, i] - 8 * pi * G_newton * T[i, i]).simplify())

# Some manual simplification
Rtt = sp.Symbol("R_{\\theta \\theta}")
eq[0] = eq[0] * r**2 / exp(2 * a)/exp(-2*b ) * (-1 )
eq[1] = eq[1] * r**2 * (-1)
eq[2] = eq[2] / r / exp(-2*b)
eq[3] = eq[3].subs(eq[2], Rtt)
for i in range(len(G)):
    print_eq(eq[i].simplify())
```

$$8\pi Gr^2\rho(r)e^{2\beta(r)} - 2r\frac{d}{dr}\beta(r) - e^{2\beta(r)} + 1 = 0$$

$$8\pi Gr^2p(r)e^{2\beta(r)} - 2r\frac{d}{dr}\alpha(r) + e^{2\beta(r)} - 1 = 0$$

$$-8\pi Grp(r)e^{2\beta(r)} + r\left(\frac{d}{dr}\alpha(r)\right)^2 - r\frac{d}{dr}\alpha(r)\frac{d}{dr}\beta(r) + r\frac{d^2}{dr^2}\alpha(r) + \frac{d}{dr}\alpha(r) - \frac{d}{dr}\beta(r) = 0$$

$$R_{\theta\theta}re^{-2\beta(r)}\sin^2(\theta) = 0$$

Define $e^{2\beta} = [1 - 2Gm(r)/r]^{-1}$

[30]:
```
m = sp.Function("m", Real=True)(r)
f = (1 - 2 * G_newton * m / r)**(-1)
eq1 = (eq[0] * exp(- 2 *a)).simplify().subs(b, Rational(1, 2) * log(f)).
 ↪simplify().expand().simplify()
s = sp.solve(eq1, m.diff(r))
eq1 = m.diff(r) - s[0]
```

Use $\nabla_\mu T^{\mu r} = 0 \implies (p+\rho)\partial_r \alpha = -\partial_r p$.

[31]:
```
eq2 = (eq[1] * r**2).subs(exp(2 * b), f).simplify()
s = sp.solve(eq2, a.diff(r))
eq2 = a.diff(r) - s[0]
eq2 = ((a.diff(r) - s[0]).subs(a.diff(r), - p.diff(r) / (p + rho))*(p + rho)).
 ↪simplify()
s = sp.solve(eq2, p.diff())
eq2 = p.diff(r) - s[0].factor()
```

The TOV-equation and equation for $m(r)$, both expressions are equal to 0.

[32]:
```
print_eq(eq1)
print_eq(eq2)
```

$$-4\pi r^2 \rho(r) + \frac{d}{dr} m(r) = 0$$

$$\frac{G\left(4\pi r^3 p(r) + m(r)\right)\left(\rho(r) + p(r)\right)}{r\left(-2Gm(r) + r\right)} + \frac{d}{dr} p(r) = 0$$

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