Master

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Todo list

Contents

Chapter 1

Introduction

1.1 Units

In this thesis, we employ *natural units*, defined by

$$\hbar = c = k_B = 1, \tag{1.1}$$

where $\hbar$ is Planck's reduced constant, c is the speed of light, and k_B is Boltzmann's constant. Dimensionfull results are often given in MeV. Uncertainties are indicated when the precision is less than four significant figures. However, the central value is always used in calculations. Uncertainties are indicated in parentheses, and the addition and subtraction of this value to the least significant result denote the confidence interval of one standard deviation. That is, for a result 123.456(7), the range 123.456 ± 0.007 cover a confidence interval of 68.3% [1]. All values in this section are from the Particle Data Group [1]. To obtain results in the SI-system, we use the following conversion factors, as given by

$$c = 2.998 \cdot 10^{8}\,\mathrm{m\,s^{-1}}, \tag{1.2}$$
$$\hbar = 1.055 \cdot 10^{-34}\,\mathrm{J\,s}, \tag{1.3}$$
$$k_B = 1.380 \cdot 10^{-23}\,\mathrm{J\,K^{-1}}, \tag{1.4}$$
$$G = 6.674 \cdot 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}, \tag{1.5}$$

where G is Newton's gravitational constant. The conversion factor between MeV and SI-units is

$$1\,\mathrm{MeV} = 1.60218\ \cdot 10^{-19}\,\mathrm{J}. \tag{1.6}$$

The fine structure constant and the elementary charge is

$$\alpha = 7.297 \cdot 10^{-3}, \tag{1.7}$$
$$e := \sqrt{4\pi\alpha} = 3.028 \cdot 10^{-1}. \tag{1.8}$$

In the calculation in section 4.4, the value for the neutron mass is

$$m_N = 939.57\,\mathrm{MeV} = 1.674 \cdot 10^{-27}\,\mathrm{kg}. \tag{1.9}$$

In astronomical calculation, the solar mass is used, which is

$$M_\odot = 1.988 \cdot 10^{30}\,\mathrm{kg}. \tag{1.10}$$

When working with chiral perturbation theory, we use

$$f_\pi = \frac{1}{\sqrt{2}} 130.2(8)\,\text{MeV} = 92.1(6)\,\text{MeV}, \tag{1.11}$$
$$m_\pi = 134.98\,\text{MeV} = 2.406 \cdot 10^{-28}\,\text{kg}, \tag{1.12}$$
$$m_{\pi_\pm} = 139.57\,\text{MeV} = 2.488 \cdot 10^{-28}\,\text{kg}, \tag{1.13}$$
$$m_{K^\pm} = 493.68\,\text{MeV}, \tag{1.14}$$
$$m_{K^0} = 497.61\,\text{MeV}, \tag{1.15}$$

where f_π is the pion decay constant, and m_π the mass of the neutral pion, π^0. The lepton masses are

$$m_e = 0.5110\,\text{MeV}, \tag{1.16}$$
$$m_\mu = 1776\,\text{MeV}. \tag{1.17}$$
$$\tag{1.18}$$

1.2 Structure of thesis

To make this thesis as self-contained as possible, we have included some parts from the earlier specialization project, with minor modifications. These sections are marked with an asterisk in the table of content.

Chapter 2

Mathematics

General relativity, and a lot of quantum field theory, is formulated in the language of *differential geometry*. Differential geometry generalizes n-dimensional calculus to more general spaces than the usual $\mathbb{R}^n$, such as curved spacetime or the more abstract space of symmetries of a quantum field theory. This chapter will survey differential geometry and Lie groups and familiarize us with the tools needed for the physics we use in this thesis.

2.1 Differential geometry

This section is based on [2, 3].

2.1.1 Manifolds and coordinates

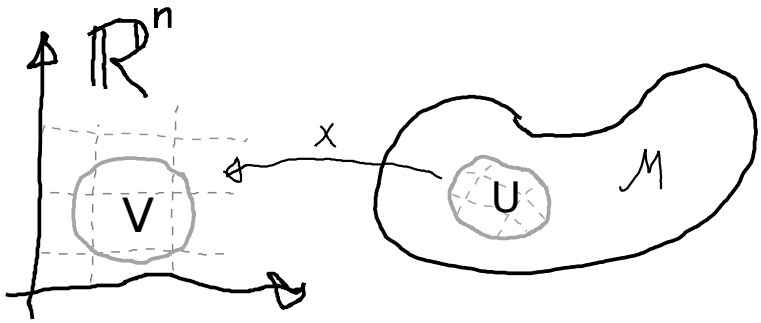


Figure 2.1: (Kladd) The coordinate function x maps a neighborhood U in the manifold $\mathcal{M}$ to a negiborhood V in $\mathbb{R}^n$.

Differential geometry is the study of *smooth manifolds*. An n-dimensional manifold, $\mathcal{M}$, is a set of points, locally homeomorphic to $\mathbb{R}^n$. That is, for all points $p \in \mathcal{M}$, there exists a neighborhood U around p, together with a corresponding set of continuous, bijective functions,

$$x : U \subseteq \mathcal{M} \longmapsto V \subseteq \mathbb{R}^n, \tag{2.1}$$

$$p \longmapsto x^\mu(p). \tag{2.2}$$

We call $x(p) = (x^0(p), \ldots, x^{n-1}(p)) = x^\mu(p)$ a coordinate function of $\mathcal{M}$. The inverse of x, x^{-1}, obeys $x^{-1}(x(p)) = p$, for all $p \in U$. A smooth manifold is one in which the coordinate functions are infinitely differentiable. To define differentiability on manifolds, consider two coordinate functions, x, and x'. The

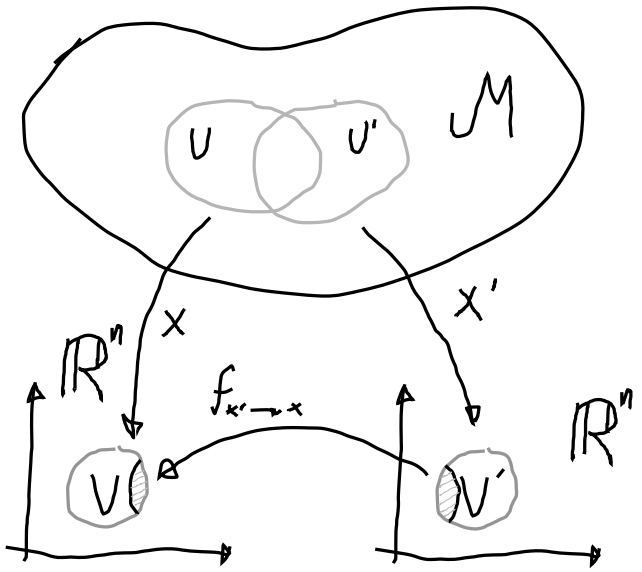


Figure 2.2: (Kladd) The transition map $f_{x'\to x}$ between two coordinate functions, x and x', maps between the images of these function, via the manifold $\mathcal{M}$. The function's domain and image are restricted to a (possibly empty) subset of the images of x and x'. This is illustrated by the shaded regions in V and V'.

corresponding domains U and U' may or may not overlap. We then define the transition function, a function between subsets of $\mathbb{R}^n$ by mapping via $\mathcal{M}$, as

$$f_{x\to x'} = x' \circ x^{-1} : \mathbb{R}^n \mapsto \mathbb{R}^n. \tag{2.3}$$

The map is illustrated in Figure 2.2. [1] A set of coordinate functions $\mathcal{A} = \{x_i\}$ whose domain cover $\mathcal{M}$ is called an *atlas* of $\mathcal{M}$. If the transition function between all pairings of coordinate functions in the atlas is smooth—that is, infinitely differentiable—we call the atlas smooth. We then define a smooth manifold as the topological manifold $\mathcal{M}$ together with a *maximal* smooth atlas $\mathcal{A}$. A smooth atlas is maximal if no coordinate function can be added while the atlas remains smooth.[2]

Consider two m- and n-dimensional smooth manifolds $\mathcal{M}$ and $\mathcal{N}$. Let x denoted the coordinates on $\mathcal{M}$, while y denotes the coordinates on $\mathcal{N}$. We can define smooth functions between these manifolds similarly to how we define smooth coordinates. Consider the function

$$F : \mathcal{M} \longmapsto \mathcal{N}. \tag{2.4}$$

It is said to be smooth if, for all points $p \in M$, there is a set of local coordinates x around p and y around $F(p)$ such that the map $\tilde{F} = y \circ F \circ x^{-1}$ is smooth. This map may be illustrated by a diagram,

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{F} & \mathcal{N} \\ \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle y} \\ \mathbb{R}^m & \xrightarrow{\tilde{F}} & \mathbb{R}^n \end{array} \tag{2.5}$$

We will not be careful with the distinction between F, the function between the abstract manifolds, and $\tilde{F}$, the function of their coordinates, but rather denote both by $F(x)$. We may take the partial derivative of such a function with respect to the coordinates x, $\partial F/\partial x^\mu$. However, this is dependent on our choice of coordinates, as a set of local coordinates can always be scaled arbitrarily. Any physical theory must be independent of our choice of coordinates, so our next task is to define the properties of a smooth manifold in a coordinate independent way.

[1] To be rigorous, one has to restrict the domains and image of the coordinate function when combining them. This is illustrated in Figure 2.2.

[2] The maximal condition ensres that two equivalent atlases correspond to the same differentiable manifold. A single manifold can be combined with different maximal atlases of smooth coordinates or differentiable structures. A set of examples are *exotic spheres*, smooth manifolds which are *homeomorphic* to S^n, but not *diffeomorphic*.

2.1.2 Vectors and tensors

A curve γ through $\mathcal{M}$ is a function from $\mathbb{R}$ to $\mathcal{M}$,

$$\gamma : \mathbb{R} \longmapsto \mathcal{M} \tag{2.6}$$
$$\lambda \longmapsto \gamma(\lambda). \tag{2.7}$$

Such curves are often denoted only by their coordinates and the parameter λ, $x^\mu(\lambda) = (x^\mu \circ \gamma)(\lambda)$. With this curve, we can take the directional derivative of a real-valued function on the manifold, $f : \mathcal{M} \mapsto \mathbb{R}$. Assume $\gamma(\lambda = 0) = p$. As we are always taking the derivative of functions between $\mathbb{R}^n$, for different n, we can use the chain rule. The directional derivative of f at p, given by this curve γ, is then

$$\frac{d}{d\lambda} f(x(\lambda))\Big|_p = \frac{dx^\mu}{d\lambda}\Big|_{\lambda=0} \frac{\partial}{\partial x^\mu} f(x)\Big|_p. \tag{2.8}$$

The set of all such directional derivatives, $d/d\lambda$ at p, form a vector space, $T_p\mathcal{M}$, called the *tangent space*. The tangent space is illustrated in Figure 2.3. The coordinates x^μ induce a basis of this vector space, namely partial derivatives with respect to the coordinate functions at p

$$e_\mu = \frac{\partial}{\partial x^\mu}\Big|_p = \partial_\mu|_p, \quad \mu \in \{0, ...n-1\}. \tag{2.9}$$

Any element $v \in T_p\mathcal{M}$ can therefore be written

$$v = v^\mu \partial_\mu|_p = \frac{dx^\mu}{d\lambda}\Big|_{\lambda=0} \frac{\partial}{\partial x^\mu}\Big|_p. \tag{2.10}$$

Here, λ is the parameter of the curve corresponding to the directional derivative v.[3] The evaluation at $\lambda = 0$ and p will often be implicit for ease of notation. This directional derivative acts on functions $f : \mathcal{M} \mapsto \mathbb{R}$ as

$$v(f) = v^\mu \partial_\mu f. \tag{2.11}$$

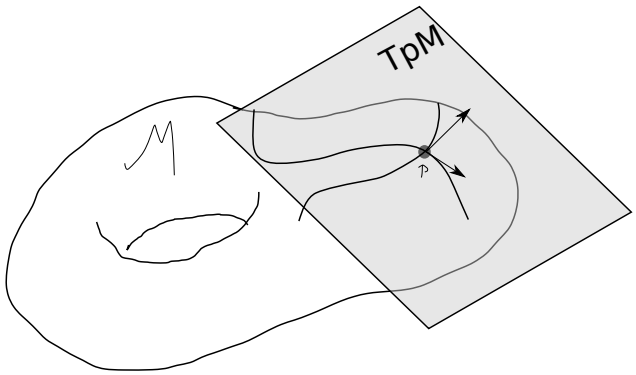


Figure 2.3: (Kladd) The tangent space $T_p\mathcal{M}$, the shaded rectangle, is the sett of all directional derivatives at $p \in \mathcal{M}$. A directional derivative is defined in terms of a curve that passes through p.

A map F between two manifolds $\mathcal{M}$ and $\mathcal{N}$ also induces a map between the tangent spaces of these manifolds. This is the *differential* of F at p,

$$\mathrm{d}F_p : T_p\mathcal{M} \longmapsto T_p\mathcal{N}, \tag{2.12}$$
$$v \longmapsto \mathrm{d}F_p(v). \tag{2.13}$$

As $\mathrm{d}F_p(v)$ is an element of $T_p\mathcal{N}$, directional derivative on $\mathcal{N}$, defined as

$$\mathrm{d}F_p(v)(g) = v(g \circ F), \tag{2.14}$$

[3]There is not only one curve corresponding to any directional derivative but rather an equivalence class. We will gloss over this technicality, as it does not affect our work.

for functions $g : \mathcal{N} \mapsto \mathbb{R}$. It thus acts on functions on $\mathcal{N}$ by "extending" the derivative v. This is a linear map between vector spaces and may be written in component form by considering the differentials of the coordinate functions. Denote the coordinates of $\mathcal{N}$ by y^μ, and $y^\mu \circ F = F^\mu$. Then,

$$\mathrm{d}F_p(\partial_\mu)(g) = \partial_\mu(g \circ F)|_p = \frac{\partial F^\nu}{\partial x^\mu}\Big|_p \frac{\partial g}{\partial y^\nu}\Big|_{F(p)}, \tag{2.15}$$

or more suggestively

$$\mathrm{d}F\left(\frac{\partial}{\partial x^\mu}\right) = \frac{\partial F^\nu}{\partial x^\mu}\frac{\partial}{\partial y_\nu}. \tag{2.16}$$

This is a linear map of vectors between two vectors by the matrix $A_\mu{}^\nu = \partial_\mu F^\nu$. The differential is thus a generalization of the Jacobian. In the case of a real valued function, $f : \mathcal{M} \mapsto \mathbb{R}$, and $g : \mathbb{R} \mapsto \mathbb{R}$, we get

$$\mathrm{d}f(v)(g) = v(g \circ f) = (v^\mu \partial_\mu f)\,\frac{dg}{dy}. \tag{2.17}$$

$\mathrm{d}f$ is thus a map from $T_p\mathcal{M}$ to $T_{f(p)}\mathbb{R}$, which is isomorphic to $\mathbb{R}$. The g be the identity function, so that $dg/dy = 1$. Then, the differential of a scalar function, also called a 1-form, is a map from vectors v to real numbers,

$$\mathrm{d}f(v) := v^\mu \partial_\mu f. \tag{2.18}$$

The set of all linear maps from a vector space V to the real numbers is called the *dual space* of V, denoted V^*. This is a new vector space with the same dimensionality as V. We denote the dual of $T_p\mathcal{M}$ as $T_p^*\mathcal{M}$. We can regard each coordinate function as a real-valued function with a corresponding differential. This differential obeys

$$\mathrm{d}x^\mu(\partial_\nu) = \frac{\partial x^\mu}{\partial x_\nu} = \delta^\mu_\nu. \tag{2.19}$$

The differentials of the coordinate functions thus form a basis for $T_p^*\mathcal{M}$, called the dual basis. Any differential $\mathrm{d}f$ can thus be written as $\mathrm{d}f = \omega_\mu \mathrm{d}x^\nu$ for some components ω_μ. We finde the components by applying the differential to the coordinate basis, $\mathrm{d}f(\partial_\mu) = \partial_\mu f = \omega_\mu$. In other words, we recover the classical expression

$$\mathrm{d}f = \frac{\partial f}{\partial x^\mu}\mathrm{d}x^\mu, \tag{2.20}$$

however we now interpret it as a covector-field instead of an "infinitesimal displacement".

Linear maps from vectors to real numbers is generalized by *tensors*. Given a vector space V, a general (n, m) tensor T is a multilinear map, which associates n elements from V and m from its dual V^* to the real numbers, i.e.,

$$T : V \times V \times \cdots \times V^* \times \ldots \longmapsto \mathbb{R}, \tag{2.21}$$

$$(v, u \ldots; \omega, \ldots) \longmapsto T(v, u, \ldots; \omega, \ldots). \tag{2.22}$$

Multilinear means that T is linear in each argument. The set of all such maps is the tensor product space $V \otimes V \otimes \cdots \otimes V^* \otimes \ldots$, a $\dim(V)^{n+m}$-dimensional vector space. If $\{e_\mu\}$ and $\{e^\mu\}$ are the basis for V and V^*, then we can write the basis of this of the tensor product space as $\{e_\mu \otimes \cdots \otimes e^\nu \otimes \ldots\}$. The tensor can thus be written

$$T = T^{\mu\nu\cdots}{}_{\rho\ldots}\, e_\mu \otimes e_\nu \otimes \ldots e^\rho \otimes \ldots, \quad T^{\mu\nu\cdots}{}_{\rho\ldots} = T(e^\mu, e^\nu, \ldots; e_\rho, \ldots). \tag{2.23}$$

We often what to decopose a tensor down into its symmetric and antisymmetric parts. To do this, we introduce the symmetrization of a tensor T,

$$T_{(\mu_1\ldots\mu_n)} = \frac{1}{n!}\sum_{\sigma\in S_n} T_{\mu_{\sigma(1)}\ldots\mu_{\sigma(n)}}, \tag{2.24}$$

where S_n is the set of all permutations of n objects. The antisymmetrization of a tensor is defined as

$$T_{[\mu_1\ldots\mu_n]} = \frac{1}{n!}\sum_{\sigma\in S_n} \mathrm{sgn}(\sigma) T_{\mu_{\sigma(1)}\ldots\mu_{\sigma(n)}}. \tag{2.25}$$

The function $\sigma = \pm 1$, depending on if σ is a even or odd permutation. We may now write

$$T_{\mu\nu} = T_{(\mu\nu)} + T_{[\mu\nu]}. \tag{2.26}$$

2.1.3 Geometry and the metric

The metric is a symmetric, non-degenerate $(0,2)$ tensor

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu. \tag{2.27}$$

It defines the geometry of the manifold $\mathcal{M}$, and is the main object of study in general relativity. As it is invertible, we can define $g^{\mu\nu} = (g^{-1})_{\mu\nu}$, which is the components of a $(2,0)$ tensor. We use this to raise and lower indices, as is done with the Minkowski metric $\eta_{\mu\nu}$ in special relativity.

Up until now, we have only considered the tangent space $T_p\mathcal{M}$ at a point p and the corresponding tensor-product spaces. We are, however, more interested in *fields* of vectors, covectors, or tensors. For each point $p \in \mathcal{M}$, a tensor field T "picks out" a tensor $T(p)$ from each tensor product space corresponding to the tangent space at p, $T_p\mathcal{M}$. We will use a vector field to illustrate. This vector field can be written as

$$v(p) = v^\mu(p)\partial_\mu|_p. \tag{2.28}$$

We will mostly be working with the components v^μ, which are functions of $\mathcal{M}$. For ease of notation, we write the vector as a function of the coordinates x. The vector field $v(x)$ is unchanged by a coordinate-transformation $x^\mu \to x'^\mu$; the coordinates are only a tool for our convenience. However, with a new set of coordinates, we get a new set of basis vectors, ∂'_μ:

$$v = v^\mu\partial_\mu = v^\mu\frac{\partial x'^\nu}{\partial x^\mu}\partial'_\nu = v'^\mu\partial'_\mu, \tag{2.29}$$

This gives us the transformation rules for the components of vectors,

$$v'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}v^\nu. \tag{2.30}$$

Tangent vectors are also called *contravariant* vectors, as their components transform contra to basis vectors. For covectors, it is

$$\omega'_\mu = \frac{\partial x^\nu}{\partial x'^\mu}\omega_\nu, \tag{2.31}$$

which is why covectors also are called *covariant* vectors.

The gradient of a scalar function f, $\mathrm{d}f = \partial_\mu f \mathrm{d}x^\mu$, is a coordinate-independent derivative, as $\partial_\mu f$ follows the transformation law for covectors. We define the covariant derivative, ∇, as a map from (n,m) tensor fields to $(n,m+1)$ tensor fields. When considering a scalar as a $(0,0)$ tensor, we see that this generalizes the scalar derivative. The components of a covariant derivative, $\nabla_\rho T^{\mu_1\cdots}{}_{\nu_1,\ldots}$, must follow the tensor transformation law. However, this is not strong enough to uniquely define ∇. We further assume

- Linearity: $\nabla(T+S) = \nabla T + \nabla S$.
- The product rule: $\nabla(T\otimes S) = (\nabla T)\otimes S + T\otimes(\nabla S)$.
- Reduces to partial derivative for scalars: $\nabla_\mu f = \partial_\mu f$.
- Kronecker delta gives zero: $\nabla_\mu \delta^\rho_\nu = 0$.

With this, we can, in general, write the covariant derivative as [2]

$$\nabla_\mu v^\nu = \partial_\mu v^\mu + \Gamma^\mu_{\nu\rho}v^\rho, \tag{2.32}$$

$$\nabla_\mu \omega_\nu = \partial_\mu\omega_\nu - \Gamma^\rho_{\mu\nu}\omega_\rho, \tag{2.33}$$

for vectors and covectors. $\Gamma^\mu_{\nu\rho}$ are called *Christoffel symbols*. The generalization for higher-order tensors is straightforward,

$$\nabla_\mu T^{\nu\cdots}{}_{\rho\ldots} = \partial_\mu T^{\nu\cdots}{}_{\rho\ldots} + \Gamma^\mu_{\nu\lambda}T^{\lambda\cdots}{}_{\rho\ldots} + \cdots - \Gamma^\lambda_{\mu\rho}T^{\mu\cdots}{}_{\lambda\ldots} - \ldots. \tag{2.34}$$

This is still not enough to uniquely determine the covariant derivative. We will furthermore assume $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$ and $\nabla_\mu g_{\nu\rho} = 0$. With these, we can find an explicit formula of the Christoffel symbols in terms of the metric,

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} - \partial_\sigma g_{\mu\nu} + \partial_\nu g_{\sigma\mu}). \tag{2.35}$$

With the notion of a covariant derivative, we may also generalize *parallel transport* to curved spaces. The notion of parallel transport of a vector in flat $\mathbb{R}^n$ is intuitive—given a line $x^\mu(\lambda)$, a vector v^μ at $x^\mu(\lambda_0)$ is parallel transported to v'^μ at $x^\mu(\lambda_1)$ if they "point in the same direction". To make this more precise, a vector field v^μ is parallel transported along $x^\mu(\lambda)$ if $\frac{d}{d\lambda} v^\mu = \frac{dx^\nu}{d\lambda}\partial_\nu v^\mu$. We generalize this to curved spaces by replacing the partial derivative with a covariant derivative, and so the criterion for parallel transport is

$$\frac{dx^\mu}{d\lambda}\nabla_\mu v^\nu = 0. \tag{2.36}$$

With this, we can imagine creating a special class of paths, called *geodesics*, namely those which parallel transport their tangent vectors $\frac{dx^\mu}{d\lambda}$. We imagine following an arrow we are holding without turning it as we walk. Using the definition of parallel transport Eq. (2.36), together with the covariant derivative Eq. (2.32), we get the geodesic equation,

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\rho}{d\lambda} = 0. \tag{2.37}$$

In a flat space, where the Christoffel symbols vanish, this reduces to the familiar criterion for straight lines, $\frac{d^2x^\mu}{d\lambda^2} = 0$.

The curvature of a manifold $\mathcal{M}$, with the metric $g_{\mu\nu}$, is encoded in the Riemann tensor. It is defined by

$$[\nabla_\mu, \nabla_\nu]v^\rho = R^\rho{}_{\sigma\mu\nu}v^\sigma, \tag{2.38}$$

which in our case gives the explicit formula

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}. \tag{2.39}$$

Although the Christoffel symbols are not tensors, the Riemann tensor is due to its definition using covariant derivatives. We can therefore contract some of its indices to get other tensor quantities. We define the Ricci tensor and Ricci scalar as

$$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}, \tag{2.40}$$

$$R = R^\mu{}_\mu = g^{\mu\nu}R_{\mu\nu}. \tag{2.41}$$

This form gives us several useful identities, such as

$$R_{\rho\sigma\mu\nu} = R_{[\rho\sigma]\mu\nu} = R_{\rho\sigma[\mu\nu]} = R_{\mu\nu\rho\sigma}. \tag{2.42}$$

Using can use the Jacobi identity of the commutator, we have

$$[\nabla_\mu, [\nabla_\nu, \nabla_\sigma]] + [\nabla_\sigma, [\nabla_\mu, \nabla_\nu]] + [\nabla_\nu, [\nabla_\sigma, \nabla_\mu]] = 0. \tag{2.43}$$

If we apply this on δ^μ_ν, we get the differential Bianchi identity, compactly written

$$\nabla_{[\mu}R_{\nu\rho]\sigma\eta} = 0. \tag{2.44}$$

To interpret the Riemann tensor, we define the parallel propagator P. A vector that is parallel transported along a curve parametrized by λ, so that $v^\mu(\lambda)$ obey the criterion of parallel transport Eq. (2.36), obey

$$v^\mu(\lambda) = P^\mu{}_\nu(\lambda)v^\nu. \tag{2.45}$$

Inserting this into the equation for parallel transport, Eq. (2.36), this operator must obey

$$\frac{d}{d\lambda}P^\mu{}_\nu = -\Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}P^\sigma{}_\nu. \tag{2.46}$$

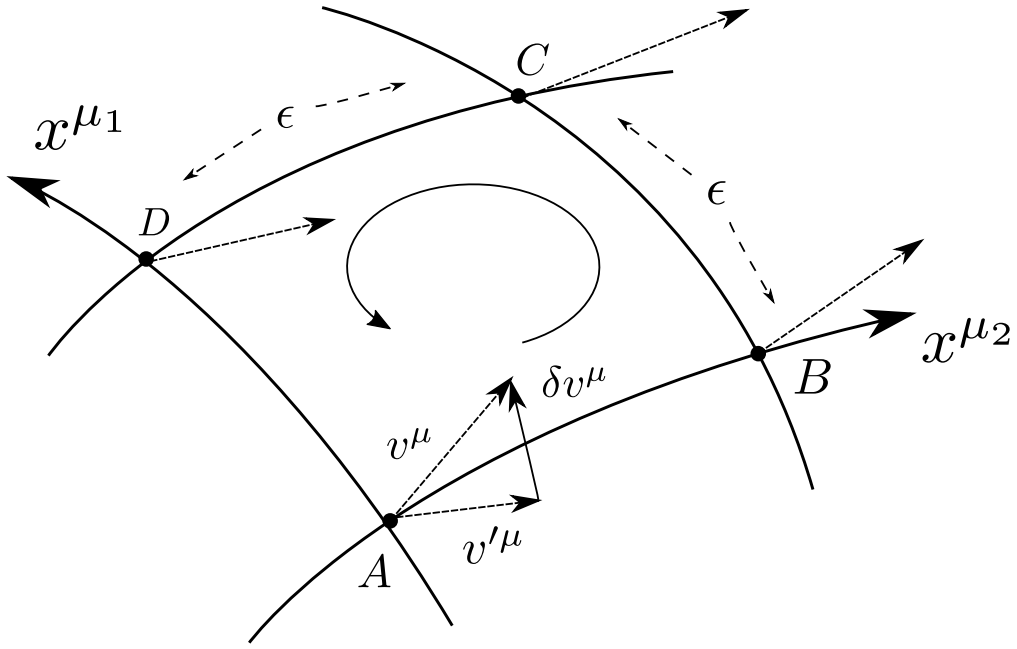


Figure 2.4: A vector v^μ is parallel transported in a small, closed loop, defined by the coordinate functions x^{μ_1} and x^{μ_2}. As a consequence of the curvature, it has changed by δv^μ by the time it arrives back at A.

This has the same form as the definition of the unitary time-evolution operator in quantum mechanics, and we could therefore write down a solution involving an exponential and a path ordering operator, $\mathcal{P}$, analogous to the time ordering operator from quantum mechanics. We may rewrite the equation as an integral equation,

$$P^\mu{}_\nu(\lambda) = \delta^\mu_\nu - \int_0^\lambda \mathrm{d}\lambda' \Gamma^\mu_{\rho\sigma} V^\rho P^\sigma{}_\nu, \tag{2.47}$$

where we denote $\frac{dx^\mu}{d\lambda} = V^\mu$. This allows us to solve the equation iteratively. If $\lambda \leq \epsilon \ll 1$, then we would expect this to converge, as long as the g is well-behaved. In that case, we can start with the zeroth-order solution $P^\mu{}_\nu = \delta^\mu_\nu$, then iterate twice to obtain

$$P^\mu{}_\nu(\lambda) = \delta^\mu_\nu - \int_0^\lambda \mathrm{d}\lambda' \, \Gamma^\mu_{\rho\nu} V^\rho + \int_0^\lambda \mathrm{d}\lambda' \int_0^{\lambda'} \mathrm{d}\lambda'' \, \Gamma^\mu_{\rho\sigma} \Gamma^\sigma_{\eta\nu} V^\rho V^\eta + \mathcal{O}(\epsilon^3). \tag{2.48}$$

With this, we will investigate how much a vector v^μ is changed by being parallel transported around in a small loop. This is illustrated in Figure 2.4. A vector v^μ, is parallel transported in a loop along the coordinate lines. These lines are where either of the coordinate functions x^{μ_1} or x^{μ_2} are equal to 0 or ϵ. Here, the indices μ_1 and μ_2 are not free, but identify the two coordinate functions which define this loop. The line from A to B, defined by $x^{\mu_1} = 0$, is parametrized by $x^\mu(\lambda) = \lambda \delta^\mu_{\mu_2}$, so $V^\mu = \delta^\mu_{\mu_2}$. The Christoffel symbol along this line is

$$\Gamma^\mu_{\nu\rho}(\lambda) = \Gamma^\mu_{\nu\rho}|_A + \lambda \partial_{\mu_2} \Gamma^\mu_{\nu\rho}|_A + \mathcal{O}(\lambda^2). \tag{2.49}$$

Inserting this into Eq. (2.48), we get

$$P^\mu{}_\nu(\epsilon) = \delta^\mu_\nu - \epsilon \Gamma^\mu_{\nu\mu_2}|_A + \frac{1}{2}\epsilon^2 \left(\Gamma^\mu_{\mu_2\sigma} \Gamma^\sigma_{\mu_2\nu}|_A - \partial_{\mu_2} \Gamma^\mu_{\nu\mu_2}|_A \right) + \mathcal{O}(\epsilon^3). \tag{2.50}$$

Next, from B to C, the line is $x^\mu(\lambda) = \epsilon \delta^\mu_{\mu_2} + \lambda \delta^\mu_{\mu_1}$, so $V^\mu = \delta^\mu_{\mu_1}$, and the Christoffel symbols are $\Gamma^\mu_{\nu\rho} = \Gamma^\mu_{\nu\rho}|_B + \lambda \partial_{\mu_1} \Gamma^\mu_{\nu\rho}|_B$ to fist order in λ. Here, we have to expand once more to evaluate the symbols at A. Then, we get

$$\Gamma^\mu_{\nu\rho} = \Gamma^\mu_{\nu\rho}|_A + \epsilon \partial_{\mu_2} \Gamma^\mu_{\nu\rho}|_A + \lambda \partial_{\mu_1} \Gamma^\mu_{\nu\rho}|_A, \tag{2.51}$$

The parallel propagator from B to C is then

$$P^\mu{}_\nu(\epsilon) = \delta^\mu_\nu - \epsilon \Gamma^\mu_{\nu\mu_1}|_A + \frac{1}{2}\epsilon^2 \left(\Gamma^\mu_{\sigma\mu_1} \Gamma^\sigma_{\nu\mu_1} - \partial_{\mu_1} \Gamma^\mu_{\nu\mu_1}|_A - 2\partial_{\mu_2} \Gamma^\mu_{\nu\mu_1}|_A \right). \tag{2.52}$$

The combined propagator from A to C is then, to second order in ϵ,

$$P_{AC}{}^{\mu}{}_{\nu} = \left[\delta^{\mu}_{\sigma} - \epsilon\Gamma^{\mu}_{\sigma\mu_1} + \frac{1}{2}\epsilon^2\left(\Gamma^{\mu}_{\eta\mu_1}\Gamma^{\eta}_{\sigma\mu_1} - \partial_{\mu_2}\Gamma^{\mu}_{\sigma\mu_2} - 2\partial_{\mu_2}\Gamma^{\mu}_{\sigma\mu_1}\right)\right]\cdot\left[\delta^{\sigma}_{\nu} - \epsilon\Gamma^{\sigma}_{\nu\mu_2} + \frac{1}{2}\epsilon^2\left(\Gamma^{\sigma}_{\eta\mu_2}\Gamma^{\eta}_{\nu\mu_2} - \partial_{\mu_2}\Gamma^{\sigma}_{\nu\mu_2}\right)\right]$$
$$= \delta^{\mu}_{\nu} - \epsilon\left(\Gamma^{\mu}_{\nu\mu_1} + \Gamma^{\mu}_{\nu\mu_2}\right) + \epsilon^2\frac{1}{2}\left(2\Gamma^{\mu}_{\sigma\mu_1}\Gamma^{\sigma}_{\nu\mu_2} + \Gamma^{\mu}_{\sigma\mu_1}\Gamma^{\sigma}_{\nu\mu_1} + \Gamma^{\mu}_{\sigma\mu_2}\Gamma^{\sigma}_{\nu\mu_2} - 2\partial_{\mu_2}\Gamma^{\mu}_{\nu\mu_1} - \partial_{\mu_1}\Gamma^{\mu}_{\nu\mu_1} - \partial_{\mu_2}\Gamma^{\mu}_{\nu\mu_2}\right).$$

The parallel propagator for CDA is the propagator for ADC with its signs flipped. The ADC propagator is the same as ABC, only with the μ_1 and μ_2 indices switched. It is thus

$$P_{CA}{}^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + \epsilon\left(\Gamma^{\mu}_{\nu\mu_2} + \Gamma^{\mu}_{\nu\mu_1}\right) + \epsilon^2\frac{1}{2}\left(2\Gamma^{\mu}_{\sigma\mu_2}\Gamma^{\sigma}_{\nu\mu_1} + \Gamma^{\mu}_{\sigma\mu_2}\Gamma^{\sigma}_{\nu\mu_2} + \Gamma^{\mu}_{\sigma\mu_1}\Gamma^{\sigma}_{\nu\mu_1} + 2\partial_{\mu_1}\Gamma^{\mu}_{\nu\mu_2} + \partial_{\mu_2}\Gamma^{\mu}_{\nu\mu_2} + \partial_{\mu_1}\Gamma^{\mu}_{\nu\mu_1}\right).$$

The full propagator, from A to A, is $P^{\mu}{}_{\nu} = P_{CA}{}^{\mu}{}_{\rho}P_{AC}{}^{\rho}{}_{\nu}$. The terms linear in ϵ vanish, and the same with the terms with two equal μ_i-indices. The change in the vector as it is rotated around the loop is therefore, to second order in ϵ,

$$\delta v^{\mu} = P^{\mu}{}_{\nu}v^{\nu} - v^{\mu} = \epsilon^2\left(\Gamma^{\mu}_{\sigma\mu_1}\Gamma^{\sigma}_{\nu\mu_2} - \Gamma^{\mu}_{\sigma\mu_2}\Gamma^{\sigma}_{\nu\mu_1} + \partial_{\mu_1}\Gamma^{\mu}_{\nu\mu_2} - \partial_{\mu_2}\Gamma^{\mu}_{\nu\mu_1}\right)v^{\nu}. \tag{2.53}$$

Comparing with Eq. (2.39), we see that this is the Riemann curvature tensor. In other words, the Riemann tensor encodes how a vector is transformed when parallel transported in a small, closed loop.

2.1.4 Integration on manifolds

The integral of a scalar function on a manifold is not a coordinate-independent notion, and we must introduce the notion of n-forms. A n-form is a antisymmetric $(0, n)$ tensor. The wedge product, $\wedge$ is a product which maps two n- and m-forms to a $n + m$-form, and is defined as

$$(A \wedge B)_{\mu_1\ldots\mu_{n+m}} = \frac{(n+m)!}{n!m!}A_{[\mu_1\ldots\mu_n}B_{\mu_{n+1}\ldots\mu_{n+m}]}, \tag{2.54}$$

A n-form ω may now be written as

$$\omega = \omega_{[\mu_1\ldots\mu_n]}\,\mathrm{d}x^{\mu_1}\otimes\cdots\otimes\mathrm{d}x^{\mu_n} = \omega_{\mu_1\ldots\mu_n}\,\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_n}. \tag{2.55}$$

Furthermore, we define the exterior derivative, a map from n-forms to $n + 1$-forms, defined by

$$(\mathrm{d}T)_{\mu_1\ldots\mu_{n+1}} = (n+1)\partial_{[\mu_1}T_{\mu_2\ldots\mu_{n+1}]}. \tag{2.56}$$

We are interested in a coordinated independent quantity that we can integrate over. To that end, we define

$$\mathrm{d}^n x := \mathrm{d}x^0\wedge\cdots\wedge\mathrm{d}x^{n-1} = \frac{1}{n!}\varepsilon_{\mu_1\ldots\mu_n}\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_n}, \tag{2.57}$$

Where $\varepsilon_{\mu_1\ldots\mu_n}$ is the Levi-Civita symbol. Given a different set of coordinates, x'^{μ}, these are related by

$$\mathrm{d}^n x = \det\left(\frac{\partial x}{\partial x'}\right)\mathrm{d}^n x', \tag{2.58}$$

where we have used the relation $\varepsilon_{\mu_1\ldots\mu_n}\det(A) = \varepsilon_{\nu_1\ldots\nu_n}A^{\nu_1}{}_{\mu_1}\ldots A^{\nu_n}{}_{\mu_n}$. We define $|g| = |\det(g_{\mu\nu})|$, which, by the transformation properties of tensors, transforms as

$$\sqrt{|g'|} = \left|\det\left(\frac{\partial x'}{\partial x}\right)\right|\sqrt{|g'|}, \tag{2.59}$$

This means that we can use this to compensate for the transformation of $\mathrm{d}^n x$, and get a volume form with a coordinate independent expression,

$$\mathrm{d}V = \sqrt{|g|}\,\mathrm{d}^n x = \sqrt{|g'|}\,\mathrm{d}^n x'. \tag{2.60}$$

With this, we can integrate scalars in a well-defined way by mapping them to a corresponding n-form, $f \to f\mathrm{d}V$. We define the integral of a scalar function f on a manifold $\mathcal{M}$ with a metric g as

$$I = \int_{\mathcal{M}} \mathrm{d}V\, f = \int_{\mathcal{M}} \mathrm{d}^n x \sqrt{|g(x)|}\, f(x). \tag{2.61}$$

Stoke's theorem generalizes the fundamental theorem of calculus and the divergence theorem to manifolds. Let $\mathcal{M}$ be a differential manifold of dimension n, with the boundary $\partial\mathcal{M}$. Stoke's theorem says that, for an $n-1$-form ω,

$$\int_{\mathcal{M}} \mathrm{d}\omega = \int_{\partial\mathcal{M}} \omega. \tag{2.62}$$

Stoke's theorem then implies a generalized divergence theorem. The boundary of $\mathcal{M}$ is a $n-1$ manifold dimensional, and a metric g on $\mathcal{M}$ will induce a new metric γ on $\partial\mathcal{M}$. This metric corresponds to the restriction of g to $\partial\mathcal{M}$. Furthermore, there will be a vector field n^μ of normalized vectors orthogonal to all elements of $T\partial\mathcal{M}$. This theorem states that for a vector field V^μ on $\mathcal{M}$,

$$\int_{\mathcal{M}} \mathrm{d}^n x \sqrt{|g|}\, \nabla_\mu V^\mu = \int_{\partial\mathcal{M}} \mathrm{d}^{n-1} y \sqrt{|\gamma|}\, n_\mu V^\mu. \tag{2.63}$$

2.2 *Lie groups

This section is based on [3–7].

2.2.1 Groups

Lie groups are a natural structure to capture the symmetries of a physical theory. A Lie group is a smooth manifold, with the additional structure of a *group*. A group is a set, G, together with a map

$$(\cdot,\cdot) : G \times G \longmapsto G, \tag{2.64}$$

$$(g_1, g_2) \longmapsto g_3, \tag{2.65}$$

called group multiplication. This map obeys the group axioms, which are the existence of an identity element $\mathbb{1}$, associativity and the existence of an inverse element g^{-1} for all $\in G$. These can be written as

$$\begin{array}{ll} \forall g \in G, & (g, \mathbb{1}) = g, \\ \forall g_1, g_2, g_3 \in G, & (g_1, (g_2, g_3)) = ((g_1, g_2), g_3), \\ \forall g \in G,\ \exists g^{-1} \in G, \text{ s.t.}, & (g, g^{-1}) = \mathbb{1}. \end{array}$$

In addition, we require that both the multiplication map and the inverse map, $g \mapsto g^{-1}$, are smooth. As we will discuss later, a symmetry transformation is a map between physical states which leave the equations governing that system unchanged. Assume the field, or set of fields, φ is governed by the equation $f(\varphi) = 0$. A symmetry transformation $\varphi \mapsto g\varphi$ will then obey $f(g\varphi) = 0$. This is what makes groups the natural structures to describe symmetries. Assume G is the set of all, or a subset closed under compositions, symmetries of a system,

$$G = \{\, g \mid f(g\varphi) = 0 \,\}, \tag{2.66}$$

as a Lie group. The group G might act on φ linearly, so $(g\varphi)_i = g_{ij}\varphi_j$, or in a more complicated matter. In this case, the group multiplication is composition, i.e., performing transformations in succession. This map is closed, as the composite of two symmetry transformations is another symmetry transformation. The identity map is a symmetry transformation, and composition is associative. This means that invertible symmetry transformations form a group.

We will focus on connected Lie groups, in which all elements $g \in G$ are in the same connected piece as the identity map $\mathbb{1}\varphi = \varphi$. This means that for each $g \in G$, one can find a continuous path $\gamma(t)$ in the manifold,

such that $\gamma(0) = \mathbb{1}$ and $\gamma(1) = g$. Given such a path, we can study transformations close to the identity element. As the Lie group is a smooth manifold, we can write[4]

$$\gamma(\epsilon) = \mathbb{1} + i\epsilon V + \mathcal{O}(\epsilon). \tag{2.67}$$

V is a generator, and is defined as

$$iV = \frac{d\gamma}{dt}\Big|_{t=0}. \tag{2.68}$$

The generator is thus a member of the tangent space of the identity element, $T_{\mathbb{1}}G$. We denote the coordinates of G by $\eta_\alpha \in \mathbb{R}^n$. As before, we can denote a path γ in a manifold G by its path through $\mathbb{R}^n$, $\gamma(t) = g(\eta(t))$. We will assume, without loss of generality, that $\eta_\alpha(0) = 0$ and $g(0) = \mathbb{1}$. We can then write the generator as

$$V = \frac{d\gamma}{dt}\Big|_{t=0} = \frac{d\eta_\alpha}{dt}\Big|_{t=0}\frac{\partial g}{\partial \eta_\alpha}\Big|_{\eta=0} = v_\alpha T_\alpha, \quad T_\alpha = \frac{d\eta_\alpha}{dt}\Big|_{t=0}, \quad \frac{\partial g}{\partial \eta_\alpha}\Big|_{\eta=0}. \tag{2.69}$$

Infinitesimal transformations can therefore be written as

$$\gamma(\epsilon) = \mathbb{1} + i\epsilon v_\alpha T_\alpha + \mathcal{O}\epsilon. \tag{2.70}$$

2.2.2 Lie algebra

The tangent space $T_{\mathbb{1}}G$, together with the additional operation

$$[T_\alpha, T_\beta] = iC^\gamma_{\alpha\beta}T_\gamma, \tag{2.71}$$

called the Lie bracket, form a Lie algebra denoted $\mathfrak{g}$. $C^\gamma_{\alpha\beta}$ are called structure constants. They obey the Jacobi identity,

FEIL! FIKS!

$$C^\gamma_{\alpha\beta} + C^\alpha_{\beta\gamma} + C^\beta_{\gamma\alpha} = 0, \tag{2.72}$$

which mean that they are totally antisymmetric. For matrix groups, which we deal with in this text, the Lie bracket is the commutator. A subset of the original Lie group, $H \subset G$, closed under the group action, is called a subgroup. H then has its own Lie algebra $\mathfrak{h}$, with a set of $m = \dim H$ generators, t_a, which is a subset of the original generators T_α. We denote the remaining set of generators x_i, such that t_a and x_i together span $\mathfrak{g}$. The commutators of t_a must be closed, which means that we can write

$$[t_a, t_b] = iC^c_{ab}t_c, \tag{2.73}$$
$$[t_a, x_i] = iC^k_{ai}x_k, \tag{2.74}$$
$$[x_i, x_j] = iC^k_{ij}x_k + iC^c_{ij}t_c, \tag{2.75}$$

where abc runs over the generators of $\mathfrak{h}$, and ijk runs over the rest. The second formula can be derived using the Jacobi identity Eq. (2.72), which implies that $C^k_{ab} = 0 = -C^b_{ak}$. This is called a Cartan decomposition.

One parameter subgroups are one special case of Lie subgroups. If a curve $\gamma(t)$ through G obey

$$\gamma(t)\gamma(s) = \gamma(t+s), \quad \gamma(0) = \mathbb{1}, \tag{2.76}$$

then all the points on this curve from a one parameter subgroup of G. This path is associated with a generator,

$$\frac{d\gamma}{dt}\Big|_{t=0} = i\eta_\alpha T_\alpha. \tag{2.77}$$

[4]The factor i is a physics convention and differs from how mathematicians define generators of a Lie group.

This association is one-to-one, and allows us to define the exponential map,

$$\exp\{i\eta_\alpha T_\alpha\} := \gamma(1). \tag{2.78}$$

For connected and compact Lie groups, all elements of the Lie group $g \in G$ can be written as an exponential of elements in the corresponding Lie algebra $\eta_\alpha T_\alpha \in \mathfrak{g}$. For matrix groups, the exponential equals the familiar series expansion

$$\exp\{X\} = \sum_n \frac{1}{n!} X^n. \tag{2.79}$$

write short about the fundamental and adjoint representation

Chapter 3

Quantum field theory

In this section, we survey some general properties of quantum field theory that are necessary for chiral perturbation theory. First, we introduce the path integral and the 1-particle irreducible effective action and the effective potential. We will derive Goldstone's theorem and present the CCWZ construction, which is the basis for χPT.

3.1 *QFT via path integrals

This section is based on [4–7]

In the path integral formalism, one evaluates quantum observable by integrating over the contributions of all possible configurations. If the system has specified initial and final states, this amounts to all possible paths the system might evolve between these, hence the name. We assume the reader has some familiarity with this formalism. However, if a refresher is needed, section C.2 contains a derivation of the closely related imaginary-time formalism and compares it with the path integral approach.

In the path integral formalism, the vacuum-to-vacuum transition amplitude, i.e., the probability that that vacuum at $t = -\infty$ evolves to the vacuum at time $t = \infty$, is given by

$$Z = \lim_{T\to\infty} \langle \Omega, T/2 | -T/2, \Omega\rangle = \lim_{T\to\infty} \left\langle \Omega \left| e^{-iHT} \right| \Omega \right\rangle = \int \mathcal{D}\pi\mathcal{D}\varphi \exp\left\{ i \int \mathrm{d}^4x \, \left(\pi\dot{\varphi} - \mathcal{H}[\pi,\varphi]\right)\right\}, \tag{3.1}$$

where $|\Omega\rangle$ is the vacuum state. The φ are the fields of the theory, and π their canonical momenta. We will work as if φ are a bosonic field. However, this can be readily generalized to fermions. By introducing a source term into the Hamiltonian density, $\mathcal{H} \to \mathcal{H} - J(x)\varphi(x)$, we get the generating functional

$$Z[J] = \int \mathcal{D}\pi\mathcal{D}\varphi \exp\left\{ i \int \mathrm{d}^4x \, \left(\pi\dot{\varphi} - \mathcal{H}[\pi,\varphi] + J\varphi\right)\right\}. \tag{3.2}$$

If $\mathcal{H}$ is quadratic in π, we can complete the square and integrate out π to obtain

$$Z[J] = C \int \mathcal{D}\varphi \exp\left\{ i \int \mathrm{d}^4x \, \left(\mathcal{L}[\varphi] + J\varphi\right)\right\}. \tag{3.3}$$

C is infinite, but constant, and will drop out of physical quantities. In scattering theory, the main objects of study are correlation functions $\langle\varphi(x_1)\varphi(x_2)...\rangle = \langle\Omega|T\left\{\varphi(x_1)\varphi(x_2)\dots\right\}|\Omega\rangle$, where T is the time ordering operator. These are given by functional derivatives of $Z[J]$,

$$\langle\varphi(x_1)\varphi(x_2)...\rangle = \frac{\int \mathcal{D}\varphi(x)\,[\varphi(x_1)\varphi(x_2)...]e^{iS[\varphi]}}{\int \mathcal{D}\varphi(x)\,e^{iS[\varphi]}} = \frac{1}{Z[0]}\prod_i \left(-i\frac{\delta}{\delta J(x_i)}\right) Z[J]\Big|_{J=0}, \tag{3.4}$$

where

$$S[\varphi] = \int \mathrm{d}^4x\, \mathcal{L}[\varphi] \tag{3.5}$$

is the action of the theory. The functional derivative is described in section A.2. In a free theory, we are able to write

$$Z_0[J] = Z_0[0]\exp\{(\}\, iW_0[J]), \quad iW_0[J] = -\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4y\, J(x)D_0(x-y)J(y), \tag{3.6}$$

where D_0 is the propagator of the free theory. Using this form of the generating functional, Eq. (3.4) becomes

$$\begin{aligned}
&\frac{1}{Z[0]}(-i)^n\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_n)}Z_0[J]\Big|_{J=0} = (-i)^n\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_n)}e^{iW_0[J]}\Big|_{J=0}\\
&= (-i)^n\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_{n-1})}\left(i\frac{\delta W_0[J]}{\delta J(x_n)}\right)e^{iW_0[J]}\Big|_{J=0}\\
&= (-i)^n\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_{n-2})}\left(i\frac{\delta^2 W_0[J]}{\delta J(x_{n-1})\,\delta J(x_n)} + i^2\frac{\delta W_0[J]}{\delta J(x_{n-1})}\frac{\delta W_0[J]}{\delta J(x_n)}\right)e^{iW_0[J]}\Big|_{J=0} = \dots\\
&= (-i)^{\lfloor n/2\rfloor}\sum_{(a,b)}\prod_{i=1}^{\lfloor n/2\rfloor}\frac{\delta^2 W_0[J]}{\delta J(x_{a(i)})\,\delta J(x_{b(i)})}\Big|_{J=0}.
\end{aligned}$$

In the last line we have introduced the functions a, b, which define a way to pair up n elements. $\lfloor\cdot\rfloor$ is the floor function. The domain of these functions are the integers between 1 and $\lfloor n/2\rfloor$, the image a subset of the integers between 1 and n of size $\lfloor n/2\rfloor$. A valid pairing is a set $\{(a(1),b(1)),\dots(a(\lfloor n/2\rfloor),b(\lfloor n/2\rfloor))\}$, where all elements $a(i)$ and $b(j)$ are different, such that all integers up to and including n are featured. A pair is not directed, so $(a(i),b(i))$ is the same pair as $(b(i),a(i))$. The sum is over the set $\{(a,b)\}$ of all possible, unique pairings. If n is odd, the expression is equal to 0. This is Wick's theorem, and it can more simply be stated as *a correlation function is the sum of all possible pairings of 2-point functions*,

$$\left\langle\prod_{i=1}^n\varphi(x_i)\right\rangle_0 = \sum_{\{(a,b)\}}\prod_{i=1}^{\lfloor n/2\rfloor}\left\langle\varphi(x_{a(i)})\varphi(x_{b(i)})\right\rangle_0. \tag{3.7}$$

The subscript on the expectation value indicates that it is evaluated in the free theory.

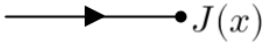
If we have an interacting theory, that is, a theory with an action $S = S_0 + S_I$, where S_0 is a free theory, the generating functional can be written

$$Z[J] = Z_0[0]\left\langle\exp\{(\}\, iS_I + i\int \mathrm{d}^4x\, J(x)\varphi(x))\right\rangle_0. \tag{3.8}$$

We can expand the exponential in power series, which means the expectation value in Eq. (3.8) becomes

$$\sum_{n,m}\frac{1}{n!m!}\left\langle(iS_I)^n\left(i\int \mathrm{d}^4x\, J(x)\varphi(x)\right)^m\right\rangle_0. \tag{3.9}$$

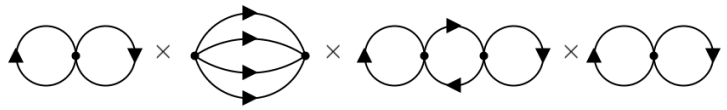
The terms in this series are represented by Feynman diagrams, constructed using the Feynman rules, and can be read from the action. We will not further detail how the Feynman rules are derived. The Feynman rules for a free scalar field in thermal field theory are derived in section C.4, and the general procedure is found in any of the main sources for this section [4–7] The source terms gives rise to an additional vertex



. (3.10)

The generating functional $Z[J]$ thus equals $Z_0[0]$ times *the sum of all diagrams with external sources* $J(x)$.

Consider a general diagram without external legs, built up of N different connected subdiagrams, where subdiagram i appears n_i times. As an illustration, a generic vacuum diagram in φ^4-theory has the form

$\mathcal{M} =$  $\dots$ (3.11)

If sub-diagram i as a stand-alone diagram equals $\mathcal{M}_i$, each copy of that subdiagram contributes a factor $\mathcal{M}_i$ to the total diagram. However, due to the symmetry of permuting identical subdiagrams, one must divide by the extra symmetry factor $s = n_i!$, the total number of permutations of all the copies of diagram i. The full diagram therefore equals

$$\mathcal{M} = \prod_{i=1}^{N} \frac{1}{n_i!} \mathcal{M}_i^{n_i}. \tag{3.12}$$

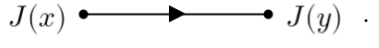
$\mathcal{M}$ is uniquely defined by a finite sequence of integers, $(n_1, n_2, \dots n_N, 0, 0, \dots)$, so the sum of all diagrams is the sum over the set S of all finite sequences of integers. This allows us to write the sum of all diagrams as

$$\sum_{(n_1,\dots)\in S} \prod_i \frac{1}{n_i!} \mathcal{M}_i^{n_i} = \prod_{i=1}^{\infty} \sum_{n_i=1}^{\infty} \frac{1}{n_i!} \mathcal{M}_i^{n_i} = \exp\{(\} \sum\nolimits_i \mathcal{M}_i). \tag{3.13}$$

We showed that the generating functional $Z[J]$ were the $Z_0[0]$ times the sum of all diagrams due to external sources. From Eq. (3.13), if we define

$$Z[J] = Z_0[0] \exp\{(\} iW[J]), \tag{3.14}$$

then $W[J]$ is the sum of all connected diagrams. This is trivially true for the free theory, where the only connected diagram is

$W_0[J] =$  . (3.15)

The two-point function in the full, interacting theory can thus be written

$$-i \frac{\delta^2 W[J]}{\delta J(x)\, \delta J(y)} = D(x-y). \tag{3.16}$$

3.2 *The 1PI effective action

This section is based on [4–7]

The generating functional for connected diagrams, $W[J]$, is dependent on the external source current J. We can define a new quantity with a different independent variable, using the Legendre transformation analogously to what is done in thermodynamics and Lagrangian mechanics. The new independent variable is

$$\varphi_J(x) := \frac{\delta W[J]}{\delta J(x)} = \langle \varphi(x) \rangle_J \,. \tag{3.17}$$

The subscript J on the expectation value indicate that it is evaluated in the presence of a source. The Legendre transformation of W is then

$$\Gamma[\varphi_J] = W[J] - \int \mathrm{d}^4x \, J(x) \varphi_J(x). \tag{3.18}$$

Using the definition of φ_J, we have that

$$\frac{\delta \Gamma[\varphi_J]}{\delta \varphi_J(x)} = \int \mathrm{d}^4y \, \frac{\delta J(y)}{\delta \varphi_J(x)} \left. \frac{\delta J(y)}{\delta W} \right|_J - \int \mathrm{d}^4y \, \frac{\delta J(y)}{\delta \varphi_J(x)} \varphi_J(y) - J(x) = -J(x). \tag{3.19}$$

If we compare this to the classical equations of motion of a field φ with the action S,

$$\frac{\delta S[\varphi]}{\delta \varphi(x)} = -J(x), \tag{3.20}$$

we see that Γ is an action that gives the equation of motion for the expectation value of the field, given a source current $J(x)$.

To interpret Γ further, we observe what happens if we treat $\Gamma[\varphi]$ as a classical action with a coupling g. The generating functional in this new theory is

$$Z[J,g] = \int \mathcal{D}\varphi \exp\left\{ig^{-1}\left(\Gamma[\varphi] + \int \mathrm{d}^4x\, \varphi(x)J(x)\right)\right\} \tag{3.21}$$

The free propagator in this theory will be proportional to g, as it is given by the inverse of the equation of motion for the free theory. All vertices in this theory, on the other hand, will be proportional to g^{-1}, as they are given by the higher-order terms in the action $g^{-1}\Gamma$. This means that a diagram with V vertices and I internal lines is proportional to g^{I-V}. Regardless of what the Feynman-diagrams in this theory are, the number of loops of a connected diagram is[1]

$$L = I - V + 1. \tag{3.22}$$

To see this, we first observe that diagrams with one single loop must have equally many internal lines as vertices, so the formula holds for $L = 1$. The formula still holds if we add a new loop to a diagram with n loops by joining two vertices. If we attach a new vertex with one line, the formula still holds, and as the diagram is connected, any more lines connecting the new vertex to the diagram will create additional loops. This ensures that the formula holds by induction. As a consequence of this, any diagram is proportional to g^{L-1}. This means that in the limit $g \to 0$, the theory is fully described at the tree-level, i.e., by only considering diagrams without loops. In this limit, we may use the stationary phase approximation, as described in section A.2, which gives

$$Z[J, g\to 0] \approx C \det\left(-\frac{\delta^2\Gamma[\varphi_J]}{\delta\varphi(x)\,\delta\varphi(y)}\right) \exp\left\{ig^{-1}\left(\Gamma[\varphi_J] + \int \mathrm{d}^4x\, J(x)\varphi_J(x)\right)\right\}. \tag{3.23}$$

This means that

$$-ig\ln(Z[J,g]) = gW[J,g] = \Gamma[\varphi_J] + \int \mathrm{d}^4x\, J(x)\varphi_J(x) + \mathcal{O}(g), \tag{3.24}$$

which is exactly the Legendre transformation we started out with, modulo the factor g. Γ is, therefore, the action that describes the full theory at the tree level. For a free theory, the classical action S equals the effective action.

As we found in the last section, the propagator $D(x,y) = \langle\varphi(x)\varphi(y)\rangle_J$ is given by $-i$ times the second functional derivative of $W[J]$. Using the chain rule, together with Eq. (3.19), we get

$$(-i)\int \mathrm{d}^4z \frac{\delta^2 W[J]}{\delta J(x)\delta J(z)}\frac{\delta^2\Gamma[\varphi_J]}{\delta\varphi_J(z)\varphi_J(y)} = (-i)\int \mathrm{d}^4z \frac{\delta\varphi_J[z]}{\delta J(x)}\frac{\delta^2\Gamma[\varphi_J]}{\delta\varphi_J(z)\varphi_J(y)} = (-i)\frac{\delta}{\delta J(x)}\frac{\delta\Gamma[\varphi_J]}{\delta\varphi_J(y)} = i\delta(x-y). \tag{3.25}$$

This is exactly the definition of the inverse propagator,

$$\frac{\delta^2\Gamma[\varphi_J]}{\delta\varphi_J(x)\,\delta\varphi_J(y)} = D^{-1}(x,y). \tag{3.26}$$

The inverse propagator is the sum of all one-particle-irreducible (1PI) diagrams, with two external vertices. More generally, Γ is the generating functional for 1PI diagrams, which is why it is called the 1PI effective action.

Γ may be viewed as an effective action as defined in the introduction. We define η as the fluctuations around the expectation value of the field, $\varphi(x) = \varphi_J(x) + \eta(x)$, and use this to change variables of integration in the path integral. The expectation value φ_J is constant with respect to the integral, so

$$\int \mathcal{D}\varphi \exp\{iS[\varphi]\} = \int \mathcal{D}\eta \exp\{iS[\varphi_J + \eta]\}. \tag{3.27}$$

By assumption, $\langle\eta\rangle_J = 0$, which means this path integral is described by only 1PI diagrams, connected or not. We can therefore write

$$\exp\{i\Gamma[\varphi_J]\} = \int \mathcal{D}\eta \exp\{iS[\varphi_J + \eta]\}. \tag{3.28}$$

[1] This is a consequence of the Euler characteristic $\chi = V - E + F$.

As we will discuss later, we interpret this form as *integrating out* the η degrees of freedom, leaving us with an effective description of the physics dependent only on the ground state solution φ_J. The disadvantage of this is that there is no bound on how complicated Γ might be—it is not restricted by the usual assumptions of the form of the action, such as locality [5]. With some simplifying assumptions, though, we can still make use of the 1PI effective action, as we will see in the next subsection.

3.2.1 Effective potential

For a constant field configuration $\varphi(x) = \varphi_0$, the effective action, a functional, becomes a regular function. We define the effective potential $\mathcal{V}_{\text{eff}}$ by

$$\Gamma[\varphi_0] = -VT\,\mathcal{V}_{\text{eff}}(\varphi_0), \tag{3.29}$$

where VT is the volume of space-time. For a constant ground state, the effective potential will equal the energy of this state. To calculate the effective potential, we can expand the action around this state to calculate the effective action, by changing variables to $\varphi(x) = \varphi_0 + \eta(x)$. $\eta(x)$ now parametrizes fluctuations around the ground state, and has by assumption a vanishing expectation value. The generating functional becomes

$$Z[J] = \int \mathcal{D}(\varphi_0 + \eta) \exp\left\{ iS[\varphi_0 + \eta] + i\int \mathrm{d}^4x\,[\varphi_0 + \eta(x)]J(x) \right\}. \tag{3.30}$$

The functional version of a Taylor expansion, as described in section A.2, is

$$S[\varphi_0 + \eta] = S[\varphi_0] + \int \mathrm{d}x\,\eta(x)\,\frac{\delta S[\varphi_0]}{\delta\varphi(x)} + \frac{1}{2}\int \mathrm{d}x\mathrm{d}y\,\eta(x)\eta(y)\,\frac{\delta^2 S[\varphi_0]}{\delta\varphi(x)\delta\varphi(y)} + \dots \tag{3.31}$$

The notation

$$\frac{\delta S[\varphi_0]}{\delta\varphi(x)} \tag{3.32}$$

indicates that the functional $S[\varphi]$ is differentiated with respect to $\varphi(x)$, then evaluated at $\varphi(x) = \varphi_0$. We define

$$S_0[\eta] := \int \mathrm{d}^4x\mathrm{d}^4y\,\eta(x)\eta(y)\,\frac{\delta^2 S[\varphi_0]}{\delta\varphi(x)\,\delta\varphi(y)}, \tag{3.33}$$

$$S_I[\eta] := \int \mathrm{d}^4x\mathrm{d}^4y\mathrm{d}^4z\,\eta(x)\eta(y)\eta(z)\,\frac{\delta^3 S[\varphi_0]}{\delta\varphi(x)\,\delta\varphi(y)\,\delta\varphi(z)} + \dots, \tag{3.34}$$

where the dots indicate higher derivatives. When we insert this expansion into the generating functional $Z[J]$ we get

$$Z[J] = \int \mathcal{D}\eta \exp\left\{ i\int \mathrm{d}^4x\,(\mathcal{L}[\varphi_0] + J\varphi_0) + i\int \mathrm{d}^4x\,\eta(x)\,\left(\frac{\delta S[\varphi_0]}{\delta\varphi(x)} + J(x)\right) + iS_0[\eta] + iS_I[\eta] \right\}. \tag{3.35}$$

The first term is constant with respect to η and may be taken outside the path integral. The second term gives rise to tadpole diagrams, which alter the expectation value of $\eta(x)$. For $J = 0$, this expectation value should vanish, and this term can be ignored. Furthermore, this means that the ground state must minimize the classical potential,

$$\frac{\partial\mathcal{V}(\varphi_0)}{\partial\varphi} = 0. \tag{3.36}$$

This leaves us with

$$-i\ln Z[J] = W[J] = \int \mathrm{d}^4x\,(\mathcal{L}[\varphi_0] + J\varphi_0) - i\ln\left(\int \mathcal{D}\eta \exp\left\{iS_0[\eta] + iS_I[\eta]\right\}\right). \tag{3.37}$$

We can now use the definition of the 1PI effective action to obtain a formula for the effective potential,

$$\mathcal{V}_{\text{eff}}(\varphi_0) = -\frac{1}{VT}\left(W[J] - \int \mathrm{d}^4x\,J(x)\varphi_0\right) = \mathcal{V}(\varphi_0) - i\ln\left(\int \mathcal{D}\eta \exp\left\{iS_0[\eta] + iS_I[\eta]\right\}\right). \tag{3.38}$$

In Eq. (3.18), we showed that the 1PI effective action describes the whole quantum theory of the original action at the tree-level. This was done by inspecting a theory with an action proportional to g^{-1}. In this theory, Feynman diagrams with L loops are proportional to g^{L-1}. We can use the same argument to expand the effective potential in loops. This is done by modifying the action $S[\varphi] \to g^{-1}S[\varphi]$, and then expand in power of g. The first term in the effective potential is modified by $\mathcal{V} \to g^{-1}\mathcal{V}$, which means that it is made up of tree-level terms. This is as expected, since the tree-level result corresponds to the classical result without any quantum corrections. The second term becomes

$$\ln\left(\int \mathcal{D}\eta\, e^{iS_0+iS_I}\right) \longrightarrow \ln\left(\int \mathcal{D}\eta\, e^{ig^{-1}S_0+ig^{-1}S_I}\right) = \ln\left(\int \mathcal{D}\eta\, e^{ig^{-1}S_0}\right) + \ln\left(\frac{\int \mathcal{D}\eta\, e^{ig^{-1}S_I}\, e^{ig^{-1}S_0}}{\int \mathcal{D}\eta\, e^{ig^{-1}S_0}}\right)$$

The first term is quadratic in η, and can therefore be evaluated as a generalized Gaussian integral, as described in section A.2,

$$\begin{aligned}&\ln\left[\int \mathcal{D}\eta \exp\left\{ig^{-1}\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4y\, \eta(x)\eta(y)\, \frac{\delta^2 S[\varphi_0]}{\delta\varphi(x)\,\delta\varphi(y)}\right\}\right]\\ &= \ln\left[\det\left(-g^{-1}\frac{\delta^2 S[\varphi_0]}{\delta\varphi(x)\,\delta\varphi(y)}\right)^{-1/2}\right] = -\frac{1}{2}\mathrm{Tr}\left\{\ln\left[-\frac{\delta^2 S[\varphi_0]}{\delta\varphi(x)\,\delta\varphi(y)}\right]\right\} + \text{const.}\end{aligned}$$

We then use the identity $\ln\{\det M\} = \mathrm{Tr}\,\{\ln M\}$. After we remove the constant, this term is proportional to g^0, i.e., it is made up of one-loop terms.

The last term can be evaluated by first expanding the exponential containing the S_I term, then using $\ln(1+x) = \sum_n \frac{1}{n}x^n$. Using

$$\langle A\rangle_0 = \frac{\int \mathcal{D}\varphi\, A\, e^{ig^{-1}S_0}}{\int \mathcal{D}\varphi\, e^{ig^{-1}S_0}}, \tag{3.39}$$

we can write

$$\ln\left[\frac{\int \mathcal{D}\eta\, e^{ig^{-1}S_I} e^{ig^{-1}S_0}}{\int \mathcal{D}\varphi\, e^{ig^{-1}S_0}}\right] = \ln\left(\sum_{n=0}^{\infty}\frac{1}{n!}\left\langle (ig^{-1}S_I)^n\right\rangle_0\right). \tag{3.40}$$

We recognize this as the sum of all connected Feynman diagrams, with Feynman rules from the interaction term S_I. We know that S_I is made up of terms that are third power or higher in the fields. Each internal line is connected to two vertices, and each vertex is connected to at least three internal lines, i.e., $I \geq 3/2V$. The number of loops is therefore $L = I - V + 1 \geq (3/2-1)V + 1$. There is at leas one vertex, i.e. $L \geq 3/2$. This shows that the first logarithm contains *all* one-loop contributions. The effective potential to one-loop order is therefore

$$\mathcal{V}_{\text{eff}}(\varphi_0) = \mathcal{V}(\varphi_0) - \frac{i}{VT}\frac{1}{2}\mathrm{Tr}\left\{\ln\left(-\frac{\delta^2 S[\varphi_0]}{\delta\varphi(x)\,\delta\varphi(y)}\right)\right\}. \tag{3.41}$$

3.3 *Symmetry and Goldstone's theorem

This section is based on [3–7].

Symmetry plays a prominent role in modern physics. If we can transform a physical state in such a way that the governing equations of this system are unchanged, we call that transformation a *symmetry transformation*. All such transformations are known as the symmetries of that theory. The symmetries of a theory encode a lot of physics, such as the presence of conserved quantities and the system's low energy behavior. We distinguish between internal and external symmetries. An external symmetry is an active coordinate transformation, such as rotations or translations. They relate degrees of freedom at different space-time points, while internal symmetries transform degrees of freedom at each space-time point independently. A further distinction is between global and local symmetry transformations. Global transformations have one rule for transforming degrees of freedom at each point, which is applied everywhere, while local transformations are functions of space-time.

In classical field theory, symmetries are encoded in the behavior of the Lagrangian when the fields are transformed. We will consider continuous transformations, which can in general be written as

$$\varphi(x) \longrightarrow \varphi'(x) = f_t[\varphi](x), \quad t \in [0,1]. \tag{3.42}$$

Here, $f_t[\varphi]$ is a functional in φ, and a smooth function of t, with the constraint that $f_0[\varphi] = \varphi$. This allows us to look at "infinitesimal" transformations,

$$\varphi'(x) = f_\epsilon[\varphi] = \varphi(x) + \epsilon \frac{df_t[\varphi]}{dt}\bigg|_{t=0} + \mathcal{O}(\epsilon). \tag{3.43}$$

When considering infinitesimal transformations, we will not always write $+\mathcal{O}(\epsilon)$, but rather consider it implicit. We will consider internal, global transformations which act linearly on φ. For N fields, φ_i, this can be written

$$\varphi_i'(x) = \varphi_i(x) + \epsilon\, iV_{ij}\varphi_j(x). \tag{3.44}$$

V_{ij} is called the generator of the transformation. A symmetry transformation of the system is then a transformation in which the Lagrangian left is unchanged, or at most differ by a 4-divergence term. That is, a transformation $\varphi \to \varphi'$ is a symmetry if

$$\mathcal{L}[\varphi'] = \mathcal{L}[\varphi] + \partial_\mu K^\mu[\varphi], \tag{3.45}$$

where $K^\mu[\varphi]$ is a functional of φ.[2] This is a requirement for symmetry in quantum field theory too. However, as physical quantities in quantum field theory are given not just by the action of a single state but the path integral, the integration measure $\mathcal{D}\varphi$ has to be invariant as well. If a classical symmetry fails due to the non-invariance of the integration measure, it is called an *anomaly*.

To investigate the symmetry properties of a quantum theory, we explore what constraints a symmetry imposes on the effective action. To that end, assume

$$\mathcal{D}\varphi'(x) = \mathcal{D}\varphi(x), \quad S[\varphi'] = S[\varphi]. \tag{3.46}$$

In the generating functional, such a transformation corresponds to a change of integration variable. Using the infinitesimal version of the transformation, we may write

$$\begin{aligned} Z[J] = \int \mathcal{D}\varphi \exp\left\{iS[\varphi] + i\int \mathrm{d}^4x\, J_i(x)\varphi_i(x)\right\} &= \int \mathcal{D}\varphi' \exp\left\{iS[\varphi'] + i\int \mathrm{d}^4x\, J_i(x)\varphi_i'(x)\right\} \\ &= Z[J] + i\epsilon \int \mathrm{d}^4x\, J_i(x) \int \mathcal{D}\varphi\, [V_{ij}\varphi_j(x)] e^{iS[\varphi]}, \end{aligned} \tag{3.47}$$

Using Eq. (3.19), we can write this as

$$\int \mathrm{d}^4x\, \frac{\delta\Gamma[\varphi_J]}{\delta\varphi_i(x)}\, V_{ij}\, \langle\varphi_j(x)\rangle_J = 0. \tag{3.48}$$

This constraint will allow us to deduce the properties of a theory close to the ground state, only using information about the symmetries of the theory.

The archetypical example of an internal, global, and continuous symmetry is the linear sigma model, which we will use as an example throughout this section. The linear sigma model is made up of N real scalar fields φ_i, whose Lagrangian is

$$\mathcal{L}[\varphi] = \frac{1}{2}\partial_\mu\varphi_i(x)\partial^\mu\varphi_i(x) - \mathcal{V}(\varphi), \quad \mathcal{V}(\varphi) = -\frac{1}{2}\mu^2\varphi_i(x)\varphi_i(x) + \frac{1}{4}\lambda[\varphi_i(x)\varphi_i(x)]^2. \tag{3.49}$$

This system is invariant under the rotation of the N fields into each other,

$$\varphi_i \longrightarrow \varphi_i' = M_{ij}\varphi_j, \quad M^{-1} = M^T. \tag{3.50}$$

The set of all such transformations forms the Lie group $\mathrm{O}(N)$. Lie groups will be discussed in the next section. For $N = 2$, and $\mu^2, \lambda > 0$ we get the ubiquitous Mexican hat potential, as illustrated in Figure 3.1.

[2] Terms of the form $\partial_\mu K^\mu$ do not affect the physics, as variational principle $\delta S = 0$ do not vary the fields at infinity. Together with the divergence theorem, this means that such terms do not influence the equations of motion.

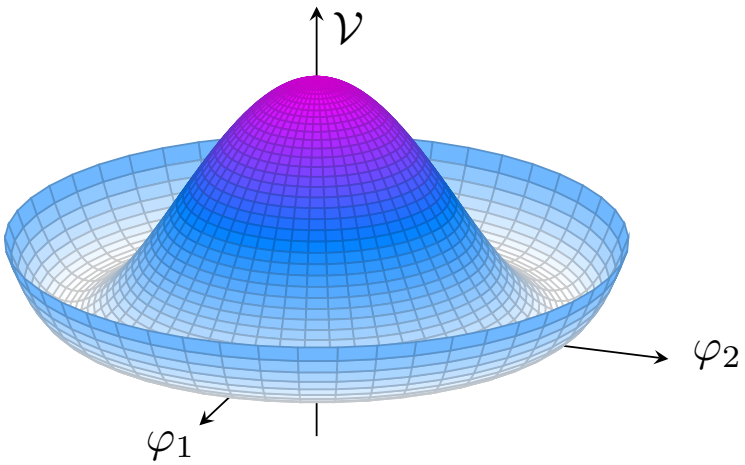


Figure 3.1: The Mexican hat potential is the classical potential $\mathcal{V}$ for the $N = 2$ linear sigma model.

3.3.1 Nöther's theorem

One of the most profound consequences of symmetry in physics is the appearance of conserved quantities. Assume we have a set of fields φ_i. Nöther's theorem tells us that if the Lagrangian $\mathcal{L}[\varphi_i]$ has a continuous symmetry, then there is a corresponding conserved current [2, 4]. Consider an infinitesimal transformation,

$$\varphi_i(x) \longrightarrow \varphi_i'(x) = \varphi_i(x) + \delta\varphi_i(x), . \tag{3.51}$$

Applying this transformation to the Lagrangian will in general change its form,

$$\mathcal{L}[\varphi] \to \mathcal{L}[\varphi'] = \mathcal{L}[\varphi] + \delta\mathcal{L}. \tag{3.52}$$

We assume this transformation is a symmetry, i.e.,

$$\delta\mathcal{L} = \partial_\mu K^\mu.$$

By considering the Lagrangian as a function of the field and its derivatives, $\mathcal{L} = \mathcal{L}(\varphi_i, \partial_\mu\varphi_i)$, we can write the difference term as a Taylor expansion around $(\varphi_i, \partial_\mu\varphi_i)$,

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\varphi_i}\delta\varphi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}\delta(\partial_\mu\varphi_i), \tag{3.53}$$

where $\delta(\partial_\mu\varphi_i) = \partial_\mu\varphi_i' - \partial_\mu\varphi_i$. By the linearlity of the derivative,

$$\delta(\partial_\mu\varphi_i) = \partial_\mu\varphi_i' - \partial_\mu\varphi_i = \partial_\mu(\varphi_i' - \varphi_i) = \partial_\mu\delta\varphi_i. \tag{3.54}$$

With this, and the Euler-Lagrange equations

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} - \frac{\partial\mathcal{L}}{\partial\varphi_i} = 0, \tag{3.55}$$

we can rewrite

$$\delta\mathcal{L} = \left(\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}\right)\delta\varphi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}(\partial_\mu\delta\varphi_i) = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}\delta\varphi_i\right) \tag{3.56}$$

If we define the current

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}\delta\varphi_i(x) - K^\mu, \tag{3.57}$$

then

$$\partial_\mu j^\mu = \delta\mathcal{L} - \delta\mathcal{L} = 0. \tag{3.58}$$

This is Nöther's theorem; a continuous symmetry implies the existence of a conserved current.

The current flux through some spacelike surface V defines a conserved charge. The surface of constant time in some reference frame has the normal vector $n_\mu = (1, 0, 0, 0)$, so the charge is

$$Q = \int_V \mathrm{d}^4x\, n_\mu j^\mu = \int_V \mathrm{d}^3x\, j^0. \tag{3.59}$$

We can then use the divergence theorem. Assume ∂V is the boundary of V, which has the space-like normal vector k_i, and that the current falls of quickly towards infinity. Then, using $\partial_\mu j^\mu = 0$,

$$\frac{dQ}{dt} = \int_V \mathrm{d}^3\, \partial_0 j^0 = -\int_V \mathrm{d}^3x\, \partial_i j^i = -\int_{\partial V} \mathrm{d}^2x\, k_i j^i = 0, \tag{3.60}$$

proving that the charge is conserved over time.

3.3.2 Goldstone's theorem

A symmetry transformation will leave the governing equation of a theory unchanged. This, however, does not imply that physical states, such as the ground state, are invariant under this transformation. The $N = 2$ linear sigma model illustrates this. If we assume the ground state φ_0 is translationally invariant, then it is given by minimizing the effective potential, of which the classical potential, $\mathcal{V}$, is the leading order approximation. This potential is illustrated in Figure 3.1. The ground state is therefore given by any of the values along the brim of the potential. If we, without loss of generality, choose $\varphi_0 = (0, v)$ as the ground state, then any rotation will change this state. We say that the symmetry has been *spontaneously broken*.

To explore this in a general context, assume a theory of N real scalar fields φ_i are invariant under the actions of some Lie group, G. A symmetry $g \in G$ is broken if the vacuum expectation value of the original fields and the transformed fields differ. That is, if

$$\langle\varphi\rangle_0 \neq \langle\varphi'\rangle_0 = \langle g\varphi\rangle_0 \tag{3.61}$$

We can now exploit what we learned about Lie groups to write the infinitesimal transformation as

$$\langle\varphi'\rangle_0 = \langle\varphi\rangle_0 + i\epsilon\eta_\alpha T_\alpha \langle\varphi\rangle_0 . \tag{3.62}$$

Let x_i be the set of generators corresponding to broken symmetries, i.e.,

$$x_i \langle\varphi\rangle_0 \neq 0. \tag{3.63}$$

These are called the *broken generators*. The remaining set of generators t_a, which obey

$$t_a \langle\varphi\rangle_0 = 0, \tag{3.64}$$

are called unbroken, and generate a subgroup $H \subset G$ as the set of symmetry transformations of the vacuum is a group.

In Eq. (3.48) we found that, if V is the generator of some symmetry, then the effective action obeys

$$\int \mathrm{d}^4x\, \frac{\delta\Gamma[\varphi_J]}{\delta\varphi_i} V_{ij} \langle\varphi_j\rangle_0 = 0, \tag{3.65}$$

We now differentiate this expression with respect to $\varphi_k(y)$, which gives

$$\int \mathrm{d}^4x\, \frac{\delta^2\Gamma[\varphi_0]}{\delta\varphi_k(y)\,\delta\varphi_i(x)} V_{ij} \langle\varphi_j\rangle_0 = 0. \tag{3.66}$$

With the assumption that the ground state is constant, we get

$$\frac{\partial^2\mathcal{V}_{\text{eff}}}{\partial\varphi_k\,\partial\varphi_i} V_{ij} \langle\varphi_j\rangle_0 = 0. \tag{3.67}$$

This is trivial for unbroken symmetries, as $t^a_{ij} \langle\varphi_j\rangle_0 = 0$ by definition. However, in the case of a broken symmetry, the second derivative of the effective potential must have an eigenvector $x^\ell_{ij} \langle\varphi_j\rangle_0$ with a zero eigenvalue for each broken generator. Here, ℓ label the set of generators, while (ij) are the indices corresponding to field-components φ_i. In Eq. (3.25), we found that the second derivative of the effective action is the inverse propagator,

$$D^{-1}_{ij}(x,y) = \frac{\delta^2\Gamma[\varphi_0]}{\delta\varphi_i(y)\,\delta\varphi_j(x)} = \int \frac{\mathrm{d}^4p}{(2\pi)^4} e^{-ip(x-y)}\, \tilde{D}^{-1}_{ij}(p). \tag{3.68}$$

Using this, we can write

$$\tilde{D}^{-1}_{ij}(p=0)\, x^\ell_{jk} \langle\varphi_k\rangle_0 = 0. \tag{3.69}$$

Zeros of the inverse propagator correspond to the physical mass of particles. In Lorentz-invariant systems, each zero-eigenvalue vector corresponds to a massless particle, called a Goldstone boson.[3] This means there are $n_G = \dim G - \dim H$ zero-mass modes. In general, the counting of massless modes is complicated and depends on the dispersion relation of the particles at low momenta. Systems with Goldstone bosons with quadratic dispersion relation, that is $E \propto |\vec{p}|^2$ when $\vec{p} \to 0$, often exhibit a lower number of massless modes. An example is ferromagnetic, where the SU(2) rotational symmetry is broken down to U(1) when they align along one axis. This corresponds to two broken generators, yet the system exhibits only one massless mode [8].

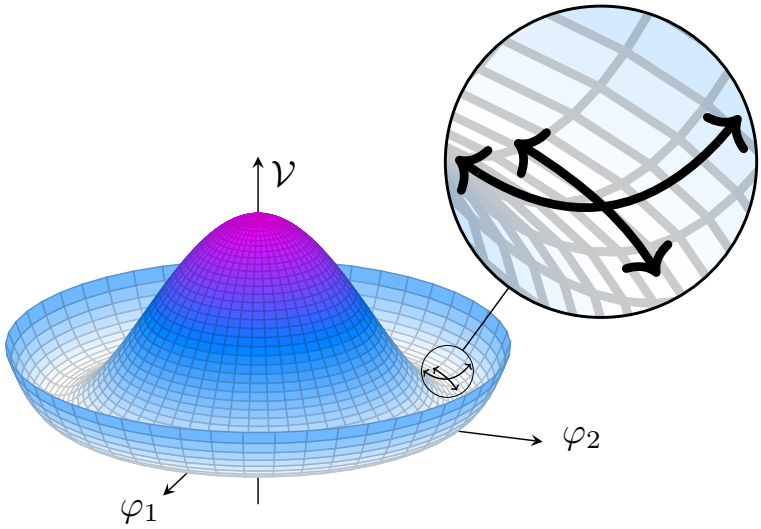


Figure 3.2: Excitations along the brim does not cost any energy, as the potential is flat, unlike excitations in the radial direction.

The linear sigma model gives an intuition for the Goldstone mode. In the case of $N = 2$, the symmetry of the Lagrangian are rotations in the plane. As the ground state is a point along the "brim" of the hat, this rotational symmetry is broken. However, any excitations in the angular direction do not cost any energy, which is indicative of a massless mode. This is illustrated in Figure 3.2. In this example, the original symmetry group is one-dimensional, so there are no unbroken symmetries. Consider instead the $N = 3$ linear sigma model, which has the three-dimensional symmetry group SO(3), rotations of the sphere. We see that the ground state is left invariant under a subgroup of the original symmetry transformations. The ground state manifold of this system, the set of all states that minimizes the effective potential, is then a sphere. When the system chooses one single ground state, this symmetry is broken, but only for two of the generators. The generator for rotations around the ground state leaves that point unchanged and is thus an unbroken symmetry. Any excitations in the direction of the broken symmetries do not cost energy, as it is in the ground state manifold. On the other hand, the unbroken symmetry does not correspond to an excitation. This is illustrated in Figure 3.3.

[3] The particles are bosons due to the bosonic nature of the transformations, g. If the generators are Grassmann numbers, the resulting particles, called goldstinos, are fermions.

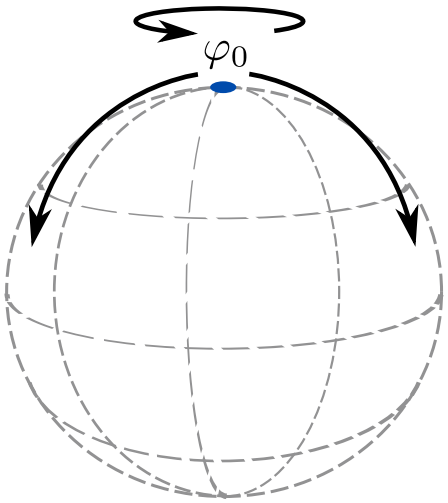


Figure 3.3: Excitations for the $N = 3$ sigma model. Two of the symmetries are broken, while rotations around the ground state leaves the system unchanged.

3.4 *CCWZ construction

As Goldstone bosons are massless, they play a crucial role in low-energy dynamics. To best describe this limit, we seek a parametrization of the theory in which they are the degrees of freedom. This can be done using the CCWZ construction, named after Callan, Coleman, Wess, and Zumino. This section is based on [7, 9–11], as well as the original papers [12, 13].

3.4.1 Parametrizing the Goldstone manifold

We saw that the Goldstone bosons correspond to excitations within the vacuum manifold. The vacuum manifold corresponds to points in field space φ that can be reached from the vacuum φ_0 with a transformation $g \in G$. Assume this group acts linearly on the fields. This means that we can write such excitations as

$$\varphi_i = (\tilde{\Sigma}\varphi_0)_i = \tilde{\Sigma}_{ij}(\varphi_0)_j, \quad \tilde{\Sigma} = \tilde{\Sigma}(\eta) = \exp\{i\eta_\alpha T_\alpha\} \tag{3.70}$$

We will drop the indices for the sake of compact notation. $\tilde{\Sigma}$ is thus a function from the parameter space, $\eta_\alpha \in \mathbb{R}^n$, to G,

$$\tilde{\Sigma} : \mathbb{R}^n \longmapsto G. \tag{3.71}$$

We then get field configurations by making the parameters dependent on space-time. We will for now assume η_α is constant. This parametrization is highly redundant. Two elements $\tilde{\Sigma}$ and $\tilde{\Sigma}'$, related by

$$\tilde{\Sigma}' = \tilde{\Sigma}e^{i\theta_a t_a} \tag{3.72}$$

results in the same φ. This is because $e^{i\theta_a t_a} = h \in H$, and $h\varphi_0 = \varphi_0$, by assumption. The set of all equivalent $\tilde{\Sigma}$'s is exactly the left coset, $gH = \{\, gh \mid h \in H \,\}$. The set of cosets forms a new manifold, G/H, called the Goldstone manifold. This is a manifold of dimension $\dim(G/H) = \dim(G) - \dim(H)$, which is the number of broken generators and thus also the number of Goldstone modes. Membership of a certain coset is an equivalence relation, $g \sim g'$ if $g' = gh$. This means that the cosets gH form a partition of G and that each element $g \in G$ belongs to one, and only one, coset. To remove the redundancy in the parametrization, we need to choose one representative element from each coset.

By the inverse function theorem, any mapping between manifolds $f : \mathcal{M} \mapsto \mathcal{N}$ that has a non-degenerate differential, that is an invertible Jacobian, at a point $p \in \mathcal{M}$, is invertible in a neighborhood of p. If we write

$$\tilde{\Sigma}(\xi, \theta) = \exp\{i\xi_i x_i\} \exp\{i\theta_a t_a\}, \tag{3.73}$$

then the map is invertible at $p = (\xi_i = 0, \theta_a = 0)$, as the Jacobian is the identity matrix. This point is mapped to the identity element of G. This means that, in a neighborhood $U \subset G$ of the identity, each element g has a unique representation $g = \tilde{\Sigma}$ [3]. Furthermore, two elements $\tilde{\Sigma}'$ and $\tilde{\Sigma}$ related by $\tilde{\Sigma}' = \tilde{\Sigma}h$, $h \in H$ have the same ξ-arguments. We see that ξ_i parametrize G/H, in the neighborhood of the identity. We therefore demand that $\tilde{\Sigma}$ always appear in the standard form

$$\Sigma(\xi) = \tilde{\Sigma}(\xi, 0) = \exp\{i\xi_i x_i\}. \tag{3.74}$$

The field $\varphi(x)$ can therefore be written as

$$\varphi(x) = \Sigma(x)\varphi_0 = \exp\{i\xi_i(x)x_i\}\,\varphi_0, \tag{3.75}$$

and $\xi_i(x)$ can be associated with the Goldstone bosons.

In the linear sigma model, the original O(N) symmetry is broken down to O($N-1$), which transforms the remaining $N-1$ fields with vanishing expectation values into each other. However, O(N) consists of two disconnected subsets, those matrices with determinant 1 and those with determinant -1. There is no continuous path that takes an element of O(N) with determinant of -1 to an element with determinant 1.[4] The set of symmetries that are connected to the identity is

$$G = SO(N) = \{\, M \in \mathrm{O}(N) \mid \det M = 1 \,\}. \tag{3.76}$$

If we choose $\varphi_0 = (0, 0, ..., v)$, then it is apparent that the ground state is invariant under the rotations of the $N-1$ first fields, so the unbroken symmetry is $H = \mathrm{SO}(N-1)$. The Goldstone manifold is $G/H = \mathrm{SO}(N)/\mathrm{SO}(N-1)$.

Consider the case of $N = 3$, which is illustrated in Figure 3.3. G is the rotations of the sphere, while H is rotations around φ_0, SO(2). The Goldstone manifold consists of the rotations of φ_0 to other points of the sphere, i.e. $G/H = \mathrm{SO}(3)/\mathrm{SO}(2) = S^2$, the 2-sphere. This is not a Lie group, as translating φ in a closed path around the sphere may result in a rotation around the z-axis. This is illustrated in Figure 3.4

To check that ξ_i, in fact, are the Goldstone modes, we study the way they appear in the Lagrangian. As they are massless, no mass term of the form $M_{ij}\xi_i\xi_j$ should appear. The original Lagrangian $\mathcal{L}[\varphi]$ was invariant under global transformations $\varphi(x) \mapsto g\varphi(x)$. However, any terms that only depend on $\varphi(x)$, and not its derivatives, will also be invariant under a *local* transformation, $\varphi(x) \mapsto g(x)\varphi(x)$. Our parametrization of the fields, $\varphi(x) = \Sigma(x)\varphi_0$ is exactly such a transformation, which means that any such terms are independent of the Goldstone bosons. We can therefore write

$$\mathcal{L}[\varphi] = \mathcal{L}_{\mathrm{kin}}[\varphi] + \mathcal{V}(\varphi_0), \tag{3.77}$$

where all terms in $\mathcal{L}_{\mathrm{kin}}$ are proportional to at least one derivative term, $\partial_\mu\varphi(x)$, while $\mathcal{V}$ is independent of $\Sigma(x)$. Inserting the parametrization into this derivative term, we get

$$\partial_\mu\varphi(x) = \partial_\mu[\Sigma(x)\varphi_0] = \Sigma(x)[\Sigma(x)^{-1}\partial_\mu\Sigma(x)]\varphi_0. \tag{3.78}$$

The Lagrangian will therefore depend on ξ_i via terms of the form $\Sigma(x)^{-1}\partial_\mu\Sigma(x)$, which is called the Mauer-Cartan form. This is a $\mathfrak{g}$-valued function, which means that it can be written as

$$i\Sigma(x)^{-1}\partial_\mu\Sigma(x) = d_\mu(x) + e_\mu(x), \tag{3.79}$$

$$d_\mu = ix_i d_{ij}(\xi)\partial_\mu\xi_j, \tag{3.80}$$

$$e_\mu = it_a e_{ai}(\xi)\partial_\mu\xi_i, \tag{3.81}$$

where d_{ij} and e_{ai} are as-of-yet unknown real valued functions of ξ [7, 14].

3.4.2 Transformation properties of Goldstone bosons

We can deduce how the Goldstone bosons transforms under G from how φ transforms. In general,

$$\varphi' = g\varphi = (g\Sigma(\xi))\varphi_0 = \Sigma(\xi')\varphi_0 \quad g \in G. \tag{3.82}$$

[4] A simple proof of this is the fact that the determinant is a continuous function, while any path $\det M(t)$ such that $\det M(1) = -1$, $\det M(0) = 1$ must make a discontinuous jump.

While $\Sigma(\xi')$ has the standard form by assumption,

$$\Sigma(\xi') = \exp\{i\xi'_i x_i\}\,, \tag{3.83}$$

$g\Sigma(\xi)$ does not, in general.

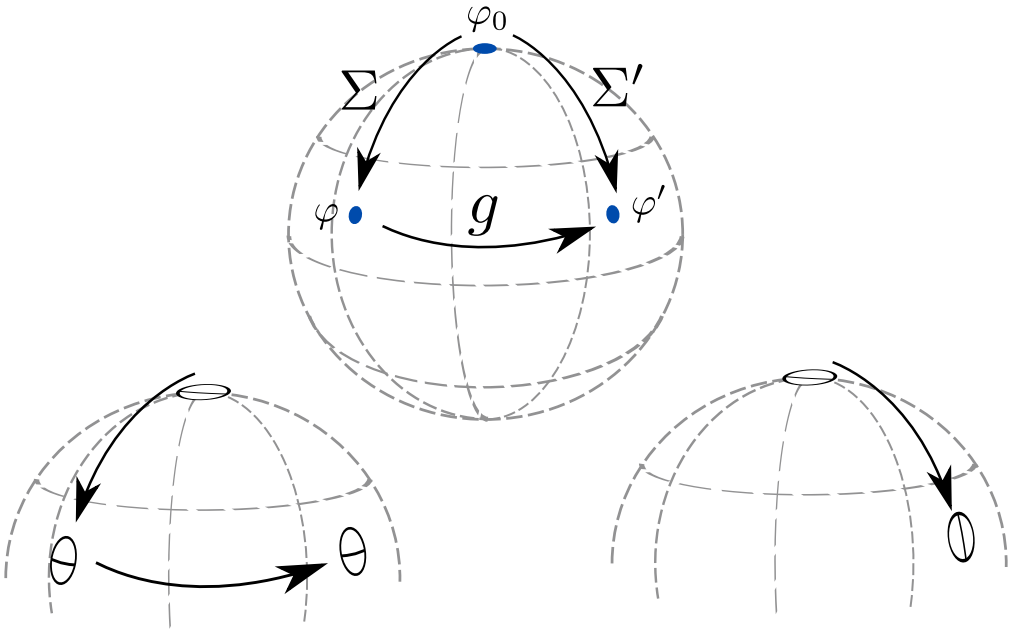


Figure 3.4: The top figure illustrates the transformation of φ_0 to φ and then φ', and the alternative, direct transformation $\varphi_0 \to \varphi'$. The bottom figure illustrates how this can rotate a neighborhood of φ_0 differently.

Figure 3.4 illustrates this in the case of $G = \mathrm{SO}(3)$. $\Sigma(\xi)$ transforms φ_0 to φ, then g transforms φ to $\varphi' = \Sigma(\xi')\varphi_0$. Assuming φ and φ' are close enough to φ_0, we can write $\Sigma(\xi)$ and $\Sigma(\xi')$ on the standard form. However, if we follow a small neighborhood around φ_0 as it is acted on by $\Sigma(\xi)$, then g, it will be rotated by the time it arrives at φ' when compared to the same neighborhood if it was acted on by $\Sigma(\xi')$.

$g\Sigma(\xi)$ and $\Sigma(\xi')$ are in the same coset, as they by assumption corresponds to the same physical state. This means that we can write $g\Sigma(\xi) = \Sigma(\xi')h(g,\xi)$, where $h(g,\xi) \in H$. The transformation rule of ξ under G is therefore implicitly defined by

$$\Sigma(\xi') = g\Sigma(\xi)[h(g,\xi)]^{-1}. \tag{3.84}$$

This is, in general, not a linear representation, which is why this construction also is called a *non-linear realization*. Using the transformation rule, we can obtain the transformation rule of the Maurer-Cartan form. We use the shorthand $\Sigma = \Sigma(\xi)$, $\Sigma' = \Sigma(\xi')$, and $h = h(g,\xi)$. This gives

$$\begin{aligned}
\Sigma^{-1}\partial_\mu\Sigma \to \Sigma'^{-1}\partial_\mu\Sigma' &\\
&= (g\Sigma h^{-1})^{-1}\partial_\mu(g\Sigma h^{-1}) \\
&= (h\Sigma^{-1}g^{-1})g[(\partial_\mu\Sigma)h^{-1} + \Sigma\partial_\mu h^{-1}] \\
&= h\Sigma^{-1}(\partial_\mu\Sigma)h^{-1} + h\partial_\mu h^{-1} \\
&= h(\Sigma^{-1}\partial_\mu\Sigma + \partial_\mu)h^{-1}.
\end{aligned}$$

In terms of d_μ and e_μ,

$$d_\mu \to h d_\mu h^{-1} \tag{3.85}$$

$$e_\mu \to h(e_\mu + i\partial_\mu)h^{-1}. \tag{3.86}$$

The transformation rule of e_μ is that of a gauge field, with the gauge group H. We will discuss gauge fields in more detail in chapter 5. If we include massive degrees of freedom and not only the Goldstone modes,

e_μ is used to create a covariant derivative of the massive modes. We are only interested in the Goldstone modes and will therefore be satisfied with d_μ. We have now greatly constrained the way the Goldstone modes may appear in the Lagrangian. However, this does not yet solve the problem of the strong force being non-perturbative. To do this, we need to create an effective field theory in which the strong force has been integrated out.

3.5 Constructing an effective field theory

One of the most powerful concepts in quantum field theory is the notion of effective field theories. The methods we have laid out for quantum field theory involve, in general, calculations where we must integrate over all possible momenta and thus all possible energies. However, we do not profess to know how physics behaves at arbitrarily high energies, which at first glance seem to render our theories moot. The fact that the standard model allows for such precise predictions suggests that the physics that happens at energies that are accessible to us can be described without knowledge of physics at the highest energies. This is a familiar concept from our everyday life—we can describe billiard balls colliding or rocks falling with high precision without having an accurate microscopic description of these objects. An effective field theory is a description of the physics of some underlying theory, some degrees of freedom φ_a governed by a Lagrangian $\mathcal{L}[\varphi]$, in terms of a smaller set of degrees of freedom, π_i, governed by an effective Lagrangian $\mathcal{L}_{\text{eff}}[\pi]$. As they describe the same physics, these two descriptions are related by

$$Z = \int \mathcal{D}\varphi \exp\left\{ i \int \mathrm{d}^4x \, \mathcal{L}[\varphi] \right\} = \int \mathcal{D}\pi \exp\left\{ i \int \mathrm{d}^4x \, \left(\mathcal{L}_{\text{eff}}[\pi]\right) \right\}. \tag{3.87}$$

We say that the degrees of freedom not present in the effective description have been *integrated out*. An effective theory can come from integrating out all degrees of freedom above some energy cut off or integrating out a particle and describing the interactions it mediates as point-like. In section 3.2, we found that the 1PI effective action resulted from integrating out all fluctuations away from the ground state, leaving us with an effective field theory in which all interactions are described entirely at the tree level. Furthermore, the modern understanding of the Standard Model is that it is an effective field theory [15].

Skrive om standarmodellen som effektiv teori?

One of the pioneers of the philosophy of effective field theories was Steven Weinberg. He proposed that quantum field theories, in themselves, have almost no content beyond some basic assumptions [16]. This means that if we try to model a system by writing down the most general possible Lagrangian, we cannot be wrong. This was formulated more precisely in—as Weinberg himself called it—a "theorem":

> "[I]f one writes down the most general possible Lagrangian, including all terms consistent with assumed symmetry principles, and then calculates matrix elements with this Lagrangian to any given order of perturbation theory, the result will simply be the most general possible S-matrix consistent with analyticity, perturbative unitary, cluster decomposition and the assumed symmetry principles." [17]

The properties of "analyticity, perturbative unitarity and cluster decomposition" are basic properties we expect of good, effective theories. Analyticity is an assumption about the poles of the S-matrix, and perturbative unitarity says that the theory should be unitary, as quantum theories should, for *any order in perturbation theory*. Cluster decomposition states that non-entangled processes far apart should be independent [6, 7]. Such a "most general possible Lagrangian" will have the form

$$\mathcal{L}_{\text{eff}}[\pi] = \sum_i \lambda_i \mathcal{O}_i, \tag{3.88}$$

where $\mathcal{O}_i$ are local functions of the effective fields and their derivatives, and λ_i are coupling constants. The coupling constants are free parameters, which parametrizes the most general S-matrix consistent with foundational assumptions and the underlying theory. A Lagrangian with an infinite amount of free parameters might seem useless. However, if we can find a consistent series expansion, then only a finite number of terms are needed to calculate quantities to any given order in perturbation theory. Furthermore, even though such a theory is called "non-renormalizable"—renormalizing an arbitrary order in perturbation theory requires an

arbitrary number of parameters—only a finite number of parameters are needed to renormalize any *given* order. Non-renormalizable theories can thus perfectly well be renormalized.

In the last section, we showed that the Goldstone modes will always appear in the Lagrangian as the terms of the Mauer-Cartan form, d_μ, and e_μ. Thus, the approach to creating an effective theory of Goldstone bosons, such as chiral perturbation theory, is to write down the most general Lagrangian, consistent with the underlying symmetries, made up of these terms. Then, using Weinberg's power counting scheme, as we discuss in subsection 5.2.3, we expand perturbatively in the Goldstone boson energies. This will give us a self-consistent description of the Goldstone bosons of the theory, the pseudoscalar mesons. The world is, of course, not only made up of pseudoscalar mesons. We also need to describe how these fields interact with other fields, or external sources. Furthermore, the global $\mathrm{SU}(N_f) \times \mathrm{SU}(N_f)$ symmetry of QCD with N_f quarks is only approximate, as we will explore further in chapter 5. We must extend the CCWZ construction to incorporate these effects, which we will do by introducing some new QFT tools in the next subsection.

3.5.1 Schwinger-Dyson equations and Ward identities

Given a system of fields φ_a governed by some action $S[\varphi]$, the expectation value of a functional of the fields, $F[\varphi]$, is given by

$$\langle F[\varphi] \rangle = \int \mathcal{D}\varphi \, e^{iS[\varphi]} F[\varphi]. \tag{3.89}$$

If we perform a *local* transformation of the field on the form $\varphi(x) \to \varphi(x) + \epsilon\eta(x)$, the integral measure will remain unchanged. The expectation value, to first order in ϵ, then changes as

$$\langle F \rangle \to \int \mathcal{D}\varphi \, e^{i(S+\epsilon\delta S)}(F + \epsilon\delta F) = \langle F \rangle + \epsilon \langle i(\delta_\eta S)F \rangle + \epsilon \langle \delta_\eta F \rangle . \tag{3.90}$$

Where the variation $\delta_\eta S = \int \mathrm{d}^n x \, \frac{\delta S[\varphi]}{\delta\varphi(x)}\eta(x)$, as defined in section A.2. As this amounts to a change of integration variable, the expectation value should remain unchanged. This gives us the important identity

$$\langle i(\delta_\eta S[\varphi])F[\varphi] \rangle + \langle \delta_\eta F[\varphi] \rangle = 0. \tag{3.91}$$

Inserting the integral form of the variation, and using the fact that η is independent of φ, we may write this identity as

$$\left\langle \frac{\delta S[\varphi]}{\delta\varphi(x)} F[\varphi] \right\rangle = i \left\langle \frac{\delta F[\varphi]}{\delta\varphi(x)} \right\rangle . \tag{3.92}$$

The Schwinger-Dyson equations are important special cases of this identity. They are the equations of motion of correlation functions. They thus incorporate the dynamics of a theory. We derive them by setting $F[\varphi] = \varphi(x_1)...\varphi(x_n)$. If we have a Lagrangian on the form $\mathcal{L} = -\frac{1}{2}\varphi(\partial^2 + m^2)\varphi - V[\varphi]$, then Eq. (3.92) becomes

$$(\partial_x^2 + m^2) \langle \varphi(x)\varphi(x_1)\dots\varphi(x_n) \rangle = - \langle \mathcal{V}'[\varphi](x)\varphi(x_1)\dots\varphi(x_n) \rangle - i \sum_i \delta(x - x_i) \left\langle \varphi(x_1)\dots\widehat{\varphi(x_i)}\dots\varphi(x_n) \right\rangle ,$$

where the hat denotes that the field is *omitted*. If $n = 1$ and $\mathcal{V} = 0$, we get the defining relation for the free Greens function,

$$(\partial_x^2 + m^2) \langle \varphi(x)\varphi(y) \rangle = -i\delta(x - y). \tag{3.93}$$

We may also consider slightly more general transformations of $\varphi(x)$, such as local phase-transformations $\varphi(x) \to e^{i\epsilon(x)}\varphi(x)$, as long as they do not affect the measure of the path integral. We will use this to derive identities related to the Schwinger-Dyson equations that incorporate the symmetries of a given theory. If $\varphi(x) \to \varphi(x) + \delta\varphi(x)$ is a global symmetry transformation, so that $\delta\mathcal{L} = \partial_\mu K^\mu$ and the integration measure is unchanged, then $\varphi(x) \to \varphi(x) + \eta(x)\delta\varphi(x)$ is a corresponding local transformation. We recover the global

transformation for $\eta = 1$. The variation of the action from this transformation will be

$$\begin{aligned}\delta S &= \int \mathrm{d}^4x \left(\frac{\partial \mathcal{L}}{\partial \varphi}\eta\delta\varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi)}\partial_\mu(\eta\delta\varphi)\right) \\ &= \int \mathrm{d}^4x \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi)}\delta\varphi\right)\partial_\mu\eta + \int \mathrm{d}^4x\,\eta(x)\left(\frac{\partial \mathcal{L}}{\partial\varphi}\delta\varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi)}\partial_\mu\delta\varphi\right) \\ &= -\int \mathrm{d}^4x\,\eta(x)\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi)}\delta\varphi - K^\mu\right)\end{aligned}$$

In the last line, we integrated by parts, and used $\delta\mathcal{L} = \partial_\mu K^\mu$. From subsection 3.3.1, we recognize the term within the parenthesis as precisely the Nöther current J^μ, so

$$\delta S = -\int \mathrm{d}^4x\,\eta(x)\partial_\mu J^\mu. \tag{3.94}$$

As φ is an integration variable in the path integral, it is not necessarily on-shell. We can therefore not use Nöther's theorem, $\partial_\mu J^\mu = 0$, as this relies on the equation of motion. However, for $F = 1$ and thus $\delta F = 0$, we can insert Eq. (3.94) into Eq. (3.89) to obtain the quantum version of the current conservation equation,

$$\partial_\mu \langle J^\mu \rangle = 0. \tag{3.95}$$

With $F = \varphi(x_1)\varphi(x_2)$, we get [4, 5]

$$\partial_{x,\mu}\langle J^\mu(x)\varphi(x_1)\varphi(x_2)\rangle = -i\delta(x-x_1)\langle\delta\varphi(x_1)\varphi(x_2)\rangle - i\delta(x-x_2)\langle\varphi(x_1)\delta\varphi(x_2)\rangle. \tag{3.96}$$

Identities of this form are called Ward-Takashi identities, often just Ward-identities, and encode the symmetries of a theory. In case symmetry is only approximate, so $\delta\mathcal{L} = \partial_\mu K^\mu + \Delta$, where Δ is some small symmetry breaking operator, or it is subject to an anomaly, so $\mathcal{D}\varphi \to \mathcal{D}\varphi(1+\Delta)$, then the conservation equation is modified to

$$\partial_\mu \langle J^\mu \rangle = \langle \Delta \rangle. \tag{3.97}$$

To create the generating functional, we must add external currents j. However, these new terms in the Lagrangian might break the invariance under a symmetry transformation $\varphi \to \varphi'$. If we transform the external currents as $j \to j'$ to counteract the transformation of the fields, then the theory should remain invariant. As before, we make both these transformations local, making sure that they leave the measure invariant. We can then perform a change of variable in the path integral, which relates generating functionals with different external currents, $Z[j] = Z[j']$. This relation must not only be obeyed by the underlying theory but also by any effective theory, which significantly constrains the form of the effective Lagrangian. As an illustration, we consider an example of spinor fields adapted from [18], as this is closely related to the construction of chiral perturbation theory. Spinors and gauge theory, which are relevant for this example, are discussed in more depth in chapter 5. Consider a massless spinor field with the Lagrangian,

$$\mathcal{L} = i\bar{\psi}\not{\partial}\psi - \mathcal{V}[\psi,\bar{\psi}]. \tag{3.98}$$

Assume this theory has a global $\mathrm{SU}(N)$ symmetry, so the $\mathcal{V}$ remains unchanged under the transformation $\psi \to U\psi$, $\bar{\psi} \to \bar{\psi}U^\dagger$. The system then has a corresponding conserved current,

$$J^\mu_\alpha = \bar{\psi}T_\alpha\gamma^\mu\psi, \tag{3.99}$$

where T_α are the generators of $\mathrm{SU}(N)$. We then include spinor sources $\eta = \eta_\alpha T_\alpha$ and vector sources $v_\mu = v^\alpha_\mu T_\alpha$ by adding the terms $\bar{\eta}\psi$, $\bar{\psi}\eta$, and $v^\alpha_\mu J^\mu_\alpha$ to the Lagrangian. Under a local $\mathrm{SU}(N)$ transformation, $\psi \to e^{i\theta_\alpha(x)T_\alpha}\psi$, the action changes as

$$S \to \int \mathrm{d}^4x\,\left[i\bar{\psi}\not{\partial}\psi - \mathcal{V}[\psi,\bar{\psi}] + \bar{\eta}U\psi + \bar{\psi}U^\dagger\eta + \bar{\psi}\gamma^\mu(U^\dagger v_\mu U + iU^\dagger\partial_\mu U)\psi\right]. \tag{3.100}$$

The last term corresponds to the change in action without sources, which we found earlier Eq. (3.94). We then define transformations of the external fields to counteract the transformation of ψ. As these transformations are local, the sources now act as gauge fields. The gauge transformation of the external sources are

$$\eta \to U\eta, \quad \bar{\eta} \to \bar{\eta}U^\dagger, \quad v_\mu \to U(v_\mu + i\partial_\mu)U^\dagger. \tag{3.101}$$

This gives the relation $S[\psi', \bar{\psi}', \eta', \bar{\eta}', v'] = S[\psi, \bar{\psi}, \eta, \bar{\eta}, v]$, where the mark indicates the gauge transformed field. As we argued in the subsection on the Dyson-Schwinger equations, we can change the integration variables inside the path integral without changing the result. Considering and infinitesimal transformation, and expanding to first order in θ, we get

$$\begin{aligned} 0 &= Z[\eta', \bar{\eta}', v'] - Z[\eta, \bar{\eta}, v] \\ &= i \int \mathrm{d}^4x \left\langle i\theta_\alpha(x)\bar{\psi}T_\alpha\eta - i\theta_\alpha(x)\bar{\eta}T_\alpha\psi + i\bar{\psi}\gamma^\mu(i\theta_\alpha(x)[T_\alpha, v_\mu] - i\partial_\mu\theta_\alpha(x)T_\alpha)\psi \right\rangle \end{aligned} \tag{3.102}$$

As the transformation, and thus θ_α, is arbitrary, the integrand must vanish. After integrating by parts, we are left with

$$\left\langle \bar{\psi}T_\alpha\eta - \bar{\eta}T_\alpha\psi + D^{\alpha\beta}_\mu J^\mu_\beta \right\rangle = 0. \tag{3.103}$$

Here, $D^{\alpha\beta}_\mu$ is the covariant derivative in the adjoint representation, $D^{\alpha\beta}_\mu J^\beta_\nu = (\delta_{\alpha\beta}\partial_\mu + iv^\gamma_\mu f^{\alpha\gamma\beta})J^\nu_\beta$, and $f^{\alpha\beta\gamma}$ are the structure constants of $\mathfrak{su}(N)$. We can get a more general expression by writing this using the generating functional W,

$$\left(\frac{\delta}{\delta\eta_\alpha(x)}\eta - \bar{\eta}\frac{\delta}{\delta\bar{\eta}_\alpha(x)} + D^{\alpha\beta}_\mu \frac{\delta}{\delta v^\beta_\mu(x)} \right) W[\eta, \bar{\eta}, v] = 0. \tag{3.104}$$

If we evaluate this at $\eta = \bar{\eta} = v_\mu = 0$, we get the quantum conservation equation $\partial_\mu \langle J^\mu_\alpha \rangle = 0$. From Eq. (3.104), we can also get more general ward identities by taking functional derivatives with respect to the external sources.

We have now seen how the Ward identities encode the global symmetries of the theory, and that they may be derived by transforming external source fields as gauge field to ensure the invariance of the action under the corresponding *local* transformations. This is the key insight behind the systematic development of chiral perturbation theory [19–21]. With the CCWZ construction, we can create the Lagrangian of Goldstone bosons alone, given only the symmetry breaking pattern $G \to G/H$. However, it does not tell us how external fields couples to them. With the constraint Eq. (3.102), however, we know that the new action must be invariant under *local* G transformations, given that we transform the external fields as gauge fields. When including external fields, the new effective Lagrangian $\mathcal{L}_{\text{eff}}$ must therefore not only be invariant under global G transformations but rather local G transformations. If we modify the Mauer-Cartan form Eq. (3.79) by introducing a covariant derivative,

$$i\Sigma(x)^{-1}\partial_\mu\Sigma(x) \to i\Sigma(x)^{-1}\nabla_\mu\Sigma(x), \tag{3.105}$$

then all terms that were globally G-invariant become locally so. This is because, as in the case of covariant derivative in subsection 2.1.3, the covariant derivative transforms as the object it acts on. In addition to new and modified terms due to the covariant derivative, we can now also combine the external currents and Σ into G-invariant terms. This will allow us to take into account approximate symmetries as well. By treating the symmetry-breaking parameter in the underlying Lagrangian, such as the mass of quarks in the case of chiral perturbation theory, as an external current, we can still apply this procedure. Such fields are called *spurion fields*. We will apply this theory to derive chiral perturbation theory in chapter 5.

Chapter 4

General relativity and the TOV equation

General relativity describes how matter and energy curve the fabric of space and time. Einstein first wrote down the theory more than a century ago, and it is still our most accurate theory of gravitational effects. It makes precise and counterintuitive predictions, which experiments have borne out. This chapter surveys the basics of general relativity. We will then use this to derive the Tolman-Oppenheimer-Volkoff (TOV) equation, a differential equation used to model stars. However, first we will shortly summarize the theory it succeeded.

4.1 Newtonian Gravity

Newton's famous law of gravity states that the force of gravity from an object of mass M acting on another object of mass m is proportional to both objects' masses and is inversely proportional to the distance between them squared. This force is directed radially inwards. Formulating this as an equation, with $\vec{r}$ as the vector from the object with mass M to that with mass m, gives

$$\vec{F}_g = -\frac{GMm}{r^2}\hat{r}. \tag{4.1}$$

Here, G is Newton's gravitation constant, and $\hat{r} = \vec{r}/|\vec{r}|$. The vector $\vec{r}$ is a purely spatial three-vector, as space is separated from time in the Newtonian picture. The law of gravitation, together with Newton's second law of motion,

$$\sum_i \vec{F}_i = m\vec{a}, \tag{4.2}$$

where $\vec{a}$ is the acceleration of an object and $\vec{F}_i$ the forces acting upon it, can account for the motion of stellar objects. These laws work well in all but the most extreme circumstances, involving very massive objects or very short distances. As we will see, such extreme circumstances can be quantified by $2GM \approx r$. Newtonian gravity works well as an approximation because G is small in everyday units. Highly precise measurements of the orbit of Mercury were needed before any deviation from it was noticed.

We restate Newtonian gravity in a field-theoretic language to better compare Newtonian gravity to its successor theory. The gravitational potential from a mass M, which gives rise to Newton's force law, obeys the equation

$$\nabla^2\Phi_g = -4\pi G\rho. \tag{4.3}$$

Here, ρ is the mass density. For a single point mass, $\rho(\vec{r}) = M\delta(\vec{r})$, this has the solution $\Phi_g = GM/r$. The acceleration due to gravity is then

$$\vec{a} = -\vec{\nabla}\Phi_g. \tag{4.4}$$

We see that mass acts as a gravitational charge. Due to the success of Newtonian gravity, we expect it to be a limit of any theory that succeeds it. This gives us the ability to theoretically test any new theory of gravity, as well as to connect parameters in the new theory to old, known quantities.

4.2 General relativity

This section is based on [2]. The derivation of the spherically symmetric metric is done using computer code, as described in Appendix D.

4.2.1 Einstein's field equations

General relativity describes spacetime as a smooth manifold $\mathcal{M}$, with a (pseudo-Riemannian) metric, $g_{\mu\nu}$. This metric is treated as a dynamical field, which is affected by the presence of matter and energy. The matter and energy contents of spacetime are encoded in the stress-energy tensor $T_{\mu\nu}$, while the behavior of $g^{\mu\nu}$ is encoded in a scalar Lagrangian density. We employ the minimal-coupling rule to reformulate laws from special relativity to curved spacetime. This rule states that laws written in an inertial frame in a coordinate-independent way remain true in curved spacetime. In an inertial frame, the Crisoffel-symbols vanish, so $\nabla_\mu = \partial_\mu$. We can thus write any laws containing partial derivatives in a coordinate impenitent way by exchanging them for covariant derivatives. To generalize Newton's second law Eq. (4.2), we must first make it relativistic by introducing a 4-force, $F^\mu = \frac{d}{d\tau}p^\mu$, where p^μ is the 4-momentum. Newton's second law then becomes, using Eq. (2.37),

$$\sum_i F_i^\mu = \frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda}. \tag{4.5}$$

In the absence of any external forces, objects will follow geodesics in spacetime. With this, we must now find the law governing $g^{\mu\nu}$. As this is a field, we will do this by assigning it a Lagrangian density. The most obvious—and correct—choice as the Lagrangian is the Ricci scalar, which results in the Einstein-Hilbert action,

$$S_{\text{EH}} = \frac{1}{2\kappa}\int_{\mathcal{M}} \mathrm{d}^n x\, \sqrt{|g|}\, R. \tag{4.6}$$

The $\sqrt{|g|}$-factor is included for the integral to be coordinate-independent, as discussed in subsection 2.1.4.[1] The κ is a constant and decides how strong the coupling of gravity to matter and energy is. This constant can then be related to Newton's constant of gravitation G by $\kappa = 8\pi G$. When including the contributions from other fields with an action S_{m}, the total action becomes

$$S = S_{\text{EH}} + S_{\text{m}}. \tag{4.7}$$

The equations of motion of the dynamical field, which in this case is the metric, are given by Hamilton's principle of stationary action. Using functional derivatives, as defined in subsection A.2.3, this is stated as

$$\frac{\delta S}{\delta g^{\mu\nu}} = 0. \tag{4.8}$$

We define the stress-energy tensor as

$$T_{\mu\nu} = -2\frac{\delta S_{\text{m}}}{\delta g^{\mu\nu}}. \tag{4.9}$$

The functional derivative of the Einstein-Hilbert action is evaluated in subsection A.2.4, and with the result, Eq. (A.58), we get the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu}, \tag{4.10}$$

[1] The gravitational action can also include a cosmological constant, modifying the Lagrangian to $R + 2\Lambda$. This constant does not affect the subject of this thesis and is therefore not included here.

The left-hand side of the Einstein field equations is called the Einstein tensor, $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$. This tensor obeys the identity

$$\nabla^\mu G_{\mu\nu} = 0, \tag{4.11}$$

as a consequence of the more general Bianchi identity, Eq. (2.44).

4.2.2 Spherically symmetric spacetime

To model stars, we will assume that the metric is spherically symmetric and time-independent. In this case, the most general metric can be written, at least locally, as [2]

$$\mathrm{d}s^2 = e^{2\alpha(r)}\mathrm{d}t^2 - e^{2\beta(r)}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2), \tag{4.12}$$

where α and β are general functions of the radial coordinate r. In matrix form, this corresponds to

$$g_{\mu\nu} = \begin{pmatrix} e^{2\alpha(r)} & 0 & 0 & 0 \\ 0 & -e^{2\beta(r)} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2(\theta) \end{pmatrix}. \tag{4.13}$$

Using Eq. (2.35), we can now compute the Christoffel symbols in terms of the unknown functions. These computations in this subsection are done using computer code, which is shown in Appendix D. The results are

$$\Gamma^t_{\mu\nu} = \begin{pmatrix} 0 & \frac{d}{dr}\alpha(r) & 0 & 0 \\ \frac{d}{dr}\alpha(r) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.14}$$

$$\Gamma^r_{\mu\nu} = \begin{pmatrix} e^{2\alpha(r)}e^{-2\beta(r)}\frac{d}{dr}\alpha(r) & 0 & 0 & 0 \\ 0 & \frac{d}{dr}\beta(r) & 0 & 0 \\ 0 & 0 & -re^{-2\beta(r)} & 0 \\ 0 & 0 & 0 & -re^{-2\beta(r)}\sin^2(\theta) \end{pmatrix}, \tag{4.15}$$

$$\Gamma^\theta_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\sin(\theta)\cos(\theta) \end{pmatrix}, \tag{4.16}$$

$$\Gamma^\phi_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \frac{\cos(\theta)}{\sin(\theta)} \\ 0 & \frac{1}{r} & \frac{\cos(\theta)}{\sin(\theta)} & 0 \end{pmatrix}. \tag{4.17}$$

The symbols not included are zero. Substituting these results into Eq. (2.39) gives the Riemann tensor curvature tensor. We can then obtain the Ricci tensor by taking the trace, as shown in Eq. (2.40). The results are

$$R_{tt} = \left[r\left(\frac{d}{dr}\alpha(r)\right)^2 - r\frac{d}{dr}\alpha(r)\frac{d}{dr}\beta(r) + r\frac{d^2}{dr^2}\alpha(r) + 2\frac{d}{dr}\alpha(r)\right]\frac{e^{2\alpha(r)}e^{-2\beta(r)}}{r}, \tag{4.18}$$

$$R_{rr} = -\frac{1}{r}\left[r\left(\frac{d}{dr}\alpha(r)\right)^2 - r\frac{d}{dr}\alpha(r)\frac{d}{dr}\beta(r) + r\frac{d^2}{dr^2}\alpha(r) - 2\frac{d}{dr}\beta(r)\right], \tag{4.19}$$

$$R_{\theta\theta} = -\left[r\frac{d}{dr}\alpha(r) - r\frac{d}{dr}\beta(r) - e^{2\beta(r)} + 1\right]e^{-2\beta(r)}, \tag{4.20}$$

$$R_{\varphi\varphi} = R_{\theta\theta}\sin^2(\theta). \tag{4.21}$$

All other components are zero. Finally, the trace of the Ricci tensor gives the Ricci scalar,

$$R = \frac{2e^{-2\beta(r)}}{r^2}\Bigg[r^2\left(\frac{d}{dr}\alpha(r)\right)^2 - r^2\frac{d}{dr}\alpha(r)\frac{d}{dr}\beta(r) + r^2\frac{d^2}{dr^2}\alpha(r) + 2r\frac{d}{dr}\alpha(r) - 2r\frac{d}{dr}\beta(r) - e^{2\beta(r)} + 1\Bigg]. \tag{4.22}$$

The unknown functions α and β are now determined by the matter and energy content of the universe, which is encoded in $T_{\mu\nu}$, through Einstein's field equation, Eq. (4.10).

4.2.3 The Schwarzschild metric

The simples case for a matter distribution in spacetime is $T_{\mu\nu} = 0$. Although this might only seem to be useful to model a non-empty universe, it can be combined with a central point particle and empty space elsewhere. In this case, the Einstein equations are simply $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0$. We can show that the trace of the Ricci tensor is zero by taking the trace of this equation, simplifying it to $R_{\mu\nu} = 0$. By combining Eq. (4.18) and Eq. (4.19), we find

$$R_{tt} + e^{2(\alpha-\beta)}R_{rr} = 2\frac{d}{dr}(\alpha+\beta) = 0, \tag{4.23}$$

which implies $\alpha = -\beta + \text{const}$. The constant corresponds to rescaling the coordinates, which allows us to set it to zero. From Eq. (4.20), we get

$$e^{2\beta}R_{\theta\theta} = -2r\frac{d}{dr}\alpha - e^{-2\alpha} + 1 = 0, \tag{4.24}$$

which may be restated as

$$\frac{d}{dr}\left(re^{2\alpha}\right) = 1, \tag{4.25}$$

This equation has the solution

$$e^{2\alpha(r)} = e^{-2\beta(r)} = \left(1 - \frac{R_s}{r}\right), \tag{4.26}$$

where R_s, the Schwarzschild radius, is a constant. With this, we should obtain Newtonian law of gravity with a small perturbation, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, and at slow speeds, $\frac{d}{d\tau}x^i \ll \frac{d}{d\tau}t$. Inserting this into the geodesic equation Eq. (4.5) with $F_i = 0$, using $\partial_0 g_{\mu\nu} = 0$ and expanding to leading order, we get

$$\frac{d^2x^i}{dt^2} = \frac{1}{2}\eta^{ij}\partial_j h_{00}. \tag{4.27}$$

We now obtain Newtonian gravity, as formulated in Eq. (4.4), if we identify $h_{00} = 1 - e^{2\alpha} = -2\Phi_g$. The Schwarzschild radius is thus $R_s = 2GM$, and we see that this solution corresponds to a point-mass M located at $\vec{r} = 0$. We will extend our discussion to finite densities in the next section. Inserting or results into the metric, we get the Schwarzschild metric

$$\mathrm{d}s^2 = \left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right). \tag{4.28}$$

4.3 The TOV equation

This section is in part based on [2, 22, 23]. We will model a star as being made up of a *perfect fulid*. A fluid has a velocity field, v_μ, and perfect fluids have zero viscosity and thus no sheer stress. Therefore, in the rest frame of the fluid, the stress-energy tensor will be diagonal. A perfect fluid is furthermore *isotropic* in its rest frame. The stress-energy tensor thus takes the form $\mathrm{diag}(u, -p, -p, -p)$, where u is the energy density and p the pressure. We can then construct the stress-energy tensor in an arbitrary frame either by a Lorentz

boost or by constructing a covariant tensor expression which gives us the correct equation in the rest frame. The result is

$$T_{\mu\nu} = (u + p)v_\mu v_\nu - p g_{\mu\nu}, \tag{4.29}$$

We now want an explicit expression for this tensor in our spherically symmetric spacetime. In the rest frame of the fluid, we may write

$$v_\mu = (v_0, 0, 0, 0)\,. \tag{4.30}$$

This, together with the normalization condition of 4-velocities, $v_\mu v^\mu = 1$, allows us to calculate that

$$v_\mu v^\mu = g^{\mu\nu} v_\mu v_\nu = g^{00}(v_0)^2 = 1. \tag{4.31}$$

Using Eq. (4.13), we see that

$$v_0 = e^{\alpha(r)}. \tag{4.32}$$

This gives us the stress-energy tensor of the perfect fluid in its rest frame of our spacetime,

$$T_{\mu\nu} = \begin{pmatrix} u(r)e^{2\alpha(r)} & 0 & 0 & 0 \\ 0 & p(r)e^{2\beta(r)} & 0 & 0 \\ 0 & 0 & p(r)r^2 & 0 \\ 0 & 0 & 0 & p(r)r^2\sin^2(\theta) \end{pmatrix}. \tag{4.33}$$

With the stress-energy tensor, we may now input it into the Einstein field equation. We will use the tt and rr components of the equations,

$$8\pi G r^2 u(r) e^{2\beta(r)} = 2r\frac{d}{dr}\beta(r) + e^{2\beta(r)} - 1, \tag{4.34}$$

$$8\pi G r^2 p(r) e^{2\beta(r)} = 2r\frac{d}{dr}\alpha(r) - e^{2\beta(r)} + 1. \tag{4.35}$$

In analogy with the Schwarzschild metric, we define the function $m(r)$ by

$$e^{2\beta(r)} = \left(1 - \frac{2Gm(r)}{r}\right)^{-1}. \tag{4.36}$$

Substituting this into Eq. (4.34) yields

$$\frac{dm(r)}{dr} = 4\pi r^2\, u(r). \tag{4.37}$$

The solution is simply

$$m(r) = 4\pi \int_0^r \mathrm{d}r'\, r'^2 u(r'). \tag{4.38}$$

In flat spacetime, we would have no qualms simply calling this the mass contained within a radius r. However, as discussed in section 2.1, the volume element of a curved geometry is $\mathrm{d}V = \mathrm{d}^n x\sqrt{|g|}$. In this case, we are interested in the mass contained in a 3-volume, and the volume form is therefore given by the metric restricted by $\mathrm{d}t = 0$. Using Eq. (4.13), $\sqrt{|g|} = e^\beta r^2 \sin\theta$, the total mass-energy contents of the star is

$$M' = 4\pi \int \mathrm{d}r'\, r'^2 \left(1 - \frac{2Gm(r')}{r'}\right)^{-1/2} u(r'). \tag{4.39}$$

However, this does not take into account the gravitational potential energy. Gravitation is self-interacting, and we must therefore include the gravitational potential energy when calculating gravitational effects. It can be shown that the definition of *gravitational mass*, Eq. (4.38), does exactly this. Furthermore, as we will see later, it matches up with what we call mass in the Newtonian limit [24].

Using the Bianchi identity, Eq. (4.11), together with Einstein's equation, we find

$$\nabla^\mu G_{\mu\nu} = \nabla^\mu T_{\mu\nu} = 0. \tag{4.40}$$

The r-component of this equation is

$$\begin{aligned}\nabla_\mu T^{\mu r} &= \partial_r T^{rr} + \Gamma^\mu_{\mu\nu} T^{\nu r} + \Gamma^r_{\mu\nu} T^{\mu\nu} \\ &= \partial_r \left(p e^{-2\beta}\right) + (2\Gamma^r_{rr} + \Gamma^t_{tr}) T^{rr} + \Gamma^r_{tt} T^{tt} \\ &= e^{-2\beta} \left(\partial_r p + p \partial_r \alpha + u \partial_r \alpha\right) = 0.\end{aligned}$$

This allows us to relate α to p and u, via

$$\frac{d\alpha}{dr} = -\frac{1}{p+u}\frac{dp}{dr} \tag{4.41}$$

When we substitute this, together with the definition of $m(r)$, into Eq. (4.35), we obtain

$$\frac{dp}{dr} = -\frac{G}{r^2}\left(4\pi r^3 p + m\right)(p+u)\left(1 - \frac{2Gm}{r}\right)^{-1}, \tag{4.42}$$

the Tolman-Oppenheimer-Volkoff (TOV) equation. This equation was first obtained by Oppenheimer and Volkoff in 1939 [25] and was based on earlier work by Tolman [26]. In their paper, Oppenheimer and Volkoff studied the properties of a star made up of cold, degenerate fermions.

With $p(r)$, $u(r)$, and $m(r)$, we can construct the metric. We already have the rr-component of the metric from Eq. (4.36). If we combine Eq. (4.41), with Eq. (4.42), we get the solution

$$\alpha(r) = G\int^r \mathrm{d}r\, \frac{1}{r^2}(4\pi r^3 p + m)\left(1 - \frac{2Gm}{r^2}\right)^{-1}. \tag{4.43}$$

Outside the star, we have $p(r) = 0$, and $m(r) = M$. This then reduces to

$$\alpha(r) = GM\int^r \mathrm{d}r\, \frac{1}{r^2}\left(1 - \frac{2GM}{r}\right)^{-1}. \tag{4.44}$$

We can evaluate this integral by making the substitution $x = (1 - 2GM/r)$, $\mathrm{d}x = -2GM/r^2\mathrm{d}r$,

$$\alpha(r) = \frac{1}{2}\int^{1-\frac{2GM}{r}} \frac{\mathrm{d}x}{x} = \frac{1}{2}\ln\left(1 - \frac{2GM}{r}\right) + \text{const.} \tag{4.45}$$

We then impose the boundary condition $\alpha(\infty) = 0$, which means setting the constant to zero. Inserting this into Eq. (4.13) gives the metric for $r < R$,

$$\mathrm{d}s^2 = \left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\mathrm{d}\varphi^2\right). \tag{4.46}$$

We recognize this as the Schwarzchild metric, Eq. (4.28). This justifies our choice of $m(r)$ as gravitational mass. As discussed earlier, the quantity M in the Schwarzschild solution maps corresponds to Newtonian gravitational mass in the weak-field limit.

In addition to the dynamics of spacetime, we to know the thermodynamic properties of the matter that make up our star. The relationship between the pressure and energy density of a substance is called the *equation of state*, or EOS, and has the form

$$f(p, u, \{\xi_i\}) = 0, \tag{4.47}$$

where $\{\xi_i\}$ are possible other thermodynamic variables. This allows us to, at least locally, express the energy density as a function of the pressure, $u = u(p, \{\xi_i\})$, which is what we in this text will call the "equation of state". Our fluid is by assumption in thermodynamic equilibrium and is therefore bound by the first law of thermodynamics,

$$\mathrm{d}U = T\mathrm{d}S - p\mathrm{d}V, \tag{4.48}$$

where $U = uV$ is the internal energy, $T = 1/\beta$ the temperature, S the entropy, and V the volume. Given a conserved charge $Q = nV$, which in our case will be particle number and thus n particle number density. We will in be concerned with stars at $T = 0$. Inserting this into Eq. (4.48), we get

$$\mathrm{d}\left(\frac{u}{n}\right) = -p\mathrm{d}\left(\frac{1}{n}\right). \tag{4.49}$$

To summarize, we have three unknown functions, $u(r)$, $p(r)$, and $m(r)$. The equation of state, Eq. (4.47), determines $u = u(p)$, eliminating one unknown. The two differential equations Eq. (4.38) and Eq. (4.42), together with the boundary conditions $p(0) = p_c$ and $m(0) = 0$, then yield $p(r)$ and $m(r)$ when integrated. As long as both the pressure and the energy density are positive, and we always are outside the Schwarzschild radius, i.e., $r < 2Gm(r)$, then $dp/dr \leq 0$ and the pressure is strictly decreasing. We define the point where the pressure vanishes as the stellar radius R, i.e., $p(R) = 0$. Given this, we can solve for all the unknown functions, either analytically or numerically.

We can gain some insight into the system without solving these equations by expressing the problem in terms of dimensionless variables. We define

$$u = u_0\tilde{u}, \quad p = p_0\tilde{p}, \quad m = m_0\tilde{m}, \quad r = r_0\tilde{r}. \tag{4.50}$$

Here, quantities with subscript 0 are dimensionful constants, which may be chosen as the characteristic quantities of the problem, while the tilde indicates a dimensionless variable. By substituting this into Eq. (4.37) and Eq. (4.42), we can collect the dimensionful constants into a smaller number of dimensionless constants, k_i. These constants will decide the nature of the solution. Any change in the dimensionful constants that leave the k_i's invariant is a scaling of the problem — it corresponds to the same solution with different units. The new differential equations are

$$\frac{d\tilde{m}}{d\tilde{r}} = 3k_2\,\tilde{r}^2\tilde{u} \tag{4.51}$$

$$\frac{d\tilde{p}}{d\tilde{r}} = -\frac{k_1}{k_3}\frac{1}{\tilde{r}^2}\left(k_3\tilde{p} + \tilde{u}\right)\left(3k_2k_3\tilde{r}^3\tilde{p} + \tilde{m}\right)\left(1 - \frac{2k_1\tilde{m}}{\tilde{r}}\right)^{-1}, \tag{4.52}$$

where the dimensionless constants are defined as

$$k_1 = G\frac{m_0}{r_0}, \quad k_2 = \frac{4\pi}{3}\frac{r_0^3u_0}{m_0}, \quad k_3 = \frac{p_0}{u_0}. \tag{4.53}$$

The energy density and pressure are of comparable magnitude in the relativistic regime. We will therefore often choose $k_3 = 1$, defining $p_0 = u_0$. If we have a non-complete set of characteristic quantities, the dimensionless constants k_i tell us something about the magnitude we should expect the solution to have. After defining the remaining dimensionful constants by setting $k_i = 1$, we expect that the dimensionless sizes of a typical solution will be of order 1. In other words, the dimensionful constants defined by $k_i = 1$ are new, characteristic quantities given to us by the form of the governing equation only.

4.3.1 Newtonian limit and polytropes

In the Newtonian limit, the rest energy, i.e., mass, gives the dominant contribution to the gravitational field, while the contribution from pressure is negligible. In other words, the characteristic pressure, p_0, is far smaller than the characteristic energy density u_0, and we can use the approximation $k_3 \ll 1$. Furthermore, the star's radius should be much larger than the Schwarzschild radius, $R_s = 2GM$. If we choose $r_0 = R$, then $k_1 \ll 1$. In this limit, the lowest-order contribution to the TOV equation is

$$\frac{d\tilde{p}}{d\tilde{r}} = -\frac{k}{\tilde{r}^2}\tilde{u}\tilde{m}, \quad k = \frac{k_1}{k_3} = G\frac{u_0m_0}{p_0r_0}. \tag{4.54}$$

Using the mass equation Eq. (4.51), we can write this as

$$4\pi\tilde{r}^2\frac{d\tilde{p}}{d\tilde{m}} = -k'\frac{\tilde{m}}{\tilde{r}^2}, \quad k' = \frac{4\pi}{3}\frac{k_1}{k_2k_3} = G\frac{m_0^2}{r_0^4p_0}. \tag{4.55}$$

It is illuminating to derive this equation directly from Newtonian gravity. Assume we have a static, gravitationally bound ball of matter, as illustrated in Figure 4.1. The force due to the pressure gradient over a thin, spherical shell, $F_p = 4\pi r^2\mathrm{d}p$, must be counteracted by the gravitational force on the same shell, $F_g = -Gm\mathrm{d}m/r^2$. Setting $F_g = F_p$, we obtain Eq. (4.55)

Both the Newtonian limit and the TOV equation are equations of *hydrostatic equilibrium*, where the forces on a small volume of the fluid cancel out. In the case of the TOV equation, we tacitly assumed hydrostatic

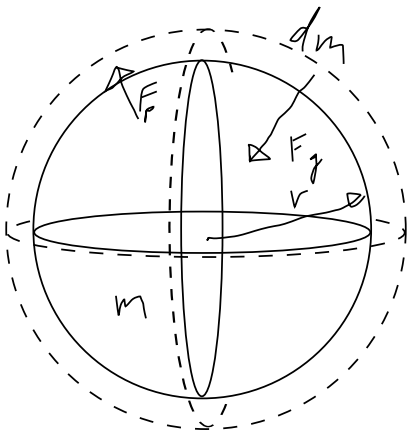


Figure 4.1: Kladd: The forces acting on a thin shell dm.

equilibrium when we gave the fluid a rest frame where we could write $v_\mu = (v_0, 0, 0, 0)$ globally. We can eliminate the equation for mass by differentiating Eq. (4.54) with respect to $\tilde{r}$. This gives us a single equation for hydrostatic equilibrium in the Newtonian limit,

$$\frac{d}{d\tilde{r}}\left(\frac{\tilde{r}^2}{\tilde{u}}\frac{d\tilde{p}}{d\tilde{r}}\right) = -k''\tilde{r}^2\tilde{u}, \quad k'' = 3\frac{k_2 k_1}{k_3} = 4\pi G\frac{u_0^2 r_0^2}{p_0}. \tag{4.56}$$

This is a second order differential equation, so we need an new boundary condition in addition to $p(0) = p_c$. Close to the center, we can see from Eq. (4.54) that for a finite energy density, we must have $p'(0) = 0$, our second boundary condition.

One important model for stars is the *polytrope*, which has an equation of state of the form $u = Kp^\gamma$ for some constant K. As we will see, this fits well as the Newtonian limit of many equations of state and can be used to make predictions such as the Chandrasekhar limit, which sets the upper limit of the mass of white dwarf stars to $M \approx 1.5\,M_\odot$ [23, 27]. To write Eq. (4.56) on the standard form, we assume $\gamma \neq 1$ and introduce

$$\tilde{u} = a\theta^n, \quad n = \frac{1}{\gamma - 1}, \quad a = \frac{u(p_c)}{u_0}, \quad a^{\frac{\gamma-2}{2}}C\xi = r, \quad C = \sqrt{\frac{K}{k''}\frac{\gamma}{\gamma-1}}. \tag{4.57}$$

n is called the *polytropic index* of the star. Inserting these new variables into the equation of hydrostatic equilibrium gives

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) + \theta^n = 0. \tag{4.58}$$

This is called the Lane-Emden equation and was first used to model stars as early as 1870 [28]. The boundary conditions $p(0) = p_c$ and $p'(0) = 0$ now read $\theta(0) = 1$ and $\theta'(0) = 0$. The stellar radius is defined by the first zero of the Lane-Emden function above $\xi = 0$, $\theta(\xi_1) = 0$, so that

$$R = C\xi_1 a^{\frac{\gamma-2}{2}}. \tag{4.59}$$

The total mass of the star can be integrated using Eq. (4.51) and Eq. (4.58),

$$M = \frac{3k_2C^3}{r_0^3}\,a^{\frac{3\gamma-4}{2}}\int_0^{\xi_1}\mathrm{d}\xi\,\xi^2\theta^n = -\frac{3k_2C^3}{r_0^3}\,\xi_1^2\theta'(\xi_1)\,a^{\frac{3\gamma-4}{2}}. \tag{4.60}$$

Thus, given a specific equation of state, and thus γ, the mass-radius relationship is given by

$$R \propto M^\beta, \quad \beta = \frac{\gamma - 2}{3\gamma - 4}. \tag{4.61}$$

Figure 4.2 illustrates this relationship, in arbitrary units, for a series of different values of γ, as well as the dependence of β on γ. For most values of γ, the stellar radius will increase as the mass increases. The only

range within which the radius decrease as the mass increase is $\gamma \in \left(\frac{4}{3}, 2\right)$. At $\gamma = \frac{4}{3}$ and $\gamma = 2$, respectively the mass and radius become independent of the central density. If included in our figure, these polytropes would correspond to a horizontal and a vertical line.

The case of $\gamma = \frac{4}{3}$ was used by Chandrasekhar to find his eponymous upper mass limit for white dwarfs. In white dwarfs, most of the energy density is due to the rest mass of nucleons, while the pressure is provided by electron degeneracy. In the ultrarelativistic limit, where the Fermi energy of the electrons is much larger than their mass, this results in a polytrope with $\gamma = \frac{4}{3}$. As we will see, a pion condensate has a non-relativistic limit with $\gamma = 2$. In chapter 7, we will use this to derive an upper limit for the radius of pion stars.

In the case of $\gamma = 1$, we can integrate Eq. (4.48), and obtain

$$u \propto n^{K+1}, \tag{4.62}$$

The non-relativistic limit of massive matter, on the other hand, is $u = mn$, where m is the rest mass of the particles. $\gamma = 1$ is thus not a realistic model of low energy matter. Rather, it corresponds to radiation, in which case $p = \frac{1}{3}u$, or the ultrarelativistic limit of matter, as we will find later in this text.

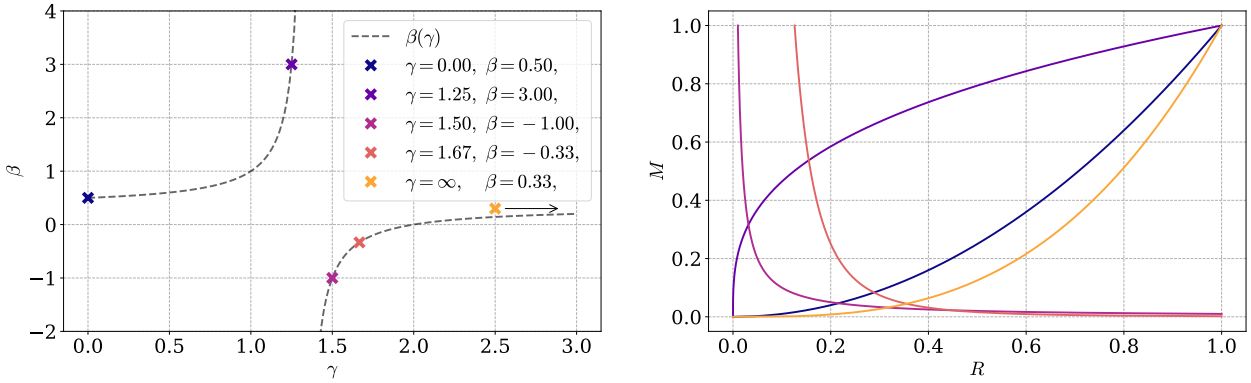


Figure 4.2: Left: The dependence of β on γ, together with a selection of points. Right: The mass-radius relation, in arbitrary units, for polytropes corresponding to the selected points on the left side. The color of the lines indicates to which point it corresponds.

4.3.2 Incompressible fluid

The simplest model for a star is one made up of an incompressible fluid, where the energy density is independent of the pressure. This corresponds to a polytrope with $\gamma = \infty$. In this case, the energy density of the star will be constant for a radius $r < R$, before it drops to zero,

$$u(r) = u_0\,\theta(R - r), \tag{4.63}$$

where u_0 is a constant and $\theta(x)$ the Heaviside step function. We choose $r_0 = R$. Inserting this into the differential equation of the mass function, Eq. (4.51), together with the boundary condition $m(0) = 0$, yields

$$\tilde{m}(\tilde{r}) = k_2\tilde{r}^3, \tag{4.64}$$

when $r < R$. For $r \geq R$, or $\tilde{r} \geq 1$, this relationship is simply constant $\tilde{m}(\tilde{r}) = \tilde{m}(1) = k_2$. We choose m_0 to be the gravitational mass of the star, $M = \frac{4\pi}{3}R^3u_0$, which is equivalent to setting $k_2 = 1$. Lastly, we choose $u_0 = p_0$, so that $k_3 = 1$. With this the TOV equation, Eq. (4.52), becomes

$$\frac{d\tilde{p}}{d\tilde{r}} = -k_1\tilde{r}\frac{(1+\tilde{p})(1+3\tilde{p})}{(1-2k_1\tilde{r}^2)}. \tag{4.65}$$

This is a separable ODE, and each variable may be integrated separately. Using

$$\int \frac{\mathrm{d}x}{(1+x)(1+3x)} = \frac{1}{2}\ln\frac{3x+1}{x+1} + \text{const.}, \quad \int \mathrm{d}x\frac{x}{1-2x^2} = \frac{1}{4}\ln\left(1-2x^2\right) + \text{const.}, \tag{4.66}$$

together with the boundary condition $p(r = R) = 0$, we get

$$\tilde{p}(\tilde{r}) = -\frac{\sqrt{1-2k_1} - \sqrt{1-2k_1\tilde{r}^2}}{3\sqrt{1-2k_1} - \sqrt{1-2k_1\tilde{r}^2}}. \tag{4.67}$$

We see that the star is entirely characterized by k_1. In Figure 4.3, we have plotted the pressure as a function of radius for some values of k_1. As k_1 approaches $0.\bar{4} = 4/9$, the pressure at the center of the star increases rapidly. From the denominator of Eq. (4.67) at $r \to 0$, we find the limit

$$k_1 = G\frac{M}{R} < \frac{4}{9} \tag{4.68}$$

for the pressure to remain finite. This is an absolute limit of the mass of an object given its radius or vice versa. Although this limit is derived for a particular, unrealistic case, the more general statement can be shown to hold. General relativity does not allow for a static solution with energy densities greater than this limit; any such configuration would collapse [2].

If we expand the solution Eq. (4.67) in powers of k_1, then the leading order contribution is

$$\tilde{p}(r) = \frac{1}{2}k_1(1-\tilde{r}^2). \tag{4.69}$$

This is the Newtonian limit. As a cross-check, we see that this solution obeys the equation of hydrostatic equillibrium in this limit, Eq. (4.54), as $\tilde{u} = 1$ and $k_2 = k_1 = 1$. This is the general solution for an incompressible fluid in Newtonian gravity. This solution does not have any upper limit for k_1; the limit $M/R < 4/9$ is a purely relativistic phenomenon. In Figure 4.4, the Newtonian approximation is compared to the full, relativistic solution. We see that the Newtonian approximation is highly accurate for k_1 less than around 0.01.

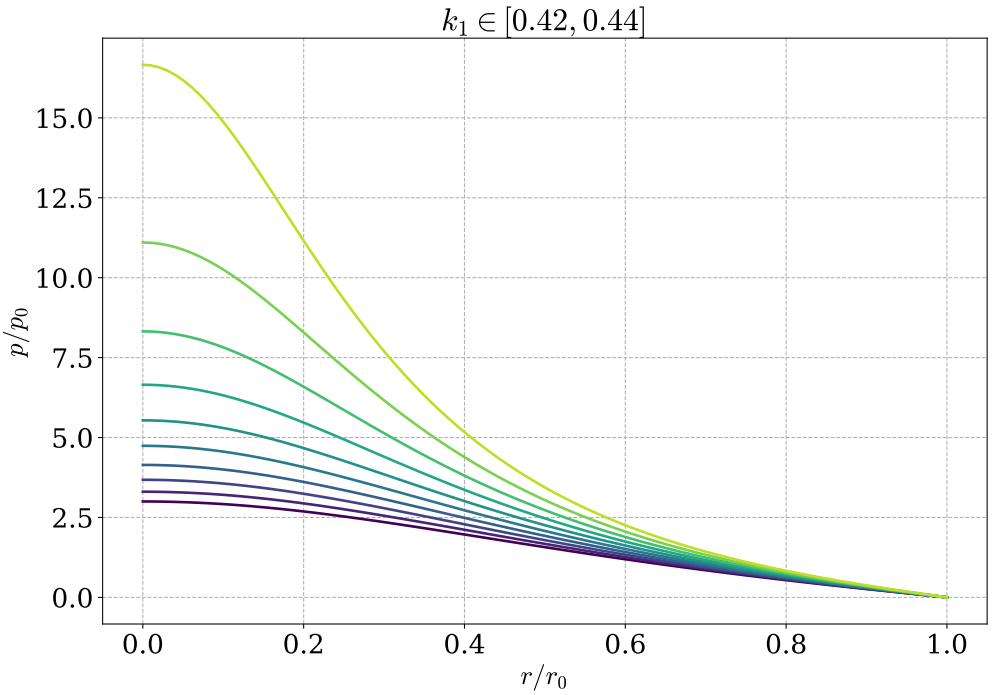


Figure 4.3: The pressure in units of p_0, as a function of the radius, in units of r_0. The graphs with lighter color and higher pressure at $r = 0$ corresponds to higher values of k_1. The values of k_1 are linearly spaced.

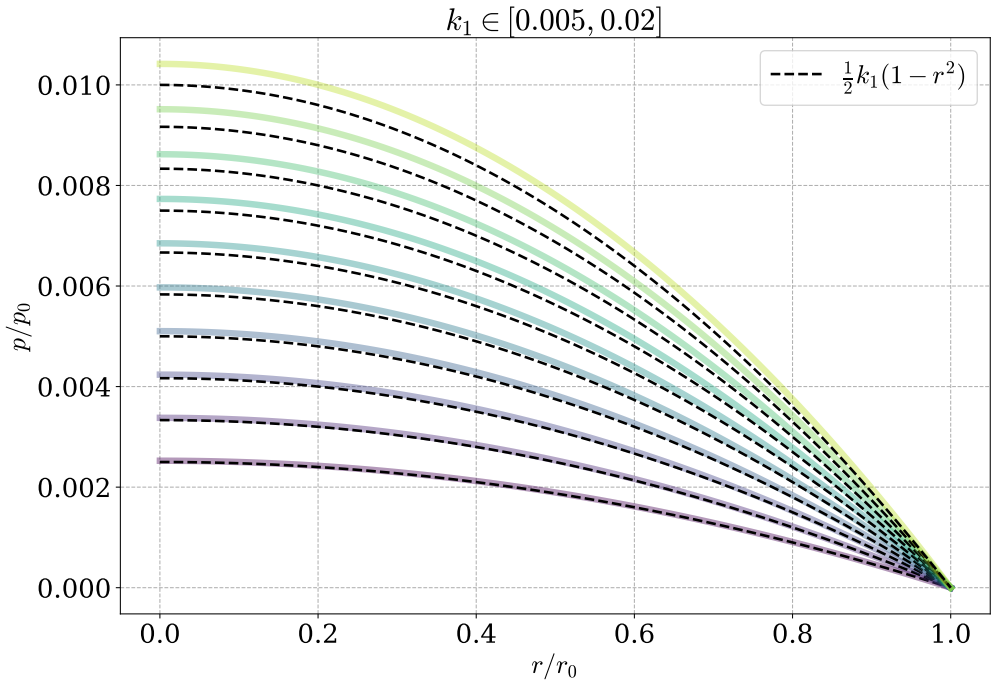


Figure 4.4: The pressure in units of p_0, as a function of the radius, in units of r_0. The wide, colored lines correspond to the full relativistic solution, while the dashed lines is the Newtonian approximation, for the same value of k_1. The values of k_1 are linearly spaced.

4.4 A star of cold, non-interacting fermions

This section is based on [23, 29, 30].

In this section, we will study a simple model of a star made up of non-interacting, cold neutrons. This is one of the earliest models used to study neutron stars, the remenants of massive stars [23]. For this model, we use results derived in section C.5.

4.4.1 Thermodynamics and the equation of state

The total energy U is related to the grand canonical free energy F by a Legendre transformation,

$$F(T, V, \mu) = U - TS - \mu Q, \quad \mathrm{d}F = -S\mathrm{d}T - p\mathrm{d}V - Q\mathrm{d}\mu. \tag{4.70}$$

Here $T = 1/\beta$ is temperature, and S entropy, p pressure, and V volume. Q is some conserved charge, in our case the number of particles minus antiparticles, and μ is the corresponding chemical potential. These thermodynamic variables are related to the free energy by

$$S = -\frac{\partial F}{\partial T} = \beta^2 \frac{\partial F}{\partial \beta}, \quad Q = -\frac{\partial F}{\partial \mu}, \quad p = -\frac{\partial F}{\partial V}. \tag{4.71}$$

When the free energy can be written as $F = V\mathcal{F}$, where the free energy density $\mathcal{F}$ is independent of the volume V, then $\mathcal{F} = -p$ and

$$\mathrm{d}(V\mathcal{F}) = V\mathrm{d}\mathcal{F} - p\mathrm{d}V, \tag{4.72}$$

allowing us to write

$$\mathcal{F}(T, \mu) = u - Ts - \mu n, \quad \mathrm{d}\mathcal{F} = -s\mathrm{d}T - n\mathrm{d}\mu, \tag{4.73}$$

where s and n are entropy and charge density, defined by

$$s = -\frac{\partial \mathcal{F}}{\partial T} = \beta^2 \frac{\partial \mathcal{F}}{\partial \beta}, \quad n = -\frac{\partial \mathcal{F}}{\partial \mu}. \tag{4.74}$$

With this, we can write the energy density as [29]

$$u = \frac{\partial}{\partial\beta}\left(\beta\mathcal{F}\right) + \mu n. \tag{4.75}$$

We calculate the free energy density of non-interacting fermions in section C.5, with the result Eq. (C.91),

$$\mathcal{F} = -\frac{2}{\beta}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\left[\beta\omega + \ln\left(1+e^{-\beta(\omega-\mu)}\right) + \ln\left(1+e^{-\beta(\omega+\mu)}\right)\right], \tag{4.76}$$

where $\omega = \sqrt{p^2+m^2}$. The first term in the integral is the divergent vacuum energy, which must be renormalized. We can drop this term; it does not have any observable effects on our results, as we are interested in relative pressure and energy density. With this, we find the charge density

$$n = \frac{1}{\pi^2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}[n_f(\omega-\mu) - n_f(\omega+\mu)], \tag{4.77}$$

where

$$n_f(\omega) = \frac{1}{e^{\beta\omega}+1}. \tag{4.78}$$

is the Fermi-Dirac distribution. Using this, we find that the energy density is

$$u = \frac{1}{\pi^2}\int_0^\infty \mathrm{d}p\, p^2\,\omega\,[n_f(\omega-\mu) + n_f(\omega+\mu)]. \tag{4.79}$$

As expected, this is the energy per mode times the density of states, integrated over all modes. To write the pressure, $p = -\mathcal{F}$ in terms of an integral over the Fermi-Dirac distribution, we integrate by parts. We have

$$\int_0^\infty \mathrm{d}p\, p^2 \ln\left[1+e^{-\beta(\omega\pm m)}\right] = \frac{1}{3}p^3\ln\left[1+e^{-\beta(\omega\pm m)}\right]\Big|_0^\infty + \frac{1}{3}\int_0^\infty \mathrm{d}p\,\frac{\beta p^4}{\omega}n_f(\omega\pm\mu), \tag{4.80}$$

where the boundary term vanish. This allows us to write the pressure as

$$p = \frac{1}{3}\int_0^\infty \mathrm{d}p\,\frac{p^4}{\omega}[n_f(\omega-\mu) + n_f(\omega+\mu)] \tag{4.81}$$

We are interested in the $T = 0$ limit. In this case, the Fermi distribution becomes a step function, $n_f(\omega) = \theta(-\omega)$. Without loss of generality, we assume that $\mu > 0$, i.e., we are dealing with an abundance of matter compared to anti-matter. The dispersion relation $\omega = \sqrt{p^2+m^2}$ is always positivive. This means that the contribution to thermodynamic quantities from anti-particles vanish, as the integral is multiplied with $n_f(\omega+\mu) = \theta(-\omega-\mu)$, where the argument $-\omega-\mu$ is strictly negative on the domain of integration. At zero temperature, the only dynamics are due to the degeneracy pressure of the fermions, that is, due to the Pauli exclusion principle. There are no thermal fluctuations that can create a particle-antiparticle pair. Thus, if the system has a positive chemical potential, it will contain no antiparticles. Furthermore, if $\mu < m$, then integrand multiplied with $n_f(\omega-\mu)$ is also zero in the whole domain of integration. It is only when $\mu \geq m$ that it is energetically favorable for the system to be in a state with particles. We define the Fermi momentum p_f by $\mu = \sqrt{{p_f}^2+m^2}$. In the zero-temperature limit, we can then rewrite any integral over the Fermi distribution as

$$\int_0^\infty \mathrm{d}p\,[f(p)n_f(\omega-\mu) + g(p)n_f(\omega-\mu)] = \int_0^{p_f}\mathrm{d}p\, f(p). \tag{4.82}$$

The charge density is thus

$$n = \frac{1}{\pi^2}\int_0^{p_f}\mathrm{d}p\,p^2 = \frac{p_f^3}{3\pi^2}. \tag{4.83}$$

At $T = 0$, this is the particle number density, as there are no antiparticles. This density is given by the chemical potential and vanishes when $\mu \leq m$, i.e. when the free energy cost of creating a particle is positive. We can write the energy density and pressure integrals, Eq. (4.79) and Eq. (4.81), as

$$u = \frac{1}{\pi^2}\int_0^{p_f}\mathrm{d}p\,p^2\sqrt{p^2+m^2} = \frac{m^4}{\pi^2}\int_0^{x_f}\mathrm{d}x\,x^2\sqrt{x^2+1}, \tag{4.84}$$

$$p = \frac{1}{3\pi^2}\int_0^{p_f}\mathrm{d}p\,\frac{p^4}{\sqrt{p^2+m^2}} = \frac{m^4}{3\pi^2}\int_0^{x_f}\frac{\mathrm{d}x\,x^4}{\sqrt{x^2+1}}. \tag{4.85}$$

We have defined $x = p/m$ and $x_f = p_f/m$. These integrals can be evaluated exactly as

$$\int_0^a \mathrm{d}x\, x^2\sqrt{x^2+1} = \frac{1}{8}\left[\sqrt{a^4+1}(2a^3+a) - \operatorname{arcsinh}(a)\right], \tag{4.86}$$

$$\int_0^a \mathrm{d}x\, \frac{x^4}{\sqrt{x^2+1}} = \frac{1}{8}\left[\sqrt{a^2+1}(2a^3-3a) + 3\operatorname{arcsinh}(a)\right]. \tag{4.87}$$

We introduce the characteristic energy and number density,

$$u_0 = \frac{m^4}{8\pi^2}, \quad n_0 = \frac{u_0}{m}, \tag{4.88}$$

which allows us to write the thermodynamic variables as

$$n = \frac{8}{3} n_0\, x_f^3 \tag{4.89}$$

$$u = u_0\left[(2x_f^3 + x_f)\sqrt{1+x_f^2} - \operatorname{arcsinh}(x_f)\right], \tag{4.90}$$

$$p = \frac{u_0}{3}\left[(2x_f^3 - 3x_f)\sqrt{1+x_f^2} + 3\operatorname{arcsinh}(x_f)\right]. \tag{4.91}$$

We have thus chosen $u_0 = p_0$, or equivalently set $k_3 = 1$. This is natural in the case of a relativistic fluid.

4.4.2 Limits

In the non-relativistic limit, as the chemical potential approaches m and thus $p_f \ll m$, the lowest order contributions to the energy density and pressure are given by the Taylor series around $x_f = 0$,

$$\tilde{u}(x_f) = \frac{8}{3}x_f^3 + \frac{4}{5}x_f^5 + \mathcal{O}(x_f^7), \tag{4.92}$$

$$\tilde{p}(x_f) = \frac{8}{15}x_f^5 + \mathcal{O}(x_f^7). \tag{4.93}$$

By neglecting terms of order x_f^7 and higher, we can write this as

$$\tilde{u} = \tilde{n} + \frac{4}{5}\left(\frac{8}{3}\tilde{n}\right)^{5/3}, \quad \tilde{p} = \frac{8}{15}\left(\frac{8}{3}\tilde{n}\right)^{5/3}. \tag{4.94}$$

The leading order contribution to the energy density is the rest mass of the particles. This term does not contribute to the pressure. As discussed earlier, the non-relativistic limit corresponds to $k_3 \ll 1$, if we chose units so that $\tilde{u} \approx \tilde{p}$, or $\tilde{u} \gg \tilde{p}$ if we demand that $k_3 = 1$. We see that $x_f \to 0$ corresponds to the latter case. By including only the leading order term, we can eliminate the Fermi momentum and write the equation of state in the non-relativistic limit as $u_{\mathrm{nr}} = kp^{\frac{3}{5}}$ where $k = 8/3 \cdot (15/8)^{3/5}$. The non-relativistic approximation of the cold fermions is thus a polytrope with $\gamma = \frac{5}{3}$. As we see from Figure 4.2, this is within the range where the mass decreases with the size of the star.

In the ultrarelativistic limit, where $p_f \gg m$, the leading order contributions to the pressure and energy density are

$$\tilde{u} = 2x_f^4, \quad \tilde{p} = \frac{2}{3}x_f^4, \tag{4.95}$$

and we get the particularly simple equation of state for the ultrarelativistic limit, $u_{\mathrm{ur}} = 3p$, which we recognize as the formula for radiation pressure. The equation of state $\tilde{u}(\tilde{p})$ in two different regimes are shown in Figure 4.5. The full equation of state is compared to the non-relativistic and ultrarelativistic approximations.

4.4.3 Units

The equation of state has given us the characteristic energy density and pressure, u_0 and p_0. If we demand

$$G\frac{m_0}{r_0} = \frac{4\pi}{3}\frac{r_0^3 u_0}{m_0} = 1, \tag{4.96}$$

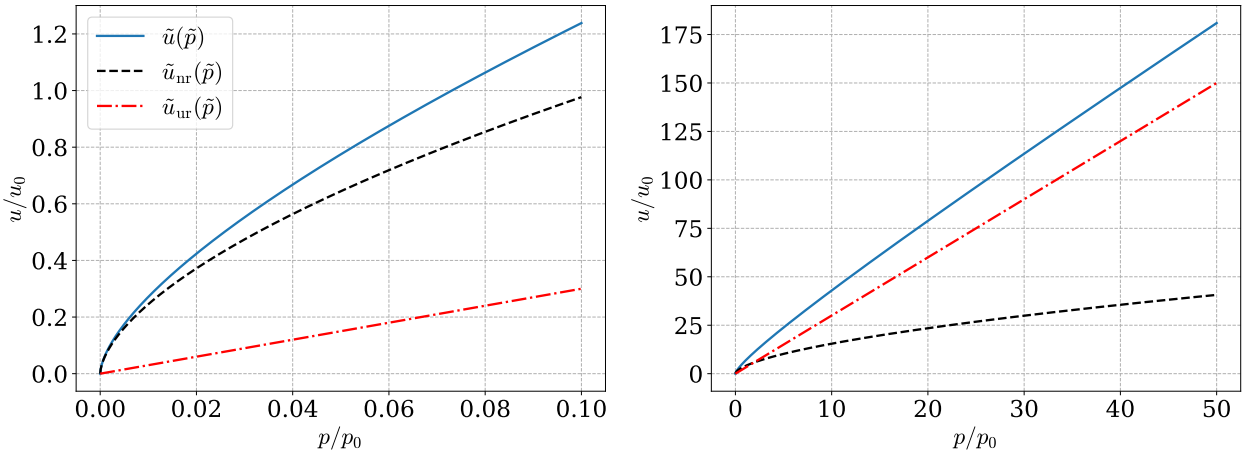


Figure 4.5: The equation of state of a cold Fermi gas. Both pressure and energy density is normalized to their characteristic quantities, p_0 and u_0. The equation of state is compared to the non-relativistic approximation, $\tilde{u}_{\mathrm{nr}}$ as well as the ultrarelativistic approximation, $\tilde{u}_{\mathrm{ur}}$, in two different regimes.

we have two equations and two unknowns, m_0 and r_0. This thus defines a complete set of units. We are using the cold Fermi-gas as a model for a neutron star, an the mass of the fermion m is therefore the neutron mass, Eq. (1.9), $m_N = 1.674 \cdot 10^{-27}\,\mathrm{kg}$. After reinstating $\hbar$ and c in metric units, we get

$$u_0 = p_0 = \frac{m^4}{8\pi^2}\frac{c^5}{\hbar^3} = 2.032 \cdot 10^{35}\,\mathrm{J\,m^{-3}}, \tag{4.97}$$

$$m_0 = \frac{c^4}{\sqrt{\frac{4\pi}{3}u_0 G^3}} = 1.608 \cdot 10^{31}\,\mathrm{kg} = 8.082\,M_\odot \tag{4.98}$$

$$r_0 = \frac{G m_0}{c^2} = 11.93\,\mathrm{km}. \tag{4.99}$$

From this, we expect our star to have a mass of the order of a solar mass, $M_\odot = 1.98841 \cdot 10^{30}\,\mathrm{kg}$ [1], and a radius of the order of kilometers, without solving the TOV equation.

4.4.4 Numerical results

With the energy density, Eq. (4.90), and pressure, Eq. (4.91), we can numerically solve the TOV equation given a central pressure p_c. This is done using an adaptive Runge-Kutta method, with the stop criterion $p(r) = 0$. Description of the code and where to find it is given in Appendix D. The top graph in Figure 4.6 shows the pressure, normalized to the central pressure p_c, as a function of radius, normalized to the corresponding stellar radius R. The boundary conditions are logarithmically spaced. The lower graph in Figure 4.6 shows the mass, normalized to the total mass $M = m(R)$, as a function of the radius, again normalized to the stellar radius. As in the case of an incompressible fluid, the pressure follows a half bell-shaped curve, with a peak that becomes narrower as the central pressure increases. The black dashed line corresponds to the solution with the maximum mass, which will discuss shortly. We see that the pressure and mass curves changes most drastically when the central pressure is higher than that corresponding to the most massive star.

In Figure 4.7, we see the relationship between the mass and radius of the star. This line is parametrized by the base-10 logarithm of the central pressure, $p(0)$, normalized by $p_0 = u_0$. The cross marks the maximum mass, $M_{\max} = 0.711\,M_\odot$, which corresponds to a radius of $R = 9.20\,\mathrm{km}$. This matches the results obtained by Oppenheimer and Volkoff [25], $M_{\max} = 0.71$. In their 1939 paper, Oppenheimer and Volkoff computed five data points in the mass-radius plane. The results are marked by blue circles in Figure 4.7. We find good agreement between the three points closest to the maximum value and our results. However, the two results of Oppenheimer and Volkoff furthest away differ significantly from our results. The black dashed line is the absolute mass-radius constraint, Eq. (4.68), and any stable configuration must be on the right side of this

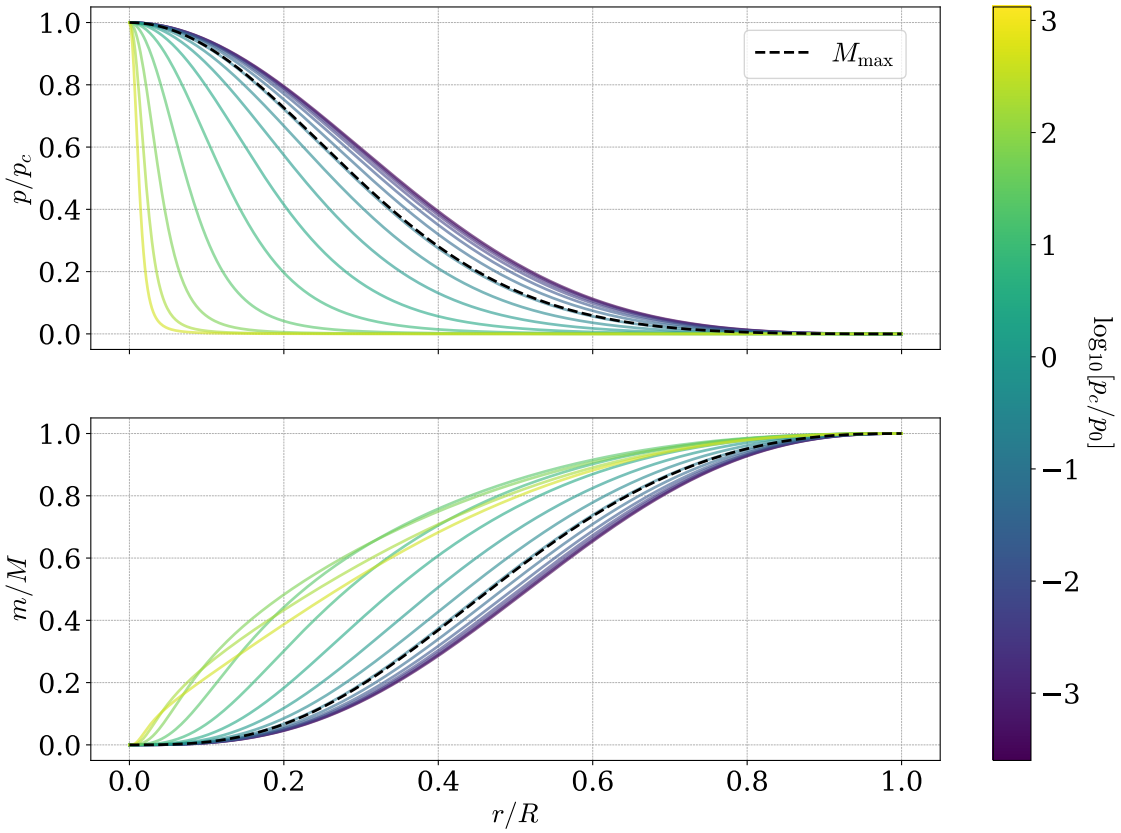


Figure 4.6: Top: The pressure normalized to central value, as a function of radius, normalized to the stellar radius. Bottom: The mass normalized to the total mass, as a function of radius, normalized to the stellar radius. This is plotted for several different values of central pressure, which is indicated by the color shceme.

line. As we predicted from looking at the non-relativistic equation of state, the mass decreases with the star's radius, at least for stars with low central pressure.

In Figure 4.8, we compare the mass-radius relationship obtained from the full theory with results from approximations. The lowest line is obtained by using both the full TOV equation and the exact equation of state. The next line above is obtained using the non-relativistic equation of state together with the full TOV equation. The second uppermost line is obtained from the exact equation of state and the Newtonian approximation for the TOV equation. The uppermost line uses both the Newtonian approximation to the TOV equation and the non-relativistic approximation for the equation of state. This last line corresponds to a polytrope in Newtonian gravity, as we studied in subsection 4.3.1. Unlike the other systems, it does not seem to have an upper limit for the mass, as expected.

4.4.5 Upper bound and stability

Utvid om stability

For any equation of state, the TOV equation will give a one-parameter family of stars, parametrized by the central pressure p_c. This leads to the possibility of an *absolute maximum* mass for a given equation of state. In the case of a non-interacting neutron, we found the limit to be $0.71\,M_\odot$, in agreement with Oppenheimer and Volkoff. To obtain a more general upper limit for the mass of neutron stars, or compact stars in general, one has to account for more general equations of state. To constrain the equation of state, we assume firstly that $dp/du \geq 0$. To justify this, take the non-relativisitc case, $u = n$, in which case the assumption is equivalent to $dp/dn \geq 0$. This says that an increase in particle density, for example, due to compression, will

Can there be multiple branches?

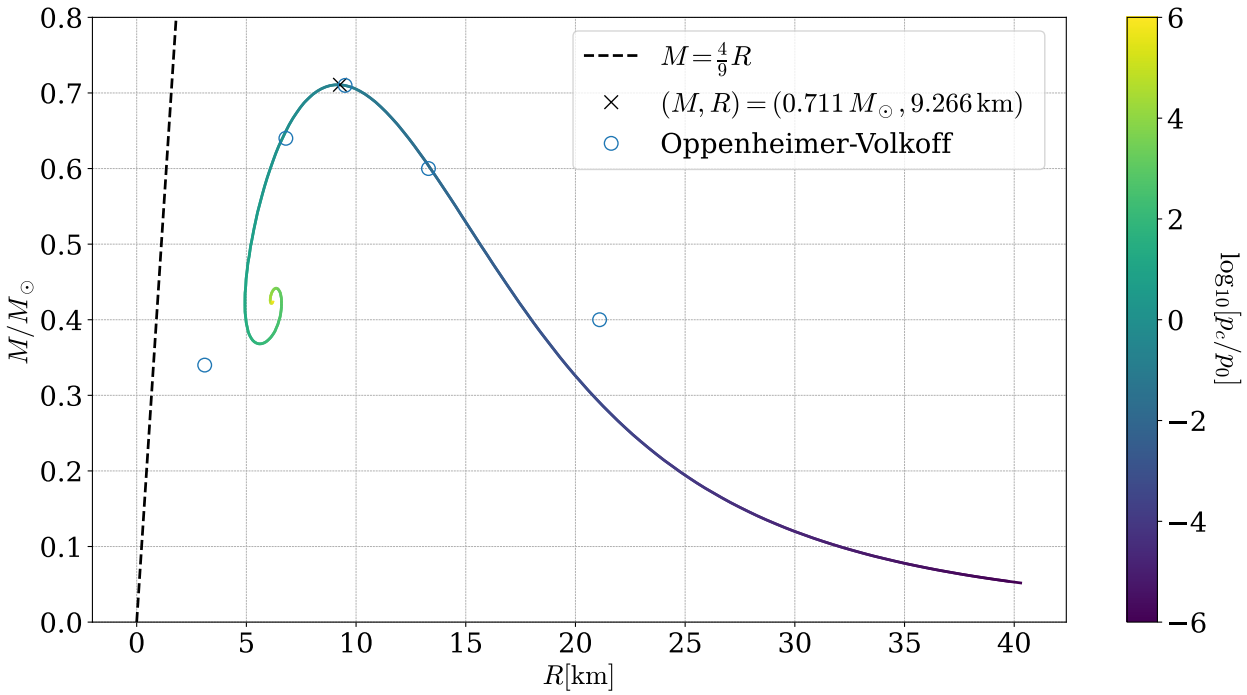


Figure 4.7: The mass-radius relationship of a star made of a cold gas of neutrons. The line is parametrized by the central pressure p_c. The corss indicate the maximum mass solution. The blue circles are results form the 1939 paper of Oppenheimer and Volkoff [25].

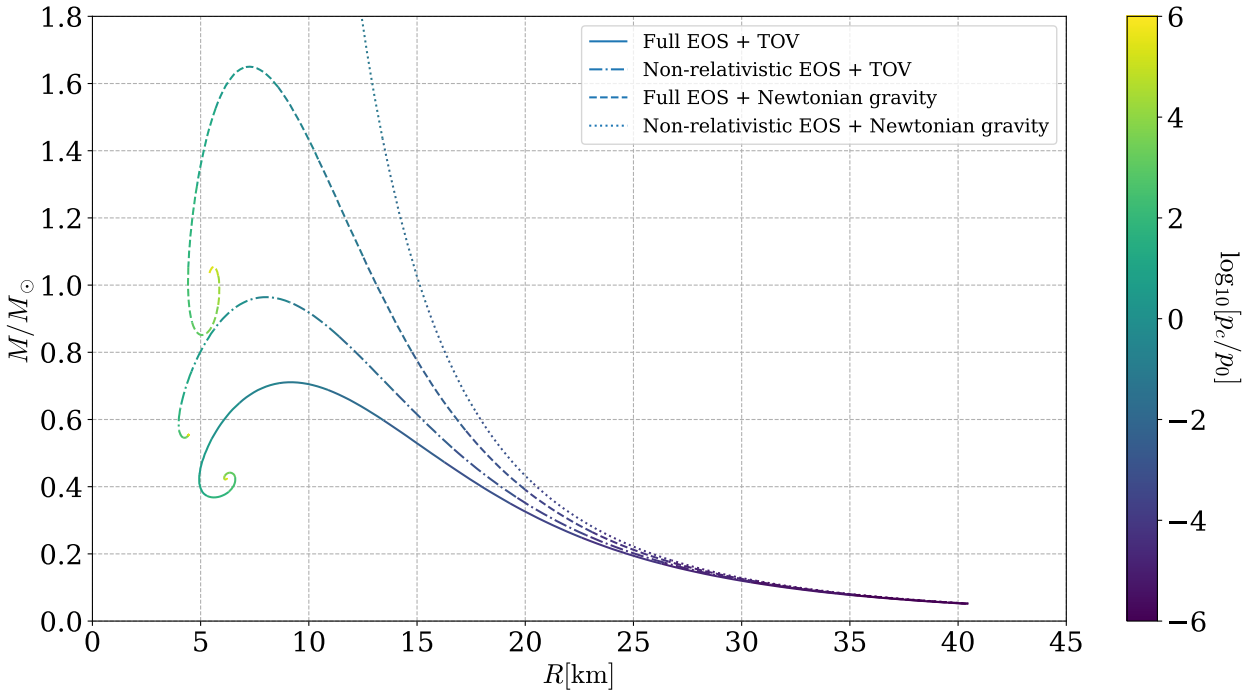


Figure 4.8: The mass-radius relationship of a cold gas of neutrons. The lowest line is obtained from the TOV equation and full equation of state. The middle line is from the TOV equation and the non-relativistic equation of state. The upper line is obtained from the Newtonian approximation of the TOV equation and the non-relativistic equation of state.

result in a rise in pressure. This is an instance of Le Chatelier's principle; nature will counteract any change forced upon it. The speed of sound in the fluid, v_s, is given by [22]

Vis dette?

$$v_s^2 = \frac{dp}{du}. \tag{4.100}$$

A realistic fluid should not have a speed of sound greater than the speed of light, leading to the constraint $dp/du < 1$. Using these general assumptions, Rhoades and Ruffini found an upper limit for neutron stars of $3.2\,M_\odot$ [31].

An equation of state with a *high* speed of sound, i.e., with a flat curve in the $p-u$-plane, is called *stiff*. From Figure 4.5, we see that the Newtonian equation of state is stiffer in the high-energy regime. The most extreme case is the incompressible model we saw earlier, which breaks causality. In general, a stiffer equation of state leads to a larger maximum mass. This is intuitive; the TOV equation describes the balancing of forces from pressure and gravity, and if the pressure raises fast as the density increases, then it can sustain a large total mass before it collapses [23].

Solutions to the TOV equation are systems in hydrostatic equilibrium. However, as a pen perfectly balances on its edge, this does not imply stability. When perturbed, a stable system returns back towards its equilibrium position as a marble in the bottom of a bowl. On the other hand, an unstable system will amplify perturbations, leading to a collapse or an explosion. We can make an intuitive argument for which configurations for a given family of stars are stable. We will again assume that the equation of state, on a microscopic level, obeys Le Chatelier's principle in the form $dp/dn > 0$. We can see that this holds for all the cases we are considering. A star in equilibrium will find itself on the line parametrized by its central pressure, such as Figure 4.7. In this case, a perturbation reducing the radius of the star will increase the central pressure as the particle density increases. This is illustrated in Figure 4.9. A star in equilibrium at point A can be compressed to an out-of-equilibrium configuration, point B. This point has a central pressure corresponding to the equilibrium state at point C. As the equilibrium configuration at point C has a *lower* mass than the configuration at B, it has a weaker gravitational effect. Therefore, we would expect it to shrink further, as the central pressure of B is not enough to support its gravitational mass. This compression will lead to an even higher central pressure. We thus have a positive feedback loop, and the initial perturbation will continue to grow. We can make a similar argument in the case where the mass increases with an increase in the central pressure. Here, a compression will lead to a state in which central pressure corresponds to an equilibrium state with a *higher* mass. This state will thus tend to expand after a compression, counteracting the perturbation. This gives us the following criterion for stability,

Er det en god antagelse?

Kan vi være helt sikre på dette?

$$\frac{dM}{dp_c} > 0. \tag{4.101}$$

As it turns out, this is only a necessary requirement for stability. To conduct a more rigorous study of stability, one must derive the equation of hydrostatic equilibrium with the addition of perturbations as time-dependent, radial oscillations. This is done by discplacing a fluid elemet at a radius r and time t by $\delta r(r,t) = \sum_n A_n \xi_n(r) e^{i\omega_n t}$. Here, ξ_n are normal modes with frequencies ω_n, $n \in \{0, 1, ...\}$. The equation for the eigenmodes was first obtained by Chandrasekhar [32], and can be written in the form of a Sturm-Liouville theory equation,

$$\left[\frac{d}{dr}\left(\Pi\frac{d}{dr}\right) + Q + \omega_n W\right]\xi_n = 0, \tag{4.102}$$

where Π, Q, and W are functions of the pressure, energy density, particle density, α, and β. These quantities are thus given by a solution to the equilibrium problem [23]. This analysis is outside the scope of this thesis, but we summarize some important conclusions. Stability is encoded in the sign of the square of the frequencies. For $\omega_n^2 > 0$, the mode will remain oscillatory, while if $\omega_n^2 < 0$, it will grow exponentially. Thus, if the system has *any* modes such that $\omega_n^2 < 0$, it is unstable. One mode ω_n will change stability at a critical point on the $M - R$ curve, where

$$\frac{dM}{du_c} = 0, \tag{4.103}$$

and this will *only* happen at critical points [33]. Here, u_c is the central energy density corresponding to p_c. This is equivaent to the criterion $dM/dp_c = 0$ as long as dp/du is finite. Whether or not the change is from

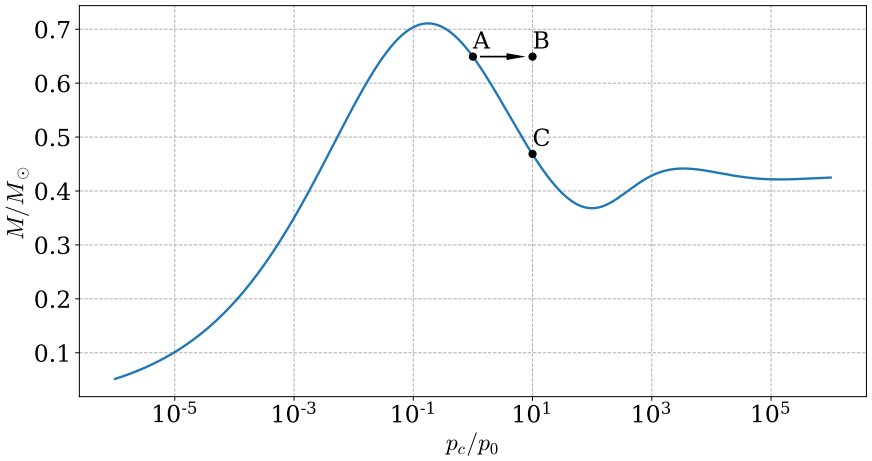


Figure 4.9: The plot shows the mass, in units of solar masses, of a star of a cold gas of neutrons, as a function of the central pressure, normalized to the characteristic pressure. Point A denotes a position of equilibrium, which can be compressed to an out-of-equilibrium point, B, which has a central pressure corresponding to an equilibrium configuration, point C.

a stable mode to an unstable one is dependent on whether or not the curve turns clockwise (a mode becomes stable) counterclockwise (a node becomes unstable) [33]. We know that very low-pressure, cold fermions are stable, which means that configuration with a radius larger than the maximum mas $0.71\,M_\odot$ will be stable. As illustrated in Figure 4.7, the curve then turns counterclockwise, and a new mode is made unstable each half turn.

Chapter 5

Chrial perturbation theory

In this chapter, we will take the general knowledge from the general theory in chapter 3 and apply it to the specific case of quantum chromodynamics, which results in *chiral perturbation theory*, or χPT.

5.1 QCD

This section is based on [4, 5, 18].

5.1.1 Yang-Mills theory and Gauge symmetry

In our discussion on global symmetries, we considered the global transformation of fields by some group G. In gauge theories, we will consider local transformations. That is, the transformations are themselves functions of spacetime, $U = U(x)$, and take on some value in G for all points in space. With this, however, we encounter a problem with comparing the value of a field at different points. As the symmetry is local, a gauge transformation will generally affect the field at two points differently. We must find a way to compare fields at different points independent of gauge transformations. This is similar to a problem we have encountered before. In differential geometry, as described in section 2.1, we needed a connection $\Gamma^{\rho}_{\mu\nu}$ to compare vectors in different tangent spaces in a coordinate independent way. In gauge theories, we generalize this by defining a connection, A_μ, to compare field values at different points in a gauge-independent way.

Consider a set of N_c fields ψ_c, which the symmetry group $\mathrm{SU}(N)$ acts linearly on as $\psi_c \to U_{cc'}\psi_{c'}$. We can write $U = \exp\{i\eta_\alpha T_\alpha\}$, where T_α are the generators of $\mathfrak{su}(N_c)_c$, and can therefore be written $A_\mu = A^\alpha_\mu T_\alpha$. The transformation is then made local by letting the coordinates of $\mathrm{SU}(N)$ be functions of spacetime, $\eta_\alpha = \eta_\alpha(x)$. As we did in section 2.1, we define the covariant derivative D_μ to transform as the thing on which it acts. It has the form

$$D^{cc'}_\mu \psi_{c'} = (\delta_{cc'}\partial_\mu - igA^{cc'}_\mu)\psi_{c'}, \tag{5.1}$$

where $A^{cc'}_\mu$ is a new, dynamic field, the gauge field. This field takes values in the Lie algebra of the gauge group, $\mathfrak{su}(N)$. We will suppress the c-indices for cleaner notation. This field also transform under the gauge group. By enforcing the transformation rule $D_\mu A_\nu \to U D_\mu A_\nu$, we can deduce the transformation properties of the gauge field,

Why? Mention it is adjoint rep., and explain.

$$A_\mu \to U\left(A_\mu + \frac{i}{g}\partial_\mu\right)U^\dagger \tag{5.2}$$

With the covariant derivative, we can create gauge-invariant terms, such as $\bar{\psi}D_\mu\psi$. In section 2.1 we introduced the Riemann tensor as the commutator of covaraint derivatives, Eq. (2.38). This ensures that it transforms as a tensor and gives us the interpretation as a quantity that measures the amount vectors curved

when parallel transported in a small loop. In analogy, we define the *field strength tensor*,

$$G_{\mu\nu} := \frac{i}{g}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]. \tag{5.3}$$

A_μ is an element of a Lie algebra, so the commutator is given by the structure constants of that algebra, Eq. (2.71). The field strength tensor transforms as $G_\mu \to UG_{\mu\nu}U^\dagger$. This allows us to create gauge-invariant terms of only this tensor, which, as with the Ricci scalar in general relativity, are the building blocks of the Lagrangian of the gauge field. The lowest order terms are

$$G^{\mu\nu}_\alpha G^\alpha_{\mu\nu}, \quad \epsilon^{\mu\nu\rho\sigma} G^\alpha_{\mu\nu} G^\alpha_{\rho\sigma}. \tag{5.4}$$

Here, α is the index in $\mathfrak{su}(N)$-space.

5.1.2 The QCD Lagrangian

Quantum chromodynamics, or QCD, is the specific gauge theory of quarks q_{fc}, spin-$\frac{1}{2}$ spinor particles, interacting via the strong force, a $\mathrm{SU}(3)_c$ gauge field denoted A_μ. There are six quarks q, called flavors and indexed by f, and the quantum number corresponding to the strong force is called color indexed by c. The quarks, labeled u, d, s, c, t, and b, have different masses. This thesis will include the two or three lightest quarks at different times. Therefore, we denote the number of flavors by N_f. The Lagrangian of QCD, including only the strong force, is

$$\mathcal{L}_{\mathrm{QCD}} = \bar{q}(i\not{D} - m)q - \frac{1}{4} G^\alpha_{\mu\nu} G^{\mu\nu}_\alpha. \tag{5.5}$$

We have suppressed color and flavor indices. $\not{D}q = \gamma^\mu(\partial_\mu - igA_\mu)q$ is the covariant derivative associated with the $\mathrm{SU}(3)_c$ gauge group with coupling constant g, and γ^μ are the Dirac matrices, as described in section A.1. The quark mass matrix, m, acts on the flavor indexes as the flavor states are mass eigenstates. There are no known symmetries that forbid an $\epsilon^{\mu\nu\rho\sigma} G^\alpha_{\mu\nu} G^\alpha_{\rho\sigma}$-term, however it is an empirical fact that it either not present or highly suppressed. Its absence is dubbed the strong CP problem [5].

5.1.3 Chiral symmetry

If we consider the massless QCD Lagrangian, $m = 0$, it has an additional symmetry of rotation in its flavour indices. We can project the quarks down to their *chiral* components by introducing projection operators

$$P_R = \frac{1}{2}\left(1 + \gamma^5\right), \quad P_L = \frac{1}{2}\left(1 - \gamma^5\right). \tag{5.6}$$

Here, γ^5 is the "fifth gamma-matrix", as described in section A.1. As good projection operators, they obey

$$P_R + P_L = 0, \quad P_R P_L = P_L P_R = 0, \quad P_I^2 = P_I,\ I = R, L. \tag{5.7}$$

By the properties of γ^5 and $\bar{q} = q^\dagger \gamma^0$, these operators project out the opposite chirality of q and q,

$$P_I q = q_I, \quad \bar{q} P_I = q_{\bar{I}}, \quad I = R, L, \quad \bar{I} = L, R. \tag{5.8}$$

With this, we can write the quark-sector of massless QCD as

$$i\bar{q}\not{D}q = i\bar{q}\not{D}(P_R + P_L)^2 q = i\bar{q}_L \not{D} q_L + i\bar{q}_R \not{D} q_R. \tag{5.9}$$

This operator is invariant under the transformations

$$q_R \to U_R q_R, \quad q_L \to U_L q_L, \tag{5.10}$$

where U_L and U_R are Hermitian matrices that act on the flavor indices. These transformations form the Lie group $\mathrm{U}(N_f)_R \times \mathrm{U}(N_f)_L = \mathrm{U}(1)_R \times \mathrm{SU}(N_f)_R \times \mathrm{U}(1)_L \times \mathrm{SU}(N_f)_L$. This transformation can also be described in terms of the diagonal subgroup. This subgroup is made up of transformations where $U_R = U_L$,

called vector transformations, and the remaining subgroup of transformations where $U_L = U_R^\dagger$, called axial transformations. These together form $\mathrm{U}(N_f)_A \times \mathrm{U}(N_f)_V = \mathrm{U}(1)_V \times \mathrm{SU}(N_f)_V \times \mathrm{U}(1)_A \times \mathrm{SU}(N_f)_A$. The currents corresponding to these transformations are

$$J_V^\mu = \bar{q}_R \gamma^\mu q_R, \quad V_\alpha^\mu = \bar{q} T_\alpha \gamma^\mu q, \quad J_A^\mu = \bar{q}_L \gamma^\mu \gamma^5 q_L, \quad A_\alpha^\mu = \bar{q} T_\alpha \gamma^\mu \gamma^5 q. \tag{5.11}$$

Here, T_α and $T_\alpha \gamma^5$ are the generators of $\mathrm{SU}(N_f)_V$ and $\mathrm{SU}(N_f)_A$. This symmetry, though, is broken in several ways. Firstly, transformations of the form $e^{i\alpha\gamma^5} \in \mathrm{U}(1)_A$ are subject to the *axial anomaly*. As mentioned in section 3.3, in a quantum theory not only the action has to be invariant but the integration measure as well, and $\mathcal{D}q\mathcal{D}\bar{q}$ is not. This is encoded in the Schwinger-Dyson equation [5]

$$\partial_\mu \langle J_A^\mu \rangle = -\frac{e^2}{(4\pi)^2} \langle \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \rangle , \tag{5.12}$$

The remaining symmetry is $\mathrm{U}(1)_V \times \mathrm{SU}(N_f)_V \times \mathrm{SU}(N_f)_A$. Next, the mass term explicitly breaks this symmetry. If we consider all quarks to have the same mass m_q, so that $m = m_q \mathbb{1}$, only $\mathrm{U}(N_f)_A$ is broken. This is called the *chiral limit*. However, when we include the fact that the masses of the quarks are different, we break the symmetry further. Other external currents, chemical potentials, and the electromagnetic interaction also break the symmetry.

Lastly, the symmetry is broken spontaneously by the ground state quark condensate, $\langle \bar{q}q \rangle \neq= 0$. In the chiral limit, one can show that the scalar bilinear for all three quarks u, d, and s must be equal, and we define $\langle \bar{q}q \rangle = \sum_f \langle \bar{q}_f q_f \rangle$ [18]. The scalar quark operator is not invariant under $\mathrm{SU}(N_f)_A$, and as discussed in section 3.3, this leads to the spontaneous symmetry breaking pattern.

$$\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \longrightarrow \mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \big/ \mathrm{SU}(N_f)_A = \mathrm{SU}(N_f)_V . \tag{5.13}$$

This pattern enables us to construct an effective low energy theory for QCD physics. We will take this symmetry breaking as an axiom and use it to construct χPT. The scalar quark condensate is quantified by the constant B_0 via the relation

$$\langle \bar{q}q \rangle = -f_\pi B_0. \tag{5.14}$$

Here, f_π is the pion decay constant. With the spontaneous symmetry breaking of the quark condensate, we get resulting Goldstone bosnos φ_a. The pion decay constant is defined by the overlap between the axial vector current A_α^μ acting on the single-particle momentum eigenstates of the Goldstone bosons, and the vacuum. By Lorentz invariance, this matrix element can be written as

$$\langle 0 | \mathrm{T} A_a^\mu(x) | \varphi_b(p) \rangle = i p_\mu f_\pi \delta_{ab} e^{i p_\mu x^\mu}. \tag{5.15}$$

We are now ready to construct χPT.

5.2 Chiral perturbation theory

We now apply the theory we developed in chapter 3. The systematics of chiral perturbation theory, or χPT, was laid out by Gasser and Leutwyler [19, 20] and is based on Weinberg's idea that quantum field theories on their own does not contain more information than the bare minimum [17]. In addition to these paper, this section is based on [18, 34, 35].

5.2.1 *Non-linear realization

To construct the Lagrangian of chiral perturbation theory we start with the Lagrangian of massless QCD,

$$\mathcal{L}_{\mathrm{QCD}}^0 = i\bar{q}\not{D}q - \frac{1}{4} G_{\mu\nu}^\alpha G_\alpha^{\mu\nu} \tag{5.16}$$

As discussed in last section, this Lagrangian is invariant under the full symmetry group $G = \mathrm{SU}(N_f)_R \times \mathrm{SU}(N_f)_L$, but the system undergoes spontaneous symmetry breaking to the smaller group $H = \mathrm{SU}(N_f)_V$.

As we found in section 3.4, the low energy dynamics will therefore be described by a $G/H = \mathrm{SU}(N_f)_A$-valued field Σ. Let $g \in G$. We write $g = (U_L, U_R)$, where $U_R \in \mathrm{SU}(N_f)_R$, $U_L \in \mathrm{SU}(N_f)_L$. Elements in H are then of the form (U, U), while elements in G are of the for $(U, U^\dagger)$. A general element g can be written as

$$g = (U_L, U_R) = (1, U_R {U_L}^\dagger)(U_L, U_L). \tag{5.17}$$

Since $(U_L, U_L) \in H$, this means that we can write the coset gH as $(1, U_R {U_L}^\dagger)H$, which gives a way to choose a representative element for each coset. We identify

$$\Sigma = U_R {U_L}^\dagger. \tag{5.18}$$

This is our standard form for elements in gH. As we saw in section 3.4, it therefore implicitly define transformation properties of the Goldstone bosons, which is given by the function $h(g, \xi)$. For $\tilde{g} \in G$, we have

$$\tilde{g}(1, \Sigma) = (\tilde{U}_L, \tilde{U}_R)(1, U_R {U_L}^\dagger) = (1, \tilde{U}_R (U_R {U_L}^\dagger) {\tilde{U}_L}^\dagger)(\tilde{U}_L, \tilde{U}_L) = (1, \tilde{U}_R \Sigma \tilde{U}_L)\tilde{h}. \tag{5.19}$$

This gives the transformation rule

$$\Sigma \to \Sigma' = U_R \Sigma {U_L}^\dagger. \tag{5.20}$$

This gives simple transformation rules for $(U, U) \in H$ and $(U, U^\dagger) \in G/H$,

$$H: \quad \Sigma \to \Sigma' = U \Sigma U^\dagger, \tag{5.21}$$

$$G/H: \quad \Sigma \to \Sigma' = U \Sigma U. \tag{5.22}$$

hvorfor?

Due to how G factors into two Lie groups, the constituents of the Mauer-Cartan form are

$$d_\mu = i\Sigma(x)^\dagger \partial_\mu \Sigma(x), \quad e_\mu = 0. \tag{5.23}$$

We can now create G-invariant terms by taking traces of d_μ's. As we will discuss in subsection 5.2.3, the order of a term in the Lagrangian will be dependent on the number of d_μ's. As $d_\mu \in \mathfrak{su}(N_f)$, which we represent by the traceless matrices, the lowest order term is trivial,

$$\mathrm{Tr}\{d_\mu\} = 0. \tag{5.24}$$

Using $\partial_\mu[\Sigma(x)^\dagger \Sigma(x)] = 0$, we can write

$$d_\mu d_\nu = -\Sigma(x)^\dagger [\partial_\mu \Sigma(x)] \Sigma(x)^\dagger [\partial_\nu \Sigma(x)] = \Sigma(x)^\dagger [\partial_\mu \Sigma(x)][\partial_\nu \Sigma(x)^\dagger] \Sigma(x). \tag{5.25}$$

This leaves us with the single Lorentz invariant leading order term,

$$\mathrm{Tr}\{d_\mu d^\mu\} = \mathrm{Tr}\left\{\partial_\mu \Sigma (\partial^\mu \Sigma)^\dagger\right\}, \tag{5.26}$$

However, constructing the effective Lagrangian out of terms invariant under G is too restrictive to get the most general effective action. This only allows for an even number of d_μ's, and observed processes such as the decay of the neutral pion through $\pi^0 \to \gamma\gamma$ would not be possible [18]. This is because we have not allowed for terms that change the Lagrangian with a divergence term, as discussed in section 3.3. Terms of this type are called Wess-Zumino-Witten (WZW) terms [7]. We will not consider these here, as they do not affect the thermodynamic quantities in question [36].

5.2.2 External currents

As discussed in section 3.5, we can incorporate external currents and symmetry breaking terms by promoting the symmetry G to a gauge symmetry, treating the external currents as gauge fields, and demanding gauge invariance of the effective Lagrangian. The external currents may couple to conserved currents, Eq. (5.11), or the other bilinears we can create out of quarks, $\bar{q}q$, $\bar{q}\gamma^5 q$, $\bar{q}T_\alpha q$, and $\bar{q}T_\alpha \gamma^5 q$. The Lagrangian of these external currens is

$$\mathcal{L}_{\text{ext}} = -\bar{q}\left(s - i\gamma^5 p\right) q + \bar{q}\gamma^\mu \left(v_\mu + \gamma^5 a_\mu\right) q. \tag{5.27}$$

Hva er grunnen til valgene av fortegn?

Here, s, p, v_μ and a_μ are all $N_f \times N_f$ matrices acting on the flavor indices. They are, respectively, the scalar, pseudo-scalar, vector, and pseudo-vector currents. We denote these currents collectively as $j = (s, p, v^\mu, a^\mu)$. The masses of the quarks are accounted for by setting the scalar current $s = m + \tilde{s}$. Here, m is the mass matrix of the quarks, while $\tilde{s}$ are possible other scalar currents. Other examples of external currents are chemical potentials, such as the isospin chemical potential, which regulate conserved charges in the system. We now need to find the transformation properties of these currents under G. We define

$$r_\mu = v_\mu + a_\mu, \quad l_\mu = v_\mu - a_\mu, \quad \chi = 2B_0(s + ip), \quad \chi^\dagger = 2B_0(s - ip). \tag{5.28}$$

By making a local G-transformation and enforcing gauge-invariance, we find that these transform as

$$r_\mu \to U_R(r_\mu + i\partial_\mu)U_R^\dagger, \tag{5.29}$$
$$l_\mu \to U_L(l_\mu + i\partial_\mu)U_L^\dagger, \tag{5.30}$$
$$\chi \to U_R \chi {U_L}^\dagger. \tag{5.31}$$

As in Yang-Mills theory, we can now create field strength tensors of the gauge fields, to build more gauge-invariant terms. We define

$$f_{\mu\nu}^{(r)} = \partial_\mu r_\nu - \partial_\nu r_\mu - i[r_\mu, r_\nu], \quad f_{\mu\nu}^{(l)} = \partial_\mu l_\nu - \partial_\nu l_\mu - i[l_\mu, l_\nu]. \tag{5.32}$$

Including dynamical fields, such as the photon field $\mathcal{A}_\mu$, is slightly more complicated. Quantum electrodynamics, or QED, is a gauge theory with a $\mathrm{U}(1)_{\mathrm{EM}}$ gauge group, where the covariant derivative acing on quarks is

$$i\bar{q}\not{D}' q = i\bar{q}\gamma^\mu \left(\mathbb{1}\partial_\mu - ieQ\mathcal{A}_\mu\right) q = i\bar{q}\not{\partial} q - e\mathcal{A}_\mu J^\mu, \tag{5.33}$$

Here, $\mathcal{A}_\mu$ is the photon field corresponding to the gauge group, $e = |e|$ is the elementary charge as given in Eq. (1.8), $J^\mu = -\bar{q}Q\gamma^\mu q$ is the electromagnetic charge current, and Q is the quark charge matrix. This matrix is the generator of $\mathrm{U}(1)_{\mathrm{EM}}$. In the case of $N_f = 3$, $Q = \mathrm{diag}(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$. From Eq. (5.33), we see that $eQ\mathcal{A}_\mu$ is a vector current. Although the transformation of the quarks under the electromagnetic gauge group can be seen as a subgroup of G, we *do not* transform external currents to enforce gauge invariance, this is instead done by $\mathcal{A}_\mu$. As $\mathcal{A}_\mu$ is a dynamical field, we can not use it to enforce G-gauge invariance. However, if we treat the charge matrix Q as an external field, then we can restore the invariance. This gives the transformation rule

$$Q_I \to U_I Q_I U_I^\dagger, \; I = R, L. \tag{5.34}$$

Here, $Q_I = P_I Q$ are the chiral charge matrices. With these external fields, we must introduce a covariant derivative acting on Σ to enforce local G invariance. This is

$$\nabla_\mu \Sigma = \partial_\mu \Sigma - i r'_\mu \Sigma + i \Sigma l'_\mu, \tag{5.35}$$

where $r'_\mu = r_\mu + eQ\mathcal{A}_\mu$ and $l'_\mu = l_\mu + eQ\mathcal{A}_\mu$. We do not strictly *need* to include the electromagnetic field in the gauge derivative, we could just build G invariant terms of eQ, $\mathcal{A}_\mu$ and Σ, however this is the most economical way to achieve this.

Lastly, we must include terms from quantum electrodynamics involving only the photon field, which are

$$\mathcal{L}^0_{\mathrm{QED}}[\mathcal{A}] = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \quad F_{\mu\nu} = 2\partial_{[\mu}\mathcal{A}_{\nu]}. \tag{5.36}$$

The full Lagrangian is then

$$\mathcal{L}_{\mathrm{QCD}}[q, \bar{q}, A, \mathcal{A}, j] = \mathcal{L}^0_{\mathrm{QCD}}[q, \bar{q}, A] + \mathcal{L}^0_{\mathrm{QED}}[\mathcal{A}] + \mathcal{L}_{\mathrm{ext}}[\mathcal{A}, j]. \tag{5.37}$$

We now define the effective Lagrangian of χPT, $\mathcal{L}_{\mathrm{eff}}$ as

$$Z[j] = \int \mathcal{D}q\mathcal{D}\bar{q}\mathcal{D}A\mathcal{D}\mathcal{A} \exp\left\{ i \int \mathrm{d}^4x \mathcal{L}_{\mathrm{QCD}}[q, \bar{q}, A, \mathcal{A}, j] \right\} = \int \mathcal{D}\pi\mathcal{D}\mathcal{A} \exp\left\{ i \int \mathrm{d}^4x \mathcal{L}_{\mathrm{eff}}[\varphi, \mathcal{A}, j] \right\}. \tag{5.38}$$

5.2.3 Weinberg's power counting scheme

Skriv/kopier tekst om Weinberg's power counting scheme

5.3 Three-flavor χPT to leading order

Using the building blocks from last section, we can build the leading order Lagrangian, which is [18–20, 37]

$$\mathcal{L}_2 = \frac{1}{4}f^2\mathrm{Tr}\left\{\nabla_\mu\Sigma\nabla^\mu\Sigma^\dagger\right\} + \frac{1}{4}f^2\mathrm{Tr}\left\{\chi\Sigma^\dagger + \Sigma\chi^\dagger\right\} + e^2C\mathrm{Tr}\left\{\Sigma Q\Sigma^\dagger Q\right\}. \tag{5.39}$$

We will work with three flavors, i.e. $N_f = 3$, so the mass matrix is now

$$m = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}. \tag{5.40}$$

When we evaluate $s = m$ and $p = 0$, the scalar term then becomes

$$\chi = 2B_0 m = \begin{pmatrix} \bar{m}^2 - \Delta m^2 & 0 & 0 \\ 0 & \bar{m}^2 + \Delta m^2 & 0 \\ 0 & 0 & m_S^2 \end{pmatrix}, \tag{5.41}$$

where we have defined

$$\bar{m}^2 = B_0(m_u + m_d), \quad \Delta m^2 = B_0(m_d - m_u), \quad m_S^2 = 2B_0 m_s. \tag{5.42}$$

The charge matrix is

$$Q = \frac{1}{3}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \frac{1}{2}\left(\lambda_3 + \frac{1}{\sqrt{3}}\lambda_8\right). \tag{5.43}$$

In the vacuum, when there are no external currents, we choose the standard, exponential parametrization of the Goldstone manifold,

$$\Sigma(x) = \exp\left\{i\frac{\varphi_a\lambda_a}{f}\right\}. \tag{5.44}$$

Here, λ_α are the Gell-Mann matrices, as shown in section A.1, are generators of SU (3), and f is the bare pion decay constant. There are eight Goldstone bosons, φ_a, which are real functions of space-time. This parametrization ensures that $\varphi = 0$ corresponds to the vacuum. Using the isospin transformation rule Eq. (5.21), we can perform an infinitesimal transformation of the Goldstone fields,

$$\Sigma \to U_V\Sigma U_V^\dagger \sim \left(1 + i\eta_a\frac{1}{2}\lambda_a\right)\left(1 + i\frac{1}{f}\varphi_b\lambda_b\right)\left(1 - i\eta_c\frac{1}{2}\lambda_c\right) \sim 1 + i\frac{\varphi_a}{f}\lambda_a + i\frac{\varphi_a}{f}\eta_b f_{abc}\lambda_c, \tag{5.45}$$

or

$$\varphi_a \to [\delta_{ab} + i\eta_\alpha(-if_{\alpha ab})]\varphi_b. \tag{5.46}$$

That is, φ_a transforms under the adjoint representation of $\mathfrak{su}(3)$, which is made up of elements of the form $\eta_\alpha(-if_{\alpha ab})$. The $\mathfrak{su}(3)$ Lie algebra has three independent $\mathfrak{su}(2)$ sub-algebras. We introduce the matrices

$$\lambda_Q = \lambda_3 + \frac{1}{\sqrt{3}}\lambda_8, \quad \lambda_K = -\lambda_3 + \frac{1}{\sqrt{3}}\lambda_8, \tag{5.47}$$

From the structure constants, Eq. (A.14), we can conclude that they commute, i.e., $[\lambda_Q, \lambda_K] = 0$. Furthermore, we find the commutation relations

$$[\lambda_i, \lambda_j] = 2i\epsilon_{ijk}\lambda_k, \quad ijk \in \{1,2,3\}, \{4,5,Q\}, \text{ or } \{6,7,K\}. \tag{5.48}$$

We here define the Levi-Civita symbol by $\epsilon_{123} = \epsilon_{34Q} = \epsilon_{67K} = 1$. This is the defining commutation relation of $\mathfrak{su}(2)$. The first subalgebra, spanned by $\{\lambda_1, \lambda_2, \lambda_3\}$, corresponds to isospin transformations, which are rotations of the up and down quark into each other. Consider the transformation where $\eta_3 \neq 0$, while $\eta_\alpha = 0$ for $\alpha \neq 3$. Acting on the quarks, this transformation is generated by λ_3, while in the adjoint representation the generator is f_{3ab}. Under this transformation, $\varphi_3\lambda_3$ is invariant. We can see this from the fact that the structure constants f_{abc} is totally antisymmetric, and thus $f_{33b} = 0$. This means that φ_3 has the quantum number $I_3 = 0$. $\varphi_1\lambda_1$ and $\varphi_2\lambda_2$ do not have definite values of the third component of isospin as they are

not eigenvectors of f_{3ab}, but they do have definite values for the first and second component. The linear combinations $\pi^{\pm}(\lambda_1 \mp \lambda_2)$, on the other hand, do. This shows the relationship between our fields φ_a, and the observed, charged pions π^+ and π^-, as they have definite values for I_3.[1] The full relationship between the φ_a-fields and the observed pseudoscalar mesons is [18]

$$\varphi_a\lambda_a = \begin{pmatrix} \varphi_3 + \frac{1}{\sqrt{3}}\varphi_8 & \varphi_1 - i\varphi_2 & \varphi_4 - i\varphi_5 \\ \varphi_1 + i\varphi_2 & \varphi_3 + \frac{1}{\sqrt{3}}\varphi_8 & \varphi_6 - i\varphi_7 \\ \varphi_4 - i\varphi_5 & \varphi_6 - i\varphi_7 & \frac{2}{\sqrt{3}}\varphi_8 \end{pmatrix} = \begin{pmatrix} \pi_0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi_0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix}. \tag{5.49}$$

5.3.1 Ground state

When we take into account chemical potentials, we need to pick a new parametrization. We will start the analysis by assuming $e = 0$ and then reintroduce electromagnetic interactions later. The covariant derivative is then

$$\nabla_\mu\Sigma = \partial_\mu\Sigma - i[v_\mu, \Sigma], \quad v_\mu = \mu\delta^0_\mu. \tag{5.50}$$

The chemical potential matrix μ has three independent degrees of freedom for each quark, an is of the form

$$\mu = \begin{pmatrix} \mu_u & 0 & 0 \\ 0 & \mu_d & 0 \\ 0 & 0 & \mu_s \end{pmatrix} = \begin{pmatrix} \frac{1}{3}\mu_B + \frac{1}{2}\mu_I & 0 & 0 \\ 0 & \frac{1}{3}\mu_B - \frac{1}{2}\mu_I & 0 \\ 0 & 0 & \frac{1}{3}\mu_B - \mu_S \end{pmatrix} = \frac{1}{3}(\mu_B - \mu_S)\mathbb{1} + \frac{1}{2}\mu_I\lambda_3 + \frac{1}{\sqrt{3}}\mu_S\lambda_8, \tag{5.51}$$

where we have defined $\mu_B = \frac{3}{2}(\mu_u + \mu_d)$, $\mu_I = \mu_u - \mu_d$ and $\mu_S = \frac{1}{2}(\mu_u + \mu_d) - \mu_s$. Here, μ_u, μ_d, and μ_s are the up, down, and strange quark chemical potentials, while μ_B, μ_I, and μ_S are the baryon number, isospin, and strangeness chemical potentials. Σ transforms as $\Sigma \to \Sigma$ under $U(1)_V$, the symmetry corresponding to the baryon number; it is a baryon number singlet. This reflects the fact that the baryon number of mesons, and thus the φ_a's, is zero. Therefore, the chemical potential corresponding to the baryon number, μ_B, should not affect the final result. We can also see this because μ_B only appears with the identity matrix $\mathbb{1}$ in μ. Any dependence on μ_B in $\nabla_\mu\Sigma$ will vanish as $\mathbb{1}$ commutes with everything.

We will assume the ground state is a spatially independent configuration, $\varphi^0_a = \text{const}$. This configuration must then minimize the free energy, too leading order, is equivalent to minimizing the static Hamiltonian, i.e., $\mathcal{H}^{(0)} = \mathcal{H}[\varphi^0]$. To this end, we define

$$\Sigma_\alpha = \exp\{i\alpha n_a\lambda_a\}, \quad \alpha = \frac{1}{f}\sqrt{\varphi^0_a\varphi^0_a}, \quad n_a = \frac{\varphi^0_a}{\sqrt{\varphi^0_b\varphi^0_b}}. \tag{5.52}$$

We show how to derive the correct parametrization of the ground state in the case of two flavors in subsection B.1.1. For $\mu_S = 0$, we expect to recover this result, in which $n_1^2 + n_2^2 = 1$, and thus $n_a = 0$ for $a > 2$. Furthermore, we showed that we may choose $n_1 = 0$ without loss of generality, in which case the ground state becomes

$$\Sigma^{\pi^\pm}_\alpha = \exp\{i\alpha\lambda_2\} = (\mathbb{1} - \lambda_2^2) + \lambda_2^2\cos\alpha + i\lambda_2\sin\alpha. \tag{5.53}$$

The ground state is thus parameterized by α only. As we will show in chapter 6, when the isospin chemical potential exedes a critical value, $\mu_I \geq \mu_I^c$, the system undergoes a phase transition from the vacuum phase to a phase in which $\alpha \neq 0$, and the charged pions form a condensate. It is only when we reach this phase that the equation of state is non-trivial at $T = 0$, which makes it possible for pion stars to form.

If we define $\mu_{K^\pm} = \mu_S + \frac{1}{2}\mu_I$ and $\mu_{K^0} = \mu_S - \frac{1}{2}\mu_I$, then we can write the terms of the QCD Lagrangian made up of μ_I and μ_S and their corresponding currents densities as

$$\bar{q}\gamma^0\left(\frac{1}{2}\mu_I\lambda_3 + \frac{1}{\sqrt{3}}\mu_S\lambda_8\right)q = \bar{q}\gamma^0\left(\frac{1}{2}\mu_{K^\pm}\lambda_Q + \frac{1}{2}\mu_{K^0}\lambda_K\right)q. \tag{5.54}$$

Analogously to how a higher μ_I leads to a condensate in the first $\mathfrak{su}(2)$ subalgebra, we can expect these chemical potentials to lead to different condensates in their respective subalgebras. If we assume $\mu_{K^0} = 0$,

[1] Authors differ if they define $\sqrt{2}\pi^\pm = \varphi_1 \pm i\varphi_2$, or with opposite signs, $\sqrt{2}\pi^\pm = \varphi_1 \mp i\varphi_2$. We choose the latter, so that $\pi_+|0\rangle$ is the state with the quantum numbers of the positive pion.

we would expect the new ground state to take the form

$$\Sigma_\alpha^{K^\pm} = \exp\{i\alpha\lambda_5\} = (\mathbb{1} - \lambda_5^2) + \lambda_5^2\cos\alpha + i\lambda_5\sin\alpha. \tag{5.55}$$

This analysis extends to all four quadrants of the $\mu_I - \mu_S$ plane. If we set $\mu_{K^\pm} = 0$, we would expect a ground state of the form $e^{i\alpha\lambda_7}$. In [38], Kogut and Toublan show that this analysis is right, and that these states are local minima of the static Lagrangian. However, the domains of the different condensates overlap, so there is a phase transition between the condensates. We will study the different condensates and the transition between them in chapter 6.

5.3.2 The pion-condensed phase

Similar to what we found for the two-flavor case in section B.1, the ground states of the different condensates are parameterized as

$$\Sigma(x) = A_\alpha^i U(x)\Sigma_0 U(x) A_\alpha^i, \quad U(x) = \exp\left\{i\frac{\varphi_a\lambda_a}{2f}\right\}, \quad A_\alpha^i = \exp\left\{i\frac{\alpha\lambda_i}{2}\right\}, \tag{5.56}$$

where $i = 2, 5, 7$ depending on which phase we are in. We start working in the pion-condensed phase, so $i = 2$, and assume $\mu_I > 0$ and $e = 0$. Inserting this into Eq. (5.39), and expanding up to and including $\mathcal{O}\left((\pi/f)^2\right)$, we get

$$\mathcal{L}_2^{(0)} = \frac{1}{2}f^2\left(\mu_I^2\sin^2\alpha + 2\bar{m}\cos\alpha + m_S\right), \tag{5.57}$$

$$\mathcal{L}_2^{(1)} = -f\mu_I\partial_0\varphi_1\sin\alpha + f\sin\alpha\left(\mu_I^2\cos\alpha - \bar{m}^2\right)\varphi_2, \tag{5.58}$$

$$\mathcal{L}_2^{(2)} = \frac{1}{2}\partial_\mu\varphi_a\partial^\mu\varphi_a + \frac{1}{2}m_{ab}\varphi_a\partial_0\varphi_b - \frac{1}{2}m_a^2\varphi_a^2 - \frac{1}{\sqrt{3}}\Delta m^2\varphi_3\varphi_8. \tag{5.59}$$

Here, we have introduced a number of mass parameters, which will in general be functions of α and the chemical potentials. The diagonal mass terms are

$$m_1^2 = \bar{m}^2\cos\alpha - \mu_I^2\cos^2\alpha, \tag{5.60}$$

$$m_2^2 = \bar{m}^2\cos\alpha - \mu_I^2\cos 2\alpha, \tag{5.61}$$

$$m_3^2 = \bar{m}^2\cos\alpha + \mu_I^2\sin^2\alpha, \tag{5.62}$$

$$m_4^2 = m_5^2 = m_-^2 - m_{\mu+}^2, \tag{5.63}$$

$$m_6^2 = m_7^2 = m_+^2 - m_{\mu-}^2, \tag{5.64}$$

$$m_8^2 = \frac{1}{3}(\bar{m}^2\cos\alpha + 2m_S^2), \tag{5.65}$$

where

$$m_\pm^2 = \frac{1}{2}(\bar{m}^2\cos\alpha \pm \Delta m^2 + m_S^2), \quad m_{\mu\pm}^2 = \frac{1}{4}\mu_I^2\cos 2\alpha \pm \mu_I\mu_S\cos\alpha + \mu_S^2. \tag{5.66}$$

At $\mu_S = \mu_I = 0$ and $\alpha = 0$, these correspond to the well known leading-order masses of the pseudoscalar mesons [34],

$$m_1^2 = m_2^2 = m_3^2 = \bar{m}^2 = B_0(m_u + m_d) = m_\pi^2, \tag{5.67}$$

$$m_4^2 = m_5^2 = \frac{1}{2}(\bar{m}^2 - \Delta m^2 + m_S^2) = B_0(m_u + m_s) = m_{K^\pm}^2, \tag{5.68}$$

$$m_6^2 = m_7^2 = \frac{1}{2}(\bar{m}^2 + \Delta m^2 + m_S^2) = B_0(m_d + m_s) = m_{K^0}^2, \tag{5.69}$$

$$m_8^2 = \frac{1}{3}(\bar{m}^2 + 2m_S^2) = \frac{1}{3}B_0(m_u + m_d + 4m_s) = m_\eta^2. \tag{5.70}$$

In addition, we have off-diagonal terms,

$$m_{12} = 2\mu_I\cos\alpha, \tag{5.71}$$

$$m_{45} = \mu_I\cos\alpha + 2\mu_S, \tag{5.72}$$

$$m_{76} = \mu_I\cos\alpha - 2\mu_S. \tag{5.73}$$

Here, $m_{ab} = -m_{ba}$, and terms not defined above are zero. These terms mean that the basis φ_a does not correspond to mass eigenstates. This is to be expected, as they are not eigenstates of isospin nor strangeness. To find the physical masses, we must analyze the propagator and the spectrum of the theory. We follow [36, 39]. The spectrum is given by the zero of the inverse propagator, which in momentum space is

$$
\begin{aligned}
D_{ab}^{-1} &= \frac{\delta^2 S}{\delta\pi_a\,\delta\pi_b} = (p^2 - m_a^2)\delta_{ab} - ip_0 m_{ab} \\
&= \begin{pmatrix}
p^2 - m_1^2 & -ip_0 m_{12} & & & & & & \\
ip_0 m_{12} & p^2 - m_1^2 & & & & & 0 & \\
 & & p^2 - m_3^2 & & & & & \\
 & & & p^2 - m_4^2 & -ip_0 m_{45} & & & \\
 & & & ip_0 m_{45} & p^2 - m_5^2 & & & \\
 & & & & & p^2 - m_6^2 & -ip_0 m_{67} & \\
 & 0 & & & & ip_0 m_{67} & p^2 - m_7^2 & \\
 & & & & & & & p^2 - m_8^2
\end{pmatrix}.
\end{aligned} \tag{5.74}
$$

We have chosen to neglect the $\pi^0 - \eta$-mixing terms $D_{83}^{-1} = D_{38}^{-1} = -\Delta m^2/\sqrt{3}$. We will, however, keep $\Delta m \neq 0$ in the other mass contribution. The spectrum of the theory is given by the zeros of the determinant of the propagator,

$$
\begin{aligned}
\det\left(D^{-1}\right) = &(p^2 - m_3^2)(p^2 - m_8^2)\left[(p^2 - m_1^2)(p^2 - m_2^2) + p_0^2 m_{12}^2\right] \\
&\times \left[(p^2 - m_4^2)(p^2 - m_5^2) + p_0^2 m_{45}^2\right]\left[(p^2 - m_6^2)(p^2 - m_7^2) + p_0^2 m_{67}^2\right] = 0.
\end{aligned} \tag{5.75}
$$

Solving $\det(D^{-1}) = 0$ for p_0^2 gives eight roots E_i^2, one for each particle. Writing the four-momentum as $p^\mu = (p_0, \vec{p})$, these zeros are

$$E_0^2 = |\vec{p}|^2 + m_3^2, \tag{5.76}$$

$$E_\eta^2 = |\vec{p}|^2 + m_8^2, \tag{5.77}$$

$$E_{\pi^\pm}^2 = |\vec{p}|^2 + \frac{1}{2}\left(m_1^2 + m_2^2 + m_{12}^2\right) \mp \frac{1}{2}\sqrt{4|\vec{p}|^2 m_{12}^2 + \left(m_1^2 + m_2^2 + m_{12}^2\right)^2 - 4m_1^2 m_2^2}, \tag{5.78}$$

$$E_{K^\pm}^2 = |\vec{p}|^2 + m_4^2 + \frac{1}{2}m_{45}^2 \mp \frac{1}{2}m_{45}\sqrt{4|\vec{p}|^2 + 4m_4^2 + m_{45}^2}, \tag{5.79}$$

$$E_{K^0}^2 = |\vec{p}|^2 + m_6^2 + \frac{1}{2}m_{67}^2 \pm \frac{1}{2}m_{67}\sqrt{4|\vec{p}|^2 + 4m_6^2 + m_{67}^2}. \tag{5.80}$$

The masses of the particles are given by the energy at $\vec{p} = 0$. For $\alpha = 0$, which we will later show this is valid for low μ's, we get a Zeeman-like splitting of the energies of the pions,

$$E_0^2 = \sqrt{|\vec{p}|^2 + \bar{m}^2}, \quad E_{\pi^\pm} = \sqrt{|\vec{p}|^2 + \bar{m}^2} \mp \mu_I. \tag{5.81}$$

The kaons have a similar splitting, only due to the kaon chemical potentials $\mu_{K^\pm}$ and μ_{K^0} instead. The masses of the various mesons are shown in Figure 5.1. These results use the relationship between μ_I and α, which we derive in next chapter. In the lower plot, we see the Zeeman splitting of the pion masses, which persists until the mass of the positive pion vanishes. As we explore further in next chapter, this happens as the exact isospin symmetry $\mathrm{U}(1)_{I_3} \subset \mathrm{SU}(3)_V$ is broken spontaneously by the pion condensate, and π^+ is the corresponding Goldstone boson. On the top, we see that the kaons also get this Zeeman splitting in the vacuum phase.

If we adjust the strangeness chemical potential instead, while the isospin chemical potential remains constant, then the kaons will get a similar splitting, as shown in Figure 5.2, while the pions are unaffected. For $\mu_I > 0$, the mass of the positive kaon will reach zero first, at the point of transition into the charged kaon condensate. At this point, the results obtained in this section will become invalid, as $\Sigma = \exp\{i\alpha\lambda_2\}$ no longer is the ground state.

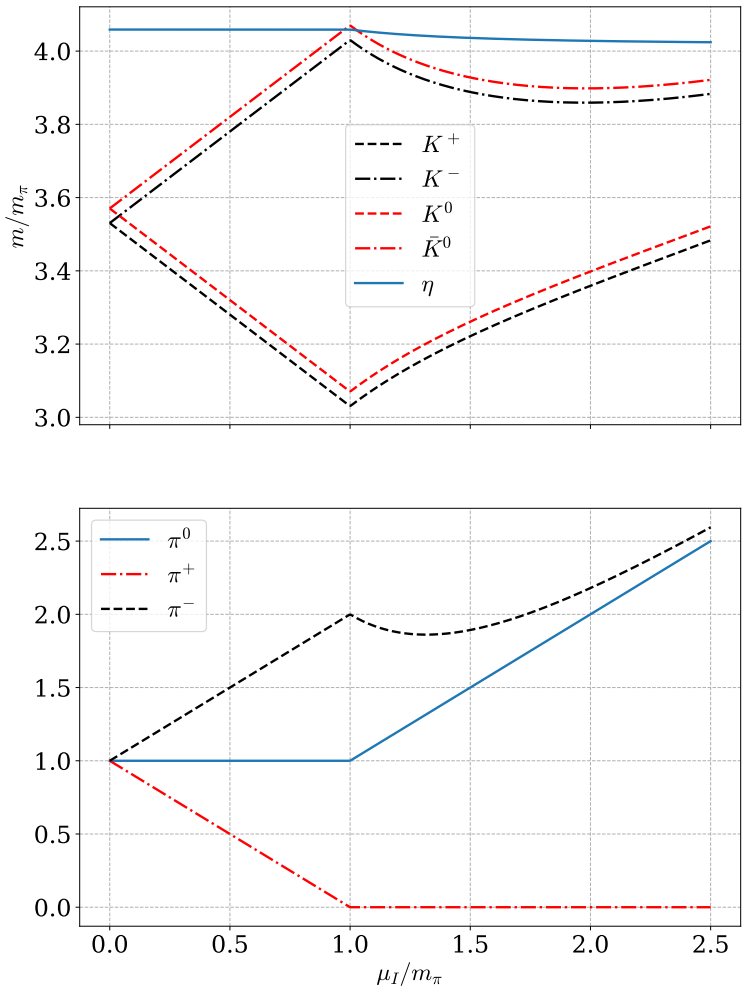


Figure 5.1: The leading order masses of the pseudoscalar mesons, as functions of the isospin chemical potential. Both the masses and chemical potential are normalized to the pion mass. These results are at $\mu_S = 0$. The difference between the kaons at $\mu_I = 0$ is a result of the fact that $\Delta m \neq 0$.

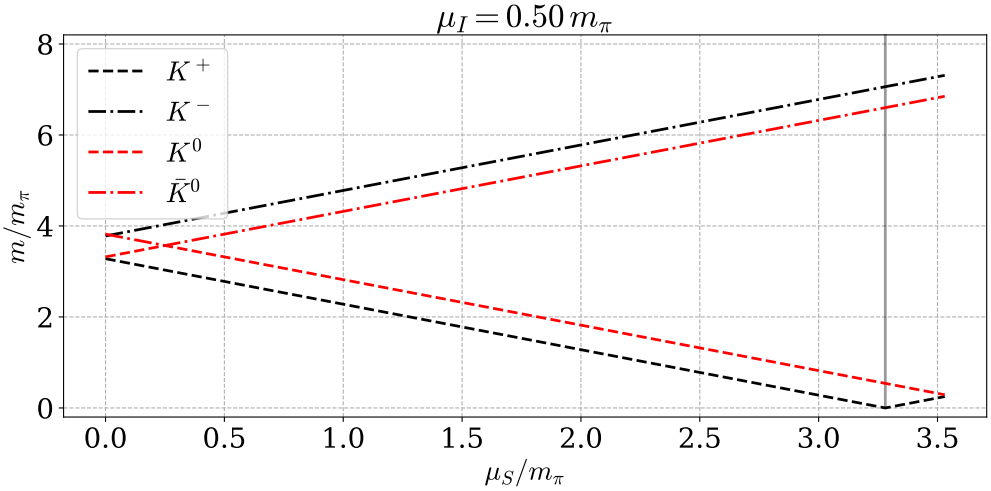


Figure 5.2: The kaon masses as a function of μ_S, both in units of m_π, evaluated at $\mu_I = 0.5\, m_\pi$. These results are only valid to the left of the gray line at $\mu_S \approx 3.3\, m_\pi$, where m_{K^+} reaches zero.

5.3.3 The kaon condensed phase

In the $K^{\pm}$-condensate, we get

$$\mathcal{L}_2^{(0)} = \frac{1}{2}f^2\left(\mu_{K^\pm}^2\sin^2\alpha + 2m_{K^\pm}^2\cos\alpha + \bar{m}^2 + \Delta m^2\right), \tag{5.82}$$

$$\mathcal{L}_2^{(1)} = -\frac{1}{2}f\mu_{K^\pm}\partial_0\varphi_4\,\sin\alpha + f\sin\alpha\left(\mu_{K^\pm}^2\cos\alpha - m_{K^\pm}^2\right)\varphi_5 \tag{5.83}$$

$$\mathcal{L}_2^{(1)} = \frac{1}{2}\partial_\mu\varphi_a\partial^\mu\varphi_a + \frac{1}{2}m'_{ab}\varphi_a\partial_0\varphi_a - \frac{1}{2}m_a'^2\varphi_a^2 - \frac{1}{2}\Delta_{\varphi\eta}\varphi_3\varphi_8 \tag{5.84}$$

where Sjekk disse

$$m'_{12} = \frac{1}{2}(\cos\alpha + 3)\mu_I + (\cos\alpha - 1)\mu_S \tag{5.85}$$

$$m'_{45} = 2\mu_{K^\pm}\cos\alpha \tag{5.86}$$

$$m'_{76} = \frac{1}{2}(3 - \cos\alpha)\mu_I - (1 + \cos\alpha)\mu_S, \tag{5.87}$$

$$m_1'^2 = m_2^2 = m_-'^2 + m_{\mu-}'^2 \tag{5.88}$$

$$m_3'^2 = \frac{1}{4}\left(\mu_{K^\pm}^2\sin^2\alpha + \bar{m}^2(\cos\alpha + 3) + \Delta m^2(\cos\alpha - 1) + m_S^2(\cos\alpha - 1)\right) \tag{5.89}$$

$$m_4'^2 = m_{K^\pm}^2\cos\alpha - \mu_{K^\pm}\cos^2\alpha \tag{5.90}$$

$$m_5'^2 = m_{K^\pm}^2\cos\alpha - \mu_{K^\pm}\cos 2\alpha \tag{5.91}$$

$$m_6'^2 = m_7^2 = m_+'^2 + m_{\mu+}'^2 \tag{5.92}$$

$$m_8'^2 = \frac{1}{12}\left[\frac{9}{4}\mu_{K^\pm}^2\sin^2\alpha + \bar{m}^2(5\cos\alpha - 1) + 5\Delta m^2(\cos\alpha - 1) + m_S^2(5\cos\alpha + 3))\right] \tag{5.93}$$

$$\Delta_{\eta\pi} = \frac{\sqrt{3}}{2}\left[\mu_{K^\pm}^2\sin^2\alpha + \frac{1}{3}\bar{m}^2(\cos\alpha - 1) + \frac{1}{3}\Delta m^2(\cos\alpha + 3) + \frac{1}{3}m_S^2(\cos\alpha - 1)\right], \tag{5.94}$$

and

$$m_\pm'^2 = \frac{1}{4}\bar{m}^2(\cos\alpha \mp 1 + 2) + \frac{1}{4}\Delta m^2(\cos\alpha \mp 1 - 2) + \frac{1}{4}m_S^2(\cos\alpha \pm 1), \tag{5.95}$$

$$m_{\mu\pm}'^2 = \frac{1}{2}(\sin^2\alpha \pm 3\cos\alpha - 5)\mu_I^2 + (\sin^2\alpha \pm \cos\alpha + 1)\mu_I\mu_s + (\sin^2\alpha \mp \cos\alpha - 1)\mu_s^2. \tag{5.96}$$

We see that both the Lagrangian and the masses have a similar structure to the pion condensate, only with φ_4 and φ_5 taking the roles of φ_1 and φ_2, and $\mu_{K^\pm}$ and $m_{K^\pm}$ the roles of μ_I and $\bar{m}$. The phase with a K^0-condensate, with the ground state $\Sigma = \exp\{i\lambda_7\alpha\}$ has a very similar structure. The static Lagrangian is

$$\mathcal{L}_2^{(0)} = \frac{1}{2}f^2\left(\mu_{K^0}^2\sin^2\alpha + 2m_{K^0}^2\cos\alpha + \bar{m}^2 - \Delta m^2\right). \tag{5.97}$$

5.3.4 Electromagnetic contributions

We now reintroduce e. First, we set $\mu_I = \mu_S = 0$, so we are in the vacuum phase, $\Sigma = U^2 = \exp\{i\varphi_a\lambda_a/f\}$, and the covariant derivative is

$$\nabla_\mu\Sigma = \partial_\mu\Sigma - ie\mathcal{A}_\mu[Q, \Sigma], \tag{5.98}$$

where Q is the charge matrix Eq. (5.43). Inserting this into the terms of the leading-order Lagrangian, Eq. (5.39), and expanding to and including $\mathcal{O}\left((\pi/f)^2\right)$ yields

$$\mathcal{L}_2 = \frac{1}{2}\partial_\mu\varphi_a\partial^\mu\varphi_a - \frac{1}{2}(m_a^2 + \Delta m_{\mathrm{EM},a}^2)\varphi_a^2 - \frac{\Delta m^2}{\sqrt{3}}\varphi_3\varphi_8 + f^2\left(\bar{m}^2 + \frac{1}{2}m_S^2 + \frac{2}{3}Ce^2\right)$$
$$+ e\mathcal{A}^\mu(\varphi_1\partial_\mu\varphi_2 - \varphi_2\partial_\mu\varphi_1 + \varphi_4\partial_\mu\varphi_5 - \varphi_5\partial_\mu\varphi_4) + \frac{1}{2}e^2\mathcal{A}^2(\varphi_1^2 + \varphi_2^2 + \varphi_4^2 + \varphi_5^2). \tag{5.99}$$

Here, m_a are the masses we found in Eq. (5.39), at $\mu_I = \mu_S = \alpha = 0$, while $\Delta m_{\text{EM},a}$ is the new electromagnetic contribution to the masses. This only affects the charged pions $\pi^\pm$, which are linear combinations φ_1 and φ_2, and the charged kaons, $K^\pm$, which are linear combinations of φ_4 and φ_5. The contribution to mass from electromagnetic effects are, to leading order, the same for these particles,

$$\Delta m^2_{\text{EM},a} = 2C\frac{e^2}{f^2} := \Delta m^2_{\text{EM}}, \quad a \in \{1,2,4,5\}. \tag{5.100}$$

This is known as Dashen's theorem [40]. We can express C in terms of the pion decay constant, and the mass and decay constant of the ρ-meson. This was first done in [41], using the then newly derive Weinberg sum rules relating the masses of heavier mesons [42]. This yields

$$C = \frac{3m_\rho^2 f_\rho^2}{32\pi^2}\ln\left(\frac{f_\rho^2}{f_\rho^2 - f_\pi^2}\right), \tag{5.101}$$

Urech, using the values $f_\pi = 93.3\,\text{MeV}$, $f_\rho = 154\text{MeV}$ and $m_\rho = 770\,\text{MeV}$, gets the numerical result $6.11 \times 10^{-6}\,(\text{GeV})^4$ [43]. With the value $f_\pi = 92.1\,\text{MeV}$ as used in the rest of this text, we obtain $C = 5.90 \times 10^{-6}\,(\text{GeV})^4$. As C is the sole source of difference in the masses of the neutral and charged pions to leading order, it can also be obtained directly from these masses. Using the values listed in section 1.1, we find

$$C = \frac{f^2}{2e^2}(m^2_{\pi_\pm} - m^2_\pi) = 5.824 \times 10^{-6}\,(\text{GeV})^4, \tag{5.102}$$

or in the characteristic units of the system, $C = 0.3771\,u_0$. This corresponds to $\Delta m^2_{\text{EM}} = 35.50\,\text{MeV}$. We see that the numerical differences between these results are small. When choosing the numerical values to use, we must take care to use a consistent set of values. Formulas such as Eq. (5.101) mean that the decay constants and masses are over constrained. In this text, we use the masses and the decay constant listed in section 1.1, and therefore choose the value in Eq. (5.102) for C.

The contribution to the pion mass from electromagnetic interactions is

$$m_{\pi^\pm} - m_\pi = \left(\sqrt{1 + \Delta m^2_{\text{EM}}/m^2_\pi} - 1\right) m_\pi = 3.401 \times 10^{-2}\, m_0 = 4.590\,\text{MeV}. \tag{5.103}$$

Even though the contribution to the *square* of the masses of the charged pion and kaons are the same, we see that the absolute difference between the neutral and charged pion depend on $\Delta m_{\text{EM}}/m_\pi$. We therefore expect the mass contribution from electromagnetic interactions to be lower for the heavier charged kaon. From the values in section 1.1, however, we see that the mass difference between the charged and neutral pions masses are very close to that of the charged and neutral kaons. This is because the difference in mass of the kaons is not only due to the electric charge at leading order, unlike the pions, but also due to Δm,

$$m^2_{K^0} - m^2_{K^\pm} = \Delta m^2 - \Delta m^2_{\text{EM}}. \tag{5.104}$$

We notice that the two contributions work in opposite directions. As we already have an independent way of determining Δm^2_{EM}, we can disentangle these contributions. To leading order,

$$\Delta m = \sqrt{m^2_{K^0} - (m^2_{K^\pm} - \Delta m^2_{\text{EM}})} = 0.5320\, m_\pi = 71.80\,\text{MeV}. \tag{5.105}$$

This is consistent with the fact that the down quark is around twice as heavy as the light quark [1]. The electromagnetic contribution to the mass of the charged kaon is

$$\left(1 - \sqrt{1 - \Delta m^2_{\text{EM}}/m^2_{K^\pm}}\right) m_{K^\pm} = 9.468 \times 10^{-3}\, m_\pi = 1.278\,\text{MeV}. \tag{5.106}$$

The difference due to Δm, on the other hand, is

$$m_{K^0} - \sqrt{m_{K^\pm} - \Delta m^2_{\text{EM}}} = 3.858\, m_\pi = 5.208\,\text{MeV}. \tag{5.107}$$

We analyze how electromagnetic interactions affect the condensed phases. In the pion condensate, the covariant derivative is

$$\nabla_\mu \Sigma = \partial_\mu \Sigma - i[v_\mu, \Sigma], \quad v_\mu = \mu\delta^0_\mu + e\mathcal{A}_\mu Q. \tag{5.108}$$

To zeroth order in π/f, the field parametrization is

$$\Sigma = (\mathbb{1} - \lambda_2^2) + \lambda_2^2 \cos\alpha + i\lambda_2 \sin\alpha. \tag{5.109}$$

Inserting this in the leading-order Lagrangian Eq. (5.39), and setting $\mathcal{A}_\mu = 0$ gives us the static Lagrangian including electromagnetic effects,

$$\mathcal{L}_2^{\mathrm{EM},(0)} = \frac{1}{2} f^2 \left(\mu_I^2 \sin^2\alpha + 2\bar{m}^2 \cos\alpha + \Delta m_{\mathrm{EM}}^2 \cos^2\alpha - \frac{1}{3}\Delta m_{\mathrm{EM}}^2 + m_S^2 \right) \tag{5.110}$$

Similarly, the static Lagrangian in the charged kaon condensate is

$$\mathcal{L}_2^{\mathrm{EM},(0)} = \frac{1}{2} f^2 \left(\mu_{K^\pm}^2 \sin^2\alpha + 2m_{K^\pm}^2 \cos\alpha + \Delta m_{\mathrm{EM}}^2 \cos^2\alpha - \frac{1}{3}\Delta m_{\mathrm{EM}}^2 + \bar{m}^2 + \Delta m^2 \right) \tag{5.111}$$

In the neutral kaon condensate, on the other hand, the static Lagrangian remains unchanged.

Chapter 6

Thermodynamics

In this chapter, we will use the results from χPT to calculate the thermodynamic properties of our system, such as pressure and energy density, and study the phase diagram of χPT. We follow the procedure used in [36, 44].

6.1 Free energy in a homogenous system

In our case, we are working with a homogenous system, in which we may write the free energy as $F = V\mathcal{F}$, where $\mathcal{F}$ is the *free energy density*. We use the grand canonical ensemble, so this is the grand canonical free energy, also called the grand canonical potential. It is related to the partition function of statistical mechanics, $\mathcal{Z}$, by

$$\mathcal{F} = -\frac{1}{V\beta} \ln \mathcal{Z}. \tag{6.1}$$

Here, V is the volume, and $\beta = 1/T$ is inverse temperature. Using imaginary-time formalism for thermal field theory, described in Appendix C, we find that the partition function is given by the path integral of the *Euclidean* Lagrange density, as shown in equation Eq. (C.12). In the zero temperature limit $\beta \to \infty$, the partition function is related to the generating functional $Z = Z[j]$, as described in section 3.1, by a Wick rotation. The free energy density at zero temperature is therefore

$$\mathcal{F} = \frac{i}{VT} \ln Z, \tag{6.2}$$

where VT is the volume of space-time. As we found in Eq. (3.18), this equals the effective potential in the ground state. In section 3.2, we found an explicit formula for this to one-loop order, Eq. (3.41). In section A.3, we show how to expand free energy density and other thermodynamic quantities in a self-consistent way.

We are after the equation of state, which the free energy density will give us access to. The free energy is defined as

$$F(T, V, \mu_I) = U - TS - \sum\nolimits_i \mu_i Q_i, \quad \mathrm{d}F = -S\mathrm{d}T - p\mathrm{d}V - \sum\nolimits_i Q_i \mathrm{d}\mu_i. \tag{6.3}$$

Here, U is the energy and S the entropy. Q_i are conserved charges, in our case the isospin and strangeness charge, and μ_i their corresponding chemical potentials. We work with $T = 0$. From Eq. (6.3), the pressure is then given by

$$p = -\left(\frac{\partial F}{\partial V}\right)_{T,\mu} = -\mathcal{F}. \tag{6.4}$$

The total charges are proportional to volume, which means that the corresponding densities are

$$n_i = \frac{Q_i}{V} = -\frac{1}{V}\left(\frac{\partial F}{\partial \mu_i}\right)_{T,V,\mu\neq\mu_i} = -\frac{\partial \mathcal{F}}{\partial \mu_i}, \quad i = I, S. \tag{6.5}$$

From Eq. (6.3) we get the energy density, $u = U/V$, at $T = 0$, is given by

$$u(\mu_I) = -p(\mu_i) + \sum\nolimits_i \mu_i n_i(\mu_i), \tag{6.6}$$

6.2 Leading order analysis

The leading order contribution to free energy is given by the static Lagrangian, to first order in Weinberg's power counting scheme. We start by assuming $e = 0$, i.e., no electromagnetic interactions. We thus get the leading order contribution to the free energy density from Eq. (5.57), and it is

$$\mathcal{F} = -\frac{1}{2} f^2 \left(\mu_I^2 \sin^2\alpha + 2\bar{m}\cos\alpha + m_S\right). \tag{6.7}$$

The parameter α is determined by minimizing $\mathcal{F}$ for a given value of μ_I,

$$\frac{\partial \mathcal{F}}{\partial \alpha} = f^2 \left(\bar{m}^2 - \mu_I^2 \cos\alpha\right) \sin\alpha = 0. \tag{6.8}$$

This gives an explicit formula for α in terms of μ_I. As long as the chemical potential is lower than the critical value $\mu_I^c = \bar{m}$, the only solution to this equation is $\alpha = 0$. As the chemical potential reaches this critical value, the system undergoes a phase transition from the vacuum phase to the *pion condensate* phase. In this new phase, the solution is

$$\cos\alpha = \frac{\bar{m}^2}{\mu_I^2}. \tag{6.9}$$

We introduce a dimensionless variable $x^2 = \cos\alpha = \bar{m}^2/\mu_I^2$. This variable has the domain $[0, 1]$, and $\cos\alpha = x^2$ implies that $\sin^2\alpha = 1 - x^4$. Substituting the dimensionless variable into the free energy density, we get

$$\mathcal{F} = -\frac{u_0}{2}\left(x^2 + \frac{1}{x^2}\right) + \text{const.} \tag{6.10}$$

We have introduced the characteristic energy density $u_0 = \bar{m}^2 f^2$. As we found in the last section, the pressure is given by negative the free energy density. We normalized the pressure to $\mu_I = \bar{m}$, or $x = 1$, and choose $p_0 = u_0$, so the dimensionless pressure is

$$\tilde{p} = -\frac{1}{p_0}\left(\mathcal{F} - \mathcal{F}_{x=1}\right) = \frac{1}{2}\left(x - \frac{1}{x}\right)^2. \tag{6.11}$$

The charge density corresponding to a chemical potential is given by minus the derivative of the free energy with respect to that chemical potential. We must, however, not assume any dependence of α on μ_I when taking this derivative. The isospin density is

$$n_I = -\frac{\partial \mathcal{F}}{\partial \mu_I} = f^2 \mu_I \sin^2\alpha = \frac{u_0}{\mu_I}\left(\frac{1}{x^2} - x^2\right), \tag{6.12}$$

while the strangeness is zero. With this, the dimensionless energy density at $T = 0$ is

$$\tilde{u} = -\tilde{p} + \frac{\mu_I n_I}{u_0} = \frac{1}{2}\left(2 + \frac{1}{x^2} - 3x^2\right). \tag{6.13}$$

The ratio of pressure to energy density is

$$\frac{p}{u} = \frac{1 - x^2}{1 + 3x^2}, \tag{6.14}$$

which matches earlier results [45]. In the ultrarelativistic limit, where $\mu_I \to \infty$ and thus $x \to 0$, we get $p/u = 1$, or $u_{\text{ur}} = p$. The non-relativistic limit is $\mu_I^2 = m_\pi^2(1 + \epsilon)$ and thus $x^{-2} = 1 + \epsilon$, $\epsilon \ll 1$. With this we get $\tilde{p} = \epsilon^2/2$, and $\tilde{u} = 2\epsilon$, so the equation of state is $\tilde{u}_{\text{nr}} = \sqrt{8}\sqrt{\tilde{p}}$. The isospin density, and thus the pion number density, is $n_I = 2\frac{u_0}{\bar{m}}\epsilon$, and we can therefore write the energy density in this limit as $u = \bar{m} n_I + \mathcal{O}(\epsilon^2)$. The energy density is thus dominated by the rest mass as $\epsilon \to 0$, as we expect from the non-relativistic limit. Figure 6.1 shows the equation of state in two different regimes and compares it with the ultrarelativistic and non-relativistic limits.

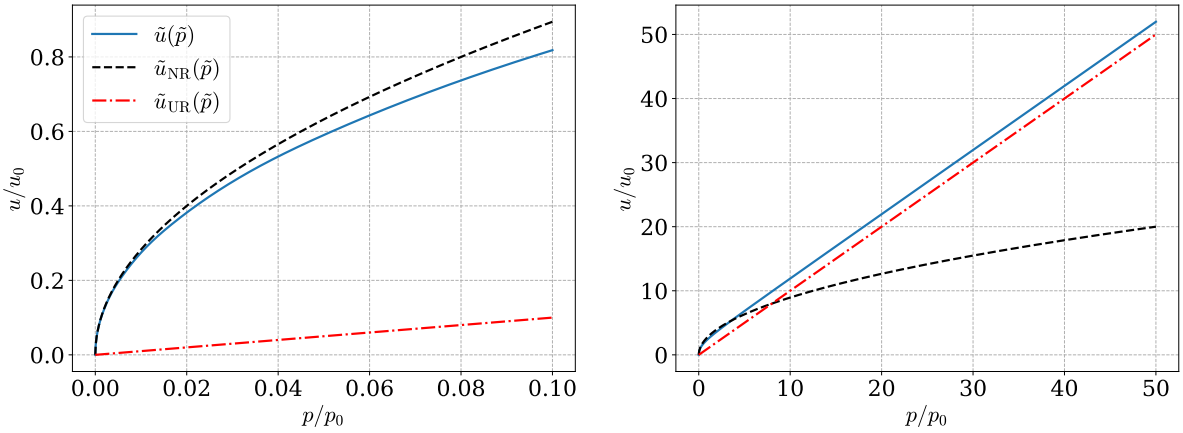


Figure 6.1: The leading order equation of state from two-flavor chiral perturbation theory, in two different regimes. The full equation is shown as a solid line and is compared to the ultrarelativistic and non-relativistic limits shown as dashed lines. The y-axis shows the energy density normalized to u_0, x-axis shows the pressure normalized to p_0. We have chosen $p_0 = u_0$.

6.2.1 Including electromagnetism

From Eq. (5.110), the free energy density, including electromagnetic interactions, is

$$\mathcal{F} = -\frac{1}{2} f^2 \left(\mu_I^2 \sin^2\alpha + 2\bar{m}^2 \cos\alpha + \Delta m_{\rm EM}^2 \cos^2\alpha - \frac{1}{3}\Delta m_{\rm EM}^2 + m_S^2 \right). \tag{6.15}$$

Free energy minimization now gives

$$\frac{1}{u_0}\frac{\partial \mathcal{F}}{\partial \alpha} = \left[\left(\frac{1}{x^2} - \Delta \right) \cos\alpha - 1 \right] \sin\alpha = 0. \tag{6.16}$$

Here, x is defined as before, and we introduced the new quantity $\Delta = \Delta m_{\rm EM}^2/\bar{m}^2 = 0.06916$. We see that the phase transition is raised, the critical chemical potential is now $\mu_I^c = \bar{m}\sqrt{1+\Delta}$, the mass of the charged pions. Below this value, $\alpha = 0$ remains the only solution. In the pion condensate phase, the solution is

$$\cos\alpha = \frac{\bar{m}^2}{\mu_I^2 - \Delta m_{\rm EM}^2} = \frac{1}{\frac{1}{x^2} - \Delta}. \tag{6.17}$$

This reduces to our old solution for $\Delta = 0$, as it should. With the same procedure as in the last section, we get the pressure and energy density

$$\tilde{p}_{\rm EM} = \frac{1}{2}\left(\frac{1}{x^2} - \Delta + \frac{x^2}{1 - x^2\Delta} - 2 \right), \tag{6.18}$$

$$\tilde{u}_{\rm EM} = \frac{1}{2}\left(\frac{1}{x^2} - x^2 \frac{3 - x^2\Delta}{(1 - x^2\Delta)^2} + 2 + \Delta \right). \tag{6.19}$$

The ratio between pressure and energy is now

$$\frac{p_{\rm EM}}{u_{\rm EM}} = \frac{1 - (2\Delta+1)x^2 + \Delta(\Delta+1)x^4}{1 + 3x^2 - \Delta(\Delta+1)x^4}. \tag{6.20}$$

In the limit $\Delta = 0$, these results reduce to those we found in the last section. In the ultra-relativistic limit, that is, for $x \ll 1$, the behavior is the same as before, and we again approach $p = u$. As the mass of the charged pions have changed, the non-relativistic limit is now obtained by substituting $x^{-2} = 1 + \Delta + \epsilon$, for $\epsilon \ll 1$. To first order in ϵ we get $\tilde{p} = \epsilon/2$, which is the same as before. However, the energy density limit is perturbed by the inclusion of electromagnetism and is now $\tilde{u} = 2(1+\Delta)\epsilon$. The non-relativistic equation of state is thus still a polytrope of the form $\tilde{p} = K\tilde{u}^2$, however the constant is now $K^{-1} = 8(1+\Delta)^2$.

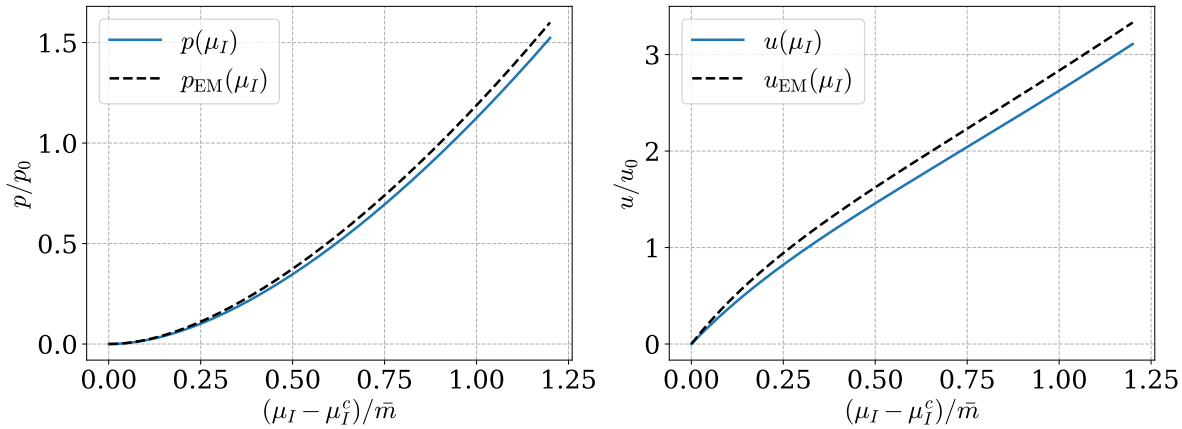


Figure 6.2: Left: The pressure, normalized to p_0, as a function of the chemical potential above the critical value, normalized to $\bar{m}$. Right: The energy density, normalized to u_0, also as a function of the chemical potential. Results with electromagnetic interaction are shown as dashed lines. The y-axis corresponds to different absolute values of isospin-chemical potential, as the critical value of the chemical is changed by the inclusion of electromagnetic interactions. See main text for details.

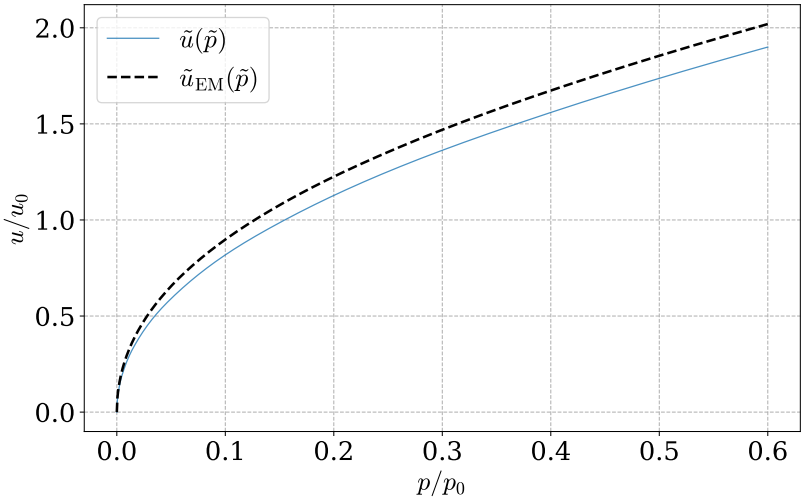


Figure 6.3: The equation of state in the pion condensate phase. Results with electromagnetic interactions are shown as dashed lines. The energy density and pressure is normalized to u_0 and $p_0 = u_0$.

6.3 Phase transitions

As the static Lagrangian has the same form in the kaon condensed phase as in the pion condensed phase, the analysis of the phase transition and α as a function of $\mu_{K^\pm}$ will be the same as in the pion condensate. The system will be in the phase whose static Lagrangian minimizes the free energy, or equivalently, maximizes the pressure. Therefore, we can find the transition line between the condensates by setting the pressure of the two condensates equal. Using Eq. (6.11), and the similar expression in the kaon condensed phase, we get

$$p = \frac{1}{2} f^2 m_{K^\pm}^2 \left(\frac{\mu_{K^\pm}}{m_{K^\pm}} - \frac{m_{K^\pm}}{\mu_{K^\pm}} \right)^2 = \frac{1}{2} f^2 m_\pi^2 \left(\frac{\mu_I}{m_\pi} - \frac{m_\pi}{\mu_I} \right)^2 . \tag{6.21}$$

Solving for $\mu_{K^\pm}$, we get

$$\mu_{K^\pm} = \frac{1}{2\mu_I} \left(\mu_I^2 - m_\pi^2 + \sqrt{(\mu_I^2 - m_\pi^2)^2 + 4\mu_I^2 m_{K^\pm}^2} \right) . \tag{6.22}$$

The transitions into the condensates are at $\mu_I = m_\pi$, and $\mu_{K^\pm} = m_{K^\pm}$. This point, $(\mu_I, \mu_S) = (m_\pi, m_{K^\pm} - m_\pi/2)$, satisfies Eq. (6.22), and is thus a triple point between the vacuum phase, $\pi^\pm$ condensate and the $K^\pm$ condensate. Similarly, the line between the charged and neutral kaon condensed phase is defined by

$$\frac{1}{2}f^2 m_{K^\pm}^2 \left(\frac{\mu_{K^\pm}}{m_{K^\pm}} - \frac{m_{K^\pm}}{\mu_{K^\pm}}\right)^2 = \frac{1}{2}f^2 m_{K^0}^2 \left(\frac{\mu_{K^0}}{m_{K^0}} - \frac{m_{K^0}}{\mu_{K^0}}\right)^2. \tag{6.23}$$

The solution is

$$2\mu_{K^\pm} = \frac{1}{\mu_{K^0}}\left(\mu_{K^0}^2 - m_{K^0}^2 + \sqrt{(\mu_{K^0}^2 - m_{K^0}^2)^2 + 4\mu_{K^0}^2 m_{K^\pm}^2}\right). \tag{6.24}$$

We can expand this to first order in the difference in mass between the kaons, $m_{K^0}^2 - m_{K^\pm}^2 = \Delta m^2$, which gives

$$\begin{aligned}\mu_{K^\pm} &= \frac{1}{\mu_{K^0}}\left(\mu_{K^0}^2 - m_{K^0}^2 + \sqrt{(\mu_{K^0}^2 + m_{K^0}^2)^2 + 4\mu_{K^0}^2 \Delta m^2}\right) \\ &= \mu_{K^0}\left(1 + \frac{\Delta m^2}{\mu_{K^0}^2 + m_{K^0}^2}\right) + \mathcal{O}\left(\frac{\Delta m^4}{(\mu_{K^0}^2 + m_{K^0}^2)^2}\right)\end{aligned} \tag{6.25}$$

This is an excellent approximation, as $\Delta m^4/4m_{K^0}^4 \approx 4\times10^{-4}$. The triple point is in this case at $\mu_{K^\pm} = m_{K^\pm}$, $\mu_{K^0} = m_{K^0}$ or $(\mu_I, \mu_S) = (m_{K^\pm} - m_{K^0}, [m_{K^\pm} + m_{K^0}]/2)$. If we consider $\Delta m = 0$, then this reduces to $\mu_{K^\pm} = \mu_{K^0}$, or $\mu_S = 0$, as obtained in [38]. A similar analysis gives the transition line between all phases. Figure 6.4 shows the phase diagram in the $\mu_I - \mu_S$-plane.

In the pion phase, the isospin and strangeness densities are

$$n_I = -\frac{\partial \mathcal{F}}{\partial \mu_I} = f_\pi^2 \mu_I \left(1 - \frac{m_\pi^4}{\mu_I^4}\right), \quad n_S = -\frac{\partial \mathcal{F}}{\partial \mu_S} = 0. \tag{6.26}$$

In the charged kaon condensed phase, they are

$$n_I = -\frac{\partial \mathcal{F}}{\partial \mu_I} = 2\frac{\partial \mathcal{F}}{\partial \mu_{K^\pm}} = 2f_\pi^2 \mu_{K^\pm}\left(1 - \frac{m_{K^\pm}^4}{\mu_{K^\pm}^4}\right), \quad n_S = -\frac{\partial \mathcal{F}}{\partial \mu_S} = f_\pi^2 \mu_{K^\pm}\left(1 - \frac{m_{K^\pm}^4}{\mu_{K^\pm}^4}\right). \tag{6.27}$$

At the line of phase transition, $\mu_{K^\pm} > m_{K^\pm}$. The strangeness density thus jumps discontinuously to a non-zero value. This phase transition is, therefore, of first order. In the neutral kaon condensed phase, the isospin density is

$$n_I = -\frac{\partial \mathcal{F}}{\partial \mu_I} = -2\frac{\partial \mathcal{F}}{\partial \mu_{K^\pm}} = -2f_\pi^2 \mu_{K^\pm}\left(1 - \frac{m_{K^\pm}^4}{\mu_{K^\pm}^4}\right). \tag{6.28}$$

This, too, will change discontinuously between the two kaon condensed phases. Similar arguments hold between all condensates. This indicates that the transitions between condensates are of *first order*.

6.3.1 Electromagnetic contribution

We can describe the effect of electromagnetic interactions on the pressure, as obtained subsection 6.2.1, by modifying the isospin chemical potential by $\mu_I^2 \to \mu_I^2 - \Delta m_{\text{EM}}^2$. From the structure of the Lagrangian Eq. (5.111), the same will happen in the case of the charged kaon, only the change will be $\mu_{K^\pm}^2 \to \mu_{K^\pm}^2 - \Delta m_{\text{EM}}^2$, while from Eq. (5.97) we see that the results will be unchanged by electromagnetic interactions. We define the effective chemical potential by $\mu'^2 = \mu^2 - \Delta m_{\text{EM}}^2$. The transitions between the vacuum phase and the condensed phase will now be $\mu_I' = \bar{m}$ and $\mu_{K^\pm}' = m_{K^\pm}$, while the line between these condensed phases is now

$$m_{K^\pm}^2 \left(\frac{\mu_{K^\pm}'}{m_{K^\pm}} - \frac{m_{K^\pm}}{\mu_{K^\pm}'}\right)^2 = m_\pi^2 \left(\frac{\mu_I'}{m_\pi} - \frac{m_\pi}{\mu_I'}\right)^2, \tag{6.29}$$

The neutral kaon condensate, on the other hand, remains unchanged by the inclusion of electromagnetic interactions. The line between the kaon condensates is thus given by

$$m_{K^\pm}^2 \left(\frac{\mu_{K^\pm}'}{m_{K^\pm}} - \frac{m_{K^\pm}}{\mu_{K^\pm}'^2}\right)^2 = m_{K^0}^2 \left(\frac{\mu_{K^0}}{m_{K^0}} - \frac{m_{K^0}}{\mu_{K^0}^2}\right)^2. \tag{6.30}$$

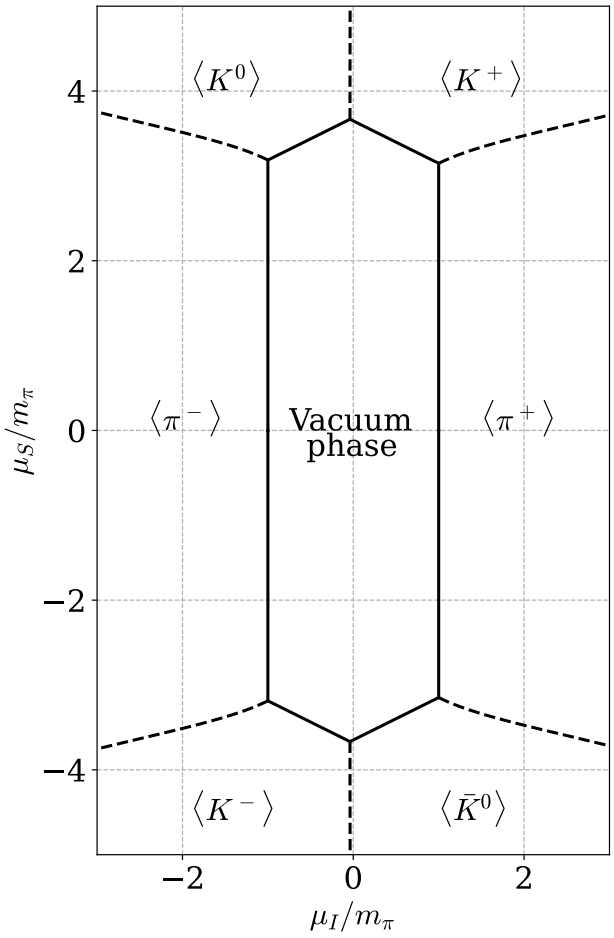


Figure 6.4: Phase diagram of χPT in the $\mu_I - \mu_S$-plane. Chemical potentials are given in units of then pion mass. The expectation values in each region indicate which particles form a condensate. The dashed lines are first-order phase transitions between the condensates, while the solid line indicates the second-order phase transition from the vacuum phase of the vacuum to the condensates.

The new phase diagram is shown in Figure 6.5 in red, where it is compared with the results without electromagnetic interactions in black. The change is very small. However, it affects the pion condensation more than the charged kaon. This is because, as we discussed earlier, the *square* of the mass is the same by Dashen's theorem, Δm^2_{EM}, while the absolute shift will depend on the ratio between Δm_{EM} and the non-electromagnetic mass. This is thus lower for the heavier kaon. The line between the kaon condensates is moved slightly closer to $\mu_I = 0$, as the electromagnetic contribution to the lighter charged kaon mass reduces the difference between its and the neutral kaon's mass.

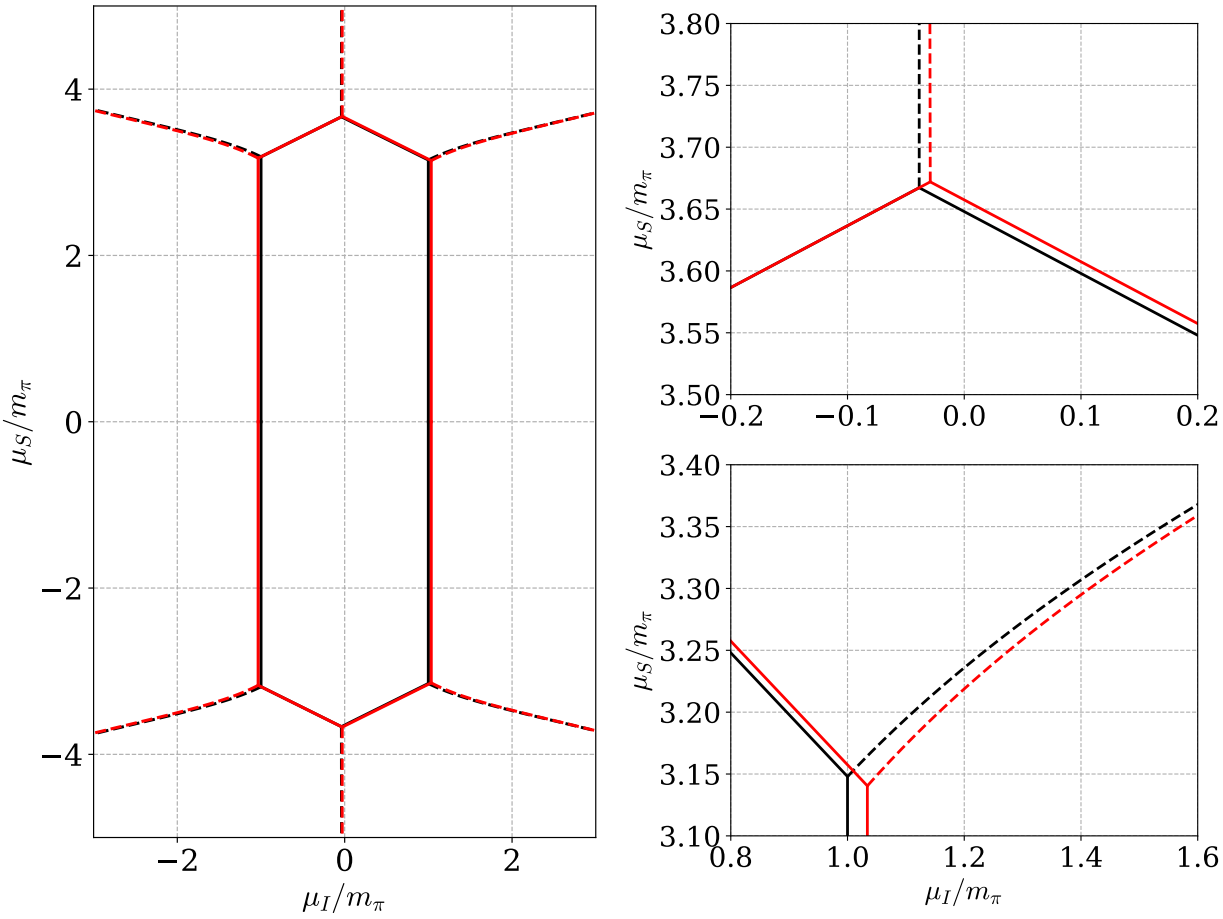


Figure 6.5: The phase diagram of χPT in the $\mu_I - \mu_S$-plane. The black lines are results without the electromagnetic interactions, while red lines are results including them. To the right, we have zoomed in on two of the triple points. At the top is the intersection of the normal, neutral kaon condensed and charged kaon condensed phases, while at the bottom is the intersection of the normal, pion condensed, and charged kaon condensed phases.

6.4 Electric charge neutrality

A pion condensate will have an electric charge. In the grand canonical ensemble, the QCD Lagrangian will have the term $\mu_Q \bar{q}\gamma^0 Q q$, where $\bar{q}\gamma^\mu Q q$ is the electric current density, Q the quark charge matrix Eq. (5.43), and $\mu_Q = \mu_I + 2\mu_S$ is the electric charge chemical potential. In the case of $\mu_S = 0$, the charge density is thus

$$n_Q = -e\frac{\partial \mathcal{F}}{\partial \mu_Q} = e n_I. \tag{6.31}$$

A realistic astrophysical object will not have a macroscopic electric charge. Therefore, we will model pion stars with the additional constraint of charge neutrality by including charged leptons in the form of muons or electrons. These leptons are free fermions, with an electric charge of $-e$. We may therefore use the results from section 4.4 and section C.5. The electric charge density of the leptons is given $-en_\ell$, where n_ℓ is the particle number and, in this case, the lepton number. In Eq. (4.89), we found the expression

$$n_\ell = \frac{8}{3}\frac{u_{\ell,0}}{m_\ell}x_f^3, \tag{6.32}$$

where $x_f = \sqrt{\mu_\ell^2/m_\ell^2 - 1}$ is the dimensionless Fermi momentum, m_ℓ the lepton mass, and μ_ℓ the lepton chemical potential. This formula is valid for $\mu_\ell \geq m_\ell$. We have introduced the characteristic energy density

of leptons,

$$u_{\ell,0} = \frac{m_\ell^4}{3\pi^2}. \tag{6.33}$$

The criterion for charge neutrality is then

$$n_I = n_\ell. \tag{6.34}$$

With this, we can determine the lepton chemical potential as a function of the isospin chemical potential, $\mu_\ell = \mu_\ell(\mu_I)$. The leading order result for the isospin density of the pion condensate is given in Eq. (6.12). Inserting the expressions for these densities into Eq. (6.34), we get

$$A\left(\frac{\mu_\ell^2}{m_\ell^2} - 1\right)^{3/2} = \frac{\mu_I}{m_\pi}\left(1 - \frac{m_\pi^4}{\mu_I^4}\right). \tag{6.35}$$

Both densities vanish at the point $(\mu_I, \mu_\ell) = (m_\pi, m_\ell)$, which we have seen earlier is the point where the pressure and energy density of both the Fermi gas and the pion condensate vanish. We have introduced the dimensionless constant

$$A = \frac{3}{8}\frac{m_\pi}{m_\ell}\frac{u_{0,\ell}}{u_0} = \frac{1}{8\pi^2}\frac{m_\ell^3}{m_\pi f_\pi^2}. \tag{6.36}$$

Setting the lepton mass equal the electron mass or muon mass gives, respectively, $A = 1.1 \times 10^{-8}$ and $A = 4.4 \times 10^2$. In this case, we can write the expression for $\mu_\ell(\mu_I)$ as an explicit function,

$$\frac{\mu_\ell}{m_\ell} = \sqrt{1 + A^{-2/3}\left[\frac{\mu_I}{m_\pi}\left(1 - \frac{m_\pi^4}{\mu_I^4}\right)\right]^{2/3}}. \tag{6.37}$$

These relationships are illustrated in Figure 6.6. The plot on the left is the electron chemical potential as a function of isospin chemical potential, both normalized to the masses of their corresponding particles. We see that the value of the electron chemical potential as a fraction of the particle mass is far greater than that of the pion condensate, as it is suppressed by the A constant in Eq. (6.35). The muon chemical potential, shown on the right and normalized to the muon mass, is promoted by A and thus much smaller than the isospin chemical potential.

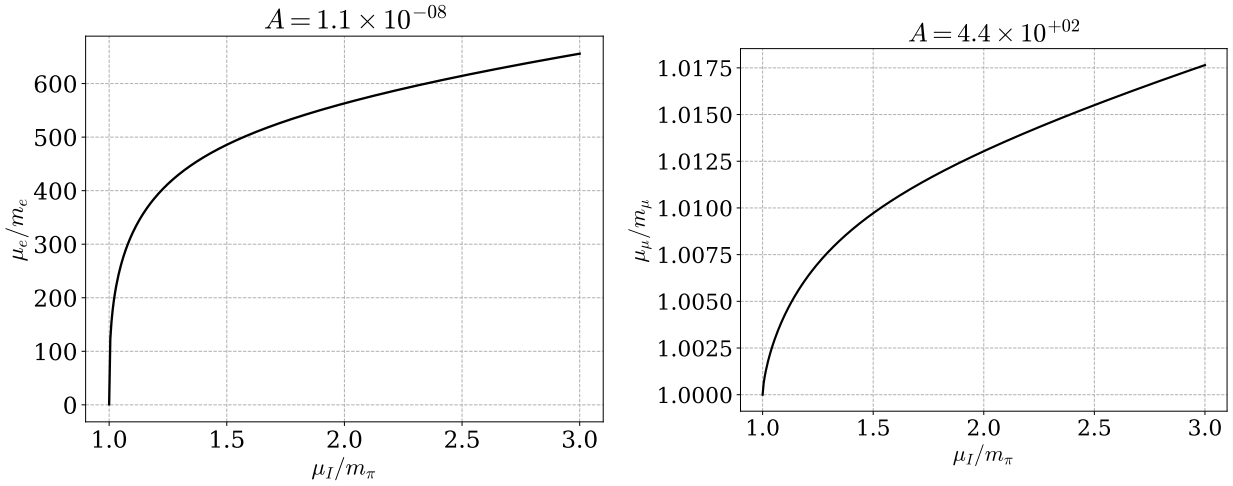


Figure 6.6: The lepton chemical potentials as functions of the isospin chemical potential normalized to their respective particles mass. The electron chemical potential is shown to the left, the muon to the right.

The total pressure and energy density will now be the sum of the contributions from the pion condensate and the leptons. The lepton contribution to these, which we found in Eq. (4.90) and Eq. (4.91), is

$$u_\ell = u_{\ell,0}\left[(2x_f^3 + x_f)\sqrt{1 + x_f^2} - \operatorname{arcsinh}(x_f)\right], \tag{6.38}$$

$$p_\ell = \frac{1}{3}u_{\ell,0}\left[(2x_f^3 - 3x_f)\sqrt{1 + x_f^2} + 3\operatorname{arcsinh}(x_f)\right]. \tag{6.39}$$

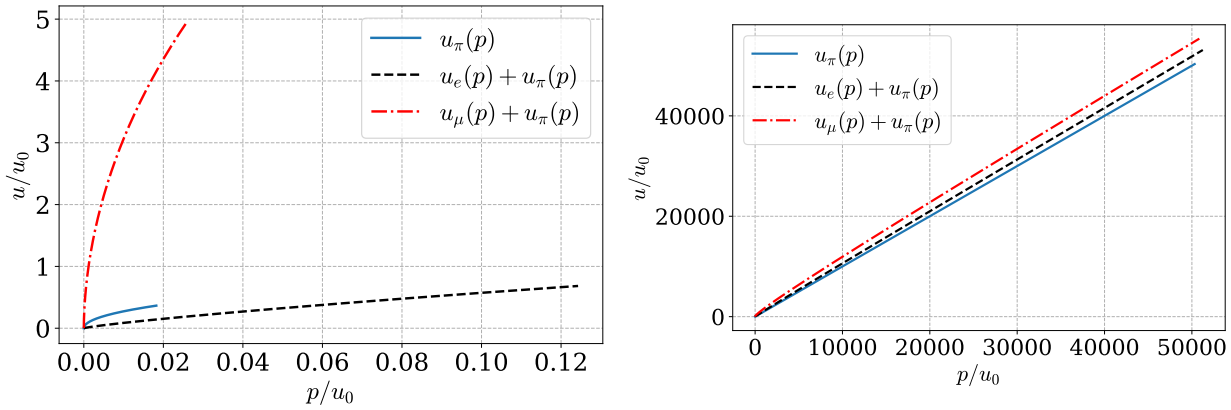


Figure 6.7: The equation of state including a lepton is compared with the equation of state of only the pion condensate in two different regimes. The pressure and energy density is normalized to $u_0 = f_\pi^2 m_\pi^2$.

The contribution from the pion condensate is, as we found in Eq. (6.11) and Eq. (6.13),

$$u_\pi = \frac{1}{2}u_0\left(\frac{\mu_I}{m_\pi} - \frac{m_\pi}{\mu_I}\right)^2 \tag{6.40}$$

$$p_\pi = \frac{1}{2}u_0\left(2 + \frac{\mu_I^2}{m_\pi^2} - 3\frac{m_\pi^2}{\mu_I^2}\right) \tag{6.41}$$

This leads to a total pressure and energy

$$p = p_\pi + p_\ell, \quad u = u_\pi + u_\ell. \tag{6.42}$$

As Eq. (6.34) gives μ_ℓ as a function of μ_I, these are both parametrized by the isospin chemical potential, and we can extract the equation of state $u = u(p)$. The full equation of state in two different regimes is shown in Figure 6.7. The plot on the left is the low-pressure regime. As the light electron almost immediately enters the ultrarelativistic regime, defined by $\mu_e \gg m_e$, it will contribute mainly to the pressure. On the other hand, the heavy muon is non-relativistic and thus contributes mainly to the energy density due to its rest mass. On the right, we see that the ultrarelativistic limit is dominated by the pion contribution.

We can study the limit of the combined system by again letting $\mu_I^2/m_\pi^2 = 1+\epsilon$. Inserting this into Eq. (6.37) and expanding to first order in ϵ, we get $\mu_\ell = 1 + (2A^{-1}\epsilon)^{2/3}$. This is equivalent to $x_f = (2A^{-1}\epsilon)^{1/3}$. In section 4.4, we found the non-relativistic limit of the pressure and energy of the Fermi gas, i.e., the lowest order contribution in x_f, as $x_f \to 0$. Inserting this new result into these limits, we get the leading low energy limits of the pressure and energy,

$$u_{\ell,\mathrm{nr}} = \frac{8}{3}\frac{2}{A}u_{\ell,0}\epsilon, \quad p_{\ell,\mathrm{nr}} = \frac{8}{15}\left(\frac{2}{A}\right)^{5/3}u_{\ell,0}\epsilon^{5/3}. \tag{6.43}$$

From section 6.2, we have the equivalent expressions for the pion condensate,

$$u_{\pi,\mathrm{nr}} = 2u_0\epsilon, \quad p_{\pi,\mathrm{nr}} = \frac{1}{2}u_0\epsilon^2. \tag{6.44}$$

As we see, the energy density of the pion condensate and the leptons are of the same order, and both will therefore contribute to the leading order energy density. However, the lepton pressure is of a lower order, and *only* this will contribute to the leading order pressure. At low enough isospin chemical potential, then, the leading order behavior of the combined system is

$$u_{\mathrm{nr}} = 2u_0\left(1 + \frac{8}{3}\frac{u_{\ell,0}}{u_0}A^{-1}\right)\epsilon, \quad p_{\mathrm{nr}} = \frac{8}{15}u_{\ell,0}\left(\frac{2}{A}\right)^{5/3}\epsilon^{5/3}. \tag{6.45}$$

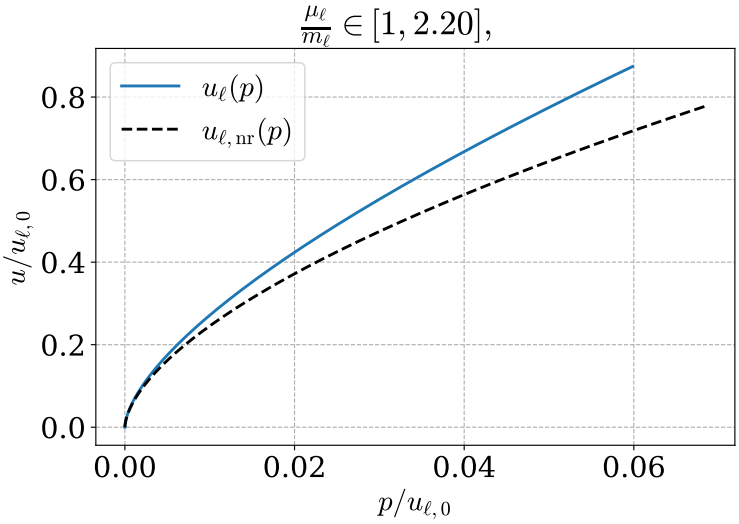


Figure 6.8: The equation of state of the lepton, compared with the non-relativisitic limit. Both energy density and pressure are normalized to the characteristic lepton density.

The equation of state is now a polytrope with $\gamma = \frac{5}{3}$, different from the $\gamma = 2$ polytrope of only the pion condensate. The equation of state of the lepton is compared with this limit in Figure 6.8 This figure is not dependent on the mass of the lepton.

In an intermediate range, however, the pressure of a heavy lepton will be suppressed by a factor $u_{\ell,0}/u_0 A^{-5/3} \propto (m_\pi^{1/3} f_\pi^{2/3})^5 (m_\pi f_\pi)^{-2} m_\ell^{-1}$, which for $m_\ell \gg m_\pi$ and $m_\ell \gg f_\pi$ is $\ll 1$, and the pion contribution might be dominant for a while. In this regime, the equation of state is still a polytrope with $\gamma = 2$, but the constant is changed due to the lepton contribution to the energy density. The pressure in the intermediate range is

$$p_i = \frac{1}{2} u_0 \epsilon^2, \tag{6.46}$$

and the equation of state is thus

$$p_i = K \left(\frac{u_{\mathrm{nr}}}{u_0} \right)^2, \quad K = \frac{1}{8} \left(1 + \frac{4}{3} \frac{u_{\ell,0}}{u_0} A^{-1} \right)^{-2}. \tag{6.47}$$

This is illustrated in Figure 6.9. In this figure, both the intermediate limit and the non-relativistic limit is compared to the full equation of state. On the top is the system with electrons, and as $m_e < m_\pi$, the intermediate limit has no validity. For $p/u_0 < 10^{-10}$, we see that the non-relativistic limit is very good. On the bottom, we see that the intermediate limit has a range of applicability, around $p/u_0 = 1$ to $p/u_0 = 1$ $p/u_0 = 10^{-3}$. The equation of state is then very well approximated by the non-relativistic limit around $p/u_0 < 10^{-5}$.

We can find the ultrarelativistic regime by letting $\mu_I^2/m_\pi^2 = y$, $y \gg 1$. From Eq. (6.37), we find that the lepton chemical potential to leading order in y is $\mu_\ell^2 \propto y^{1/3}$. In section 4.4, we found that the ultrarelativistic limit of the Fermi, both the pressure and energy is proportional to $x_f^4 \propto y^{2/3}$. In the case of the pion, however, both are proportional to $\mu_I^2 \propto y$. Therefore, the ultrarelativistic limit of the combined system is to leading order given by the ultrarelativistic limit of the pion condensate alone.

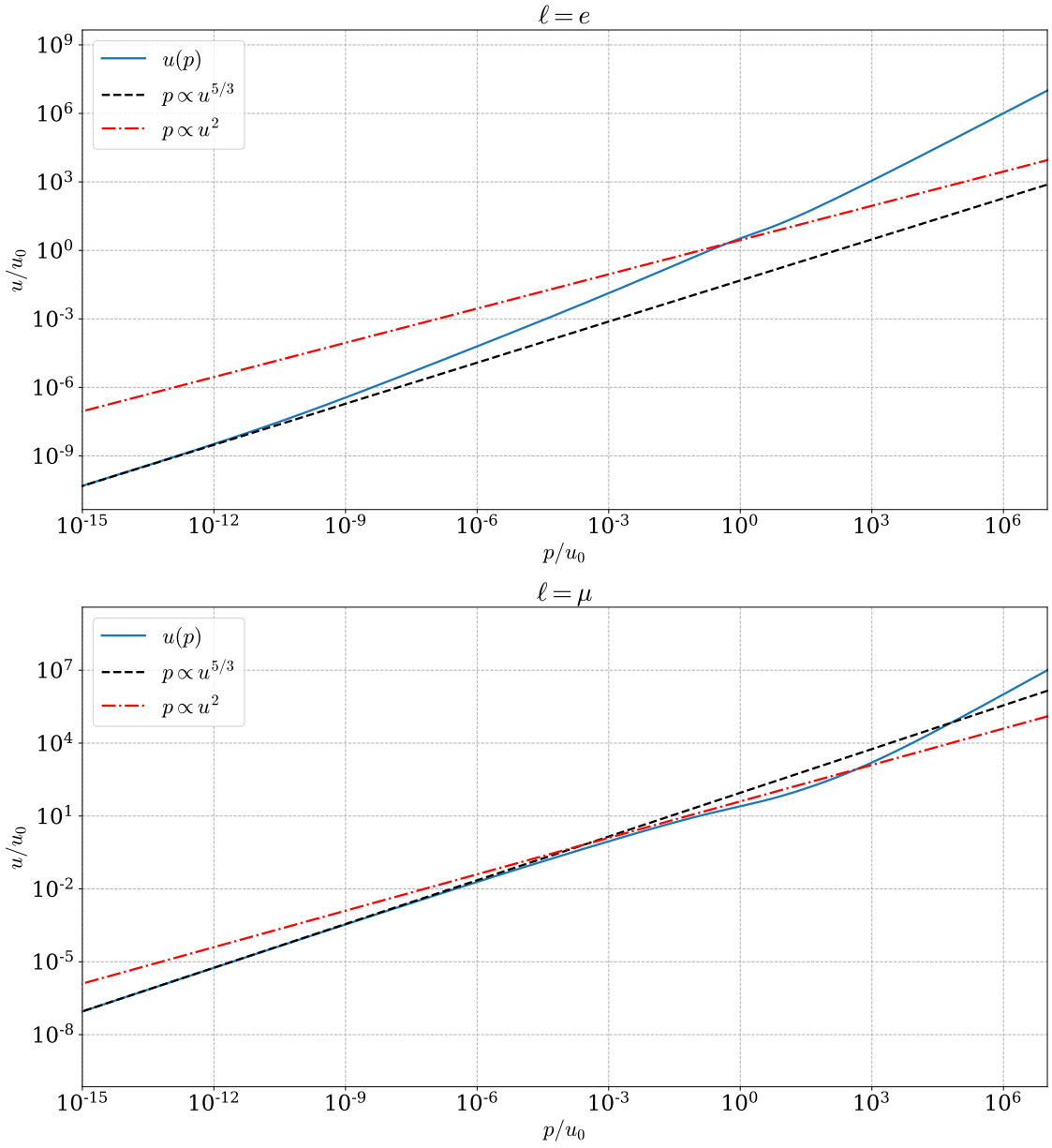


Figure 6.9: The full equation of state of the pion + lepton system, compared with intermediate and non-relativistic limit. On the top, the lepton is the electron, while on the bottom it is the muon. The full equation of state is compared to two different limits.

Chapter 7

Pion stars

We apply the results from chapter 6 to study the properties of pion stars.

7.1 Leading-order results

We start with the most simple model, using the leading order equation of state with $e = 0$. We can gain some insights by reviewing the units of the problem. The characteristic mass and length, as discussed in section 4.3, are found by setting $k_1 = k_2 = k_3 = 1$. These are the dimensionless constants of the TOV equation, Eq. (4.53). At leading order, the bare constants f and $\bar{m}$ are related to physical constants by $f = f_\pi$ and $m = m_\pi$, the pion decay constant and the pion mass. Using the values for f_π and m_π as given in section 1.1 and reinstating c and $\hbar$, these quantities are give in by

$$u_0 = m_\pi^2 f_\pi^2 \frac{c}{\hbar^3} = 3.216 \cdot 10^{33}\,\mathrm{J\,m^{-3}}, \tag{7.1}$$

$$m_0 = \frac{c^4}{\sqrt{\frac{4\pi}{3} u_0 G^3}} = 64.21\,M_\odot, \tag{7.2}$$

$$r_0 = \frac{G}{c^2} m_0 = 94.79\,\mathrm{km}. \tag{7.3}$$

We, therefore, expect both the radius and mass of the pion star to be around one order of magnitude larger than the star made up of cold neutrons.

Can we make a better argument by setting gravitational + internal energy equal 0?

7.1.1 Limiting radius

We found that the non-relativisitc limit of the equation of state is $\tilde{p} = 8^{-1}\tilde{u}^2$, i.e., it is a polytrope with $\gamma = 2$. As discussed in subsection 4.3.1, this corresponds to a situation where the radius of the star is independent of the central pressure, at least in the Newtonian limit of gravity. When simulating the Newtonian, non-relativistic limit of the pion star, we should expect the radius to be constant. From Eq. (4.59), the radius is $R = C\xi_1$, where

$$C = \frac{1}{\sqrt{4(4\pi)Gu_0}} = \frac{1}{\sqrt{12}} r_0, \tag{7.4}$$

and ξ_1 is the root of the Lane-Emden function $\theta(\xi)$ for polytrope index $n = 1$, the solution to

$$\theta'' + \frac{2}{\xi}\theta' + \theta = 0. \tag{7.5}$$

By substituting θ for its power series expansion, $\theta = \sum_n a_n \xi^n$, we get

$$\sum_n \left[(n+2)(n+1)a_{n+2} + 2(n+1)a_{n+1}\xi^{-1} + a_n\right] \xi^n = 0. \tag{7.6}$$

This must be obeyed for arbitrary ξ. We therefore get the recursion relation $a_{n+2} = -a_n/(n+1)(n+2)$. With our boundary condition, the solution is

$$\theta(\xi) = \frac{\sin(\xi)}{\xi}, \tag{7.7}$$

and the first root is therefore $\xi_1 = \pi$. With this, we get a closed-form expression for the stellar radius of this non-relativistic and Newtonian limit—which we expect the full theory to approach as the central pressure decreases—namely

$$R = \frac{\pi}{\sqrt{12}} r_0 = 85.97\,\text{km}. \tag{7.8}$$

7.1.2 Results

The code used for obtaining numerical results is discussed in Appendix D.

Figure 7.1 show the pressure and mass as a function of radius for varying values of central pressure. The quantities are normalized to the stellar radius, stellar mass, and central pressure, respectively. The black dashed line corresponds to the configuration with the maximum mass. We see that both the pressure and mass distribution are very similar for stars with a mass less than the maximum. As the central pressure increase beyond that of the star with maximum mass, the pressure gradient close to the center grows sharply. This is similar to what we saw in the case of an incompressible fluid, subsection 4.3.2.

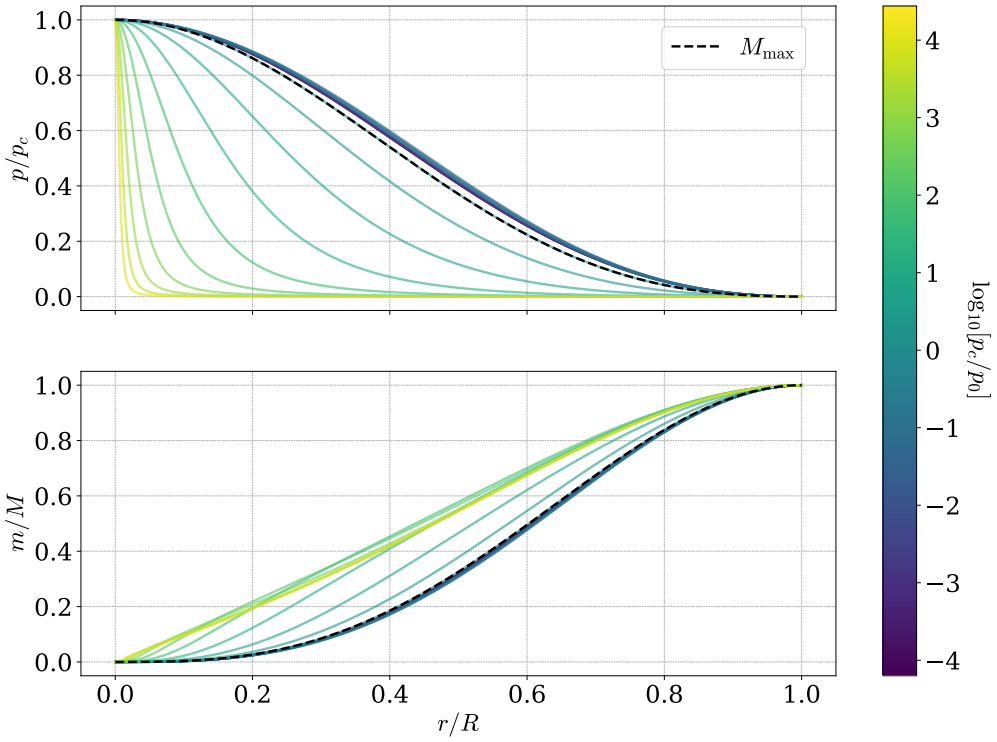


Figure 7.1: Top: The pressure normalized to the central pressure, as a function of radius, normalized to the stellar radius. Bottom: The mass, normalized to the stellar mass, within a radius r, normalized to the stellar radius. Both plots show a range of stars with different central pressures, indicated by the color. The black dashed line corresponds to the star with the largest mass.

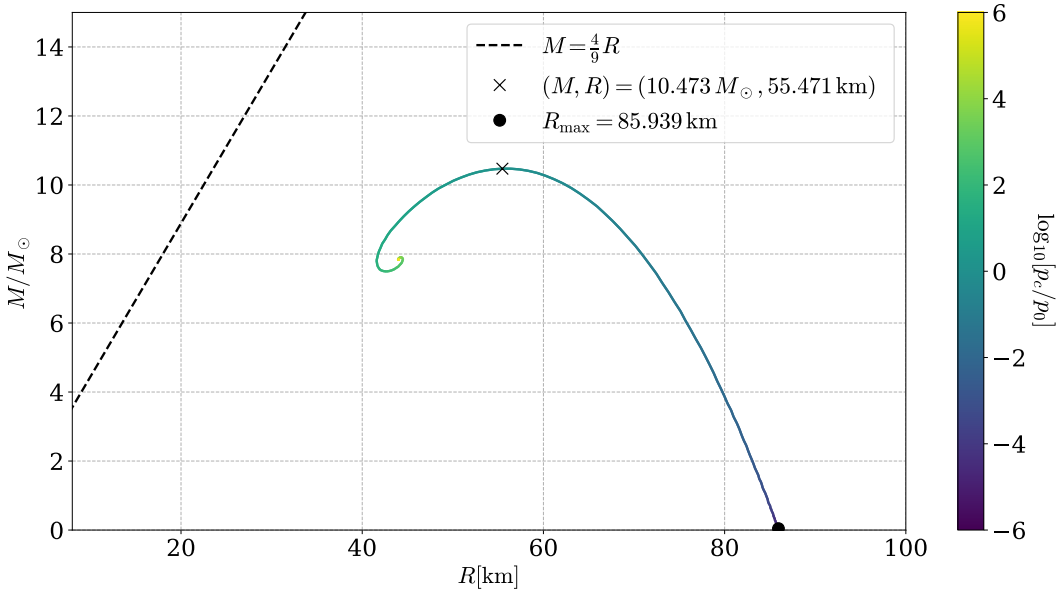


Figure 7.2: The lowest order mass-radius relation a pion star using two-flavor chiral perturbation theory. The mass is given in units of solar masses, while the radius is measured in kilometers. This line is parameterized by the central pressure p_c of the star, as indicated by the color gradient. The dashed black line indicates the theoretical maximum mass for a given radius, and any configuration above it will collapse to a black hole.

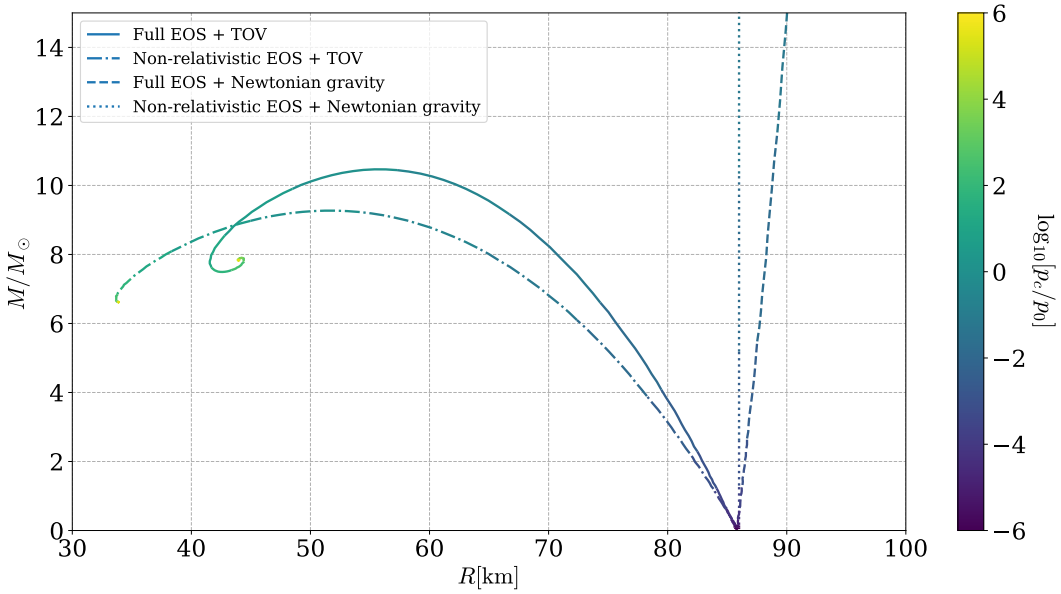


Figure 7.3: The mass-radius relationship of the pion star from the full, leading-order equation of state from two-flavor chiral perturbation and the TOV equation, compared with results in various limits.

Figure 7.2 shows the mass-radius relation for the pion star. As in the case of the neutron star, it has a maximum mass, in this case of $M_{\max} = 10.47\,M_\odot$. However, in contrast to the case of the neutron star, the stellar radius approaches a maximum radius as the central pressure decreases. This matches our expectation from the non-relativistic, Newtonian limit. We see that the largest radius in our results, corresponding to $p_c = 10^{-6}\,p_0$, is $R = 85.82\,\mathrm{km}$, which is in good agreement with our earlier analysis, Eq. (7.8).

Figure 7.3 compares the mass-radius relation from the full equation of state and TOV equation with various

limits. In the non-relativistic, Newtonian limit, the stellar radius is independent of the mass, as we found in our earlier analysis.

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7.1.3 Including electromagnetic contributions

With the results fromsubsection 6.2.1, the radius of the polytrope and the limiting radius of the full system changes and is now

$$R = \frac{\pi}{\sqrt{12}(1+\Delta)} r_0 = 80.40\,\text{km}. \tag{7.9}$$

Figure 6.2 shows the pressure and energy density, normalized to their characteristic quantities, as a function of chemical potential above the critical value, normalized to $\bar{m}$. Figure 6.3 shows the equation of state. The results with and without electromagnetic results are compared. We see that the inclusion of electromagnetic contributions results in a less stiff equation of state; a given pressure correspond to a higher energy density when including electromagnetic interactions.

Figure 7.4 shows the mass-radius reaction of the pion star when the electromagnetic interaction is taken into account. We see that the shape of the curve has not changed much from our earlier result. Both the maximum mass and radius are slightly smaller. The new result for maximum radius, $R_{\text{max}} = 80.35\,\text{km}$, is in excellent agreement with our expectation, Eq. (7.9). The result with and without electromagnetic interaction is compared in Figure 7.5. As discussed in section 4.4, we expect a stiffer equation of state to correspond to a more massive star, as happens in this case.

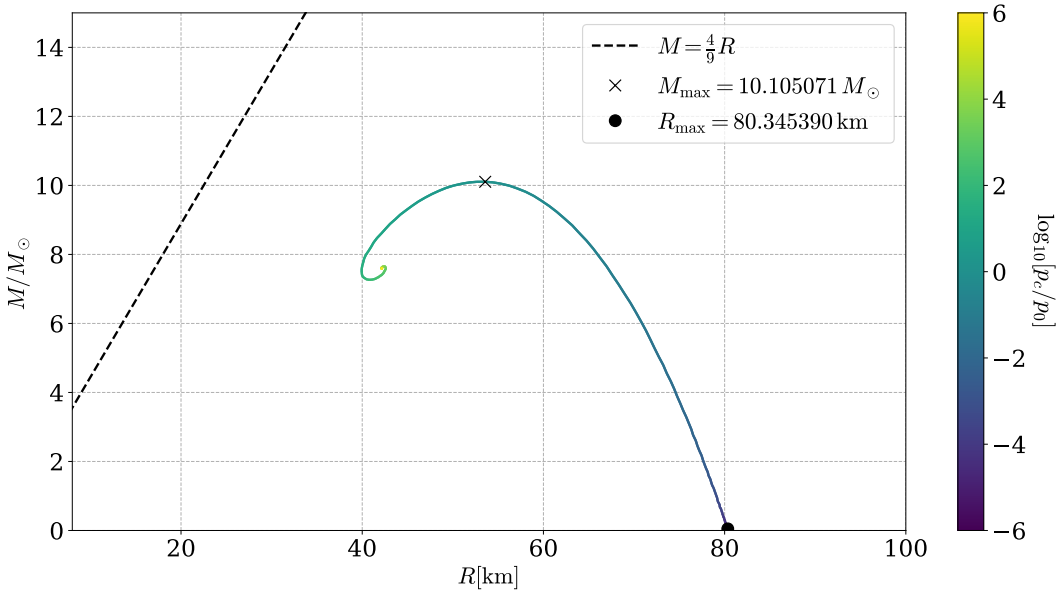


Figure 7.4: The mass-radius relation of a pion star including electromagnetic interactions, parameterized by the logarithm of the central pressure. The dashed line shows the absolute limiting mass for a given radius. The cross indicates the maximum mass configuration, and the dot the maximum radius configuration. The mass is given in units of solar masses, while the radius is in kilometers.

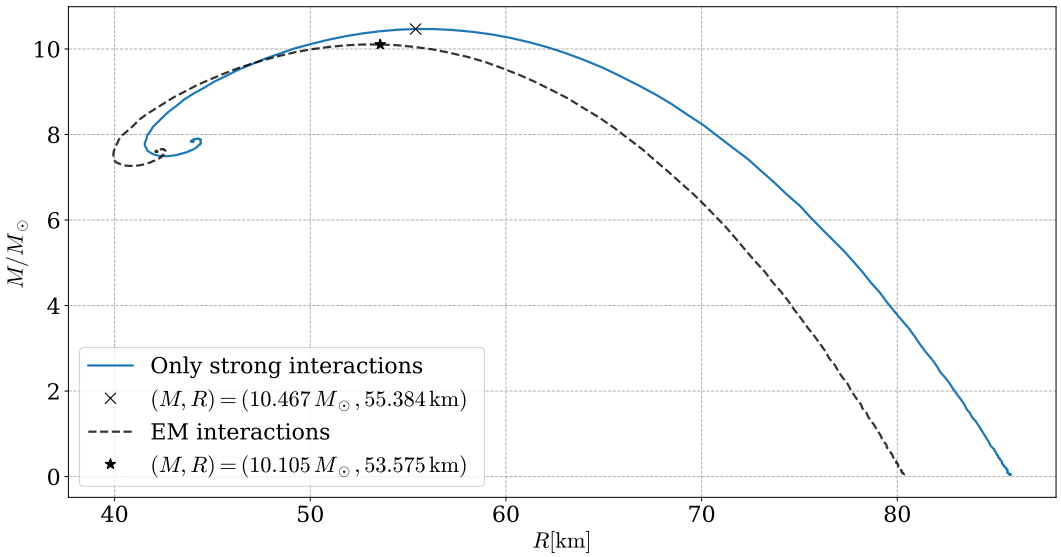


Figure 7.5: The mass-radius relation of pion stars with and without the effects of electromagnetism included. The radius is given in kilometers and the mass in units of the solar mass. The marked points are the maximum mass and corresponding radius of the stars.

7.1.4 Charge neutral stars

We now apply the results from section 6.4, where we added a lepton to enforce charge neutrality. The star with electrons is shown in Figure 7.6. We see that this star is much larger than those made of only pions. This is because the light electrons make the equation of state much stiffer. Furthermore, the non-relativisitic equation of state is now a polytrope with $\gamma = \frac{5}{3}$, instead of $\gamma = 2$, and there is therefore no upper mass limit. The maximum mass is now $146\,M_\odot$, and the corresponding maximum radius is 1.87×10^4 km.

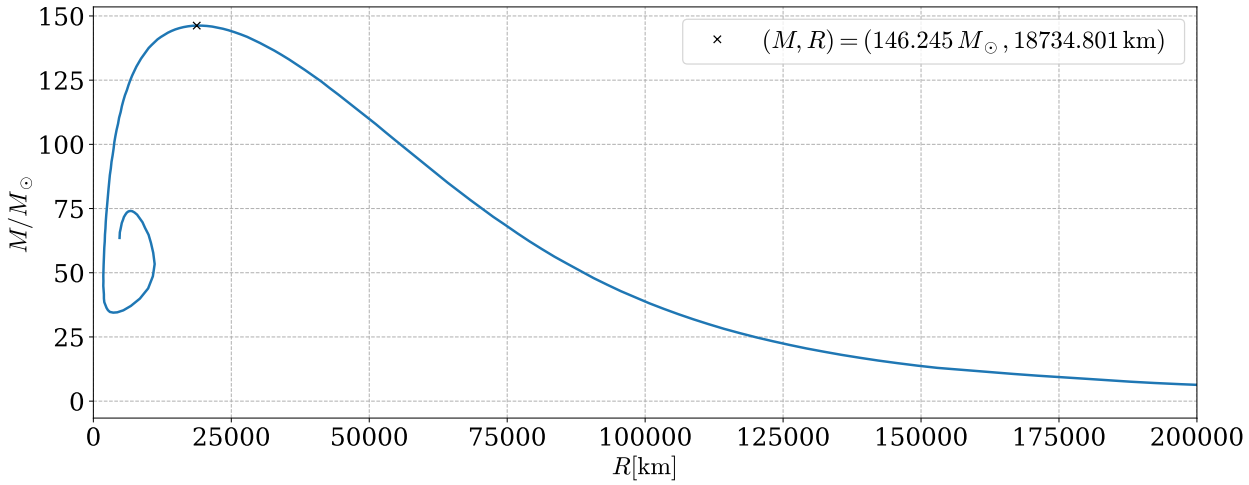


Figure 7.6: The mass-radius relation of pion stars with electrons to enforce charge neutrality. The radius is given in kilometers and the mass in units of solar masses.

The mass radius relation for a star where the lepton is a muon is shown in Figure 7.7. We see that the two different leptons affect the mass-radius relation in very different ways. The heavy muon results in a much *less* stiff equation of state, and thus smaller and lighter stars. In this case, the maximum mass is $1.08\,M_\odot$, and the corresponding maximum radius is 6.66 km. As in the case wit the electron, there is no upper limit to the radius as the central pressure decreases. This is because also here, the non-relativisitic limit is a polytrope with $\gamma = \frac{5}{3}$. However, as we found in section 6.4, there is an intermediate range where the equation of state behaves as a polytrope with $\gamma = 2$. Likewise, there is an intermediate range of the mass-radius relation, where it seems to approach a limiting mass, before the mass quickly start to grow again. We can get a rough estimate for this "seeming" limit, by using the polytrope constant of the intermediate limit, which is $K^{-1} = 8(1 + \Delta_\ell)^2$, where

$$\Delta_\ell = \frac{4}{3}\frac{u_{\ell,0}}{u_0}A^{-1}. \tag{7.10}$$

Following our earlier analysis, this leads to the limiting radius

$$R = \frac{\pi}{\sqrt{12}(1 + \Delta_\mu)} = 6.072\,\text{km}. \tag{7.11}$$

The results using both electrons and muons are compared to the results with only pions in Figure 7.8.

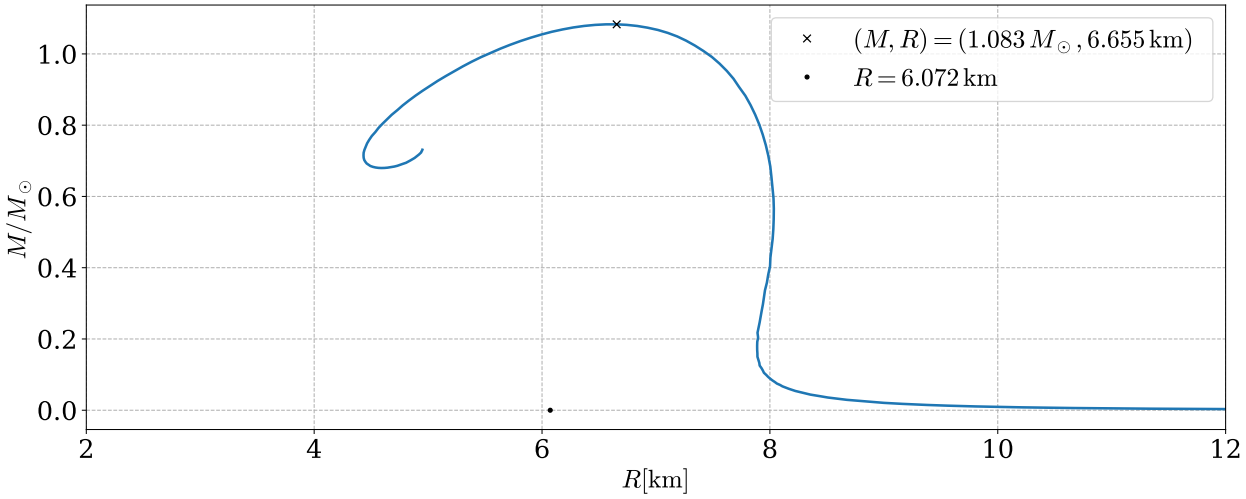


Figure 7.7: The mass-radius relation of pion stars, including leptons to enforce charge neutrality, is compared with pion stars of only pions. The radius is given in kilometers and the mass in units of solar masses.

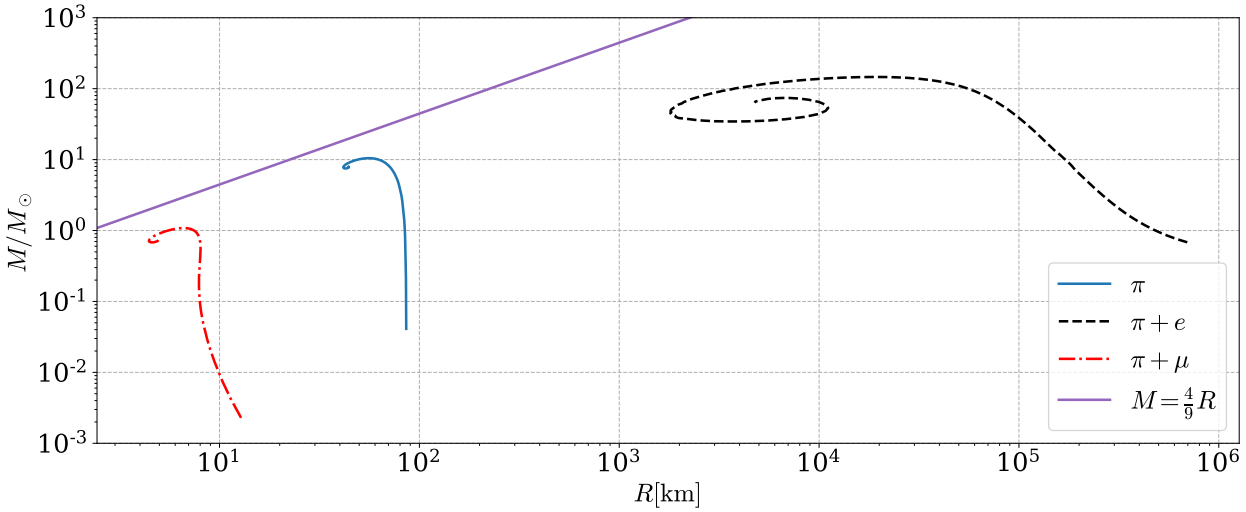


Figure 7.8: The mass-radius relation of pion stars, including leptons to enforce charge neutrality, is compared with pion stars of only pions. The radius is given in kilometers and the mass in units of solar masses.

Appendix A

A.1 Algebra bases

A.1.1 Pauli matrices

The Pauli matrices are

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.1}$$

They obey

$$[\tau_a, \tau_b] = 2i\varepsilon_{abc}\tau_c, \tag{A.2}$$
$$\{\tau_a, \tau_b\} = 2\delta_{ab}\mathbb{1}, \tag{A.3}$$
$$\mathrm{Tr}\{\tau_a\} = 0, \tag{A.4}$$
$$\mathrm{Tr}\{\tau_a\tau_b\} = 2\delta_{ab}, \tag{A.5}$$
$$\mathrm{Tr}\{\tau_a\tau_b\tau_c\tau_d\} = 2(\delta_{ab}\delta_{cd} - \delta_{ac}\delta_{cb} + \delta_{ad}\delta_{cb}). \tag{A.6}$$

Together with the identity matrix $\mathbb{1}$, the Pauli matrices form a basis for the vector space of all 2-by-2 matrices. An arbitrary 2-by-2 matrix M may be written

$$M = M_0\mathbb{1} + M_a\tau_a, \quad M_0 = \frac{1}{2}\mathrm{Tr}\{M\}, \; M_a = \frac{1}{2}\mathrm{Tr}\{\tau_a M\}. \tag{A.7}$$

A.1.2 Gell-Mann matrices

The Gell-Mann matrices are

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$
$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

They obey

$$[\lambda_a, \lambda_b] = 2if^{abc}\lambda_c, \tag{A.8}$$
$$\{\lambda_a, \lambda_b\} = \frac{4}{3}\mathbb{1}\delta_{ab} + 2d_{abc}\lambda_c, \tag{A.9}$$
$$\mathrm{Tr}\{\lambda_a\} = 0, \tag{A.10}$$
$$\mathrm{Tr}\{\lambda_a\lambda_b\} = 2\delta_{ab}, \tag{A.11}$$
$$\mathrm{Tr}\{\lambda_a\lambda_b\lambda_c\lambda_d\} = \frac{4}{3}\delta_{ab}\delta_{cd} + 2(d_{abe} + if_{abe})(d_{cde} + if_{cde}). \tag{A.12}$$

where

$$f_{abc} = -\frac{i}{4}\mathrm{Tr}\left\{\lambda_a[\lambda_b, \lambda_c]\right\}, \quad d_{abc} = -\frac{i}{4}\mathrm{Tr}\left\{\lambda_a\{\lambda_b, \lambda_c\}\right\}. \tag{A.13}$$

where the non-zero elements of f_{abc} and d_{abc} are

$$f_{123} = 1, \quad f_{147} = f_{246} = f_{257} = f_{345} = -f_{156} = -f_{367} = \frac{1}{2}, \quad f_{458} = f_{678} = \frac{\sqrt{3}}{2}, \tag{A.14}$$

$$d_{146} = d_{157} = d_{256} = -d_{247} = d_{355} = -d_{366} = -d_{377} = \frac{1}{2}$$

$$d_{118} = d_{228} = d_{338} = -d_{888} = \frac{1}{\sqrt{3}}, \quad d_{448} = d_{558} = d_{668} = d_{778} = -\frac{1}{2\sqrt{3}}, \tag{A.15}$$

or a permutation of the indices. The indices of f are totally antisymmetric, while those of d are totally symmetric [46].

A.1.3 Gamma matrices

The gamma matrices γ^μ, $\mu \in \{0, 1, 2, 3\}$, obey

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}, \tag{A.16}$$

$$\gamma^{0\dagger} = \gamma^0, \quad \gamma^{i\dagger} = -\gamma^i. \tag{A.17}$$

These matrices, together with

$$\sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\mu], \tag{A.18}$$

$$\gamma^\mu_A = \gamma^\mu\gamma^5, \tag{A.19}$$

$$\gamma^5 = \frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma, \tag{A.20}$$

form the Clifford algebra $\mathrm{Cl}_{1,3}$, also known as the *space-time algebra*. The subscripts $(1, 3)$ denotes the signature of the metric. The "fifth γ-matrix", which can be expressed as $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$, obey

$$\{\gamma^5, \gamma^\mu\} = 0, \quad (\gamma^5)^2 = \mathbb{1}. \tag{A.21}$$

The Euclidian counterpart of the space-time algebra, Cl_4, is defined by the "Euclidian gamma matrices", which obey

$$\{\tilde{\gamma}_a, \tilde{\gamma}_b\} = 2\delta_{ab}\mathbb{1}. \tag{A.22}$$

These can be related to the regular Minkowski-matrices by

$$\tilde{\gamma}_0 = \gamma^0, \quad \tilde{\gamma}_j = -i\gamma^j. \tag{A.23}$$

These then obey

$$\tilde{\gamma}_a^\dagger = \tilde{\gamma}_a. \tag{A.24}$$

The Euclidean $\tilde{\gamma}_5$ is defined as

$$\tilde{\gamma}_5 = \gamma_0\gamma_1\gamma_2\gamma_3 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5. \tag{A.25}$$

It thus also anti-commutes with the Euclidean γ-matrices,

$$\{\tilde{\gamma}_5, \tilde{\gamma}_a\} = 0. \tag{A.26}$$

A.2 Functionals

(TODO: INKLUDER KILER!!!!)

The principle of stationary action and the path integral method relies on functional calculus, where ordinary, n-dimensional calculus is generalized to an infinite-dimensional calculus on a space of functions. A functional, S, takes in a function $\varphi(x)$, and returns a real number $S[\varphi]$. We will be often be dealing with functionals of the form

$$S[\varphi] = \int_{\mathcal{M}} \mathrm{d}^n x \, \mathcal{L}[\varphi](x), \tag{A.27}$$

Here, $\mathcal{L}[\varphi](x)$, the Lagrangian density, is a functional which takes in a function φ, and returns a real number $\mathcal{L}[\varphi](x)$ *for each point* $x \in \mathcal{M}$. Thus, $\mathcal{L}$ does, in fact, return a real-valued function, not just a number. $\mathcal{M}$ is the manifold, in our case space-time, of which both $\varphi(x)$ and $\mathcal{L}[\varphi](x)$ are functions. The function φ can, in general, take on the value of a scalar, complex number, spinor, vector, etc…, while $\mathcal{L}[\varphi](x)$ must be a scalar-valued function. This strongly constraints the form of any Lagrangian and is an essential tool in constructing quantum field theories. Although this section is written with a single scalar-valued function φ, this can easily be generalized by adding an index, $\varphi \to \varphi_\alpha$, enumerating all the degrees of freedom, then restating the arguments [2, 5].

A.2.1 Functional derivative

The functional derivative is base on an arbitrary *variation* η of the function φ. The variation η, often written $\delta\varphi$ is an arbitrary function only constrained to vanish *quickly enough* at the boundary $\partial\mathcal{M}$.[1] The variation of the functional S is defined as

$$\delta_\eta S[\varphi] = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(S[\varphi + \epsilon\eta] - S[\varphi] \right) = \frac{d}{d\epsilon} S[\varphi + \epsilon\eta]|_{\epsilon=0}. \tag{A.28}$$

We can regard the variation of a functional as the generalization of the differential of a function, Eq. (2.18), as the best linear approximation around a point. In regular differential geometry, a function f can be approximated around a point x by

$$f(x + \epsilon v) = f(x) + \epsilon \mathrm{d}f(v), \tag{A.29}$$

where v is a vector in the tangent space at x. In functional calculus, the functional S is analogous to f, φ to x, and η to v. We can more clearly see the resemblance by writing

$$\frac{d}{d\epsilon} f(x + \epsilon v) = \mathrm{d}f(v) = \frac{\partial f}{\partial x^\mu} v^\mu. \tag{A.30}$$

In the last line we expanded the differential using the basis-representation, $v = v^\mu \partial_\mu$. To generalize this to functional, we define the *functional derivative*, by

$$\delta_\eta S[\varphi] = \int_{\mathcal{M}} \mathrm{d}^n x \, \frac{\delta S[\varphi]}{\delta \eta(x)} \eta(x). \tag{A.31}$$

If we let $S[\varphi] = \varphi(x)$, for some fixed x, the variation becomes

$$\delta_\eta S[\varphi] = \eta(x) = \int \mathrm{d}^n y \, \delta(x - y)\eta(y), \tag{A.32}$$

which leads to the identity

$$\frac{\delta\varphi(x)}{\delta\varphi(y)} = \delta(x - y). \tag{A.33}$$

There is also a generalized chain rule for functional derivatives. If ψ is some new functional variable, then

$$\frac{\delta S[\varphi]}{\delta\varphi(x)} = \int_{\mathcal{M}} \mathrm{d}^n y \, \frac{\delta S[\varphi]}{\delta\psi(y)} \frac{\delta\psi(y)}{\delta\varphi(x)}. \tag{A.34}$$

[1] The condition of "quickly enough" is to ensure that we can integrate by parts and ignore the boundary condition, which we will do without remorse.

Higher functional derivatives are defined in terms of higher-order variations,

$$\delta_\eta^m S[\varphi] = \frac{d}{d\epsilon}\delta_\eta^{m-1} S[\varphi+\epsilon\eta]|_{\epsilon=0} = \int_{\mathcal{M}} \left(\prod_{i=1}^{m} \mathrm{d}^n x_i \eta(x_i)\right) \frac{\delta^m S[\varphi]}{\delta\varphi(x_1)...\delta\varphi(x_m)}. \tag{A.35}$$

With this, we can write the functional Taylor expansion,

$$S[\varphi_0+\varphi] = S[\varphi_0] + \int_{\mathcal{M}} \mathrm{d}^n x\, \varphi(x) \frac{\delta S[\varphi_0]}{\delta\varphi(x)} + \frac{1}{2}\int_{\mathcal{M}} \mathrm{d}^n x \mathrm{d}^n y\, \varphi(x)\varphi(y) \frac{\delta^2 S[\varphi_0]}{\delta\varphi(x)\,\delta\varphi(y)} + \dots \tag{A.36}$$

Here, the notation $\frac{\delta S[\varphi_0]}{\delta\varphi}$ indicate that $S[\varphi]$ is first differentiated with respect to φ, then evaluated at $\varphi = \varphi_0$ [4].

A.2.2 The Euler-Lagrange equation

The Lagrangian may also be written as a scalar function of the field-values at x, $\varphi(x)$, as well as its derivatives, $\partial_\mu\varphi(x)$, for example

$$\mathcal{L}(\varphi, \partial_\mu\varphi) = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{1}{4!}\lambda\varphi^4 + \dots \tag{A.37}$$

We have omitted the evaluation at x for the brevity of notation. We use this to evaluate the variation of a functional in the of Eq. (A.27),

$$\delta_\eta S[\varphi] = \frac{d}{d\epsilon}\int_{\mathcal{M}} \mathrm{d}^n x\, \mathcal{L}[\varphi+\epsilon\eta](x), \tag{A.38}$$

by Taylor expanding the Lagrangian density as a function of φ and its derivatives,

$$\mathcal{L}[\varphi+\epsilon\eta] = \mathcal{L}\left(\varphi+\epsilon\eta, \partial_\mu\{\varphi+\epsilon\eta\}\right) = \mathcal{L}[\varphi] + \epsilon\left(\frac{\partial\mathcal{L}}{\partial\varphi}\eta + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\partial_\mu\eta\right) + \mathcal{O}(\epsilon^2). \tag{A.39}$$

Inserting this into Eq. (A.38) and partially integrating the last term allows us to write the variation in the form Eq. (A.31), and the functional derivative is

$$\frac{\delta S}{\delta\varphi} = \frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}. \tag{A.40}$$

The principle of stationary action says that the equation of motion of a field obeys $\delta_\eta S = 0$. As η is arbitrary, this is equivalent to setting the functional derivative of S equal to zero. The result is the Euler-Lagrange equations of motion [5],

$$\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} = 0. \tag{A.41}$$

A.2.3 Functional calculus on a curved manifold

As discussed in subsection 2.1.4, when integrating a scalar on a curved manifold, we must include the $\sqrt{|g|}$-factor to get a coordinate-independent result. The action in curved spacetime is therefore [2]

$$S[g,\varphi] = \int_{\mathcal{M}} \mathrm{d}^n x\, \sqrt{|g|}\mathcal{L}[g,\varphi], \tag{A.42}$$

where the action and the Lagrangian now is a functional of both the matter-field φ and the metric $g_{\mu\nu}$. Our example Lagrangian from last section now takes the form

$$\mathcal{L}(g_{\mu\nu}, \varphi, \nabla_\mu\varphi) = \frac{1}{2}g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{1}{4!}\lambda\varphi^4\dots, \tag{A.43}$$

where partial derivatives are substituted with covariant derivatives. We define the functional derivative as

$$\delta_\eta S = \int_{\mathcal{M}} \mathrm{d}^n x\sqrt{|g|}\frac{\delta S}{\delta\eta(x)}\eta(x). \tag{A.44}$$

If this is a variation in φ only, this gives the same result as before. However, in general relativity, the metric itself is a dynamic field, and we may thereofre vary it. Consider $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$. The variation of the action is then assuming $\mathcal{L}$ only depends on g and not its derivatives, we get

$$\delta_g S = \int_{\mathcal{M}} \mathrm{d}^n x \left[\left(\delta\sqrt{|g|} \right) \mathcal{L}[g] + \sqrt{|g|}\delta\mathcal{L}[g] \right] \tag{A.45}$$

We have used

$$\delta\sqrt{|g|} = -\frac{1}{2}\sqrt{|g|} g_{\mu\nu}\delta g^{\mu\nu} \tag{A.46}$$

The variation of the $\sqrt{|g|}$-factor can be evaluated using Using the Levi-Civita symbol $\varepsilon_{\mu_1\dots\mu_n}$, a determinant of a $n \times n$-matrix may be written as

$$\det(A) = \frac{1}{n!}\varepsilon_{\mu_1\dots\mu_n}\varepsilon^{\nu_1\dots\nu_n} A^{\mu_1}{}_{\nu_1} \dots A^{\mu_n}{}_{\nu_n}. \tag{A.47}$$

Using this, we can write for a matrix M

$$\begin{aligned} \det(\mathbb{1} + \varepsilon M) &= \frac{1}{n!}\varepsilon_{\mu_1\dots}\varepsilon^{\nu_1\cdots}(\mathbb{1} + \epsilon M)^{\mu_1}{}_{\nu_1}(\mathbb{1} + \epsilon M)^{\mu_2}{}_{\nu_2}\dots & \text{(A.48)} \\ &= \frac{1}{n!}\varepsilon_{\mu_1\dots}\varepsilon^{\nu_1\cdots}[\delta^{\mu_1}_{\nu_1}\cdots + \epsilon(M^{\mu_1}{}_{\nu_1}\delta^{\mu_2}_{\nu_1}\cdots + M^{\mu_2}{}_{\nu_2}\delta^{\mu_1}_{\nu_1}\cdots + \dots) + \dots] & \text{(A.49)} \\ &= 1 + M^{\mu}{}_{\mu} + \mathcal{O}(\epsilon^2) & \text{(A.50)} \end{aligned}$$

Thus,

$$\delta\sqrt{|g|} = \sqrt{|\det[g_{\mu\nu}(\delta^\nu_\rho + g^{\nu\sigma}\delta g_{\sigma\rho})]|} - \sqrt{|g|} = \sqrt{|g|}\left(\sqrt{|1 + g^{\mu\nu}\delta g_{\mu\nu}|} - 1\right) = -\frac{1}{2}\sqrt{|g|}g^{\mu\nu}\delta g_{\mu\nu}. \tag{A.51}$$

The minus sign is included as the determinant of a Lorentzian metric is negative. Assuming the Lagrangian only depneds on the metric directly, and not its derivatives, the variation of the action is

$$\delta_g S = \int_{\mathcal{M}} \mathrm{d}^n x \sqrt{|g|}\left(\frac{\partial\mathcal{L}}{\partial g^{\mu\nu}} - \frac{1}{2}g_{\mu\nu}\mathcal{L}\right)\delta g^{\mu\nu}. \tag{A.52}$$

With the Lagrangian in Eq. (A.43), we get

$$\frac{\delta S}{\delta g^{\mu\nu}} = \frac{\partial\mathcal{L}}{\partial g^{\mu\nu}} - \frac{1}{2}g_{\mu\nu}\mathcal{L} = -\frac{1}{2}\left(\frac{1}{2}\nabla_\mu\varphi\nabla_\nu\varphi + \frac{1}{2}m^2\varphi^2 + \dots\right). \tag{A.53}$$

We recognize the $(\mu,\nu) = (0,0)$-component as negative half the Hamiltonian density, which supports the definition of the definition of the stress-energy tensor Eq. (4.9).

A.2.4 Functional derivative of the Einstein-Hilbert action

(NEEDS MORE CLEANUP)

In the Einstein-Hilbert action, Eq. (4.6), the Lagrangian density is $\mathcal{L} = kR = kg^{\mu\nu}R_{\mu\nu}$, where k is a constant and $R_{\mu\nu}$ the Ricci tensor, Eq. (2.40). As the Ricci tensor is dependent on both the derivative and second derivative of the metric, we can not use Eq. (A.53) directly. Instead, we use the variation

$$\delta S_{\mathrm{EH}} = k\int_{\mathcal{M}} \mathrm{d}^n x \sqrt{|g|}\left(\delta R - \frac{1}{2}g_{\mu\nu}R\delta g^{\mu\nu}\right). \tag{A.54}$$

The variation of the Ricci scalar is

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}, \tag{A.55}$$

We can write the variation of the Ricci scalar, and thus the Riemann curvature tensor, in terms of variations in Christoffel symbols, $\delta\Gamma^\rho_{\mu\nu}$ using the explicit formula for a symmetric, metric-compatible covariant derivative, Eq. (2.39). As $\delta\Gamma = \Gamma - \Gamma'$, it is a tensor, and we may write

clean up

$$\begin{aligned}\delta R^{\rho}{}_{\sigma\mu\nu} &= \delta(\partial_{[\mu}\Gamma^{\rho}_{\nu]\sigma} + \Gamma^{\rho}_{\lambda[\mu}\Gamma^{\lambda}_{\nu]\sigma}) = \partial_{[\mu}\delta\Gamma^{\rho}_{\nu]\sigma} + (\delta\Gamma^{\rho}_{\lambda[\mu})\Gamma^{\lambda}_{\nu]\sigma} + \Gamma^{\rho}_{\lambda[\mu}(\delta\Gamma^{\lambda}_{\nu]\sigma}) \\ &= \partial_{\mu}\delta\Gamma^{\rho}_{\nu\sigma} + \Gamma^{\rho}_{\lambda\mu}(\delta\Gamma^{\lambda}_{\nu\sigma}) - \Gamma^{\lambda}_{\mu\sigma}(\delta\Gamma^{\rho}_{\lambda\nu}) - \Big(\partial_{\nu}\delta\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\lambda\nu}(\delta\Gamma^{\lambda}_{\mu\sigma}) - \Gamma^{\lambda}_{\nu\sigma}(\delta\Gamma^{\rho}_{\lambda\mu})\Big) + (\Gamma^{\lambda}_{\mu\nu}\delta\Gamma^{\rho}_{\lambda\sigma} - \Gamma^{\lambda}_{\mu\nu}\delta\Gamma^{\rho}_{\lambda\sigma}) \\ &= \nabla_{\mu}\delta\Gamma^{\rho}_{\nu\sigma} - \nabla_{\nu}\delta\Gamma^{\rho}_{\mu\sigma} = \nabla_{\eta}\left(g^{\eta}{}_{\mu}\delta\Gamma^{\rho}_{\nu\sigma} - g^{\eta}{}_{\nu}\delta\Gamma^{\rho}_{\mu\sigma}\right) = \nabla_{\eta}(K^{\rho}{}_{\sigma\mu\nu})^{\eta},\end{aligned}$$

where K is a tensorial quantity, which vanish at the boundary of our spacetime. Using the generalized divergence theorem, Eq. (2.63), we see that the contribution to the action from this quantity vansih. The contribution comes from an integral over $g^{\mu\nu}\delta R_{\mu\nu} = g^{\mu\nu}\delta R^{\rho}_{\mu\rho\nu} = g^{\mu\nu}\nabla_{\eta}(K^{\rho}{}_{\mu\rho\nu})^{\eta}$ Using metric compatibility, we can exchange the covariant derivative and the metric, and we have $g^{\mu\nu}\delta R_{\mu\nu} = \nabla_{\eta}[g^{\mu\nu}K^{\eta\rho}{}_{\mu\rho\nu}]$. The contribution to the action therefore becomes

$$\int_{\mathcal{M}} \mathrm{d}^4x\,\sqrt{|g|}g^{\mu\nu}\delta R_{\mu\nu} = \int_{\mathcal{M}} \mathrm{d}^4x\,\sqrt{|g|}\nabla_{\eta}[g^{\mu\nu}K^{\eta\rho}{}_{\mu\rho\nu}] = \int_{\partial\mathcal{M}} \mathrm{d}^3y\,\sqrt{|\gamma|}n_{\eta}[g^{\mu\nu}K^{\eta\rho}{}_{\mu\rho\nu}] = 0, \tag{A.56}$$

where we used the fact that $\delta g_{\mu\nu}$, and thus K, vanish at $\partial\mathcal{M}$. The variation of the action is therefore

$$\delta S_{\mathrm{EH}} = k\int_{\mathcal{M}} \mathrm{d}^nx\sqrt{|g|}\left[R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right]\delta g^{\mu\nu}, \tag{A.57}$$

and by the definition of the functional derivative,

$$\frac{\delta S_{\mathrm{EH}}}{\delta g^{\mu\nu}} = k(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}). \tag{A.58}$$

A.3 *Consistent series expansion o thermodynamic quantities

As with all other quantities, we calculate using χPT, the free energy density $\mathcal{F}$ must be expanded in chiral dimension, as explained in subsection 5.2.3, as well as an expansion in loops. We write

$$\mathcal{F} = \mathcal{F}^{(0)}_2 + \mathcal{F}^{(1)}_2 + \mathcal{F}^{(2)}_2 + \mathcal{F}^{(0)}_4, \ldots \tag{A.59}$$

where $\mathcal{F}^{(n)}_D$ is the n-loop contribution with chiral dimension D. In chapter 6, we found a relationship between α and μ_I, using the leading-order result for $\mathcal{F}$. To calculate any thermodynamic quantities to leading order, at tree-level, we must use this result. When using the NLO result for the free energy, we must consistently calculate this and other quantities to the same order. As we have seen in section 3.2, when replacing the action by $S[\varphi] \to g^{-1}S[\varphi]$, the L-loop contribution is proportional to g^{L-1}. In Weinberg's power counting scheme, we scale $p \to tp$ and $m_q \to t^2m_q$. Then, the nth term in the expansion is proportional to t^{2n}. Using both these scalings, the expansion of the free energy becomes

$$\mathcal{F} = t^2g^{-1}\mathcal{F}^{(0)}_2 + t^2\mathcal{F}^{(1)}_2 + t^4g^{-1}\mathcal{F}^{(0)}_4 + ... \tag{A.60}$$

We consider terms where $k = L + n$ has the same value to be of same order. This expansion can be written as

$$\mathcal{F} = \sum_{k=0}^{\infty}\sum_{n+L=k} t^{2n}g^{L-1}\mathcal{F}^{(L)}_{2n}. \tag{A.61}$$

If we now define

$$\tilde{\mathcal{F}}_k = \sum_{n+L=k} t^{2n}g^{L-1}\mathcal{F}^{(L)}_{2n}, \tag{A.62}$$

then scale $t \to \sqrt{s}\,t$ and $g \to sg$, where s is some real number, then $t^{2n}g^{L-1}$ scales as $s^{n+L-1} = s^{m-1}$. All expansions are now done in this new parameter s. The free energy expansion is

$$\mathcal{F} = s^{-1}\sum_{k=0}^{\infty}\tilde{\mathcal{F}}_k\,s^k. \tag{A.63}$$

As argued earlier, α must minimize the free energy and therefore satisfy

$$\frac{\partial\mathcal{F}}{\partial\alpha} = 0, \tag{A.64}$$

to all orders. We expand this solution in s,

$$\alpha = \alpha_0 + \alpha_1 s + \dots. \tag{A.65}$$

Combining this, we get

$$\begin{aligned} 0 &= \frac{\partial}{\partial \alpha}\left[s^{-1}\tilde{\mathcal{F}}_0 + \tilde{\mathcal{F}}_1 + \mathcal{O}(s)\right]\Big|_{\alpha=\alpha_0+\alpha_1 s+\mathcal{O}s} \\ &= s^{-1}\left[\tilde{\mathcal{F}}_0'(\alpha_0) + (\alpha'-\alpha_0)\tilde{\mathcal{F}}_0''(\alpha_0) + \mathcal{O}(s^2)\right] + \tilde{\mathcal{F}}_1'(\alpha_0) + \mathcal{O}(s) \\ &= s^{-1}\tilde{\mathcal{F}}_1'(\alpha_0) + s^0\left[\alpha_1\tilde{\mathcal{F}}_0''(\alpha_0) + \tilde{\mathcal{F}}_1'(\alpha_0)\right] + \mathcal{O}(s). \end{aligned} \tag{A.66}$$

Here, the prime indicates partial derivatives with respect to α. The equality in Eq. (A.66) has to hold term for term. After setting $s = g = t = 1$, we get

$$\frac{\partial \tilde{\mathcal{F}}_0}{\partial \alpha}\Big|_{\alpha=\alpha_0} = 0, \quad \tilde{\mathcal{F}}_0 = \mathcal{F}_2^{(0)},$$

which is what we have used as the leading-order result. The first correction is on this result is

$$\alpha_1 = -\frac{\tilde{\mathcal{F}}_1'(\alpha_0)}{\tilde{\mathcal{F}}_0''(\alpha_0)}, \quad \tilde{\mathcal{F}}_1 = \mathcal{F}_4^{(0)} + \mathcal{F}_2^{(1)}. \tag{A.67}$$

The next to leading order results for the free energy and α are

$$\mathcal{F}_{\text{NLO}} = \tilde{\mathcal{F}}_0 + \tilde{\mathcal{F}}_1, \quad \alpha_{\text{NLO}} = \alpha_0 + \alpha_1. \tag{A.68}$$

We have that

$$\mathcal{F}_{\text{NLO}}'(\alpha_{\text{NLO}}) = \tilde{\mathcal{F}}_0'(\alpha_0) + \alpha_1\tilde{\mathcal{F}}_0''(\alpha_0) + \dots + \tilde{\mathcal{F}}_1'(\alpha_0) + \dots, \tag{A.69}$$

where the excluded terms are beyond next-to-leading order. Using Eq. (A.67), we see that this vanishes to next-to-leading order. We therefore use the criterion

$$\frac{\partial \mathcal{F}_{\text{NLO}}}{\partial \alpha}\Big|_{\alpha=\alpha_{\text{NLO}}} = 0 \tag{A.70}$$

to calculate α_{NLO}.

We can use the expansion in s to consistently evaluate any observable to any power in perturbation theory. Assume that f is some observable, and a function of α. We then expand in s,

$$f(\alpha) = f_0(\alpha) + sf_1(\alpha) + s^2 f_2(\alpha) + \dots \tag{A.71}$$

When Taylor expanding around the leading order result for α, we get

$$f(\alpha) = f_0(\alpha_0) + s\alpha_1 f_0'(\alpha_0) + sf_1(\alpha_0) + \mathcal{O}(s^2) = f_0(\alpha_0 + s\alpha_1) + sf_1(\alpha_0) + \mathcal{O}(s^2).$$

To get a consistent expansion, we must evaluate the leading order result for the function f_0 at next-to-leading order in α, while next-to-leading order correction can be evaluated at leading order. However, as

$$f_1(\alpha_0 + s\alpha_1) = f_1(\alpha_0) + \mathcal{O}(s), \tag{A.72}$$

we also get a consistent expansion if we evaluate the leading-order result and its correction at next-to-leading order in α. This is what we do in all results in this thesis.

Appendix B

*Two flavor results

B.1 *Two-flavour χPT to leading order

B.1.1 Leading order Lagrangian

In this section, we will assume $N_f = 2$, which means the generators are $T_a = \frac{1}{2}\tau_a$, where τ_a are the Pauli Matrices, as described in section A.1. The leading order Lagrangian in Winberg's power counting scheme, with $e = 0$, is

$$\mathcal{L}_2 = \frac{1}{4}f^2 \mathrm{Tr}\left\{\nabla_\mu \Sigma (\nabla^\mu \Sigma)^\dagger\right\} + \frac{1}{4}f^2 \mathrm{Tr}\left\{\chi^\dagger \Sigma + \Sigma^\dagger \chi\right\}. \tag{B.1}$$

The external source currents are

$$\nabla_\mu \Sigma = \partial_\mu \Sigma - i[v_\mu, \Sigma], \quad v_\mu = \frac{1}{2}\mu_I \delta^0_\mu \tau_3. \tag{B.2}$$

To incorporate a finite isospin density, we must parametrize the Goldstone manifold differently than in the vacuum. We follow the analysis in [36]. We assume the ground state is independent of space, $\pi_a(x) = \pi^0_a$, and write it as

$$\Sigma_\alpha := \exp\{i\alpha n_a \tau_a\} = \cos\alpha + i n_a \tau_a \sin\alpha, \tag{B.3}$$

where

$$\alpha = \frac{1}{f}\sqrt{\pi^0_a \pi^0_a}, \quad n_a = \frac{\pi^0_a}{\sqrt{\pi^0_a \pi^0_a}}. \tag{B.4}$$

With this, the covariant derivative is $\nabla_\mu \Sigma_\alpha = -i v^a_\mu [\tau_a, \Sigma_\alpha]$, and the two terms in the first order Lagrangian are

$$\mathrm{Tr}\left\{\nabla_\mu \Sigma_\alpha (\nabla^\mu \Sigma_\alpha)^\dagger\right\} = 2\mu_I^2 (n_1^2 + n_2^2)\sin^2\alpha, \quad \mathrm{Tr}\left\{\chi^\dagger \Sigma_\alpha + \Sigma_\alpha^\dagger \chi\right\} = 4\bar{m}^2 \cos\alpha. \tag{B.5}$$

We see that, to first order, all results are independent of Δm. To find the new ground state, we minimize the Hamiltonian density. With the assumption that the fields are constant, the first order Hamiltonian density is

$$\mathcal{H}_2 = -\mathcal{L}_2 = -f^2\left[\bar{m}^2 \cos\alpha + \frac{1}{2}\mu_I^2 (n_1^2 + n_2^2)\sin^2\alpha\right] \tag{B.6}$$

For $\mu_I = 0$, this is independent of n_a, and minimized by $\alpha = 0$. Now, as $n_i n_i = 1$, we have that $n_1^2 + n_2^2 = 1 - n_3^2$. This means that, for $\mu_I \neq 0$, the energy is minimized by $n_3 = 0$. We can write $n_1 = \cos\phi$, $n_2 = \sin\phi$, for some real number ϕ, which gives the ground state

$$\Sigma_\alpha = \mathbb{1}\cos\alpha + i(\tau_1 \cos\phi + \tau_2 \sin\phi)\sin\alpha. \tag{B.7}$$

We can choose, without loss of generality, $\phi = 0$ [47]. This corresponds to a change of basis of $\mathfrak{su}(2)$, $\tau_1 \to \tilde{\tau}_1 = \tau_1 \cos\phi + \tau_2 \sin\phi$ and $\tau_2 \to \tilde{\tau}_2 = -\tau_1 \sin\phi + \tau_2 \cos\phi$. With this, the new ground state is

$$\Sigma_\alpha = \exp\{i\alpha\tau_1\} \tag{B.8}$$

Any exited state is a transformation of the ground state by $\mathrm{SU}(2)_A$. For $\mu_I = 0$, this corresponds to

$$\Sigma(x) = U_R(x)\Sigma_0 U_L^\dagger(x) = U(x)\Sigma_0 U(x). \tag{B.9}$$

where

$$U(x) = \exp\left\{i\frac{\tau_a \pi_a(x)}{2f}\right\}. \tag{B.10}$$

We see that this recovers the parametrization **??**. For $\mu_I \neq 0$, the ground state may be shifted, and so $U(x)$ must be too. The groundstate transforms as

$$\Sigma_0 \to \Sigma_\alpha = \hat{U}_L \Sigma_0 \hat{U}_R^\dagger = A_\alpha \Sigma_0 A_\alpha. \tag{B.11}$$

where

$$A_\alpha := \exp\left\{i\frac{1}{2}\alpha\tau_1\right\} = \cos\frac{\alpha}{2} + i\tau_1 \sin\frac{\alpha}{2}. \tag{B.12}$$

This induces the following transformations for the fluctuations,

$$U_L \to \hat{U}_L U_L \hat{U}_L^\dagger = A_\alpha U_L A_\alpha^\dagger, \tag{B.13}$$

$$U_R \to \hat{U}_R U_R \hat{U}_R^\dagger = A_\alpha^\dagger U_R A_\alpha. \tag{B.14}$$

The new parametrization is thus

$$\Sigma(x) = A_\alpha [U(x)\Sigma_0 U(x)] A_\alpha. \tag{B.15}$$

With this, we can expand the first order Lagrangian, Eq. (B.1), in powers of π/f We will use this expansion to calculate the free energy density. Expanding Σ to $\mathcal{O}\left((\pi/f)^5\right)$, we get

$$\Sigma = \left(1 - \frac{\pi_a^2}{2f^2} + \frac{\pi_a^2\pi_b^2}{24f^4}\right)(\cos\alpha + i\tau_1\sin\alpha) + \left(\frac{\pi_a}{f} - \frac{\pi_b^2\pi_a}{6f^3}\right)\left(i\tau_a - 2i\delta_{a1}\tau_1\sin^2\frac{\alpha}{2} - \delta_{a1}\sin\alpha\right). \tag{B.16}$$

The kinetic term in the χPT Lagrangian is

$$\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger = \partial_\mu\Sigma\partial^\mu\Sigma^\dagger - i\left(\partial_\mu\Sigma[v^\mu,\Sigma^\dagger] - \text{h.c.}\right) - [v_\mu,\Sigma][v_\mu,\Sigma^\dagger]. \tag{B.17}$$

Using Eq. (B.16) we find the expansion of the constitutive parts of the kinetic term to be

$$\begin{aligned}\partial_\mu\Sigma = &\left[\left(\frac{-1}{f^2} + \frac{\pi_b^2}{6f^4}\right)(\pi_a\partial_\mu\pi_a)\cos\alpha - \left(\frac{\partial_\mu\pi_1}{f} - \frac{\pi_b^2\partial_\mu\pi_1 + 2\pi_1\pi_b\partial_\mu\pi_b}{6f^3}\right)\sin\alpha\right] \\ &- \left[\left(\frac{-1}{f^2} + \frac{\pi_b^2}{6f^4}\right)(\pi_a\partial_\mu\pi_a)\sin\alpha - \left(\frac{\partial_\mu\pi_1}{f} - \frac{\pi_b^2\partial_\mu\pi_1 + 2\pi_1\pi_b\partial_\mu\pi_b}{6f^3}\right)2\sin^2\frac{\alpha}{2}\right] i\tau_1 \\ &+ \left(\frac{\partial_\mu\pi_a}{f} - \frac{\pi_b^2\partial_\mu\pi_a + 2\pi_a\pi_b\partial_\mu\pi_b}{6f^3}\right)i\tau_a,\end{aligned} \tag{B.18}$$

and

$$[v_\mu,\Sigma] = -\mu_I\delta_\mu^0\left\{\left[\left(1 - \frac{\pi_a^2}{2f^2} + \frac{\pi_a^2\pi_b^2}{24f^4}\right)\sin\alpha + \left(\frac{\pi_1}{f} - \frac{\pi_b^2\pi_1}{6f^3}\right)\cos\alpha\right]\tau_2 - \left(\frac{\pi_2}{f} - \frac{\pi_b^2\pi_2}{6f^3}\right)\tau_1\right\}. \tag{B.19}$$

Combining Eq. (B.18) and Eq. (B.19) gives the following terms

$$\mathrm{Tr}\left\{\partial_\mu\Sigma\partial^\mu\Sigma^\dagger\right\} = \frac{2}{f^2}\partial_\mu\pi_a\partial^\mu\pi_a + \frac{2}{3f^4}\left[(\pi_a\partial_\mu\pi_a)(\pi_b\partial^\mu\pi_b) - (\pi_a\partial_\mu\pi_b)(\pi_b\partial^\mu\pi_a)\right], \tag{B.20}$$

$$\begin{aligned}-i\mathrm{Tr}\left\{\partial^\mu\Sigma[v_\mu,\Sigma^\dagger] - \text{h.c.}\right\} = &\,4\mu_I\frac{\partial_0\pi_2}{f} + 8\mu_I\frac{\pi_3}{3f^3}\sin\alpha(\pi_2\partial_0\pi_3 - \pi_3\partial_0\pi_2)\sin\alpha \\ &+ \left(\frac{4\mu_I}{f^2}\cos\alpha - \frac{8\mu_I\pi_1}{3f^3}\sin\alpha - \frac{4\mu_I\pi_a\pi_a}{3f^4}\cos\alpha\right)(\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1),\end{aligned} \tag{B.21}$$

$$-\mathrm{Tr}\left\{[v_\mu,\Sigma][v^\mu,\Sigma^\dagger]\right\} = {\mu_I}^2\left[2\sin^2\alpha + \left(\frac{2}{f} - \frac{4\pi_a\pi_a}{3f^3}\right)\pi_1\sin 2\alpha + \left(\frac{2}{f^2} - \frac{2\pi_a\pi_a}{3f^4}\right)\pi_a\pi_b k_{ab}\right], \tag{B.22}$$

$$\mathrm{Tr}\left\{\chi^\dagger\Sigma + \Sigma^\dagger\chi\right\} = \bar{m}^2\left(4\cos\alpha - \frac{4\pi_1}{f}\sin\alpha - \frac{2\pi_a\pi_a}{f^2}\cos\alpha + \frac{2\pi_1\pi_a\pi_a}{3f^3}\sin\alpha + \frac{(\pi_a\pi_a)^2}{6f^4}\cos\alpha\right), \tag{B.23}$$

where $k_{ab} = \delta_{a1}\delta_{b1}\cos 2\alpha + \delta_{a2}\delta_{b2}\cos^2\alpha - \delta_{a3}\delta_{b3}\sin^2\alpha$. Notice that the mass term is independent of the difference in quark masses, Δm. If we write the Lagrangian Eq. (B.1) as $\mathcal{L}_2 = \mathcal{L}_2^{(0)} + \mathcal{L}_2^{(1)} + \mathcal{L}_2^{(2)} + ...$, where $\mathcal{L}_2^{(n)}$ contains all terms of order $(\pi/f)^n$, then the result of the series expansion is

$$\mathcal{L}_2^{(0)} = f^2\left(\bar{m}^2\cos\alpha + \frac{1}{2}\mu^2\sin^2\alpha\right), \tag{B.24}$$

$$\mathcal{L}_2^{(1)} = f(\mu_I^2\cos\alpha - \bar{m}^2)\pi_1\sin\alpha + f\mu_I\partial_0\pi_2\sin\alpha, \tag{B.25}$$

$$\mathcal{L}_2^{(2)} = \frac{1}{2}\partial_\mu\pi_a\partial^\mu\pi_a + \mu_I\cos\alpha\,(\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1) - \frac{1}{2}\bar{m}^2\pi_a\pi_a\cos\alpha + \frac{1}{2}\mu_I^2\pi_a\pi_b k_{ab}, \tag{B.26}$$

$$\begin{aligned}\mathcal{L}_2^{(3)} &= \frac{\pi_a\pi_a\pi_1}{6f}(\bar{m}^2\sin\alpha - 2{\mu_I}^2\sin 2\alpha)\\ &\quad - \frac{2\mu_I}{3f}\left[\pi_1(\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1) + \pi_3(\pi_3\partial_0\pi_2 - \pi_2\partial_0\pi_3)\right]\sin\alpha,\end{aligned} \tag{B.27}$$

$$\begin{aligned}\mathcal{L}_2^{(4)} &= \frac{1}{6f^2}\left\{\frac{1}{4}\bar{m}^2(\pi_a\pi_a)^2\cos\alpha - \left[(\pi_a\pi_a)(\partial_\mu\pi_b\partial^\mu\pi_b) - (\pi_a\partial_\mu\pi_a)(\pi_b\partial^\mu\pi_b)\right]\right\}\\ &\quad - \frac{\mu_I\pi_a\pi_a}{3f^2}\left[(\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1)\cos\alpha + \frac{1}{2}\mu_I\pi_a\pi_b k_{ab}\right].\end{aligned} \tag{B.28}$$

B.1.2 Propagator

We may write the quadratic part of the Lagrangian Eq. (B.26) as[1]

$$\mathcal{L}_2^{(2)} = \frac{1}{2}\sum_a \partial_\mu\pi_a\partial^\mu\pi_a + \frac{1}{2}m_{12}(\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1) - \frac{1}{2}\sum_a m_a^2\pi_a^2, \tag{B.29}$$

where

$$m_1^2 = \bar{m}^2\cos\alpha - \mu_I^2\cos 2\alpha, \tag{B.30}$$

$$m_2^2 = \bar{m}^2\cos\alpha - \mu_I^2\cos^2\alpha, \tag{B.31}$$

$$m_3^2 = \bar{m}^2\cos\alpha + \mu_I^2\sin^2\alpha, \tag{B.32}$$

$$m_{12} = 2\mu_I\cos\alpha. \tag{B.33}$$

The inverse propagator is given by the functional derivative,

$$D_{ab}^{-1}(x-y) = \frac{\delta^2 S[\pi]}{\delta\pi_a(x)\,\delta\pi_b(y)} = \left[-\delta_{ab}(\partial_x^2 + m_a^2) + m_{12}(\delta_{a1}\delta_{b2} - \delta_{a2}\delta_{b1})\partial_{x,0}\right]\delta(x-y). \tag{B.34}$$

The momentum space inverse propagator is

$$D_{ab}^{-1}(p) = \delta_{ab}(p^2 - m_a^2) + ip_0 m_{12}(\delta_{a1}\delta_{b2} - \delta_{a2}\delta_{b1}). \tag{B.35}$$

The spectrum of the particles is given by solving $\det(D^{-1}) = 0$ for p^0. With $p = (p_0, \vec{p})$ as the four momentum, this gives

$$\det(D^{-1}) = D_{33}^{-1}\left(D_{11}^{-1}D_{22}^{-1} + (D_{12}^{-1})^2\right) = \left(p^2 - m_3^2\right)\left[\left(p^2 - m_1^2\right)\left(p^2 - m_2^2\right) - p_0^2 m_{12}^2\right] = 0.$$

This equation has the solutions

$$E_0^2 = |\vec{p}|^2 + m_3^2, \tag{B.36}$$

$$E_\pm^2 = |\vec{p}|^2 + \frac{1}{2}\left(m_1^2 + m_2^2 + m_{12}^2\right) \pm \frac{1}{2}\sqrt{4|\vec{p}|^2 m_{12}^2 + \left(m_1^2 + m_2^2 + m_{12}^2\right)^2 - 4m_1^2 m_2^2}. \tag{B.37}$$

[1] Summation over isospin index (a, b, c) will be explicit in this section.

These are the energies of three particles π_0, π_+ and π_-. π_0 is π_3, while $\pi_\pm$ are linear combinations of π_1 and π_2. [2] We will show that for $\mu_I < m_\pi$, $\alpha = 0$, before it starts to increase for $\mu_I \geq m_\pi$. This result is presented in chapter 7. For $\alpha = 0$, we get

$$\frac{1}{2}(m_1^2 + m_2^2 + m_{12}^2) = \bar{m}^2 + \mu_I^2, \quad m_1^2 m_2^2 = (\bar{m}^2 - \mu_I^2)^2, \quad m_3^2 = \bar{m}^2,$$
$$\implies E_\pm^2 = |\vec{p}|^2 + \bar{m}^2 + \mu_I^2 \pm 2\mu_I \sqrt{|\vec{p}|^2 + \bar{m}^2}.$$

This corresponds to a Zeeman-like splitting of the energies,

$$E_0 = \sqrt{|\vec{p}|^2 + \bar{m}^2}, \tag{B.38}$$
$$E_\pm = \pm\mu_I + \sqrt{|\vec{p}|^2 + \bar{m}^2}. \tag{B.39}$$

The (tree-level) masses of these particles are found by setting $\vec{p} = 0$ and are

$$m_0^2 = m_3^2, \tag{B.40}$$
$$m_\pm^2 = \frac{1}{2}\left[m_1^2 + m_2^2 + m_{12}^2\right] \pm \frac{1}{2}\sqrt{\left(m_1^2 + m_2^2 + m_{12}^2\right)^2 - 4m_1^2 m_2^2}. \tag{B.41}$$

Using the result for α, Figure B.1 shows the masses as functions of μ_I. We observe that the mass of the π_--particle goes to zero at $\mu_I = m_\pi$. This is indicative of spontaneous symmetry breaking, which we will investigate in the next chapter.

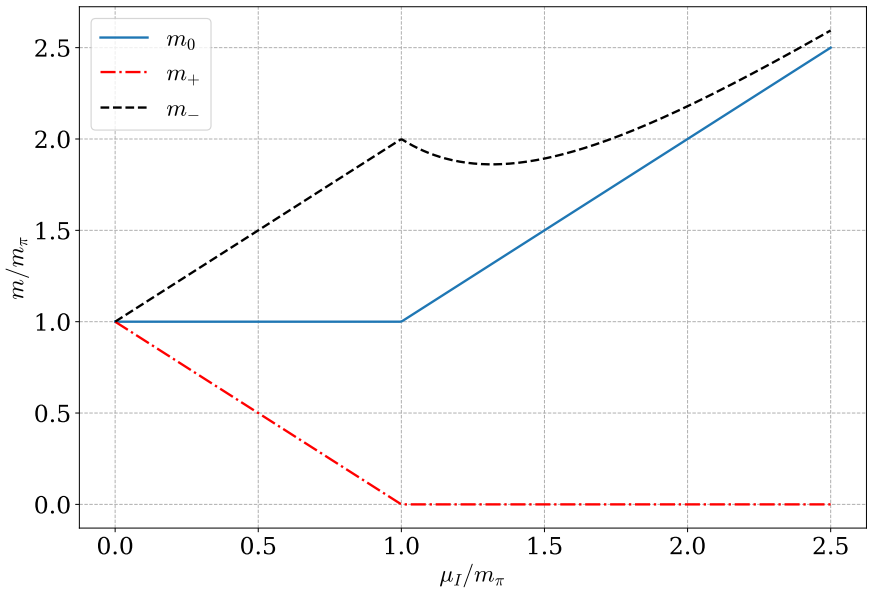


Figure B.1: The masses of the three particles as functions of isosopin chemical potential. Rresults are given in units of the pion mass, m_π.

With the energies of the pions, we can write the determinant of the inverse propagator as

$$\det(D^{-1}) = (p_0^2 - E_0^2)(p_0^2 - E_+^2)(p_0^2 - E_-^2). \tag{B.42}$$

The propagator and the inverse propagator in momentum space obey[3]

$$\sum_c D_{ac}(p) D_{cb}^{-1}(p) = i\delta_{ab} \tag{B.43}$$

[2] An unfortunate notational convention is that E_+ is the energy of π^--particle, and E_+ for the π^+-particle. This is because the positivley charged pion, π^+, has isospin $I_3 = +1$, so that the mass will decrease as μ_I increseas, and hence the negative sign.

[3] One has to be careful regarding the factor i in the physicist's definition of propagators. It has the consequence that D^{-1} is not strictly the operator inverse of the propagator D.

Using this, we can solve for the propagator

$$D = (-iD^{-1})^{-1} = i \begin{pmatrix} \frac{p^2-m_2^2}{(p_0^2-E_+^2)(p_0^2-E_-^2)} & \frac{-ip_0 m_{12}}{(p_0^2-E_+^2)(p_0^2-E_-^2)} & 0 \\ \frac{ip_0 m_{12}}{(p_0^2-E_+^2)(p_0^2-E_-^2)} & \frac{p^2-m_1^2}{(p_0^2-E_+^2)(p_0^2-E_-^2)} & 0 \\ 0 & 0 & \frac{1}{p_0^2-E_0^2} \end{pmatrix}. \tag{B.44}$$

B.2 *Next-to-leading order Lagrangian

Constructing the next-to-leading order (NLO) Lagrangian is now a business of combining the building blocks we found in section 5.2. We must include all terms that obey all symmetries and that are fourth-order in Weinberg's power counting scheme and remove possible redundant terms, as discussed in section B.3. We will quote the result from [18],

$$\begin{aligned} \mathcal{L}_4 = &\frac{l_1}{4}\mathrm{Tr}\left\{\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\right\}^2 + \frac{l_2}{4}\mathrm{Tr}\left\{\nabla_\mu\Sigma(\nabla_\nu\Sigma)^\dagger\right\}\mathrm{Tr}\left\{\nabla^\mu\Sigma(\nabla^\nu\Sigma)^\dagger\right\} + \frac{l_3+l_4}{16}\mathrm{Tr}\left\{\chi\Sigma^\dagger+\Sigma\chi^\dagger\right\}^2 \\ &+ \frac{l_4}{8}\mathrm{Tr}\left\{\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\right\}\mathrm{Tr}\left\{\chi\Sigma^\dagger+\Sigma\chi^\dagger\right\} - \frac{l_7}{16}\mathrm{Tr}\left\{\chi\Sigma^\dagger-\Sigma\chi^\dagger\right\}^2 + \frac{h_1+h_3-l_4}{4}\mathrm{Tr}\left\{\chi\chi^\dagger\right\} \\ &+ \frac{h_1-h_3-l_4}{16}\left[\mathrm{Tr}\left\{\chi\Sigma^\dagger+\Sigma\chi^\dagger\right\}^2 + \mathrm{Tr}\left\{\chi\Sigma^\dagger-\Sigma\chi^\dagger\right\}^2 - 2\mathrm{Tr}\left\{\left(\chi\Sigma^\dagger\right)^2+\left(\Sigma\chi^\dagger\right)^2\right\}\right]. \end{aligned} \tag{B.45}$$

We have ignored terms containing the field strength tensors for external fields, as they vanish in our case. The parameters l_i and h_i are called low energy constants, or LEO. In section B.3, we show how to rewrite the Lagrangian to match the one used in [36, 44]. To obtain $\mathcal{L}_4$ to $\mathcal{O}\big((\pi/f)^3\big)$, we use the result from Eq. (B.18) and Eq. (B.19), which gives

$$\begin{aligned} \mathrm{Tr}\left\{\partial_\mu\Sigma\partial_\nu\Sigma^\dagger\right\} &= 2\frac{\partial_\mu\pi_a\partial_\nu\pi_a}{f^2} \\ -i\mathrm{Tr}\left\{\partial_\mu\Sigma[v_\nu,\Sigma^\dagger] - \mathrm{h.c.}\right\} &= \frac{2\mu_I\pi_2}{f}(\delta^0_\mu\partial_\nu+\delta^0_\nu\partial_\mu)\sin\alpha + \frac{2\mu_I}{f^2}[\pi_1(\delta^0_\mu\partial_\nu+\delta^0_\nu\partial_\mu)\pi_2 - \pi_2(\delta^0_\mu\partial_\nu+\delta^0_\nu\partial_\mu)\pi_1]\cos\alpha \\ -\mathrm{Tr}\left\{[v_\nu,\Sigma][v_\nu,\Sigma^\dagger]\right\} &= 2{\mu_I}^2\delta^0_\mu\delta^0_\nu\left[\sin^2\alpha + \frac{\pi_1}{f}\sin 2\alpha + \frac{\pi_a\pi_b}{f^2}k_{ab}\right], \end{aligned}$$

up to $\mathcal{O}\big((\pi/f)^3\big)$. Inserting $\chi = 2B_0 m = \bar{m}^2\mathbb{1} + \Delta m^2\tau_3$ gives

$$\begin{aligned} \chi\Sigma^\dagger + \Sigma\chi^\dagger &= 2(\bar{m}^2+\Delta m^2\tau_3)\left[\left(1-\frac{\pi_a^2}{2f^2}\right)\cos\alpha - \frac{\pi_1}{f}\sin\alpha\right] \\ &\quad + 2\Delta m^2\left[\left(1-\frac{\pi_a^2}{2f^2}\right)\tau_2\sin\alpha + \frac{\pi_a}{f}\left(\delta_{a1}\tau_2\cos\alpha - \delta_{a2}\tau_1\right)\right], \\ \chi\Sigma^\dagger - \Sigma\chi^\dagger &= -2i\bar{m}^2\left[\left(1-\frac{\pi_a^2}{2f^2}\right)\tau_1\sin\alpha + \frac{\pi_a}{f}\left(\tau_a - 2\delta_{1a}\tau_1\sin^2\frac{\alpha}{2}\right)\right] - 2i\Delta m^2\frac{\pi_3}{f}. \end{aligned}$$

Combining these results gives all the terms in $\mathcal{L}_4$, to $\mathcal{O}\big((\pi/f)^3\big)$:

$$\begin{aligned} &\mathrm{Tr}\left\{\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\right\}^2 = \mathrm{Tr}\left\{\partial_\mu\Sigma\partial^\mu\Sigma^\dagger - i\left(\partial_\mu\Sigma[v^\mu,\Sigma^\dagger] - \mathrm{h.c.}\right) - [v_\mu,\Sigma][v^\mu,\Sigma^\dagger]\right\}^2 \\ &\quad = \frac{8\mu_I^2}{f^2}(\partial_\mu\pi_a\partial^\mu\pi_a + 2\partial_\mu\pi_2\partial^\mu\pi_2)\sin^2\alpha \\ &\quad + 16\mu_I^3\left[\frac{\partial_0\pi_2}{f}\sin^3\alpha + \frac{1}{f^2}\left(3\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1\right)\cos\alpha\sin^2\alpha\right] \\ &\quad + 4\mu_I^4\left\{\sin^4\alpha + 2\sin^2\alpha\left[\frac{\pi_1}{f}\sin 2\alpha + \frac{\pi_a\pi_b}{f^2}\left(k_{ab} + 2\delta_{a1}\delta_{a2}\cos^2\alpha\right)\right]\right\}, \end{aligned} \tag{B.46}$$

$$\begin{aligned} &\mathrm{Tr}\left\{\nabla_\mu\Sigma(\nabla_\nu\Sigma)^\dagger\right\}\mathrm{Tr}\left\{\nabla^\mu\Sigma(\nabla^\nu\Sigma)^\dagger\right\} \\ &\quad = \frac{4{\mu_I}^2}{f^2}\left(\partial_0\pi_a\partial_0\pi_a + \partial_0\pi_2\partial_0\pi_2 + \partial_\mu\pi_2\partial^\mu\pi_2\right)\sin^2\alpha \end{aligned}$$

$$+ 16\mu_I^3 \left[\frac{\partial_0 \pi_2}{f} \sin^3\alpha + \frac{1}{f^2}\left(3\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1\right)\cos\alpha\sin^2\alpha \right]$$
$$+ 4\mu_I^4 \left\{ \sin^4\alpha + 2\sin^2\alpha \left[\frac{\pi_1}{f}\sin 2\alpha + \frac{\pi_a\pi_b}{f^2}\left(k_{ab} + 2\delta_{a1}\delta_{a2}\cos^2\alpha\right)\right]\right\}, \tag{B.47}$$

$$\mathrm{Tr}\left\{\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\right\}\mathrm{Tr}\left\{\chi\Sigma^\dagger + \Sigma\chi^\dagger\right\}$$
$$= 4\bar{m}^2\left\{ 2\frac{\partial_\mu\pi_a\partial^\mu\pi_a}{f^2}\cos\alpha + 4\mu_I\left[\frac{\partial_0\pi_2}{2f}\sin 2\alpha + \frac{1}{f^2}\left(\pi_1\partial_0\pi_2\cos 2\alpha - \pi_2\partial_0\pi_1\cos^2\alpha\right)\right] \right. \tag{B.48}$$
$$\left. + {\mu_I}^2\left[2\cos\alpha\sin^2\alpha - 2\frac{\pi_1}{f}\sin\alpha\left(2-3\sin^2\alpha\right) + \frac{1}{f^2}\left(\pi_1^2[2-9\sin^2\alpha] + \pi_2^2[2-3\sin^2\alpha] - 3\pi_3^2\sin^2\alpha\right)\cos\alpha\right]\right\},$$

$$\mathrm{Tr}\left\{\chi\Sigma^\dagger + \Sigma\chi^\dagger\right\}^2 = 16\bar{m}^4\left[\cos^2\alpha - \frac{\pi_1}{f}\sin 2\alpha + \frac{1}{f^2}\left(\pi_1^2\sin^2\alpha - \pi_a\pi_a\cos^2\alpha\right)\right], \tag{B.49}$$

$$\mathrm{Tr}\left\{\chi\Sigma^\dagger - \Sigma\chi^\dagger\right\}^2 = -16\left(\frac{\Delta m^2\pi_3}{f}\right)^2, \tag{B.50}$$

$$\mathrm{Tr}\left\{\left(\chi\Sigma^\dagger\right)^2 + \left(\Sigma\chi^\dagger\right)^2\right\}$$
$$= 4\bar{m}^4\left(\cos 2\alpha - 2\frac{\pi_1}{f}\sin 2\alpha - 2\frac{\pi_a\pi_a}{f^2}\cos^2\alpha + 2\frac{\pi_1^2}{f^2}\sin^2\alpha\right) + 4\Delta m^4\left(1 - 2\frac{\pi_3^2}{f^2}\right), \tag{B.51}$$

$$\mathrm{Tr}\left\{\chi^\dagger\chi\right\} = 2\bar{m}^4 + 2\Delta m^4. \tag{B.52}$$

The different terms of the NLO Lagrangian is

$$\mathcal{L}_4^{(0)} = (l_1+l_2)\mu_I^4\sin^4\alpha + (l_3+l_4)\bar{m}^2\cos^2\alpha + l_4\bar{m}{\mu_I}^2\cos\alpha\sin^2\alpha + (h_1-l_4)\bar{m}^4 + h_3\Delta m^4, \tag{B.53}$$

$$\mathcal{L}_4^{(1)} = 4\mu_I^3\frac{l_1+l_2}{f}\left(\partial_0\pi_2 + \mu_I\cos\alpha\pi_1\right)\sin^3\alpha - \frac{l_3+l_4}{f}\bar{m}^4\pi_1\sin 2\alpha$$
$$+ \bar{m}^2\frac{l_4}{f}\left[\mu_I\partial_0\pi_2\sin 2\alpha - {\mu_I}^2\pi_1\sin\alpha\left(3\sin^2\alpha - 2\right)\right], \tag{B.54}$$

$$\mathcal{L}_4^{(2)} = 2\mu_I^2\frac{l_1}{f^2}(\partial_\mu\pi_a\partial^\mu\pi_a + 2\partial_0\pi_2\partial_0\pi_2)\sin^2\alpha + {\mu_I}^2\frac{l_2}{f^2}\left(\partial_\mu\pi_2\partial^\mu\pi_2 + 2\partial_0\pi_a\partial_0\pi_a + 2\partial_0\pi_2\partial_0\pi_2\right)\sin^2\alpha$$
$$+ 2\frac{l_1+l_2}{f^2}\left[2\mu_I^3\left(3\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1\right)\cos\alpha + \mu_I^4\pi_a\pi_b\left(k_{ab} + 2\delta_{a1}\delta_{a2}\cos^2\alpha\right)\right]\sin^2\alpha$$
$$+ \frac{l_3+l_4}{f^2}\bar{m}^2\left(\pi_1^2\sin^2\alpha - \pi_a\pi_a\cos^2\alpha\right) + \frac{l_4}{f^2}\bar{m}^2\left[\partial_\mu\pi_a\partial^\mu\pi_a\cos\alpha + 4\mu_I\left(\pi_1\partial_0\pi_2\cos 2\alpha - \pi_2\partial_0\pi_1\cos^2\alpha\right)\right.$$
$$\left. + \frac{1}{2}{\mu_I}^2\left(\pi_1^2[2-9\sin^2\alpha] + \pi_2^2[2-3\sin^2\alpha] - 3\pi_3^2\sin^2\alpha\right)\cos\alpha\right] + \frac{l_7}{f^2}\Delta m^2\pi_3^2. \tag{B.55}$$

B.3 *Rewriting terms

This section shows how to rewrite terms in the Lagrangian of Chiral perturbation theory. These techniques and more are used to reduce the total number of terms and to change between different conventions. Changing the field parametrization that appears in the Lagrangian does not affect any of the physics, as it corresponds to a change of variables in the path integral [18, 48, 49]. However, a change of variables can result in new terms in the Lagrangian. As a result of this, terms that appear independent on their face may be redundant. These terms can be eliminated by using the classical equations of motion. In this section, we show first the derivation of the equations of motion, then use this result to identify redundant terms which need not be included in the most general Lagrangian.

We derive the equations of motion for the leading order Lagrangian using the principle of least action. Choosing the parametrization $\Sigma = \exp\{i\pi_a\tau_a\}$, a variation $\pi_a \to \pi_a + \delta\pi_a$ results in a variation in Σ,

$\delta\Sigma = i\tau_a\delta\pi_a\Sigma$. The variation of the leading order action,

$$S_2 = \int_\Omega \mathrm{d}^4x\,\mathcal{L}_2, \tag{B.56}$$

when varying π_a is

$$\delta S = \int_\Omega \mathrm{d}x\,\frac{f^2}{4}\mathrm{Tr}\left\{(\nabla_\mu\delta\Sigma)(\nabla^\mu\Sigma)^\dagger + (\nabla_\mu\Sigma)(\nabla^\mu\delta\Sigma)^\dagger + \chi\delta\Sigma^\dagger + \delta\Sigma\chi^\dagger\right\}.$$

Using the properties of the covariant derivative to do partial integration, as shown in section B.4, as well as $\delta(\Sigma\Sigma^\dagger) = (\delta\Sigma)\Sigma^\dagger + \Sigma(\delta\Sigma)^\dagger = 0$, the variation of the action can be written

$$\begin{aligned}\delta S &= \frac{f^2}{4}\int_\Omega \mathrm{d}x\,\mathrm{Tr}\left\{-\delta\Sigma\nabla^2\Sigma^\dagger + (\nabla^2\Sigma)(\Sigma^\dagger\delta\Sigma\Sigma^\dagger) - \chi(\Sigma^\dagger\delta\Sigma\Sigma^\dagger) + \delta\Sigma\chi^\dagger\right\}\\ &= \frac{f^2}{4}\int_\Omega \mathrm{d}x\,\mathrm{Tr}\left\{\delta\Sigma\Sigma^\dagger\left[(\nabla^2\Sigma)\Sigma^\dagger - \Sigma\nabla^2\Sigma^\dagger - \chi\Sigma^\dagger + \Sigma\chi^\dagger\right]\right\}\\ &= i\frac{f^2}{4}\int_\Omega \mathrm{d}x\,\mathrm{Tr}\left\{\tau_a\left[(\nabla^2\Sigma)\Sigma^\dagger - \Sigma\nabla^2\Sigma^\dagger - \chi\Sigma^\dagger + \Sigma\chi^\dagger\right]\right\}\delta\pi_a = 0.\end{aligned}$$

As the variation is arbitrary, the equations of motion to leading order is

$$\mathrm{Tr}\left\{\tau_a\left[(\nabla^2\Sigma)\Sigma^\dagger - \Sigma\nabla^2\Sigma^\dagger - \chi\Sigma^\dagger + \Sigma\chi^\dagger\right]\right\} = 0. \tag{B.57}$$

We define

$$\mathcal{O}^{(2)}_{\mathrm{EOM}} = (\nabla^2\Sigma)\Sigma^\dagger - \Sigma\nabla^2\Sigma^\dagger - \chi\Sigma^\dagger + \Sigma\chi^\dagger. \tag{B.58}$$

The next step in eliminating redundant terms is to change the parametrization of Σ by $\Sigma(x) \to \Sigma'(x)$. Here, $\Sigma(x) = e^{iS(x)}\Sigma'(x)$, $S(x) \in \mathfrak{su}(2)$. This change leads to a new Lagrange density, $\mathcal{L}[\Sigma] = \mathcal{L}[\Sigma'] + \Delta\mathcal{L}[\Sigma']$. We are free to choose $S(x)$, as long Σ' still obey the required transformation properties. Any terms in the Lagrangian $\Delta\mathcal{L}$ due to a reparametrization can be neglected, as argued earlier. When demanding that Σ' obey the same symmetries as Σ, the most general transformation to second order in Weinberg's power counting scheme is [18]

$$S_2 = i\alpha_2\left[(\nabla^2\Sigma')\Sigma^\dagger - \Sigma'(\nabla^2\Sigma')^\dagger\right] + i\alpha_2\left[\chi\Sigma'^\dagger - \Sigma'\chi^\dagger - \frac{1}{2}\mathrm{Tr}\left\{\chi\Sigma'^\dagger - \Sigma'\chi^\dagger\right\}\right]. \tag{B.59}$$

α_1 and α_2 are arbitrary real numbers. As Eq. (B.59) is to second order, $\Delta\mathcal{L}$ is fourth order in Weinberg's power counting scheme. Inserting this gives

$$\begin{aligned}\mathcal{L}_2\left[e^{iS_2}\Sigma'\right] &= \frac{f^2}{4}\mathrm{Tr}\left\{[\nabla_\mu(1+iS_2)\Sigma'][\nabla^\mu\Sigma'^\dagger(1-iS_2)]\right\} + \frac{f^2}{4}\mathrm{Tr}\left\{\chi\Sigma'^\dagger(1-iS_2) + (1+iS_2)\Sigma'\chi^\dagger\right\}\\ &= \mathcal{L}[\Sigma'] + i\frac{f^2}{4}\mathrm{Tr}\left\{[\nabla_\mu(S_2\Sigma')][\nabla^\mu\Sigma']^\dagger - [\nabla_\mu\Sigma'][\nabla^\mu(\Sigma'^\dagger S_2)]\right\} - i\frac{f^2}{4}\mathrm{Tr}\left\{\chi\Sigma'^\dagger S_2 - S_2\Sigma'\chi^\dagger\right\}\end{aligned}$$

Using the properties of the covariant derivative, we may use the product rule and partial integration to write the difference between the two Lagrangians to fourth-order as

$$\begin{aligned}\Delta\mathcal{L}[\Sigma'] &= i\frac{f^2}{4}\mathrm{Tr}\left\{(\nabla_\mu S_2)(\Sigma'\nabla^\mu\Sigma'^\dagger - (\nabla^\mu\Sigma')\Sigma'^\dagger)\right\} - i\frac{f^2}{4}\mathrm{Tr}\left\{\chi\Sigma'^\dagger S_2 - S_2\Sigma'\chi^\dagger\right\}\\ &= \frac{f^2}{4}\mathrm{Tr}\left\{iS_2\mathcal{O}^{(2)}_{\mathrm{EOM}}\right\}.\end{aligned} \tag{B.60}$$

Any term that can be written in the form of Eq. (B.60) for arbitrary $\alpha_1, \alpha_2 \in \mathbb{R}$ is redundant, as we argued earlier, and may therefore be discarded. $\Delta\mathcal{L}_2$ is of fourth order, and it can thus be used to remove terms from $\mathcal{L}_4$ or higher order.

B.3.1 Rewriting NLO Lagrangian

The NLO Lagrangian used in this text is given in Eq. (B.45), and is

$$\begin{aligned}\mathcal{L}_4 = &\frac{l_1}{4}\mathrm{Tr}\left\{\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\right\}^2 + \frac{l_2}{4}\mathrm{Tr}\left\{\nabla_\mu\Sigma(\nabla_\nu\Sigma)^\dagger\right\}\mathrm{Tr}\left\{\nabla^\mu\Sigma(\nabla^\nu\Sigma)^\dagger\right\} + \frac{l_3+l_4}{16}\mathrm{Tr}\left\{\chi\Sigma^\dagger+\Sigma\chi^\dagger\right\}^2 \\ &+ \frac{l_4}{8}\mathrm{Tr}\left\{\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\right\}\mathrm{Tr}\left\{\chi\Sigma^\dagger+\Sigma\chi^\dagger\right\} - \frac{l_7}{16}\mathrm{Tr}\left\{\chi\Sigma^\dagger-\Sigma\chi^\dagger\right\}^2 + \frac{h_1+h_3-l_4}{4}\mathrm{Tr}\left\{\chi\chi^\dagger\right\} \\ &+ \frac{h_1-h_3-l_4}{16}\left[\mathrm{Tr}\left\{\chi\Sigma^\dagger+\Sigma\chi^\dagger\right\}^2 + \mathrm{Tr}\left\{\chi\Sigma^\dagger-\Sigma\chi^\dagger\right\}^2 - 2\mathrm{Tr}\left\{\left(\chi\Sigma^\dagger\right)^2+\left(\Sigma\chi^\dagger\right)^2\right\}\right]. \end{aligned} \tag{B.61}$$

We can rewrite it to match the one used in [36, 44], starting with

$$\begin{aligned}&\frac{h_1-h_3-l_4}{16}\left(\mathrm{Tr}\left\{\chi\Sigma^\dagger-\Sigma\chi^\dagger\right\}^2 - 2\mathrm{Tr}\left\{(\chi\Sigma^\dagger)^2+(\Sigma\chi^\dagger)^2\right\}\right) \\ &= \frac{h_1-h_3-l_4}{16}\left(\mathrm{Tr}\left\{\chi\Sigma^\dagger\right\}^2 - 2\mathrm{Tr}\left\{\chi\Sigma^\dagger\right\}\mathrm{Tr}\left\{\Sigma\chi^\dagger\right\} + \mathrm{Tr}\left\{\Sigma\chi^\dagger\right\}^2 - 2\mathrm{Tr}\left\{(\chi\Sigma^\dagger)^2\right\} - 2\mathrm{Tr}\left\{(\Sigma\chi^\dagger)^2\right\}\right).\end{aligned}$$

Using $\mathrm{Tr}\left\{A^2\right\} = \mathrm{Tr}\left\{A\right\}^2 - \det(A)\mathrm{Tr}\left\{\mathbb{1}\right\}$, we get

$$\begin{aligned}&= -\frac{h_1-h_3-l_4}{16}\left(\mathrm{Tr}\left\{\chi\Sigma^\dagger\right\}^2 + 2\mathrm{Tr}\left\{\chi\Sigma^\dagger\right\}\mathrm{Tr}\left\{\Sigma\chi^\dagger\right\} + \mathrm{Tr}\left\{\Sigma\chi^\dagger\right\}^2 - 4\det(\chi\Sigma^\dagger) - 4\det(\Sigma\chi^\dagger)\right) \\ &= -\frac{h_1-h_3-l_4}{16}\left(\mathrm{Tr}\left\{\chi\Sigma^\dagger+\Sigma\chi^\dagger\right\}^2 - 4\det(\chi\Sigma^\dagger) - 4\det(\Sigma\chi^\dagger)\right).\end{aligned}$$

Furthermore, as $\det(\Sigma) = 1$,

$$\begin{aligned}\mathcal{L}_4 = &\frac{l_1}{4}\mathrm{Tr}\left\{\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\right\}^2 + \frac{l_2}{4}\mathrm{Tr}\left\{\nabla_\mu\Sigma(\nabla_\nu\Sigma)^\dagger\right\}\mathrm{Tr}\left\{\nabla^\mu\Sigma(\nabla^\nu\Sigma)^\dagger\right\} + \frac{l_3+l_4}{16}\mathrm{Tr}\left\{\chi\Sigma^\dagger+\Sigma\chi^\dagger\right\}^2 \\ &+ \frac{l_4}{8}\mathrm{Tr}\left\{\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\right\}\mathrm{Tr}\left\{\chi\Sigma^\dagger+\Sigma\chi^\dagger\right\} - \frac{l_7}{16}\mathrm{Tr}\left\{\chi\Sigma^\dagger-\Sigma\chi^\dagger\right\}^2 + \frac{h_1+h_3-l_4}{4}\mathrm{Tr}\left\{\chi\chi^\dagger\right\} \\ &+ \frac{h_1-h_3-l_4}{4}(\det\chi+\det\chi^\dagger). \end{aligned} \tag{B.62}$$

For real χ, we have $\mathrm{Tr}\left\{\chi\chi^\dagger\right\} = \det(\chi) + \det(\chi^\dagger)$, and we can define $h_1' = h_1 - l_4$ to get

$$\begin{aligned}\mathcal{L}_4 = &\frac{l_1}{4}\mathrm{Tr}\left\{\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\right\}^2 + \frac{l_2}{4}\mathrm{Tr}\left\{\nabla_\mu\Sigma(\nabla_\nu\Sigma)^\dagger\right\}\mathrm{Tr}\left\{\nabla^\mu\Sigma(\nabla^\nu\Sigma)^\dagger\right\} + \frac{l_3+l_4}{16}\mathrm{Tr}\left\{\chi\Sigma^\dagger+\Sigma\chi^\dagger\right\}^2 \\ &+ \frac{l_4}{8}\mathrm{Tr}\left\{\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\right\}\mathrm{Tr}\left\{\chi\Sigma^\dagger+\Sigma\chi^\dagger\right\} - \frac{l_7}{16}\mathrm{Tr}\left\{\chi\Sigma^\dagger-\Sigma\chi^\dagger\right\}^2 + \frac{h_1'}{2}\mathrm{Tr}\left\{\chi\chi^\dagger\right\}. \end{aligned} \tag{B.63}$$

If one assumes $\Delta m = 0$, i.e., what is called the chiral limit, then the term l_7 falls away, as $\chi = \chi^\dagger$.

B.4 *Covariant derivative

In χPT at finite isospin chemical potential μ_I, the covariant derivative acts on functions $A(x) : \mathcal{M}_4 \to \mathrm{SU}(2)$, where $\mathcal{M}_4$ is the space-time manifold. It is defined as

$$\nabla_\mu A(x) = \partial_\mu A(x) - i[v_\mu, A(x)], \quad v_\mu = \frac{1}{2}\mu_I\delta^0_\mu\tau_3. \tag{B.64}$$

The covariant derivative obeys the product rule, as

$$\nabla_\mu(AB) = (\partial_\mu A)B + A(\partial_\mu B) - i[v_\mu, AB] = (\partial_\mu A - i[v_\mu, A])B + A(\partial_\mu B - i[v_\mu, B]) = (\nabla_\mu A)B + A(\nabla_\mu B).$$

Decomposing a 2-by-2 matrix M, as described in section A.1, shows that the trace of the commutator of τ_b and M is zero:

$$\mathrm{Tr}\left\{[\tau_a, M]\right\} = M_b\mathrm{Tr}\left\{[\tau_a, \tau_b]\right\} = 0.$$

Together with the fact that $\mathrm{Tr}\{\partial_\mu A\} = \partial_\mu \mathrm{Tr}\{A\}$, this gives the product rule for invariant traces:

$$\mathrm{Tr}\{A\nabla_\mu B\} = \partial_\mu \mathrm{Tr}\{AB\} - \mathrm{Tr}\{(\nabla_\mu A)B\}.$$

This allows for the use of the divergence theorem when doing partial integration. Let $\mathrm{Tr}\{K^\mu\}$ be a space-time vector, and $\mathrm{Tr}\{A\}$ scalar. Let Ω be the domain of integration, with coordinates x and $\partial\Omega$ its boundary, with coordinates y. Then,

$$\int_\Omega \mathrm{d}x\, \mathrm{Tr}\{A\nabla_\mu K^\mu\} = \int_{\partial\Omega} \mathrm{d}y\, n_\mu \mathrm{Tr}\{AK^\mu\} - \int_\Omega \mathrm{d}x\, \mathrm{Tr}\{(\nabla_\mu A)K^\mu\},$$

where n_μ is the normal vector of $\partial\Omega$ [2]. This makes it possible to do partial integration and discard surface terms in the χPT Lagrangian, given the assumption of no variation on the boundary.

B.5 *NLO analysis

The one-loop contribution to the free energy density is

$$\mathcal{F}^{(1)} = -\frac{i}{VT}\frac{1}{2}\mathrm{Tr}\left\{\ln\left(-\frac{\delta^2 S[\pi=0]}{\delta\pi_a(x)\,\delta\pi_b(y)}\right)\right\}. \tag{B.65}$$

This can be evaluated using the rules for functional differentiation given in section A.2. To leading order,

$$\frac{\delta^2 S[\pi=0]}{\delta\pi_a(x)\,\delta\pi_b(y)} = \frac{\delta^2}{\delta\pi_a(x)\,\delta\pi_b(y)}\int \mathrm{d}^4x\, \mathcal{L}_2^{(2)} = D^{-1}_{ab}(x-y). \tag{B.66}$$

Here, $\mathcal{L}_2^{(2)}$ is the quadratic part of the Lagrangian, as given in Eq. (B.29), and D^{-1} is the corresponding inverse propagator of the pion fields,

$$D^{-1}_{ab}(x-y) = \left[-\delta_{ab}(\partial_x^2 + m_a^2) + m_{12}(\delta_{a1}\delta_{b2} - \delta_{a2}\delta_{b1})\partial_{x,0}\right]\delta(x-y) \tag{B.67}$$

The inverse propagator is a matrix, which means that the determinant in Eq. (B.65) is both a matrix determinant, over the three pion indices, and a functional determinant. In subsection B.1.2 we found the matrix part of the determinant in momentum space, which we can write using the dispersion relations of the pion fields

$$\det(-D^{-1}) = \det(-p_0^2 + E_0^2)\det(-p_0^2 + E_+^2)\det(-p_0^2 + E_-^2). \tag{B.68}$$

These dispersion relations are functions of the three-momentum $\vec{p}$, and are given in Eq. (B.36) and Eq. (B.37). The functional determinant can therefore be evaluated as

$$\begin{aligned}
\mathrm{Tr}\left\{\ln\left(-\frac{\delta^2 S[\pi=0]}{\delta\pi_a(x)\,\delta\pi_b(y)}\right)\right\} &= \ln\det(-p_0^2 + E_0^2) + \ln\det(-p_0^2 + E_+^2) + \ln\det(-p_0^2 + E_-^2) \\
&= \mathrm{Tr}\left\{\ln(-p_0^2 + E_0^2) + \ln(-p_0^2 + E_+^2) + \ln(-p_0^2 + E_-^2)\right\} \\
&= (VT)\int \frac{\mathrm{d}^4p}{(2\pi)^4}\left[\ln(-p_0^2 + E_0^2) + \ln(-p_0^2 + E_+^2) + \ln(-p_0^2 + E_-^2)\right],
\end{aligned} \tag{B.69}$$

where we have used the identity $\ln\det M = \mathrm{Tr}\{\ln\} M$. These terms all have the form

$$I = \int \frac{\mathrm{d}^4p}{(2\pi)^2}\ln(-p_0^2 + E^2), \tag{B.70}$$

where E is some function of the 3-momentum $\vec{p}$, but not p_0. We use the trick

$$\frac{\partial}{\partial\alpha}\left(-p_0^2 + E^2\right)^{-\alpha}\Big|_{\alpha=0} = \frac{\partial}{\partial\alpha}\exp\left\{-\alpha\ln\left(-p_0^2 + E^2\right)\right\}\Big|_{\alpha=0} = \ln\left(-p_0^2 + E^2\right), \tag{B.71}$$

and then perform a Wick-rotation of the p_0-integral to write the integral on the form

$$I = i\frac{\partial}{\partial\alpha}\int \frac{\mathrm{d}^4p}{(2\pi)^4}\left(p_0^2 + E^2\right)^{-\alpha}\Big|_{\alpha=0}, \tag{B.72}$$

where p now is a Euclidean four-vector. The p_0 integral equals $\Phi_1(E, 1, \alpha)$, as defined in Eq. (C.25). The result is therefore given by Eq. (C.32),

$$\int \frac{\mathrm{d}p_0}{2\pi}(p_0^2 + E)^{-\alpha} = \frac{E^{1-2\alpha}}{\sqrt{4\pi}} \frac{\Gamma(\alpha - \frac{1}{2})}{\Gamma(\alpha)}. \tag{B.73}$$

The derivative of the Gamma function is $\Gamma'(\alpha) = \psi(\alpha)\Gamma(\alpha)$, where $\psi(\alpha)$ is the digamma function. Using

$$\frac{\partial}{\partial\alpha} \frac{\Gamma(\alpha - \frac{1}{2})}{\Gamma(\alpha)}\Big|_{\alpha=0} = \Gamma\left(\alpha - \frac{1}{2}\right) \frac{\psi(\alpha - \frac{1}{2}) - \psi(\alpha)}{\Gamma(\alpha)}\Big|_{\alpha=0} = \sqrt{4\pi}, \tag{B.74}$$

$$\frac{\Gamma(\alpha - \frac{1}{2})}{\Gamma(\alpha)}\Big|_{\alpha=0} = 0, \tag{B.75}$$

we get

$$I = i \int \frac{\mathrm{d}^3 p}{(2\pi)^3} E. \tag{B.76}$$

We see that the result is what we would expect physically; the total energy is the integral of each mode's energy. This also agrees with the result from Appendix C in the zero-temperature limit $\beta \to \infty$. The one-loop contribution can therefore be written

$$\mathcal{F}^{(1)} = \frac{1}{2}\left[\int \frac{\mathrm{d}^3 p}{(2\pi)^3} E_0 + \int \frac{\mathrm{d}^3 p}{(2\pi)^3}(E_+ + E_-)\right] = \mathcal{F}^{(1)}_{\pi_0} + \mathcal{F}^{(1)}_{\pi_\pm}. \tag{B.77}$$

The first integral is identical to what we find for a free field in section C.3, in the zero-temperature limit. These terms are all divergent and must be regularized. We will use dimensional regularization, in which the integral is generalized to d dimensions, and the $\overline{\text{MS}}$-scheme, as described in subsection C.3.2. Using the result for a free field Eq. (C.39), we get

$$\mathcal{F}^{(1)}_{\pi_0} = -\mu^{-2\epsilon} \frac{1}{4} \frac{m_3^4}{(4\pi)^2}\left(\frac{1}{\epsilon} + \frac{3}{2} + \ln\frac{\tilde{\mu}^2}{m_3^2}\right) + \mathcal{O}(\epsilon), \tag{B.78}$$

where μ is the renormalization scale, a parameter with mass-dimension 1, introduced to ensure the action integral remains dimensionless during dimensional regularization. $\tilde{\mu}$ is a related to μ as described in Eq. (C.38).

The contribution to the free energy from the π_+ and π_- particles is more complicated, as the dispersion relation is given by

$$E_\pm = \sqrt{|\vec{p}|^2 + \frac{1}{2}(m_1^2 + m_2^2 + m_{12}^2) \pm \frac{1}{2}\sqrt{4|\vec{p}|^2 m_{12}^2 + (m_1^2 + m_2^2 + m_{12}^2)^2 - 4m_1^2 m_2^2}}. \tag{B.79}$$

This is not an integral we can easily do in dimensional regularization. Instead, we will seek a function $f(|\vec{p}|)$ with the same UV-behavior, that is, behavior for large $\vec{p}$, as $E_+ + E_-$. If we then add $0 = f(|\vec{p}|) - f(|\vec{p}|)$ to the integrand, we can isolate the divergent behavior

$$\mathcal{F}^{(1)}_{\pi_\pm} = \frac{1}{2}\int \frac{\mathrm{d}^3 p}{(2\pi)^3}[E_+ + E_- + f(|\vec{p}|) - f(|\vec{p}|)] = \mathcal{F}^{(1)}_{\text{fin},\pi_\pm} + \mathcal{F}^{(1)}_{\text{div},\pi_\pm}. \tag{B.80}$$

This results in a finite integral,

$$\mathcal{F}^{(1)}_{\text{fin},\pi_\pm} = \frac{1}{2}\int \frac{\mathrm{d}^3 p}{(2\pi)^3}[E_+ + E_- - f(|\vec{p}|)], \tag{B.81}$$

which we can evaluate numerically, and a divergent integral

$$\mathcal{F}^{(1)}_{\text{div},\pi_\pm} = \frac{1}{2}\int \frac{\mathrm{d}^3 p}{(2\pi)^3} f(|\vec{p}|), \tag{B.82}$$

which we hopefully will be able to do in dimensional regularization. We can explore the UV-behavior of $E_+ + E_-$ by expanding it in powers of $1/|\vec{p}|$,

$$\begin{aligned} E_+ + E_- &= 2|\vec{p}| + \frac{m_{12} + 2(m_1^2 + m_2^2)}{4}|\vec{p}|^{-1} - \frac{m_{12}^4 + 4m_{12}^2(m_1^2 + m_2^2) + 8(m_1^4 + m_2^4)}{64}|\vec{p}|^{-3} + \mathcal{O}(|\vec{p}|^{-5}) \\ &= a_1|\vec{p}| + a_2|\vec{p}|^{-1} + a_3|\vec{p}|^{-3} + \mathcal{O}(|\vec{p}|^{-5}). \end{aligned} \tag{B.83}$$

We have defined new constants a_i for brevity of notation. As

$$\int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} |\vec{p}|^n = C \int_0^\infty \mathrm{d}p\, p^{2+n} \tag{B.84}$$

is UV divergent for $n \geq -3$, f need to match the expansion of $E_+ + E_-$ up to and including $\mathcal{O}(|\vec{p}|^{-3})$ for $\mathcal{F}^{(1)}_{\mathrm{fin},\pi_\pm}$ to be finite. The most obvious choice for f is

$$f(|\vec{p}|) = a_1 |\vec{p}| + a_2 |\vec{p}|^{-1} + a_3 |\vec{p}|^{-3}. \tag{B.85}$$

However, this introduces a new problem. f has the same UV-behavior as $E_+ + E_-$, but the last term diverges in the IR, that is, for low $|\vec{p}|$. This can be amended by introducing a mass term. Let

$$|\vec{p}|^{-3} = \left(\frac{1}{\sqrt{|\vec{p}|^2}} \right)^3 \longrightarrow \left(\frac{1}{\sqrt{|\vec{p}|^2 + m^2}} \right)^3. \tag{B.86}$$

For $|\vec{p}|^2 \to \infty$, this is asymptotic to $|\vec{p}|^{-3}$, so it retains its UV behavior. However, for $|\vec{p}| \to 0$, it now approaches m^{-3}, so the IR-divergence is gone. The cost of this technique is that we have introduced an arbitrary mass parameter. Any final result must thus be independent of the value of m to be acceptable.

We will instead regularize the integral by defining $E_i = \sqrt{|\vec{p}|^2 + \tilde{m}_i^2}$, and $\tilde{m}_i^2 = m_i^2 + \frac{1}{4} m_{12}^2$. Using the definition of the masses, Eq. (B.30), Eq. (B.31), Eq. (B.32), and Eq. (B.33), we get

$$m_3^2 = \bar{m}^2 \cos\alpha + \mu_I^2 \sin^2\alpha, \tag{B.87}$$

$$\tilde{m}_1^2 = m_1^2 + \mu^2 \cos\alpha^2 = \bar{m}^2 \cos\alpha + \mu_I^2 \sin^2\alpha = m_3^2 \tag{B.88}$$

$$\tilde{m}_2^2 = m_2^2 + \mu^2 \cos\alpha^2 = \bar{m}^2 \cos\alpha. \tag{B.89}$$

Finally, we define $f(|\vec{p}|) = E_1 + E_2$, which differ from $E_+ + E_-$ by $\mathcal{O}\left(|\vec{p}|^{-5}\right)$ and is well-behaved in the IR. This leads to a divergent integral the same form as in the case of a free scalar. Thus, in the $\overline{\mathrm{MS}}$-scheme,

$$\mathcal{F}^{(1)}_{\mathrm{div},\pi_\pm} = -\mu^{-2\epsilon} \frac{1}{4} \frac{\tilde{m}_1^4}{(4\pi)^2} \left(\frac{1}{\epsilon} + \frac{3}{2} + \ln \frac{\tilde{\mu}^2}{\tilde{m}_1^2} \right) - \mu^{-2\epsilon} \frac{1}{4} \frac{\tilde{m}_2^4}{(4\pi)^2} \left(\frac{1}{\epsilon} + \frac{3}{2} + \ln \frac{\tilde{\mu}^2}{\tilde{m}_2^2} \right) + \mathcal{O}(\epsilon). \tag{B.90}$$

We define

$$\mathcal{F}^{(1)}_{\mathrm{fin},\pi_\pm} = \frac{1}{2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} (E_+ + E_- - E_1 - E_2), \tag{B.91}$$

which is a finite integral. The total one-loop contribution is then, using Eq. (B.88) and Eq. (B.89),

$$\mathcal{F}^{(2)} = \mathcal{F}^{(1)}_{\mathrm{fin},\pi_\pm} - \mu^{-2\epsilon} \frac{1}{2} \frac{1}{(4\pi)^2} \left[\left(\frac{1}{\epsilon} + \frac{3}{2} + \ln \frac{\tilde{\mu}^2}{m_3^2} \right) m_3^4 + \frac{1}{2} \left(\frac{1}{\epsilon} + \frac{3}{2} + \ln \frac{\tilde{\mu}^2}{\tilde{m}_2^2} \right) \tilde{m}_2^4 \right] + \mathcal{O}(\epsilon). \tag{B.92}$$

B.6 *Next-to-leading order and renormalization

We have now regularized the divergences, which allows them to be handled in a well-defined way. However, they are still there. To get rid of them, we need to use renormalization. As laid out in section 5.2, the power counting scheme ensures that all terms in $\mathcal{L}_{2n}$ scales as t^{2n} when the momenta p are scaled as $p \to tp$.[4] The tree-level free energy from $\mathcal{L}_{2n}$ is thus of order p^{2n}. The m-loop correction to the tree level result is then suppressed by p^{2m} [17, 19]. Our one-loop calculation using $\mathcal{L}_2$ therefore contains divergences of order p^4. Since $\mathcal{L}_4$ is, by construction, the most general possible Lagrangian at order p^4, it contains coupling constants that can be renormalized to absorb all these divergences.

The renormalized coupling constants in $\mathcal{L}_4$ have been calculated for $\mu_I = 0$[19]. They are independent of μ_I, and we can therefore use them in this calculation. The renormalized coupling constants in the $\overline{\mathrm{MS}}$-scheme

[4] Remember that we scale pion mass $\bar{m} = B_0(m_u + m_d)$ as t^2, and the chemical potential as t.

are related to the bare couplings through

$$l_i = l_i^r - \mu^{-2\epsilon}\frac{1}{2}\frac{\gamma_i}{(4\pi)^2}\left(\frac{1}{\epsilon}+1\right), \qquad i \in \{1,...7\}, \tag{B.93}$$

$$h_i = h_i^r - \mu^{-2\epsilon}\frac{1}{2}\frac{\delta_i}{(4\pi)^2}\left(\frac{1}{\epsilon}+1\right), \qquad i \in \{1,...3\}. \tag{B.94}$$

Here, γ_i and δ_i are numerical constants which are used to match the divergences. The relevant terms are[5]

$$\gamma_1 = \frac{1}{3}, \quad \gamma_2 = \frac{2}{3}, \quad \gamma_3 = -\frac{1}{2}, \quad \gamma_4 = 2, \tag{B.95}$$

$$\delta_1 = 2, \quad \delta_3 = 0. \tag{B.96}$$

The bare coupling constants l_i and h_i, though infinite, are independent of our renormalization scale μ. From this we obtain the renormalization group equations for the running coupling constants,

$$\mu\frac{dl_i^r}{d\mu} = -\mu^{-2\epsilon}\frac{\gamma_i}{(4\pi)^2} + \mathcal{O}(\epsilon), \quad \mu\frac{dh_i^r}{d\mu} = -\mu^{-2\epsilon}\frac{\delta_i}{(4\pi)^2} + \mathcal{O}(\epsilon). \tag{B.97}$$

These have the general solutions

$$l_i^r = \frac{1}{2}\mu^{-2\epsilon}\frac{\gamma_i}{(4\pi)^2}\left(\bar{l}_i - \ln\frac{\tilde{\mu}^2}{M^2}\right) + \mathcal{O}(\epsilon), \quad h_i^r = \frac{1}{2}\mu^{-2\epsilon}\frac{\gamma_i}{(4\pi)^2}\left(\bar{h}_i - \ln\frac{\tilde{\mu}^2}{M^2}\right) + \mathcal{O}(\epsilon), \tag{B.98}$$

where $\bar{l}_i$ and $\bar{h}_i$ are the values of the coupling constants (times a constant) measured at the energy M. This only applies if the numerical constants γ_i/δ_i are non-zero. If they are zero, then the coupling is not running, and the measured value can be applied at all energies. The bare couplings are thus given by

$$l_i = \mu^{-2\epsilon}\frac{1}{2}\frac{\gamma_i}{(4\pi)^2}\left(\bar{l}_i - 1 - \frac{1}{\epsilon} - \ln\frac{\tilde{\mu}^2}{M^2}\right) + \mathcal{O}(\epsilon), \tag{B.99}$$

$$h_i = \mu^{-2\epsilon}\frac{1}{2}\frac{\delta_i}{(4\pi)^2}\left(\bar{h}_i - 1 - \frac{1}{\epsilon} - \ln\frac{\tilde{\mu}^2}{M^2}\right) + \mathcal{O}(\epsilon). \tag{B.100}$$

The next-to-leading contribution to the free energy at tree-level is $\mathcal{F}_4^0 = -\mathcal{L}_4^{(0)}$, which is given by Eq. (B.53). When substituting Eq. (B.99) into the bare couplings, we get

$$\begin{aligned}
\mathcal{F}_4^{(0)} &= -(l_1+l_2)\mu_I^4\sin^4\alpha - (l_3+l_4)\bar{m}^4\cos^2\alpha - l_4\bar{m}^2{\mu_I}^2\cos\alpha\sin^2\alpha - (h_1-l_4)\bar{m}^4 - h_3\Delta m^4 \\
&= -\mu^{-2\epsilon}\frac{1}{2}\frac{1}{(4\pi)^2}\Bigg[\frac{1}{3}\left(\bar{l}_1+2\bar{l}_2-3\right)\mu_I^4\sin^4\alpha + \frac{1}{2}\left(-\bar{l}_3+4\bar{l}_4-3\right)\bar{m}^4\cos^2\alpha \\
&\qquad + 2\left(\bar{l}_4-1\right)\bar{m}^2\mu_I^2\cos\alpha\sin^2\alpha + 2(\bar{l}_4-\bar{h}_1)\bar{m}^4 + \bar{h}_3\Delta m^4 \\
&\qquad - \left(\frac{1}{\epsilon}+\ln\frac{\tilde{\mu}^2}{M^2}\right)\left(\mu_I^4\sin^4\alpha + \frac{3}{2}\bar{m}^4\cos^2\alpha + 2\bar{m}^2\mu_I^2\cos\alpha\sin^2\alpha\right)\Bigg] + \mathcal{O}(\epsilon).
\end{aligned}$$

Notice that the term proportional to ϵ^{-1} cancel exactly with the divergent term from $\mathcal{F}^{(2)}$, as we expected. Adding all the contribution to the free energy density, and taking the limit $\epsilon \to 0$, we get the next-to-leading order free energy density,

$$\begin{aligned}
\mathcal{F}_{\text{NLO}} &= -f^2\left(\bar{m}^2\cos\alpha + \frac{1}{2}\mu_I^2\sin^2\alpha\right) + \mathcal{F}^{(1)}_{\text{fin},\pi_\pm} \\
&\quad - \frac{1}{2}\frac{1}{(4\pi)^2}\Bigg[\frac{1}{3}\left(\bar{l}_1+2\bar{l}_2+\frac{3}{2}+3\ln\frac{M^2}{m_3^2}\right)\mu_I^4\sin^4\alpha + \frac{1}{2}\left(-\bar{l}_3+4\bar{l}_4+\frac{3}{2}+2\ln\frac{M^2}{m_3^2}+\ln\frac{M^2}{\tilde{m}_2^2}\right)\bar{m}^4\cos^2\alpha \\
&\qquad + 2\left(\bar{l}_4+\frac{1}{2}+\ln\frac{M^2}{m_3^2}\right)\bar{m}^2\mu_I^2\cos\alpha\sin^2\alpha\Bigg].
\end{aligned} \tag{B.101}$$

[5]Some authors [36, 50] instead use $h_1' = h_1 - l_4$, with a corresponding $\delta_1' = \delta_1 - \gamma_1 = 0$.

We have dropped the terms proportional to $\bar{l}_4 - \bar{h}_1$ and $\bar{h}_3$, as they only add an unobservable constant value to the free energy. With next to leading order results, we must use the next-to-leading order values of the masses and pion decay constants, given by [19]

$$m_\pi^2 = \bar{m}^2 + \frac{1}{2}\bar{l}_3 \frac{\bar{m}^4}{(4\pi)^2 f^2}, \tag{B.102}$$

$$f_\pi^2 = f^2 + 2\bar{l}_4 \frac{\bar{m}^2}{(4\pi)^2 f^2}. \tag{B.103}$$

Appendix C

Thermal field theory

This section is based on [30, 51].

C.1 *Statistical mechanics

In classical mechanics, a thermal system at temperature $T = 1/\beta$ is described as an ensemble state, which have a probability P_n of being in state n, with energy E_n. In the canonical ensemble, the probability is proportional to $e^{-\beta E_n}$. The expectation value of some quantity A, with value A_n in state n is

$$\langle A \rangle = \sum_n A_n P_n = \frac{1}{Z} \sum_n A_n e^{-\beta E_n}, \quad Z = \sum_n e^{-\beta A_n}.$$

Z is called the partition function. In quantum mechanics, an ensemble configuration is described by a non-pure density operator,

$$\hat{\rho} = \sum_n P_n \left| n \right\rangle \left\langle n \right|,$$

where $|n\rangle$ is some basis for the relevant Hilbert space. Assuming $|n\rangle$ are energy eigenvectors, i.e., $\hat{H} |n\rangle = E_n |n\rangle$, the density operator for the canonical ensemble, where $P_n = Ce^{-\beta E_n}$, is

$$\hat{\rho} = \sum_n Ce^{-\beta E_n} \left| n \right\rangle \left\langle n \right| = Ce^{-\beta \hat{H}} \sum_n \left| n \right\rangle \left\langle n \right| = Ce^{-\beta \hat{H}}.$$

The expectation value in the ensemble state of a quantity corresponding to the operator $\hat{A}$ is given by

$$\langle A \rangle = \frac{\mathrm{Tr}\left\{\hat{\rho}\hat{A}\right\}}{\mathrm{Tr}\left\{\hat{\rho}\right\}} = \frac{1}{Z} \mathrm{Tr}\left\{\hat{A} e^{-\beta \hat{H}}\right\}. \tag{C.1}$$

The partition function Z ensures that the probabilities adds up to 1, and is defined as

$$Z = \mathrm{Tr}\left\{e^{-\beta \hat{H}}\right\}. \tag{C.2}$$

The grand canonical ensemble takes into account the conserved charges of the system, which are a result of Nöther's theorem, as discussed in In the grand canonical ensemble, a system with n conserved charges Q_i has probability proportional to $e^{-\beta(H - \mu_i Q_i)}$. Here, μ_i are the chemical potentials corresponding to conserved charge Q_i. This leads to the partition function

$$Z = \mathrm{Tr}\left\{e^{-\beta(\hat{H} - \mu_i \hat{Q}_i)}\right\}. \tag{C.3}$$

C.2 *Imaginary-time formalism

The partition function may be calculated similarly to the path integral approach, in what is called the imaginary-time formalism. This formalism is restricted to time independent problems, and is used to study fields in a volume V. This volume is taken to infinity in the thermodynamic limit. As an example, take a scalar quantum field theory with the Hamiltonian

$$\hat{H} = \int_V \mathrm{d}^3x\, \hat{\mathcal{H}}\left[\hat{\varphi}(\vec{x}), \hat{\pi}(\vec{x})\right], \tag{C.4}$$

where $\hat{\varphi}(\vec{x})$ is the field operator, and $\hat{\pi}(\vec{x})$ is the corresponding canonical momentum operator. These field operators have time independent eigenvectors, $|\varphi\rangle$ and $|\pi\rangle$, defined by

$$\hat{\varphi}(\vec{x})\,|\varphi\rangle = \varphi(\vec{x})\,|\varphi\rangle\,, \quad \hat{\pi}(\vec{x})\,|\pi\rangle = \pi(\vec{x})\,|\pi\rangle\,. \tag{C.5}$$

In analogy with regular quantum mechanics, they obey the relations [1]

$$\mathbb{1} = \int \mathcal{D}\varphi(\vec{x})\,|\varphi\rangle\,\langle\varphi| = \int \mathcal{D}\pi(\vec{x})\,|\pi\rangle\,\langle\pi|\,, \tag{C.6}$$

$$\langle\varphi|\pi\rangle = \exp\left\{i\int_V \mathrm{d}^3x\,\varphi(\vec{x})\pi(\vec{x})\right\}, \tag{C.7}$$

$$\langle\pi_a|\pi_b\rangle = \delta(\phi_a - \phi_b), \quad \langle\varphi_a|\varphi_b\rangle = \delta(\varphi_a - \varphi_b). \tag{C.8}$$

The functional integral is defined by starting with M degrees of freedom, $\{\varphi_m\}_{m=1}^M$ located at a finite grid $\{\vec{x}_m\}_{m=1}^M \subset V$. The integral is then the limit of the integral over all degrees of freedom, as $M \to \infty$:

$$\int \mathcal{D}\varphi(\vec{x}) = \lim_{M\to\infty} \int \left(\prod_{m=1}^{M} \mathrm{d}\varphi_m\right).$$

The functional Dirac-delta $\delta(f) = \prod_x \delta(f(x))$ is generalization of the familiar Dirac delta function. Given a functional $\mathcal{F}[f]$, it is defined by the relation

$$\int \mathcal{D}f(x)\,\mathcal{F}[f]\delta(f-g) = \mathcal{F}[g]. \tag{C.9}$$

The Hamiltonian is the limit of a sum of Hamiltonians $\hat{H}_m$ for each point $\vec{x}_m$

$$\hat{H} = \lim_{M\to\infty} \sum_{m=1}^{M} \frac{V}{M} \hat{H}_m(\{\hat{\varphi}_m\}, \{\hat{\pi}_m\}).$$

H_m may depend on the local degrees of freedom $\hat{\varphi}_m$, $\hat{\pi}_m$ as well as those at neighboring points. By inserting the completeness relations N times into the definition of the partition function, it may be written as

$$Z = \int \mathcal{D}\varphi(\vec{x})\,\left\langle\varphi\middle|e^{-\beta\hat{H}}\middle|\varphi\right\rangle = \prod_{n=1}^{N}\left(\int \mathcal{D}\varphi_n(\vec{x}) \int \mathcal{D}\pi_n(\vec{x})\right) \prod_{n=1}^{N} \langle\varphi_n\rangle\,\pi_n\left\langle\pi_n\middle|e^{-\epsilon\hat{H}}\middle|\varphi_{n+1}\right\rangle\langle\varphi_1\rangle\,\varphi_{N+1},$$

where $\epsilon = \beta/N$. The last term ensures that $\varphi_1 = \varphi_{N+1}$. Bosons such as the scalar field φ, follow the periodic boundary condition $\varphi(0,\vec{x}) = \varphi(\beta,\vec{x})$. Fermions, as we will show later, follow the anti-periodic boundary condition $\psi(0,\vec{x}) = -\psi(\beta,\vec{x})$. We now want to exploit the fact that $|\pi\rangle$ and $|\varphi\rangle$ are the eigenvectors of the operators that define the Hamiltonian. In our case, as the Hamiltonian density $\mathcal{H}$ can be written as a sum of functions of φ and π separately, $\mathcal{H}[\varphi(\vec{x}),\pi(\vec{x})] = \mathcal{F}_1[\varphi(\vec{x})] + \mathcal{F}_2[\pi(\vec{x})]$ we may evaluate it as $\langle\pi_n|\mathcal{H}[\hat{\varphi}(\vec{x}),\hat{\pi}(\vec{x})]|\varphi_{n+1}\rangle = \mathcal{H}[\varphi_{n+1}(\vec{x}),\pi_n(\vec{x})]\,\langle\pi_n\rangle\,\varphi_{n+1}$. This relationship does not, however, hold for more general functions of the field operators. In that case, one has to be more careful about the ordering of the operators, for example, by using *Weyl ordering* By series expanding $e^{-\epsilon\hat{H}}$ and exploiting this relationship, the partition function can be written as, to second order in ϵ,

$$Z = \prod_{n=1}^{N}\left(\int \mathcal{D}\varphi_n(\vec{x}) \int \mathcal{D}\pi_n(\vec{x})\right) \exp\left\{-\epsilon\sum_{n=1}^{N}\int_V \mathrm{d}^3x\,\left(\mathcal{H}[\varphi_n(\vec{x}),\pi_n(\vec{x})] - i\pi_n(\vec{x})\frac{\varphi_n(\vec{x}) - \varphi_{n+1}(\vec{x})}{\epsilon}\right)\right\}.$$

[1] Some authors write $\mathcal{D}\pi/2\pi$. This extra factor 2π is a convention which in this text is left out for notational clarity.

We denote $\varphi_n(\vec{x}) = \varphi(\tau_n, \vec{x})$, $\tau \in [0, \beta]$ and likewise with $\pi_n(\vec{x})$. In the limit $N \to \infty$, the expression for the partition function becomes

$$Z = \int_S \mathcal{D}\varphi(\tau,\vec{x}) \int \mathcal{D}\pi(\tau,\vec{x}) \exp\left\{ -\int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}\vec{x}\, \{\mathcal{H}[\varphi(\tau,\vec{x}),\pi(\tau,\vec{x})] - i\pi(\tau,\vec{x})\dot{\varphi}(\tau,\vec{x})\} \right\}, \tag{C.10}$$

where S is the set of field configurations φ such that $\varphi(\beta,\vec{x}) = \varphi(0,\vec{x})$. With a Hamiltonian density of the form $\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\varphi)^2 + \mathcal{V}(\varphi)$, we can evaluate the integral over the canonical momentum π by discretizing $\pi(\tau_n, \vec{x}_m) = \pi_{n,m}$,

$$\begin{aligned}
&\int \mathcal{D}\pi \exp\left\{ -\int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}^3x \left(\frac{1}{2}\pi^2 - i\pi\dot{\varphi} \right) \right\} \\
&= \lim_{M,N\to\infty} \int \left(\prod_{m,n=1}^{M,N} \frac{\mathrm{d}\pi_{m,n}}{2\pi} \right) \exp\left\{ -\sum_{m,n} \frac{V\beta}{MN} \left[\frac{1}{2}(\pi_{m,n} - i\dot{\varphi}_{m,n})^2 + \frac{1}{2}\dot{\varphi}_{m,n}^2 \right] \right\} \\
&= \lim_{M,N\to\infty} \left(\frac{MN}{2\pi V\beta} \right)^{MN/2} \exp\left\{ -\int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}^3x\, \frac{1}{2}\dot{\varphi}^2 \right\},
\end{aligned}$$

where $\dot{\varphi}_{m,n} = (\varphi_{m,n+1} - \varphi_{m,n})/\epsilon$. The partition function is then,

$$Z = C \int \mathcal{D}\varphi \exp\left\{ -\int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}^3x \left[\frac{1}{2}\left(\dot{\varphi}^2 + \nabla\varphi^2 \right) + \mathcal{V}(\varphi) \right] \right\}. \tag{C.11}$$

Here, C is the divergent constant that results from the π-integral. In the last line, we exploited the fact that the variable of integration $\pi_{n,m}$ may be shifted by a constant without changing the integral, and used the Gaussian integral

$$\int_{-\infty}^{\infty} \mathrm{d}x\, e^{-ax^2/2} = \sqrt{\frac{2\pi}{a}}.$$

The partition function resulting from this procedure may also be obtained by starting with the ground state path integral

$$Z_g = \int \mathcal{D}\varphi\mathcal{D}\pi \exp\left\{ i\int_{\Omega'} \mathrm{d}^4x\, (\pi\dot{\varphi} - \mathcal{H}[\varphi,\pi]) \right\} = C' \int \mathcal{D}\varphi(x) \exp\left\{ i\int_{\Omega'} \mathrm{d}^4x\, \mathcal{L}[\varphi, \partial_\mu\varphi] \right\},$$

and follow a formal procedure. First, the action integral is modified by performing a Wick-rotation of the time coordinate t. This involves changing the domain of t from the real line to the imaginary line by closing the contour at infinity and changing variable $it \to \tau$. The new variable is then restricted to the interval $\tau \in [0, \beta]$, and the domain of the functional integral $\int \mathcal{D}\varphi$ is restricted from *all* (smooth enough) field configurations $\varphi(t, \vec{x})$, to only those that obey $\varphi(\beta, \vec{x}) = e^{i\theta}\varphi(0, \vec{x})$, which is denoted S. Here, $\theta \in \{0, \pi\}$, depending on if the particle is a boson of fermion. This procedure motivates the introduction of the Euclidean Lagrange density, $\mathcal{L}_E(\tau, \vec{x}) = -\mathcal{L}(-i\tau, \vec{x})$, as well as the name "imaginary-time formalism". The result is the same partition function as before,

$$\begin{aligned}
Z &= C \int_S \mathcal{D}\varphi \int \mathcal{D}\pi \exp\left\{ -\int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}^3x\, [-i\dot{\varphi}\pi + \mathcal{H}(\varphi,\pi)] \right\} \\
&= C' \int_S \mathcal{D}\varphi \exp\left\{ -\int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}^3x\, \mathcal{L}_E(\varphi,\pi) \right\}.
\end{aligned} \tag{C.12}$$

C.2.1 Fourier series

Due to the finite range of the imaginary-time coordinate $\tau \in [0, \beta]$, the momentum-space fields in imaginary-time formalism have a discrete coordinate. We define the Matsubara-frequencies as $\omega_n = 2n\pi/\beta$ for bosons and $\omega_n = (2n+1)\pi/\beta$ for fermions. They together form the reciprocal space $\tilde{\Omega} = \{\omega_n\} \times \tilde{V}$, where $\tilde{V}$

is reciprocal to V. To get a more economical notation, we denote the Euclidean real-space coordinates as $X = (\tau, \vec{x})$ and the reciprocal space coordinates as $K = (\omega_n, \vec{k})$. The dot product is $X \cdot K = \omega_n \tau + \vec{k} \cdot \vec{x}$. In the limit $V \to \infty$, we follow the prescription

$$\frac{1}{V} \sum_{\vec{p} \in \tilde{V}} \to \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3}.$$

The sum over all degrees of freedom, and the corresponding integrals for the thermodynamic limit are

$$\frac{\beta V}{NM} \sum_{n=1}^{N} \sum_{\vec{x}_m \in V} \xrightarrow{N,M \to \infty} \int_0^\beta \mathrm{d}\tau \int_{\mathbb{R}^3} \mathrm{d}^3 x = \int_\Omega \mathrm{d}X,$$
$$\frac{1}{V} \sum_{n=-\infty}^{\infty} \sum_{\vec{k} \in \tilde{V}} \xrightarrow{V \to \infty} \sum_{n=-\infty}^{\infty} \int_{R^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} = \int_{\tilde{\Omega}} \mathrm{d}K.$$

The convention used for the Fourier expansion of thermal fields is in accordance with The prefactor is chosen to make the Fourier components of the field dimensionless, which makes it easier to evaluate the trace correctly. For bosons, the Fourier expansion is

$$\varphi(X) = \sqrt{V\beta} \int_{\tilde{\Omega}} \mathrm{d}K \, \tilde{\varphi}(K) e^{iX \cdot K} = \sqrt{\frac{\beta}{V}} \sum_{n=-\infty}^{\infty} \sum_{\vec{k} \in \tilde{V}} \tilde{\varphi}_n(\vec{p}) \exp\left\{ i(\omega_n \tau + \vec{x} \cdot \vec{k}) \right\},$$
$$\tilde{\varphi}(K) = \sqrt{\frac{1}{V\beta^3}} \int_{\tilde{\Omega}} \mathrm{d}X \, \tilde{\varphi}(X) e^{-iX \cdot K},$$

while for Fermions it is

$$\psi(X) = \sqrt{V} \int_{\tilde{\Omega}} \mathrm{d}K \, \tilde{\psi}(K) e^{iX \cdot K} = \frac{1}{\sqrt{V}} \sum_{n=-\infty}^{\infty} \sum_{\vec{k} \in \tilde{V}} \psi(\omega_n, \vec{k}) \exp\left\{ i(\omega_n \tau + \vec{x} \cdot \vec{k}) \right\}. \tag{C.13}$$

Two often used identities are

$$\int_\Omega \mathrm{d}X e^{iX \cdot (K - K')} = \beta \delta_{nn'} (2\pi)^3 \delta^3(\vec{k} - \vec{k}') := \beta \delta(K - K'), \tag{C.14}$$
$$\int_{\tilde{\Omega}} \mathrm{d}K \, e^{iK(X - X')} = \beta \delta(\tau - \tau') \delta^3(\vec{x} - \vec{x}') := \beta \delta(X - X'). \tag{C.15}$$

C.3 *Free scalar field

The procedure for obtaining the thermal properties of an interacting scalar field is similar to that used in scattering theory. One starts with a free theory, which can be solved exactly. Then an interaction term is added, which is accounted for perturbatively by using Feynman diagrams. The Euclidean Lagrangian for a free scalar gas is, after integrating by parts,

$$\mathcal{L}_E = \frac{1}{2} \varphi(X) \left(-\partial_E^2 + m^2 \right) \varphi(X) \tag{C.16}$$

Here, $X = (\tau, \vec{x})$ is the Euclidean coordinate resulting from the Wick-rotation as described in the last section. We have also introduced the Euclidean Laplace operator, $\partial_E^2 = \partial_\tau^2 + \nabla^2$. Following the procedure to obtain the thermal partition function yields

$$Z = C \int_S \mathcal{D}\varphi(X) \exp\left\{ - \int_\Omega \mathrm{d}X \frac{1}{2} \varphi(X) \left(-\partial_E^2 + m^2 \right) \varphi(X) \right\}. \tag{C.17}$$

Here, Ω is the domain $[0, \beta] \times V$. We then insert the Fourier expansion of φ and change the functional integration variable to the Fourier components. The integration measures are related by

$$\mathcal{D}\varphi(X) = \det\left(\frac{\delta \varphi(X)}{\delta \tilde{\varphi}(K)} \right) \mathcal{D}\tilde{\varphi}(K),$$

where $K = (\omega_n, \vec{k})$ is the Euclidean Fourier-space coordinate. The determinant factor which appears may be absorbed into the constant C, as the integration variables are related by a linear transform. The action becomes

$$\begin{aligned} S &= -\int_\Omega \mathrm{d}X\, \mathcal{L}_E = -\frac{1}{2} V\beta \int_\Omega \mathrm{d}X \int_{\tilde{\Omega}} \mathrm{d}K \int_{\tilde{\Omega}} \mathrm{d}K'\, \tilde{\varphi}(K') \left(\omega_n^2 + \vec{k}^2 + m^2\right) \tilde{\varphi}(K) e^{iX\cdot(K-K')} \\ &= -\frac{1}{2} V\beta^2 \int_{\tilde{\Omega}} \mathrm{d}K\, \tilde{\varphi}(K)^* \left(\omega_n^2 + \omega_k^2\right) \tilde{\varphi}(K), \end{aligned}$$

where $\omega_k^2 = \vec{k}^2 + m^2$. $\tilde{\Omega}$ is the reciprocal space corresponding to Ω. We used the fact that φ is real, which implies that $\tilde{\varphi}(-K) = \tilde{\varphi}(K)^*$, as well as the identity This gives the partition function

$$Z = C \int_{\tilde{S}} \mathcal{D}\tilde{\varphi}(K) \exp\left\{ -\frac{1}{2} V \int_{\tilde{\Omega}} \mathrm{d}K\, \tilde{\varphi}(K)^* \left[\beta^2(\omega_n^2 + \omega_k^2)\right] \tilde{\varphi}(K) \right\}, \tag{C.18}$$

Going back to before the continuum limit, this integral can be written as a product of Gaussian integrals and may therefore be evaluated

$$Z = C \prod_{n=-\infty}^{\infty} \prod_{k\in\tilde{V}} \left(\int \mathrm{d}\tilde{\varphi}_{n,\vec{k}} \exp\left\{ -\frac{1}{2} \tilde{\varphi}^*_{n,\vec{k}} \left[\beta^2(\omega_n^2 + \omega_k^2)\right] \tilde{\varphi}_{n,\vec{k}} \right\} \right) = C \prod_{n=-\infty}^{\infty} \prod_{k\in\tilde{V}} \sqrt{\frac{2\pi}{\beta^2(\omega_n^2 + \omega_k^2)}}.$$

The partition function is related to free energy F through

$$\frac{F}{TV} = -\frac{\ln(Z)}{V} = \frac{1}{2} \int_{\tilde{\Omega}} \mathrm{d}K \ln[\beta^2(\omega_n^2 + \omega_k^2)] + \frac{F_0}{TV}, \tag{C.19}$$

where F_0 is a constant.

A faster and more formal way to get to this result is to compare the partition function to the multidimensional version of the Gaussian integral The partition function has the form

$$I_n = \int_{\mathbb{R}^n} \mathrm{d}^n x \exp\left\{ -\frac{1}{2} \langle x, D_0^{-1} x\rangle \right\},$$

where D_0^{-1} is a linear operator, and $\langle\cdot,\cdot\rangle$ an inner product on the corresponding vector space. By diagonalizing D_0^{-1}, we get the result

$$I_n = \sqrt{\frac{(2\pi)^n}{\det(D_0^{-1})}}.$$

We may then use the identity

$$\det(D_0^{-1}) = \prod_i \lambda_i = \exp\left\{ \mathrm{Tr}\left\{ [] \ln(D_0^{-1}) \right\} \right\}, \tag{C.20}$$

where λ_i are the eigenvalues of D_0^{-1}. The trace in this context is defined by the vector space D_0^{-1} acts on. For given an orthonormal basis x_n, such that $\langle x_n, x'_n\rangle = \delta_{nn'}$ the trace can be evaluated as $\mathrm{Tr}\left\{D_0^{-1}\right\} = \sum_n \langle x_n, D_0^{-1} x_n\rangle$. Identifying

$$\langle x, D_0^{-1} x\rangle = \int_\Omega \mathrm{d}X \varphi(X) \left(-\partial_E^2 + m^2\right) \varphi(X),$$

we get the formal result

$$Z = \det(-\partial_E^2 + m^2)^{-1/2},$$

and

$$\beta F = \frac{1}{2} \mathrm{Tr}\left\{ \ln(-\partial_E^2 + m^2) \right\}.$$

The logarithm may then be evaluated by using the eigenvalues of the linear operator. This is found by diagonalizing the operator,

$$\langle x, D_0^{-1} x\rangle = \int_\Omega \mathrm{d}X \varphi(X) \left(-\partial_E^2 + m^2\right) \varphi(X) = V \int_{\tilde{\Omega}} \mathrm{d}K \tilde{\varphi}(K)^* [\beta^2(\omega_k^2 + \omega_n^2)] \tilde{\varphi}(K),$$

leaving us with the same result as we obtained in

$$\beta F = \frac{1}{2}\mathrm{Tr}\left\{\ln(-\partial_E^2 + m^2)\right\} = \frac{1}{2}V\int_{\tilde{\Omega}} \mathrm{d}K \ln[\beta^2(\omega_n^2 + \omega_k^2)].$$

Sums similar to this show up a lot, and we show how to evaluate them in the next section.

C.3.1 Low-temperature limit

Using the result from subsection C.3.4 on the result for the free energy density of the free scalar field, Eq. (C.12), we get

$$\mathcal{F} = \frac{\ln(Z)}{\beta V} = \frac{1}{2}\int_{\tilde{V}} \frac{\mathrm{d}^3k}{(2\pi)^3}\left[\omega_k + \frac{2}{\beta}\ln(1 - e^{-\beta\omega_k})\right]. \tag{C.21}$$

The free energy density thus has two contributions from parts; the first part is dependent on temperature, the other is a temperature-independent vacuum contribution. Noticing that the integral is spherically symmetric, we may write the two contributions as

$$\mathcal{F}_0 = \frac{1}{2}\frac{1}{2\pi^2}\int_{\mathbb{R}} \mathrm{d}k\, k^2\sqrt{k^2 + m^2}, \quad \mathcal{F}_T = \frac{T^4}{2\pi^2}\int_{\mathbb{R}} \mathrm{d}x\, x^2 \ln(1 - e^{-\sqrt{x^2 + (m/T)^2}}), \tag{C.22}$$

The temperature-independent part, $\mathcal{F}_0$, is divergent, and we must impose a regulator and then add counter-terms. $\mathcal{F}_T$, however, is convergent. To see this, we use the series expansion $\ln(1+\epsilon) \sim \epsilon + \mathcal{O}\epsilon$ to find the leading part of the integrand for large k's,

$$x^2 \ln(1 - e^{-\sqrt{x^2 + (\beta m)^2}}) \sim -x^2 e^{-x}, \tag{C.23}$$

which is exponentially suppressed, making the integral convergent. In the limit of $T \to 0$, we get

$$\mathcal{F}_T \sim \frac{T^4}{2\pi^2}\int_{\mathbb{R}} \mathrm{d}x\, x^2 \ln(1 - e^{-x}) = -\frac{T^4}{2\pi^2}\sum_{n=1}\frac{1}{n}\frac{\partial^2}{\partial n^2}\int \mathrm{d}x e^{-nx} = -\frac{T^4}{2\pi^2}\sum_{n=1}\frac{2}{n^4} = -\frac{T^4}{\pi^2}\zeta(4),$$

where ζ is the Riemann-zeta function. Using $\zeta(4) = \frac{\pi^4}{90}$, we get

$$\mathcal{F}_T \sim -\frac{\pi^2}{90}T^4, \quad T \to 0. \tag{C.24}$$

C.3.2 Regularization

Returning to the temperature-independent part, we use dimensional regularization to control its divergent behavior. To that end, we define

$$\Phi_n(m, d, \alpha) = \int_{\tilde{\Omega}} \frac{\mathrm{d}^d k}{(2\pi)^d}(k^2 + m^2)^{-\alpha}, \tag{C.25}$$

so that $\mathcal{F}_0 = \Phi_3(m, 3, -1/2)/2$. We will use the formula for integration of spherically symmetric function in d-dimensions,

$$\int_{\mathbb{R}^d} \mathrm{d}^d x\, f(r) = \frac{2\pi^{d/2}}{\Gamma(d/2)}\int_{\mathbb{R}} \mathrm{d}r\, r^{d-1} f(r), \tag{C.26}$$

where $r = \sqrt{x_i x_i}$ is the radial distance, and Γ is the Gamma function. The factor in the front of the integral is the solid angle. By extending this formula from integer-valued d to real numbers, we get

$$\Phi_n = \frac{2\pi^{d/2}}{\Gamma(d/2)}\int_{\mathbb{R}} \mathrm{d}k\, \frac{k^{d-1}}{(k^2 + m^2)^\alpha} = \frac{m^{n-2\alpha}m^{d-n}}{(4\pi)^{d/2}\Gamma(d/2)}2\int_{\mathbb{R}} \mathrm{d}z\, \frac{z^{d-1}}{(1+z)^\alpha}, \tag{C.27}$$

where we have made the change of variables $mz = k$. We make one more change of variable to the integral,

$$I = 2\int_{\mathbb{R}} \mathrm{d}z\, \frac{z^{d-1}}{(1+z)^\alpha}. \tag{C.28}$$

Let

$$z^2 = \frac{1}{s} - 1 \implies 2z\mathrm{d}z = -\frac{\mathrm{d}s}{s^2} \tag{C.29}$$

Thus,

$$I = \int_0^a \mathrm{d}s\, s^{\alpha - d/2 - 1}(1-z)^{d/2-1}. \tag{C.30}$$

This is the beta function, which can be written in terms of Gamma functions [4]

$$I = B\left(\alpha - \frac{d}{2}, \frac{d}{2}\right) = \frac{\Gamma\left(\alpha - \frac{d}{2}\right)\Gamma\left(\frac{d}{2}\right)}{\Gamma(\alpha)}. \tag{C.31}$$

Combining this gives

$$\Phi_n(m,d,\alpha) = \mu^{n-d}\frac{m^{n-2\alpha}}{(4\pi)^{d/2}}\frac{\Gamma\left(\alpha - \frac{d}{2}\right)}{\Gamma(\alpha)}\left(\frac{m^2}{\mu^2}\right)^{(d-n)/2}. \tag{C.32}$$

In the last step, we have introduced a parameter μ with mass dimension 1, that is, $[\mu] = [m]$. This is done to be able to series expand around $d - n$ in a dimensionless variable. This parameter is arbitrary, and all physical quantities should therefore be independent of it. We will shortly justify this parameter further.

Inserting $n = 3$, $d = 3 - 2\epsilon$ and $\alpha = -1/2$, we get

$$\Phi_3(m, 3-2\epsilon, -1/2) = \frac{m^4\mu^{-2\epsilon}}{(4\pi)^{d/2}\Gamma(-1/2)}\Gamma(-2+\epsilon)\left(\frac{m^2}{\mu^2}\right)^{-\epsilon} = -\mu^{-2\epsilon}\frac{m^4}{(4\pi)^2}\left(\frac{m^2}{4\pi\mu^2}\right)^{-\epsilon}\frac{\Gamma(\epsilon)}{(\epsilon-2)(\epsilon-1)}, \tag{C.33}$$

where we have used the defining property $\Gamma(z+1) = z\Gamma(z)$ and $\Gamma(1/2) = \sqrt{\pi}$. Expanding around $\epsilon = 0$ gives

$$\left(\frac{m^2}{4\pi\mu^2}\right)^{-\epsilon} \sim 1 + \epsilon\ln\left(4\pi\frac{\mu^2}{m^2}\right), \tag{C.34}$$

$$\Gamma(\epsilon) \sim \frac{1}{\epsilon} - \gamma, \tag{C.35}$$

$$\frac{1}{(\epsilon-2)(\epsilon-1)} \sim \frac{1}{2}\left(1 + \frac{3}{2}\epsilon\right). \tag{C.36}$$

The divergent behavior of the temperature-independent term is therefore

$$\mathcal{F}_0 \sim -\mu^{-2\epsilon}\frac{1}{4}\frac{m^4}{(4\pi)^2}\left[\frac{1}{\epsilon} - \gamma + \frac{3}{2} + \ln\left(4\pi\frac{\mu^2}{m^2}\right)\right]. \tag{C.37}$$

With this regulator, one can then add counter-terms to cancel the ϵ^{-1}-divergence. The exact form of the counter-term is convention. One may also cancel the finite contribution due to the regulator. The minimal subtraction (MS) scheme involves only subtracting the divergent term, as the name suggests. We will use the modified minimal subtraction, or $\overline{\mathrm{MS}}$, scheme. In this scheme, one also removes the $-\gamma$ and $\ln(4\pi)$ term, by defining a new mass scale $\tilde{\mu}$ by

$$-\gamma + \ln(4\pi\frac{\mu^2}{m^2}) = \ln(4\pi e^{-\gamma}\frac{\mu^2}{m^2}) = \ln(\frac{\tilde{\mu}^2}{m^2}), \tag{C.38}$$

which leads to the expression

$$\mathcal{F}_0 = -\mu^{-2\epsilon}\frac{1}{4}\frac{m^4}{(4\pi)^2}\left(\frac{1}{\epsilon} + \frac{3}{2} + \ln\frac{\tilde{\mu}^2}{m^2}\right) + \mathcal{O}\epsilon. \tag{C.39}$$

C.3.3 Renormalization

Now that we have applied a regulator, we are able to handle the divergence in a well-defined way. When $\epsilon \neq 0$, we can subtract terms which are proportional to ϵ^{-1}, and be left with a term that is finite in the limit $\epsilon \to 0$. Consider an arbitrary Lagrangian,

$$\mathcal{L}[\varphi] = \sum_n \lambda_n \mathcal{O}_n[\varphi]. \tag{C.40}$$

Here, $\mathcal{O}_n[\varphi]$ are operators consisting of φ and $\partial_\mu\varphi$, and λ_n are coupling constants. In d dimensions, the action integral is

$$S[\varphi] = \sum_n \int \mathrm{d}^d x\, \lambda_n \mathcal{O}_n[\varphi]. \tag{C.41}$$

The action has mass dimension 0. This means that all terms $\lambda_n\mathcal{O}_n$ must have mass dimension d, as $[\mathrm{d}^d x] = -d$. We are free to choose the coupling constant corresponding to $\mathcal{O}_0 = \partial_\mu\varphi\partial^\mu\varphi$ to be of mass dimension 0, and therefore set $\lambda_0 = 1/2$ to get canonical normalization. This allows us to deduce the dimensionality of φ. As $[\partial_\mu] = 1$, we have that $[\varphi] = (d-2)/2$. Assume $\mathcal{O}_n$ consists of k_n factors of φ, and l_n factors of $\partial_\mu\varphi$. We must then have

$$[\lambda_n] + [\mathcal{O}_n] - d = [\lambda_n] + (k_n + l_n)(d-2)/2 + l_n - d = 0, \tag{C.42}$$

$$\implies D_n := [\lambda_n] = d - k_n\frac{d-2}{2} - l_n\frac{d}{2}. \tag{C.43}$$

From this formula, we recover that $[\lambda_0] = 0$, and if $\mathcal{O}_1 = \varphi^2$, then $[\lambda_1] = 2$, which we recognize as the mass term squared. The mass dimensions of these coupling constants are independent of d. However, the coupling constant for the interaction term

$$-\frac{1}{4!}\lambda_3\varphi^4 \tag{C.44}$$

has mass dimensions $[\lambda_3] = d - 4(d-2)/2 = 4 - 2d$. Our goal now is to exchange the bare coupling constants λ_n with renormalized ones, λ_n^r, and remove the divergent terms proportional to $(d-4)^{-m}$. We can always define the renormalized coupling constants as dimensionless, i.e., $[\lambda_n^r] = 0$, by measuring them in units of a mass scale. We therefore write

$$\lambda_n = \mu^{4-D_n}\left[\lambda_n^r + \sum_{m=1}\frac{a_m(\lambda_n^r)}{(d-4)^m}\right],$$

where we have introduced the dimensionfull parameter μ to ensure that λ_n has the correct mass dimension, so that the action integral stays dimensionless. The functions a_m are then determined to each order in perturbation theory by calculating Feynman diagrams. As μ again is arbitrary, λ_4' should not depend on this parameter. In this case, we chose the same renormalization scale as we did when regulating the one-loop integral. This is only for our own convenience. This means that if we change $\mu \to \mu'$, then λ_i^r and a_m must adjust to compensate and keep λ_n constant

The vacuum energy term absorbs the divergence in the one loop contribution to the free energy density. It is

$$\lambda_4\mathcal{O}_4 = \lambda_4 = m^4\lambda_4'. \tag{C.45}$$

Using the expansion in terms of the renormalized coupling, we have,

$$\lambda_4' = \mu^{-2\epsilon}\left[\lambda_4^r + \frac{1}{2\epsilon}a_1(\lambda_r^4) + ...\right], \tag{C.46}$$

where $d = 4 - 2\epsilon$. After adding Eq. (C.45) to the Lagrangian of the free scalar, the temperature independent free energy density becomes

$$\mathcal{F}_0 \sim -\mu^{-2\epsilon}\frac{1}{4}\frac{m^4}{(4\pi)^2}\left[\frac{1}{\epsilon} + \frac{3}{2} + \ln\frac{\tilde{\mu}^2}{m^2} + (4(4\pi)^2)\left(\lambda_4^r + \frac{1}{2\epsilon}a_1(\lambda_4^r)\right)\right], \quad \epsilon \to 0. \tag{C.47}$$

Thus, if we choose $a_1 = -8(4\pi)^2 + \mathcal{O}(\lambda_4^r)$, and define $\lambda_4'^r = 4(4\pi)^2\lambda_4^r$, we are able to cancel the divergence, and may take the limit $\epsilon \to 0$ safely. The free energy is now

$$\mathcal{F} = -\frac{1}{4}\frac{m^4}{(4\pi)^2}\left(\frac{3}{2} + \lambda_4'^r + \ln\frac{\tilde{\mu}^2}{m^2}\right) + \frac{T^4}{2\pi^2}\int \mathrm{d}x\, x^2 \ln(1 - \exp\left\{-\sqrt{x^2 + \beta^2 m^2}\right\}). \tag{C.48}$$

Notice that all choices we have made up until now, such as defining $\lambda_4 = m^4\lambda_4'$ and using the same renormalization scale μ, has no impact on this result. Different choices would force us to define λ_4^r and a_4 differently.

C.3.4 Thermal sum

When evaluating thermal integral, we will often encounter sums of the form

$$j(\omega,\mu) = \frac{1}{2\beta}\sum_{\omega_n} \ln\{\beta^2[(\omega_n + i\mu) + \omega^2]\} + g(\beta), \tag{C.49}$$

where the sum is over either the bosonic Matsubara frequencies $\omega_n = 2n\pi/\beta$, $n \in \mathbb{Z}$, or the fermionic ones, $\omega_n = (2n+1)\pi/\beta$, $n \in \mathbb{Z}$. $\mu \in \mathbb{R}$ is a chemical potential. g may be a function of β, but we assume it is independent of ω. Thus, the factor β^2 could strictly be dropped, but it is kept to make the argument within the logarithm dimensionless. We define the function

$$i(\omega,\mu) = \frac{1}{\omega}\frac{d\omega}{dj}(\omega,\mu) = \frac{1}{\beta}\sum_{\omega_n}\frac{1}{(\omega_n + i\mu)^2 + \omega^2}. \tag{C.50}$$

We will first work with the sum over bosonic Matsubara frequencies. Assume $f(z)$ is an analytic function, except perhaps on a set of isolated poles $\{z_i\}$ located outside the real line. We can exploit this using the properties of the Bose-distribution $n_B(z)$. The Bose distribution is defined as

$$n_B(\omega) = \frac{1}{e^{\beta\omega} - 1}. \tag{C.51}$$

This function obeys

$$n_B(-i\omega) = -1 - n_B(i\omega). \tag{C.52}$$

We can expand it around the Bose Matsubara frequencies on the imaginary line:

$$in_B[i(\omega_n + \epsilon)] = \frac{i}{e^{i\beta\epsilon + 2\pi i n} - 1} = i[i\beta\epsilon + \mathcal{O}(\epsilon)]^{-1} \sim \frac{1}{\epsilon\beta}. \tag{C.53}$$

This means that $in_B(i\omega)$ has a pole on all Matsubara-frequencies, with residue $1/\beta$. Using this, we can rewrite the sum over Matsubara frequencies as a contour integral

$$\frac{1}{\beta}\sum_{\omega_n} f(\omega_n) = \oint_\gamma \frac{\mathrm{d}z}{2\pi i} f(z) i n_B(iz),$$

where γ is a contour that goes from $-\infty - i\epsilon$ to $+\infty - i\epsilon$, crosses the real line at ∞, goes from $+\infty - i\epsilon$ to $-\infty + i\epsilon$ before closing the curve. The contour γ and the new contours are illustrated in Figure C.1. This result exploits Cauchy's integral formula by letting the poles of $in_B(iz)$ at the Matsubara frequencies "pick out" the necessary residues. The integral over γ is equivalent to two integrals along $\mathbb{R} \pm i\epsilon$,

$$\begin{aligned}
\frac{1}{\beta}\sum_{\omega_n} f(\omega_n) &= \left(\int_{\infty+i\epsilon}^{-\infty+i\epsilon} \frac{\mathrm{d}z}{2\pi} + \int_{-\infty-i\epsilon}^{\infty-i\epsilon}\frac{\mathrm{d}z}{2\pi}\right) f(z) n_B(iz), \\
&= \int_{-\infty-i\epsilon}^{\infty-i\epsilon}\frac{\mathrm{d}z}{2\pi}\left\{-f(-z) + [f(z) - f(-z)]\, n_B(iz)\right\} \\
&= \int_{-\infty}^{\infty}\frac{\mathrm{d}z}{2\pi} f(z) + \int_{-\infty-i\epsilon}^{\infty-i\epsilon}\frac{\mathrm{d}z}{2\pi}\left[f(z) + f(-z)\right] n_B(iz).
\end{aligned} \tag{C.54}$$

In the second line, we have changed variables $z \to -z$ in the first integral, and exploited the property $n_B(-iz) = -1 - n_B(iz)$. In the last line, we use the assumption that $f(z)$ is analytic on the real line, and therefore also in a neighborhood of it. This allows us to shift the first integral back to the real line. As $n_B(iz)$ is analytic outside the real line, the result of the second integral is the sum of residues of $f(z) + f(-z)$ in the lower half-plane. The function

$$f(z) = \frac{1}{(z + i\mu)^2 + \omega^2} = \frac{i}{2\omega}\left(\frac{1}{z + i(\mu + \omega)} - \frac{1}{z + i(\mu - \omega)}\right) \tag{C.55}$$

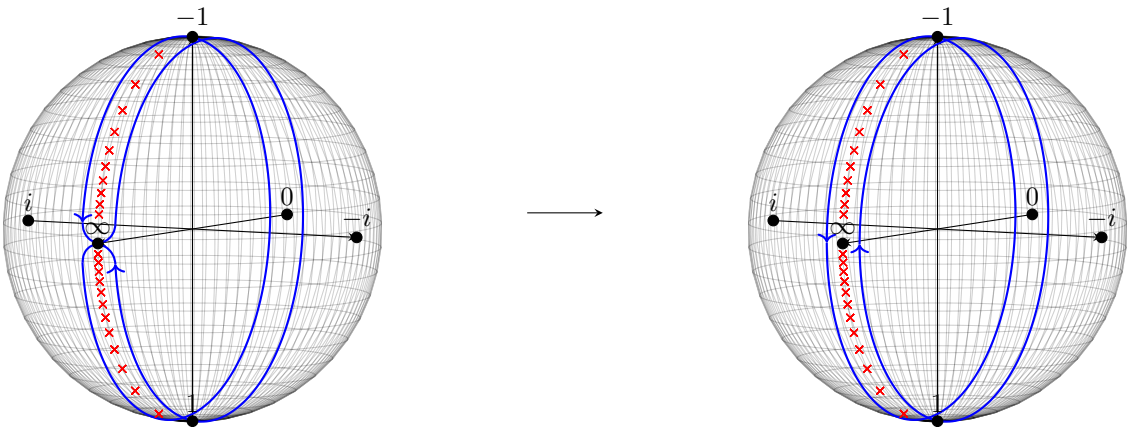


Figure C.1: The integral contour γ, and the result of deforming it into to contours close to the real line. The red crosses illustrate the poles of n_B.

obeys the assumed properties, as it has poles at $z = -i(\mu \pm \omega)$, with residue $1/(2\omega)$, so the function defined in Eq. (C.50) may be written

$$i(\omega, \mu) = \frac{1}{2\omega}[1 + n_B(\omega - \mu) + n_B(\omega + \mu)]. \tag{C.56}$$

Using the antiderivative of the Bose distribution,

$$\frac{d\omega}{d\ln}(1 - e^{-\beta\omega}) = \beta n_B(\omega), \tag{C.57}$$

we get the final form of Eq. (C.49)

$$j(\omega, \mu) = \int \mathrm{d}\omega'\, \omega' i(\omega', \mu) = \frac{1}{2}\omega + \frac{1}{2\beta}\left[\ln\left(1 - e^{-\beta(\omega-\mu)}\right) + \ln\left(1 - e^{-\beta(\omega+\mu)}\right)\right] + g'(\beta). \tag{C.58}$$

The extra ω-independent term $g'(\beta)$ is an integration constant. We see there are temperature dependent terms, one due to the particle and one due to the anti-particle, and one due to the antiparticle, as they have opposite chemical potentials.

We now consider the sum over fermionic frequencies, which we for clarity denote $\tilde{\omega}_n$ in this chapter. The procedure, in this case, is the same, except that we have to use a function with poles at the fermionic Matsubara frequencies. This is done by the Fermi distribution, $n_F(z)$. The Fermi distribution is

$$n_F(\omega) = \frac{1}{e^{\beta\omega} + 1}. \tag{C.59}$$

It obeys

$$\frac{d}{d\omega}\ln(1 + e^{-\beta\omega}) = -\beta n_F(\omega), \tag{C.60}$$

$$n_F(-i\omega) = 1 - n_F(i\omega). \tag{C.61}$$

With this, the sum over fermionic Matsubara frequencies gives

$$\frac{1}{\beta}\sum_{\tilde{\omega}_n} f(\tilde{\omega}_n) = \left(\int_{\infty+i\epsilon}^{-\infty+i\epsilon} \frac{\mathrm{d}z}{2\pi} + \int_{-\infty-i\epsilon}^{\infty-i\epsilon} \frac{\mathrm{d}z}{2\pi}\right) f(z) n_B(iz) \tag{C.62}$$

$$= \int_{-\infty}^{\infty} \frac{\mathrm{d}z}{2\pi} f(z) - \int_{-\infty-i\epsilon}^{\infty-i\epsilon} \frac{\mathrm{d}z}{2\pi} \left[f(z) - f(-z)\right] n_F(iz), \tag{C.63}$$

and

$$i(\omega, \mu) = \frac{1}{2\omega}[1 - n_F(\omega - \mu) - n_F(\omega + \mu)]. \tag{C.64}$$

Using the antiderivative of the Fermi-distribution, we get

$$j(\omega, \mu) = \frac{1}{2}\omega + \frac{1}{2\beta}\left[\ln\left(1 + e^{-\beta(\omega-\mu)}\right) + \ln\left(1 + e^{-\beta(\omega+\mu)}\right)\right]. \tag{C.65}$$

C.4 *Interacting scalar

We now study a scalar field with a $\lambda\varphi^4$ interaction term. We write the Lagrangian in the form

$$\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(I)}, \quad \mathcal{L}^{(0)} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - m^2\varphi^2, \quad \mathcal{L}^{(I)} = -\frac{\lambda}{4!}\varphi^4$$

$\mathcal{L}^{(I)}$ is called the interaction term, and makes it impossible to exactly solve for the partition function. Instead, we turn to perturbation theory. The canonical partition function in this theory is

$$Z = \mathrm{Tr}\left\{e^{-\beta\hat{H}}\right\} = \int_S \mathcal{D}\varphi \exp\left\{-\int_\Omega \mathrm{d}X \left(\mathcal{L}_E^{(0)} + \mathcal{L}_E^{(I)}\right)\right\} = \int_S \mathcal{D}\varphi\, e^{-S_0} e^{-S_I}. \tag{C.66}$$

Here, S_0 and S_I denote the Euclidean action due to the free and interacting Lagrangian, respectively. The domain of integration S is again periodic field configurations $\varphi(\beta,\vec{x}) = \varphi(0,\vec{x})$. We may write the free energy as

$$-\beta F = \ln\left[\int_S \mathcal{D}\varphi\, e^{-S_0} \sum_n \frac{1}{n!}(-S_I)^n\right] = \ln Z_0 + \ln Z_I,$$

where Z_0 is the partition function of the free theory. The correction to the partition function is thus given by

$$Z_I = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left\langle {S_I}^n \right\rangle_0, \tag{C.67}$$

where

$$\langle A\rangle_0 = \frac{\int_S \mathcal{D}\varphi\, A\, e^{-S_0}}{\int_S \mathcal{D}\varphi\, e^{-S_0}}. \tag{C.68}$$

To evaluate expectation values of the form $\langle\varphi(X_1)...\rangle_0$, we introduce the partition function with a source term

$$Z[J] = \int_S \mathcal{D}\varphi \exp\left\{-\frac{1}{2}\int_\Omega \mathrm{d}X\, \varphi(-\partial_E^2 + m^2)\varphi + \int_\Omega \mathrm{d}X\, J\varphi\right\}. \tag{C.69}$$

Thermal propagators are the generalization of the time-ordered two-point functions $\langle T\{\varphi(x)\varphi(y)\}\rangle$ of the vacuum formalism. For some differential operator D^{-1}, the thermal propagator is defined as

$$D^{-1}D(X,Y) = \beta\delta(X-Y). \tag{C.70}$$

The Fourier transformed propagator is, assuming $D(X,Y) = D(X-Y,0)$,

$$\begin{aligned}
\tilde{D}(K,K') &= \frac{1}{V\beta^3}\int_\Omega \mathrm{d}X\mathrm{d}Y\, D(X,Y)\exp\{(\} - i[X\cdot K + Y\cdot K']) \\
&= \frac{1}{V\beta^3}\int_\Omega \mathrm{d}X'\mathrm{d}Y'\, D(X',0)\exp\{(\} - i[X'\cdot\frac{1}{2}(K-K') + Y\cdot(K+K')]) \\
&= \frac{1}{V\beta^2}\tilde{D}(K)\delta(K+K'),
\end{aligned} \tag{C.71}$$

where

$$\tilde{D}(K) = \int \mathrm{d}X e^{iK\cdot X} D(X,0). \tag{C.72}$$

We write the thermal propagator of the free field as $D_0(X,Y)$. With this, we may complete the square,

$$Z[J] = Z[0]\exp\left\{\frac{1}{2}\int_\Omega \mathrm{d}X\mathrm{d}Y\, J(X)D_0(X,Y)J(Y)\right\} = Z[0]\exp\{(\}\, W[J]). \tag{C.73}$$

We can now write

$$\langle\varphi(X)\varphi(Y)\rangle_0 = \frac{1}{Z[0]}\frac{\delta}{\delta J(X)}\frac{\delta}{\delta J(Y)} Z[J]\Big|_{J=0} = D_0(X,Y). \tag{C.74}$$

This generalizes to higher order expectation values,

$$\langle\varphi(X_i)\dots\varphi(X_n)\rangle_0 = \frac{1}{Z[0]}\left(\prod_{i=1}^{n}\frac{\delta}{\delta J(X_i)}\right)Z[J]\Big|_{J=0}, \tag{C.75}$$

Using Wick's theorem, as described in section 3.1, the expectation values we are evaluating can be written

$$\begin{aligned}\langle {S_I}^m\rangle_0 &= \left(-\frac{\lambda}{4!}\right)^m\int_\Omega \mathrm{d}X_1\dots\mathrm{d}X_m\left\langle\varphi^4(X_1)\dots\varphi^4(X_m)\right\rangle_0\\ &= \left(-\frac{\lambda}{4!}\right)^m\int_\Omega \mathrm{d}X_1\dots\mathrm{d}X_m\sum_{\{a,b\}}\left\langle\varphi(X_{a(1)})\varphi(X_{b(1)})\right\rangle_0\dots\left\langle\varphi(X_{a(2m)})\varphi(X_{b(2m)})\right\rangle_0,\end{aligned}$$

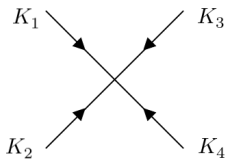
where X_i for $i > m$ is defined to equal X_j, where $j = i \mod m$. More simpliy, $X_{m+i} = X_i$. The functions a, b represents a possible pairing, as described in section 3.1. Inserting the Fourier expansions of the field gives

$$\begin{aligned}&\langle {S_I}^m\rangle_0\\ &= \left(-\frac{\lambda}{4!}\right)^m\int_\Omega \mathrm{d}X_1\dots\mathrm{d}X_m(V\beta)^2\int_{\tilde\Omega}\mathrm{d}K_1\dots\mathrm{d}K_{2m}\sum_{\{a,b\}}\\ &\qquad\left\langle\varphi(K_{a(1)})\varphi(K_{b(1)})\right\rangle_0\dots\left\langle\varphi(K_{a(2m)})\varphi(K_{b(2m)})\right\rangle_0\exp\{(\}\,i\textstyle\sum_{i=1}^m X_i\cdot K_i)\\ &= \left(-\frac{\lambda}{4!}\right)^m\frac{(V\beta)^{2m}\beta^m}{(V\beta^2)^{2m}}\int_{\tilde\Omega}\mathrm{d}K_1{...}\mathrm{d}K_{2m}\sum_{\{a,b\}}\\ &\qquad\tilde D(K_{a(1)})\delta(K_{a(1)}+K_{b(1)})\dots\tilde D(K_{a(2m)})\delta(K_{a(2m)}+K_{b(2m)})\prod_{i=1}^m\delta\left(X_i\cdot\textstyle\sum_{j=0}^3 K_{i+jm}\right)\\ &= \left(-\frac{\lambda\beta}{4!}\right)^m\prod_{i=1}^{2m}\int_{\tilde\Omega}\left(\mathrm{d}K_i\frac{1}{\beta}\tilde D(K_i)\right)\prod_{i=1}^m\delta\left(X_i\cdot\textstyle\sum_{j=0}^3 K_{i+jm}\right)\sum_{\{a,b\}}\prod_{n=1}^{2m}\delta(K_{a(k)}+K_{b(k)}).\end{aligned}$$

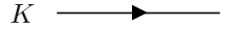
Here we have used that $V\beta^2\tilde D_0(K,P) = \tilde D_0(K)\delta(P+K)$, where $\tilde D_0(K)$ is the thermal propagator for the free field. In this case, it is

$$\tilde D_0(K) = \tilde D_0(\omega_n,\vec k) = \frac{1}{\omega_k^2+\omega_n^2}. \tag{C.76}$$

This expectation value can be represented graphically using Feynman diagrams. The thermal $\lambda\varphi^2$-theory gets the prescription



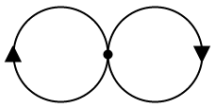
$$= -\lambda\beta\delta\left(\textstyle\sum_i K_i\right), \tag{C.77}$$



$$= \frac{1}{\beta}D_0(K). \tag{C.78}$$

Lastly, one has to integrate over internal momenta and divide by the symmetry factor of the diagram s, which is described in detail in

Calculating $\langle {S_I}^n\rangle_0$ boils down to the sum of all possible Feynman diagrams with n vertices. The first example is

$\langle S_I\rangle_0 = \frac{1}{8}$  . (C.79)

In section 3.1, we saw that the sum of all vacuum diagrams is the exponential of the sum of all *connected* diagrams, so the free energy of the interacting theory is given by

$$-\beta F = \ln Z_0 + \Sigma(\text{all connected diagrams}). \tag{C.80}$$

C.5 Fermions

fix: intro til grassmantall, utled path integral

The anti-periodic nature of fermion-fields, as mentioned in section C.2, can be determined by studying the properties of the thermal Greens function. The thermal Greens function may be written

$$D(X_1, X_2) = D(\vec{x}, \vec{y}, \tau_1, \tau_2) = \langle\Omega| e^{-\beta\hat{H}} \mathrm{T}\left\{\hat{\varphi}(X_1)\hat{\varphi}(X_2)\right\} |\Omega\rangle .$$

$\mathrm{T}\{...\}$ is time-ordering operator, defined as

$$\mathrm{T}\left\{\varphi(\tau_1)\varphi(\tau_2)\right\} = \theta(\tau_1 - \tau_2)\varphi(\tau_1)\varphi(\tau_2) + \nu\theta(\tau_2 - \tau_1)\varphi(\tau_2)\varphi(\tau_1),$$

where $\nu = \pm 1$ for bosons and fermions respectively, and $\theta(\tau)$ is the Heaviside step function. In the same way that $i\hat{H}$ generates the time translation of a quantum field operator through $\hat{\varphi}(x) = \hat{\varphi}(t, \vec{x}) = e^{it\hat{H}}\hat{\varphi}(0, \vec{x})e^{-it\hat{H}}$, the imaginary-time formalism implies the relation

$$\hat{\varphi}(X) = \hat{\varphi}(\tau, \vec{x}) = e^{\tau\hat{H}}\hat{\varphi}(0, \vec{x})e^{-\tau\hat{H}}. \tag{C.81}$$

Using $\mathbb{1} = e^{\tau\hat{H}}e^{-\tau\hat{H}}$ and the cyclic property of the trace, we show that, assuming $\beta > \tau > 0$,

$$\begin{aligned}
D(\vec{x}, \vec{y}, \tau, 0) &= \langle\Omega| e^{-\beta\hat{H}} \mathrm{T}\left\{\varphi(\tau, \vec{x})\varphi(0, \vec{y})\right\} |\Omega\rangle \\
&= \frac{1}{Z}\mathrm{Tr}\left\{e^{-\beta\hat{H}}\varphi(\tau, \vec{x})\varphi(0, \vec{y})\right\} \\
&= \frac{1}{Z}\mathrm{Tr}\left\{\varphi(0, \vec{y})e^{-\beta\hat{H}}\varphi(\tau, \vec{x})\right\} \\
&= \frac{1}{Z}\mathrm{Tr}\left\{e^{-\beta\hat{H}}e^{\beta\hat{H}}\varphi(0, \vec{y})e^{-\beta\hat{H}}\varphi(\tau, \vec{x})\right\} \\
&= \frac{1}{Z}\mathrm{Tr}\left\{e^{-\beta\hat{H}}\varphi(\vec{y}, \beta)\varphi(\tau, \vec{x})\right\} \\
&= \nu \langle\Omega| e^{-\beta\hat{H}} \mathrm{T}\left\{\varphi(\tau, \vec{x})\varphi(\beta, \vec{y})\right\} |\Omega\rangle .
\end{aligned}$$

This implies that $\varphi(0, x) = \nu\varphi(\beta, \varphi)$, which shows that bosons are periodic in time, as stated earlier, while fermions are anti-periodic.

The Lagrangian density of a free fermion is

$$\mathcal{L} = \bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi. \tag{C.82}$$

This Lagrangian is invariant under the transformation $\psi \to e^{-i\alpha}\psi$, which by Nöther's theorem results in a conserved current

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta\psi = \bar{\psi}\gamma^\mu\psi. \tag{C.83}$$

The corresponding conserved charge is

$$Q = \int_V \mathrm{d}^3x\, j^0 = \int_V \mathrm{d}^3x\, \bar{\psi}\gamma^0\psi. \tag{C.84}$$

We can now use our earlier result for the thermal partition function, Eq. (C.10), only with the substitution $\mathcal{H} \to \mathcal{H} - \mu\bar{\psi}\gamma^0\psi$, and integrate over anti-periodic ψ's:

$$Z = \mathrm{Tr}\left\{e^{-\beta(\hat{H}-\mu\hat{Q})}\right\} = \prod_{ab}\int \mathcal{D}\psi_a\mathcal{D}\pi_b \exp\left\{\int_\Omega \mathrm{d}X\, \left(i\dot{\psi}\pi - \mathcal{H}(\psi, \pi) + \mu\bar{\psi}\gamma^0\psi\right)\right\},$$

where a, b are the spinor indices. The canonical momentum corresponding to ψ is

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} = i\bar{\psi}\gamma^0, \tag{C.85}$$

and the Hamiltonian density is

$$\mathcal{H} = \pi\dot{\psi} - \mathcal{L} = \bar{\psi}(-i\gamma^i\partial_i + m)\psi \tag{C.86}$$

which gives

$$\mathcal{L}_E = -i\pi\dot{\psi} + \mathcal{H}(\psi, \pi) - \mu\bar{\psi}\gamma^0\psi = \bar{\psi}[\gamma^0(\partial_\tau - \mu) - i\gamma^i\partial_i + m]\psi, \tag{C.87}$$

By using the Grassman-version of the Gaussian integral formula, the partition function can be written

$$\begin{aligned} Z &= \prod_{ab}\int \mathcal{D}\psi_a \mathcal{D}\bar{\psi}_b \exp\left\{-\int_\Omega \mathrm{d}X\, \bar{\psi}\left[\tilde{\gamma}_0(\partial_\tau - \mu) - i\gamma^i\partial_i + m\right]\psi\right\} \\ &= C\prod_{ab}\int \mathcal{D}\tilde{\psi}_a \mathcal{D}\tilde{\bar{\psi}}_b \exp\left\{-\int_{\tilde{\Omega}} \mathrm{d}K\, \tilde{\bar{\psi}}\left[i\tilde{\gamma}_0(\omega_n + i\mu) + i\gamma_i p_i + m\right]\tilde{\psi}\right\} \\ &= C\prod_{ab}\int \mathcal{D}\tilde{\psi}_a \mathcal{D}\tilde{\bar{\psi}}_b e^{-\langle\tilde{\bar{\psi}}, D^{-1}\tilde{\psi}\rangle} = \det(D^{-1}). \end{aligned}$$

In the second line, we have inserted the Fourier expansion of the field, as defined in section C.2, and changed variable of integration, as we did for the scalar field. The linear operator in this case is

$$D^{-1} = i\gamma^0(-i\partial_\tau + i\mu) - (-i\gamma^i)\partial_i + m = \beta[i\tilde{\gamma}_a p_a + m]. \tag{C.88}$$

This equality must be understood as an equality between linear operators, which are represented in different bases. We introduced the notation $p_a = (\omega_n + i\mu, p_i)$ and use the Euclidean gamma matrices, as defined in

include gamma + pauli matrices

We use the fact that

$$\det(i\tilde{\gamma}_a p_a + m) = \det(\gamma^5\gamma^5)\det(i\tilde{\gamma}_a p_a + m) = \det[\gamma^5(i\tilde{\gamma}_a p_a + m)\gamma^5] = \det(-i\tilde{\gamma}_a p_a + m),$$

Hva skjer med beta?

Let $\tilde{D}^{-1} = \beta[-i\tilde{\gamma}_a p_a + m]$, which means we can write

$$Z = \sqrt{\det(D^{-1})\det(\tilde{D}^{-1})} = \sqrt{\det(D^{-1}\tilde{D}^{-1})} = \det[\mathbb{1}\beta^2(p_a p_a + m^2)]^{1/2}, \tag{C.89}$$

where we have used the anti-commutation rule for the Euclidean gamma-matrices, $\{\gamma_a, \gamma_b\} = 2\delta_{ab}$. It is important to keep in mind that the determinant here refers to linear operators on the space of spinor functions. Thus

Vær konsekvent på bruk av k og p

$$\begin{aligned} \ln(Z) &= \ln\left\{\det[\mathbb{1}\beta^2(p_a p_a + m^2)]^{1/2}\right\} = \frac{1}{2}\mathrm{Tr}\left\{\ln[\mathbb{1}\beta^2(p_a p_a + m^2)]\right\} \\ &= 2\int_{\tilde{\Omega}} \mathrm{d}K\, \ln\{\beta^2[(\omega_n + i\mu)^2 + \omega_k^2]\}. \end{aligned} \tag{C.90}$$

In the last step, we used the fact that the matrix within the logarithm is diagonal. The matrix-part of the trace is trivial therefore trivial. Using the fermionic version of the thermal sum from subsection C.3.4 gives the answer

$$\mathcal{F} = -\frac{2}{\beta}\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\left[\beta\omega_k + \ln\left(1 + e^{-\beta(\omega_k - \mu)}\right) + \ln\left(1 + e^{-\beta(\omega_k + \mu)}\right)\right]. \tag{C.91}$$

We see again that the temperature-independent part of the integral diverges, and must be regulated. There are two temperature-dependent terms, one from the particle and one from the anti-particle.

Appendix D

Code

All code is available at: `https://github.com/martkjoh/master`.

D.1 Integrating the TOV equations

For numerical integration of the TOV equations, we use SciPy's `integrate.solve_ivp`.[1] Equations of state are evaluated either as explicit functions if a closed-form is available or as an interpolating function is created using a cubic spline without smoothing. All code is written using dimensionless variables, and setting $k_1 = k_2 = k_3$. The TOV equation is then Eq. (4.52)

$$\frac{d\tilde{m}}{d\tilde{r}} = 3\tilde{r}^2\tilde{u}, \quad \frac{d\tilde{p}}{d\tilde{r}} = -\frac{1}{\tilde{r}^2}\left(\tilde{p}+\tilde{u}\right)\left(3\tilde{r}^3\tilde{p}+\tilde{m}\right)\left(1-\frac{2\tilde{m}}{\tilde{r}}\right)^{-1}. \tag{D.1}$$

As $r \to 0$, parts of the TOV equation Eq. (4.52) approaches a 0/0-limit, and we must make use of an approximation for numeric evaluation. The Taylor-expansion of the mass function around $\tilde{r} = 0$ is

$$\tilde{m}(r) = \tilde{m}(0) + \tilde{m}'(0)\,\tilde{r} + \frac{1}{2!}\tilde{m}''(0)\tilde{r}^2 + \frac{1}{3!}\tilde{m}'''(0)\tilde{r}^3 + \mathcal{O}\left(\tilde{r}^4\right). \tag{D.2}$$

One of the boundary conditions is $\tilde{m}(0) = 0$. We then use the differential equation for $\tilde{m}$, Eq. (4.37), to find

$$\tilde{m}'(0) = 0, \quad \tilde{m}''(0) = 0, \quad \tilde{m}'''(0) = 6k_2\tilde{u}_0, \tag{D.3}$$

where $\tilde{u}_0 = \tilde{u}(r=0)$. We get an approximation of the TOV equation for $\tilde{r} \ll 1$ by substituting the $\tilde{m}$ for its Taylor expansion and including only the leading-order term, which gives

$$\frac{d\tilde{p}}{d\tilde{r}} \sim -\tilde{r}\,\left(\tilde{p}+\tilde{u}\right)\left(3\tilde{p}+\tilde{u}_0\right)\left(1-2\tilde{u}_0\tilde{r}^2\right)^{-1}, \quad r \to 0 \tag{D.4}$$

For the Newtonian approximation to the TOV equation, we get

$$\frac{d\tilde{p}}{d\tilde{r}} = -\frac{\tilde{u}\tilde{m}}{\tilde{r}^2} \sim -\tilde{u}\tilde{u}_0\tilde{r}, \quad r \to 0. \tag{D.5}$$

D.2 Spherically symmetric metric

The calculations in chapter 4 were done using a CAS system. The code is written in Python in a Jupyter notebook. The full `.ipynb` file with executable code is available in the repository, at `https://github.com/martkjoh/master/blob/main/scripts/TOV.ipynb` Below is some of the code, which illustrates the main functions and the outputs.

[1] Reference available here: `https://docs.scipy.org/doc/scipy/reference/generated/scipy.integrate.solve_ivp.html`.

1 Metric $g_{\mu\nu}$ for spherically symmetric spacetime

[8]:
```
t, r, th, ph = symbols("t, r, \\theta, \\phi")
x1 = r * cos(ph) * sin(th)
x2 = r * sin(ph) * sin(th)
x3 = r * cos(th)

one = Rational(1)
eta = sp.diag(one, -one, -one, -one)
var = (t, r, th, ph)
J = Matrix([t, x1, x2, x3]).jacobian(var)
g = np.array(simplify(J.T *eta* J))

a = sp.Function("\\alpha", real=True)(r)
b = sp.Function("\\beta", real=True)(r)
g[0, 0] *= exp(2 * a)
g[1, 1] *= exp(2 * b)
g_inv = get_g_inv(g)

print_matrix(g)
print_matrix(g_inv)
```

$$\begin{bmatrix} e^{2\alpha(r)} & 0 & 0 & 0 \\ 0 & -e^{2\beta(r)} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2(\theta) \end{bmatrix}$$

$$\begin{bmatrix} e^{-2\alpha(r)} & 0 & 0 & 0 \\ 0 & -e^{-2\beta(r)} & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{r^2 \sin^2(\theta)} \end{bmatrix}$$

[9]:
```
C = Christoffel(g, g_inv, var)
c = print_christoffel(C, var)
```

$$\Gamma^{t}_{\mu\nu} = \begin{bmatrix} 0 & \frac{d}{dr}\alpha(r) & 0 & 0 \\ \frac{d}{dr}\alpha(r) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Gamma^{r}_{\mu\nu} = \begin{bmatrix} e^{2\alpha(r)}e^{-2\beta(r)}\frac{d}{dr}\alpha(r) & 0 & 0 & 0 \\ 0 & \frac{d}{dr}\beta(r) & 0 & 0 \\ 0 & 0 & -re^{-2\beta(r)} & 0 \\ 0 & 0 & 0 & -re^{-2\beta(r)}\sin^2(\theta) \end{bmatrix}$$

$$\Gamma^{\theta}_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\sin(\theta)\cos(\theta) \end{bmatrix}$$

$$\Gamma^{\phi}_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \frac{\cos(\theta)}{\sin(\theta)} \\ 0 & \frac{1}{r} & \frac{\cos(\theta)}{\sin(\theta)} & 0 \end{bmatrix}$$

[10]:
```
Rie = Riemann_tensor(C, var)
Ricci = contract(Rie, num_indx=4, upper=1, indx=(0, 2))

for i in range(4):
    print_scalar(Ricci[i, i].factor())
```

$$\frac{\left(r\left(\frac{d}{dr}\alpha(r)\right)^2 - r\frac{d}{dr}\alpha(r)\frac{d}{dr}\beta(r) + r\frac{d^2}{dr^2}\alpha(r) + 2\frac{d}{dr}\alpha(r)\right)e^{2\alpha(r)}e^{-2\beta(r)}}{r}$$

$$-\frac{r\left(\frac{d}{dr}\alpha(r)\right)^2 - r\frac{d}{dr}\alpha(r)\frac{d}{dr}\beta(r) + r\frac{d^2}{dr^2}\alpha(r) - 2\frac{d}{dr}\beta(r)}{r}$$

$$-\left(r\frac{d}{dr}\alpha(r) - r\frac{d}{dr}\beta(r) - e^{2\beta(r)} + 1\right)e^{-2\beta(r)}$$

$$-\left(r\frac{d}{dr}\alpha(r) - r\frac{d}{dr}\beta(r) - e^{2\beta(r)} + 1\right)e^{-2\beta(r)}\sin^2(\theta)$$

[11]:
```
R = contract(Ricci, g_inv=g_inv, upper=0).simplify()
print_scalar(R)
```

$$\frac{2\left(r^2\left(\frac{d}{dr}\alpha(r)\right)^2 - r^2\frac{d}{dr}\alpha(r)\frac{d}{dr}\beta(r) + r^2\frac{d^2}{dr^2}\alpha(r) + 2r\frac{d}{dr}\alpha(r) - 2r\frac{d}{dr}\beta(r) - e^{2\beta(r)} + 1\right)e^{-2\beta(r)}}{r^2}$$

[12]:
```
G = Ricci - Rational(1, 2) * R * g
for i in range(4):
    G[i, i] = G[i, i].simplify().factor()
    print_scalar(G[i, i])
```

$$\frac{\left(2r\frac{d}{dr}\beta(r) + e^{2\beta(r)} - 1\right)e^{2\alpha(r)}e^{-2\beta(r)}}{r^2}$$

$$\frac{2r\frac{d}{dr}\alpha(r) - e^{2\beta(r)} + 1}{r^2}$$

$$r\left(r\left(\frac{d}{dr}\alpha(r)\right)^2 - r\frac{d}{dr}\alpha(r)\frac{d}{dr}\beta(r) + r\frac{d^2}{dr^2}\alpha(r) + \frac{d}{dr}\alpha(r) - \frac{d}{dr}\beta(r)\right)e^{-2\beta(r)}$$

$$r\left(r\left(\frac{d}{dr}\alpha(r)\right)^2 - r\frac{d}{dr}\alpha(r)\frac{d}{dr}\beta(r) + r\frac{d^2}{dr^2}\alpha(r) + \frac{d}{dr}\alpha(r) - \frac{d}{dr}\beta(r)\right)e^{-2\beta(r)}\sin^2(\theta)$$

1.0.1 Stress-energy tensor $T_{\mu\nu}$ for perfect fluid

[13]:
```
p = sp.Function("p")(r)
u = sp.Function("u")(r)

UU = np.zeros((4, 4), dtype=sp.Rational)
UU[0, 0] = exp(2 * a)

T = (p + u) * UU - p * g
for i in range(4):
    T[i, i] = T[i, i].simplify()
print_matrix(T)
```

$$\begin{bmatrix} u(r)e^{2\alpha(r)} & 0 & 0 & 0 \\ 0 & p(r)e^{2\beta(r)} & 0 & 0 \\ 0 & 0 & r^2p(r) & 0 \\ 0 & 0 & 0 & r^2p(r)\sin^2(\theta) \end{bmatrix}$$

2 Einstin's field equations

$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}$

[14]:
```
G_newton = sp.Symbol("G")

eq = []
for i in range(len(G)):
    eq.append((G[i, i] - 8 * pi * G_newton * T[i, i]).simplify())

# Some manual simplification
Rtt = sp.Symbol("R_{\\theta \\theta}")
eq[0] = eq[0] * r**2 / exp(2 * a)/exp(-2*b ) * (-1 )
eq[1] = eq[1] * r**2 * (-1)
eq[2] = eq[2] / r / exp(-2*b)
eq[3] = eq[3].subs(eq[2], Rtt)
for i in range(len(G)):
    print_eq(eq[i].simplify())
```

$$8\pi G r^2 u(r) e^{2\beta(r)} - 2r\frac{d}{dr}\beta(r) - e^{2\beta(r)} + 1 = 0$$

$$8\pi G r^2 p(r) e^{2\beta(r)} - 2r\frac{d}{dr}\alpha(r) + e^{2\beta(r)} - 1 = 0$$

$$-8\pi G r p(r) e^{2\beta(r)} + r\left(\frac{d}{dr}\alpha(r)\right)^2 - r\frac{d}{dr}\alpha(r)\frac{d}{dr}\beta(r) + r\frac{d^2}{dr^2}\alpha(r) + \frac{d}{dr}\alpha(r) - \frac{d}{dr}\beta(r) = 0$$

$$R_{\theta\theta} r e^{-2\beta(r)}\sin^2(\theta) = 0$$

Define $e^{2\beta} = [1 - 2Gm(r)/r]^{-1}$

[15]:
```
m = sp.Function("m", Real=True)(r)
f = (1 - 2 * G_newton * m / r)**(-1)
eq1 = (eq[0] * exp(- 2 *a)).simplify().subs(b, Rational(1, 2) * log(f)).
 ↪simplify().expand().simplify()
s = sp.solve(eq1, m.diff(r))
eq1 = m.diff(r) - s[0]
```

Use $\nabla_\mu T^{\mu r} = 0 \implies (p+\rho)\partial_r \alpha = -\partial_r p$.

[16]:
```
eq2 = (eq[1] * r**2).subs(exp(2 * b), f).simplify()
s = sp.solve(eq2, a.diff(r))
eq2 = a.diff(r) - s[0]
eq2 = ((a.diff(r) - s[0]).subs(a.diff(r), - p.diff(r) / (p + u))*(p + u)).
 ↪simplify()
s = sp.solve(eq2, p.diff())
eq2 = p.diff(r) - s[0].factor()
```

The TOV-equation and equation for $m(r)$, both expressions are equal to 0.

[17]:
```
print_eq(eq1)
print_eq(eq2)
```

$$-4\pi r^2 u(r) + \frac{d}{dr} m(r) = 0$$

$$\frac{G\left(4\pi r^3 p(r) + m(r)\right)\left(p(r) + u(r)\right)}{r\left(-2Gm(r) + r\right)} + \frac{d}{dr} p(r) = 0$$

Bibliography

[1] Particle Data Group et al. "Review of Particle Physics". In: *Progress of Theoretical and Experimental Physics* 2020.8 (Aug. 2020), p. 083C01. ISSN: 2050-3911. DOI: 10.1093/ptep/ptaa104.

[2] S. M. Carroll. *Spacetime and Geometry: An Introduction to General Relativity*. First. Cambridge University Press, Aug. 2019. ISBN: 978-1-108-48839-6 978-1-108-77038-5. DOI: 10.1017/9781108770385.

[3] J. M. Lee. *Introduction to Smooth Manifolds*. Springer Science & Business Media, 2003. ISBN: 978-0-387-95448-6.

[4] M. E. Peskin and D. V. Schroeder. *An Introduction to Quantum Field Theory*. Reading, USA: Addison-Wesley, 1995. ISBN: 978-0-201-50397-5.

[5] M. D. Schwartz. *Quantum Field Theory and the Standard Model*. https://www.cambridge.org/highereducation/books/quantum-field-theory-and-the-standard-model/A4CD66B998F2C696DCC75B984A7D5799. Dec. 2013. DOI: 10.1017/9781139540940.

[6] S. Weinberg. *The Quantum Theory of Fields: Volume 1: Foundations*. Vol. 1. Cambridge: Cambridge University Press, 1995. ISBN: 978-0-521-67053-1. DOI: 10.1017/CBO9781139644167.

[7] S. Weinberg. *The Quantum Theory of Fields: Volume 2: Modern Applications*. Vol. 2. Cambridge: Cambridge University Press, 1996. ISBN: 978-0-521-55002-4. DOI: 10.1017/CBO9781139644174.

[8] T. Brauner. "Spontaneous Symmetry Breaking and Nambu-Goldstone Bosons in Quantum Many-Body Systems". In: *Symmetry* 2.2 (Apr. 2010), pp. 609–657. ISSN: 2073-8994. DOI: 10.3390/sym2020609. arXiv: 1001.5212.

[9] L. A. Morrison. *Coleman-Callan-Wess-Zumino Construction*. http://scipp.ucsc.edu/~haber/archives/physics251_17/PH sentation_L_Morrison. July 2017.

[10] G. Panico and A. Wulzer. "The Composite Nambu-Goldstone Higgs". In: *arXiv:1506.01961 [hep-ph]* 913 (2016). DOI: 10.1007/978-3-319-22617-0. arXiv: 1506.01961 [hep-ph].

[11] A. Pich. "Effective Field Theory with Nambu-Goldstone Modes". In: *arXiv:1804.05664 [hep-ph, physics:hep-th]* (Jan. 2020). arXiv: 1804.05664 [hep-ph, physics:hep-th].

[12] C. G. Callan et al. "Structure of Phenomenological Lagrangians. II". In: *Phys. Rev.* 177.5 (Jan. 1969), pp. 2247–2250. DOI: 10.1103/PhysRev.177.2247.

[13] S. Coleman, J. Wess, and B. Zumino. "Structure of Phenomenological Lagrangians. I". In: *Phys. Rev.* 177.5 (Jan. 1969), pp. 2239–2247. DOI: 10.1103/PhysRev.177.2239.

[14] H. Watanabe and H. Murayama. "Effective Lagrangian for Nonrelativistic Systems". In: *Phys. Rev. X* 4.3 (Sept. 2014), p. 031057. ISSN: 2160-3308. DOI: 10.1103/PhysRevX.4.031057. arXiv: 1402.7066.

[15] R. Penco. "An Introduction to Effective Field Theories". In: *arXiv:2006.16285 [hep-th]* (June 2020). arXiv: 2006.16285 [hep-th].

[16] S. Weinberg. "On the Development of Effective Field Theory". In: *EPJ H* 46.1 (Mar. 2021), p. 6. ISSN: 2102-6467. DOI: 10.1140/epjh/s13129-021-00004-x.

[17] S. Weinberg. "Phenomenological Lagrangians". In: *Physica A: Statistical Mechanics and its Applications* 96.1 (Apr. 1979), pp. 327–340. ISSN: 0378-4371. DOI: 10.1016/0378-4371(79)90223-1.

[18] S. Scherer. "Introduction to Chiral Perturbation Theory". In: *arXiv:hep-ph/0210398* (Oct. 2002). arXiv: hep-ph/0210398.

[19] J. Gasser and H. Leutwyler. "Chiral Perturbation Theory to One Loop". In: *Annals of Physics* 158.1 (Nov. 1984), pp. 142–210. ISSN: 0003-4916. DOI: 10.1016/0003-4916(84)90242-2.

[20] J. Gasser and H. Leutwyler. "Chiral Perturbation Theory: Expansions in the Mass of the Strange Quark". In: *Nuclear Physics B* 250.1 (Jan. 1985), pp. 465–516. ISSN: 0550-3213. DOI: 10.1016/0550-3213(85)90492-4.

[21] H. Leutwyler. "On the Foundations of Chiral Perturbation Theory". In: *Annals of Physics* 235.1 (Oct. 1994), pp. 165–203. ISSN: 0003-4916. DOI: 10.1006/aphy.1994.1094.

[22] S. Weinberg. *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*. New York: Wiley, 1972. ISBN: 978-0-471-92567-5.

[23] N. K. Glendenning. *Compact Stars: Nuclear Physics, Particle Physics and General Relativity*. Springer Science & Business Media, Dec. 2012. ISBN: 978-1-4684-0491-3.

[24] C. W. Misner, K. S. Thorne, and J. A. Wheeler. "Gravitation". In: (Dec. 2009).

[25] J. R. Oppenheimer and G. M. Volkoff. "On Massive Neutron Cores". In: *Phys. Rev.* 55.4 (Feb. 1939), pp. 374–381. DOI: 10.1103/PhysRev.55.374.

[26] R. C. Tolman. *Relativity, Thermodynamics, and Cosmology*. The International Series of Monographs on Physics. Oxford: Clarendon Press, 1934.

[27] S. Chandrasekhar. "The Highly Collapsed Configurations of a Stellar Mass. (Second Paper.)" In: *Monthly Notices of the Royal Astronomical Society* 95.3 (Jan. 1935), pp. 207–225. ISSN: 0035-8711. DOI: 10.1093/mnras/95.3.207.

[28] H. J. Lane. "On the Theoretical Temperature of the Sun, under the Hypothesis of a Gaseous Mass Maintaining Its Volume by Its Internal Heat, and Depending on the Laws of Gases as Known to Terrestrial Experiment". In: *American Journal of Science* s2-50.148 (July 1870), pp. 57–74. ISSN: 0002-9599, 1945-452X. DOI: 10.2475/ajs.s2-50.148.57.

[29] J. O. Andersen. *Introduction to Statistical Mechanics*. Akademika forlag, 2012. ISBN: 978-82-321-0105-4.

[30] J. I. Kapusta and C. Gale. *Finite-Temperature Field Theory: Principles and Applications*. Second. Cambridge Monographs on Mathematical Physics. Cambridge: Cambridge University Press, 2006. ISBN: 978-0-521-82082-0. DOI: 10.1017/CBO9780511535130.

[31] C. E. Rhoades and R. Ruffini. "Maximum Mass of a Neutron Star". In: *Phys. Rev. Lett.* 32.6 (Feb. 1974), pp. 324–327. DOI: 10.1103/PhysRevLett.32.324.

[32] S. Chandrasekhar. "Dynamical Instability of Gaseous Masses Approaching the Schwarzschild Limit in General Relativity". In: *Phys. Rev. Lett.* 12.4 (Jan. 1964), pp. 114–116. DOI: 10.1103/PhysRevLett.12.114.

[33] K. S. Thorne. "The General-Relativistic Theory of Stellar Structure and Dynamics". In: *pp 166-280 of High-Energy Astrophysics. Gratton, L. (ed.). New York and London, Academic Press, 1966.* (Oct. 1968).

[34] G. Ecker. "Chiral Perturbation Theory". In: *Progress in Particle and Nuclear Physics* 35 (Jan. 1995), pp. 1–80. ISSN: 01466410. DOI: 10.1016/0146-6410(95)00041-G. arXiv: hep-ph/9501357.

[35] H. W. Fearing and S. Scherer. "Extension of the Chiral Perturbation Theory Meson Lagrangian to Order ${\mathit{p}}{6}$". In: *Phys. Rev. D* 53.1 (Jan. 1996), pp. 315–348. DOI: 10.1103/PhysRevD.53.315.

[36] P. Adhikari, J. O. Andersen, and P. Kneschke. "Two-Flavor Chiral Perturbation Theory at Nonzero Isospin: Pion Condensation at Zero Temperature". In: *Eur. Phys. J. C* 79.10 (Oct. 2019), p. 874. ISSN: 1434-6044, 1434-6052. DOI: 10.1140/epjc/s10052-019-7381-4. arXiv: 1904.03887.

[37] G. Ecker et al. "The Role of Resonances in Chiral Perturbation Theory". In: *Nuclear Physics B* 321.2 (July 1989), pp. 311–342. ISSN: 0550-3213. DOI: 10.1016/0550-3213(89)90346-5.

[38] J. B. Kogut and D. Toublan. "QCD at Small Nonzero Quark Chemical Potentials". In: *Phys. Rev. D* 64.3 (July 2001), p. 034007. DOI: 10.1103/PhysRevD.64.034007.

[39] P. Adhikari, J. O. Andersen, and M. A. Mojahed. "Quark, Pion and Axial Condensates in Three-Flavor Finite Isospin Chiral Perturbation Theory". In: *Eur. Phys. J. C* 81.5 (May 2021), p. 449. ISSN: 1434-6052. DOI: 10.1140/epjc/s10052-021-09212-7.

[40] R. Dashen. "Chiral $\mathrm{SU(3)\ensuremath{\bigotimes}\mathrm{SU(3)$ as a Symmetry of the Strong Interactions". In: *Phys. Rev.* 183.5 (July 1969), pp. 1245–1260. DOI: 10.1103/PhysRev.183.1245.

[41] T. Das et al. "Electromagnetic Mass Difference of Pions". In: *Phys. Rev. Lett.* 18.18 (May 1967), pp. 759–761. DOI: 10.1103/PhysRevLett.18.759.

[42] S. Weinberg. "Precise Relations between the Spectra of Vector and Axial-Vector Mesons". In: *Phys. Rev. Lett.* 18.13 (Mar. 1967), pp. 507–509. DOI: 10.1103/PhysRevLett.18.507.

[43] R. Urech. "Virtual Photons in Chiral Perturbation Theory". In: *Nuclear Physics B* 433.1 (Jan. 1995), pp. 234–254. ISSN: 05503213. DOI: 10.1016/0550-3213(95)90707-N. arXiv: hep-ph/9405341.

[44] M. Martin Aria. "Two-Flavor Chiral Perturbation Theory at Finite Isospin beyond Leading Order". Master's Thesis. NTNU, 2020.

[45] D. T. Son and M. A. Stephanov. "QCD at a Finite Isospin Density: From the Pion to Quark-Antiquark Condensation". In: *Phys. Atom. Nuclei* 64.5 (May 2001), pp. 834–842. ISSN: 1562-692X. DOI: 10.1134/1.1378872.

[46] V. I. Borodulin, R. N. Rogalyov, and S. R. Slabospitskii. "CORE 3.1 (COmpendium of RElations, Version 3.1)". In: *arXiv:1702.08246 [hep-ph]* (Mar. 2017). arXiv: 1702.08246 [hep-ph].

[47] D. T. Son and M. A. Stephanov. "QCD at Finite Isospin Density". In: (May 2000). DOI: 10.1103/PhysRevLett.86.592.

[48] J. S. R. Chisholm. "Change of Variables in Quantum Field Theories". In: *Nuclear Physics* 26.3 (Aug. 1961), pp. 469–479. ISSN: 0029-5582. DOI: 10.1016/0029-5582(61)90106-7.

[49] S. Kamefuchi, L. O'Raifeartaigh, and A. Salam. "Change of Variables and Equivalence Theorems in Quantum Field Theories". In: *Nuclear Physics* 28.4 (Dec. 1961), pp. 529–549. ISSN: 0029-5582. DOI: 10.1016/0029-5582(61)90056-6.

[50] P. Gerber and H. Leutwyler. "Hadrons below the Chiral Phase Transition". In: *Nuclear Physics B* 321.2 (July 1989), pp. 387–429. ISSN: 0550-3213. DOI: 10.1016/0550-3213(89)90349-0.

[51] M. Laine and A. Vuorinen. "Basics of Thermal Field Theory – a Tutorial on Perturbative Computations". In: *arXiv:1701.01554 [hep-ph]* 925 (2016). DOI: 10.1007/978-3-319-31933-9. arXiv: 1701.01554 [hep-ph].