Master

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Todo list

Contents

Chapter 1

Introduction

1.1 Units

In this thesis, we employ *natural units*, defined by

$$\hbar = c = k_B = 1, \tag{1.1}$$

where $\hbar$ is Planck's reduced constant, c is the speed of light, and k_B is Boltzmann's constant. Dimensionfull results are often given in MeV. Uncertainties are indicated when the preciscion is less than four significant figures, however, the central value is always used in calcualtions. Uncertainties are indicated in parantethesis, and the addition and subtraction of this value to the least significant result denote the confidence interval of one standard deviation. That is, for a result 123.456(7), the range 123.456 ± 0.007 cover a confidence interval of 68.3% [1]. All values in this section are from the Particle Data Group [1]. To obtain results in the SI-system, we use the following conversion factors, as given by

$$c = 2.998 \cdot 10^{8}\,\mathrm{m\,s^{-1}}, \tag{1.2}$$

$$\hbar = 1.055 \cdot 10^{-34}\,\mathrm{J\,s}, \tag{1.3}$$

$$k_B = 1.380 \cdot 10^{-23}\,\mathrm{J\,K^{-1}}, \tag{1.4}$$

$$G = 6.674 \cdot 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}, \tag{1.5}$$

where G is Newton's gravitational constant. The conversion factor between MeV and SI-units is

$$1\,\mathrm{MeV} = 1.60218\ \cdot 10^{-19}\,\mathrm{J}. \tag{1.6}$$

The fine structure constant and the elementary charge is

$$\alpha = 7.297 \cdot 10^{-3}, \tag{1.7}$$

$$e := \sqrt{4\pi\alpha} = 3.028 \cdot 10^{-1}. \tag{1.8}$$

In the calculation in section 4.3, the value for the neutron mass is

$$m_N = 939.57\,\mathrm{MeV} = 1.674 \cdot 10^{-27}\,\mathrm{kg}. \tag{1.9}$$

In astronomical calculation, the solar mass is used, which is

$$M_\odot = 1.988 \cdot 10^{30}\,\mathrm{kg}. \tag{1.10}$$

When working with chiral perturbation theory, we use

$$f_\pi = \frac{1}{\sqrt{2}} 130.2(8)\,\mathrm{MeV} = 92.1(6)\,\mathrm{MeV}, \tag{1.11}$$

$$m_\pi = 134.98\,\mathrm{MeV} = 2.406 \cdot 10^{-28}\,\mathrm{kg}, \tag{1.12}$$

$$m_{\pi_\pm} = 139.57\,\mathrm{MeV} = 2.488 \cdot 10^{-28}\,\mathrm{kg}, \tag{1.13}$$

where f_π is the pion decay constant, and m_π the mass of the neutral pion, π^0.

1.2 Structure of thesis

To make this thesis as self-contained as possible, we have included some parts from the earlier specialization project, with minor modifications. These sections are marked with an asterisk in the table of content.

Chapter 2

Mathematics

General relativity, and a lot of quantum field theory, is formulated in the language of *differential geometry*. Differential geometry generalizes n-dimensional calculus to more general spaces than the usual $\mathbb{R}^n$, such as curved spacetime or the more abstract space of symmetries of a quantum field theory. This chapter will survey differential geometry and Lie groups and familiarize us with the tools needed for the physics we use in this thesis.

2.1 Differential geometry

This section is based on [2, 3].

2.1.1 Manifolds and coordinates

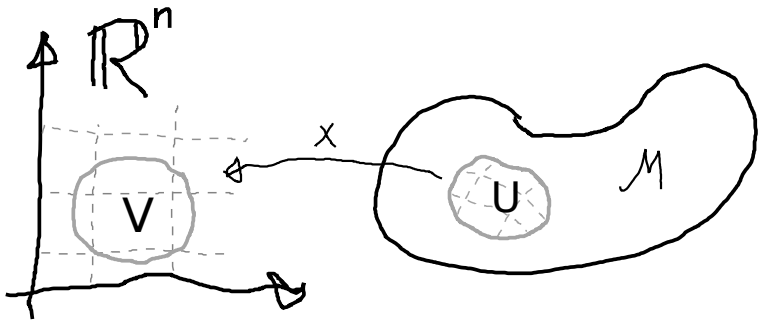


Figure 2.1: (Kladd) The coordinate function x maps a neighborhood U in the manifold $\mathcal{M}$ to a negiborhood V in $\mathbb{R}^n$.

Differential geometry is the study of *smooth manifolds*. An n-dimensional manifold, $\mathcal{M}$, is a set of points, locally homeomorphic to $\mathbb{R}^n$. That is, for all points $p \in \mathcal{M}$, there exists a neighborhood U around p, together with a corresponding set of continuous, bijective functions,

$$x : U \subseteq \mathcal{M} \longmapsto V \subseteq \mathbb{R}^n, \tag{2.1}$$

$$p \longmapsto x^\mu(p). \tag{2.2}$$

We call $x(p) = (x^0(p), \ldots, x^{n-1}(p)) = x^\mu(p)$ a coordinate function of $\mathcal{M}$. The inverse of x, x^{-1}, obeys $x^{-1}(x(p)) = p$, for all $p \in U$. A smooth manifold is one in which the coordinate functions are infinitely differentiable. To define differentiability on manifolds, consider two coordinate functions, x, and x'. The

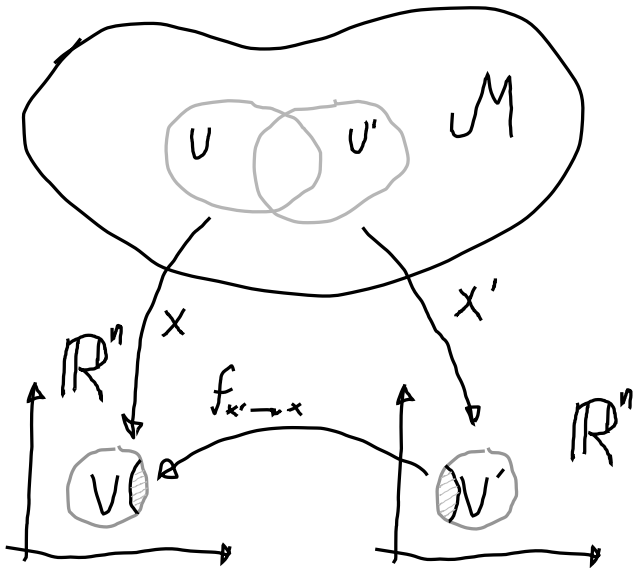


Figure 2.2: (Kladd) The transition map $f_{x'\to x}$ between two coordinate functions, x and x', maps between the images of these function, via the manifold $\mathcal{M}$. The function's domain and image are restricted to a (possibly empty) subset of the images of x and x'. This is illustrated by the shaded regions in V and V'.

corresponding domains U and U' may or may not overlap. We then define the transition function, a function between subsets of $\mathbb{R}^n$ by mapping via $\mathcal{M}$, as

$$f_{x\to x'} = x' \circ x^{-1} : \mathbb{R}^n \mapsto \mathbb{R}^n. \tag{2.3}$$

The map is illustrated in Figure 2.2. [1] A set of coordinate functions $\mathcal{A} = \{x_i\}$ whose domain cover $\mathcal{M}$ is called an *atlas* of $\mathcal{M}$. If the transition function between all pairings of coordinate functions in the atlas is smooth—that is, infinitely differentiable—we call the atlas smooth. We then define a smooth manifold as the topological manifold $\mathcal{M}$ together with a *maximal* smooth atlas $\mathcal{A}$. A smooth atlas is maximal if no coordinate function can be added while the atlas remains smooth.[2]

Consider two m- and n-dimensional smooth manifolds $\mathcal{M}$ and $\mathcal{N}$. Let x denoted the coordinates on $\mathcal{M}$, while y denotes the coordinates on $\mathcal{N}$. We can define smooth functions between these manifolds similarly to how we define smooth coordinates. Consider the function

$$F : \mathcal{M} \longmapsto \mathcal{N}. \tag{2.4}$$

It is said to be smooth if, for all points $p \in M$, there is a set of local coordinates x around p and y around $F(p)$ such that the map $\tilde{F} = y \circ F \circ x^{-1}$ is smooth. This map may be illustrated by a diagram,

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{F} & \mathcal{N} \\ \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle y} \\ \mathbb{R}^m & \xrightarrow{\tilde{F}} & \mathbb{R}^n \end{array} \tag{2.5}$$

We will not be careful with the distinction between F, the function between the abstract manifolds, and $\tilde{F}$, the function of their coordinates, but rather denote both by $F(x)$. We may take the partial derivative of such a function with respect to the coordinates x, $\partial F/\partial x^\mu$. However, this is dependent on our choice of coordinates, as a set of local coordinates can always be scaled arbitrarily. Any physical theory must be independent of our choice of coordinates, so our next task is to define the properties of a smooth manifold in a coordinate independent way.

[1] To be rigorous, one has to restrict the domains and image of the coordinate function when combining them. This is illustrated in Figure 2.2.

[2] The maximal condition ensres that two equivalent atlases correspond to the same differentiable manifold. A single manifold can be combined with different maximal atlases of smooth coordinates or differentiable structures. A set of examples are *exotic spheres*, smooth manifolds which are *homeomorphic* to S^n, but not *diffeomorphic*.

2.1.2 Vectors and tensors

A curve γ through $\mathcal{M}$ is a function from $\mathbb{R}$ to $\mathcal{M}$,

$$\gamma : \mathbb{R} \longmapsto \mathcal{M} \tag{2.6}$$
$$\lambda \longmapsto \gamma(\lambda). \tag{2.7}$$

Such curves are often denoted only by their coordinates and the parameter λ, $x^\mu(\lambda) = (x^\mu \circ \gamma)(\lambda)$. With this curve, we can take the directional derivative of a real-valued function on the manifold, $f : \mathcal{M} \mapsto \mathbb{R}$. Assume $\gamma(\lambda = 0) = p$. As we are always taking the derivative of functions between $\mathbb{R}^n$, for different n, we can use the chain rule. The directional derivative of f at p, given by this curve γ, is then

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} f(x(\lambda))\Big|_p = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\Big|_{\lambda=0} \frac{\partial}{\partial x^\mu} f(x)\Big|_p . \tag{2.8}$$

The set of all such directional derivatives, $\mathrm{d}/\mathrm{d}\lambda$ at p, form a vector space, $T_p\mathcal{M}$, called the *tangent space*. The tangent space is illustrated in Figure 2.3. The coordinates x^μ induce a basis of this vector space, namely partial derivatives with respect to the coordinate functions at p

$$e_\mu = \frac{\partial}{\partial x^\mu}\Big|_p = \partial_\mu|_p, \quad \mu \in \{0, ...n-1\}. \tag{2.9}$$

Any element $v \in T_p\mathcal{M}$ can therefore be written

$$v = v^\mu \partial_\mu|_p = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\Big|_{\lambda=0} \frac{\partial}{\partial x^\mu}\Big|_p . \tag{2.10}$$

Here, λ is the parameter of the curve corresponding to the directional derivative v.[3] The evaluation at $\lambda = 0$ and p will often be implicit for ease of notation. This directional derivative acts on functions $f : \mathcal{M} \mapsto \mathbb{R}$ as

$$v(f) = v^\mu \partial_\mu f. \tag{2.11}$$

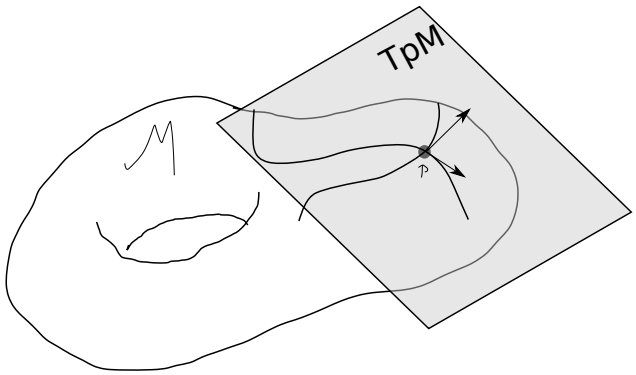


Figure 2.3: (Kladd) The tangent space $T_p\mathcal{M}$, the shaded rectangle, is the sett of all directional derivatives at $p \in \mathcal{M}$. A directional derivative is defined in terms of a curve that passes through p.

A map F between two manifolds $\mathcal{M}$ and $\mathcal{N}$ also induces a map between the tangent spaces of these manifolds. This is the *differential* of F at p,

$$\mathrm{d}F_p : T_p\mathcal{M} \longmapsto T_p\mathcal{N}, \tag{2.12}$$
$$v \longmapsto \mathrm{d}F_p(v). \tag{2.13}$$

As $\mathrm{d}F_p(v)$ is an element of $T_p\mathcal{N}$, directional derivative on $\mathcal{N}$, defined as

$$\mathrm{d}F_p(v)(g) = v(g \circ F), \tag{2.14}$$

[3] There is not only one curve corresponding to any directional derivative but rather an equivalence class. We will gloss over this technicality, as it does not affect our work.

for functions $g : \mathcal{N} \mapsto \mathbb{R}$. It thus acts on functions on $\mathcal{N}$ by "extending" the derivative v. This is a linear map between vector spaces and may be written in component form by considering the differentials of the coordinate functions. Denote the coordinates of $\mathcal{N}$ by y^μ, and $y^\mu \circ F = F^\mu$. Then,

$$\mathrm{d}F_p(\partial_\mu)(g) = \partial_\mu(g \circ F)|_p = \frac{\partial F^\nu}{\partial x^\mu}\Big|_p \frac{\partial g}{\partial y^\nu}\Big|_{F(p)}, \tag{2.15}$$

or more suggestively

$$\mathrm{d}F\left(\frac{\partial}{\partial x^\mu}\right) = \frac{\partial F^\nu}{\partial x^\mu}\frac{\partial}{\partial y_\nu}. \tag{2.16}$$

This is a linear map of vectors between two vectors by the matrix $A_\mu{}^\nu = \partial_\mu F^\nu$. The differential is thus a generalization of the Jacobian. In the case of a real valued function, $f : \mathcal{M} \mapsto \mathbb{R}$, and $g : \mathbb{R} \mapsto \mathbb{R}$, we get

$$\mathrm{d}f(v)(g) = v(g \circ f) = (v^\mu \partial_\mu f)\,\frac{\mathrm{d}g}{\mathrm{d}y}. \tag{2.17}$$

$\mathrm{d}f$ is thus a map from $T_p\mathcal{M}$ to $T_{f(p)}\mathbb{R}$, which is isomorphic to $\mathbb{R}$. The g be the identity function, so that $\mathrm{d}g/\mathrm{d}y = 1$. Then, the differential of a scalar function, also called a 1-form, is a map from vectors v to real numbers,

$$\mathrm{d}f(v) := v^\mu \partial_\mu f. \tag{2.18}$$

The set of all linear maps from a vector space V to the real numbers is called the *dual space* of V, denoted V^*. This is a new vector space with the same dimensionality as V. We denote the dual of $T_p\mathcal{M}$ as $T_p^*\mathcal{M}$. We can regard each coordinate function as a real-valued function with a corresponding differential. This differential obeys

$$\mathrm{d}x^\mu(\partial_\nu) = \frac{\partial x^\mu}{\partial x_\nu} = \delta^\mu_\nu. \tag{2.19}$$

The differentials of the coordinate functions thus form a basis for $T_p^*\mathcal{M}$, called the dual basis. Any differential $\mathrm{d}f$ can thus be written as $\mathrm{d}f = \omega_\mu \mathrm{d}x^\nu$ for some components ω_μ. We finde the components by applying the differential to the coordinate basis, $\mathrm{d}f(\partial_\mu) = \partial_\mu f = \omega_\mu$. In other words, we recover the classical expression

$$\mathrm{d}f = \frac{\partial f}{\partial x^\mu}\mathrm{d}x^\mu, \tag{2.20}$$

however we now interpret it as a covector-field instead of an "infinitesimal displacement".

Linear maps from vectors to real numbers is generalized by *tensors*. Given a vector space V, a general (n, m) tensor T is a multilinear map, which associates n elements from V and m from its dual V^* to the real numbers, i.e.,

$$T : V \times V \times \cdots \times V^* \times \ldots \longmapsto \mathbb{R}, \tag{2.21}$$

$$(v, u \ldots; \omega, \ldots) \longmapsto T(v, u, \ldots; \omega, \ldots). \tag{2.22}$$

Multilinear means that T is linear in each argument. The set of all such maps is the tensor product space $V \otimes V \otimes \cdots \otimes V^* \otimes \ldots$, a $\dim(V)^{n+m}$-dimensional vector space. If $\{e_\mu\}$ and $\{e^\mu\}$ are the basis for V and V^*, then we can write the basis of this of the tensor product space as $\{e_\mu \otimes \cdots \otimes e^\nu \otimes \ldots\}$. The tensor can thus be written

$$T = T^{\mu\nu\cdots}{}_{\rho\ldots}\, e_\mu \otimes e_\nu \otimes \ldots e^\rho \otimes \ldots, \quad T^{\mu\nu\cdots}{}_{\rho\ldots} = T(e^\mu, e^\nu, \ldots; e_\rho, \ldots). \tag{2.23}$$

We often what to decopose a tensor down into its symmetric and antisymmetric parts. To do this, we introduce the symmetrization of a tensor T,

$$T_{(\mu_1\ldots\mu_n)} = \frac{1}{n!}\sum_{\sigma\in S_n} T_{\mu_{\sigma(1)}\ldots\mu_{\sigma(n)}}, \tag{2.24}$$

where S_n is the set of all permutations of n objects. The antisymmetrization of a tensor is defined as

$$T_{[\mu_1\ldots\mu_n]} = \frac{1}{n!}\sum_{\sigma\in S_n} \mathrm{sgn}(\sigma) T_{\mu_{\sigma(1)}\ldots\mu_{\sigma(n)}}. \tag{2.25}$$

The function $\sigma = \pm 1$, depending on if σ is a even or odd permutation. We may now write

$$T_{\mu\nu} = T_{(\mu\nu)} + T_{[\mu\nu]}. \tag{2.26}$$

2.1.3 Geometries and the metric

The metric is a symmetric, non-degenerate $(0,2)$ tensor

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu. \tag{2.27}$$

It defines the geometry of the manifold $\mathcal{M}$, and is the main object of study in general relativity. As it is invertible, we can define $g^{\mu\nu} = (g^{-1})_{\mu\nu}$, which is the components of a $(2,0)$ tensor. We use this to raise and lower indices, as is done with the Minkowski metric $\eta_{\mu\nu}$ in special relativity.

Up until now, we have only considered the tangent space $T_p\mathcal{M}$ at a point p and the corresponding tensor-product spaces. We are, however, more interested in *fields* of vectors, covectors, or tensors. For each point $p \in \mathcal{M}$, a tensor field T "picks out" a tensor $T(p)$ from each tensor product space corresponding to the tangent space at p, $T_p\mathcal{M}$. We will use a vector field to illustrate. This vector field can be written as

$$v(p) = v^\mu(p)\partial_\mu|_p. \tag{2.28}$$

We will mostly be working with the components v^μ, which are functions of $\mathcal{M}$. For ease of notation, we write the vector as a function of the coordinates x. The vector field $v(x)$ is unchanged by a coordinate-transformation $x^\mu \to x'^\mu$; the coordinates are only a tool for our convenience. However, with a new set of coordinates, we get a new set of basis vectors, ∂'_μ:

$$v = v^\mu\partial_\mu = v^\mu\frac{\partial x'^\nu}{\partial x^\mu}\partial'_\nu = v'^\mu\partial'_\mu, \tag{2.29}$$

This gives us the transformation rules for the components of vectors,

$$v'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}v^\nu. \tag{2.30}$$

Tangent vectors are also called *contravariant* vectors, as their components transform contra to basis vectors. For covectors, it is

$$\omega'_\mu = \frac{\partial x^\nu}{\partial x'^\mu}\omega_\nu, \tag{2.31}$$

which is why covectors also are called *covariant* vectors.

The gradient of a scalar function f, $\mathrm{d}f = \partial_\mu f \mathrm{d}x^\mu$, is a coordinate-independent derivative, as $\partial_\mu f$ follows the transformation law for covectors. We define the covariant derivative, ∇, as a map from (n,m) tensor fields to $(n,m+1)$ tensor fields. When considering a scalar as a $(0,0)$ tensor, we see that this generalizes the scalar derivative. The components of a covariant derivative, $\nabla_\rho T^{\mu_1\cdots}{}_{\nu_1,\ldots}$, must follow the tensor transformation law. However, this is not strong enough to uniquely define ∇. We further assume

- Linearity: $\nabla(T+S) = \nabla T + \nabla S$.
- The product rule: $\nabla(T\otimes S) = (\nabla T)\otimes S + T\otimes(\nabla S)$.
- Reduces to partial derivative for scalars: $\nabla_\mu f = \partial_\mu f$.
- Kronecker delta gives zero: $\nabla_\mu \delta^\rho_\nu = 0$.

With this, we can, in general, write the covariant derivative as [2]

$$\nabla_\mu v^\nu = \partial_\mu v^\mu + \Gamma^\mu_{\nu\rho}v^\rho, \tag{2.32}$$

$$\nabla_\mu \omega_\nu = \partial_\mu\omega_\nu - \Gamma^\rho_{\mu\nu}\omega_\rho, \tag{2.33}$$

for vectors and covectors. $\Gamma^\mu_{\nu\rho}$ are called *Christoffel symbols*. The generalization for higher-order tensors is straightforward,

$$\nabla_\mu T^{\nu\cdots}{}_{\rho\ldots} = \partial_\mu T^{\nu\cdots}{}_{\rho\ldots} + \Gamma^\mu_{\nu\lambda}T^{\lambda\cdots}{}_{\rho\ldots} + \cdots - \Gamma^\lambda_{\mu\rho}T^{\mu\cdots}{}_{\lambda\ldots} - \ldots. \tag{2.34}$$

This is still not enough to uniquely determine the covariant derivative. We will furthermore assume $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$ and $\nabla_\mu g_{\nu\rho} = 0$. With these, we can find an explicit formula of the Christoffel symbols in terms of the metric,

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} - \partial_\sigma g_{\mu\nu} + \partial_\nu g_{\sigma\mu}). \tag{2.35}$$

With the notion of a covariant derivative, we may also generalize *parallel transport* to curved spaces. The notion of parallel transport of a vector in flat $\mathbb{R}^n$ is intuitive—given a line $x^\mu(\lambda)$, a vector v^μ at $x^\mu(\lambda_0)$ is parallel transported to v'^μ at $x^\mu(\lambda_1)$ if they "point in the same direction". To make this more precise, a vector field v^μ is parallel transported along $x^\mu(\lambda)$ if $\frac{\mathrm{d}}{\mathrm{d}\lambda} v^\mu = \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\partial_\nu v^\mu$. We generalize this to curved spaces by replacing the partial derivative with a covariant derivative, and so the criterion for parallel transport is

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \nabla_\mu v^\nu = 0. \tag{2.36}$$

With this, we can imagine creating a special class of paths, called *geodesics*, namely those which parallel transport their tangent vectors $\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}$. We imagine following an arrow we are holding without turning it as we walk. Using the definition of parallel transport Eq. (2.36), together with the covariant derivative Eq. (2.32), we get the geodesic equation,

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu_{\rho\sigma} \frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda} = 0. \tag{2.37}$$

In a flat space, where the Christoffel symbols vanish, this reduces to the familiar criterion for straight lines, $\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\lambda^2} = 0$.

The curvature of a manifold $\mathcal{M}$, with the metric $g_{\mu\nu}$, is encoded in the Riemann tensor. It is defined by

$$[\nabla_\mu, \nabla_\nu] v^\rho = R^\rho{}_{\sigma\mu\nu} v^\sigma, \tag{2.38}$$

which in our case gives the explicit formula

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}. \tag{2.39}$$

Although the Christoffel symbols are not tensors, the Riemann tensor is due to its definition using covariant derivatives. We can therefore contract some of its indices to get other tensor quantities. We define the Ricci tensor and Ricci scalar as

$$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}, \tag{2.40}$$

$$R = R^\mu{}_\mu = g^{\mu\nu} R_{\mu\nu}. \tag{2.41}$$

This form gives us several useful identities, such as

$$R_{\rho\sigma\mu\nu} = R_{[\rho\sigma]\mu\nu} = R_{\rho\sigma[\mu\nu]} = R_{\mu\nu\rho\sigma}. \tag{2.42}$$

Using can use the Jacobi identity of the commutator, we have

$$[\nabla_\mu, [\nabla_\nu, \nabla_\sigma]] + [\nabla_\sigma, [\nabla_\mu, \nabla_\nu]] + [\nabla_\nu, [\nabla_\sigma, \nabla_\mu]] = 0. \tag{2.43}$$

If we apply this on δ^μ_ν, we get the differential Bianchi identity, compactly written

$$\nabla_{[\mu} R_{\nu\rho]\sigma\eta} = 0. \tag{2.44}$$

To interpret the Riemann tensor, we define the parallel propagator P. A vector that is parallel transported along a curve parametrized by λ, so that $V^\mu(\lambda)$ obey the criterion of parallel transport Eq. (2.36), obey

$$V^\mu(\lambda) = P^\mu{}_\nu(\lambda) V^\nu. \tag{2.45}$$

Inserting this into the equation for parallel transport, Eq. (2.36), this operator must obey

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} P^\mu{}_\nu = -\Gamma^\mu_{\rho\sigma} \frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda} P^\sigma{}_\nu. \tag{2.46}$$

This has the same form as the definition of the unitary time-evolution operator in quantum mechanics, and we could therefore write down a solution involving an exponential and a path ordering operator, $\mathcal{P}$, analogous

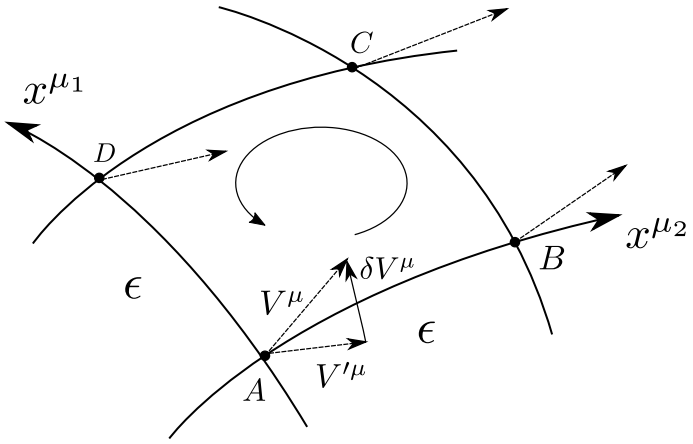


Figure 2.4: fig

to the time ordering operator from quantum mechanics. However, we only need the first approximation for $\lambda = \epsilon \ll 1$. In that case, the parallel propagator takes the form

$$P^\mu{}_\nu(\epsilon) = \delta^\mu_\nu - \int_0^\epsilon \mathrm{d}\eta\, \Gamma^\mu_{\nu\rho} \frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda} + \mathcal{O}(\epsilon^2). \tag{2.47}$$

With this, we will investigate how much a vector V^μ is changed by its motion around in a small loop. This is illustrated in Figure 2.4. A vector V^μ, is parallel transported in a loop along the coordinate lines. These lines are where either the coordinates x^{μ_1} or x^{μ_2} are equal to 0 or ϵ. Here, the indices μ_1 and μ_2 are not free but identify the two coordinate functions which define this loop.

Using the parallel propagator, the vector at point B is, to first order in ϵ

$$V^\mu|_B = \left(\delta^\mu_\nu - \int_0^\epsilon \mathrm{d}\eta\, \Gamma^\mu_{\nu\rho} \frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda}\right) V^\nu|_A. \tag{2.48}$$

The line from A to B, defined by $x^{\mu_1} = 0$, is parametrized by $x^\mu(\lambda) = \lambda\delta^\mu_{\mu_2}$. The Christoffel symbol along this line is

$$\Gamma^\mu_{\nu\rho}(\lambda) = \Gamma^\mu_{\nu\rho}|_A + \lambda\partial_{\mu_2}\Gamma^\mu_{\nu\rho}|_A + \mathcal{O}(\lambda^2) \tag{2.49}$$

we can therefore evaluate the integral as

$$\int_0^\epsilon \mathrm{d}\eta\, \Gamma^\mu_{\nu\mu_2} = \epsilon\Gamma^\mu_{\nu\mu_2}|_A + \frac{1}{2}\epsilon^2\partial_{\mu_2}\Gamma^\mu_{\nu\rho}|_A + \mathcal{O}(\epsilon^3). \tag{2.50}$$

Next, from B to C, the line is $x^\mu(\lambda) = \epsilon\delta^\mu_{\mu_2} + \lambda\delta^\mu_{\mu_1}$, and the Christoffel symbols are $\Gamma^\mu_{\nu\rho} = \Gamma^\mu_{\nu\rho}|_B + \lambda\partial_{\mu_1}\Gamma^\mu_{\nu\rho}|_B$ to fist order in λ. Here, we have to expand once more to evaluate the symbols at A. Then, we get

$$\Gamma^\mu_{\nu\rho} = \Gamma^\mu_{\nu\rho}|_A + \epsilon\partial_{\mu_2}\Gamma^\mu_{\nu\rho}|_A + \lambda\partial_{\mu_1}\Gamma^\mu_{\nu\mu_2}|_A, \tag{2.51}$$

where we have expanded to first order in ϵ, and treat λ as order ϵ. The integral in the parallel propagator from B to C is then

$$\int_0^\epsilon \mathrm{d}\eta\, \Gamma^\mu_{\nu\mu_1} = \epsilon\Gamma^\mu_{\nu\mu_1}|_A + \epsilon^2\partial_{\mu_2}\Gamma^\mu_{\nu\mu_1}|_A + \frac{1}{2}\epsilon^2\partial_{\mu_1}\Gamma^\mu_{\nu\mu_1}|_A. \tag{2.52}$$

The combined propagator, from A to C is then

$$P^\mu{}_\nu = \left[\delta^\mu_\sigma - \epsilon\Gamma^\mu_{\sigma\mu_1} - \epsilon^2\partial_{\mu_2}\Gamma^\mu_{\sigma\mu_1} - \frac{1}{2}\epsilon^2\partial_{\mu_1}\Gamma^\mu_{\sigma\mu_1}\right] \cdot \left[\delta^\sigma_\nu - \epsilon\Gamma^\sigma_{\nu\mu_2} - \frac{1}{2}\epsilon^2\partial_{\mu_2}\Gamma^\sigma_{\nu\mu_2}\right] \tag{2.53}$$

$$= \delta^\mu_\nu - \epsilon\left(\Gamma^\mu_{\nu\mu_1} + \Gamma^\mu_{\nu\mu_2}\right) + \epsilon^2\left(\Gamma^\mu_{\sigma\mu_1}\Gamma^\sigma_{\nu\mu_2} - \partial_{\mu_2}\Gamma^\mu_{\nu\mu_1} - \frac{1}{2}\partial_{\mu_1}\Gamma^\mu_{\nu\mu_1} - \frac{1}{2}\partial_{\mu_2}\Gamma^\mu_{\nu\mu_2}\right) + \mathcal{O}(\epsilon^3) \tag{2.54}$$

To create the parallel propagator for CDA is the propagator for ADC with its signs flipped, and the ADC propagator is the same as ABC only with the μ_1 and μ_2 indices exchanged in the derivative terms. Propagator for the full loop is, therefore, to and including second order in ϵ,

$$P^{\mu}{}_{\nu} = \left[\delta^{\mu}_{\sigma} + \epsilon\left(\Gamma^{\mu}_{\sigma\mu_2} + \Gamma^{\mu}_{\sigma\mu_1}\right) + \epsilon^2\left(\Gamma^{\mu}_{\rho\mu_1}\Gamma^{\rho}_{\sigma\mu_2} + \partial_{\mu_1}\Gamma^{\mu}_{\sigma\mu_2} + \frac{1}{2}\partial_{\mu_2}\Gamma^{\mu}_{\sigma\mu_2} + \frac{1}{2}\partial_{\mu_1}\Gamma^{\mu}_{\sigma\mu_1}\right)\right] \tag{2.55}$$

$$\left[\delta^{\sigma}_{\nu} - \epsilon\left(\Gamma^{\sigma}_{\nu\mu_1} + \Gamma^{\sigma}_{\nu\mu_2}\right) + \epsilon^2\left(\Gamma^{\sigma}_{\rho\mu_1}\Gamma^{\rho}_{\nu\mu_2} - \partial_{\mu_2}\Gamma^{\sigma}_{\nu\mu_1} - \frac{1}{2}\partial_{\mu_1}\Gamma^{\sigma}_{\nu\mu_1} - \frac{1}{2}\partial_{\mu_2}\Gamma^{\sigma}_{\nu\mu_2}\right)\right] \tag{2.56}$$

The terms linear in ϵ vanish, and the same with the terms with two equal μ_i-indices. The change in the vector as it is rotated around the loop is therefore, to second order in ϵ,

$$\begin{aligned}\delta V^{\mu} &= P^{\mu}{}_{\nu}V^{\nu} - V^{\mu} \\ &= \epsilon^2\left(\Gamma^{\mu}_{\sigma\mu_1}\Gamma^{\sigma}_{\nu\mu_2} - \Gamma^{\mu}_{\sigma\mu_2}\Gamma^{\sigma}_{\nu\mu_1} - \Gamma^{\mu}_{\sigma\mu_2}\Gamma^{\sigma}_{\nu\mu_2} - \Gamma^{\mu}_{\sigma\mu_1}\Gamma^{\sigma}_{\nu\mu_1} + \partial_{\mu_1}\Gamma^{\mu}_{\nu\mu_2} - \partial_{\mu_2}\Gamma^{\mu}_{\nu\mu_1}\right)V^{\nu}.\end{aligned} \tag{2.57}$$

find why these extra terms appear

Comparing with Eq. (2.39), we see that this is the Riemann curvature tensor. In other words, this tensor encodes how a vector is transformed by transporting it in small, closed loops.

2.1.4 Integration on manifolds

The integral of a scalar function on a manifold is not a coordinate-independent notion, and we must introduce the notion of n-forms. A n-form is a antisymmetric $(0, n)$ tensor. The wedge product, $\wedge$ is a product which maps two n- and m-forms to a $n + m$-form, and is defined as

$$(A \wedge B)_{\mu_1\ldots\mu_{n+m}} = \frac{(n+m)!}{n!m!}A_{[\mu_1\ldots\mu_n}B_{\mu_{n+1}\ldots\mu_{n+m}]}, \tag{2.58}$$

A n-form ω may now be written as

$$\omega = \omega_{[\mu_1\ldots\mu_n]}\,\mathrm{d}x^{\mu_1} \otimes \cdots \otimes \mathrm{d}x^{\mu_n} = \omega_{\mu_1\ldots\mu_n}\,\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_n}. \tag{2.59}$$

Furthermore, we define the exterior derivative, a map from n-forms to $n + 1$-forms, defined by

$$(\mathrm{d}T)_{\mu_1\ldots\mu_{n+1}} = (n+1)\partial_{[\mu_1}T_{\mu_2\ldots\mu_{n+1}]}. \tag{2.60}$$

We are interested in a coordinated independent quantity that we can integrate over. To that end, we define

$$\mathrm{d}^n x := \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{n-1} = \frac{1}{n!}\varepsilon_{\mu_1\ldots\mu_n}\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_n}, \tag{2.61}$$

Where $\varepsilon_{\mu_1\ldots\mu_n}$ is the Levi-Civita symbol. Given a different set of coordinates, x'^{μ}, these are related by

$$\mathrm{d}^n x = \det\left(\frac{\partial x}{\partial x'}\right)\mathrm{d}^n x', \tag{2.62}$$

where we have used the relation $\varepsilon_{\mu_1\ldots\mu_n}\det(A) = \varepsilon_{\nu_1\ldots\nu_n}A^{\nu_1}{}_{\mu_1}\ldots A^{\nu_n}{}_{\mu_n}$. We define $|g| = |\det(g_{\mu\nu})|$, which, by the transformation properties of tensors, transforms as

$$\sqrt{|g'|} = \left|\det\left(\frac{\partial x'}{\partial x}\right)\right|\sqrt{|g'|}, \tag{2.63}$$

This means that we can use this to compensate for the transformation of $\mathrm{d}^n x$, and get a volume form with a coordinate independent expression,

$$\mathrm{d}V = \sqrt{|g|}\,\mathrm{d}^n x = \sqrt{|g'|}\,\mathrm{d}^n x'. \tag{2.64}$$

With this, we can integrate scalars in a well-defined way by mapping them to a corresponding n-form, $f \to f\mathrm{d}V$. We define the integral of a scalar function f on a manifold $\mathcal{M}$ with a metric g as

$$I = \int_{\mathcal{M}} \mathrm{d}V\, f = \int_{\mathcal{M}} \mathrm{d}^n x \sqrt{|g(x)|}\, f(x). \tag{2.65}$$

Stoke's theorem generalizes the fundamental theorem of calculus and the divergence theorem to manifolds. Let $\mathcal{M}$ be a differential manifold of dimension n, with the boundary $\partial\mathcal{M}$. Stoke's theorem says that, for an $n-1$-form ω,

$$\int_{\mathcal{M}} \mathrm{d}\omega = \int_{\partial\mathcal{M}} \omega. \tag{2.66}$$

Stoke's theorem then implies a generalized divergence theorem. The boundary of $\mathcal{M}$ is a $n-1$ manifold dimensional, and a metric g on $\mathcal{M}$ will induce a new metric γ on $\partial\mathcal{M}$. This metric corresponds to the restriction of g to $\partial\mathcal{M}$. Furthermore, there will be a vector field n^μ of normalized vectors orthogonal to all elements of $T\partial\mathcal{M}$. This theorem states that for a vector field V^μ on $\mathcal{M}$,

$$\int_{\mathcal{M}} \mathrm{d}^n x \sqrt{|g|}\, \nabla_\mu V^\mu = \int_{\partial\mathcal{M}} \mathrm{d}^{n-1} y \sqrt{|\gamma|}\, n_\mu V^\mu. \tag{2.67}$$

2.2 *Lie groups

This section is based on [3–7].

2.2.1 Groups

Lie groups are a natural structure to capture the symmetries of a physical theory. A Lie group is a smooth manifold, with the additional structure of a *group*. A group is a set, G, together with a map

$$(\cdot,\cdot) : G \times G \longmapsto G, \tag{2.68}$$

$$(g_1, g_2) \longmapsto g_3, \tag{2.69}$$

called group multiplication. This map obeys the group axioms, which are the existence of an identity element $\mathbb{1}$, associativity and the existence of an inverse element g^{-1} for all $\in G$. These can be written as

$$\begin{array}{ll} \forall g \in G, & (g, \mathbb{1}) = g, \\ \forall g_1, g_2, g_3 \in G, & (g_1, (g_2, g_3)) = ((g_1, g_2), g_3), \\ \forall g \in G,\ \exists g^{-1} \in G, \text{ s.t.}, & (g, g^{-1}) = \mathbb{1}. \end{array}$$

In addition, we require that both the multiplication map and the inverse map, $g \mapsto g^{-1}$, are smooth. As we will discuss later, a symmetry transformation is a map between physical states which leave the equations governing that system unchanged. Assume the field, or set of fields, φ is governed by the equation $f(\varphi) = 0$. A symmetry transformation $\varphi \mapsto g\varphi$ will then obey $f(g\varphi) = 0$. This is what makes groups the natural structures to describe symmetries. Assume G is the set of all, or a subset closed under compositions, symmetries of a system,

$$G = \{\, g \mid f(g\varphi) = 0 \,\}, \tag{2.70}$$

as a Lie group. The group G might act on φ linearly, so $(g\varphi)_i = g_{ij}\varphi_j$, or in a more complicated matter. In this case, the group multiplication is composition, i.e., performing transformations in succession. This map is closed, as the composite of two symmetry transformations is another symmetry transformation. The identity map is a symmetry transformation, and composition is associative. This means that invertible symmetry transformations form a group.

We will focus on connected Lie groups, in which all elements $g \in G$ are in the same connected piece as the identity map $\mathbb{1}\varphi = \varphi$. This means that for each $g \in G$, one can find a continuous path $\gamma(t)$ in the manifold,

such that $\gamma(0) = \mathbb{1}$ and $\gamma(1) = g$. Given such a path, we can study transformations close to the identity element. As the Lie group is a smooth manifold, we can write[4]

$$\gamma(\epsilon) = \mathbb{1} + i\epsilon V + \mathcal{O}(\epsilon). \tag{2.71}$$

V is a generator, and is defined as

$$iV = \frac{\mathrm{d}\gamma}{\mathrm{d}t}\Big|_{t=0}. \tag{2.72}$$

The generator is thus a member of the tangent space of the identity element, $T_{\mathbb{1}}G$. We denote the coordinates of G by $\eta_\alpha \in \mathbb{R}^n$. As before, we can denote a path γ in a manifold G by its path through $\mathbb{R}^n$, $\gamma(t) = g(\eta(t))$. We will assume, without loss of generality, that $\eta_\alpha(0) = 0$ and $g(0) = \mathbb{1}$. We can then write the generator as

$$V = \frac{\mathrm{d}\gamma}{\mathrm{d}t}\Big|_{t=0} = \frac{\mathrm{d}\eta_\alpha}{\mathrm{d}t}\Big|_{t=0}\frac{\partial g}{\partial \eta_\alpha}\Big|_{\eta=0} = v_\alpha T_\alpha, \quad T_\alpha = \frac{\mathrm{d}\eta_\alpha}{\mathrm{d}t}\Big|_{t=0}, \quad \frac{\partial g}{\partial \eta_\alpha}\Big|_{\eta=0}. \tag{2.73}$$

Infinitesimal transformations can therefore be written as

$$\gamma(\epsilon) = \mathbb{1} + i\epsilon v_\alpha T_\alpha + \mathcal{O}\epsilon. \tag{2.74}$$

2.2.2 Lie algebra

The tangent space $T_{\mathbb{1}}G$, together with the additional operation

$$[T_\alpha, T_\beta] = iC^\gamma_{\alpha\beta}T_\gamma, \tag{2.75}$$

called the Lie bracket, form a Lie algebra denoted $\mathfrak{g}$. $C^\gamma_{\alpha\beta}$ are called structure constants. They obey the Jacobi identity,

$$C^\gamma_{\alpha\beta} + C^\alpha_{\beta\gamma} + C^\beta_{\gamma\alpha} = 0, \tag{2.76}$$

which mean that they are totally antisymmetric. For matrix groups, which we deal with in this text, the Lie bracket is the commutator. A subset of the original Lie group, $H \subset G$, closed under the group action, is called a subgroup. H then has its own Lie algebra $\mathfrak{h}$, with a set of $m = \dim H$ generators, t_a, which is a subset of the original generators T_α. We denote the remaining set of generators x_i, such that t_a and x_i together span $\mathfrak{g}$. The commutators of t_a must be closed, which means that we can write

$$[t_a, t_b] = iC^c_{ab}t_c, \tag{2.77}$$

$$[t_a, x_i] = iC^k_{ai}x_k, \tag{2.78}$$

$$[x_i, x_j] = iC^k_{ij}x_k + iC^c_{ij}t_c, \tag{2.79}$$

where abc runs over the generators of $\mathfrak{h}$, and ijk runs over the rest. The second formula can be derived using the Jacobi identity Eq. (2.76), which implies that $C^k_{ab} = 0 = -C^b_{ak}$. This is called a Cartan decomposition.

One parameter subgroups are one special case of Lie subgroups. If a curve $\gamma(t)$ through G obey

$$\gamma(t)\gamma(s) = \gamma(t+s), \quad \gamma(0) = \mathbb{1}, \tag{2.80}$$

then all the points on this curve from a one parameter subgroup of G. This path is associated with a generator,

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t}\Big|_{t=0} = i\eta_\alpha T_\alpha. \tag{2.81}$$

This association is one-to-one, and allows us to define the exponential map,

$$\exp\{i\eta_\alpha T_\alpha\} := \gamma(1). \tag{2.82}$$

For connected and compact Lie groups, all elements of the Lie group $g \in G$ can be written as an exponential of elements in the corresponding Lie algebra $\eta_\alpha T_\alpha \in \mathfrak{g}$. For matrix groups, the exponential equals the familiar series expansion

$$\exp\{X\} = \sum_n \frac{1}{n!}X^n. \tag{2.83}$$

[4]The factor i is a physics convention and differs from how mathematicians define generators of a Lie group.

Chapter 3

Quantum field theory

In this section, we survey some general properties of quantum field theory that are necessary for chiral perturbation theory. First, we introduce the path integral and the 1-particle irreducible effective action and the effective potential. We will derive Goldstone's theorem and present the CCWZ construction, which is the basis for χPT.

3.1 *QFT via path integrals

This section is based on [4–7]

In the path integral formalism, one evaluates quantum observable by integrating over the contributions of all possible configurations. If the system has specified initial and final states, this amounts to all possible paths the system might evolve between these, hence the name. We assume the reader has some familiarity with this formalism. However, if a refresher is needed, section B.2 contains a derivation of the closely related imaginary-time formalism and compares it with the path integral approach.

In the path integral formalism, the vacuum-to-vacuum transition amplitude, i.e., the probability that that vacuum at $t = -\infty$ evolves to the vacuum at time $t = \infty$, is given by

$$Z = \lim_{T\to\infty} \langle \Omega, T/2 | -T/2, \Omega \rangle = \lim_{T\to\infty} \langle \Omega | e^{-iHT} | \Omega \rangle = \int \mathcal{D}\pi \mathcal{D}\varphi \, \exp\left\{ i \int \mathrm{d}^4x \, (\pi\dot{\varphi} - \mathcal{H}[\pi, \varphi]) \right\}, \tag{3.1}$$

where $|\Omega\rangle$ is the vacuum state. The φ are the fields of the theory, and π their canonical momenta. We will work as if φ are a bosonic field. However, this can be readily generalized to fermions. By introducing a source term into the Hamiltonian density, $\mathcal{H} \to \mathcal{H} - J(x)\varphi(x)$, we get the generating functional

$$Z[J] = \int \mathcal{D}\pi \mathcal{D}\varphi \, \exp\left\{ i \int \mathrm{d}^4x \, (\pi\dot{\varphi} - \mathcal{H}[\pi, \varphi] + J\varphi) \right\}. \tag{3.2}$$

If $\mathcal{H}$ is quadratic in π, we can complete the square and integrate out π to obtain

$$Z[J] = C \int \mathcal{D}\varphi \, \exp\left\{ i \int \mathrm{d}^4x \, (\mathcal{L}[\varphi] + J\varphi) \right\}. \tag{3.3}$$

C is infinite, but constant, and will drop out of physical quantities. In scattering theory, the main objects of study are correlation functions $\langle \varphi(x_1)\varphi(x_2)... \rangle = \langle \Omega | T \{ \varphi(x_1)\varphi(x_2) \dots \} | \Omega \rangle$, where T is the time ordering operator. These are given by functional derivatives of $Z[J]$,

$$\langle \varphi(x_1)\varphi(x_2)... \rangle = \frac{\int \mathcal{D}\varphi(x) \, [\varphi(x_1)\varphi(x_2)...] e^{iS[\varphi]}}{\int \mathcal{D}\varphi(x) \, e^{iS[\varphi]}} = \frac{1}{Z[0]} \prod_i \left(-i \frac{\delta}{\delta J(x_i)} \right) Z[J] \Big|_{J=0}, \tag{3.4}$$

where

$$S[\varphi] = \int \mathrm{d}^4x\, \mathcal{L}[\varphi] \tag{3.5}$$

is the action of the theory. The functional derivative is described in section A.2. In a free theory, we are able to write

$$Z_0[J] = Z_0[0]\exp(iW_0[J]), \quad iW_0[J] = -\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4y\, J(x)D_0(x-y)J(y), \tag{3.6}$$

where D_0 is the propagator of the free theory. Using this form of the generating functional, Eq. (3.4) becomes

$$\begin{aligned}
&\frac{1}{Z[0]}(-i)^n\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_n)}Z_0[J]\Big|_{J=0} = (-i)^n\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_n)}e^{iW_0[J]}\Big|_{J=0}\\
&= (-i)^n\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_{n-1})}\left(i\frac{\delta W_0[J]}{\delta J(x_n)}\right)e^{iW_0[J]}\Big|_{J=0}\\
&= (-i)^n\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_{n-2})}\left(i\frac{\delta W_0[J]}{\delta J(x_{n-1}),J(x_n)} + i^2\frac{\delta W_0[J]}{\delta J(x_{n-1})}\frac{\delta W_0[J]}{\delta J(x_n)}\right)e^{iW_0[J]}\Big|_{J=0} = \dots\\
&= (-i)^{\lfloor n/2\rfloor}\sum_{(a,b)}\prod_{i=1}^{\lfloor n/2\rfloor}\frac{\delta W_0[J]}{\delta J(x_{a(i)}),J(x_{b(i)})}\Big|_{J=0}.
\end{aligned}$$

In the last line we have introduced the functions a, b, which define a way to pair up n elements. $\lfloor\cdot\rfloor$ is the floor function. The domain of these functions are the integers between 1 and $\lfloor n/2\rfloor$, the image a subset of the integers between 1 and n of size $\lfloor n/2\rfloor$. A valid pairing is a set $\{(a(1),b(1)),\dots(a(\lfloor n/2\rfloor),b(\lfloor n/2\rfloor))\}$, where all elements $a(i)$ and $b(j)$ are different, such that all integers up to and including n are featured. A pair is not directed, so $(a(i),b(i))$ is the same pair as $(b(i),a(i))$. The sum is over the set $\{(a,b)\}$ of all possible, unique pairings. If n is odd, the expression is equal to 0. This is Wick's theorem, and it can more simply be stated as *a correlation function is the sum of all possible pairings of 2-point functions*,

$$\left\langle\prod_{i=1}^n\varphi(x_i)\right\rangle_0 = \sum_{\{(a,b)\}}\prod_{i=1}^{\lfloor n/2\rfloor}\left\langle\varphi(x_{a(i)})\varphi(x_{b(i)})\right\rangle_0. \tag{3.7}$$

The subscript on the expectation value indicates that it is evaluated in the free theory.

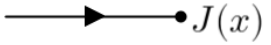
If we have an interacting theory, that is, a theory with an action $S = S_0 + S_I$, where S_0 is a free theory, the generating functional can be written

$$Z[J] = Z_0[0]\left\langle\exp\left(iS_I + i\int\mathrm{d}^4x\, J(x)\varphi(x)\right)\right\rangle_0. \tag{3.8}$$

We can expand the exponential in power series, which means the expectation value in Eq. (3.8) becomes

$$\sum_{n,m}\frac{1}{n!m!}\left\langle(iS_I)^n\left(i\int\mathrm{d}^4x\, J(x)\varphi(x)\right)^m\right\rangle_0. \tag{3.9}$$

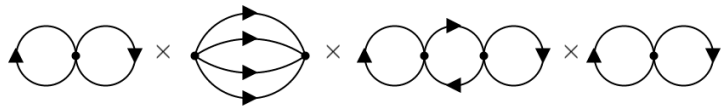
The terms in this series are represented by Feynman diagrams, constructed using the Feynman rules, and can be read from the action. We will not further detail how the Feynman rules are derived. The Feynman rules for a free scalar field in thermal field theory are derived in section B.4, and the general procedure is found in any of the main sources for this section [4–7] The source terms gives rise to an additional vertex



. (3.10)

The generating functional $Z[J]$ thus equals $Z_0[0]$ times *the sum of all diagrams with external sources* $J(x)$.

Consider a general diagram without external legs, built up of N different connected subdiagrams, where subdiagram i appears n_i times. As an illustration, a generic vacuum diagram in φ^4-theory has the form

$\mathcal{M} =$  ... (3.11)

If sub-diagram i as a stand-alone diagram equals $\mathcal{M}_i$, each copy of that subdiagram contributes a factor $\mathcal{M}_i$ to the total diagram. However, due to the symmetry of permuting identical subdiagrams, one must divide by the extra symmetry factor $s = n_i!$, the total number of permutations of all the copies of diagram i. The full diagram therefore equals

$$\mathcal{M} = \prod_{i=1}^{N} \frac{1}{n_i!} \mathcal{M}_i^{n_i}. \tag{3.12}$$

$\mathcal{M}$ is uniquely defined by a finite sequence of integers, $(n_1, n_2, \dots n_N, 0, 0, \dots)$, so the sum of all diagrams is the sum over the set S of all finite sequences of integers. This allows us to write the sum of all diagrams as

$$\sum_{(n_1,\dots)\in S} \prod_i \frac{1}{n_i!} \mathcal{M}_i^{n_i} = \prod_{i=1}^{\infty} \sum_{n_i=1}^{\infty} \frac{1}{n_i!} \mathcal{M}_i^{n_i} = \exp\Big(\sum\nolimits_i \mathcal{M}_i\Big). \tag{3.13}$$

We showed that the generating functional $Z[J]$ were the $Z_0[0]$ times the sum of all diagrams due to external sources. From Eq. (3.13), if we define

$$Z[J] = Z_0[0] \exp(iW[J]), \tag{3.14}$$

then $W[J]$ is the sum of all connected diagrams. This is trivially true for the free theory, where the only connected diagram is

$W_0[J] =$  . (3.15)

The two-point function in the full, interacting theory can thus be written

$$-i \frac{\delta W[J]}{\delta J(x), J(y)} = D(x-y). \tag{3.16}$$

3.2 *The 1PI effective action

This section is based on [4–7]

The generating functional for connected diagrams, $W[J]$, is dependent on the external source current J. We can define a new quantity with a different independent variable, using the Legendre transformation analogously to what is done in thermodynamics and Lagrangian mechanics. The new independent variable is

$$\varphi_J(x) := \frac{\delta W[J]}{\delta J(x)} = \langle \varphi(x) \rangle_J \,. \tag{3.17}$$

The subscript J on the expectation value indicate that it is evaluated in the presence of a source. The Legendre transformation of W is then

$$\Gamma[\varphi_J] = W[J] - \int \mathrm{d}^4x \, J(x)\varphi_J(x). \tag{3.18}$$

Using the definition of φ_J, we have that

$$\frac{\delta\varphi_J(x)}{\delta\Gamma}[\varphi_J] = \int \mathrm{d}^4y \, \frac{\delta J(y)}{\delta\varphi_J(x)} \frac{\delta J(y)}{\delta W}[J] - \int \mathrm{d}^4y \, \frac{\delta J(y)}{\delta\varphi_J(x)} \varphi_J(y) - J(x) = -J(x). \tag{3.19}$$

If we compare this to the classical equations of motion of a field φ with the action S,

$$\frac{\delta S[\varphi]}{\delta\varphi(x)} = -J(x), \tag{3.20}$$

we see that Γ is an action that gives the equation of motion for the expectation value of the field, given a source current $J(x)$.

To interpret Γ further, we observe what happens if we treat $\Gamma[\varphi]$ as a classical action with a coupling g. The generating functional in this new theory is

$$Z[J,g] = \int \mathcal{D}\varphi \exp\left\{ ig^{-1}\left(\Gamma[\varphi] + \int \mathrm{d}^4x\, \varphi(x)J(x)\right)\right\} \tag{3.21}$$

The free propagator in this theory will be proportional to g, as it is given by the inverse of the equation of motion for the free theory. All vertices in this theory, on the other hand, will be proportional to g^{-1}, as they are given by the higher-order terms in the action $g^{-1}\Gamma$. This means that a diagram with V vertices and I internal lines is proportional to g^{I-V}. Regardless of what the Feynman-diagrams in this theory are, the number of loops of a connected diagram is[1]

$$L = I - V + 1. \tag{3.22}$$

To see this, we first observe that diagrams with one single loop must have equally many internal lines as vertices, so the formula holds for $L = 1$. The formula still holds if we add a new loop to a diagram with n loops by joining two vertices. If we attach a new vertex with one line, the formula still holds, and as the diagram is connected, any more lines connecting the new vertex to the diagram will create additional loops. This ensures that the formula holds by induction. As a consequence of this, any diagram is proportional to g^{L-1}. This means that in the limit $g \to 0$, the theory is fully described at the tree-level, i.e., by only considering diagrams without loops. In this limit, we may use the stationary phase approximation, as described in section A.2, which gives

$$Z[J, g\to 0] \approx C \det\left(-\frac{\delta\Gamma[\varphi_J]}{\delta\varphi(x),\varphi(y)}\right) \exp\left\{ ig^{-1}\left(\Gamma[\varphi_J] + \int \mathrm{d}^4x\, J(x)\varphi_J(x)\right)\right\}. \tag{3.23}$$

This means that

$$-ig\ln(Z[J,g]) = gW[J,g] = \Gamma[\varphi_J] + \int \mathrm{d}^4x\, J(x)\varphi_J(x) + \mathcal{O}(g), \tag{3.24}$$

which is exactly the Legendre transformation we started out with, modulo the factor g. Γ is, therefore, the action that describes the full theory at the tree level. For a free theory, the classical action S equals the effective action.

As we found in the last section, the propagator $D(x,y) = \langle\varphi(x)\varphi(y)\rangle_J$ is given by $-i$ times the second functional derivative of $W[J]$. Using the chain rule, together with Eq. (3.19), we get

$$(-i)\int \mathrm{d}^4z \frac{\delta^2 W[J]}{\delta J(x)\delta J(z)}\frac{\delta^2\Gamma[\varphi_J]}{\delta\varphi_J(z)\varphi_J(y)} = (-i)\int \mathrm{d}^4z \frac{\delta\varphi_J[z]}{\delta J(x)}\frac{\delta^2\Gamma[\varphi_J]}{\delta\varphi_J(z)\varphi_J(y)} = \frac{\delta}{\delta J(x)}\frac{\delta\Gamma[\varphi_J]}{\delta\varphi_J(y)} = i\delta(x-y). \tag{3.25}$$

This is exactly the definition of the inverse propagator,

$$\frac{\delta\Gamma[\varphi_J]}{\delta\varphi_J(x),\varphi_J(y)} = D^{-1}(x,y). \tag{3.26}$$

The inverse propagator is the sum of all one-particle-irreducible (1PI) diagrams, with two external vertices. More generally, Γ is the generating functional for 1PI diagrams, which is why it is called the 1PI effective action.

Γ may be viewed as an effective action as defined in the introduction. We define η as the fluctuations around the expectation value of the field, $\varphi(x) = \varphi_J(x) + \eta(x)$, and use this to change variables of integration in the path integral. The expectation value φ_J is constant with respect to the integral, so

$$\int \mathcal{D}\varphi\ \exp\{iS[\varphi]\} = \int \mathcal{D}\eta\ \exp\{iS[\varphi_J+\eta]\}. \tag{3.27}$$

By assumption, $\langle\eta\rangle_J = 0$, which means this path integral is described by only 1PI diagrams, connected or not. We can therefore write

$$\exp\{i\Gamma[\varphi_J]\} = \int \mathcal{D}\eta\ \exp\{iS[\varphi_J+\eta]\}. \tag{3.28}$$

skriv om integrating out dof

Comparing this to integrating out degrees of freedom, we see that the 1PI effective potential corresponds to integrating out *all* degrees of freedom, and let the expectation value appear as a static background field,

[1] This is a consequence of the Euler characteristic $\chi = V - E + F$.

3.2.1 Effective potential

For a constant field configuration $\varphi(x) = \varphi_0$, the effective action, which is a functional, becomes a regular function. We define the effective potential $\mathcal{V}_{\text{eff}}$ by

$$\Gamma[\varphi_0] = -VT\,\mathcal{V}_{\text{eff}}(\varphi_0), \tag{3.29}$$

where VT is the volume of space-time. For a constant ground state, the effective potential will equal the energy of this state. To calculate the effective potential, we can expand the action around this state to calculate the effective action, by changing variables to $\varphi(x) = \varphi_0 + \eta(x)$. $\eta(x)$ now parametrizes fluctuations around the ground state, and has by assumption a vanishing expectation value. The generating functional becomes

$$Z[J] = \int \mathcal{D}(\varphi_0 + \eta)\, \exp\left\{ iS[\varphi_0 + \eta] + i \int \mathrm{d}^4x\, [\varphi_0 + \eta(x)]J(x) \right\}. \tag{3.30}$$

The functional version of a Taylor expansion, as described in section A.2, is

$$S[\varphi_0 + \eta] = S[\varphi_0] + \int \mathrm{d}x\, \eta(x)\, \frac{\delta S[\varphi_0]}{\delta\varphi(x)} + \frac{1}{2} \int \mathrm{d}x\mathrm{d}y\, \eta(x)\eta(y)\, \frac{\delta^2 S[\varphi_0]}{\delta\varphi(x)\delta\varphi(y)} + \dots \tag{3.31}$$

The notation

$$\frac{\delta S[\varphi_0]}{\delta\varphi(x)} \tag{3.32}$$

indicates that the functional $S[\varphi]$ is differentiated with respect to $\varphi(x)$, then evaluated at $\varphi(x) = \varphi_0$. We define

$$S_0[\eta] := \int \mathrm{d}^4x \mathrm{d}^4y\, \eta(x)\eta(y)\, \frac{\delta S[\varphi_0]}{\delta\varphi(x), \varphi(y)}, \tag{3.33}$$

$$S_I[\eta] := \int \mathrm{d}^4x \mathrm{d}^4y \mathrm{d}^4z\, \eta(x)\eta(y)\eta(z)\, \frac{\delta S[\varphi_0]}{\delta\varphi(x), \varphi(y), \varphi(z)} + \dots, \tag{3.34}$$

where the dots indicate higher derivatives. When we insert this expansion into the generating functional $Z[J]$ we get

$$Z[J] = \int \mathcal{D}\eta \exp\left\{ i \int \mathrm{d}^4x\, (\mathcal{L}[\varphi_0] + J\varphi_0) + i \int \mathrm{d}^4x\, \eta(x) \left(\frac{\delta S[\varphi_0]}{\delta\varphi(x)} + J(x) \right) + iS_0[\eta] + iS_I[\eta] \right\} \tag{3.35}$$

The first term is constant with respect to η and may be taken outside the path integral. The second term gives rise to tadpole diagrams, which alter the expectation value of $\eta(x)$. For $J = 0$, this expectation value should vanish, and this term can be ignored. Furthermore, this means that the ground state must minimize the classical potential,

$$\frac{\partial \mathcal{V}(\varphi_0)}{\partial\varphi} = 0. \tag{3.36}$$

This leaves us with

$$-i \ln Z[J] = W[J] = \int \mathrm{d}^4x\, (\mathcal{L}[\varphi_0] + J\varphi_0) - i \ln \left(\int \mathcal{D}\eta \exp\{iS_0[\eta] + iS_I[\eta]\} \right) \tag{3.37}$$

We can now use the definition of the 1PI effective action to obtain a formula for the effective potential,

$$\mathcal{V}_{\text{eff}}(\varphi_0) = -\frac{1}{VT} \left(W[J] - \int \mathrm{d}^4x\, J(x)\varphi_0 \right) = \mathcal{V}(\varphi_0) - i \ln \left(\int \mathcal{D}\eta \exp\{iS_0[\eta] + iS_I[\eta]\} \right). \tag{3.38}$$

In Eq. (3.18), we showed that the 1PI effective action describes the whole quantum theory of the original action at the tree-level. This was done by inspecting a theory with an action proportional to g^{-1}. In this theory, Feynman diagrams with L loops are proportional to g^{L-1}. We can use the same argument to expand the effective potential in loops. This is done by modifying the action $S[\varphi] \to g^{-1}S[\varphi]$, and then expand in

power of g. The first term in the effective potential is modified by $\mathcal{V} \to g^{-1}\mathcal{V}$, which means that it is made up of tree-level terms. This is as expected, since the tree-level result corresponds to the classical result without any quantum corrections. The second term becomes

$$\ln\left(\int \mathcal{D}\eta\, e^{iS_0+iS_I}\right) \longrightarrow \ln\left(\int \mathcal{D}\eta\, e^{ig^{-1}S_0+ig^{-1}S_I}\right) = \ln\left(\int \mathcal{D}\eta\, e^{ig^{-1}S_0}\right) + \ln\left(\frac{\int \mathcal{D}\eta\, e^{ig^{-1}S_I}\, e^{ig^{-1}S_0}}{\int \mathcal{D}\eta\, e^{ig^{-1}S_0}}\right)$$

The first term is quadratic in η, and can therefore be evaluated as a generalized Gaussian integral, as described in section A.2,

$$\begin{aligned}
&\ln\left\{\int \mathcal{D}\eta\, \exp\left(ig^{-1}\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4y\, \eta(x)\eta(y)\, \frac{\delta S[\varphi_0]}{\delta\varphi(x),\varphi(y)}\right)\right\} \\
&= \ln\left\{\det\left(-g^{-1}\frac{\delta S[\varphi_0]}{\delta\varphi(x),\varphi(y)}\right)^{-1/2}\right\} = -\frac{1}{2}\,\mathrm{Tr}\left\{\ln\left(-\frac{\delta S[\varphi_0]}{\delta\varphi(x),\varphi(y)}\right)\right\} + \text{const.}
\end{aligned}$$

We then use the identity $\ln\det M = \mathrm{Tr}\ln M$. After we remove the constant, this term is proportional to g^0, i.e., it is made up of one-loop terms.

The last term can be evaluated by first expanding the exponential containing the S_I term, then using $\ln(1+x) = \sum_n \frac{1}{n}x^n$. Using

$$\langle A\rangle_0 = \frac{\int \mathcal{D}\varphi\, A\, e^{ig^{-1}S_0}}{\int \mathcal{D}\varphi\, e^{ig^{-1}S_0}}, \tag{3.39}$$

we can write

$$\ln\left[\frac{\int \mathcal{D}\eta\, e^{ig^{-1}S_I} e^{ig^{-1}S_0}}{\int \mathcal{D}\varphi\, e^{ig^{-1}S_0}}\right] = \ln\left(\sum_{n=0}^{\infty}\frac{1}{n!}\left\langle (ig^{-1}S_I)^n\right\rangle_0\right). \tag{3.40}$$

We recognize this as the sum of all connected Feynman diagrams, with Feynman ruels from the interaction term S_I. We know that S_I is made up of terms that are third power or higher in the fields. Each internal line is connected to two vertices, and each vertex is connected to at least three internal lines, i.e., $I \geq 3/2V$. The number of loops is therefore $L = I - V + 1 \geq (3/2-1)V + 1$. There is at leas one vertex, i.e. $L \geq 3/2$. This shows that the first logarith contains *all* one-loop contributions. The effective potential to one-loop order is therefore

$$\mathcal{V}_{\text{eff}}(\varphi_0) = \mathcal{V}(\varphi_0) - \frac{i}{VT}\frac{1}{2}\,\mathrm{Tr}\left\{\ln\left(-\frac{\delta S[\varphi_0]}{\delta\varphi(x),\varphi(y)}\right)\right\}. \tag{3.41}$$

3.3 *Symmetry and Goldstone's theorem

This section is based on [3–7].

Symmetry plays a prominent role in modern physics. If we can transform a physical state in such a way that the governing equations of this system are unchanged, we call that transformation a *symmetry transformation*. All such transformations are known as the symmetries of that theory. The symmetries of a theory encode a lot of physics, such as the presence of conserved quantities and the system's low energy behavior. We distinguish between internal and external symmetries. An external symmetry is an active coordinate transformation, such as rotations or translations. They relate degrees of freedom at different space-time points, while internal symmetries transform degrees of freedom at each space-time point independently. A further distinction is between global and local symmetry transformations. Global transformations have one rule for transforming degrees of freedom at each point, which is applied everywhere, while local transformations are functions of space-time.

In classical field theory, symmetries are encoded in the behavior of the Lagrangian when the fields are transformed. We will consider continuous transformations, which can in general be written as

$$\varphi(x) \longrightarrow \varphi'(x) = f_t[\varphi](x), \quad t \in [0,1]. \tag{3.42}$$

Here, $f_t[\varphi]$ is a functional in φ, and a smooth function of t, with the constraint that $f_0[\varphi] = \varphi$. This allows us to look at "infinitesimal" transformations,

$$\varphi'(x) = f_\epsilon[\varphi] = \varphi(x) + \epsilon \frac{\mathrm{d}f_t[\varphi]}{\mathrm{d}t}\bigg|_{t=0} + \mathcal{O}\epsilon. \tag{3.43}$$

When considering infinitesimal transformations, we will not always write $+\mathcal{O}\epsilon$, but rather consider it implicit. We will consider internal, global transformations which act linearly on φ. For N fields, φ_i, this can be written

$$\varphi_i'(x) = \varphi_i(x) + \epsilon\, iV_{ij}\varphi_j(x). \tag{3.44}$$

V_{ij} is called the generator of the transformation. A symmetry transformation of the system is then a transformation in which the Lagrangian left is unchanged, or at most differ by a 4-divergence term. That is, a transformation $\varphi \to \varphi'$ is a symmetry if

$$\mathcal{L}[\varphi'] = \mathcal{L}[\varphi] + \partial_\mu K^\mu[\varphi], \tag{3.45}$$

where $K^\mu[\varphi]$ is a functional of φ.[2] This is a requirement for symmetry in quantum field theory too. However, as physical quantities in quantum field theory are given not just by the action of a single state but the path integral, the integration measure $\mathcal{D}\varphi$ has to be invariant as well. If a classical symmetry fails due to the non-invariance of the integration measure, it is called an *anomaly*.

To investigate the symmetry properties of a quantum theory, we explore what constraints a symmetry imposes on the effective action. To that end, assume

$$\mathcal{D}\varphi'(x) = \mathcal{D}\varphi(x), \quad S[\varphi'] = S[\varphi]. \tag{3.46}$$

In the generating functional, such a transformation corresponds to a change of integration variable. Using the infinitesimal version of the transformation, we may write

$$\begin{aligned} Z[J] = \int \mathcal{D}\varphi\, \exp\!\left\{ iS[\varphi] + i\int \mathrm{d}^4x\, J_i(x)\varphi_i(x) \right\} &= \int \mathcal{D}\varphi'\, \exp\!\left\{ iS[\varphi'] + i\int \mathrm{d}^4x\, J_i(x)\varphi_i'(x) \right\} \\ &= Z[J] + i\epsilon \int \mathrm{d}^4x\, J_i(x) \int \mathcal{D}\varphi\, [V_{ij}\varphi_j(x)] e^{iS[\varphi]}, \end{aligned} \tag{3.47}$$

Using Eq. (3.19), we can write this as

$$\int \mathrm{d}^4x\, \frac{\delta\Gamma[\varphi_J]}{\delta\varphi_i(x)}\, V_{ij}\, \langle\varphi_j(x)\rangle_J = 0. \tag{3.48}$$

This constraint will allow us to deduce the properties of a theory close to the ground state, only using information about the symmetries of the theory.

The archetypical example of an internal, global, and continuous symmetry is the linear sigma model, which we will use as an example throughout this section. The linear sigma model is made up of N real scalar fields φ_i, whose Lagrangian is

$$\mathcal{L}[\varphi] = \frac{1}{2}\partial_\mu\varphi_i(x)\partial^\mu\varphi_i(x) - \mathcal{V}(\varphi), \quad \mathcal{V}(\varphi) = -\frac{1}{2}\mu^2\varphi_i(x)\varphi_i(x) + \frac{1}{4}\lambda[\varphi_i(x)\varphi_i(x)]^2. \tag{3.49}$$

This system is invariant under the rotation of the N fields into each other,

$$\varphi_i \longrightarrow \varphi_i' = M_{ij}\varphi_j, \quad M^{-1} = M^T. \tag{3.50}$$

The set of all such transformations forms the Lie group $\mathrm{O}(N)$. Lie groups will be discussed in the next section. For $N = 2$, and $\mu^2, \lambda > 0$ we get the ubiquitous Mexican hat potential, as illustrated in Figure 3.1.

[2] Terms of the form $\partial_\mu K^\mu$ does not affect the physics, as variational principle $\delta S = 0$ do not vary the fields at infinity. Together with the divergence theorem, this means that such terms do not influence the equations of motion.

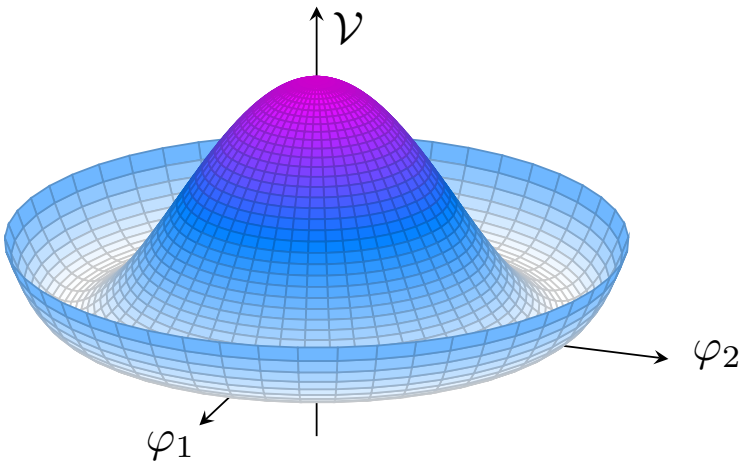


Figure 3.1: The Mexican hat potential is the classical potential $\mathcal{V}$ for the $N = 2$ linear sigma model.

3.3.1 Nöther's theorem

One of the most profound consequences of symmetry in physics is the appearance of conserved quantities. Assume we have a set of fields φ_i. Nöther's theorem tells us that if the Lagrangian $\mathcal{L}[\varphi_i]$ has a continuous symmetry, then there is a corresponding conserved current [2, 4]. Consider an infinitesimal transformation,

$$\varphi_i(x) \longrightarrow \varphi_i'(x) = \varphi_i(x) + \delta\varphi_i(x), . \tag{3.51}$$

Applying this transformation to the Lagrangian will in general change its form,

$$\mathcal{L}[\varphi] \to \mathcal{L}[\varphi'] = \mathcal{L}[\varphi] + \delta\mathcal{L}. \tag{3.52}$$

We assume this transformation is a symmetry, i.e.,

$$\delta\mathcal{L} = \partial_\mu K^\mu.$$

By considering the Lagrangian as a function of the field and its derivatives, $\mathcal{L} = \mathcal{L}(\varphi_i, \partial_\mu\varphi_i)$, we can write the difference term as a Taylor expansion around $(\varphi_i, \partial_\mu\varphi_i)$,

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\varphi_i}\delta\varphi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}\delta(\partial_\mu\varphi_i), \tag{3.53}$$

where $\delta(\partial_\mu\varphi_i) = \partial_\mu\varphi_i' - \partial_\mu\varphi_i$. By the linearlity of the derivative,

$$\delta(\partial_\mu\varphi_i) = \partial_\mu\varphi_i' - \partial_\mu\varphi_i = \partial_\mu(\varphi_i' - \varphi_i) = \partial_\mu\delta\varphi_i. \tag{3.54}$$

With this, and the Euler-Lagrange equations

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} - \frac{\partial\mathcal{L}}{\partial\varphi_i} = 0, \tag{3.55}$$

we can rewrite

$$\delta\mathcal{L} = \left(\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}\right)\delta\varphi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}(\partial_\mu\delta\varphi_i) = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}\delta\varphi_i\right) \tag{3.56}$$

If we define the current

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}\delta\varphi_i(x) - K^\mu, \tag{3.57}$$

then

$$\partial_\mu j^\mu = \delta\mathcal{L} - \delta\mathcal{L} = 0. \tag{3.58}$$

This is Nöther's theorem; a continuous symmetry implies the existence of a conserved current.

The current flux through some spacelike surface V defines a conserved charge. The surface of constant time in some reference frame has the normal vector $n_\mu = (1, 0, 0, 0)$, so the charge is

$$Q = \int_V \mathrm{d}^4x\, n_\mu j^\nu = \int_V \mathrm{d}^3x\, j^0. \tag{3.59}$$

We can then use the divergence theorem. Assume ∂V is the boundary of V, which has the space-like normal vector k_i, and that the current falls of quickly towards infinity. Then

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = -\int_V \mathrm{d}^3x\, \partial_i j^i = -\int_{\partial V} \mathrm{d}^2x\, k_i j^i = 0, \tag{3.60}$$

proving that the charge is conserved.

3.3.2 Goldstone's theorem

A symmetry transformation will leave the governing equation of a theory unchanged. This, however, does not imply that physical states, such as the ground state, are invariant under this transformation. The $N = 2$ linear sigma model illustrates this. If we assume the ground state φ_0 is translationally invariant, then it is given by minimizing the effective potential, of which the classical potential, $\mathcal{V}$, is the leading order approximation. This potential is illustrated in Figure 3.1. The ground state is therefore given by any of the values along the brim of the potential. If we, without loss of generality, choose $\varphi_0 = (0, v)$ as the ground state, then any rotation will change this state. We say that the symmetry has been *spontaneously broken*.

To explore this in a general context, assume a theory of N real scalar fields φ_i are invariant under the actions of some Lie group, G. A symmetry $g \in G$ is broken if the vacuum expectation value of the original fields and the transformed fields differ. That is, if

$$\langle\varphi\rangle_0 \neq \langle\varphi'\rangle_0 = \langle g\varphi\rangle_0 \tag{3.61}$$

We can now exploit what we learned about Lie groups to write the infinitesimal transformation as

$$\langle\varphi'\rangle_0 = \langle\varphi\rangle_0 + i\epsilon\eta_\alpha T_\alpha \langle\varphi\rangle_0 . \tag{3.62}$$

Let x_i be the set of generators corresponding to broken symmetries, i.e.,

$$x_i \langle\varphi\rangle_0 \neq 0. \tag{3.63}$$

These are called the *broken generators*. The remaining set of generators t_a, which obey

$$t_a \langle\varphi\rangle_0 = 0, \tag{3.64}$$

are called unbroken, and generate a subgroup $H \subset G$ as the set of symmetry transformations of the vacuum is a group.

In Eq. (3.48) we found that, if V is the generator of some symmetry, then the effective action obeys

$$\int \mathrm{d}^4x\, \frac{\delta\Gamma[\varphi_J]}{\delta\varphi_i} V_{ij} \langle\varphi_j\rangle_0 = 0, \tag{3.65}$$

We now differentiate this expression with respect to $\varphi_k(y)$ and evaluate it in the vacuum, which gives

$$\int \mathrm{d}^4x\, \frac{\delta\Gamma[\varphi_0]}{\delta\varphi_k(y), \varphi_i(x)} V_{ij} \langle\varphi_j\rangle_0 = 0. \tag{3.66}$$

With the assumption that the ground state is constant, we get

$$\frac{\partial^2\mathcal{V}_{\text{eff}}}{\partial\varphi_k\, \partial\varphi_i} V_{ij} \langle\varphi_j\rangle_0 = 0. \tag{3.67}$$

This is trival for unbroken symmetries, as $t^a_{ij} \langle \varphi_j \rangle_0 = 0$ by definition. However, in the case of a broken symmetry, the second derivative of the effective potential has an eigenvector $x^\ell_{ij} \langle \varphi_j \rangle_0$ with a zero eigenvalue for each broken generator. Here, ℓ label the set of generators, while (ij) are the indices corresponding to field-components φ_i. In Eq. (3.25), we found that the second derivative of the effective action is the inverse propagator,

$$D^{-1}_{ij}(x,y) = \frac{\delta\Gamma[\varphi_0]}{\delta\varphi_i(y),\varphi_j(x)} = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} e^{-ip(x-y)}\, \tilde{D}^{-1}_{ij}(p). \tag{3.68}$$

Using this, we can write

$$\tilde{D}^{-1}_{ij}(p=0)\, x^\ell_{jk} \langle \varphi_k \rangle_0 = 0. \tag{3.69}$$

Zeros of the inverse propagator correspond to the physical mass of particles. In Lorentz-invariant systems, each zero-eigenvalue vector corresponds to a masses particle, called a Goldstone boson.[3] This means there are $n_G = \dim G - \dim H$ zero-mass modes. In general, the counting of massless modes is complicated and depends on the dispersion relation of the particles at low momenta. Systems with Goldstone bosons with quadratic dispersion relation, that is $E \propto |\vec{p}|^2$ when $\vec{p} \to 0$, often exhibit a lower number of massless modes. An example is ferromagnets, where the SU(2) rotational symmetry is broken down to U(1) when they align along one axis. This corresponds to two broken generators, yet the system exhibits only one massless mode [8].

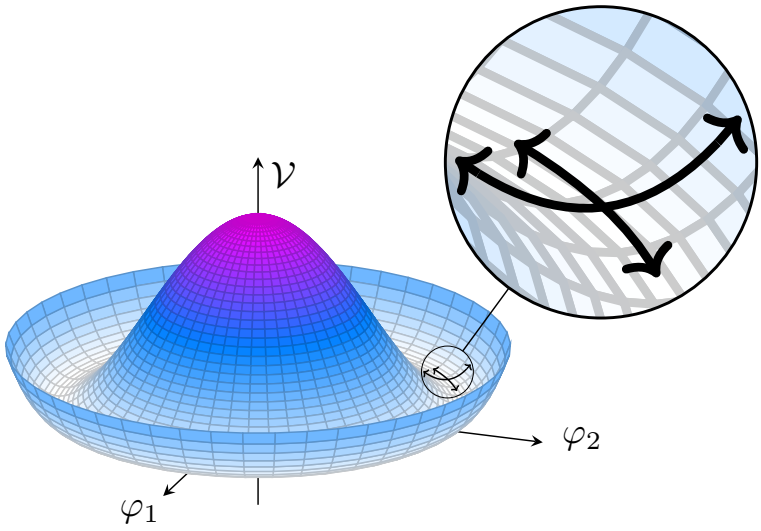


Figure 3.2: Excitations along the brim does not cost any energy, as the potential is flat, unlike excitations in the radial direction.

The linear sigma model gives an intuition for the Goldstone mode. In the case of $N = 2$, the symmetry of the Lagrangian are rotations in the plane. As the ground state is a point along the "brim" of the hat, this rotational symmetry is broken. However, any excitations in the angular direction do not cost any energy, which is indicative of a massless mode. This is illustrated in Figure 3.2. In this example, the original symmetry group is one-dimensional, so there are no unbroken symmetries. Consider instead the $N = 3$ linear sigma model, which has the three-dimensional symmetry group SO(3), rotations of the sphere. We see that the ground state is left invariant under a subgroup of the original symmetry transformations. The ground state manifold of this system, the set of all states that minimizes the effective potential, is then a sphere. When the system chooses one single ground state, this symmetry is broken, but only for two of the generators. The generator for rotations around the ground state leaves that point unchanged and is thus an unbroken symmetry. Any excitations in the direction of the broken symmetries do not cost energy, as it is in the ground state manifold. On the other hand, the unbroken symmetry does not correspond to an excitation. This is illustrated in Figure 3.3.

skriv om schwinger-dyson/Ward-takashi i symmetri

[3] The particles are bosons due to the bosonic nature of the transformations, g. If the generators are Grassmann numbers, the resulting particles, called goldstinos, are fermions.

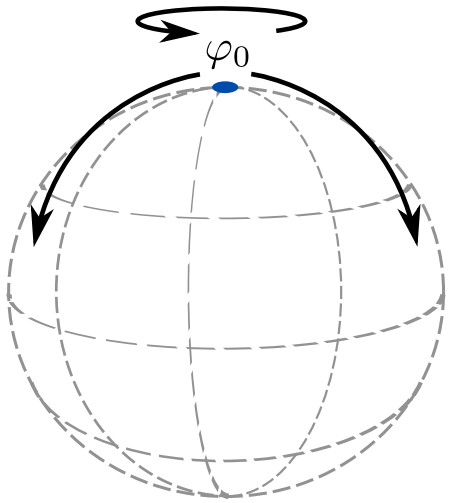


Figure 3.3: Excitations for the $N = 3$ sigma model. Two of the symmetries are broken, while rotations around the groundstate leaves the system unchanged.

3.4 *CCWZ construction

As Goldstone bosons are massless, they play a crucial role in low-energy dynamics. To best describe this limit, we seek a parametrization of the theory in which they are the degrees of freedom. This can be done using the CCWZ construction, named after Callan, Coleman, Wess, and Zumino. This section is based on [7, 9–11], as well as the original papers [12, 13].

We saw that the Goldstone bosons correspond to excitations within the vacuum manifold. The vacuum manifold corresponds to points in field space φ that can be reached from the vacuum φ_0 with a transformation $g \in G$. Assume this group acts linearly on the fields. This means that we can write such excitations as

$$\varphi_i = (\tilde{\Sigma}\varphi_0)_i = \tilde{\Sigma}_{ij}(\varphi_0)_j, \quad \tilde{\Sigma} = \tilde{\Sigma}(\eta) = \exp\{i\eta_\alpha T_\alpha\} \tag{3.70}$$

We will drop the indices for the sake of compact notation. $\tilde{\Sigma}$ is thus a function from the parameter space, $\eta_\alpha \in \mathbb{R}^n$, to G,

$$\tilde{\Sigma} : \mathbb{R}^n \longmapsto G. \tag{3.71}$$

We then get space-time-dependent field configurations by making the parameters dependent on space-time. We will for now assume η_α is constant. This parametrization is highly redundant. Two elements $\tilde{\Sigma}$ and $\tilde{\Sigma}'$, related by

$$\tilde{\Sigma}' = \tilde{\Sigma}e^{i\theta_a t_a} \tag{3.72}$$

results in the same φ. This is because $e^{i\theta_a t_a} = h \in H$, and $h\varphi_0 = \varphi_0$, by assumption. The set of all equivalent $\tilde{\Sigma}$'s is exactly the left coset, $gH = \{\, gh \mid h \in H \,\}$. The set of cosets forms a new manifold, G/H, called the Goldstone manifold. This is a manifold of dimension $\dim(G/H) = \dim(G) - \dim(H)$, which is the number of broken generators and thus also the number of Goldstone modes. Membership of a certain coset form an equivalence relation, $g \sim g'$ if $g' = gh$. This means that the cosets gH form a partition of G and that each element $g \in G$ belongs to one, and only one, coset. To remove the redundancy in the parametrization, we need to choose one representative element from each coset.

By the inverse function theorem, any mapping between manifolds $f : \mathcal{M} \mapsto \mathcal{N}$ that has a non-degenerate differential, that is an invertible Jacobian, at a point $p \in \mathcal{M}$, is invertible in a neighborhood of p. If we write

$$\tilde{\Sigma}(\xi, \theta) = \exp\{i\xi_i x_i\} \exp\{i\theta_a t_a\}, \tag{3.73}$$

then the map is invertible at $p = (\xi_i = 0, \theta_a = 0)$, as the Jacobian is the identity matrix. This point is mapped to the identity element of G. This means that, in a neighborhood $U \subset G$ of the identity, each element g has a unique representation $g = \tilde{\Sigma}$ [**leeSmoothManifolds2012**]. Furthermore, two elements $\tilde{\Sigma}'$

and $\tilde{\Sigma}$ related by $\tilde{\Sigma}' = \tilde{\Sigma}h$, $h \in H$ have the same ξ-arguments. We see that ξ_i parametrize G/H, in the neighborhood of the identity. We therefore demand that $\tilde{\Sigma}$ always appear in the standard form

$$\Sigma(\xi) = \tilde{\Sigma}(\xi, 0) = \exp\{i\xi_i x_i\}. \tag{3.74}$$

The field $\varphi(x)$ can therefore be written as

$$\varphi(x) = \Sigma(x)\varphi_0 = \exp\{i\xi_i(x)x_i\}\varphi_0, \tag{3.75}$$

and $\xi_i(x)$ can be associated with the Goldstone bosons.

In the linear sigma model, the original O(N) symmetry is broken down to O($N-1$), which transforms the remaining $N-1$ fields with vanishing expectation values into each other. However, O(N) consists of two disconnected subsets, those matrices with determinant 1 and those with determinant -1. There is no continuous path that takes an element of O(N) with determinant of -1 to an element with determinant 1.[4] The set of symmetries that are connected to the identity is

$$G = SO(N) = \{\, M \in \mathrm{O}(N) \mid \det M = 1 \,\}. \tag{3.76}$$

If we choose $\varphi_0 = (0, 0, ..., v)$, then it is apparent that the ground state is invariant under the rotations of the $N-1$ first fields, so the unbroken symmetry is $H = \mathrm{SO}(N-1)$. The Goldstone manifold is $G/H = \mathrm{SO}(N)/\mathrm{SO}(N-1)$.

Consider the case of $N = 3$, which is illustrated in Figure 3.3. G is the rotations of the sphere, while H is rotations around φ_0, SO(2). The Goldstone manifold consists of the rotations of φ_0 to other points of the sphere, i.e. $G/H = \mathrm{SO}(3)/\mathrm{SO}(2) = S^2$, the 2-sphere. This is not a Lie group, as translating φ in a closed path around the sphere may result in a rotation around the z-axis. This is illustrated in Figure 3.4

To check that ξ_i, in fact, are the Goldstone modes, we study the way they appear in the Lagrangian. As they are massless, no mass term of the form $M_{ij}\xi_i\xi_j$ should appear. The original Lagrangian $\mathcal{L}[\varphi]$ was invariant under global transformations $\varphi(x) \mapsto g\varphi(x)$. However, any terms that only depend on $\varphi(x)$, and not its derivatives, will also be invariant under a *local* transformation, $\varphi(x) \mapsto g(x)\varphi(x)$. Our parametrization of the fields, $\varphi(x) = \Sigma(x)\varphi_0$ is exactly such a transformation, which means that any such terms are independent of the Goldstone bosons. We can therefore write

$$\mathcal{L}[\varphi] = \mathcal{L}_{\mathrm{kin}}[\varphi] + V(\varphi_0), \tag{3.77}$$

where all terms in $\mathcal{L}_{\mathrm{kin}}$ are proportional to at least one derivative term, $\partial_\mu\varphi(x)$. Inserting the parametrization into this derivative term, we get

$$\partial_\mu\varphi(x) = \partial_\mu[\Sigma(x)\varphi_0] = \Sigma(x)[\Sigma(x)^{-1}\partial_\mu\Sigma(x)]\varphi_0. \tag{3.78}$$

The Lagrangian will therefore depend on ξ_i via terms of the form $\Sigma(x)^{-1}\partial_\mu\Sigma(x)$, which is called the Mauer-Cartan form. This is a $\mathfrak{g}$-valued function, which means that it can be written as

$$i\Sigma(x)^{-1}\partial_\mu\Sigma(x) = d_\mu(x) + e_\mu(x), \tag{3.79}$$

$$d_\mu = ix_i d_{ij}(\xi)\partial_\mu\xi_j, \tag{3.80}$$

$$e_\mu = it_a e_{ai}(\xi)\partial_\mu\xi_i, \tag{3.81}$$

where d_{ij} and e_{ai} are as-of-yet unknown real valued functions of ξ [7, 14].

3.4.1 Transformation properties of Goldstone bosons

We can deduce how the Goldstone bosons transforms under G from how φ transforms. In general,

$$\varphi' = g\varphi = (g\Sigma(\xi))\varphi_0 = \Sigma(\xi')\varphi_0 \quad g \in G. \tag{3.82}$$

While $\Sigma(\xi')$ has the standard form by assumption,

$$\Sigma(\xi') = \exp\{i\xi_i' x_i\}, \tag{3.83}$$

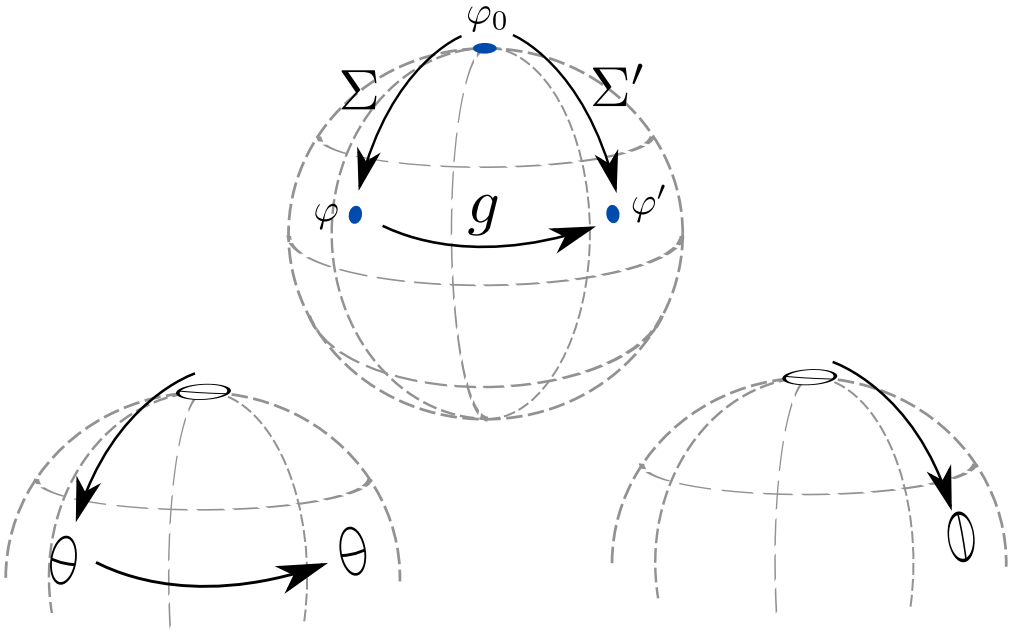


Figure 3.4: The top figure illustrates the transformation of φ_0 to φ and then φ', and the alternative, direct transformation $\varphi_0 \to \varphi'$. The bottom figure illustrates how this can rotate a neighborhood of φ_0 differently.

$g\Sigma(\xi)$ does not, in general.

Figure 3.4 illustrates this in the case of $G = \mathrm{SO}(3)$. $\Sigma(\xi)$ transforms φ_0 to φ, then g transforms φ to $\varphi' = \Sigma(\xi')\varphi_0$. Assuming φ and φ' are close enough to φ_0, we can write $\Sigma(\xi)$ and $\Sigma(\xi')$ on the standard form. However, if we follow a small neighborhood around φ_0 as it is acted on by $\Sigma(\xi)$, then g, it will be rotated by the time it arrives at φ' when compared to the same neighborhood if it was acted on by $\Sigma(\xi')$.

$g\Sigma(\xi)$ and $\Sigma(\xi')$ are in the same coset, as they by assumption corresponds to the same physical state. This means that we can write $g\Sigma(\xi) = \Sigma(\xi')h(g,\xi)$, where $h(g,\xi) \in H$. The transformation rule of ξ under G is therefore implicitly defined by

$$\Sigma(\xi') = g\Sigma(\xi)[h(g,\xi)]^{-1}. \tag{3.84}$$

This is, in general, not a linear representation, which is why this construction also is called a *non-linear realization*. Using the transformation rule, we can obtain the transformation rule of the Maurer-Cartan form. We use the shorthand $\Sigma = \Sigma(\xi)$, $\Sigma' = \Sigma(\xi')$, and $h = h(g,\xi)$. This gives

$$\begin{aligned}
\Sigma^{-1}\partial_\mu\Sigma \to \Sigma'^{-1}\partial_\mu\Sigma' &\\
&= (g\Sigma h^{-1})^{-1}\partial_\mu(g\Sigma h^{-1}) \\
&= (h\Sigma^{-1}g^{-1})g[(\partial_\mu\Sigma)h^{-1} + \Sigma\partial_\mu h^{-1}] \\
&= h\Sigma^{-1}(\partial_\mu\Sigma)h^{-1} + h\partial_\mu h^{-1} \\
&= h(\Sigma^{-1}\partial_\mu\Sigma + \partial_\mu)h^{-1}.
\end{aligned}$$

In terms of d_μ and e_μ,

$$d_\mu \to h d_\mu h^{-1} \tag{3.85}$$

$$e_\mu \to h(e_\mu + i\partial_\mu)h^{-1}. \tag{3.86}$$

These are our building blocks for constructing a general, G-invariant effective Lagrangian. The trace of a product of d_μ's are invariant under G,

$$\mathrm{Tr}\{d_\mu d_\nu \dots d_\rho\} \to \mathrm{Tr}\{h d_\mu h^{-1} h d_\nu h^{-1} h \dots d_\rho h^{-1}\} = \mathrm{Tr}\{d_\mu d_\nu \dots d_\rho\}, \tag{3.87}$$

[4] A simple proof of this is the fact that the determinant is a continuous function, while any path $\det M(t)$ such that $\det M(1) = -1$, $\det M(0) = 1$ must make a discontinuous jump.

where we have used the cyclic property of trace. However, the terms must also obey the other symmetries of the Lagrangian, such as C or P-parity and Lorentz invariance. The last criterion excludes any terms with free space-time indices. In section 5.2, we will construct an effective Lagrangian in powers of d. The lowest order terms are therefore

$$\mathrm{Tr}\{d_\mu\}\,\mathrm{Tr}\{d^\mu\}, \quad \mathrm{Tr}\{d_\mu d^\mu\}. \tag{3.88}$$

We see that e_μ transforms like a gauge field, with the gauge group H. If we include massive degrees of freedom and not only the Goldtone modes, e_μ is used to create a covariant derivative of the massive modes. We are only interested in the Goldstone modes and will therefore be satisfied with d_μ. With these tools, we can create an effective theory of quantum chromodynamics at low energies.

Chapter 4

General relativity and the TOV equation

General relativity describes how matter and energy curve the fabric of space and time. Einstein first wrote down the theory more than a century ago, and it is still our most accurate theory of gravitational effects. It makes accurate and counterintuitive predictions, which experiments have borne out. This chapter surveys the basics of general relativity. We will then use this to derive the Tolman-Oppenheimer-Volkoff (TOV) equation, a differential equation used to model stars.

(Newtonian gravity??)

Short about newtonian gravity/motivation

4.1 General relativity

This section is based on [2]. The derivation of the spherically symmetric metric is done using computer code, as descirbed in Appendix C.

4.1.1 Einstein's field equations

General relativity describes spacetime as a smooth manifold $\mathcal{M}$, with a (pseudo-Riemannian) metric, $g_{\mu\nu}$. This metric is treated as a dynamical field, which is affected by the presence of matter and energy. The matter and energy contents of spacetime are encoded in the stress-energy tensor $T_{\mu\nu}$, while the behavior of $g^{\mu\nu}$ is encoded in a scalar Lagrangian density. Some of the mathematics used in this section, such as functional derivatives, are covered in section A.2.

The most obvious—and correct—choice as the Lagrangian for $g^{\mu\nu}$ is the Ricci scalar, which results in the Einstein-Hilbert action,

$$S_{\mathrm{EH}} = \frac{1}{2\kappa} \int_{\mathcal{M}} \mathrm{d}^n x \, \sqrt{|g|} \, R. \tag{4.1}$$

The $\sqrt{|g|}$-factor is included for the integral to be coordinate-independent, as discussed in subsection 2.1.4.[1] The κ is a constant and decides how strong the coupling of gravity to matter and energy is. This constant can then be related to Newton's constant of gravitation G by $\kappa = 8\pi G$. When including the contributions from other fields with an action S_{m}, the total action becomes

$$S = S_{\mathrm{EH}} + S_{\mathrm{m}}. \tag{4.2}$$

[1] The gravitational action can also include a cosmological constant, modifying the Lagrangian to $R + 2\Lambda$. This constant does not affect the subject of this thesis and is therefore not included here.

The equations of motion of the dynamical field, which in this case is the metric, are given by Hamilton's principle of stationary action. Using functional derivatives, as defined in subsection A.2.3, this is stated as

$$\frac{\delta S}{\delta g^{\mu\nu}} = 0, \tag{4.3}$$

We define the stress-energy tensor as

$$T_{\mu\nu} = -2\frac{\delta S_{\mathrm{m}}}{\delta g^{\mu\nu}}. \tag{4.4}$$

The functional derivative of the Einstein-Hilbert action is evaluated in subsection A.2.4, and with the result, Eq. (A.50), we get the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu}, \tag{4.5}$$

The left-hand side of the Einstein field equations is called the Einstein tensor, $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$. This tensor obeys the identity

$$\nabla^{\mu} G_{\mu\nu} = 0, \tag{4.6}$$

as a consequence of the more general Bianchi identity, Eq. (2.44).

4.1.2 Spherically symmetric spacetime

To model stars, we will assume that the metric is spherically symmetric and time-independent. In this case, the most general metric can be written, at least locally, as [2]

$$\mathrm{d}s^2 = e^{2\alpha(r)}\mathrm{d}t^2 - e^{2\beta(r)}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2), \tag{4.7}$$

where α and β are general functions of the radial coordinate r. In matrix form, this corresponds to

$$g_{\mu\nu} = \begin{pmatrix} e^{2\alpha(r)} & 0 & 0 & 0 \\ 0 & -e^{2\beta(r)} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2(\theta) \end{pmatrix}. \tag{4.8}$$

Using Eq. (2.35), we can now compute the Christoffel symbols in terms of the unknown functions. These computations in this subsection are done using computer code, which is shown in Appendix C. The results are

$$\Gamma^{t}_{\mu\nu} = \begin{pmatrix} 0 & \frac{\mathrm{d}}{\mathrm{d}r}\alpha(r) & 0 & 0 \\ \frac{\mathrm{d}}{\mathrm{d}r}\alpha(r) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.9}$$

$$\Gamma^{r}_{\mu\nu} = \begin{pmatrix} e^{2\alpha(r)}e^{-2\beta(r)}\frac{\mathrm{d}}{\mathrm{d}r}\alpha(r) & 0 & 0 & 0 \\ 0 & \frac{\mathrm{d}}{\mathrm{d}r}\beta(r) & 0 & 0 \\ 0 & 0 & -re^{-2\beta(r)} & 0 \\ 0 & 0 & 0 & -re^{-2\beta(r)}\sin^2(\theta) \end{pmatrix}, \tag{4.10}$$

$$\Gamma^{\theta}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\sin(\theta)\cos(\theta) \end{pmatrix}, \tag{4.11}$$

$$\Gamma^{\phi}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \frac{\cos(\theta)}{\sin(\theta)} \\ 0 & \frac{1}{r} & \frac{\cos(\theta)}{\sin(\theta)} & 0 \end{pmatrix}. \tag{4.12}$$

The symbols not included are zero. Substituting these results into Eq. (2.39) gives the Riemann tensor curvature tensor. We can then obtain the Ricci tensor by taking the trace, as shown in Eq. (2.40). The results are

$$R_{tt} = \left[r\left(\frac{\mathrm{d}}{\mathrm{d}r}\alpha(r)\right)^2 - r\frac{\mathrm{d}}{\mathrm{d}r}\alpha(r)\frac{\mathrm{d}}{\mathrm{d}r}\beta(r) + r\frac{\mathrm{d}^2}{\mathrm{d}r^2}\alpha(r) + 2\frac{\mathrm{d}}{\mathrm{d}r}\alpha(r)\right]\frac{e^{2\alpha(r)}e^{-2\beta(r)}}{r}, \tag{4.13}$$

$$R_{rr} = -\frac{1}{r}\left[r\left(\frac{\mathrm{d}}{\mathrm{d}r}\alpha(r)\right)^2 - r\frac{\mathrm{d}}{\mathrm{d}r}\alpha(r)\frac{\mathrm{d}}{\mathrm{d}r}\beta(r) + r\frac{\mathrm{d}^2}{\mathrm{d}r^2}\alpha(r) - 2\frac{\mathrm{d}}{\mathrm{d}r}\beta(r)\right], \tag{4.14}$$

$$R_{\theta\theta} = -\left[r\frac{\mathrm{d}}{\mathrm{d}r}\alpha(r) - r\frac{\mathrm{d}}{\mathrm{d}r}\beta(r) - e^{2\beta(r)} + 1\right]e^{-2\beta(r)}, \tag{4.15}$$

$$R_{\varphi\varphi} = R_{\theta\theta}\sin^2(\theta). \tag{4.16}$$

All other components are zero. Finally, the trace of the Ricci tensor gives the Ricci scalar,

$$R = \frac{2e^{-2\beta(r)}}{r^2}\left[r^2\left(\frac{\mathrm{d}}{\mathrm{d}r}\alpha(r)\right)^2 - r^2\frac{\mathrm{d}}{\mathrm{d}r}\alpha(r)\frac{\mathrm{d}}{\mathrm{d}r}\beta(r) + r^2\frac{d^2}{dr^2}\alpha(r) + 2r\frac{\mathrm{d}}{\mathrm{d}r}\alpha(r) - 2r\frac{\mathrm{d}}{\mathrm{d}r}\beta(r) - e^{2\beta(r)} + 1\right]. \tag{4.17}$$

The unknown functions α and β are now determined by the matter and energy content of the universe, which is encoded in $T_{\mu\nu}$, through Einstein's field equation, Eq. (4.5).

4.1.3 The Schwarzschild metric

The simples case for a matter distribution in spacetime is $T_{\mu\nu} = 0$. Although this might only seem to be useful to model a non-empty universe, it can be combined with a central point particle and empty space elsewhere. In this case, the Einstein equations are simply $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0$. We can show that the trace of the Ricci tensor is zero by taking the trace of this equation, simplifying it to $R_{\mu\nu} = 0$. By combining Eq. (4.13) and Eq. (4.14), we find

$$R_{tt} + e^{2(\alpha-\beta)}R_{rr} = 2\frac{\mathrm{d}}{\mathrm{d}r}(\alpha+\beta) = 0, \tag{4.18}$$

which implies $\alpha = -\beta + \text{const}$. The constant corresponds to rescaling the coordinates, which allows us to set it to zero. From Eq. (4.15), we get

$$e^{2\beta}R_{\theta\theta} = -2r\frac{\mathrm{d}}{\mathrm{d}r}\alpha - e^{-2\alpha} + 1 = 0, \tag{4.19}$$

which may be restated as

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(re^{2\alpha}\right) = 1, \tag{4.20}$$

This equation has the solution

$$e^{2\alpha(r)} = e^{-2\beta(r)} = \left(1 - \frac{R_s}{r}\right), \tag{4.21}$$

where R_s, the Schwarzschild radius, is a constant. Using the weak field limit, we can match the solution to Newtonian gravity and show that $R_s = 2GM$, where G is Newton's constant of gravitation and M is the mass of the point particle. The metric is then

Gjøre weak field limit?

$$\mathrm{d}s^2 = \left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right). \tag{4.22}$$

4.2 The TOV equation

This section is in part based on [2, 15]. We will model a star as being made up of a *perfect fulid*, which is entirely described by its energy density u and pressure p. The relationship between the pressure and energy density of a substance is called the *equation of state*, or EOS, and has the form

$$f(p, u, \{\xi_i\}) = 0, \tag{4.23}$$

where $\{\xi_i\}$ are possible other thermodynamic variables. We will be working at zero temperature, in which case there are no other free thermodynamic variables. This allows us to, at least locally, express the energy density as a function of the pressure, $u = u(p)$. The stress-energy tensor of a perfect fluid is

Forklar

$$T_{\mu\nu} = (u + p)u_\mu u_\nu - p g_{\mu\nu}, \tag{4.24}$$

where u_μ is the 4-velocity of the fluid. In the rest frame of the fluid, we may write

$$v_\mu = (v_0, 0, 0, 0)\,. \tag{4.25}$$

This, together with the normalization condition of 4-velocities, $v_\mu v^\mu = 1$, allows us to calculate that

$$v_\mu v^\mu = g^{\mu\nu} v_\mu v_\nu = g^{00}(v_0)^2 = 1. \tag{4.26}$$

Using Eq. (4.8), we see that

$$v_0 = e^{\alpha(r)}. \tag{4.27}$$

This gives us the stress-energy tensor of the perfect fluid in its rest frame,

$$T_{\mu\nu} = \begin{pmatrix} u(r)e^{2\alpha(r)} & 0 & 0 & 0 \\ 0 & p(r)e^{2\beta(r)} & 0 & 0 \\ 0 & 0 & p(r)r^2 & 0 \\ 0 & 0 & 0 & p(r)r^2\sin^2(\theta) \end{pmatrix}. \tag{4.28}$$

We will use the tt and rr components of the Einstein field equations, which are

$$8\pi G r^2 u(r) e^{2\beta(r)} = 2r\frac{\mathrm{d}}{\mathrm{d}r}\beta(r) + e^{2\beta(r)} - 1, \tag{4.29}$$

$$8\pi G r^2 p(r) e^{2\beta(r)} = 2r\frac{\mathrm{d}}{\mathrm{d}r}\alpha(r) - e^{2\beta(r)} + 1. \tag{4.30}$$

In analogy with the Schwarzschild metric, we define the function $m(r)$ by

$$e^{2\beta(r)} = \left(1 - \frac{2Gm(r)}{r}\right)^{-1}. \tag{4.31}$$

Substituting this into Eq. (4.29) yields

$$\frac{\mathrm{d}m(r)}{\mathrm{d}r} = 4\pi r^2\, u(r). \tag{4.32}$$

The solution is simply

$$m(r) = 4\pi \int_0^r \mathrm{d}r'\, r'^2 u(r'). \tag{4.33}$$

In flat spacetime, we would have no qualms simply calling this the mass contained within a radius r. However, as discussed in section 2.1, the volume element of a curved geometry is $\mathrm{d}V = \mathrm{d}^n x\sqrt{|g|}$. In this case, we are interested in the mass contained in a 3-volume, and the volume form is therefore given by the metric restricted by $\mathrm{d}t = 0$. Using Eq. (4.8), $\sqrt{|g|} = e^\beta r^2 \sin\theta$, the total mass-energy contents of the star is

$$M' = 4\pi \int \mathrm{d}r'\, r'^2 \left(1 - \frac{2Gm(r')}{r'}\right)^{-1/2} u(r'). \tag{4.34}$$

However, this does not take into account the gravitational potential energy. Gravitation is self-interacting, and we must therefore include the gravitational potential energy when calculating gravitational effects. It

can be shown that the definition of *gravitational mass*, Eq. (4.33), does exactly this. Furthermore, as we will see later, it matches up with what we call mass in the Newtonian limit [16].

Using the Bianchi identity, Eq. (4.6), together with Einstein's equation, we find

$$\nabla^\mu G_{\mu\nu} = \nabla^\mu T_{\mu\nu} = 0. \tag{4.35}$$

The r-component of this equation is

$$\begin{aligned}\nabla_\mu T^{\mu r} &= \partial_r T^{rr} + \Gamma^\mu_{\mu\nu} T^{\nu r} + \Gamma^r_{\mu\nu} T^{\mu\nu} \\ &= \partial_r \left(p e^{-2\beta}\right) + (2\Gamma^r_{rr} + \Gamma^t_{tr}) T^{rr} + \Gamma^r_{tt} T^{tt} \\ &= e^{-2\beta} \left(\partial_r p + p \partial_r \alpha + u \partial_r \alpha\right) = 0.\end{aligned}$$

This allows us to relate α to p and u, via

$$\frac{\mathrm{d}\alpha}{\mathrm{d}r} = -\frac{1}{p+u}\frac{\mathrm{d}p}{\mathrm{d}r} \tag{4.36}$$

When we substitute this, together with the definition of $m(r)$, into Eq. (4.30), we obtain

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{G}{r^2}\left(4\pi r^3 p + m\right)(p+u)\left(1 - \frac{2Gm}{r}\right)^{-1}, \tag{4.37}$$

the Tolman-Oppenheimer-Volkoff (TOV) equation. This equation was first obtained by Oppenheimer and Volkoff in 1939 [17] and was based on earlier work by Tolman [18]. In their paper, Oppenheimer and Volkoff studied the properties of a star made up of cold, degenerate fermions.

To summarize, we have three unknown functions, $u(r)$, $p(r)$, and $m(r)$. The equation of state, Eq. (4.23), determines $u = u(p)$, eliminating one unknown. The two differential equations Eq. (4.33) and Eq. (4.37), together with the boundary conditions $p(0) = p_c$ and $m(0) = 0$, then yield $p(r)$ and $m(r)$ when integrated. As long as both the pressure and the energy density are positive, and we always are outside the Schwarzschild radius, i.e., $r < 2Gm(r)$, then $\mathrm{d}p/\mathrm{d}r \leq 0$ and the pressure is strictly decreasing. We define the point where the pressure vanishes as the stellar radius R, i.e., $p(R) = 0$. Given this, we can solve for all the unknown functions, either analytically or numerically.

With $p(r)$, $u(r)$, and $m(r)$, we can construc the metric. We already have the rr-component of the metric from Eq. (4.31). If we combine Eq. (4.36), with Eq. (4.37), we get the solution

$$\alpha(r) = G\int^r \mathrm{d}r\, \frac{1}{r^2}(4\pi r^3 p + m)\left(1 - \frac{2Gm}{r^2}\right)^{-1}. \tag{4.38}$$

Outside the star, we have $p(r) = 0$, and $m(r) = M$. This then reduces to

$$\alpha(r) = GM\int^r \mathrm{d}r\, \frac{1}{r^2}\left(1 - \frac{2GM}{r}\right)^{-1}. \tag{4.39}$$

We can evaluate this integral by making the substitution $x = (1 - 2GM/r)$, $\mathrm{d}x = -2GM/r^2\mathrm{d}r$,

$$\alpha(r) = \frac{1}{2}\int^{1-\frac{2GM}{r}} \frac{\mathrm{d}x}{x} = \frac{1}{2}\ln\left(1 - \frac{2GM}{r}\right) + \text{const.} \tag{4.40}$$

We then impose the boundary condition $\alpha(\infty) = 0$, which means setting the constant to zero. Inserting this into Eq. (4.8) gives the metric for $r < R$,

$$\mathrm{d}s^2 = \left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\mathrm{d}\varphi^2\right). \tag{4.41}$$

We recognize this as the Schwarzchild metric, Eq. (4.22). This justifies our choice of $m(r)$ as gravitational mass. As discussed earlier, the quantity M in the Schwarzschild solution maps onto what we know as mass in the weak-field limit.

We can gain some insight into the system without solving these equations by expressing the problem in terms of dimensionless variables. We define

$$u = u_0\tilde{u}, \quad p = p_0\tilde{p}, \quad m = m_0\tilde{m}, \quad r = r_0\tilde{r}. \tag{4.42}$$

Here, quantities with subscript 0 are dimensionful constants, which may be chosen as the characteristic quantities of the problem, while the tilde indicates a dimensionless variable. By substituting this into Eq. (4.32) and Eq. (4.37), we can collect the dimensionful constants into a smaller number of dimensionless constants, k_i. These constants will decide the nature of the solution. Any change in the dimensionful constants that leave the k_i's invariant is a scaling of the problem — it corresponds to the same solution with different units. The new differential equations are

$$\frac{\mathrm{d}\tilde{m}}{\mathrm{d}\tilde{r}} = 3k_2\,\tilde{r}^2\tilde{u} \tag{4.43}$$

$$\frac{\mathrm{d}\tilde{p}}{\mathrm{d}\tilde{r}} = -\frac{k_1}{k_3}\frac{1}{\tilde{r}^2}\left(k_3\tilde{p}+\tilde{u}\right)\left(3k_2k_3\tilde{r}^3\tilde{p}+\tilde{m}\right)\left(1-\frac{2k_1\tilde{m}}{\tilde{r}}\right)^{-1}, \tag{4.44}$$

where the dimensionless constants are defined as

$$k_1 = G\frac{m_0}{r_0}, \quad k_2 = \frac{4\pi}{3}\frac{r_0^3u_0}{m_0}, \quad k_3 = \frac{p_0}{u_0}. \tag{4.45}$$

The energy density and pressure are of comparable magnitude in the relativistic regime. We will therefore often choose $k_3 = 1$, defining $p_0 = u_0$. If we have a non-complete set of characteristic quantities, the dimensionless constants k_i tell us something about the magnitude we should expect the solution to have. After defining the remaining dimensionful constants by setting $k_i = 1$, we expect that the dimensionless sizes of a typical solution will be of order 1. In other words, the dimensionful constants defined by $k_i = 1$ are new, characteristic quantities given to us by the form of the governing equation only.

4.2.1 Newtonian limit and polytropes

In the Newtonian limit, the rest energy, i.e., mass, gives the dominant contribution to the gravitational field, while the contribution from pressure is negligible. In other words, the characteristic pressure, p_0, is far smaller than the characteristic energy density u_0, and we can use the approximation $k_3 \ll 1$. Furthermore, the star's radius should be much larger than the Schwarzschild radius, $R_s = 2GM$. If we choose $r_0 = R$, then $k_1 \ll 1$. In this limit, the lowest-order contribution to the TOV equation is

$$\frac{\mathrm{d}\tilde{p}}{\mathrm{d}\tilde{r}} = -\frac{k}{\tilde{r}^2}\tilde{u}\tilde{m}, \quad k = \frac{k_1}{k_3} = G\frac{u_0m_0}{p_0r_0}. \tag{4.46}$$

Using the mass equation Eq. (4.43), we can write this as

$$4\pi\tilde{r}^2\frac{\mathrm{d}\tilde{p}}{\mathrm{d}\tilde{m}} = -k'\frac{\tilde{m}}{\tilde{r}^2}, \quad k' = \frac{4\pi}{3}\frac{k_1}{k_2k_3} = G\frac{m_0^2}{r_0^4p_0}. \tag{4.47}$$

We may derive this equation directly from Newtonian gravity. Assume we have a static, gravitationally bound ball of matter, as illustrated in Figure 4.1. The force due to the pressure gradient over a thin, spherical shell, $F_p = 4\pi r^2\mathrm{d}p$, must be counteracted by the gravitational force on the same shell, $F_g = -Gm\mathrm{d}m/r^2$.

Both the Newtonian limit and the TOV equation are equations of *hydrostatic equilibrium*, where the forces on a small volume of the fluid cancel out. In the case of the TOV equation, we tacitly assumed hydrostatic equilibrium when we gave the fluid a rest frame where we could write $v_\mu = (v_0, 0, 0, 0)$ globally. We can eliminate the equation for mass by differentiating Eq. (4.46) with respect to $\tilde{r}$. This gives us a single equation for hydrostatic equilibrium in the Newtonian limit,

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{r}}\left(\frac{\tilde{r}^2}{\tilde{u}}\frac{\mathrm{d}\tilde{p}}{\mathrm{d}\tilde{r}}\right) = -k''\tilde{r}^2\tilde{u}, \quad k'' = 3\frac{k_2k_1}{k_3} = 4\pi G\frac{u_0^2r_0^2}{p_0}. \tag{4.48}$$

This is a second order differential equation, so we need an new boundary condition in addition to $p(0) = p_c$. Close to the center, we can see from Eq. (4.46) that for a finite energy density, we must have $p'(0) = 0$, our second boundary condition.

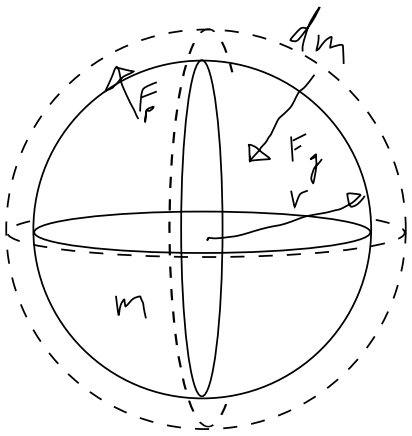


Figure 4.1: Kladd: The forces acting on a thin shell dm.

One important model for stars is the *polytrope*, which has an equation of state of the form $u = Kp^\gamma$ for some constant K. As we will see, this fits well as the Newtonian limit of many equations of state and can be used to make predictions such as the Chandrasekhar limit, which sets the upper limit of the mass of white dwarf stars to $M \approx 1.5\,M_\odot$ [15, 19]. To write Eq. (4.48) on the standard form, we assume $\gamma \neq 1$ and introduce

$$\tilde{u} = a\theta^n, \quad n = \frac{1}{\gamma - 1}, \quad a = \frac{u(p_c)}{u_0}, \quad a^{\frac{\gamma-2}{2}} C\xi = r, \quad C = \sqrt{\frac{K}{k''}\frac{\gamma}{\gamma - 1}}. \tag{4.49}$$

n is called the *polytropic index* of the star. Inserting these new variables into the equation of hydrostatic equilibrium gives

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2 \frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) + \theta^n = 0. \tag{4.50}$$

This is called the Lane-Emden equation and was first used to model stars as early as 1870 [20]. The boundary conditions $p(0) = p_c$ and $p'(0) = 0$ now read $\theta(0) = 1$ and $\theta'(0) = 0$. The stellar radius is defined by the first zero of the Lane-Emden function above $\xi = 0$, $\theta(\xi_1) = 0$, so that

$$\frac{R}{r_0} = C\xi_1 a^{\frac{\gamma-2}{2}}. \tag{4.51}$$

The total mass of the star can be integrated using Eq. (4.43) and Eq. (4.50),

$$\frac{M}{m_0} = 3k_2 a^{\frac{3\gamma-4}{2}} C^3 \int_0^{\xi_1} \mathrm{d}\xi\, \xi^2 \theta^n = [-3k_2 \xi_1^2 \theta'(\xi_1)] a^{\frac{3\gamma-4}{2}}. \tag{4.52}$$

Thus, given a specific equation of state, and thus γ, the mass-radius relationship is given by

$$R \propto M^\beta, \quad \beta = \frac{\gamma - 2}{3\gamma - 4}. \tag{4.53}$$

Figure 4.2 illustrates this relationship, in arbitrary units, for a series of different values of γ, as well as the dependence of β on γ. For most values of γ, the stellar radius will increase as the mass increases. The only range within which the radius decrease as the mass increase is $\gamma \in \left(\frac{4}{3}, 2\right)$. At $\gamma = \frac{4}{3}$ and $\gamma = 2$, respectively the mass and radius become independent of the central density. If included in our figure, these polytropes would correspond to a horizontal and a vertical line.

what about $\gamma = 1$?

4.2.2 Incompressible fluid

The simplest model for a star is one made up of an incompressible fluid, where the energy density is independent of the pressure. This corresponds to a polytrope with $\gamma = \infty$. In this case, the energy density of

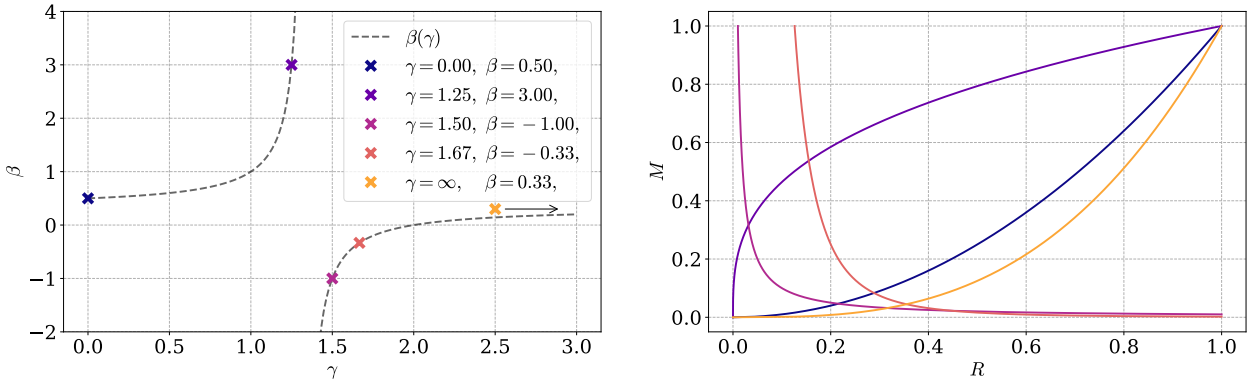


Figure 4.2: Left: The dependence of β on γ, together with a selection of points. Top: The mass-radius relation, in arbitrary units, for polytropes corresponding to the selected points on the left side. The color of the lines indicate which point it corresponds to.

the star will be constant for a radius $r < R$, before it drops to zero,

$$u(r) = u_0\,\theta(R - r), \tag{4.54}$$

where u_0 is a constant and $\theta(x)$ the Heaviside step function. We choose $r_0 = R$. Inserting this into the differential equation of the mass function, Eq. (4.43), together with the boundary condition $m(0) = 0$, yields

$$\tilde{m}(\tilde{r}) = k_2\tilde{r}^3, \tag{4.55}$$

when $r < R$. For $r \geq R$, or $\tilde{r} \geq 1$, this relationship is simply constant $\tilde{m}(\tilde{r}) = \tilde{m}(1) = k_2$. We choose m_0 to be the gravitational mass of the star, $M = \frac{4\pi}{3}R^3u_0$, which is equivalent to setting $k_2 = 1$. Lastly, we choose $u_0 = p_0$, so that $k_3 = 1$. With this the TOV equation, Eq. (4.44), becomes

$$\frac{\mathrm{d}\tilde{p}}{\mathrm{d}\tilde{r}} = -k_1\tilde{r}\frac{(1+\tilde{p})(1+3\tilde{p})}{(1-2k_1\tilde{r}^2)}. \tag{4.56}$$

This is a separable ODE, and each variable may be integrated separately. Using

$$\int \frac{\mathrm{d}x}{(1+x)(1+3x)} = \frac{1}{2}\ln\frac{3x+1}{x+1} + \text{const.}, \quad \int \mathrm{d}x\frac{x}{1-2x^2} = \frac{1}{4}\ln\left(1-2x^2\right) + \text{const.}, \tag{4.57}$$

together with the boundary condition $p(r = R) = 0$, we get

$$\tilde{p}(\tilde{r}) = -\frac{\sqrt{1-2k_1} - \sqrt{1-2k_1\tilde{r}^2}}{3\sqrt{1-2k_1} - \sqrt{1-2k_1\tilde{r}^2}}. \tag{4.58}$$

We see that the star is entirely characterized by k_1. In Figure 4.3, we have plotted the pressure as a function of radius for some values of k_1. As k_1 approaches $0.\bar{4} = 4/9$, the pressure at the center of the star increases rapidly. From the denominator of Eq. (4.58) at $r \to 0$, we find the limit

$$k_1 = G\frac{M}{R} < \frac{4}{9} \tag{4.59}$$

for the pressure to remain finite. This is an absolute limit of the mass of an object given its radius or vice versa. Although this limit is derived for a particular, unrealistic case, the more general statement can be shown to hold. General relativity does not allow for a static solution with energy densities greater than this limit; any such configuration would collapse [2].

If we expand the solution Eq. (4.58) in powers of k_1, then the leading order contribution is

$$\tilde{p}(r) = \frac{1}{2}k_1(1-\tilde{r}^2). \tag{4.60}$$

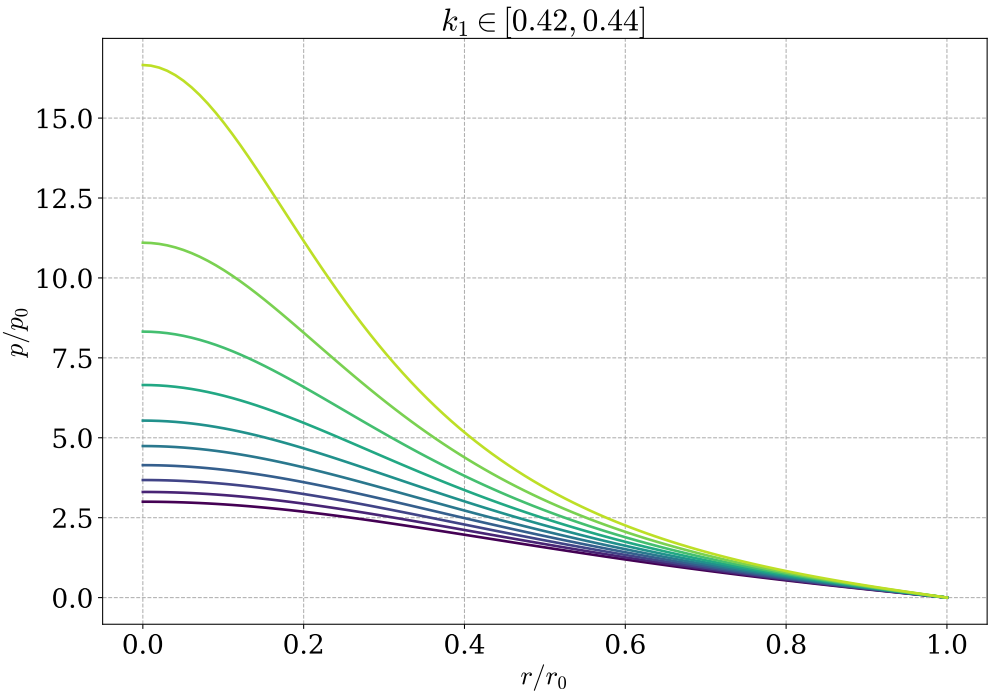


Figure 4.3: The pressure in units of p_0, as a function of the radius, in units of r_0. The graphs with lighter color and higher pressure at $r = 0$ corresponds to higher values of k_1. The values of k_1 are linearly spaced.

This is the Newtonian limit. As a cross-check, we see that this solution obeys the equation of hydrostatic equillibrium in this limit, Eq. (4.46), as $\tilde{u} = 1$ and $k_2 = k_1 = 1$. This is the general solution for an incompressible fluid in Newtonian gravity. This solution does not have any upper limit for k_1; the limit $M/R < 4/9$ is purely relativistic phenomenon. In Figure 4.4, the Newtonian approximation is compared to the full, relativistic solution. We see that the Newtonian approximation is highly accurate for k_1 less than around 0.01.

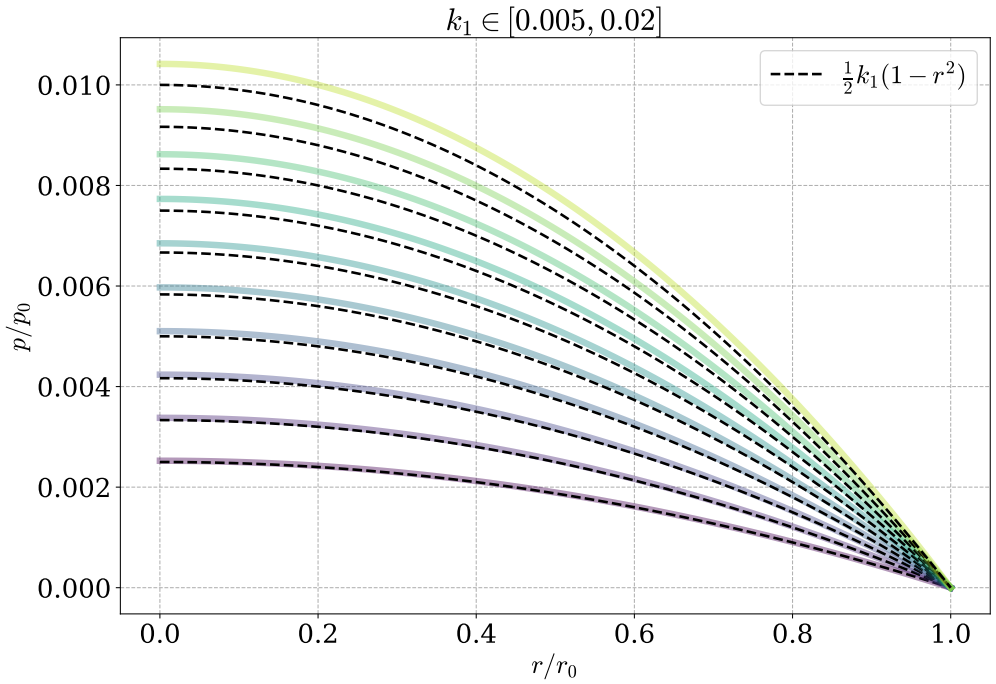


Figure 4.4: The pressure in units of p_0, as a function of the radius, in units of r_0. The wide, colored lines correspond to the full relativistic solution, while the dashed lines is the Newtonian approximation, for the same value of k_1. The values of k_1 are linearly spaced.

4.3 A star of cold, non-interacting fermions

This section is based on [15, 21, 22].

In this section, we will study a simple model of a star made up of non-interacting, cold neutrons. This is one of the earliest models used to study neutron stars, the remenants of massive stars [15]. For this model, we use results derived in section B.5.

4.3.1 Thermodynamics and the equation of state

The total energy U is related to the grand canonical free energy F by a Legendre transformation,

$$F(T, V, \mu) = U - TS - \mu Q, \quad \mathrm{d}F = -S\mathrm{d}T - p\mathrm{d}V - Q\mathrm{d}\mu. \tag{4.61}$$

Here $T = 1/\beta$ is temperature, and S entropy, p pressure, and V volume. Q is some conserved charge, in our case the number of particles minus antiparticles, and μ is the corresponding chemical potential. These thermodynamic variables are related to the free energy by

$$S = -\frac{\partial F}{\partial T} = \beta^2 \frac{\partial F}{\partial \beta}, \quad Q = -\frac{\partial F}{\partial \mu}, \quad p = -\frac{\partial F}{\partial V}. \tag{4.62}$$

When the free energy can be written as $F = V\mathcal{F}$, where the free energy density $\mathcal{F}$ is independent of the volume V, then $\mathcal{F} = -p$ and

$$\mathrm{d}(V\mathcal{F}) = V\mathrm{d}\mathcal{F} - p\mathrm{d}V, \tag{4.63}$$

allowing us to write

$$\mathcal{F}(T, \mu) = u - Ts - \mu n, \quad \mathrm{d}\mathcal{F} = -s\mathrm{d}T - n\mathrm{d}\mu, \tag{4.64}$$

where s and n are entropy and charge density, defined by

$$s = -\frac{\partial \mathcal{F}}{\partial T} = \beta^2 \frac{\partial \mathcal{F}}{\partial \beta}, \quad n = -\frac{\partial \mathcal{F}}{\partial \mu}. \tag{4.65}$$

With this, we can write the energy density as [21]

$$u = \frac{\partial}{\partial\beta}\left(\beta\mathcal{F}\right) + \mu n. \tag{4.66}$$

We calculate the free energy density of non-interacting fermions in section B.5, with the result Eq. (B.91),

$$\mathcal{F} = -\frac{2}{\beta}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\left[\beta\omega + \ln\left(1+e^{-\beta(\omega-\mu)}\right) + \ln\left(1+e^{-\beta(\omega+\mu)}\right)\right], \tag{4.67}$$

where $\omega = \sqrt{p^2+m^2}$. The first term in the integral is the divergent vacuum energy, which must be renormalized. We can drop this term; it does not have any observable effects on our results, as we are interested in relative pressure and energy density. With this, we find the charge density

$$n = \frac{1}{\pi^2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}[n_f(\omega-\mu) - n_f(\omega+\mu)], \tag{4.68}$$

where

$$n_f(\omega) = \frac{1}{e^{\beta\omega}+1}. \tag{4.69}$$

is the Fermi-Dirac distribution. Using this, we find that the energy density is

$$u = \frac{1}{\pi^2}\int_0^\infty \mathrm{d}p\, p^2\,\omega\,[n_f(\omega-\mu) + n_f(\omega+\mu)]. \tag{4.70}$$

As expected, this is the energy per mode times the density of states, integrated over all modes. To write the pressure, $p = -\mathcal{F}$ in terms of an integral over the Fermi-Dirac distribution, we integrate by parts. We have

$$\int_0^\infty \mathrm{d}p\, p^2 \ln\left[1+e^{-\beta(\omega\pm m)}\right] = \frac{1}{3}p^3\ln\left[1+e^{-\beta(\omega\pm m)}\right]\Big|_0^\infty + \frac{1}{3}\int_0^\infty \mathrm{d}p\,\frac{\beta p^4}{\omega}n_f(\omega\pm\mu), \tag{4.71}$$

where the boundary term vanish. This allows us to write the pressure as

$$p = \frac{1}{3}\int_0^\infty \mathrm{d}p\,\frac{p^4}{\omega}[n_f(\omega-\mu) + n_f(\omega+\mu)] \tag{4.72}$$

We are interested in the $T = 0$ limit. In this case, the Fermi distribution becomes a step function, $n_f(\omega) = \theta(-\omega)$. Without loss of generality, we assume that $\mu > 0$, i.e., we are dealing with an abundance of matter compared to anti-matter. The dispersion relation $\omega = \sqrt{p^2+m^2}$ is always positivive. This means that the contribution to thermodynamic quantities from anti-particles vanish, as the integral is multiplied with $n_f(\omega+\mu) = \theta(-\omega-\mu)$, where the argument $-\omega-\mu$ is strictly negative on the domain of integration. At zero temperature, the only dynamics are due to the degeneracy pressure of the fermions, that is, due to the Pauli exclusion principle. There are no thermal fluctuations that can create a particle-antiparticle pair. Thus, if the system has a positive chemical potential, it will contain no antiparticles. Furthermore, if $\mu < m$, then integrand multiplied with $n_f(\omega-\mu)$ is also zero in the whole domain of integration. It is only when $\mu \geq m$ that it is energetically favorable for the system to be in a state with particles. We define the Fermi momentum p_f by $\mu = \sqrt{{p_f}^2+m^2}$. In the zero-temperature limit, we can then rewrite any integral over the Fermi distribution as

$$\int_0^\infty \mathrm{d}p\,[f(p)n_f(\omega-\mu) + g(p)n_f(\omega-\mu)] = \int_0^{p_f}\mathrm{d}p\, f(p). \tag{4.73}$$

The charge density is thus

$$n = \frac{1}{\pi^2}\int_0^{p_f}\mathrm{d}p\,p^2 = \frac{p_f^3}{3\pi^2}. \tag{4.74}$$

At $T = 0$, this is the particle number density, as there are no antiparticles. This density is given by the chemical potential and vanishes when $\mu \leq m$, i.e. when the free energy cost of creating a particle is positive. We can write the energy density and pressure integrals, Eq. (4.70) and Eq. (4.72), as

$$u = \frac{1}{\pi^2}\int_0^{p_f}\mathrm{d}p\,p^2\sqrt{p^2+m^2} = \frac{m^4}{\pi^2}\int_0^{x_f}\mathrm{d}x\,x^2\sqrt{x^2+1}, \tag{4.75}$$

$$p = \frac{1}{3\pi^2}\int_0^{p_f}\mathrm{d}p\,\frac{p^4}{\sqrt{p^2+m^2}} = \frac{m^4}{3\pi^2}\int_0^{x_f}\frac{\mathrm{d}x\,x^4}{\sqrt{x^2+1}}. \tag{4.76}$$

We have defined $x = p/m$ and $x_f = p_f/m$. These integrals can be evaluated exactly as

$$\int_0^a \mathrm{d}x\, x^2\sqrt{x^2+1} = \frac{1}{8}\left[\sqrt{a^4+1}(2a^3+a) - \operatorname{arcsinh}(a)\right], \tag{4.77}$$

$$\int_0^a \mathrm{d}x\, \frac{x^4}{\sqrt{x^2+1}} = \frac{1}{8}\left[\sqrt{a^2+1}(2a^3-3a) + 3\operatorname{arcsinh}(a)\right]. \tag{4.78}$$

We introduce the characteristic energy and number density,

$$u_0 = \frac{m^4}{8\pi^2}, \quad n_0 = \frac{u_0}{m}, \tag{4.79}$$

which allows us to write the thermodynamic variables as

$$n = \frac{8}{3} n_0\, x_f^3 \tag{4.80}$$

$$u = u_0\left[(2x_f^3 + x_f)\sqrt{1+x_f^2} - \operatorname{arcsinh}(x_f)\right], \tag{4.81}$$

$$p = \frac{u_0}{3}\left[(2x_f^3 - 3x_f)\sqrt{1+x_f^2} + 3\operatorname{arcsinh}(x_f)\right]. \tag{4.82}$$

We have thus chosen $u_0 = p_0$, or equivalently set $k_3 = 1$. This is natural in the case of a relativistic fluid.

4.3.2 Limits

In the non-relativistic limit, as the chemical potential approaches m and thus $p_f \ll m$, the lowest order contributions to the energy density and pressure are given by the Taylor series around $x_f = 0$,

$$\tilde{u}(x_f) = \frac{8}{3}x_f^3 + \frac{4}{5}x_f^5 + \mathcal{O}(x_f^7), \tag{4.83}$$

$$\tilde{p}(x_f) = \frac{8}{15}x_f^5 + \mathcal{O}(x_f^7). \tag{4.84}$$

By neglecting terms of order x_f^7 and higher, we can write this as

$$\tilde{u} = \tilde{n} + \frac{4}{5}\left(\frac{8}{3}\tilde{n}\right)^{5/3}, \quad \tilde{p} = \frac{8}{15}\left(\frac{8}{3}\tilde{n}\right)^{5/3}. \tag{4.85}$$

The leading order contribution to the energy density is the rest mass of the particles. This term does not contribute to the pressure. As discussed earlier, the non-relativistic limit corresponds to $k_3 \ll 1$, if we chose units so that $\tilde{u} \approx \tilde{p}$, or $\tilde{u} \gg \tilde{p}$ if we demand that $k_3 = 1$. We see that $x_f \to 0$ corresponds to the latter case. By including only the leading order term, we can eliminate the Fermi momentum and write the equation of state in the non-relativistic limit as $u_{\mathrm{nr}} = kp^{\frac{3}{5}}$ where $k = 8/3 \cdot (15/8)^{3/5}$. The non-relativistic approximation of the cold fermions is thus a polytrope with $\gamma = \frac{5}{3}$. As we see from Figure 4.2, this is within the range where the mass decreases with the size of the star.

In the ultrarelativistic limit, where $p_f \gg m$, the leading order contributions to the pressure and energy density are

$$\tilde{u} = 2x_f^4, \quad \tilde{p} = \frac{2}{3}x_f^4, \tag{4.86}$$

and we get the particularly simple equation of state for the ultrarelativistic limit, $u_{\mathrm{ur}} = 3p$, which we recognize as the formula for radiation pressure. The equation of state $\tilde{u}(\tilde{p})$ in two different regimes are shown in Figure 4.5. The full equation of state is compared to the non-relativistic and ultrarelativistic approximations.

With our choice of units, the form of the TOV equation is

$$\frac{\mathrm{d}\tilde{m}}{\mathrm{d}\tilde{r}} = 3\tilde{r}^2\tilde{u}, \quad \frac{\mathrm{d}\tilde{p}}{\mathrm{d}\tilde{r}} = -\frac{1}{\tilde{r}^2}(\tilde{p}+\tilde{u})\left(3\tilde{r}^3\tilde{p}+\tilde{m}\right)\left(1-\frac{2\tilde{m}}{\tilde{r}}\right)^{-1}. \tag{4.87}$$

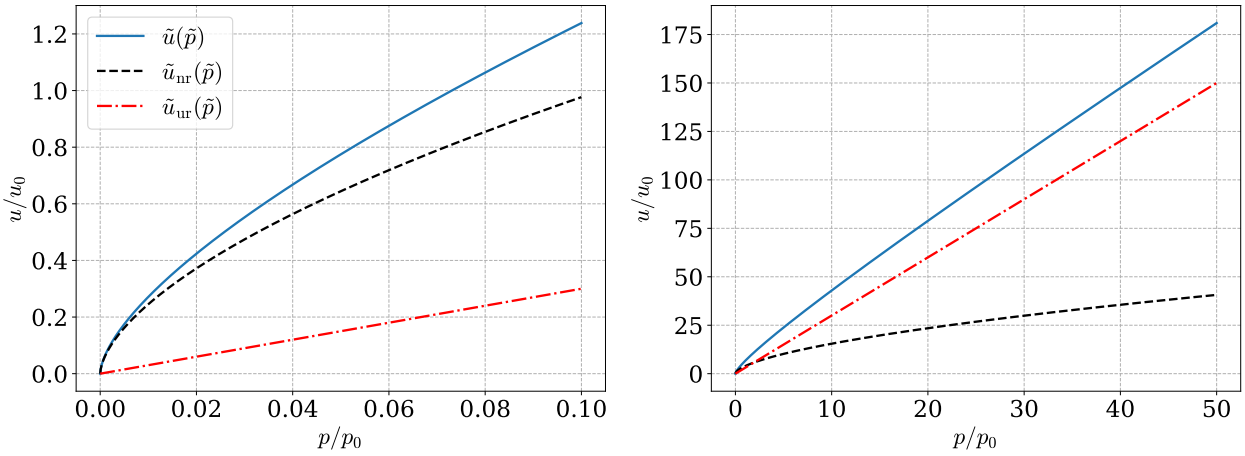


Figure 4.5: The equation of state of a cold Fermi gas. Both pressure and energy density is normalized to their characteristic quantities, p_0 and u_0. The equation of state is compared to the non-relativistic approximation, $\tilde{u}_{\mathrm{nr}}$ as well as the ultrarelativistic approximation, $\tilde{u}_{\mathrm{ur}}$, in two different regimes.

As $r \to 0$, parts of the TOV equation Eq. (4.44) approaches a 0/0-limit, and we must make use of an approximation for numeric evaluation. The Taylor-expansion of the mass function around $\tilde{r} = 0$ is

$$\tilde{m}(r) = \tilde{m}(0) + \tilde{m}'(0)\,\tilde{r} + \frac{1}{2!}\tilde{m}''(0)\tilde{r}^2 + \frac{1}{3!}\tilde{m}'''(0)\tilde{r}^3 + \mathcal{O}\left(\tilde{r}^4\right). \tag{4.88}$$

One of the boundary conditions is $\tilde{m}(0) = 0$. We then use the differnetial equation for $\tilde{m}$, Eq. (4.32), to find

$$\tilde{m}'(0) = 0, \quad \tilde{m}''(0) = 0, \quad \tilde{m}'''(0) = 6k_2\tilde{u}_0, \tag{4.89}$$

where $\tilde{u}_0 = \tilde{u}(r = 0)$. We get an approximation of the TOV equation for $\tilde{r} \ll 1$ by substituting the $\tilde{m}$ for its Taylor expansion and including only the leading-order term, which gives

$$\frac{\mathrm{d}\tilde{p}}{\mathrm{d}\tilde{r}} \sim -\tilde{r}\,\left(\tilde{p} + \tilde{u}\right)\left(3\tilde{p} + \tilde{u}_0\right)\left(1 - 2\tilde{u}_0\tilde{r}^2\right)^{-1}, \quad r \to 0 \tag{4.90}$$

For the Newtonian approximation to the TOV equation, we get

$$\frac{\mathrm{d}\tilde{p}}{\mathrm{d}\tilde{r}} = -\frac{\tilde{u}\tilde{m}}{\tilde{r}^2} \sim -\tilde{u}\tilde{u}_0\tilde{r}, \quad r \to 0. \tag{4.91}$$

4.3.3 Units

The equation of state has given us the characteristic energy density and pressure, u_0 and p_0. If we demand

$$G\frac{m_0}{r_0} = \frac{4\pi}{3}\frac{r_0^3 u_0}{m_0} = 1, \tag{4.92}$$

we have two equations and two unknowns, m_0 and r_0. This thus defines a complete set of units. We are using the cold Fermi-gas as a model for a neutron star, an the mass of the fermion m is therefore the neutron mass, Eq. (1.9), $m_N = 1.674 \cdot 10^{-27}\,\mathrm{kg}$. After reinstating $\hbar$ and c in metric units, we get

$$u_0 = p_0 = \frac{m^4}{8\pi^2}\frac{c^5}{\hbar^3} = 2.032 \cdot 10^{35}\,\mathrm{J\,m^{-3}}, \tag{4.93}$$

$$m_0 = \frac{c^4}{\sqrt{\frac{4\pi}{3}u_0 G^3}} = 1.608 \cdot 10^{31}\,\mathrm{kg} = 8.082\,M_\odot \tag{4.94}$$

$$r_0 = \frac{Gm_0}{c^2} = 11.93\,\mathrm{km}. \tag{4.95}$$

From this, we expect our star to have a mass of the order of a solar mass, $M_\odot = 1.98841 \cdot 10^{30}\,\mathrm{kg}$ [1], and a radius of the order of kilometers, without solving the TOV equation.

4.3.4 Numerical results

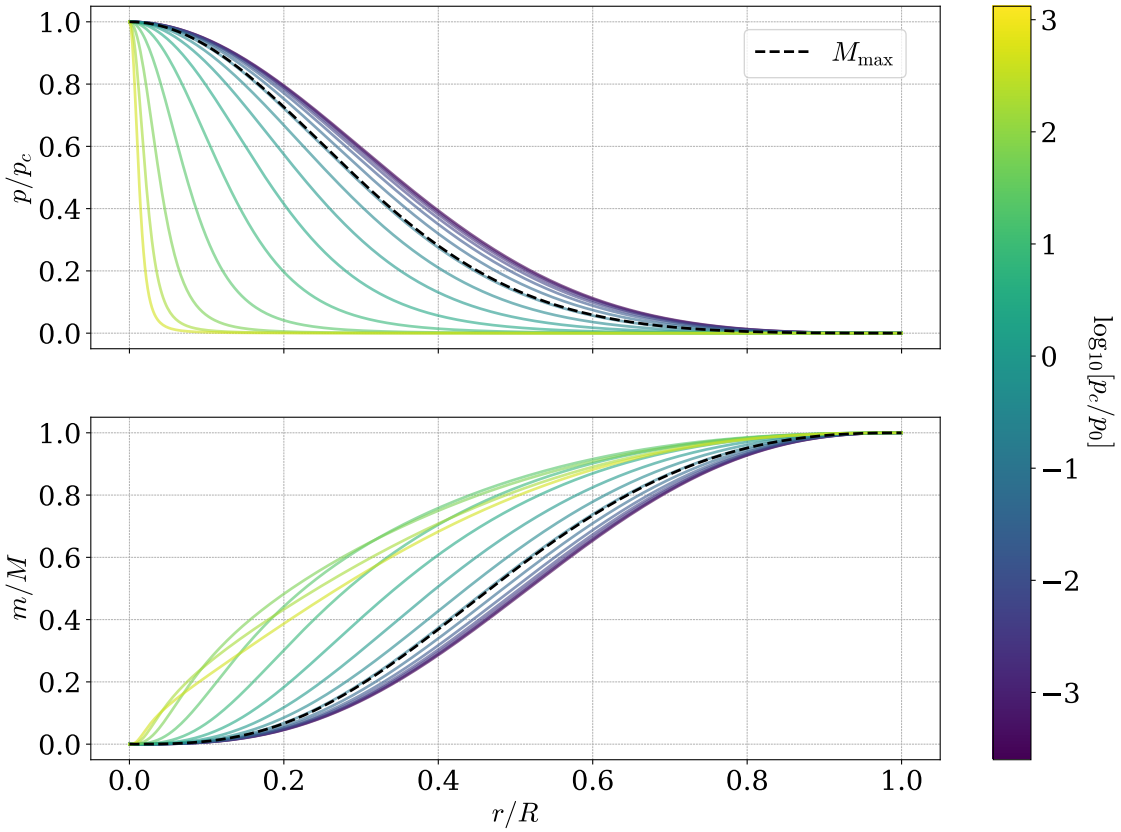


Figure 4.6: Top: The pressure normalized to central value, as a function of radius, normalized to the stellar radius. Bottom: The mass normalized to the total mass, as a function of radius, normalized to the stellar radius. This is plotted for several different values of central pressure, which is indicated by the color shceme.

With the energy density, Eq. (4.81), and pressure, Eq. (4.82), we can numerically solve the TOV equation given a central pressure p_c. This is done using an adaptive Runge-Kutta method, with the stop criterion $p(r) = 0$. Description of the code and where to find it is given in Appendix C. The top graph in Figure 4.6 shows the pressure, normalized to the central pressure p_c, as a function of radius, normalized to the corresponding stellar radius R. The boundary conditions are logarithmically spaced. The lower graph in Figure 4.6 shows the mass, normalized to the total mass $M = m(R)$, as a function of the radius, again normalized to the stellar radius. As in the case of an incompressible fluid, the pressure follows a half bell-shaped curve, with a peak that becomes narrower as the central pressure increases. The black dashed line corresponds to the solution with the maximum mass, which will discuss shortly. We see that the pressure and mass curves changes most drastically when the central pressure is higher than that corresponding to the most massive star.

In Figure 4.7, we see the relationship between the mass and radius of the star. This line is parametrized by the base-10 logarithm of the central pressure, $p(0)$, normalized by $p_0 = u_0$. The cross marks the maximum mass, $M_{\max} = 0.711\,M_\odot$, which corresponds to a radius of $R = 9.20\,\text{km}$. This matches the results obtained by Oppenheimer and Volkoff [17], $M_{\max} = 0.71$. In their 1939 paper, Oppenheimer and Volkoff computed five data points in the mass-radius plane. The results are marked by blue circles in Figure 4.7. We find good agreement between the three points closest to the maximum value and our results. However, the two results of Oppenheimer and Volkoff furthest away differ significantly from our results. The black dashed line is the absolute mass-radius constraint, Eq. (4.59), and any stable configuration must be on the right side of this line. As we predicted from looking at the non-relativistic equation of state, the mass is decreasing with the size of the star, at least for stars with a low central pressure.

In Figure 4.8, we compare the mass-radius relationship obtained from the full theory with results from

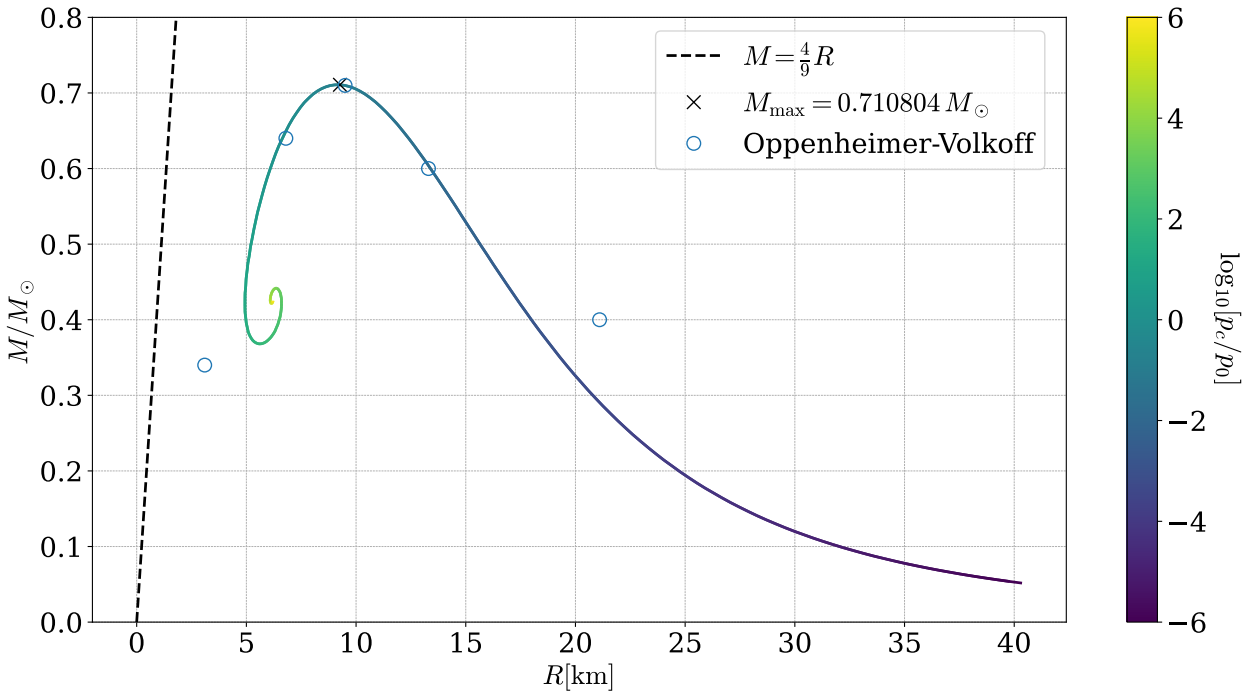


Figure 4.7: The mass-radius relationship of a star made of a cold gas of neutrons. The line is parametrized by the central pressure p_c. The corss indicate the maximum mass solution. The blue circles are results form the 1939 paper of Oppenheimer and Volkoff [17].

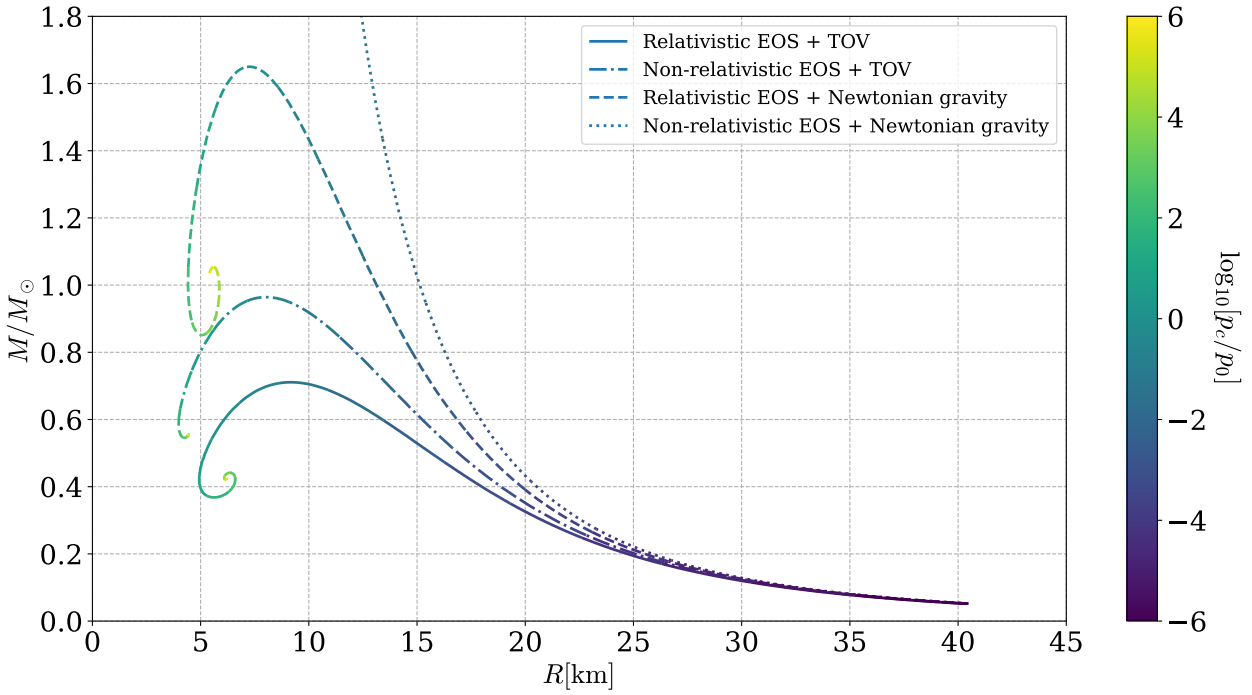


Figure 4.8: The mass-radius relationship of a cold gas of neutrons. The lowest line is obtained from the TOV equation and full equation of state. The middle line is from the TOV equation and the non-relativistic equation of state. The upper line is obtained from the Newtonian approximation of the TOV equation and the non-relativistic equation of state.

approximations. The lowest line is obtained by using both the full TOV equation and the exact equation of state. The next line above is obtained using the non-relativistic equation of state together with the full TOV equation. The second uppermost line is obtained from the exact equation of state and the Newtonian approximation for the TOV equation. The uppermost line uses both the Newtonian approximation to the TOV equation and the non-relativistic approximation for the equation of state. This last line corresponds to a polytrope in Newtonian gravity, as we studied in subsection 4.2.1. Unlike the other systems, it does not seem to have an upper limit for the mass, as expected.

4.3.5 Upper bound and stability

Utvid om stability

Can there be multiple branches?

For any equation of state, the TOV equation will give a one-parameter family of stars, parametrized by the central pressure p_c. This leads to the possibility of an *absolute maximum* mass for a given equation of state. In the case of a non-interacting neutron, we found the limit to be $0.71\,M_\odot$, in agreement with Oppenheimer and Volkoff. To obtain a more general upper limit for the mass of neutron stars, or compact stars in general, one has to account for more general equations of state. To constrain the equation of state, we assume firstly that $\mathrm{d}p/\mathrm{d}u \geq 0$. To justify this, take the non-relativisitc case, $u = n$, in which case the assumption is equivalent to $\mathrm{d}p/\mathrm{d}n \geq 0$. This says that an increase in particle density, for example, due to compression, will result in a rise in pressure. This is an instance of Le Chatelier's principle; nature will counteract any change forced upon it. The speed of sound in the fluid, v_s, is given by [23]

Vis dette?

$$v_s^2 = \frac{\mathrm{d}p}{\mathrm{d}u}. \tag{4.96}$$

A realistic fluid should not have a speed of sound greater than the speed of light, leading to the constraint $\mathrm{d}p/\mathrm{d}u < 1$. Using these general assumptions, Rhoades and Ruffini found an upper limit for neutron stars of $3.2\,M_\odot$ [24].

An equation of state with a *high* speed of sound, i.e., with a flat curve in the $p-u$-plane, is called *stiff*. From Figure 4.5, we see that the Newtonian equation of state is stiffer in the high-energy regime The most extreme case is the incompressible model we saw earlier, which breaks causality. In general, a stiffer equation of state leads to a larger maximum mass. This is intuitive; the TOV equation describes the balancing of forces from pressure and gravity, and if the pressure raises fast as the density increases, then it can sustain a large total mass before it collapses [15].

Solutions to the TOV equation are systems in hydrostatic equilibrium. However, as a pen perfectly balances on its edge, this does not imply stability. When perturbed, a stable system returns back towards its equilibrium position as a marble in the bottom of a bowl. On the other hand, an unstable system will amplify perturbations, leading to a collapse or an explosion. We can make an intuitive argument for which configurations for a given family of stars are stable. We will again assume that the equation of state, on a microscopic level, obeys Le Chatelier's principle in the form $\mathrm{d}p/\mathrm{d}n > 0$. We can see that this holds for all the cases we are considering. A star in equilibrium will find itself on the line parametrized by its central pressure, such as Figure 4.7. In this case, a perturbation reducing the radius of the star will increase the central pressure as the particle density increases. This is illustrated in Figure 4.9. A star in equilibrium at point A can be compressed to an out-of-equilibrium configuration, point B. This point has a central pressure corresponding to the equilibrium state at point C. As the equilibrium configuration at point C has a *lower* mass than the configuration at B, it has a weaker gravitational effect. Therefore, we would expect it to shrink further, as the central pressure of B is not enough to support its gravitational mass. This compression will lead to an even higher central pressure. We thus have a positive feedback loop, and the initial perturbation will continue to grow. We can make a similar argument in the case where the mass increases with an increase in the central pressure. Here, a compression will lead to a state in which central pressure corresponds to an equilibrium state with a *higher* mass. This state will thus tend to expand after a compression, counteracting the perturbation. This gives us the following criterion for stability,

Er det en god antagelse?

Kan vi være helt sikre på dette?

$$\frac{\mathrm{d}M}{\mathrm{d}p_c} > 0. \tag{4.97}$$

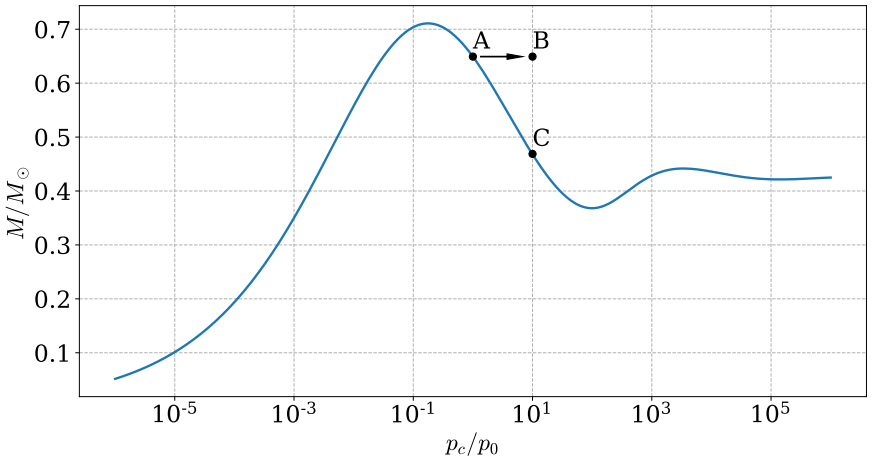


Figure 4.9: The plot shows the mass, in units of solar masses, of a star of a cold gas of neutrons, as a function of the central pressure, normalized to the characteristic pressure. Point A denotes a position of equilibrium, which can be compressed to an out-of-equilibrium point, B, which has a central pressure corresponding to an equilibrium configuration, point C.

As it turns out, this is only a necessary requirement for stability. To conduct a more rigorous study of stability, one must derive the equation of hydrostatic equilibrium with the addition of perturbations as time-dependent, radial oscillations. This is done by discplacing a fluid elemet at a radius r and time t by $\delta r(r,t) = \sum_n A_n \xi_n(r) e^{i\omega_n t}$. Here, ξ_n are normal modes with frequencies ω_n, $n \in \{0, 1, ...\}$. The equation for the eigenmodes was first obtained by Chandrasekhar [25], and can be written in the form of a Sturm-Liouville theory equation,

$$\left[\frac{\mathrm{d}}{\mathrm{d}r}\left(\Pi\frac{\mathrm{d}}{\mathrm{d}r}\right) + Q + \omega_n W\right]\xi_n = 0, \tag{4.98}$$

where Π, Q, and W are functions of the pressure, energy density, particle density, α, and β. These quantities are thus given by a solution to the equilibrium problem [15]. This analysis is outside the scope of this thesis, but we summarize some important conclusions. Stability is encoded in the sign of the square of the frequencies. For $\omega_n^2 > 0$, the mode will remain oscillatory, while if $\omega_n^2 < 0$, it will grow exponentially. Thus, if the system has *any* modes such that $\omega_n^2 < 0$, it is unstable. One mode ω_n will change stability at a critical point on the $M - R$ curve, where

$$\frac{\mathrm{d}M}{\mathrm{d}u_c} = 0, \tag{4.99}$$

and this will *only* happen at critical points [26]. Here, u_c is the central energy density corresponding to p_c. This is equivaent to the criterion $\mathrm{d}M/\mathrm{d}p_c = 0$ as long as $\mathrm{d}p/\mathrm{d}u$ is finite. Whether or not the change is from a stable mode to an unstable one is dependent on whether or not the curve turns clockwise (a mode becomes stable) counterclockwise (a node becomes unstable) [26]. We know that very low-pressure, cold fermions are stable, which means that configuration with a radius larger than the maximum mas $0.71\,M_\odot$ will be stable. As illustrated in Figure 4.7, the curve then turns counterclockwise, and a new mode is made unstable each half turn.

Chapter 5

Chrial perturbation theory

In this chapter, we will take the general knowledge from the general theory in chapter 3 and apply it to the specific case of quantum chromodynamics, which results in *chiral perturbation theory*, or χPT.

5.1 QCD

5.1.1 Yang-Mills theory and Gauge symmetry

In our discussion on global symmetries, we considered the global transformation of fields by some group G. In gauge theories, we will consider local transformations. That is, the transformations are themselves functions of spacetime, $U = U(x)$, and take on some value in G for all points in space. With this, however, we encounter a problem with comparing the value of a field at different points. As the symmetry is local, a gauge transformation will generally affect the field at two points differently. We must find a way to compare fields at different points independent of gauge transformations. This is similar to a problem we have encountered before. In differential geometry, as described in section 2.1, we needed a connection $\Gamma^{\rho}_{\mu\nu}$ to compare vectors in different tangent spaces in a coordinate independent way. In gauge theories, we generalize this by defining a connection, A_μ, to compare field values at different points in a gauge-independent way.

Consider a set of N_c fields ψ_c, which the symmetry group $\mathrm{SU}(N)$ acts linearly on as $\psi_c \to U_{cc'}\psi_{c'}$. We can write $U = \exp(i\eta_\alpha T_\alpha)$, where T_α are the generators of $\mathfrak{su}(N_c)_c$, and can therefore be written $A_\mu = A^\alpha_\mu T_\alpha$. The transformation is then made local by letting the coordinates of $\mathrm{SU}(N)$ be functions of spacetime, $\eta_\alpha = \eta_\alpha(x)$. As we did in section 2.1, we define the covariant derivative D_μ to transform as the thing it acts on. It has the form

$$D^{cc'}_\mu \psi_{c'} = (\delta_{cc'}\partial_\mu - igA^{cc'}_\mu)\psi_{c'}, \tag{5.1}$$

where $A^{cc'}_\mu$ is a new, dynamic field, the gauge field. This field takes values in the Lie algebra of the gauge group, $\mathfrak{su}(N)$. We will suppress the c-indices for cleaner notation. This field also transform under the gauge group. By enforcing the transformation rule $D_\mu A_\nu \to U D_\mu A_\nu$, we can deduce the transformation properties of the gauge field,

$$A_\mu \to U\left(A_\mu + \frac{i}{g}\partial_\mu\right)U^\dagger \tag{5.2}$$

With the covariant derivative, we can create gauge-invariant terms, such as $\bar{\psi}D_\mu\psi$. In section 2.1 we introduced the Riemann tensor as the commutator of covaraint derivatives, Eq. (2.38). This ensures that it transforms as a tensor and gives us the interpretation as a quantity that measures the amount vectors curved when parallel transported in a small loop. In analogy, we define the *field strength tensor*,

$$G_{\mu\nu} := \frac{i}{g}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]. \tag{5.3}$$

A_μ is an element of a Lie algebra, so the commutator is given by the structure constants of that algebra, Eq. (2.75). The field strength tensor transforms as $G_\mu \to UG_{\mu\nu}U^\dagger$. This allows us to create gauge-invariant terms of only this tensor, which, as with the Ricci scalar in general relativity, are the building blocks of the Lagrangian of only the gauge field. The lowest order terms are

$$G^{\mu\nu}_\alpha G^\alpha_{\mu\nu}, \quad \epsilon^{\mu\nu\rho\sigma} G^\alpha_{\mu\nu} G^\alpha_{\rho\sigma}. \tag{5.4}$$

Here, α is the index in $\mathfrak{su}(N)$-space.

5.1.2 The QCD Lagrangian

Quantum chromodynamics, or QCD, is the specific gauge theory of quarks q_{fc}, spin-$\frac{1}{2}$ particles, interacting via the strong force, a $\mathrm{SU}(3)_c$ gauge field denoted A_μ. There are six quarks q, called flavors and indexed by f, with an additional quantum number called color indexed by c. The quarks, labeled u, d, s, c, t, and b, have different masses. In this thesis, we will only include the 2 or 3 lightest quarks and denote the number of flavors by N_f. The Lagrangian of QCD, including only the strong force, is

$$\mathcal{L}_{\mathrm{QCD}} = \bar{q}(i\not{D} - m)q - \frac{1}{4} G^\alpha_{\mu\nu} G^{\mu\nu}_\alpha. \tag{5.5}$$

We have suppressed color and flavor indices. $\not{D}q = \gamma^\mu(\partial_\mu - igA_\mu)q$ is the covariant derivative associated with the $\mathrm{SU}(3)_c$ gauge group with coupling constant g, and γ^μ are the Dirac matrices, as described in section A.1. The quark mass matrix, m, acts on the flavor indexes as the flavor states are mass eigenstates. There are no known symmetries that forbid a $\epsilon^{\mu\nu\rho\sigma} G^\alpha_{\mu\nu} G^\alpha_{\rho\sigma}$-term, and its absence is dubbed the strong CP problem [5].

5.1.3 Chiral symmetry

If we consider the massless QCD Lagrangian, $m = 0$, it has an additional symmetry of rotation in its flavour indices. We can project the quarks down to their *chiral* components by introducing projection operators

$$P_R = \frac{1}{2}\left(1 + \gamma^5\right), \quad P_L = \frac{1}{2}\left(1 - \gamma^5\right). \tag{5.6}$$

Here, γ^5 is the "fifth gamma-matrix", as described in section A.1. As good projection operators, they obey

$$P_R + P_L = 0, \quad P_R P_L = P_L P_R = 0, \quad P_I^2 = P_I,\ I = R, L. \tag{5.7}$$

By the properties of γ^5 and $\bar{q} = q^\dagger\gamma^0$, these operators project out the opposite chirality of q and q,

$$P_I q = q_I, \quad \bar{q}P_I = q_{\bar{I}}, \quad I = R, L, \quad \bar{I} = L, R. \tag{5.8}$$

With this, we can write the quark-sector of massless QCD as

$$i\bar{q}\not{D}q = i\bar{q}\not{D}(P_R + P_L)^2 q = i\bar{q}_L\not{D}q_L + i\bar{q}_R\not{D}q_R. \tag{5.9}$$

This operator is invariant under the transformations

$$q \to g_R(q) = (P_L + P_R U_R)q, \quad q \to g_L(q) = (P_R + P_L U_L)q, \tag{5.10}$$

where U_L and U_R are Hermitian matrices that act on the flavor indices. These transformations form the Lie group $\mathrm{U}(N_f)_R \times \mathrm{U}(N_f)_L = \mathrm{U}(1)_R \times \mathrm{SU}(N_f)_R \times \mathrm{U}(1)_L \times \mathrm{SU}(N_f)_L$. This transformation can also be described in terms of the diagonal subgroup. This subgroup is made up of transformations where $U_R = U_L$, called vector transformations, and the remaining subgroup of transformations where $U_L = U_R^\dagger$, called axial transformations. These together form $\mathrm{U}(N_f)_A \times \mathrm{U}(N_f)_V = \mathrm{U}(1)_V \times \mathrm{SU}(N_f)_V \times \mathrm{U}(1)_A \times \mathrm{SU}(N_f)_A$ The currents corresponding to these transformations are

$$J^\mu_V = \bar{q}_R\gamma^\mu q_R, \quad V^\mu_\alpha = \bar{q}T_\alpha\gamma^\mu q, \quad J^\mu_A = \bar{q}_L\gamma^\mu\gamma^5 q_L, \quad A^\mu_\alpha = \bar{q}T_\alpha\gamma^\mu\gamma^5 q. \tag{5.11}$$

Here, T_α and $T_\alpha\gamma^5$ are the generators of $\mathrm{SU}(N_f)_V$ and $\mathrm{SU}(N_f)_A$. This symmetry, though, is broken in several ways. Firstly, transformations of the form $e^{i\gamma^5} \in \mathrm{U}(1)_A$ are subject to the *axial anomaly*. As mentioned in section 3.3, in a quantum theory not only the action has to be invariant but the integration measure as well, and $\mathcal{D}q\mathcal{D}\bar{q}$ is not. This is encoded in the Schwinger-Dyson equation

$$\partial_\mu \langle J^\mu_A \rangle = -\frac{e^2}{(4\pi)^2} \langle \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \rangle , \tag{5.12}$$

whose right side would vanish if the quantum theory was invariant under $\mathrm{U}(1)_A$. The remaining symmetry is $G = \mathrm{U}(1)_V \times \mathrm{SU}(N_f)_V \times \mathrm{SU}(N_f)_A$. Next, the mass term explicitly breaks the symmetry. In the chiral limit, where $m = m_q\mathbb{1}$, only $\mathrm{SU}(N_f)_A$ is broken, however when we include the fact that the masses of the quarks are different, we break the symmetry further. Other external currents, chemical potentials or the electromagnetic interaction break the symmetry further. We will discuss how to incorporate this in the next chapter. Lastly, the G-symmetry is broken spontaneously by the ground state quark condensate,

$$\langle \bar{q}_f q_f \rangle = -f^2 B_0 \neq 0,\ f \in \{u, d, s\}. \tag{5.13}$$

The scalar quark operator is not invariant under $\mathrm{U}(1)_A$, and as discussed in section 3.3, this leads to the spontanous symmetry breaking $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R / \mathrm{SU}(N_f)_A = \mathrm{SU}(N_f)_V$. We will take this symmetry breaking as an axiom, and use it to describe the low energy physics of QCD.

5.2 Chiral perturbation theory

Chiral perturbation theory, or χPT, is an effective field theory which exploits the chiral symmetries of QCD to describe its low energy dynamics. The basis of χPT for chiral perturbation theory is the massless QCD Lagrangian,

$$\mathcal{L}^0_{\mathrm{QCD}} = i\bar{q}\not{D}q - \frac{1}{4} G^\alpha_{\mu\nu} G^{\mu\nu}_\alpha \tag{5.14}$$

This Lagrangian is invariant under the full symmetry group $G = \mathrm{U}(1)_V \times \mathrm{SU}(N_f)_V \times \mathrm{SU}(N_f)_A$. To incorporate other fields or terms that break G, such as the quark masses, we add a Lagrangian containing external currents. These can include either couple to the conserved currents, Eq. (5.11), or the other bilinears we can create out of quarks, $\bar{q}q$, $\bar{q}\gamma^5 q$, $\bar{q}\tau_a q$, and $\bar{q}\tau_a\gamma^5 q$. The Lagrangian is

Hva er grunnen til valgene av fortegn?

$$\mathcal{L}_{\mathrm{ext}} = -\bar{q}\left(s - i\gamma^5 p\right) q + \bar{q}\gamma^\mu \left(v_\mu + \gamma^5 a_\mu\right) q. \tag{5.15}$$

The external sources are defined as

$$s = s_0\mathbb{1} + s_a\tau_a, \quad p = p_0\mathbb{1} + p_a\tau_a, \quad v^\mu = v^\mu_0\mathbb{1} + \frac{1}{2}v^\mu_a\tau_a, \quad a^\mu = a^\mu_0\mathbb{1} + \frac{1}{2}a^\mu_a\tau_a. \tag{5.16}$$

which are respectively the scalar, pseudo-scalar, vector, and pseudo-vector currents. We denote these currents collectively as $j = (s, p, v^\mu_a, a^\mu_a)$.

Generaliser til SU(N)

fiks resten av seksjon

The masses of the quarks are accounted for by setting the scalar current s_0 equal the mass matrix of the quarks,

$$m_q = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} \tag{5.17}$$

These field can be static background fields, as is the case for the mass contribution to s_0, or a dynamical field such as the photon field. Let j_s denote static fields, while j_s denote dynamical fields, so that $j = j_s + j_d$. The dynamical fields migh have their own Lagrangian with terms independent of quarks, $\mathcal{L}_d[j_d]$. In the case where the photon field is included as a dynamical field, this will have a $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ term. The full Lagrangian is then

Er dette riktig?

$$\mathcal{L}_{\mathrm{QCD}}[q, \bar{q}, A, j] = \mathcal{L}^0_{\mathrm{QCD}}[q, \bar{q}, A] + \mathcal{L}_{\mathrm{ext}}[j] + \mathcal{L}_d[j_d]. \tag{5.18}$$

In the grand canonical ensemble, as discussed in (ref termisk feltteori), we introduce a chemical potential μ and couple it to a conserved charge. We are interested in the case where the chemical potential of the third component of isospin is, denoted μ_I, is non-zero. This corresponds to a modification of the Lagrangian by

$$\mathcal{L} \to \mathcal{L} + \mu_I \frac{1}{2} \bar{q} \gamma_0 \tau_3 q, \tag{5.19}$$

which corresponds to an external vector current

$$v_I^\mu = \frac{1}{2} \mu_I \delta_0^\mu \tau_3. \tag{5.20}$$

The electromagnetic interactions is a gauge theory as well, and the electromagnetic covariant derivative acing on quarks is

$$i\bar{q} \not{D}' q = i\bar{q}\gamma^\mu \left(\mathbb{1}\partial_\mu - ieQ\mathcal{A}_\mu\right) q = i\bar{q}\not{\partial} q - e\mathcal{A}_\mu J^\mu, \tag{5.21}$$

where $\mathcal{A}_\mu$ is the photon field corresponding to the $\mathrm{U}(1)_{\mathrm{EM}}$ gauge group, $e = |e|$ is the elementary charge as given in Eq. (1.8), $J^\mu = -\bar{q} Q \gamma^\mu q$ is the electromagnetic charge current, and Q is the quark charge matrix,

$$Q = \frac{1}{3}\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{6}\mathbb{1} + \frac{1}{2}\tau_3. \tag{5.22}$$

Er dette riktig fortegn

As with the chemical potential, this is accounted for by an external current vector current,

$$v_{\mathrm{EM}}^\mu = eQ\mathcal{A}^\mu. \tag{5.23}$$

We define the right handed and left handed currents as

$$r_\mu = v_\mu + a_\mu, \quad l_\mu = v_\mu - a_\mu \tag{5.24}$$

We now define the effective Lagrangian of χPT as

$$Z[j_s] = \int \mathcal{D}q\mathcal{D}\bar{q}\mathcal{D}A\mathcal{D}j_d \exp\left\{ i \int \mathrm{d}^4x \mathcal{L}_{\mathrm{QCD}}[q, \bar{q}, A, j] \right\} = \int \mathcal{D}\pi\mathcal{D}j_d \exp\left\{ i \int \mathrm{d}^4x \mathcal{L}_{\mathrm{eff}}[\pi, \mathcal{A}, j] \right\}. \tag{5.25}$$

5.2.1 Winberg's power counting scheme

Skriv/kopier tekst om Weinberg's power counting scheme

5.2.2 Building blocks

Covariant derivative

$$\nabla_\mu \Sigma = \partial_\mu \Sigma - i r_\mu \Sigma + i \Sigma l_\mu. \tag{5.26}$$

Scalar

$$\chi = 2B_0(s + ip), \quad s = m,\ p = 0. \tag{5.27}$$

Field strength tensor

$$f_{\mu\nu}^{(r)} = \partial_\mu r_\nu - \partial_\nu r_\mu - i[r_\mu, r_\nu], \quad r \to l. \tag{5.28}$$

EM + chemical potential:

$$r_\mu = l_\mu = v_\mu = \frac{1}{2}\mu_I \delta_\mu^0 \tau_3 + eQ\mathcal{A}_\mu. \tag{5.29}$$

Transformations under $g \in G = \mathrm{SU}(N_f)_R \times \mathrm{SU}(N_f)_L \times \mathrm{U}(1)_V$

$$g \in G, \quad g = g_R \times g_L \times g_V, \tag{5.30}$$

$$g_I(q) = (P_I U_I + P_{\bar{I}})q = U_I q_I \quad g_I(\bar{q}) = \bar{q}(P_{\bar{I}} U_I^\dagger + P_I), \tag{5.31}$$

$$g_V(q) = U_V q,\ g_V(\bar{q}) = \bar{q} U_V^\dagger \tag{5.32}$$

$$U_I = P_I \exp\left\{ -i\eta_\alpha \frac{\tau_\alpha}{2} \right\}, \quad I = R, L, \tag{5.33}$$

$$U_V = \exp(-i\theta) \tag{5.34}$$

Promoting G to gauge group, to get gauge invariance we must have

$$\Sigma \to U_R \Sigma U_L^\dagger, \tag{5.35}$$

$$r_\mu \to U_V U_R (r_\mu + i\partial_\mu) U_R^\dagger U_V^\dagger = U_R^\dagger (r_\mu + i\partial_\mu) U_R^\dagger - \partial_\mu \theta, \quad r, R \to l, L. \tag{5.36}$$

$$\chi \to U_R \chi U_L^\dagger \tag{5.37}$$

$$Q_I \to U_I Q_I U_I^\dagger,\ I = R, L. \tag{5.38}$$

We count χ as order 2, e as order 2 and $\nabla_\mu \Sigma$ as order 1. Notice that e and Q must always appear as eQ, as the orignal Lagrangian Eq. (5.25) is invariant under the transformation $e \to e/\lambda$ and $Q \to \lambda Q$ [27].

5.3 Two flavour χPTto leading order

5.3.1 *Paramterization

In this section, we will assume $N_f = 2$, which means the generators are $T_a = \frac{1}{2}\tau_a$, where τ_a are the Pauli Matrices, as described in section A.1. The quark mass matrix is

$$m = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}. \tag{5.39}$$

We define $\bar{m}^2 = B_0(m_u + m_d)$ and $\Delta m^2 = B_0(m_u - m_d)$, so that when we set the scalar current s equal to the quark masses, and the pseudoscalar current to zero, we get

$$\chi = \bar{m}^2 \mathbb{1} + \Delta m^2 \tau_3, \tag{5.40}$$

For $j = 0$, the ground state is by assumption $\Sigma = \mathbb{1}$, the vacuum, and we can use the paramterization

$$\Sigma(x) = \exp\left\{ i \frac{\pi_a \tau_a}{f} \right\}, \tag{5.41}$$

where f is the bare pion decay constant, π_a are the three Goldstone bosons, a set of real functions of space-time. This ensures that $\pi = 0$ corresponds to the vacuum. If we perform an infinitesimal isospin-transformation, and assume π/f small, then

$$\Sigma \to U_V \Sigma U_V^\dagger = \left(1 + i\eta_a \frac{1}{2}\tau_a\right)\left(1 + i\frac{1}{f}\pi_b\tau_b\right)\left(1 - i\eta_c \frac{1}{2}\tau_c\right) = 1 + i\frac{1}{f}\pi_a(\delta_{ac} + i\eta_b \epsilon_{abc})\tau_c, \tag{5.42}$$

or

$$\pi_a \to (\delta_{ac} + i\eta_b \epsilon_{abc})\pi_c. \tag{5.43}$$

The generators of π_a under isospin-transformations are thus the adjoint representation of $\mathfrak{su}(2)$, and they form an isospin triplet. For $\eta_1 = \eta_2 = 0$, i.e. transformations generated by τ_3, π_3 is invariant, which means that it has quantum number $I_3 = 0$.[1] π_1 and π_2 do not have a definite value of the third component of isospin, but rather for the first and second component. They are related to the observed, charged pions π_+ and π_- by [28]

$$\pi_a \tau_a = \begin{pmatrix} \pi_3 & \pi_1 - i\pi_2 \\ \pi_1 + i\pi_2 & -\pi_3 \end{pmatrix} = \begin{pmatrix} \pi_0 & \sqrt{2}\pi_- \\ \sqrt{2}\pi_+ & -\pi_0 \end{pmatrix}, \tag{5.44}$$

where $\pi_\pm$ has a third isospin-component of $I_3 = \pm 1$. For non-zero isospin chemical potential, however, we expect that the ground state may be rotated away from the vacuum. To find what the new ground state is, we have to minimize the Hamiltonian.

[1] Authors differe if they define $\sqrt{2}\pi_\pm = \pi_1 \pm i\pi_2$, or with opposite signs.We choose the former, so that $\pi_+ |0\rangle$ is the state with the quantum numbers of the positive pion.

5.3.2 *Leading order Lagrangian

The leading order Lagrangian in Winberg's power counting scheme is

$$\mathcal{L}_2 = \frac{1}{4}f^2 \operatorname{Tr}\{\nabla_\mu \Sigma (\nabla^\mu \Sigma)^\dagger\} + \frac{1}{4}f^2 \operatorname{Tr}\{\chi^\dagger \Sigma + \Sigma^\dagger \chi\}. \tag{5.45}$$

The external source currents are

$$\nabla_\mu \Sigma = \partial_\mu \Sigma - i[v_\mu, \Sigma], \quad v_\mu = \frac{1}{2}\mu_I \delta^0_\mu \tau_3. \tag{5.46}$$

To incorporate a finite isospin density, we must parametrize the Goldstone manifold differently than in the vacuum. We follow the analysis in [29]. We assume the ground state is independent of space, $\pi_a(x) = \pi^0_a$, and write it as

$$\Sigma_\alpha := \exp\{i\alpha n_a \tau_a\} = \cos\alpha + i n_a \tau_a \sin\alpha, \tag{5.47}$$

where

$$\alpha = \frac{1}{f}\sqrt{\pi^0_a \pi^0_a}, \quad n_a = \frac{\pi^0_a}{\sqrt{\pi^0_a \pi^0_a}}. \tag{5.48}$$

With this, the covariant derivative is $\nabla_\mu \Sigma_\alpha = -i v^a_\mu [\tau_a, \Sigma_\alpha]$, and the two terms in the first order Lagrangian are

$$\operatorname{Tr}\{\nabla_\mu \Sigma_\alpha (\nabla^\mu \Sigma_\alpha)^\dagger\} = 2\mu_I^2 (n_1^2 + n_2^2)\sin^2\alpha, \quad \operatorname{Tr}\{\chi^\dagger \Sigma_\alpha + \Sigma_\alpha^\dagger \chi\} = 4\bar{m}^2 \cos\alpha. \tag{5.49}$$

We see that, to first order, all results are independent of Δm. To find the new ground state, we minimize the Hamiltonian density. With the assumption that the fields are constant, the first order Hamiltonian density is

$$\mathcal{H}_2 = -\mathcal{L}_2 = -f^2\left[\bar{m}^2 \cos^2\alpha + \frac{1}{2}\mu_I^2 (n_1^2 + n_2^2)\sin^2\alpha\right] \tag{5.50}$$

For $\mu_I = 0$, this is independent of n_a, and minimized by $\alpha = 0$. Now, as $n_i n_i = 1$, we have that $n_1^2 + n_2^2 = 1 - n_3^2$. This means that, for $\mu_I \neq 0$, the energy is minimized by $n_3 = 0$. We can write $n_1 = \cos\phi$, $n_2 = \sin\phi$, for some real number ϕ, which gives the ground state

$$\Sigma_\alpha = \mathbb{1}\cos\alpha + i(\tau_1 \cos\phi + \tau_2 \sin\phi)\sin\alpha. \tag{5.51}$$

We can choose, without loss of generality, $\phi = 0$ [30]. This corresponds to a change of basis of $\mathfrak{su}(2)$, $\tau_1 \to \tilde{\tau}_1 = \tau_1 \cos\phi + \tau_2 \sin\phi$ and $\tau_2 \to \tilde{\tau}_2 = -\tau_1 \sin\phi + \tau_2 \cos\phi$. With this, the new ground state is

$$\Sigma_\alpha = \exp\{i\alpha\tau_1\} \tag{5.52}$$

Any exited state is a transformation of the ground state by $\mathrm{SU}(2)_A$. For $\mu_I = 0$, this corresponds to

$$\Sigma(x) = U_R(x)\Sigma_0 U_L^\dagger(x) = U(x)\Sigma_0 U(x). \tag{5.53}$$

where

$$U(x) = \exp\left\{i\frac{\tau_a \pi_a(x)}{2f}\right\}. \tag{5.54}$$

We see that this recovers the parametrization Eq. (5.41). For $\mu_I \neq 0$, the ground state may be shifted, and so $U(x)$ must be too. The groundstate transforms as

$$\Sigma_0 \to \Sigma_\alpha = \hat{U}_L \Sigma_0 \hat{U}_R^\dagger = A_\alpha \Sigma_0 A_\alpha. \tag{5.55}$$

where

$$A_\alpha := \exp\left\{i\frac{1}{2}\alpha\tau_1\right\} = \cos\frac{\alpha}{2} + i\tau_1 \sin\frac{\alpha}{2}. \tag{5.56}$$

This induces the following transformations for the fluctuations,

$$U_L \to \hat{U}_L U_L \hat{U}_L^\dagger = A_\alpha U_L A_\alpha^\dagger, \tag{5.57}$$

$$U_R \to \hat{U}_R U_R \hat{U}_R^\dagger = A_\alpha^\dagger U_R A_\alpha. \tag{5.58}$$

The new parametrization is thus

$$\Sigma(x) = A_\alpha[U(x)\Sigma_0 U(x)]A_\alpha. \tag{5.59}$$

With this, we can expand the first order Lagrangian, Eq. (5.45), in powers of π/f We will use this expansion to calculate the free energy density. Expanding Σ to $\mathcal{O}\left((\pi/f)^5\right)$, we get

$$\Sigma = \left(1 - \frac{\pi_a^2}{2f^2} + \frac{\pi_a^2\pi_b^2}{24f^4}\right)(\cos\alpha + i\tau_1\sin\alpha) + \left(\frac{\pi_a}{f} - \frac{\pi_b^2\pi_a}{6f^3}\right)\left(i\tau_a - 2i\delta_{a1}\tau_1\sin^2\frac{\alpha}{2} - \delta_{a1}\sin\alpha\right). \tag{5.60}$$

The kinetic term in the χPT Lagrangian is

$$\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger = \partial_\mu\Sigma\partial^\mu\Sigma^\dagger - i\left(\partial_\mu\Sigma[v^\mu,\Sigma^\dagger] - \text{h.c.}\right) - [v_\mu,\Sigma][v_\mu,\Sigma^\dagger]. \tag{5.61}$$

Using Eq. (5.60) we find the expansion of the constitutive parts of the kinetic term to be

$$\begin{aligned}\partial_\mu\Sigma = &\left[\left(\frac{-1}{f^2} + \frac{\pi_b^2}{6f^4}\right)(\pi_a\partial_\mu\pi_a)\cos\alpha - \left(\frac{\partial_\mu\pi_1}{f} - \frac{\pi_b^2\partial_\mu\pi_1 + 2\pi_1\pi_b\partial_\mu\pi_b}{6f^3}\right)\sin\alpha\right]\\ &- \left[\left(\frac{-1}{f^2} + \frac{\pi_b^2}{6f^4}\right)(\pi_a\partial_\mu\pi_a)\sin\alpha - \left(\frac{\partial_\mu\pi_1}{f} - \frac{\pi_b^2\partial_\mu\pi_1 + 2\pi_1\pi_b\partial_\mu\pi_b}{6f^3}\right)2\sin^2\frac{\alpha}{2}\right]i\tau_1\\ &+ \left(\frac{\partial_\mu\pi_a}{f} - \frac{\pi_b^2\partial_\mu\pi_a + 2\pi_a\pi_b\partial_\mu\pi_b}{6f^3}\right)i\tau_a,\end{aligned} \tag{5.62}$$

and

$$[v_\mu,\Sigma] = -\mu_I\delta^0_\mu\left\{\left[\left(1 - \frac{\pi_a^2}{2f^2} + \frac{\pi_a^2\pi_b^2}{24f^4}\right)\sin\alpha + \left(\frac{\pi_1}{f} - \frac{\pi_b^2\pi_1}{6f^3}\right)\cos\alpha\right]\tau_2 - \left(\frac{\pi_2}{f} - \frac{\pi_b^2\pi_2}{6f^3}\right)\tau_1\right\}. \tag{5.63}$$

Combining Eq. (5.62) and Eq. (5.63) gives the following terms

$$\text{Tr}\left\{\partial_\mu\Sigma\partial^\mu\Sigma^\dagger\right\} = \frac{2}{f^2}\partial_\mu\pi_a\partial^\mu\pi_a + \frac{2}{3f^4}\left[(\pi_a\partial_\mu\pi_a)(\pi_b\partial^\mu\pi_b) - (\pi_a\partial_\mu\pi_b)(\pi_b\partial^\mu\pi_a)\right], \tag{5.64}$$

$$\begin{aligned}-i\,\text{Tr}\left\{\partial^\mu\Sigma[v_\mu,\Sigma^\dagger] - \text{h.c.}\right\} = &\,4\mu_I\frac{\partial_0\pi_2}{f} + 8\mu_I\frac{\pi_3}{3f^3}\sin\alpha(\pi_2\partial_0\pi_3 - \pi_3\partial_0\pi_2)\sin\alpha\\ &+ \left(\frac{4\mu_I}{f^2}\cos\alpha - \frac{8\mu_I\pi_1}{3f^3}\sin\alpha - \frac{4\mu_I\pi_a\pi_a}{3f^4}\cos\alpha\right)(\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1),\end{aligned} \tag{5.65}$$

$$-\text{Tr}\left\{[v_\mu,\Sigma][v^\mu,\Sigma^\dagger]\right\} = {\mu_I}^2\left[2\sin^2\alpha + \left(\frac{2}{f} - \frac{4\pi_a\pi_a}{3f^3}\right)\pi_1\sin 2\alpha + \left(\frac{2}{f^2} - \frac{2\pi_a\pi_a}{3f^4}\right)\pi_a\pi_b k_{ab}\right], \tag{5.66}$$

$$\text{Tr}\left\{\chi^\dagger\Sigma + \Sigma^\dagger\chi\right\} = \bar{m}^2\left(4\cos\alpha - \frac{4\pi_1}{f}\sin\alpha - \frac{2\pi_a\pi_a}{f^2}\cos\alpha + \frac{2\pi_1\pi_a\pi_a}{3f^3}\sin\alpha + \frac{(\pi_a\pi_a)^2}{6f^4}\cos\alpha\right), \tag{5.67}$$

where $k_{ab} = \delta_{a1}\delta_{b1}\cos 2\alpha + \delta_{a2}\delta_{b2}\cos^2\alpha - \delta_{a3}\delta_{b3}\sin^2\alpha$. Notice that the mass term is independent of the difference in quark masses, Δm. If we write the Lagrangian Eq. (5.45) as $\mathcal{L}_2 = \mathcal{L}_2^{(0)} + \mathcal{L}_2^{(1)} + \mathcal{L}_2^{(2)} + ...$, where $\mathcal{L}_2^{(n)}$ contains all terms of order $(\pi/f)^n$, then the result of the series expansion is

$$\mathcal{L}_2^{(0)} = f^2\left(\bar{m}^2\cos\alpha + \frac{1}{2}\mu^2\sin^2\alpha\right), \tag{5.68}$$

$$\mathcal{L}_2^{(1)} = f(\mu_I^2\cos\alpha - \bar{m}^2)\pi_1\sin\alpha + f\mu_I\partial_0\pi_2\sin\alpha, \tag{5.69}$$

$$\mathcal{L}_2^{(2)} = \frac{1}{2}\partial_\mu\pi_a\partial^\mu\pi_a + \mu_I\cos\alpha\,(\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1) - \frac{1}{2}\bar{m}^2\pi_a\pi_a\cos\alpha + \frac{1}{2}\mu_I^2\pi_a\pi_b k_{ab}, \tag{5.70}$$

$$\begin{aligned}\mathcal{L}_2^{(3)} = &\frac{\pi_a\pi_a\pi_1}{6f}(\bar{m}^2\sin\alpha - 2{\mu_I}^2\sin 2\alpha)\\ &- \frac{2\mu_I}{3f}\left[\pi_1(\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1) + \pi_3(\pi_3\partial_0\pi_2 - \pi_2\partial_0\pi_3)\right]\sin\alpha,\end{aligned} \tag{5.71}$$

$$\begin{aligned}\mathcal{L}_2^{(4)} = &\frac{1}{6f^2}\left\{\frac{1}{4}\bar{m}^2(\pi_a\pi_a)^2\cos\alpha - \left[(\pi_a\pi_a)(\partial_\mu\pi_b\partial^\mu\pi_b) - (\pi_a\partial_\mu\pi_a)(\pi_b\partial^\mu\pi_b)\right]\right\}\\ &- \frac{\mu_I\pi_a\pi_a}{3f^2}\left[(\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1)\cos\alpha + \frac{1}{2}\mu_I\pi_a\pi_b k_{ab}\right].\end{aligned} \tag{5.72}$$

5.3.3 *Propagator

We may write the quadratic part of the Lagrangian Eq. (5.70) as[2]

$$\mathcal{L}_2^{(2)} = \frac{1}{2}\sum_a \partial_\mu \pi_a \partial^\mu \pi_a + \frac{1}{2} m_{12}(\pi_1 \partial_0 \pi_2 - \pi_2 \partial_0 \pi_1) - \frac{1}{2}\sum_a m_a^2 \pi_a^2, \tag{5.73}$$

where

$$m_1^2 = \bar{m}^2 \cos\alpha - \mu_I^2 \cos 2\alpha, \tag{5.74}$$
$$m_2^2 = \bar{m}^2 \cos\alpha - \mu_I^2 \cos^2\alpha, \tag{5.75}$$
$$m_3^2 = \bar{m}^2 \cos\alpha + \mu_I^2 \sin^2\alpha, \tag{5.76}$$
$$m_{12} = 2\mu_I \cos\alpha. \tag{5.77}$$

The inverse propagator is given by the functional derivative,

$$D_{ab}^{-1}(x-y) = \frac{\delta S[\pi]}{\delta \pi_a(x)\pi_b(y)} = \left[-\delta_{ab}(\partial_x^2 + m_a^2) + m_{12}(\delta_{a1}\delta_{b2} - \delta_{a2}\delta_{b1})\partial_{x,0}\right]\delta(x-y). \tag{5.78}$$

The momentum space inverse propagator is

$$D_{ab}^{-1}(p) = \delta_{ab}(p^2 - m_a^2) + ip_0 m_{12}(\delta_{a1}\delta_{b2} - \delta_{a2}\delta_{b1}). \tag{5.79}$$

The spectrum of the particles is given by solving $\det\left(D^{-1}\right) = 0$ for p^0. With $p = (p_0, \vec{p})$ as the four momentum, this gives

$$\det\left(D^{-1}\right) = D_{33}^{-1}\left(D_{11}^{-1}D_{22}^{-1} + (D_{12}^{-1})^2\right) = \left(p^2 - m_3^2\right)\left[\left(p^2 - m_1^2\right)\left(p^2 - m_2^2\right) - p_0^2 m_{12}^2\right] = 0.$$

This equation has the solutions

$$E_0^2 = |\vec{p}|^2 + m_3^2, \tag{5.80}$$
$$E_\pm^2 = |\vec{p}|^2 + \frac{1}{2}\left(m_1^2 + m_2^2 + m_{12}^2\right) \pm \frac{1}{2}\sqrt{4|\vec{p}|^2 m_{12}^2 + \left(m_1^2 + m_2^2 + m_{12}^2\right)^2 - 4m_1^2 m_2^2}. \tag{5.81}$$

These are the energies of three particles π_0, π_+ and π_-. π_0 is π_3, while $\pi_\pm$ are linear combinations of π_1 and π_2. [3] We will show that for $\mu_I < m_\pi$, $\alpha = 0$, before it starts to increase for $\mu_I \geq m_\pi$. This result is presented in chapter 6. For $\alpha = 0$, we get

$$\frac{1}{2}(m_1^2 + m_2^2 + m_{12}^2) = \bar{m}^2 + \mu_I^2, \quad m_1^2 m_2^2 = (\bar{m}^2 - \mu_I^2)^2, \quad m_3^2 = \bar{m}^2,$$
$$\implies E_\pm^2 = |\vec{p}|^2 + \bar{m}^2 + \mu_I^2 \pm 2\mu_I\sqrt{|\vec{p}|^2 + \bar{m}^2}.$$

This corresponds to a Zeeman-like splitting of the energies,

$$E_0 = \sqrt{|\vec{p}|^2 + \bar{m}^2}, \tag{5.82}$$
$$E_\pm = \pm\mu_I + \sqrt{|\vec{p}|^2 + \bar{m}^2}. \tag{5.83}$$

The (tree-level) masses of these particles are found by setting $\vec{p} = 0$ and are

$$m_0^2 = m_3^2, \tag{5.84}$$
$$m_\pm^2 = \frac{1}{2}\left[m_1^2 + m_2^2 + m_{12}^2\right] \pm \frac{1}{2}\sqrt{\left(m_1^2 + m_2^2 + m_{12}^2\right)^2 - 4m_1^2 m_2^2}. \tag{5.85}$$

Using the result for α, Figure 5.1 shows the masses as functions of μ_I. We observe that the mass of the π_--particle goes to zero at $\mu_I = m_\pi$. This is indicative of spontaneous symmetry breaking, which we will investigate in the next chapter.

[2] Summation over isospin index (a, b, c) will be explicit in this section.

[3] An unfortunate notational convention is that E_+ is the energy of π_--particle, and E_+ for the π_+-particle. This is because the positivley charged pion, π_+, has isospin $I_3 = +1$, so that the mass will decrease as μ_I increseas, and hence the negative sign.

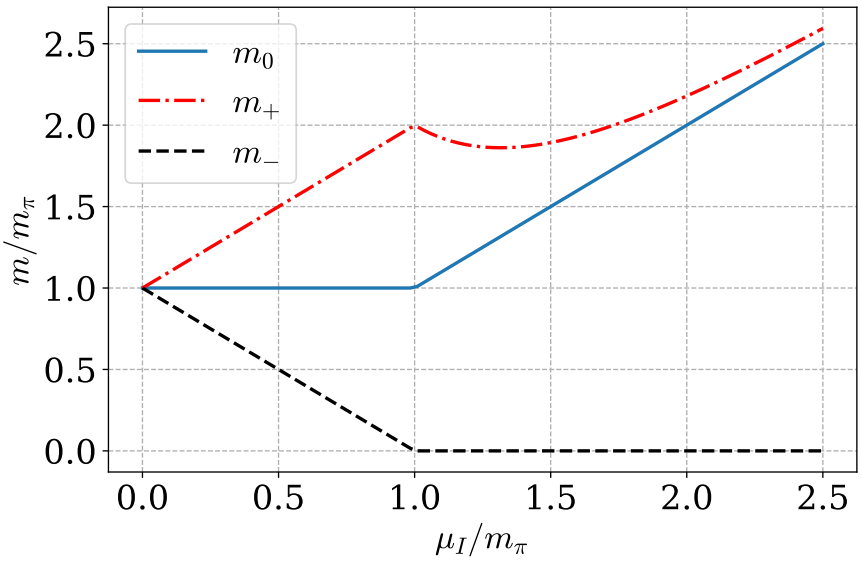


Figure 5.1: The masses of the three particles as functions of isosopin chemical potential. Rresults are given in units of the pion mass, m_π.

With the energies of the pions, we can write the determinant of the inverse propagator as

$$\det\left(D^{-1}\right) = (p_0^2 - E_0^2)(p_0^2 - E_+^2)(p_0^2 - E_-^2). \tag{5.86}$$

The propagator and the inverse propagator in momentum space obey[4]

$$\sum_c D_{ac}(p) D_{cb}^{-1}(p) = i\delta_{ab} \tag{5.87}$$

Using this, we can solve for the propagator

$$D = (-iD^{-1})^{-1} = i \begin{pmatrix} \frac{p^2 - m_2^2}{(p_0^2 - E_+^2)(p_0^2 - E_-^2)} & \frac{-ip_0 m_{12}}{(p_0^2 - E_+^2)(p_0^2 - E_-^2)} & 0 \\ \frac{ip_0 m_{12}}{(p_0^2 - E_+^2)(p_0^2 - E_-^2)} & \frac{p^2 - m_1^2}{(p_0^2 - E_+^2)(p_0^2 - E_-^2)} & 0 \\ 0 & 0 & \frac{1}{p_0^2 - E_0^2} \end{pmatrix}. \tag{5.88}$$

With these results, we can start calculating thermodynamic quantities.

5.3.4 Electromagnetic effects

When including contribution from a dynamical photon field, the leading order Lagrangian is [31, 32]

$$\mathcal{L}_2^{\mathrm{EM}} = \frac{1}{4} f^2 \operatorname{Tr}\{\nabla_\mu \Sigma \nabla^\mu \Sigma^\dagger\} + \frac{1}{4} f^2 \operatorname{Tr}\{\chi \Sigma^\dagger + \Sigma \chi^\dagger\} + e^2 C \operatorname{Tr}\{Q \Sigma Q \Sigma^\dagger\} \tag{5.89}$$

Q is the quark charge matrix, Eq. (5.22), C and dimensionfull constant, and $\chi = 2B_0 m$, where m is the quark mass matrix Eq. (5.17). To find the electromagnetic effect on the pion mass, we assume $\mu_I = 0$. We use the parametrization $\Sigma = \exp\{i\pi_a \tau_a / f\}$, and the covariant derivative is in this case

$$\nabla_\mu \Sigma = \partial_\mu \Sigma - ie\mathcal{A}_\mu [Q, \Sigma]. \tag{5.90}$$

[4]One has to be carefull regarding the factor i in the physicist's definition of propagators. It has the consequence that D^{-1} is not strictly the operator inverse of the propagator D.

We expand to second order in π_a/f, which gives

$$f^2 \operatorname{Tr}\{\nabla_\mu \Sigma \nabla^\mu \Sigma\} = 2\partial_\mu \pi_a \partial^\mu \pi_a + 4e\mathcal{A}^\mu(\pi_1 \partial_\mu \pi_2 - \pi_2 \partial_\mu \pi_1) + 4e^2 \mathcal{A}^2(\pi_1^2 + \pi_2^2), \tag{5.91}$$

$$\operatorname{Tr}\{\chi\Sigma^\dagger + \Sigma\chi^\dagger\} = 4\bar{m}^2 \left(1 - \frac{1}{2}\frac{\pi_a \pi_a}{f^2}\right), \tag{5.92}$$

$$\operatorname{Tr}\{Q\Sigma Q\Sigma^\dagger\} = \frac{5}{9} - \frac{\pi_1^2 + \pi_2^2}{f^2}. \tag{5.93}$$

Inserting this into Eq. (5.89), we get

$$\mathcal{L}_2^{\mathrm{EM}} = \bar{m}^2 f^2 + \frac{5}{9}e^2 C + \frac{1}{2}\partial_\mu \pi_a \partial^\mu \pi_a - \frac{1}{2}\bar{m}_\pm^2(\pi_1^2 + \pi_2^2) - \frac{1}{2}\bar{m}^2 \pi_3^2 + e\mathcal{A}^\mu(\pi_1 \partial_\mu \pi_2 - \pi_2 \partial_\mu \pi_1) + e^2 \mathcal{A}^2(\pi_1^2 + \pi_2^2). \tag{5.94}$$

where

$$\bar{m}_\pm^2 = \bar{m}^2 + 2\frac{e^2}{f^2}C. \tag{5.95}$$

This is the leading order electromagnetic contribution to the mass. It only affects the π_1, π_2 pions, which are a linear combination of $\pi_\pm$, the charged pions. To leading order, $\bar{m} = m_\pi$, the neutral pion mass, and $\bar{m}_\pm = m_{\pi_\pm}$ From the values listed in section 1.1, we find

$$\Delta m_\pm := \frac{e}{f}\sqrt{2C} = \sqrt{m_{\pi_\pm}^2 - m_\pi^2} = 35.50\,\mathrm{MeV}. \tag{5.96}$$

This corresponds to $C = 0.3771\,u_0 = 5.824 \cdot 10^{-5}\,\mathrm{GeV}^4$. We now no longer assume $\mu_I = 0$. The zeroth-order expansion in π/f is

$$\Sigma = e^{i\alpha\tau_1} = \sin\alpha + i\tau_1 \cos\alpha. \tag{5.97}$$

This gives the contributions

$$\operatorname{Tr}\{\nabla_\mu \Sigma \nabla^\mu \Sigma^\dagger\} = 2\sin^2\alpha\left(\mu_I^2 + 2e\mu\mathcal{A}_0 + e^2\mathcal{A}^2\right), \tag{5.98}$$

$$\operatorname{Tr}\{\chi\Sigma^\dagger + \Sigma\chi^\dagger\} = 4\bar{m}^2\cos\alpha, \tag{5.99}$$

$$\operatorname{Tr}\{Q\Sigma Q\Sigma^\dagger\} = \cos^2\alpha - \frac{4}{9}. \tag{5.100}$$

We are interested in the static Lagrangian, that is $\pi_a = \mathcal{A}_\mu = 0$. Inserting these terms into Eq. (5.89), we get

$$\mathcal{L}_0^{\mathrm{EM}} = f^2\left[\frac{1}{2}\mu_I^2 \sin^2\alpha + \bar{m}^2\cos\alpha + \frac{1}{2}\Delta m_{\pi_\pm}^2\left(\cos^2\alpha - \frac{4}{9}\right)\right]. \tag{5.101}$$

Chapter 6

Pion stars

6.1 Leading order, two flavor pion stars

6.1.1 Equation of state

The free energy density of two-flavor chiral perturbation theory, to leading-order and at $T = 0$, is

$$\mathcal{F} = -f^2 \left(\bar{m}^2 \cos\alpha + \frac{1}{2}\mu_I^2 \sin^2\alpha \right). \tag{6.1}$$

The α parameter is determined by minimizing $\mathcal{F}$ for a given value of μ_I,

$$\frac{\partial \mathcal{F}}{\partial \alpha} = f^2 \left(\bar{m}^2 - \mu_I^2 \cos\alpha \right) \sin(\alpha) = 0. \tag{6.2}$$

This gives an explicit formula for α in terms of μ_I. As long as the chemical potential is lower than the critical value $\mu_I^c = \bar{m}$, the only solution to this equation is $\alpha = 0$. As the chemical potential reaches this critical value, the system undergoes a phase transition from the vacuum phase to the *pion condensate* phase. In this new phase, the solution is

$$\cos\alpha = \frac{\bar{m}^2}{\mu_I^2}. \tag{6.3}$$

We introduce a dimensionless variable $x^2 = \cos\alpha = \bar{m}^2/\mu_I^2$. This variable has the domain $[0, 1]$. By an argument using a right triangle, we can verify that $\cos\alpha = x^2$ implies that $\sin^2\alpha = 1 - x^4$. Substituting the dimensionless variable into the free energy density, we get

$$\mathcal{F} = -\frac{u_0}{2}\left(x^2 + \frac{1}{x^2} \right). \tag{6.4}$$

We have introduced the characteristic energy density $u_0 = \bar{m}^2 f^2$. As we found in section 4.3, the pressure is given by negative the free energy density, normalized to $\mu_I = \bar{m}$, or $x = 1$. We choose $p_0 = u_0$, so the dimensionless pressure is

$$\tilde{p} = -\frac{1}{p_0}\left(\mathcal{F} - \mathcal{F}_{x=1} \right) = \frac{1}{2}\left(x^2 + \frac{1}{x^2} - 2 \right). \tag{6.5}$$

The charge density corresponding to a chemical potential is given by minus the derivative of the free energy with respect to that chemical potential. We must, however, not assume any dependence of α on μ_I when taking this derivative. The isospin density therefore is

$$n_I = -\frac{\partial \mathcal{F}}{\partial \mu_I} =^2 \mu_I^2 \sin^2\alpha = \frac{u_0}{\mu_I}\left(\frac{1}{x^2} - x^2 \right). \tag{6.6}$$

With this, the dimensionless energy density at $T = 0$ is

$$\tilde{u} = -\tilde{p} + \frac{\mu_I n_I}{u_0} = \frac{1}{2}\left(2 + \frac{1}{x^2} - 3x^2\right). \tag{6.7}$$

The ratio of pressure to energy density is

$$\frac{p}{u} = \frac{1 - 2x^2 + x^4}{1 + 2x^2 - 3x^4}. \tag{6.8}$$

In the ultrarelativistic limit, where $\mu_I \to \infty$ and thus $x \to 0$, we get $p/u = 1$, or $u_{\text{ur}} = p$. In the non-relativisitc limit, that is $x^{-2} = 1 + \epsilon$, $\epsilon \ll 1$, we get $\tilde{p} = \epsilon^2/2$, and $\tilde{u} = 2\epsilon$, so the equation of state is $\tilde{u}_{\text{nr}} = 2\sqrt{2}\sqrt{\tilde{p}}$. Figure 6.1 shows the equation of state in two different regimes and compares it with the ultrarelativistic and non-relativistic limit.

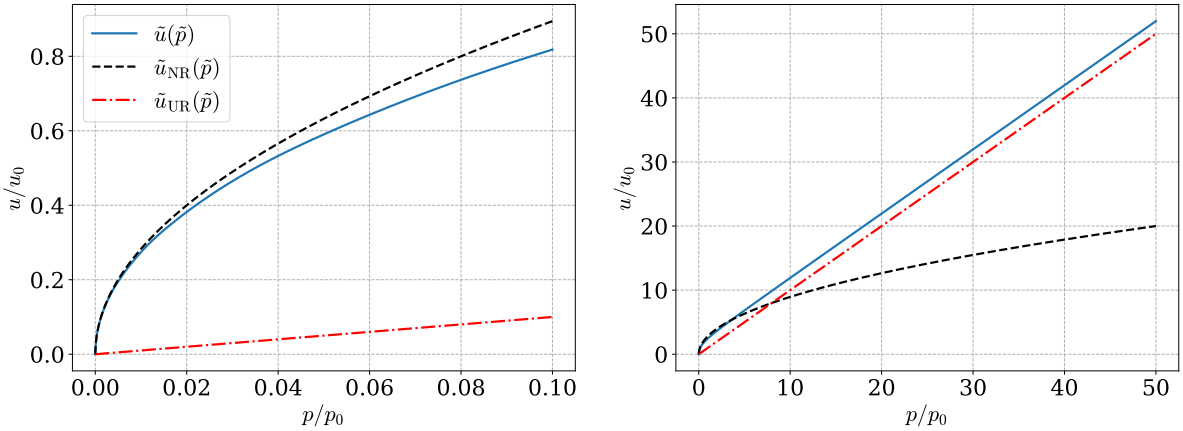


Figure 6.1: This plot shows the leading order equation of state of two-flavor chiral perturbation theory and compares it with the ultrarelativistic and non-relativistic limit, shown as dashed lines. The x-axis shows the pressure normalized to p_0, while the y-axis shows the energy density normalized to u_0.

6.1.2 Units

The characteristic mass and length, as discussed in section 4.2, are found by setting $k_1 = k_2 = k_3 = 1$. These are the dimensionless constants of the TOV equation, Eq. (4.45). At tree-level, the bare constants f and $\bar{m}$ are related to physical constants by $f = f_\pi$ and $m = m_\pi$, the pion decay constant and the pion mass. Using the values for f_π and m_π as given in section 1.1 and reinstating c and $\hbar$, these quantities are give in by

$$u_0 = m_\pi^2 f_\pi^2 \frac{c}{\hbar^3} = 3.216 \cdot 10^{33}\,\text{J}\,\text{m}^{-3}, \tag{6.9}$$

$$m_0 = \frac{c^4}{\sqrt{\frac{4\pi}{3} u_0 G^3}} = 64.21\,M_\odot, \tag{6.10}$$

$$r_0 = \frac{G}{c^2} m_0 = 94.79\,\text{km}. \tag{6.11}$$

We, therefore, expect both the radius and mass of the pion star to be around one order of magnitude larger than the star made up of cold neutrons.

6.1.3 Limiting radius

We found that the non-relativisitc limit of the equation of state is $\tilde{p} = 2^{-3}\tilde{u}^2$, i.e., it is a polytrope with $\gamma = 2$. As discussed in subsection 4.2.1, this corresponds to a situation where the radius of the star is independent

of the central pressure, at least in the Newtonian limit of gravity. When simulating the Newtonian, non-relativistic limit of the pion star, we should expect the radius to be constant. From Eq. (4.51), the radius is $R = r_0 C \xi_1$, where ξ_1 is the first zero of the Lane-Emden equation with $n = 1$, and

$$C = \frac{1}{\sqrt{4(4\pi)Gu_0}} = \frac{1}{\sqrt{12}} r_0. \tag{6.12}$$

To find ξ_1, we must find the root of

$$\theta'' + \frac{2}{\xi}\theta' + \theta = 0. \tag{6.13}$$

By substituting θ for its power series expansion, $\theta = \sum_n a_n \theta^n$, we get

$$\sum_n \left[(n+2)(n+1)a_{n+2} + 2(n+1)a_{n+1}\xi^{-1} + a_n\right]\xi^n = 0, \tag{6.14}$$

and we get the recursion relation $a_{n+2} = -a_n/(n+1)(n+2)$. With our boundary condition, the solution is

$$\theta(\xi) = \frac{\sin(\xi)}{\xi}, \tag{6.15}$$

and the first root is therefore $\xi_1 = \pi$. With this, we get a closed-form expression for the stellar radius of this non-relativistic and Newtonian limit—which we expect the full theory to approach as the central pressure decreases—namely

$$R = \frac{\pi}{\sqrt{12}} r_0 = 85.97\,\text{km}. \tag{6.16}$$

6.1.4 Results

The code used for obtaining numerical results is discussed in Appendix C.

Figure 6.2 show the pressure and mass as a function of radius for varying values of central pressure. The quantities are normalized to the stellar radius, stellar mass, and central pressure, respectively. The black dashed line corresponds to the configuration with the maximum mass. We see that both the pressure and mass distribution are very similar for stars with a mass less than the maximum. As the central pressure increase beyond that of the star with maximum mass, the pressure gradient close to the center grows sharply. This is similar to what we saw in the case of an incompressible fluid, subsection 4.2.2.

Figure 6.3 shows the mass-radius relation for the pion star. As in the case of the neutron star, it has a maximum mass, in this case of $M_{\text{max}} = 10.47\,M_\odot$. However, in contrast to the case of the neutron star, the stellar radius approaches a maximum radius as the central pressure decreases. This matches our expectation from the non-relativistic, Newtonian limit. We see that the largest radius in our results, corresponding to $p_c = 10^{-6}\,p_0$, is $R = 85.82\,\text{km}$, which is in good agreement with our earlier analysis, Eq. (6.16).

Figure 6.4 compares the mass-radius relation from the full equation of state and TOV equation with various limits. In the non-relativistic, Newtonian limit, the stellar radius is independent of the mass, as we found in our earlier analysis.

Vis kjemisk potensial

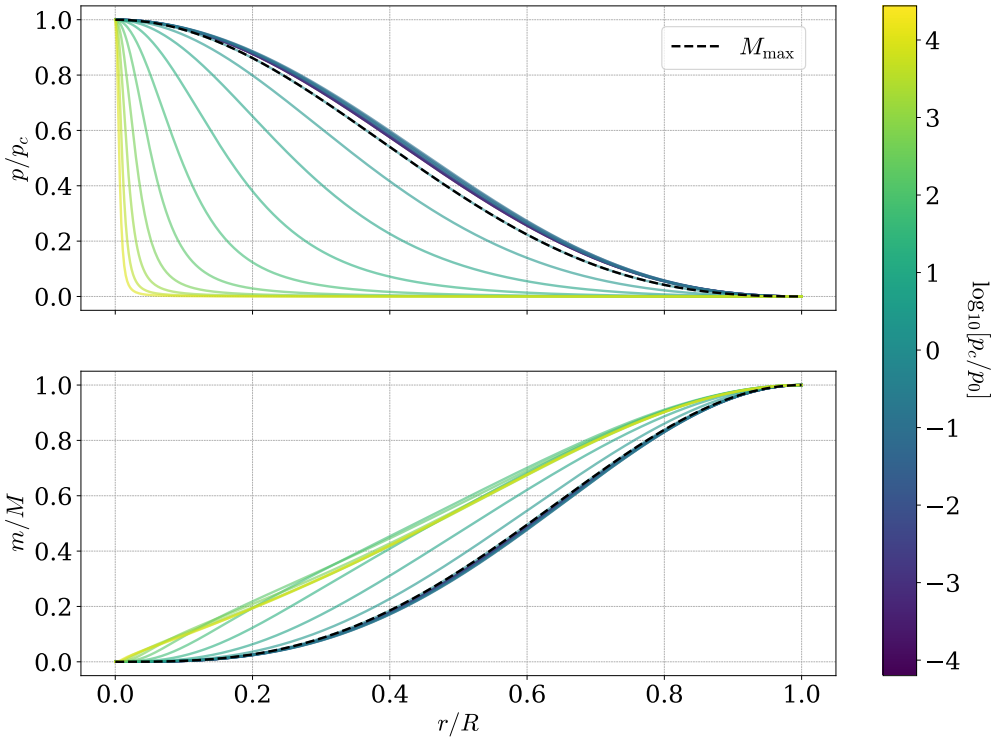


Figure 6.2: Top: The pressure normalized to the central pressure, as a function of radius, normalized to the stellar radius. Bottom: The mass, normalized to stellar mass, within a radius r, normalized to the stellar radius. Both plots show a range of stars with different central pressures, indicated by the color. The black dashed line corresponds to the star with the largest mass.

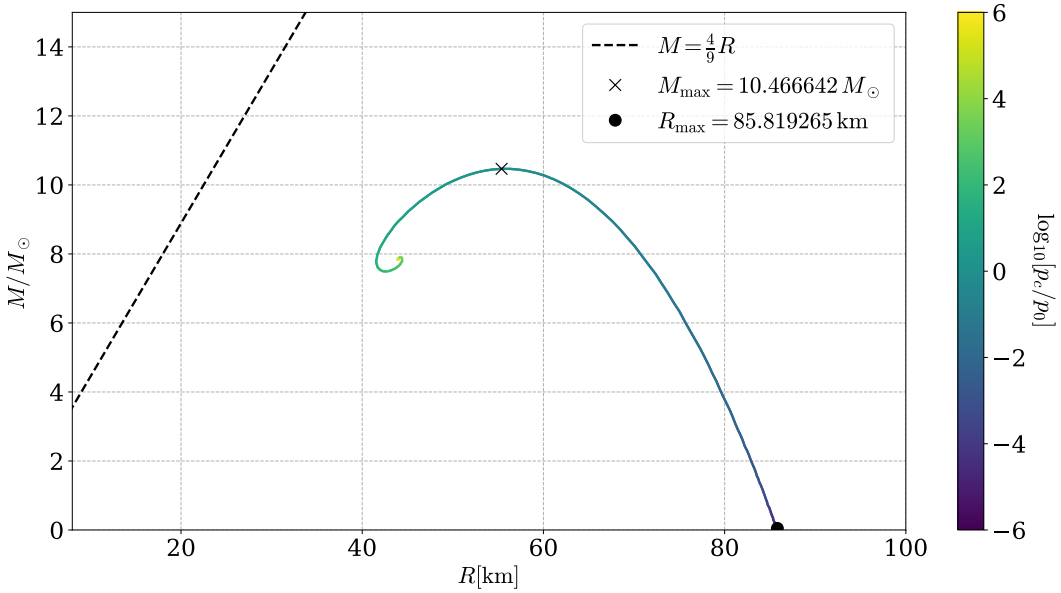


Figure 6.3: The plot shows the relationship between the mass and radius of a pion star. Mass is given in units of solar masses, while the radius is measured in kilometers. This line is parameterized by the central pressure p_c of the star, as indicated by the color gradient. The dashed black line indicates the theoretical maximum mass for a given radius, and any configuration above it will collapse to a black hole.

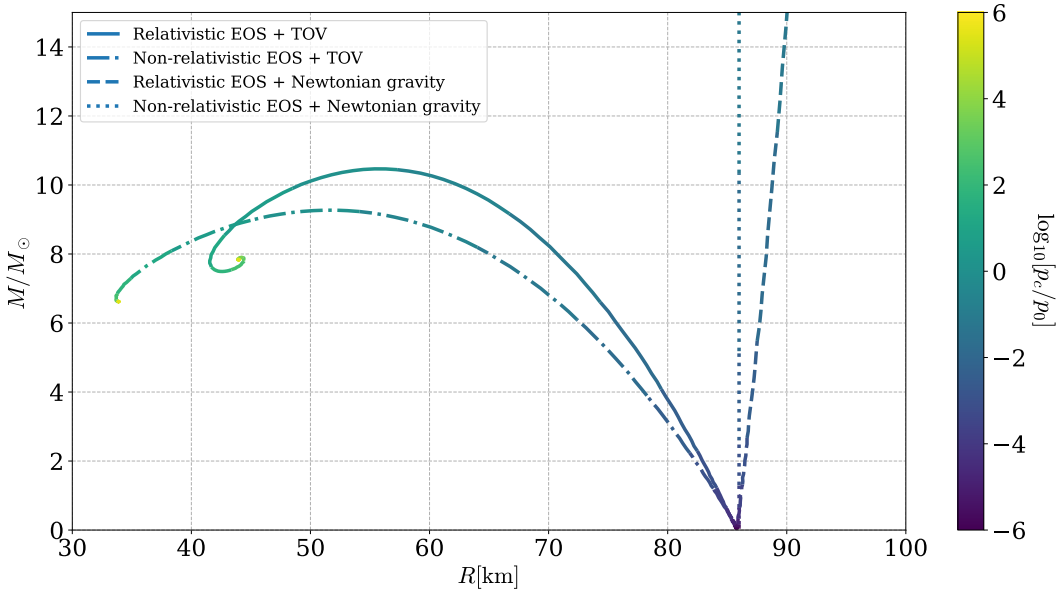


Figure 6.4: The plot compares the mass-radius relationship of the pion star from the full equation of state and the TOV-equation with various limits.

6.1.5 Including electromagnetic contributions

From Eq. (5.101), the free energy density, including electromagnetic interactions, is

$$\mathcal{F} = -f^2\left[\bar{m}^2\cos\alpha + \frac{1}{2}\mu_I^2\sin^2\alpha + \frac{1}{2}\Delta m_\pm^2\left(\cos^2\alpha - \frac{4}{9}\right)\right]. \tag{6.17}$$

Free energy minimization now gives

$$\frac{1}{u_0}\frac{\partial\mathcal{F}}{\partial\alpha} = \left[\left(\frac{1}{x^2} - \Delta\right)\cos\alpha - 1\right]\sin\alpha = 0. \tag{6.18}$$

Here, x is defined as before, and we introduced the new quantity $\Delta = \Delta m_\pm^2/\bar{m}^2 = 0.06916$. We see that the phase transition is raised, the critical chemical potential is now $\mu_I^c = \bar{m}\sqrt{1+\Delta}$, the mass of the charged pions. Below this value, $\alpha = 0$ remains the only solution. In the pion condensate phase, the solution is

$$\cos\alpha = \frac{x^2}{1-\Delta x^2}. \tag{6.19}$$

This reduces to our old solution for $\Delta = 0$, as it should. With the same procedure as in the last section, we get the pressure and energy density

$$\tilde{p}_{\rm EM} = \frac{1}{2}\left[\frac{1}{x^2} + \frac{x^2}{1-x^2\Delta} - 2 - \Delta\right], \tag{6.20}$$

$$\tilde{u}_{\rm EM} = \frac{1}{2}\left[\frac{1}{x^2} - x^2\frac{3-\Delta x^2}{(1-\Delta x^2)^2} + 2 + \Delta)\right]. \tag{6.21}$$

In the limit $\Delta = 0$, these reduce to Eq. (6.5) and Eq. (6.7). In the ultra-relativistic limit, that is, for $x \ll 1$, the behavior is the same as before, and we again approach $p = u$. We find the non-relativistic limit by substituting $x^{-2} = 1 + \Delta + \epsilon$. To first order in ϵ we get $\tilde{p} = \epsilon/2$, which is the same as before. However, the limit of the energy density is slightly perturbed by the inclusion of electromagnetism and is now $\tilde{u} = 2(1+\Delta)\tilde{\epsilon}$. The non-relativistic equation of state is thus still a polytrope of the form $p = Ku^2$, however the constant is now $K^{-1} = 8(1+\Delta)^2$. With this, the radius of the polytrope and the limiting radius of the full system changes and is now

$$R = \frac{\pi}{\sqrt{12}(1+\Delta)}r_0 = 80.40\,\text{km}. \tag{6.22}$$

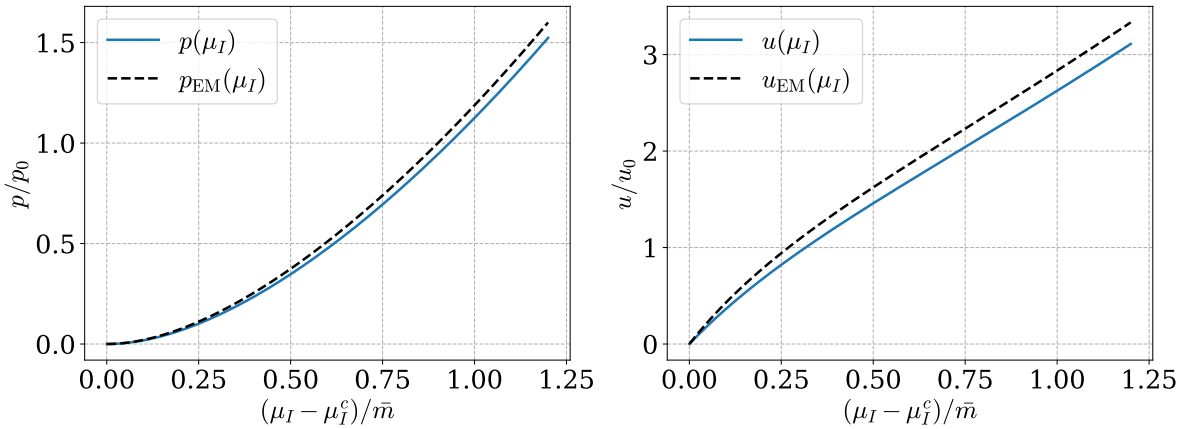


Figure 6.5: Left: The pressure, normalized to p_0, as a function of the chemical potential above the critical value, normalized to $\bar{m}$. Right: The energy density, normalized to u_0, also as a function of the chemical potential. Results with electromagnetic interaction are shown as dashed lines.

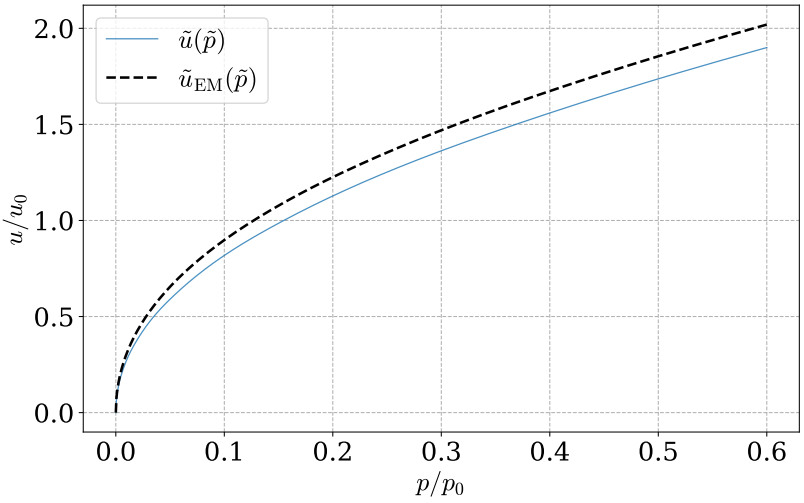


Figure 6.6: The equation of state in the pion condensate phase. Results with electromagnetic interactions are shown as dashed lines.

Figure 6.5 shows the pressure and energy density, normalized to their characteristic quantities, as a function of chemical potential above the critical value, normalized to $\bar{m}$. Figure 6.6 shows the equation of state. The results with and without electromagnetic results are compared.

Figure 6.7 shows the mass-radius reaction of the pion star when the electromagnetic interaction is taken into account. We see that the shape of the curve has not changed much from our earlier result. Both the maximum mass and radius are slightly smaller. The result with and without electromagnetic interaction is compared in Figure 6.8.

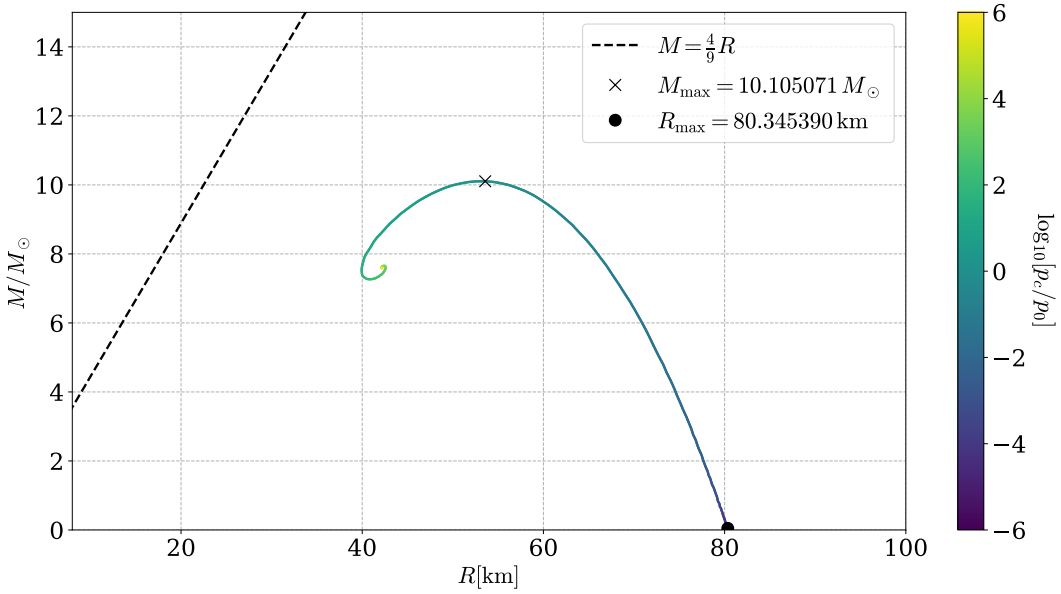


Figure 6.7: The mass-radius relation of a pion star when the electromagnetic interactions is included. The relation is parameterized by the logarithm of the central pressure. The dashed line shows the absolute limiting mass for a given radius. The cross indicates the maximum mass configuration, and the dot the maximum radius configuration.

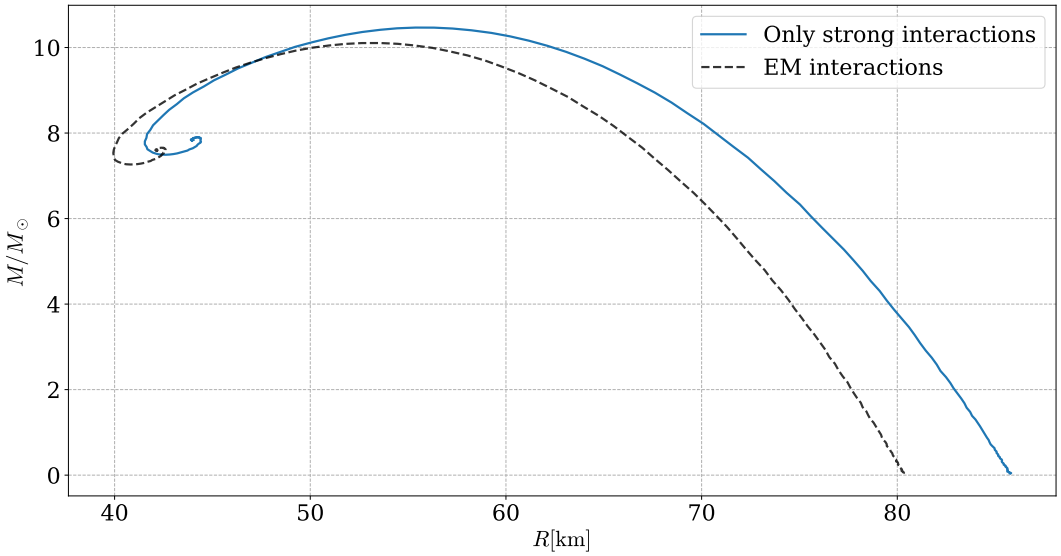


Figure 6.8: This plot compares the mass-radius relation of pion stars with and without the inclusion of electromagnetism.

Appendix A

A.1 Algebra bases

A.1.1 Pauli matrices

The $\mathfrak{su}(2)$ basis used is the Pauli matrices,

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.1}$$

They obey

$$[\tau_a, \tau_b] = 2i\varepsilon_{abc}\tau_c, \tag{A.2}$$
$$\{\tau_a, \tau_b\} = 2\delta_{ab}\mathbb{1}, \tag{A.3}$$
$$\mathrm{Tr}\{\tau_a\} = 0, \tag{A.4}$$
$$\mathrm{Tr}\{\tau_a\tau_b\} = 2\delta_{ab}, \tag{A.5}$$
$$\mathrm{Tr}\{\tau_a\tau_b\tau_c\tau_d\} = 2(\delta_{ab}\delta_{cd} - \delta_{ac}\delta_{cb} + \delta_{ad}\delta_{cb}). \tag{A.6}$$

Together with the identity matrix $\mathbb{1}$, the Pauli matrices form a basis for the vector space of all 2-by-2 matrices. An arbitrary 2-by-2 matrix M may be written

$$M = M_0\mathbb{1} + M_a\tau_a, \quad M_0 = \frac{1}{2}\mathrm{Tr}\{M\},\ M_a = \frac{1}{2}\mathrm{Tr}\{\tau_a M\}. \tag{A.7}$$

A.1.2 Gell-Mann matrices

write down the Gell-Mann matrices and their properties

A.1.3 Gamma matrices

The gamma matrices γ^μ, $\mu \in \{0, 1, 2, 3\}$, obey

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}, \tag{A.8}$$
$$\gamma^{0\dagger} = \gamma^0, \quad \gamma^{i\dagger} = -\gamma^i. \tag{A.9}$$

These matrices, together with

$$\sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\mu], \tag{A.10}$$
$$\gamma^\mu_A = \gamma^\mu\gamma^5, \tag{A.11}$$
$$\gamma^5 = \frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma, \tag{A.12}$$

form the Clifford algebra $\mathrm{Cl}_{1,3}$, also known as the *space-time algebra*. The subscripts $(1,3)$ denotes the signature of the metric. The "fifth γ-matrix", which can be expressed as $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$, obey

$$\{\gamma^5, \gamma^\mu\} = 0, \quad (\gamma^5)^2 = \mathbb{1}. \tag{A.13}$$

The Euclidian counterpart of the space-time algebra, Cl_4, is defined by the "Euclidian gamma matrices", which obey

$$\{\tilde{\gamma}_a, \tilde{\gamma}_b\} = 2\delta_{ab}\mathbb{1}. \tag{A.14}$$

These can be related to the regular Minkowski-matrices by

$$\tilde{\gamma}_0 = \gamma^0, \quad \tilde{\gamma}_j = -i\gamma^j. \tag{A.15}$$

These then obey

$$\tilde{\gamma}_a^\dagger = \tilde{\gamma}_a. \tag{A.16}$$

The Euclidean $\tilde{\gamma}_5$ is defined as

$$\tilde{\gamma}_5 = \gamma_0\gamma_1\gamma_2\gamma_3 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5. \tag{A.17}$$

It thus also anti-commutes with the Euclidean γ-matrices,

$$\{\tilde{\gamma}_5, \tilde{\gamma}_a\} = 0. \tag{A.18}$$

A.2 Functionals

(TODO: INKLUDER KILER!!!!)

The principle of stationary action and the path integral method relies on functional calculus, where ordinary, n-dimensional calculus is generalized to an infinite-dimensional calculus on a space of functions. A functional, S, takes in a function $\varphi(x)$, and returns a real number $S[\varphi]$. We will be often be dealing with functionals of the form

$$S[\varphi] = \int_{\mathcal{M}} \mathrm{d}^n x\, \mathcal{L}[\varphi](x), \tag{A.19}$$

Here, $\mathcal{L}[\varphi](x)$, the Lagrangian density, is a functional which takes in a function φ, and returns a real number $\mathcal{L}[\varphi](x)$ *for each point* $x \in \mathcal{M}$. Thus, $\mathcal{L}$ does, in fact, return a real-valued function, not just a number. $\mathcal{M}$ is the manifold, in our case space-time, of which both $\varphi(x)$ and $\mathcal{L}[\varphi](x)$ are functions. The function φ can, in general, take on the value of a scalar, complex number, spinor, vector, etc…, while $\mathcal{L}[\varphi](x)$ must be a scalar-valued function. This strongly constraints the form of any Lagrangian and is an essential tool in constructing quantum field theories. Although this section is written with a single scalar-valued function φ, this can easily be generalized by adding an index, $\varphi \to \varphi_\alpha$, enumerating all the degrees of freedom, then restating the arguments [2, 5].

A.2.1 Functional derivative

The functional derivative is base on an arbitrary *variation* η of the function φ. The variation η, often written $\delta\varphi$ is an arbitrary function only constrained to vanish *quickly enough* at the boundary $\partial\mathcal{M}$.[1] The variation of the functional S is defined as

$$\delta_\eta S[\varphi] = \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left(S[\varphi + \epsilon\eta] - S[\varphi]\right) = \frac{\mathrm{d}}{\mathrm{d}\epsilon} S[\varphi + \epsilon\eta]|_{\epsilon=0}. \tag{A.20}$$

We can regard the variation of a functional as the generalization of the differential of a function, Eq. (2.18), as the best linear approximation around a point. In regular differential geometry, a function f can be approximated around a point x by

$$f(x + \epsilon v) = f(x) + \epsilon \mathrm{d}f(v), \tag{A.21}$$

[1] The condition of "quickly enough" is to ensure that we can integrate by parts and ignore the boundary condition, which we will do without remorse.

where v is a vector in the tangent space at x. In functional calculus, the functional S is analogous to f, φ to x, and η to v. We can more clearly see the resemblance by writing

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon} f(x+\epsilon v) = \mathrm{d}f(v) = \frac{\partial f}{\partial x^\mu} v^\mu. \tag{A.22}$$

In the last line we expanded the differential using the basis-representation, $v = v^\mu \partial_\mu$. To generalize this to functional, we define the *functional derivative*, by

$$\delta_\eta S[\varphi] = \int_{\mathcal{M}} \mathrm{d}^n x \, \frac{\delta S[\varphi]}{\delta \eta(x)} \eta(x). \tag{A.23}$$

If we let $S[\varphi] = \varphi(x)$, for some fixed x, the variation becomes

$$\delta_\eta S[\varphi] = \eta(x) = \int \mathrm{d}^n y \, \delta(x-y)\eta(y), \tag{A.24}$$

which leads to the identity

$$\frac{\delta \varphi(x)}{\delta \varphi(y)} = \delta(x-y). \tag{A.25}$$

There is also a generalized chain rule for functional derivatives. If ψ is some new functional variable, then

$$\frac{\delta S[\varphi]}{\delta \varphi(x)} = \int_{\mathcal{M}} \mathrm{d}^n y \, \frac{\delta S[\varphi]}{\delta \psi(y)} \frac{\delta \psi(y)}{\delta \varphi(x)}. \tag{A.26}$$

Higher functional derivatives are defined in terms of higher-order variations,

$$\delta_\eta^m S[\varphi] = \frac{\mathrm{d}}{\mathrm{d}\epsilon} \delta_\eta^{m-1} S[\varphi + \epsilon\eta]|_{\epsilon=0} = \int_{\mathcal{M}} \left(\prod_{i=1}^m \mathrm{d}^n x_i \eta(x_i) \right) \frac{\delta^m S[\varphi]}{\delta\varphi(x_1)...\delta\varphi(x_m)}. \tag{A.27}$$

With this, we can write the functional Taylor expansion,

$$S[\varphi_0 + \varphi] = S[\varphi_0] + \int_{\mathcal{M}} \mathrm{d}^n x \, \varphi(x) \frac{\delta S[\varphi_0]}{\delta \varphi(x)} + \frac{1}{2} \int_{\mathcal{M}} \mathrm{d}^n x \mathrm{d}^n y \, \varphi(x)\varphi(y) \frac{\delta S[\varphi_0]}{\delta \varphi(x), \varphi(y)} + \dots \tag{A.28}$$

Here, the notation $\delta S[\varphi_0]/\delta\varphi$ indicate that $S[\varphi]$ is first differentiated with respect to φ, then evaluated at $\varphi = \varphi_0$ [4].

A.2.2 The Euler-Lagrange equation

The Lagrangian may also be written as a scalar function of the field-values at x, $\varphi(x)$, as well as its derivatives, $\partial_\mu \varphi(x)$, for example

$$\mathcal{L}(\varphi, \partial_\mu \varphi) = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} m^2 \varphi^2 - \frac{1}{4!} \lambda \varphi^4 + \dots \tag{A.29}$$

We have omitted the evaluation at x for the brevity of notation. We use this to evaluate the variation of a functional in the of Eq. (A.19),

$$\delta_\eta S[\varphi] = \frac{\mathrm{d}}{\mathrm{d}\epsilon} \int_{\mathcal{M}} \mathrm{d}^n x \, \mathcal{L}[\varphi + \epsilon\eta](x), \tag{A.30}$$

by Taylor expanding the Lagrangian density as a function of φ and its derivatives,

$$\mathcal{L}[\varphi + \epsilon\eta] = \mathcal{L}\left(\varphi + \epsilon\eta, \partial_\mu\{\varphi + \epsilon\eta\}\right) = \mathcal{L}[\varphi] + \epsilon \left(\frac{\partial \mathcal{L}}{\partial \varphi} \eta + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\mu \eta \right) + \mathcal{O}(\epsilon^2). \tag{A.31}$$

Inserting this into Eq. (A.30) and partially integrating the last term allows us to write the variation in the form Eq. (A.23), and the functional derivative is

$$\frac{\delta S}{\delta \varphi} = \frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}. \tag{A.32}$$

The principle of stationary action says that the equation of motion of a field obeys $\delta_\eta S = 0$. As η is arbitrary, this is equivalent to setting the functional derivative of S equal to zero. The result is the Euler-Lagrange equations of motion [5],

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} = 0. \tag{A.33}$$

A.2.3 Functional calculus on a curved manifold

As discussed in subsection 2.1.4, when integrating a scalar on a curved manifold, we must include the $\sqrt{|g|}$-factor to get a coordinate-independent result. The action in curved spacetime is therefore [2]

$$S[g,\varphi] = \int_{\mathcal{M}} \mathrm{d}^n x \, \sqrt{|g|}\mathcal{L}[g,\varphi], \tag{A.34}$$

where the action and the Lagrangian now is a functional of both the matter-field φ and the metric $g_{\mu\nu}$. Our example Lagrangian from last section now takes the form

$$\mathcal{L}(g_{\mu\nu},\varphi,\nabla_\mu\varphi) = \frac{1}{2}g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{1}{4!}\lambda\varphi^4\ldots, \tag{A.35}$$

where partial derivatives are substituted with covariant derivatives. We define the functional derivative as

$$\delta_\eta S = \int_{\mathcal{M}} \mathrm{d}^n x \sqrt{|g|}\frac{\delta S}{\delta\eta(x)}\eta(x). \tag{A.36}$$

If this is a variation in φ only, this gives the same result as before. However, in general relativity, the metric itself is a dynamic field, and we may thereofre vary it. Consider $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$. The variation of the action is then assuming $\mathcal{L}$ only depends on g and not its derivatives, we get

$$\delta_g S = \int_{\mathcal{M}} \mathrm{d}^n x \, \left[\left(\delta\sqrt{|g|}\right)\mathcal{L}[g] + \sqrt{|g|}\delta\mathcal{L}[g]\right] \tag{A.37}$$

We have used

$$\delta\sqrt{|g|} = -\frac{1}{2}\sqrt{|g|}g_{\mu\nu}\delta g^{\mu\nu} \tag{A.38}$$

The variation of the $\sqrt{|g|}$-factor can be evaluated using Using the Levi-Civita symbol $\varepsilon_{\mu_1\ldots\mu_n}$, a determinant of a $n\times n$-matrix may be written as

$$\det(A) = \frac{1}{n!}\varepsilon_{\mu_1\ldots\mu_n}\varepsilon^{\nu_1\ldots\nu_n}A^{\mu_1}{}_{\nu_1}\ldots A^{\mu_n}{}_{\nu_n}. \tag{A.39}$$

Using this, we can write for a matrix M

$$\begin{aligned}
\det(\mathbb{1}+\varepsilon M) &= \frac{1}{n!}\varepsilon_{\mu_1\ldots}\varepsilon^{\nu_1\cdots}(\mathbb{1}+\epsilon M)^{\mu_1}{}_{\nu_1}(\mathbb{1}+\epsilon M)^{\mu_2}{}_{\nu_2}\ldots && \text{(A.40)}\\
&= \frac{1}{n!}\varepsilon_{\mu_1\ldots}\varepsilon^{\nu_1\cdots}[\delta^{\mu_1}_{\nu_1}\cdots+\epsilon(M^{\mu_1}{}_{\nu_1}\delta^{\mu_2}_{\nu_1}\cdots+M^{\mu_2}{}_{\nu_2}\delta^{\mu_1}_{\nu_1}\cdots+\ldots)+\ldots] && \text{(A.41)}\\
&= 1 + M^\mu{}_\mu + \mathcal{O}(\epsilon^2) && \text{(A.42)}
\end{aligned}$$

Thus,

$$\delta\sqrt{|g|} = \sqrt{|\det\left[g_{\mu\nu}(\delta^\nu_\rho + g^{\nu\sigma}\delta g_{\sigma\rho})\right]|} - \sqrt{|g|} = \sqrt{|g|}\left(\sqrt{|1+g^{\mu\nu}\delta g_{\mu\nu}|}-1\right) = -\frac{1}{2}\sqrt{|g|}g^{\mu\nu}\delta g_{\mu\nu}. \tag{A.43}$$

The minus sign is included as the determinant of a Lorentzian metric is negative. Assuming the Lagrangian only depneds on the metric directly, and not its derivatives, the variation of the action is

$$\delta_g S = \int_{\mathcal{M}} \mathrm{d}^n x \, \sqrt{|g|}\left(\frac{\partial\mathcal{L}}{\partial g^{\mu\nu}} - \frac{1}{2}g_{\mu\nu}\mathcal{L}\right)\delta g^{\mu\nu}. \tag{A.44}$$

With the Lagrangian in Eq. (A.35), we get

$$\frac{\delta S}{\delta g^{\mu\nu}} = \frac{\partial\mathcal{L}}{\partial g^{\mu\nu}} - \frac{1}{2}g_{\mu\nu}\mathcal{L} = -\frac{1}{2}\left(\frac{1}{2}\nabla_\mu\varphi\nabla_\nu\varphi + \frac{1}{2}m^2\varphi^2 + \ldots\right). \tag{A.45}$$

We recognize the $(\mu,\nu)=(0,0)$-component as negative half the Hamiltonian density, which supports the definition of the definition of the stress-energy tensor Eq. (4.4).

A.2.4 Functional derivative of the Einstein-Hilbert action

(NEEDS MORE CLEANUP)

In the Einstein-Hilbert action, Eq. (4.1), the Lagrangian density is $\mathcal{L} = kR = kg^{\mu\nu}R_{\mu\nu}$, where k is a constant and $R_{\mu\nu}$ the Ricci tensor, Eq. (2.40). As the Ricci tensor is dependent on both the derivative and second derivative of the metric, we can not use Eq. (A.45) directly. Instead, we use the variation

$$\delta S_{\text{EH}} = k \int_{\mathcal{M}} \mathrm{d}^n x \, \sqrt{|g|} \left(\delta R - \frac{1}{2} g_{\mu\nu} R \delta g^{\mu\nu} \right). \tag{A.46}$$

The variation of the Ricci scalar is

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}, \tag{A.47}$$

We can write the variation of the Ricci scalar, and thus the Riemann curvature tensor, in terms of variations in Christoffel symbols, $\delta\Gamma^{\rho}_{\mu\nu}$ using the explicit formula for a symmetric, metric-compatible covariant derivative, Eq. (2.39). As $\delta\Gamma = \Gamma - \Gamma'$, it is a tensor, and we may write clean up

$$\begin{aligned}
\delta R^{\rho}{}_{\sigma\mu\nu} &= \delta(\partial_{[\mu}\Gamma^{\rho}_{\nu]\sigma} + \Gamma^{\rho}_{\lambda[\mu}\Gamma^{\lambda}_{\nu]\sigma}) = \partial_{[\mu}\delta\Gamma^{\rho}_{\nu]\sigma} + (\delta\Gamma^{\rho}_{\lambda[\mu})\Gamma^{\lambda}_{\nu]\sigma} + \Gamma^{\rho}_{\lambda[\mu}(\delta\Gamma^{\lambda}_{\nu]\sigma}) \\
&= \partial_{\mu}\delta\Gamma^{\rho}_{\nu\sigma} + \Gamma^{\rho}_{\lambda\mu}(\delta\Gamma^{\lambda}_{\nu\sigma}) - \Gamma^{\lambda}_{\mu\sigma}(\delta\Gamma^{\rho}_{\lambda\nu}) - \left(\partial_{\nu}\delta\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\lambda\nu}(\delta\Gamma^{\lambda}_{\mu\sigma}) - \Gamma^{\lambda}_{\nu\sigma}(\delta\Gamma^{\rho}_{\lambda\mu})\right) + (\Gamma^{\lambda}_{\mu\nu}\delta\Gamma^{\rho}_{\lambda\sigma} - \Gamma^{\lambda}_{\mu\nu}\delta\Gamma^{\rho}_{\lambda\sigma}) \\
&= \nabla_{\mu}\delta\Gamma^{\rho}_{\nu\sigma} - \nabla_{\nu}\delta\Gamma^{\rho}_{\mu\sigma} = \nabla_{\eta}\left(g^{\eta}{}_{\mu}\delta\Gamma^{\rho}_{\nu\sigma} - g^{\eta}{}_{\nu}\delta\Gamma^{\rho}_{\mu\sigma}\right) = \nabla_{\eta}(K^{\rho}{}_{\sigma\mu\nu})^{\eta},
\end{aligned}$$

where K is a tensorial quantity, which vanish at the boundary of our spacetime. Using the generalized divergence theorem, Eq. (2.67), we see that the contribution to the action from this quantity vansih. The contribution comes from an integral over $g^{\mu\nu}\delta R_{\mu\nu} = g^{\mu\nu}\delta R^{\rho}_{\mu\rho\nu} = g^{\mu\nu}\nabla_{\eta}(K^{\rho}{}_{\mu\rho\nu})^{\eta}$ Using metric compatibility, we can exchange the covariant derivative and the metric, and we have $g^{\mu\nu}\delta R_{\mu\nu} = \nabla_{\eta}[g^{\mu\nu}K^{\eta\rho}{}_{\mu\rho\nu}]$. The contribution to the action therefore becomes

$$\int_{\mathcal{M}} \mathrm{d}^4 x \, \sqrt{|g|} g^{\mu\nu}\delta R_{\mu\nu} = \int_{\mathcal{M}} \mathrm{d}^4 x \, \sqrt{|g|}\nabla_{\eta}[g^{\mu\nu}K^{\eta\rho}{}_{\mu\rho\nu}] = \int_{\partial\mathcal{M}} \mathrm{d}^3 y \, \sqrt{|\gamma|} n_{\eta}[g^{\mu\nu}K^{\eta\rho}{}_{\mu\rho\nu}] = 0, \tag{A.48}$$

where we used the fact that $\delta g_{\mu\nu}$, and thus K, vanish at $\partial\mathcal{M}$. The variation of the action is therefore

$$\delta S_{\text{EH}} = k \int_{\mathcal{M}} \mathrm{d}^n x \sqrt{|g|} \left[R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right] \delta g^{\mu\nu}, \tag{A.49}$$

and by the definition of the functional derivative,

$$\frac{\delta S_{\text{EH}}}{\delta g^{\mu\nu}} = k(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}). \tag{A.50}$$

Appendix B

Thermal field theory

This section is based on

B.1 Statistical mechanics

In classical mechanics, a thermal system at temperature $T = 1/\beta$ is described as an ensemble state, which have a probability P_n of being in state n, with energy E_n. In the canonical ensemble, the probability is proportional to $e^{-\beta E_n}$. The expectation value of some quantity A, with value A_n in state n is

$$\langle A \rangle = \sum_n A_n P_n = \frac{1}{Z} \sum_n A_n e^{-\beta E_n}, \quad Z = \sum_n e^{-\beta A_n}.$$

Z is called the partition function. In quantum mechanics, an ensemble configuration is described by a non-pure density operator,

$$\hat{\rho} = \sum_n P_n \left| n \right\rangle \!\left\langle n \right|,$$

where $|n\rangle$ is some basis for the relevant Hilbert space. Assuming $|n\rangle$ are energy eigenvectors, i.e., $\hat{H} |n\rangle = E_n |n\rangle$, the density operator for the canonical ensemble, where $P_n = Ce^{-\beta E_n}$, is

$$\hat{\rho} = \sum_n Ce^{-\beta E_n} \left| n \right\rangle \!\left\langle n \right| = Ce^{-\beta \hat{H}} \sum_n \left| n \right\rangle \!\left\langle n \right| = Ce^{-\beta \hat{H}}.$$

The expectation value in the ensemble state of a quantity corresponding to the operator $\hat{A}$ is given by

$$\langle A \rangle = \frac{\mathrm{Tr}\left\{ \hat{\rho} \hat{A} \right\}}{\mathrm{Tr}\{\hat{\rho}\}} = \frac{1}{Z} \mathrm{Tr}\left\{ \hat{A} e^{-\beta \hat{H}} \right\}. \tag{B.1}$$

The partition function Z ensures that the probabilities adds up to 1, and is defined as

$$Z = \mathrm{Tr}\left\{ e^{-\beta \hat{H}} \right\}. \tag{B.2}$$

The grand canonical ensemble takes into account the conserved charges of the system, which are a result of Nöther's theorem, as discussed in In the grand canonical ensemble, a system with n conserved charges Q_i has probability proportional to $e^{-\beta(H - \mu_i Q_i)}$. Here, μ_i are the chemical potentials corresponding to conserved charge Q_i. This leads to the partition function

$$Z = \mathrm{Tr}\left\{ e^{-\beta(\hat{H} - \mu_i \hat{Q}_i)} \right\}. \tag{B.3}$$

B.2 Imaginary-time formalism

The partition function may be calculated similarly to the path integral approach, in what is called the imaginary-time formalism. This formalism is restricted to time independent problems, and is used to study fields in a volume V. This volume is taken to infinity in the thermodynamic limit. As an example, take a scalar quantum field theory with the Hamiltonian

$$\hat{H} = \int_V \mathrm{d}^3x\, \hat{\mathcal{H}}\left[\hat{\varphi}(\vec{x}), \hat{\pi}(\vec{x})\right], \tag{B.4}$$

where $\hat{\varphi}(\vec{x})$ is the field operator, and $\hat{\pi}(\vec{x})$ is the corresponding canonical momentum operator. These field operators have time independent eigenvectors, $|\varphi\rangle$ and $|\pi\rangle$, defined by

$$\hat{\varphi}(\vec{x})\,|\varphi\rangle = \varphi(\vec{x})\,|\varphi\rangle\,, \quad \hat{\pi}(\vec{x})\,|\pi\rangle = \pi(\vec{x})\,|\pi\rangle\,. \tag{B.5}$$

In analogy with regular quantum mechanics, they obey the relations [1]

$$\mathbb{1} = \int \mathcal{D}\varphi(\vec{x})\,|\varphi\rangle\langle\varphi| = \int \mathcal{D}\pi(\vec{x})\,|\pi\rangle\langle\pi|\,, \tag{B.6}$$

$$\langle\varphi|\pi\rangle = \exp\!\left(i\int_V \mathrm{d}^3x\,\varphi(\vec{x})\pi(\vec{x})\right), \tag{B.7}$$

$$\langle\pi_a|\pi_b\rangle = \delta(\phi_a - \phi_b), \quad \langle\varphi_a|\varphi_b\rangle = \delta(\varphi_a - \varphi_b). \tag{B.8}$$

The functional integral is defined by starting with M degrees of freedom, $\{\varphi_m\}_{m=1}^M$ located at a finite grid $\{\vec{x}_m\}_{m=1}^M \subset V$. The integral is then the limit of the integral over all degrees of freedom, as $M \to \infty$:

$$\int \mathcal{D}\varphi(\vec{x}) = \lim_{M\to\infty} \int \left(\prod_{m=1}^M \mathrm{d}\varphi_m\right).$$

The functional Dirac-delta $\delta(f) = \prod_x \delta(f(x))$ is generalization of the familiar Dirac delta function. Given a functional $\mathcal{F}[f]$, it is defined by the relation

$$\int \mathcal{D}f(x)\,\mathcal{F}[f]\delta(f-g) = \mathcal{F}[g]. \tag{B.9}$$

The Hamiltonian is the limit of a sum of Hamiltonians $\hat{H}_m$ for each point $\vec{x}_m$

$$\hat{H} = \lim_{M\to\infty} \sum_{m=1}^M \frac{V}{M}\hat{H}_m(\{\hat{\varphi}_m\}, \{\hat{\pi}_m\}).$$

H_m may depend on the local degrees of freedom $\hat{\varphi}_m$, $\hat{\pi}_m$ as well as those at neighboring points. By inserting the completeness relations N times into the definition of the partition function, it may be written as

$$Z = \int \mathcal{D}\varphi(\vec{x})\,\left\langle\varphi\middle|e^{-\beta\hat{H}}\middle|\varphi\right\rangle = \prod_{n=1}^N \left(\int \mathcal{D}\varphi_n(\vec{x}) \int \mathcal{D}\pi_n(\vec{x})\right) \prod_{n=1}^N \langle\varphi_n|\pi_n\rangle \left\langle\pi_n\middle|e^{-\epsilon\hat{H}}\middle|\varphi_{n+1}\right\rangle \langle\varphi_1|\varphi_{N+1}\rangle\,,$$

where $\epsilon = \beta/N$. The last term ensures that $\varphi_1 = \varphi_{N+1}$. Bosons such as the scalar field φ, follow the periodic boundary condition $\varphi(0,\vec{x}) = \varphi(\beta,\vec{x})$. Fermions, as we will show later, follow the anti-periodic boundary condition $\psi(0,\vec{x}) = -\psi(\beta,\vec{x})$. We now want to exploit the fact that $|\pi\rangle$ and $|\varphi\rangle$ are the eigenvectors of the operators that define the Hamiltonian. In our case, as the Hamiltonian density $\mathcal{H}$ can be written as a sum of functions of φ and π separately, $\mathcal{H}[\varphi(\vec{x}),\pi(\vec{x})] = \mathcal{F}_1[\varphi(\vec{x})] + \mathcal{F}_2[\pi(\vec{x})]$ we may evaluate it as $\langle\pi_n|\mathcal{H}[\hat{\varphi}(\vec{x}),\hat{\pi}(\vec{x})]|\varphi_{n+1}\rangle = \mathcal{H}[\varphi_{n+1}(\vec{x}),\pi_n(\vec{x})]\,\langle\pi_n|\varphi_{n+1}\rangle$. This relationship does not, however, hold for more general functions of the field operators. In that case, one has to be more careful about the ordering of the operators, for example, by using *Weyl ordering* By series expanding $e^{-\epsilon\hat{H}}$ and exploiting this relationship, the partition function can be written as, to second order in ϵ,

$$Z = \prod_{n=1}^N \left(\int \mathcal{D}\varphi_n(\vec{x}) \int \mathcal{D}\pi_n(\vec{x})\right) \exp\left[-\epsilon\sum_{n=1}^N \int_V \mathrm{d}^3x\,\left(\mathcal{H}[\varphi_n(\vec{x}),\pi_n(\vec{x})] - i\pi_n(\vec{x})\frac{\varphi_n(\vec{x}) - \varphi_{n+1}(\vec{x})}{\epsilon}\right)\right].$$

[1] Some authors write $\mathcal{D}\pi/2\pi$. This extra factor 2π is a convention which in this text is left out for notational clarity.

We denote $\varphi_n(\vec{x}) = \varphi(\tau_n, \vec{x})$, $\tau \in [0, \beta]$ and likewise with $\pi_n(\vec{x})$. In the limit $N \to \infty$, the expression for the partition function becomes

$$Z = \int_S \mathcal{D}\varphi(\tau,\vec{x}) \int \mathcal{D}\pi(\tau,\vec{x}) \exp\left\{ -\int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}\vec{x}\, \{\mathcal{H}[\varphi(\tau,\vec{x}),\pi(\tau,\vec{x})] - i\pi(\tau,\vec{x})\dot{\varphi}(\tau,\vec{x})\} \right\}, \tag{B.10}$$

where S is the set of field configurations φ such that $\varphi(\beta,\vec{x}) = \varphi(0,\vec{x})$. With a Hamiltonian density of the form $\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\varphi)^2 + \mathcal{V}(\varphi)$, we can evaluate the integral over the canonical momentum π by discretizing $\pi(\tau_n, \vec{x}_m) = \pi_{n,m}$,

$$\begin{aligned}
&\int \mathcal{D}\pi \exp\left\{ -\int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}^3x \left(\frac{1}{2}\pi^2 - i\pi\dot{\varphi} \right) \right\} \\
&= \lim_{M,N\to\infty} \int \left(\prod_{m,n=1}^{M,N} \frac{\mathrm{d}\pi_{m,n}}{2\pi} \right) \exp\left\{ -\sum_{m,n} \frac{V\beta}{MN} \left[\frac{1}{2}(\pi_{m,n} - i\dot{\varphi}_{m,n})^2 + \frac{1}{2}\dot{\varphi}_{m,n}^2 \right] \right\} \\
&= \lim_{M,N\to\infty} \left(\frac{MN}{2\pi V\beta} \right)^{MN/2} \exp\left\{ -\int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}^3x\, \frac{1}{2}\dot{\varphi}^2 \right\},
\end{aligned}$$

where $\dot{\varphi}_{m,n} = (\varphi_{m,n+1} - \varphi_{m,n})/\epsilon$. The partition function is then,

$$Z = C \int \mathcal{D}\varphi \exp\left\{ -\int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}^3x \left[\frac{1}{2}\left(\dot{\varphi}^2 + \nabla\varphi^2 \right) + \mathcal{V}(\varphi) \right] \right\}. \tag{B.11}$$

Here, C is the divergent constant that results from the π-integral. In the last line, we exploited the fact that the variable of integration $\pi_{n,m}$ may be shifted by a constant without changing the integral, and used the Gaussian integral

$$\int_{-\infty}^{\infty} \mathrm{d}x\, e^{-ax^2/2} = \sqrt{\frac{2\pi}{a}}.$$

The partition function resulting from this procedure may also be obtained by starting with the ground state path integral

$$Z_g = \int \mathcal{D}\varphi\mathcal{D}\pi \exp\left\{ i \int_{\Omega'} \mathrm{d}^4x\, (\pi\dot{\varphi} - \mathcal{H}[\varphi,\pi]) \right\} = C' \int \mathcal{D}\varphi(x) \exp\left\{ i \int_{\Omega'} \mathrm{d}^4x\, \mathcal{L}[\varphi, \partial_\mu\varphi] \right\},$$

and follow a formal procedure. First, the action integral is modified by performing a Wick-rotation of the time coordinate t. This involves changing the domain of t from the real line to the imaginary line by closing the contour at infinity and changing variable $it \to \tau$. The new variable is then restricted to the interval $\tau \in [0, \beta]$, and the domain of the functional integral $\int \mathcal{D}\varphi$ is restricted from *all* (smooth enough) field configurations $\varphi(t, \vec{x})$, to only those that obey $\varphi(\beta, \vec{x}) = e^{i\theta}\varphi(0, \vec{x})$, which is denoted S. Here, $\theta \in \{0, \pi\}$, depending on if the particle is a boson of fermion. This procedure motivates the introduction of the Euclidean Lagrange density, $\mathcal{L}_E(\tau, \vec{x}) = -\mathcal{L}(-i\tau, \vec{x})$, as well as the name "imaginary-time formalism". The result is the same partition function as before,

$$\begin{aligned}
Z &= C \int_S \mathcal{D}\varphi \int \mathcal{D}\pi \exp\left\{ -\int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}^3x\, [-i\dot{\varphi}\pi + \mathcal{H}(\varphi,\pi)] \right\} \\
&= C' \int_S \mathcal{D}\varphi \exp\left\{ -\int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}^3x\, \mathcal{L}_E(\varphi,\pi) \right\}.
\end{aligned} \tag{B.12}$$

B.2.1 Fourier series

Due to the finite range of the imaginary-time coordinate $\tau \in [0, \beta]$, the momentum-space fields in imaginary-time formalism have a discrete coordinate. We define the Matsubara-frequencies as $\omega_n = 2n\pi/\beta$ for bosons and $\omega_n = (2n+1)\pi/\beta$ for fermions. They together form the reciprocal space $\tilde{\Omega} = \{\omega_n\} \times \tilde{V}$, where $\tilde{V}$

is reciprocal to V. To get a more economical notation, we denote the Euclidean real-space coordinates as $X = (\tau, \vec{x})$ and the reciprocal space coordinates as $K = (\omega_n, \vec{k})$. The dot product is $X \cdot K = \omega_n \tau + \vec{k} \cdot \vec{x}$. In the limit $V \to \infty$, we follow the prescription

$$\frac{1}{V} \sum_{\vec{p} \in \tilde{V}} \to \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3}.$$

The sum over all degrees of freedom, and the corresponding integrals for the thermodynamic limit are

$$\frac{\beta V}{NM} \sum_{n=1}^{N} \sum_{\vec{x}_m \in V} \xrightarrow{N,M \to \infty} \int_0^\beta \mathrm{d}\tau \int_{\mathbb{R}^3} \mathrm{d}^3 x = \int_\Omega \mathrm{d}X,$$
$$\frac{1}{V} \sum_{n=-\infty}^{\infty} \sum_{\vec{k} \in \tilde{V}} \xrightarrow{V \to \infty} \sum_{n=-\infty}^{\infty} \int_{R^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} = \int_{\tilde{\Omega}} \mathrm{d}K.$$

The convention used for the Fourier expansion of thermal fields is in accordance with The prefactor is chosen to make the Fourier components of the field dimensionless, which makes it easier to evaluate the trace correctly. For bosons, the Fourier expansion is

$$\varphi(X) = \sqrt{V\beta} \int_{\tilde{\Omega}} \mathrm{d}K\, \tilde{\varphi}(K) e^{iX \cdot K} = \sqrt{\frac{\beta}{V}} \sum_{n=-\infty}^{\infty} \sum_{\vec{k} \in \tilde{V}} \tilde{\varphi}_n(\vec{p}) \exp\Big\{ i(\omega_n \tau + \vec{x} \cdot \vec{k}) \Big\},$$
$$\tilde{\varphi}(K) = \sqrt{\frac{1}{V\beta^3}} \int_{\tilde{\Omega}} \mathrm{d}X\, \tilde{\varphi}(X) e^{-iX \cdot K},$$

while for Fermions it is

$$\psi(X) = \sqrt{V} \int_{\tilde{\Omega}} \mathrm{d}K\, \tilde{\psi}(K) e^{iX \cdot K} = \frac{1}{\sqrt{V}} \sum_{n=-\infty}^{\infty} \sum_{\vec{k} \in \tilde{V}} \psi(\omega_n, \vec{k}) \exp\Big\{ i(\omega_n \tau + \vec{x} \cdot \vec{k}) \Big\}. \tag{B.13}$$

Two often used identities are

$$\int_\Omega \mathrm{d}X e^{iX \cdot (K - K')} = \beta \delta_{nn'} (2\pi)^3 \delta^3(\vec{k} - \vec{k}') := \beta \delta(K - K'), \tag{B.14}$$
$$\int_{\tilde{\Omega}} \mathrm{d}K\, e^{iK(X - X')} = \beta \delta(\tau - \tau') \delta^3(\vec{x} - \vec{x}') := \beta \delta(X - X'). \tag{B.15}$$

B.3 Free scalar field

The procedure for obtaining the thermal properties of an interacting scalar field is similar to that used in scattering theory. One starts with a free theory, which can be solved exactly. Then an interaction term is added, which is accounted for perturbatively by using Feynman diagrams. The Euclidean Lagrangian for a free scalar gas is, after integrating by parts,

$$\mathcal{L}_E = \frac{1}{2} \varphi(X) \left(-\partial_E^2 + m^2 \right) \varphi(X) \tag{B.16}$$

Here, $X = (\tau, \vec{x})$ is the Euclidean coordinate resulting from the Wick-rotation as described in the last section. We have also introduced the Euclidean Laplace operator, $\partial_E^2 = \partial_\tau^2 + \nabla^2$. Following the procedure to obtain the thermal partition function yields

$$Z = C \int_S \mathcal{D}\varphi(X) \exp\Big\{ - \int_\Omega \mathrm{d}X \frac{1}{2} \varphi(X) \left(-\partial_E^2 + m^2 \right) \varphi(X) \Big\}. \tag{B.17}$$

Here, Ω is the domain $[0, \beta] \times V$. We then insert the Fourier expansion of φ and change the functional integration variable to the Fourier components. The integration measures are related by

$$\mathcal{D}\varphi(X) = \det\left(\frac{\delta \varphi(X)}{\delta \tilde{\varphi}(K)} \right) \mathcal{D}\tilde{\varphi}(K),$$

where $K = (\omega_n, \vec{k})$ is the Euclidean Fourier-space coordinate. The determinant factor which appears may be absorbed into the constant C, as the integration variables are related by a linear transform. The action becomes

$$\begin{aligned} S &= -\int_\Omega \mathrm{d}X\, \mathcal{L}_E = -\frac{1}{2} V\beta \int_\Omega \mathrm{d}X \int_{\tilde{\Omega}} \mathrm{d}K \int_{\tilde{\Omega}} \mathrm{d}K'\, \tilde{\varphi}(K') \left(\omega_n^2 + \vec{k}^2 + m^2\right) \tilde{\varphi}(K) e^{iX\cdot(K-K')} \\ &= -\frac{1}{2} V\beta^2 \int_{\tilde{\Omega}} \mathrm{d}K\, \tilde{\varphi}(K)^* \left(\omega_n^2 + \omega_k^2\right) \tilde{\varphi}(K), \end{aligned}$$

where $\omega_k^2 = \vec{k}^2 + m^2$. $\tilde{\Omega}$ is the reciprocal space corresponding to Ω. We used the fact that φ is real, which implies that $\tilde{\varphi}(-K) = \tilde{\varphi}(K)^*$, as well as the identity This gives the partition function

$$Z = C \int_{\tilde{S}} \mathcal{D}\tilde{\varphi}(K) \exp\left\{-\frac{1}{2} V \int_{\tilde{\Omega}} \mathrm{d}K\, \tilde{\varphi}(K)^* \left[\beta^2(\omega_n^2 + \omega_k^2)\right] \tilde{\varphi}(K)\right\}, \tag{B.18}$$

Going back to before the continuum limit, this integral can be written as a product of Gaussian integrals and may therefore be evaluated

$$Z = C \prod_{n=-\infty}^{\infty} \prod_{k\in\tilde{V}} \left(\int \mathrm{d}\tilde{\varphi}_{n,\vec{k}}\, \exp\left\{-\frac{1}{2} \tilde{\varphi}^*_{n,\vec{k}} \left[\beta^2(\omega_n^2 + \omega_k^2)\right] \tilde{\varphi}_{n,\vec{k}}\right\}\right) = C \prod_{n=-\infty}^{\infty} \prod_{k\in\tilde{V}} \sqrt{\frac{2\pi}{\beta^2(\omega_n^2 + \omega_k^2)}}.$$

The partition function is related to free energy F through

$$\frac{F}{TV} = -\frac{\ln(Z)}{V} = \frac{1}{2} \int_{\tilde{\Omega}} \mathrm{d}K \ln[\beta^2(\omega_n^2 + \omega_k^2)] + \frac{F_0}{TV}, \tag{B.19}$$

where F_0 is a constant.

A faster and more formal way to get to this result is to compare the partition function to the multidimensional version of the Gaussian integral The partition function has the form

$$I_n = \int_{\mathbb{R}^n} \mathrm{d}^n x\, \exp\left\{-\frac{1}{2}\langle x, D_0^{-1} x\rangle\right\},$$

where D_0^{-1} is a linear operator, and $\langle\cdot,\cdot\rangle$ an inner product on the corresponding vector space. By diagonalizing D_0^{-1}, we get the result

$$I_n = \sqrt{\frac{(2\pi)^n}{\det\left(D_0^{-1}\right)}}.$$

We may then use the identity

$$\det\left(D_0^{-1}\right) = \prod_i \lambda_i = \exp\left\{\mathrm{Tr}\left[\ln\left(D_0^{-1}\right)\right]\right\}, \tag{B.20}$$

where λ_i are the eigenvalues of D_0^{-1}. The trace in this context is defined by the vector space D_0^{-1} acts on. For given an orthonormal basis x_n, such that $\langle x_n, x'_n\rangle = \delta_{nn'}$ the trace can be evaluated as $\mathrm{Tr}\left\{D_0^{-1}\right\} = \sum_n \langle x_n, D_0^{-1} x_n\rangle$. Identifying

$$\langle x, D_0^{-1} x\rangle = \int_\Omega \mathrm{d}X \varphi(X) \left(-\partial_E^2 + m^2\right) \varphi(X),$$

we get the formal result

$$Z = \det\left(-\partial_E^2 + m^2\right)^{-1/2},$$

and

$$\beta F = \frac{1}{2} \mathrm{Tr}\left\{\ln\left(-\partial_E^2 + m^2\right)\right\}.$$

The logarithm may then be evaluated by using the eigenvalues of the linear operator. This is found by diagonalizing the operator,

$$\langle x, D_0^{-1} x\rangle = \int_\Omega \mathrm{d}X \varphi(X) \left(-\partial_E^2 + m^2\right) \varphi(X) = V \int_{\tilde{\Omega}} \mathrm{d}K \tilde{\varphi}(K)^* [\beta^2(\omega_k^2 + \omega_n^2)] \tilde{\varphi}(K),$$

leaving us with the same result as we obtained in

$$\beta F = \frac{1}{2}\,\mathrm{Tr}\big\{\ln\big(-\partial_E^2 + m^2\big)\big\} = \frac{1}{2}V\int_{\tilde{\Omega}} \mathrm{d}K\,\ln[\beta^2(\omega_n^2+\omega_k^2)].$$

Sums similar to this show up a lot, and we show how to evaluate them in the next section.

B.3.1 Low-temperature limit

Using the result from subsection B.3.4 on the result for the free energy density of the free scalar field, Eq. (B.12), we get

$$\mathcal{F} = \frac{\ln(Z)}{\beta V} = \frac{1}{2}\int_{\tilde{V}} \frac{\mathrm{d}^3k}{(2\pi)^3}\left[\omega_k + \frac{2}{\beta}\ln\big(1-e^{-\beta\omega_k}\big)\right]. \tag{B.21}$$

The free energy density thus has two contributions from parts; the first part is dependent on temperature, the other is a temperature-independent vacuum contribution. Noticing that the integral is spherically symmetric, we may write the two contributions as

$$\mathcal{F}_0 = \frac{1}{2}\frac{1}{2\pi^2}\int_{\mathbb{R}} \mathrm{d}k\,k^2\sqrt{k^2+m^2}, \quad \mathcal{F}_T = \frac{T^4}{2\pi^2}\int_{\mathbb{R}} \mathrm{d}x\,x^2\ln\Big(1-e^{-\sqrt{x^2+(m/T)^2}}\Big), \tag{B.22}$$

The temperature-independent part, $\mathcal{F}_0$, is divergent, and we must impose a regulator and then add counter-terms. $\mathcal{F}_T$, however, is convergent. To see this, we use the series expansion $\ln(1+\epsilon) \sim \epsilon + \mathcal{O}\epsilon$ to find the leading part of the integrand for large k's,

$$x^2\ln\Big(1-e^{-\sqrt{x^2+(\beta m)^2}}\Big) \sim -x^2e^{-x}, \tag{B.23}$$

which is exponentially suppressed, making the integral convergent. In the limit of $T \to 0$, we get

$$\mathcal{F}_T \sim \frac{T^4}{2\pi^2}\int_{\mathbb{R}} \mathrm{d}x\,x^2\ln\big(1-e^{-x}\big) = -\frac{T^4}{2\pi^2}\sum_{n=1}\frac{1}{n}\frac{\partial^2}{\partial n^2}\int \mathrm{d}x e^{-nx} = -\frac{T^4}{2\pi^2}\sum_{n=1}\frac{2}{n^4} = -\frac{T^4}{\pi^2}\zeta(4),$$

where ζ is the Riemann-zeta function. Using $\zeta(4) = \frac{\pi^4}{90}$, we get

$$\mathcal{F}_T \sim -\frac{\pi^2}{90}T^4, \quad T \to 0. \tag{B.24}$$

B.3.2 Regularization

Returning to the temperature-independent part, we use dimensional regularization to control its divergent behavior. To that end, we define

$$\Phi_n(m,d,\alpha) = \int_{\tilde{\Omega}} \frac{\mathrm{d}^dk}{(2\pi)^d}(k^2+m^2)^{-\alpha}, \tag{B.25}$$

so that $\mathcal{F}_0 = \Phi_3(m,3,-1/2)/2$. We will use the formula for integration of spherically symmetric function in d-dimensions,

$$\int_{\mathbb{R}^d} \mathrm{d}^dx\,f(r) = \frac{2\pi^{d/2}}{\Gamma(d/2)}\int_{\mathbb{R}} \mathrm{d}r\,r^{d-1}f(r), \tag{B.26}$$

where $r = \sqrt{x_ix_i}$ is the radial distance, and Γ is the Gamma function. The factor in the front of the integral is the solid angle. By extending this formula from integer-valued d to real numbers, we get

$$\Phi_n = \frac{2\pi^{d/2}}{\Gamma(d/2)}\int_{\mathbb{R}} \mathrm{d}k\,\frac{k^{d-1}}{(k^2+m^2)^\alpha} = \frac{m^{n-2\alpha}m^{d-n}}{(4\pi)^{d/2}\Gamma(d/2)}2\int_{\mathbb{R}} \mathrm{d}z\,\frac{z^{d-1}}{(1+z)^\alpha}, \tag{B.27}$$

where we have made the change of variables $mz = k$. We make one more change of variable to the integral,

$$I = 2\int_{\mathbb{R}} \mathrm{d}z\,\frac{z^{d-1}}{(1+z)^\alpha}. \tag{B.28}$$

Let

$$z^2 = \frac{1}{s} - 1 \implies 2z\mathrm{d}z = -\frac{\mathrm{d}s}{s^2} \tag{B.29}$$

Thus,

$$I = \int_0^a \mathrm{d}s\, s^{\alpha-d/2-1}(1-z)^{d/2-1}. \tag{B.30}$$

This is the beta function, which can be written in terms of Gamma functions

$$I = B\left(\alpha - \frac{d}{2}, \frac{d}{2}\right) = \frac{\Gamma\left(\alpha - \frac{d}{2}\right)\Gamma\left(\frac{d}{2}\right)}{\Gamma(\alpha)}. \tag{B.31}$$

Combining this gives

$$\Phi_n(m,d,\alpha) = \mu^{n-d}\frac{m^{n-2\alpha}}{(4\pi)^{d/2}}\frac{\Gamma\left(\alpha-\frac{d}{2}\right)}{\Gamma(\alpha)}\left(\frac{m^2}{\mu^2}\right)^{(d-n)/2}. \tag{B.32}$$

In the last step, we have introduced a parameter μ with mass dimension 1, that is, $[\mu] = [m]$. This is done to be able to series expand around $d - n$ in a dimensionless variable. This parameter is arbitrary, and all physical quantities should therefore be independent of it. We will shortly justify this parameter further.

Inserting $n = 3$, $d = 3 - 2\epsilon$ and $\alpha = -1/2$, we get

$$\Phi_3(m, 3-2\epsilon, -1/2) = \frac{m^4\mu^{-2\epsilon}}{(4\pi)^{d/2}\Gamma(-1/2)}\Gamma(-2+\epsilon)\left(\frac{m^2}{\mu^2}\right)^{-\epsilon} = -\mu^{-2\epsilon}\frac{m^4}{(4\pi)^2}\left(\frac{m^2}{4\pi\mu^2}\right)^{-\epsilon}\frac{\Gamma(\epsilon)}{(\epsilon-2)(\epsilon-1)}, \tag{B.33}$$

where we have used the defining property $\Gamma(z+1) = z\Gamma(z)$ and $\Gamma(1/2) = \sqrt{\pi}$. Expanding around $\epsilon = 0$ gives

$$\left(\frac{m^2}{4\pi\mu^2}\right)^{-\epsilon} \sim 1 + \epsilon\ln\left(4\pi\frac{\mu^2}{m^2}\right), \tag{B.34}$$

$$\Gamma(\epsilon) \sim \frac{1}{\epsilon} - \gamma, \tag{B.35}$$

$$\frac{1}{(\epsilon-2)(\epsilon-1)} \sim \frac{1}{2}\left(1 + \frac{3}{2}\epsilon\right). \tag{B.36}$$

The divergent behavior of the temperature-independent term is therefore

$$\mathcal{F}_0 \sim -\mu^{-2\epsilon}\frac{1}{4}\frac{m^4}{(4\pi)^2}\left[\frac{1}{\epsilon} - \gamma + \frac{3}{2} + \ln\left(4\pi\frac{\mu^2}{m^2}\right)\right]. \tag{B.37}$$

With this regulator, one can then add counter-terms to cancel the ϵ^{-1}-divergence. The exact form of the counter-term is convention. One may also cancel the finite contribution due to the regulator. The minimal subtraction (MS) scheme involves only subtracting the divergent term, as the name suggests. We will use the modified minimal subtraction, or $\overline{\mathrm{MS}}$, scheme. In this scheme, one also removes the $-\gamma$ and $\ln(4\pi)$ term, by defining a new mass scale $\tilde{\mu}$ by

$$-\gamma + \ln\left(4\pi\frac{\mu^2}{m^2}\right) = \ln\left(4\pi e^{-\gamma}\frac{\mu^2}{m^2}\right) = \ln\left(\frac{\tilde{\mu}^2}{m^2}\right), \tag{B.38}$$

which leads to the expression

$$\mathcal{F}_0 = -\mu^{-2\epsilon}\frac{1}{4}\frac{m^4}{(4\pi)^2}\left(\frac{1}{\epsilon} + \frac{3}{2} + \ln\frac{\tilde{\mu}^2}{m^2}\right) + \mathcal{O}\epsilon. \tag{B.39}$$

B.3.3 Renormalization

Now that we have applied a regulator, we are able to handle the divergence in a well-defined way. When $\epsilon \neq 0$, we can subtract terms which are proportional to ϵ^{-1}, and be left with a term that is finite in the limit $\epsilon \to 0$. Consider an arbitrary Lagrangian,

$$\mathcal{L}[\varphi] = \sum_n \lambda_n \mathcal{O}_n[\varphi]. \tag{B.40}$$

Here, $\mathcal{O}_n[\varphi]$ are operators consisting of φ and $\partial_\mu\varphi$, and λ_n are coupling constants. In d dimensions, the action integral is

$$S[\varphi] = \sum_n \int \mathrm{d}^d x\, \lambda_n \mathcal{O}_n[\varphi]. \tag{B.41}$$

The action has mass dimension 0. This means that all terms $\lambda_n\mathcal{O}_n$ must have mass dimension d, as $[\mathrm{d}^d x] = -d$. We are free to choose the coupling constant corresponding to $\mathcal{O}_0 = \partial_\mu\varphi\partial^\mu\varphi$ to be of mass dimension 0, and therefore set $\lambda_0 = 1/2$ to get canonical normalization. This allows us to deduce the dimensionality of φ. As $[\partial_\mu] = 1$, we have that $[\varphi] = (d-2)/2$. Assume $\mathcal{O}_n$ consists of k_n factors of φ, and l_n factors of $\partial_\mu\varphi$. We must then have

$$[\lambda_n] + [\mathcal{O}_n] - d = [\lambda_n] + (k_n + l_n)(d-2)/2 + l_n - d = 0, \tag{B.42}$$

$$\implies D_n := [\lambda_n] = d - k_n\frac{d-2}{2} - l_n\frac{d}{2}. \tag{B.43}$$

From this formula, we recover that $[\lambda_0] = 0$, and if $\mathcal{O}_1 = \varphi^2$, then $[\lambda_1] = 2$, which we recognize as the mass term squared. The mass dimensions of these coupling constants are independent of d. However, the coupling constant for the interaction term

$$-\frac{1}{4!}\lambda_3\varphi^4 \tag{B.44}$$

has mass dimensions $[\lambda_3] = d - 4(d-2)/2 = 4 - 2d$. Our goal now is to exchange the bare coupling constants λ_n with renormalized ones, λ_n^r, and remove the divergent terms proportional to $(d-4)^{-m}$. We can always define the renormalized coupling constants as dimensionless, i.e., $[\lambda_n^r] = 0$, by measuring them in units of a mass scale. We therefore write

$$\lambda_n = \mu^{4-D_n}\left[\lambda_n^r + \sum_{m=1}\frac{a_m(\lambda_n^r)}{(d-4)^m}\right],$$

where we have introduced the dimensionfull parameter μ to ensure that λ_n has the correct mass dimension, so that the action integral stays dimensionless. The functions a_m are then determined to each order in perturbation theory by calculating Feynman diagrams. As μ again is arbitrary, λ_4' should not depend on this parameter. In this case, we chose the same renormalization scale as we did when regulating the one-loop integral. This is only for our own convenience. This means that if we change $\mu \to \mu'$, then λ_i^r and a_m must adjust to compensate and keep λ_n constant

The vacuum energy term absorbs the divergence in the one loop contribution to the free energy density. It is

$$\lambda_4\mathcal{O}_4 = \lambda_4 = m^4\lambda_4'. \tag{B.45}$$

Using the expansion in terms of the renormalized coupling, we have,

$$\lambda_4' = \mu^{-2\epsilon}\left[\lambda_4^r + \frac{1}{2\epsilon}a_1(\lambda_r^4) + ...\right], \tag{B.46}$$

where $d = 4 - 2\epsilon$. After adding Eq. (B.45) to the Lagrangian of the free scalar, the temperature independent free energy density becomes

$$\mathcal{F}_0 \sim -\mu^{-2\epsilon}\frac{1}{4}\frac{m^4}{(4\pi)^2}\left[\frac{1}{\epsilon} + \frac{3}{2} + \ln\frac{\tilde{\mu}^2}{m^2} + (4(4\pi)^2)\left(\lambda_4^r + \frac{1}{2\epsilon}a_1(\lambda_4^r)\right)\right], \quad \epsilon \to 0. \tag{B.47}$$

Thus, if we choose $a_1 = -8(4\pi)^2 + \mathcal{O}(\lambda_4^r)$, and define $\lambda_4'^r = 4(4\pi)^2\lambda_4^r$, we are able to cancel the divergence, and may take the limit $\epsilon \to 0$ safely. The free energy is now

$$\mathcal{F} = -\frac{1}{4}\frac{m^4}{(4\pi)^2}\left(\frac{3}{2} + \lambda_4'^r + \ln\frac{\tilde{\mu}^2}{m^2}\right) + \frac{T^4}{2\pi^2}\int \mathrm{d}x\, x^2 \ln\Big(1 - \exp\Big\{-\sqrt{x^2 + \beta^2 m^2}\Big\}\Big). \tag{B.48}$$

Notice that all choices we have made up until now, such as defining $\lambda_4 = m^4\lambda_4'$ and using the same renormalization scale μ, has no impact on this result. Different choices would force us to define λ_4^r and a_4 differently.

B.3.4 Thermal sum

When evaluating thermal integral, we will often encounter sums of the form

$$j(\omega,\mu) = \frac{1}{2\beta}\sum_{\omega_n} \ln\{\beta^2[(\omega_n + i\mu) + \omega^2]\} + g(\beta), \tag{B.49}$$

where the sum is over either the bosonic Matsubara frequencies $\omega_n = 2n\pi/\beta$, $n \in \mathbb{Z}$, or the fermionic ones, $\omega_n = (2n+1)\pi/\beta$, $n \in \mathbb{Z}$. $\mu \in \mathbb{R}$ is a chemical potential. g may be a function of β, but we assume it is independent of ω. Thus, the factor β^2 could strictly be dropped, but it is kept to make the argument within the logarithm dimensionless. We define the function

$$i(\omega,\mu) = \frac{1}{\omega}\frac{\mathrm{d}}{\mathrm{d}\omega} j(\omega,\mu) = \frac{1}{\beta}\sum_{\omega_n} \frac{1}{(\omega_n + i\mu)^2 + \omega^2}. \tag{B.50}$$

We will first work with the sum over bosonic Matsubara frequencies. Assume $f(z)$ is an analytic function, except perhaps on a set of isolated poles $\{z_i\}$ located outside the real line. We can exploit this using the properties of the Bose-distribution $n_B(z)$. The Bose distribution is defined as

$$n_B(\omega) = \frac{1}{e^{\beta\omega} - 1}. \tag{B.51}$$

This function obeys

$$n_B(-i\omega) = -1 - n_B(i\omega). \tag{B.52}$$

We can expand it around the Bose Matsubara frequencies on the imaginary line:

$$i n_B[i(\omega_n + \epsilon)] = \frac{i}{e^{i\beta\epsilon + 2\pi i n} - 1} = i[i\beta\epsilon + \mathcal{O}(\epsilon)]^{-1} \sim \frac{1}{\epsilon\beta}. \tag{B.53}$$

This means that $i n_B(i\omega)$ has a pole on all Matsubara-frequencies, with residue $1/\beta$. Using this, we can rewrite the sum over Matsubara frequencies as a contour integral

$$\frac{1}{\beta}\sum_{\omega_n} f(\omega_n) = \oint_\gamma \frac{\mathrm{d}z}{2\pi i} f(z) i n_B(iz),$$

where γ is a contour that goes from $-\infty - i\epsilon$ to $+\infty - i\epsilon$, crosses the real line at ∞, goes from $+\infty - i\epsilon$ to $-\infty + i\epsilon$ before closing the curve. The contour γ and the new contours are illustrated in Figure B.1. This result exploits Cauchy's integral formula by letting the poles of $i n_B(iz)$ at the Matsubara frequencies "pick out" the necessary residues. The integral over γ is equivalent to two integrals along $\mathbb{R} \pm i\epsilon$,

$$\begin{aligned}
\frac{1}{\beta}\sum_{\omega_n} f(\omega_n) &= \left(\int_{\infty+i\epsilon}^{-\infty+i\epsilon} \frac{\mathrm{d}z}{2\pi} + \int_{-\infty-i\epsilon}^{\infty-i\epsilon} \frac{\mathrm{d}z}{2\pi}\right) f(z) n_B(iz), \\
&= \int_{-\infty-i\epsilon}^{\infty-i\epsilon} \frac{\mathrm{d}z}{2\pi} \left\{-f(-z) + [f(z) - f(-z)]\, n_B(iz)\right\} \\
&= \int_{-\infty}^{\infty} \frac{\mathrm{d}z}{2\pi} f(z) + \int_{-\infty-i\epsilon}^{\infty-i\epsilon} \frac{\mathrm{d}z}{2\pi} \left[f(z) + f(-z)\right] n_B(iz).
\end{aligned} \tag{B.54}$$

In the second line, we have changed variables $z \to -z$ in the first integral, and exploited the property $n_B(-iz) = -1 - n_B(iz)$. In the last line, we use the assumption that $f(z)$ is analytic on the real line, and therefore also in a neighborhood of it. This allows us to shift the first integral back to the real line. As $n_B(iz)$ is analytic outside the real line, the result of the second integral is the sum of residues of $f(z) + f(-z)$ in the lower half-plane. The function

$$f(z) = \frac{1}{(z + i\mu)^2 + \omega^2} = \frac{i}{2\omega}\left(\frac{1}{z + i(\mu + \omega)} - \frac{1}{z + i(\mu - \omega)}\right) \tag{B.55}$$

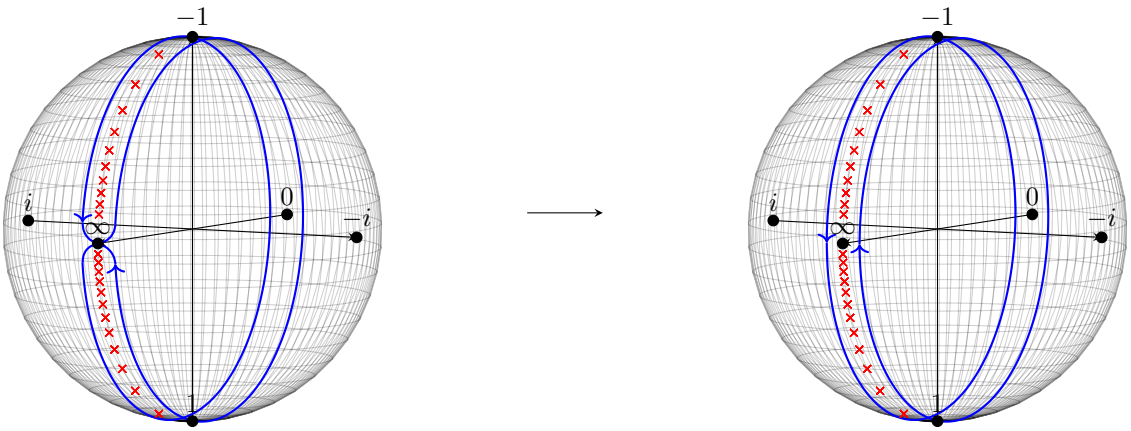


Figure B.1: The integral contour γ, and the result of deforming it into to contours close to the real line. The red crosses illustrate the poles of n_B.

obeys the assumed properties, as it has poles at $z = -i(\mu \pm \omega)$, with residue $1/(2\omega)$, so the function defined in Eq. (B.50) may be written

$$i(\omega, \mu) = \frac{1}{2\omega}[1 + n_B(\omega - \mu) + n_B(\omega + \mu)]. \tag{B.56}$$

Using the antiderivative of the Bose distribution,

$$\frac{\mathrm{d}}{\mathrm{d}\omega} \ln\left(1 - e^{-\beta\omega}\right) = \beta n_B(\omega), \tag{B.57}$$

we get the final form of Eq. (B.49)

$$j(\omega, \mu) = \int \mathrm{d}\omega'\, \omega' i(\omega', \mu) = \frac{1}{2}\omega + \frac{1}{2\beta}\left[\ln\left(1 - e^{-\beta(\omega-\mu)}\right) + \ln\left(1 - e^{-\beta(\omega+\mu)}\right)\right] + g'(\beta). \tag{B.58}$$

The extra ω-independent term $g'(\beta)$ is an integration constant. We see there are temperature dependent terms, one due to the particle and one due to the anti-particle, and one due to the antiparticle, as they have opposite chemical potentials.

We now consider the sum over fermionic frequencies, which we for clarity denote $\tilde{\omega}_n$ in this chapter. The procedure, in this case, is the same, except that we have to use a function with poles at the fermionic Matsubara frequencies. This is done by the Fermi distribution, $n_F(z)$. The Fermi distribution is

$$n_F(\omega) = \frac{1}{e^{\beta\omega} + 1}. \tag{B.59}$$

It obeys

$$\frac{\mathrm{d}}{\mathrm{d}\omega} \ln\left(1 + e^{-\beta\omega}\right) = -\beta n_F(\omega), \tag{B.60}$$

$$n_F(-i\omega) = 1 - n_F(i\omega). \tag{B.61}$$

With this, the sum over fermionic Matsubara frequencies gives

$$\frac{1}{\beta}\sum_{\tilde{\omega}_n} f(\tilde{\omega}_n) = \left(\int_{\infty+i\epsilon}^{-\infty+i\epsilon} \frac{\mathrm{d}z}{2\pi} + \int_{-\infty-i\epsilon}^{\infty-i\epsilon} \frac{\mathrm{d}z}{2\pi}\right) f(z) n_B(iz) \tag{B.62}$$

$$= \int_{-\infty}^{\infty} \frac{\mathrm{d}z}{2\pi} f(z) - \int_{-\infty-i\epsilon}^{\infty-i\epsilon} \frac{\mathrm{d}z}{2\pi} \left[f(z) - f(-z)\right] n_F(iz), \tag{B.63}$$

and

$$i(\omega, \mu) = \frac{1}{2\omega}[1 - n_F(\omega - \mu) - n_F(\omega + \mu)]. \tag{B.64}$$

Using the antiderivative of the Fermi-distribution, we get

$$j(\omega, \mu) = \frac{1}{2}\omega + \frac{1}{2\beta}\left[\ln\left(1 + e^{-\beta(\omega-\mu)}\right) + \ln\left(1 + e^{-\beta(\omega+\mu)}\right)\right]. \tag{B.65}$$

B.4 Interacting scalar

We now study a scalar field with a $\lambda\varphi^4$ interaction term. We write the Lagrangian in the form

$$\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(I)}, \quad \mathcal{L}^{(0)} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - m^2\varphi^2, \quad \mathcal{L}^{(I)} = -\frac{\lambda}{4!}\varphi^4$$

$\mathcal{L}^{(I)}$ is called the interaction term, and makes it impossible to exactly solve for the partition function. Instead, we turn to perturbation theory. The canonical partition function in this theory is

$$Z = \mathrm{Tr}\Big\{e^{-\beta\hat{H}}\Big\} = \int_S \mathcal{D}\varphi\, \exp\Big\{-\int_\Omega \mathrm{d}X\, \Big(\mathcal{L}_E^{(0)} + \mathcal{L}_E^{(I)}\Big)\Big\} = \int_S \mathcal{D}\varphi\, e^{-S_0}e^{-S_I}. \tag{B.66}$$

Here, S_0 and S_I denote the Euclidean action due to the free and interacting Lagrangian, respectively. The domain of integration S is again periodic field configurations $\varphi(\beta,\vec{x}) = \varphi(0,\vec{x})$. We may write the free energy as

$$-\beta F = \ln\left[\int_S \mathcal{D}\varphi\, e^{-S_0}\sum_n \frac{1}{n!}(-S_I)^n\right] = \ln Z_0 + \ln Z_I,$$

where Z_0 is the partition function of the free theory. The correction to the partition function is thus given by

$$Z_I = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left\langle {S_I}^n\right\rangle_0, \tag{B.67}$$

where

$$\langle A\rangle_0 = \frac{\int_S \mathcal{D}\varphi\, A\, e^{-S_0}}{\int_S \mathcal{D}\varphi\, e^{-S_0}}. \tag{B.68}$$

To evaluate expectation values of the form $\langle\varphi(X_1)...\rangle_0$, we introduce the partition function with a source term

$$Z[J] = \int_S \mathcal{D}\varphi\, \exp\Big\{-\frac{1}{2}\int_\Omega \mathrm{d}X\, \varphi(-\partial_E^2 + m^2)\varphi + \int_\Omega \mathrm{d}X\, J\varphi\Big\}. \tag{B.69}$$

Thermal propagators are the generalization of the time-ordered two-point functions $\langle T\{\varphi(x)\varphi(y)\}\rangle$ of the vacuum formalism. For some differential operator D^{-1}, the thermal propagator is defined as

$$D^{-1}D(X,Y) = \beta\delta(X-Y). \tag{B.70}$$

The Fourier transformed propagator is, assuming $D(X,Y) = D(X-Y,0)$,

$$\begin{aligned}\tilde{D}(K,K') &= \frac{1}{V\beta^3}\int_\Omega \mathrm{d}X\mathrm{d}Y\, D(X,Y)\exp(-i[X\cdot K + Y\cdot K']) \\ &= \frac{1}{V\beta^3}\int_\Omega \mathrm{d}X'\mathrm{d}Y'\, D(X',0)\exp\Big(-i[X'\cdot\frac{1}{2}(K-K') + Y\cdot(K+K')]\Big) \\ &= \frac{1}{V\beta^2}\tilde{D}(K)\delta(K+K'),\end{aligned} \tag{B.71}$$

where

$$\tilde{D}(K) = \int \mathrm{d}X e^{iK\cdot X}D(X,0). \tag{B.72}$$

We write the thermal propagator of the free field as $D_0(X,Y)$. With this, we may complete the square,

$$Z[J] = Z[0]\exp\Big\{\frac{1}{2}\int_\Omega \mathrm{d}X\mathrm{d}Y J(X)D_0(X,Y)J(Y)\Big\} = Z[0]\exp(W[J]). \tag{B.73}$$

We can now write

$$\langle\varphi(X)\varphi(Y)\rangle_0 = \frac{1}{Z[0]}\frac{\delta}{\delta J(X)}\frac{\delta}{\delta J(Y)}Z[J]\Big|_{J=0} = D_0(X,Y). \tag{B.74}$$

This generalizes to higher order expectation values,

$$\langle\varphi(X_i)\dots\varphi(X_n)\rangle_0 = \frac{1}{Z[0]}\left(\prod_{i=1}^{n}\frac{\delta}{\delta J(X_i)}\right)Z[J]\Big|_{J=0}, \tag{B.75}$$

Using Wick's theorem, as described in section 3.1, the expectation values we are evaluating can be written

$$\begin{aligned}\langle {S_I}^m\rangle_0 &= \left(-\frac{\lambda}{4!}\right)^m\int_\Omega \mathrm{d}X_1\dots\mathrm{d}X_m\left\langle\varphi^4(X_1)\dots\varphi^4(X_m)\right\rangle_0\\ &= \left(-\frac{\lambda}{4!}\right)^m\int_\Omega \mathrm{d}X_1\dots\mathrm{d}X_m\sum_{\{a,b\}}\left\langle\varphi(X_{a(1)})\varphi(X_{b(1)})\right\rangle_0\dots\left\langle\varphi(X_{a(2m)})\varphi(X_{b(2m)})\right\rangle_0,\end{aligned}$$

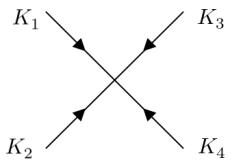
where X_i for $i > m$ is defined to equal X_j, where $j = i \mod m$. More simpliy, $X_{m+i} = X_i$. The functions a, b represents a possible pairing, as described in section 3.1. Inserting the Fourier expansions of the field gives

$$\begin{aligned}&\langle {S_I}^m\rangle_0\\ &= \left(-\frac{\lambda}{4!}\right)^m\int_\Omega \mathrm{d}X_1\dots\mathrm{d}X_m(V\beta)^2\int_{\tilde\Omega}\mathrm{d}K_1\dots\mathrm{d}K_{2m}\sum_{\{a,b\}}\\ &\qquad\left\langle\varphi(K_{a(1)})\varphi(K_{b(1)})\right\rangle_0\dots\left\langle\varphi(K_{a(2m)})\varphi(K_{b(2m)})\right\rangle_0\exp\Big(i\textstyle\sum_{i=1}^m X_i\cdot K_i\Big)\\ &= \left(-\frac{\lambda}{4!}\right)^m\frac{(V\beta)^{2m}\beta^m}{(V\beta^2)^{2m}}\int_{\tilde\Omega}\mathrm{d}K_1\dots\mathrm{d}K_{2m}\sum_{\{a,b\}}\\ &\qquad\tilde D(K_{a(1)})\delta(K_{a(1)}+K_{b(1)})\dots\tilde D(K_{a(2m)})\delta(K_{a(2m)}+K_{b(2m)})\prod_{i=1}^m\delta\left(X_i\cdot\textstyle\sum_{j=0}^3 K_{i+jm}\right)\\ &= \left(-\frac{\lambda\beta}{4!}\right)^m\prod_{i=1}^{2m}\int_{\tilde\Omega}\left(\mathrm{d}K_i\frac{1}{\beta}\tilde D(K_i)\right)\prod_{i=1}^m\delta\left(X_i\cdot\textstyle\sum_{j=0}^3 K_{i+jm}\right)\sum_{\{a,b\}}\prod_{n=1}^{2m}\delta(K_{a(k)}+K_{b(k)}).\end{aligned}$$

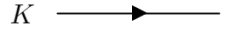
Here we have used that $V\beta^2\tilde D_0(K,P) = \tilde D_0(K)\delta(P+K)$, where $\tilde D_0(K)$ is the thermal propagator for the free field. In this case, it is

$$\tilde D_0(K) = \tilde D_0(\omega_n,\vec k) = \frac{1}{\omega_k^2+\omega_n^2}. \tag{B.76}$$

This expectation value can be represented graphically using Feynman diagrams. The thermal $\lambda\varphi^2$-theory gets the prescription



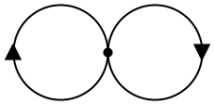
$$= -\lambda\beta\delta\left(\textstyle\sum_i K_i\right), \tag{B.77}$$



$$= \frac{1}{\beta}D_0(K). \tag{B.78}$$

Lastly, one has to integrate over internal momenta and divide by the symmetry factor of the diagram s, which is described in detail in

Calculating $\langle {S_I}^n\rangle_0$ boils down to the sum of all possible Feynman diagrams with n vertices. The first example is

$$\langle S_I\rangle_0 = \frac{1}{8}$$  . (B.79)

In section 3.1, we saw that the sum of all vacuum diagrams is the exponential of the sum of all *connected* diagrams, so the free energy of the interacting theory is given by

$$-\beta F = \ln Z_0 + \Sigma(\text{all connected diagrams}). \tag{B.80}$$

B.5 Fermions

The anti-periodic nature of fermion-fields, as mentioned in section B.2, can be determined by studying the properties of the thermal Greens function. The thermal Greens function may be written

$$D(X_1, X_2) = D(\vec{x}, \vec{y}, \tau_1, \tau_2) = \langle\Omega| e^{-\beta\hat{H}} \mathrm{T}\left\{\hat{\varphi}(X_1)\hat{\varphi}(X_2)\right\} |\Omega\rangle .$$

$\mathrm{T}\{...\}$ is time-ordering operator, defined as

$$\mathrm{T}\left\{\varphi(\tau_1)\varphi(\tau_2)\right\} = \theta(\tau_1 - \tau_2)\varphi(\tau_1)\varphi(\tau_2) + \nu\theta(\tau_2 - \tau_1)\varphi(\tau_2)\varphi(\tau_1),$$

where $\nu = \pm 1$ for bosons and fermions respectively, and $\theta(\tau)$ is the Heaviside step function. In the same way that $i\hat{H}$ generates the time translation of a quantum field operator through $\hat{\varphi}(x) = \hat{\varphi}(t, \vec{x}) = e^{it\hat{H}}\hat{\varphi}(0, \vec{x})e^{-it\hat{H}}$, the imaginary-time formalism implies the relation

$$\hat{\varphi}(X) = \hat{\varphi}(\tau, \vec{x}) = e^{\tau\hat{H}}\hat{\varphi}(0, \vec{x})e^{-\tau\hat{H}}. \tag{B.81}$$

Using $\mathbb{1} = e^{\tau\hat{H}}e^{-\tau\hat{H}}$ and the cyclic property of the trace, we show that, assuming $\beta > \tau > 0$,

$$\begin{aligned}
D(\vec{x}, \vec{y}, \tau, 0) &= \langle\Omega| e^{-\beta\hat{H}} \mathrm{T}\left\{\varphi(\tau, \vec{x})\varphi(0, \vec{y})\right\} |\Omega\rangle \\
&= \frac{1}{Z} \mathrm{Tr}\Big\{e^{-\beta\hat{H}}\varphi(\tau, \vec{x})\varphi(0, \vec{y})\Big\} \\
&= \frac{1}{Z} \mathrm{Tr}\Big\{\varphi(0, \vec{y})e^{-\beta\hat{H}}\varphi(\tau, \vec{x})\Big\} \\
&= \frac{1}{Z} \mathrm{Tr}\Big\{e^{-\beta\hat{H}}e^{\beta\hat{H}}\varphi(0, \vec{y})e^{-\beta\hat{H}}\varphi(\tau, \vec{x})\Big\} \\
&= \frac{1}{Z} \mathrm{Tr}\Big\{e^{-\beta\hat{H}}\varphi(\vec{y}, \beta)\varphi(\tau, \vec{x})\Big\} \\
&= \nu \langle\Omega| e^{-\beta\hat{H}} \mathrm{T}\left\{\varphi(\tau, \vec{x})\varphi(\beta, \vec{y})\right\} |\Omega\rangle .
\end{aligned}$$

This implies that $\varphi(0, x) = \nu\varphi(\beta, \varphi)$, which shows that bosons are periodic in time, as stated earlier, while fermions are anti-periodic.

The Lagrangian density of a free fermion is

$$\mathcal{L} = \bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi. \tag{B.82}$$

This Lagrangian is invariant under the transformation $\psi \to e^{-i\alpha}\psi$, which by Nöther's theorem results in a conserved current

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta\psi = \bar{\psi}\gamma^\mu\psi. \tag{B.83}$$

The corresponding conserved charge is

$$Q = \int_V \mathrm{d}^3x\, j^0 = \int_V \mathrm{d}^3x\, \bar{\psi}\gamma^0\psi. \tag{B.84}$$

We can now use our earlier result for the thermal partition function, Eq. (B.10), only with the substitution $\mathcal{H} \to \mathcal{H} - \mu\bar{\psi}\gamma^0\psi$, and integrate over anti-periodic ψ's:

$$Z = \mathrm{Tr}\Big\{e^{-\beta(\hat{H} - \mu\hat{Q})}\Big\} = \prod_{ab}\int \mathcal{D}\psi_a\mathcal{D}\pi_b \exp\Big\{\int_\Omega \mathrm{d}X\, \Big(i\dot{\psi}\pi - \mathcal{H}(\psi, \pi) + \mu\bar{\psi}\gamma^0\psi\Big)\Big\},$$

where a, b are the spinor indices. The canonical momentum corresponding to ψ is

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} = i\bar{\psi}\gamma^0, \tag{B.85}$$

and the Hamiltonian density is

$$\mathcal{H} = \pi\dot{\psi} - \mathcal{L} = \bar{\psi}(-i\gamma^i\partial_i + m)\psi \tag{B.86}$$

which gives

$$\mathcal{L}_E = -i\pi\dot{\psi} + \mathcal{H}(\psi, \pi) - \mu\bar{\psi}\gamma^0\psi = \bar{\psi}[\gamma^0(\partial_\tau - \mu) - i\gamma^i\partial_i + m]\psi, \tag{B.87}$$

By using the Grassman-version of the Gaussian integral formula, the partition function can be written

$$\begin{aligned} Z &= \prod_{ab} \int \mathcal{D}\psi_a \mathcal{D}\bar{\psi}_b \exp\Big\{ -\int_\Omega \mathrm{d}X\, \bar{\psi}\left[\tilde{\gamma}_0(\partial_\tau - \mu) - i\gamma^i\partial_i + m\right]\psi \Big\} \\ &= C \prod_{ab} \int \mathcal{D}\tilde{\psi}_a \mathcal{D}\tilde{\bar{\psi}}_b \exp\Big\{ -\int_{\tilde{\Omega}} \mathrm{d}K\, \tilde{\bar{\psi}}\left[i\tilde{\gamma}_0(\omega_n + i\mu) + i\gamma_i p_i + m\right]\tilde{\psi} \Big\} \\ &= C \prod_{ab} \int \mathcal{D}\tilde{\psi}_a \mathcal{D}\tilde{\bar{\psi}}_b e^{-\langle \tilde{\bar{\psi}}, D^{-1}\tilde{\psi}\rangle} = \det\big(D^{-1}\big). \end{aligned}$$

In the second line, we have inserted the Fourier expansion of the field, as defined in section B.2, and changed variable of integration, as we did for the scalar field. The linear operator in this case is

$$D^{-1} = i\gamma^0(-i\partial_\tau + i\mu) - (-i\gamma^i)\partial_i + m = \beta[i\tilde{\gamma}_a p_a + m]. \tag{B.88}$$

This equality must be understood as an equality between linear operators, which are represented in different bases. We introduced the notation $p_a = (\omega_n + i\mu, p_i)$ and use the Euclidean gamma matrices, as defined in

include gamma + pauli matrices

We use the fact that

$$\det(i\tilde{\gamma}_a p_a + m) = \det\big(\gamma^5\gamma^5\big)\det(i\tilde{\gamma}_a p_a + m) = \det\big[\gamma^5(i\tilde{\gamma}_a p_a + m)\gamma^5\big] = \det(-i\tilde{\gamma}_a p_a + m),$$

Hva skjer med beta?

Let $\tilde{D}^{-1} = \beta[-i\tilde{\gamma}_a p_a + m]$, which means we can write

$$Z = \sqrt{\det(D^{-1})\det\Big(\tilde{D}^{-1}\Big)} = \sqrt{\det\Big(D^{-1}\tilde{D}^{-1}\Big)} = \det\big[\mathbb{1}\beta^2(p_a p_a + m^2)\big]^{1/2}, \tag{B.89}$$

where we have used the anti-commutation rule for the Euclidean gamma-matrices, $\{\gamma_a, \gamma_b\} = 2\delta_{ab}$. It is important to keep in mind that the determinant here refers to linear operators on the space of spinor functions. Thus

Vær konsekvent på bruk av k og p

$$\begin{aligned} \ln(Z) &= \ln\Big\{\det\big[\mathbb{1}\beta^2(p_a p_a + m^2)\big]^{1/2}\Big\} = \frac{1}{2}\,\mathrm{Tr}\big\{\ln[\mathbb{1}\beta^2(p_a p_a + m^2)]\big\} \\ &= 2\int_{\tilde{\Omega}} \mathrm{d}K\, \ln\{\beta^2[(\omega_n + i\mu)^2 + \omega_k^2]\}. \end{aligned} \tag{B.90}$$

In the last step, we used the fact that the matrix within the logarithm is diagonal. The matrix-part of the trace is trivial therefore trivial. Using the fermionic version of the thermal sum from subsection B.3.4 gives the answer

$$\mathcal{F} = -\frac{2}{\beta}\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\left[\beta\omega_k + \ln\left(1 + e^{-\beta(\omega_k - \mu)}\right) + \ln\left(1 + e^{-\beta(\omega_k + \mu)}\right)\right]. \tag{B.91}$$

We see again that the temperature-independent part of the integral diverges, and must be regulated. There are two temperature-dependent terms, one from the particle and one from the anti-particle.

Appendix C

Code

All code is available at: `https://github.com/martkjoh/master`.

C.1 Integrating the TOV equations

Describe the numerical code for the TOV equations

C.2 Spherically symmetric metric

The calculations in chapter 4 were done using a CAS system. The code is written in Python in a Jupyter notebook. The full `.ipynb` file with executable code is available in the repository, at `https://github.com/martkjoh/master/blob/main/scripts/TOV.ipynb` Below is some of the code, which illustrates the main functions and the outputs.

1 Metric $g_{\mu\nu}$ for spherically symmetric spacetime

[8]:
```
t, r, th, ph = symbols("t, r, \\theta, \\phi")
x1 = r * cos(ph) * sin(th)
x2 = r * sin(ph) * sin(th)
x3 = r * cos(th)

one = Rational(1)
eta = sp.diag(one, -one, -one, -one)
var = (t, r, th, ph)
J = Matrix([t, x1, x2, x3]).jacobian(var)
g = np.array(simplify(J.T *eta* J))

a = sp.Function("\\alpha", real=True)(r)
b = sp.Function("\\beta", real=True)(r)
g[0, 0] *= exp(2 * a)
g[1, 1] *= exp(2 * b)
g_inv = get_g_inv(g)

print_matrix(g)
print_matrix(g_inv)
```

$$\begin{bmatrix} e^{2\alpha(r)} & 0 & 0 & 0 \\ 0 & -e^{2\beta(r)} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2(\theta) \end{bmatrix}$$

$$\begin{bmatrix} e^{-2\alpha(r)} & 0 & 0 & 0 \\ 0 & -e^{-2\beta(r)} & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{r^2 \sin^2(\theta)} \end{bmatrix}$$

[9]:
```
C = Christoffel(g, g_inv, var)
c = print_christoffel(C, var)
```

$$\Gamma^{t}_{\mu\nu} = \begin{bmatrix} 0 & \frac{d}{dr}\alpha(r) & 0 & 0 \\ \frac{d}{dr}\alpha(r) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Gamma^{r}_{\mu\nu} = \begin{bmatrix} e^{2\alpha(r)}e^{-2\beta(r)}\frac{d}{dr}\alpha(r) & 0 & 0 & 0 \\ 0 & \frac{d}{dr}\beta(r) & 0 & 0 \\ 0 & 0 & -re^{-2\beta(r)} & 0 \\ 0 & 0 & 0 & -re^{-2\beta(r)}\sin^2(\theta) \end{bmatrix}$$

$$\Gamma^{\theta}_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\sin(\theta)\cos(\theta) \end{bmatrix}$$

$$\Gamma^{\phi}_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \frac{\cos(\theta)}{\sin(\theta)} \\ 0 & \frac{1}{r} & \frac{\cos(\theta)}{\sin(\theta)} & 0 \end{bmatrix}$$

[10]:
```
Rie = Riemann_tensor(C, var)
Ricci = contract(Rie, num_indx=4, upper=1, indx=(0, 2))

for i in range(4):
    print_scalar(Ricci[i, i].factor())
```

$$\frac{\left(r\left(\frac{d}{dr}\alpha(r)\right)^2 - r\frac{d}{dr}\alpha(r)\frac{d}{dr}\beta(r) + r\frac{d^2}{dr^2}\alpha(r) + 2\frac{d}{dr}\alpha(r)\right)e^{2\alpha(r)}e^{-2\beta(r)}}{r}$$

$$-\frac{r\left(\frac{d}{dr}\alpha(r)\right)^2 - r\frac{d}{dr}\alpha(r)\frac{d}{dr}\beta(r) + r\frac{d^2}{dr^2}\alpha(r) - 2\frac{d}{dr}\beta(r)}{r}$$

$$-\left(r\frac{d}{dr}\alpha(r) - r\frac{d}{dr}\beta(r) - e^{2\beta(r)} + 1\right)e^{-2\beta(r)}$$

$$-\left(r\frac{d}{dr}\alpha(r) - r\frac{d}{dr}\beta(r) - e^{2\beta(r)} + 1\right)e^{-2\beta(r)}\sin^2(\theta)$$

[11]:
```
R = contract(Ricci, g_inv=g_inv, upper=0).simplify()
print_scalar(R)
```

$$\frac{2\left(r^2\left(\frac{d}{dr}\alpha(r)\right)^2 - r^2\frac{d}{dr}\alpha(r)\frac{d}{dr}\beta(r) + r^2\frac{d^2}{dr^2}\alpha(r) + 2r\frac{d}{dr}\alpha(r) - 2r\frac{d}{dr}\beta(r) - e^{2\beta(r)} + 1\right)e^{-2\beta(r)}}{r^2}$$

[12]:
```
G = Ricci - Rational(1, 2) * R * g
for i in range(4):
    G[i, i] = G[i, i].simplify().factor()
    print_scalar(G[i, i])
```

$$\frac{\left(2r\frac{d}{dr}\beta(r) + e^{2\beta(r)} - 1\right)e^{2\alpha(r)}e^{-2\beta(r)}}{r^2}$$

$$\frac{2r\frac{d}{dr}\alpha(r) - e^{2\beta(r)} + 1}{r^2}$$

$$r\left(r\left(\frac{d}{dr}\alpha(r)\right)^2 - r\frac{d}{dr}\alpha(r)\frac{d}{dr}\beta(r) + r\frac{d^2}{dr^2}\alpha(r) + \frac{d}{dr}\alpha(r) - \frac{d}{dr}\beta(r)\right)e^{-2\beta(r)}$$

$$r\left(r\left(\frac{d}{dr}\alpha(r)\right)^2 - r\frac{d}{dr}\alpha(r)\frac{d}{dr}\beta(r) + r\frac{d^2}{dr^2}\alpha(r) + \frac{d}{dr}\alpha(r) - \frac{d}{dr}\beta(r)\right)e^{-2\beta(r)}\sin^2(\theta)$$

1.0.1 Stress-energy tensor $T_{\mu\nu}$ for perfect fluid

[13]:
```
p = sp.Function("p")(r)
u = sp.Function("u")(r)

UU = np.zeros((4, 4), dtype=sp.Rational)
UU[0, 0] = exp(2 * a)

T = (p + u) * UU - p * g
for i in range(4):
    T[i, i] = T[i, i].simplify()
print_matrix(T)
```

$$\begin{bmatrix} u(r)e^{2\alpha(r)} & 0 & 0 & 0 \\ 0 & p(r)e^{2\beta(r)} & 0 & 0 \\ 0 & 0 & r^2p(r) & 0 \\ 0 & 0 & 0 & r^2p(r)\sin^2(\theta) \end{bmatrix}$$

2 Einstin's field equations

$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}$

[14]:
```
G_newton = sp.Symbol("G")

eq = []
for i in range(len(G)):
    eq.append((G[i, i] - 8 * pi * G_newton * T[i, i]).simplify())

# Some manual simplification
Rtt = sp.Symbol("R_{\\theta \\theta}")
eq[0] = eq[0] * r**2 / exp(2 * a)/exp(-2*b ) * (-1 )
eq[1] = eq[1] * r**2 * (-1)
eq[2] = eq[2] / r / exp(-2*b)
eq[3] = eq[3].subs(eq[2], Rtt)
for i in range(len(G)):
    print_eq(eq[i].simplify())
```

$$8\pi G r^2 u(r) e^{2\beta(r)} - 2r\frac{d}{dr}\beta(r) - e^{2\beta(r)} + 1 = 0$$

$$8\pi G r^2 p(r) e^{2\beta(r)} - 2r\frac{d}{dr}\alpha(r) + e^{2\beta(r)} - 1 = 0$$

$$-8\pi G r p(r) e^{2\beta(r)} + r\left(\frac{d}{dr}\alpha(r)\right)^2 - r\frac{d}{dr}\alpha(r)\frac{d}{dr}\beta(r) + r\frac{d^2}{dr^2}\alpha(r) + \frac{d}{dr}\alpha(r) - \frac{d}{dr}\beta(r) = 0$$

$$R_{\theta\theta} r e^{-2\beta(r)}\sin^2(\theta) = 0$$

Define $e^{2\beta} = [1 - 2Gm(r)/r]^{-1}$

[15]:
```
m = sp.Function("m", Real=True)(r)
f = (1 - 2 * G_newton * m / r)**(-1)
eq1 = (eq[0] * exp(- 2 *a)).simplify().subs(b, Rational(1, 2) * log(f)).
 ↪simplify().expand().simplify()
s = sp.solve(eq1, m.diff(r))
eq1 = m.diff(r) - s[0]
```

Use $\nabla_\mu T^{\mu r} = 0 \implies (p+\rho)\partial_r \alpha = -\partial_r p$.

[16]:
```
eq2 = (eq[1] * r**2).subs(exp(2 * b), f).simplify()
s = sp.solve(eq2, a.diff(r))
eq2 = a.diff(r) - s[0]
eq2 = ((a.diff(r) - s[0]).subs(a.diff(r), - p.diff(r) / (p + u))*(p + u)).
 ↪simplify()
s = sp.solve(eq2, p.diff())
eq2 = p.diff(r) - s[0].factor()
```

The TOV-equation and equation for $m(r)$, both expressions are equal to 0.

[17]:
```
print_eq(eq1)
print_eq(eq2)
```

$$-4\pi r^2 u(r) + \frac{d}{dr} m(r) = 0$$

$$\frac{G\left(4\pi r^3 p(r) + m(r)\right)\left(p(r) + u(r)\right)}{r\left(-2Gm(r) + r\right)} + \frac{d}{dr} p(r) = 0$$

Bibliography

[1] Particle Data Group et al. "Review of Particle Physics". In: *Progress of Theoretical and Experimental Physics* 2020.8 (Aug. 2020), p. 083C01. ISSN: 2050-3911. DOI: 10.1093/ptep/ptaa104.

[2] S. M. Carroll. *Spacetime and Geometry: An Introduction to General Relativity*. First. Cambridge University Press, Aug. 2019. ISBN: 978-1-108-48839-6 978-1-108-77038-5. DOI: 10.1017/9781108770385.

[3] J. M. Lee. *Introduction to Smooth Manifolds*. Springer Science & Business Media, 2003. ISBN: 978-0-387-95448-6.

[4] M. E. Peskin and D. V. Schroeder. *An Introduction to Quantum Field Theory*. Reading, USA: Addison-Wesley, 1995. ISBN: 978-0-201-50397-5.

[5] M. D. Schwartz. *Quantum Field Theory and the Standard Model*. https://www.cambridge.org/highereducation/books/quantum-field-theory-and-the-standard-model/A4CD66B998F2C696DCC75B984A7D5799. Dec. 2013. DOI: 10.1017/9781139540940.

[6] S. Weinberg. *The Quantum Theory of Fields: Volume 1: Foundations*. Vol. 1. Cambridge: Cambridge University Press, 1995. ISBN: 978-0-521-67053-1. DOI: 10.1017/CBO9781139644167.

[7] S. Weinberg. *The Quantum Theory of Fields: Volume 2: Modern Applications*. Vol. 2. Cambridge: Cambridge University Press, 1996. ISBN: 978-0-521-55002-4. DOI: 10.1017/CBO9781139644174.

[8] T. Brauner. "Spontaneous Symmetry Breaking and Nambu-Goldstone Bosons in Quantum Many-Body Systems". In: *Symmetry* 2.2 (Apr. 2010), pp. 609–657. ISSN: 2073-8994. DOI: 10.3390/sym2020609. arXiv: 1001.5212.

[9] L. A. Morrison. *Coleman-Callan-Wess-Zumino Construction*. http://scipp.ucsc.edu/~haber/archives/physics251_17/PH sentation_L_Morrison. July 2017.

[10] G. Panico and A. Wulzer. "The Composite Nambu-Goldstone Higgs". In: *arXiv:1506.01961 [hep-ph]* 913 (2016). DOI: 10.1007/978-3-319-22617-0. arXiv: 1506.01961 [hep-ph].

[11] A. Pich. "Effective Field Theory with Nambu-Goldstone Modes". In: *arXiv:1804.05664 [hep-ph, physics:hep-th]* (Jan. 2020). arXiv: 1804.05664 [hep-ph, physics:hep-th].

[12] C. G. Callan et al. "Structure of Phenomenological Lagrangians. II". In: *Phys. Rev.* 177.5 (Jan. 1969), pp. 2247–2250. DOI: 10.1103/PhysRev.177.2247.

[13] S. Coleman, J. Wess, and B. Zumino. "Structure of Phenomenological Lagrangians. I". In: *Phys. Rev.* 177.5 (Jan. 1969), pp. 2239–2247. DOI: 10.1103/PhysRev.177.2239.

[14] H. Watanabe and H. Murayama. "Effective Lagrangian for Nonrelativistic Systems". In: *Phys. Rev. X* 4.3 (Sept. 2014), p. 031057. ISSN: 2160-3308. DOI: 10.1103/PhysRevX.4.031057. arXiv: 1402.7066.

[15] N. K. Glendenning. *Compact Stars: Nuclear Physics, Particle Physics and General Relativity*. Springer Science & Business Media, Dec. 2012. ISBN: 978-1-4684-0491-3.

[16] C. W. Misner, K. S. Thorne, and J. A. Wheeler. "Gravitation". In: (Dec. 2009).

[17] J. R. Oppenheimer and G. M. Volkoff. "On Massive Neutron Cores". In: *Phys. Rev.* 55.4 (Feb. 1939), pp. 374–381. DOI: 10.1103/PhysRev.55.374.

[18] R. C. Tolman. *Relativity, Thermodynamics, and Cosmology*. The International Series of Monographs on Physics. Oxford: Clarendon Press, 1934.

[19] S. Chandrasekhar. "The Highly Collapsed Configurations of a Stellar Mass. (Second Paper.)" In: *Monthly Notices of the Royal Astronomical Society* 95.3 (Jan. 1935), pp. 207–225. ISSN: 0035-8711. DOI: 10.1093/mnras/95.3.207.

[20] H. J. Lane. "On the Theoretical Temperature of the Sun, under the Hypothesis of a Gaseous Mass Maintaining Its Volume by Its Internal Heat, and Depending on the Laws of Gases as Known to Terrestrial Experiment". In: *American Journal of Science* s2-50.148 (July 1870), pp. 57–74. ISSN: 0002-9599, 1945-452X. DOI: 10.2475/ajs.s2-50.148.57.

[21] J. O. Andersen. *Introduction to Statistical Mechanics*. Akademika forlag, 2012. ISBN: 978-82-321-0105-4.

[22] J. I. Kapusta and C. Gale. *Finite-Temperature Field Theory: Principles and Applications*. Second. Cambridge Monographs on Mathematical Physics. Cambridge: Cambridge University Press, 2006. ISBN: 978-0-521-82082-0. DOI: 10.1017/CBO9780511535130.

[23] S. Weinberg. *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*. New York: Wiley, 1972. ISBN: 978-0-471-92567-5.

[24] C. E. Rhoades and R. Ruffini. "Maximum Mass of a Neutron Star". In: *Phys. Rev. Lett.* 32.6 (Feb. 1974), pp. 324–327. DOI: 10.1103/PhysRevLett.32.324.

[25] S. Chandrasekhar. "Dynamical Instability of Gaseous Masses Approaching the Schwarzschild Limit in General Relativity". In: *Phys. Rev. Lett.* 12.4 (Jan. 1964), pp. 114–116. DOI: 10.1103/PhysRevLett.12.114.

[26] K. S. Thorne. "The General-Relativistic Theory of Stellar Structure and Dynamics". In: *pp 166-280 of High-Energy Astrophysics. Gratton, L. (ed.). New York and London, Academic Press, 1966.* (Oct. 1968).

[27] R. Penco. "An Introduction to Effective Field Theories". In: *arXiv:2006.16285 [hep-th]* (June 2020). arXiv: 2006.16285 [hep-th].

[28] S. Scherer. "Introduction to Chiral Perturbation Theory". In: *arXiv:hep-ph/0210398* (Oct. 2002). arXiv: hep-ph/0210398.

[29] P. Adhikari, J. O. Andersen, and P. Kneschke. "Two-Flavor Chiral Perturbation Theory at Nonzero Isospin: Pion Condensation at Zero Temperature". In: *Eur. Phys. J. C* 79.10 (Oct. 2019), p. 874. ISSN: 1434-6044, 1434-6052. DOI: 10.1140/epjc/s10052-019-7381-4. arXiv: 1904.03887.

[30] D. T. Son and M. A. Stephanov. "QCD at Finite Isospin Density". In: (May 2000). DOI: 10.1103/PhysRevLett.86.592.

[31] G. Ecker et al. "The Role of Resonances in Chiral Perturbation Theory". In: *Nuclear Physics B* 321.2 (July 1989), pp. 311–342. ISSN: 0550-3213. DOI: 10.1016/0550-3213(89)90346-5.

[32] R. Urech. "Virtual Photons in Chiral Perturbation Theory". In: *Nuclear Physics B* 433.1 (Jan. 1995), pp. 234–254. ISSN: 05503213. DOI: 10.1016/0550-3213(95)90707-N. arXiv: hep-ph/9405341.