Master

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February 1, 2022

Todo list

Contents

Chapter 1

General relativity and the TOV-equation

General relativity describes how matter and energy curve the fabric of space and time. Einstein first wrote down the theory more than a century ago, and it is still our most accurate theory of gravitational effects. It makes accurate and counterintuitive predictions, which experiments have borne out. This chapter surveys the basics of general relativity and some mathematical prerequisites. We will then use this to derive the Tolman-Oppenheimer-Volkoff (TOV) equation, a differential equation used to model stars. This chapter is based on [1, 2].

1.1 Differential geometry

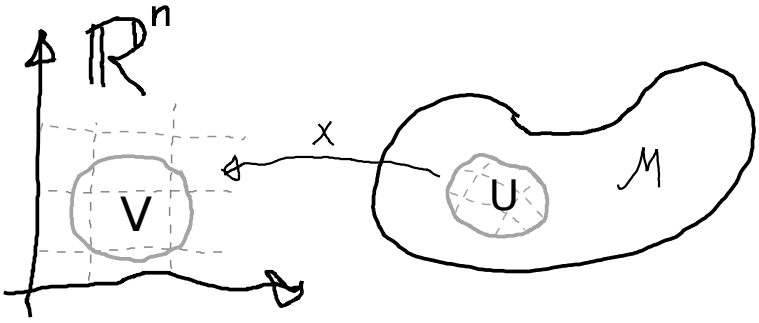

Figure 1.1: (Kladd) The coordinate function x maps a neighborhood U in the manifold $\mathcal{M}$ to a negiborhood V in $\mathbb{R}^n$.

General relativity is formulated in the language of *differential geometry*, which generalizes multivariable calculus to more general spaces than $\mathbb{R}^n$. These spaces are *smooth manifolds*. A n-dimensional manifold, $\mathcal{M}$, is a set of points, locally homeomorphic to $\mathbb{R}^n$. That is, for all points $p \in \mathcal{M}$, there exists a neighborhood U around p, together with a corresponding set of continuous, bijective functions,

$$x : U \subseteq \mathcal{M} \longmapsto V \subseteq \mathbb{R}^n, \tag{1.1}$$

$$p \longmapsto x^\mu(p). \tag{1.2}$$

We call $x(p) = (x^0(p), \ldots, x^{n-1}(p)) = x^\mu(p)$ a coordinate function of $\mathcal{M}$. The inverse of x, x^{-1}, obeys $x^{-1}(x(p)) = p$, for all $p \in U$. A smooth manifold is one in which the coordinate functions are infinitely differentiable. Differentiability is defined by considering two different coordinate functions, x, and x'. The corresponding domains U and U' may or may not overlap. We then define the transition function, a function

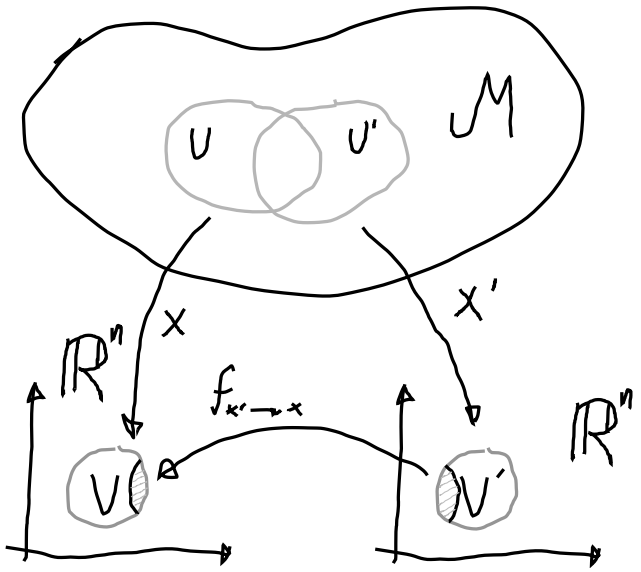

Figure 1.2: (Kladd) The transition map $f_{x'\to x}$ between two coordinate functions, x and x', maps between the images of these function, via the manifold $\mathcal{M}$. The function's domain and image are restricted to a (possibly empty) subset of the images of x and x'. This is illustrated by the shaded regions in V and V'.

between subsets of $\mathbb{R}^n$ by mapping via $\mathcal{M}$, as

$$f_{x'\to x} = x' \circ x^{-1} : \mathbb{R}^n \mapsto \mathbb{R}^n. \tag{1.3}$$

The map is illustrated in Figure 1.2. [1] A set of coordinate functions $\mathcal{A} = \{x_i\}$ whose domain cover $\mathcal{M}$ is called an *atlas* of $\mathcal{M}$. If the transition function between all pairings of coordinate functions in the atlas is smooth—that is, infinitely differentiable—we call the atlas smooth. We then define a smooth manifold as the topological manifold $\mathcal{M}$ together with a *maximal* smooth atlas $\mathcal{A}$. A smooth atlas is maximal if no coordinate function can be added while the atlas remains smooth.[2]

Consider two m- and n-dimensional smooth manifolds $\mathcal{M}$ and $\mathcal{N}$. Let x denoted the coordinates on $\mathcal{M}$, while y denotes the coordinates on $\mathcal{N}$. We can define smooth functions between these manifolds similarly to how we define smooth coordinates. Consider the function

$$F : \mathcal{M} \longmapsto \mathcal{N}. \tag{1.4}$$

It is said to be smooth if, for all points $p \in M$, there is a set of local coordinates x around p and y around $F(p)$ such that the map $\tilde{F} = y \circ F \circ x^{-1}$ is smooth. This map is defined by the diagram

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{F} & \mathcal{N} \\ \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle y} \\ \mathbb{R}^m & \xrightarrow{\tilde{F}} & \mathbb{R}^n \end{array} \tag{1.5}$$

We will not be careful with the distinction between F, the function between the abstract manifolds, and $\tilde{F}$, the function of their coordinates, but rather denote both by $F(x)$. We may take the partial derivative of such a function with respect to the coordinates x, $\partial F/\partial x^\mu$. However, this is obviously dependent on our choice of coordinates, as a set of local coordinates can always be scaled. Any physical theory must be independent of our choice of coordinates, so our next task is to define the properties of a smooth manifold in a coordinate independent way.

[1] To be rigorous, one has to restrict the domains and image of the coordinate function when combining them.

[2] The maximal condition is to ensure that two equivalent atlases correspond to the same differentiable manifold. A single manifold can be combined with different maximal atlases, also called differentiable structures.

Vectors and tensors

A curve γ through $\mathcal{M}$ is a function from $\mathbb{R}$ to $\mathcal{M}$,

$$\gamma : \mathbb{R} \longmapsto \mathcal{M} \tag{1.6}$$
$$\lambda \longmapsto \gamma(\lambda). \tag{1.7}$$

Such curves are often denoted only by their coordinates and the parameter λ, $x^\mu(\lambda) = (x^\mu \circ \gamma)(\lambda)$. With this curve, we can take the directional derivative of a real-valued function on the manifold, $f : \mathcal{M} \mapsto \mathbb{R}$. Assume $\gamma(\lambda = 0) = p$. As we are always taking the derivative of functions between $\mathbb{R}^n$, for different n, we can use the chain rule. The directional derivative of f at p, given by this curve γ, is then

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} f(x(\lambda))\Big|_p = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\Big|_{\lambda=0} \frac{\partial}{\partial x^\mu} f(x)\Big|_p . \tag{1.8}$$

The set of all such directional derivatives, $\mathrm{d}/\mathrm{d}\lambda$ at p, form a vector space, $T_p\mathcal{M}$, called the *tangent space*. The tangent space is illustrated in Figure 1.3. The coordinates x^μ induce a basis of this vector space,

Find the best notation here

$$e_\mu = \frac{\partial}{\partial x^\mu}\Big|_p = \partial_\mu|_p, \quad \mu \in \{0, ...n-1\}, \tag{1.9}$$

so any element $v \in T_p\mathcal{M}$ can be written

$$v = v^\mu \partial_\mu|_p = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\Big|_{\lambda=0} \frac{\partial}{\partial x^\mu}\Big|_p . \tag{1.10}$$

Here, λ is the parameter of the curve corresponding to the directional derivative v.[3] We assume $\lambda = 0$ corresponds to p. The evaluation at $\lambda = 0$ and p will often be implicit for ease of notation. This directional derivative acts on functions $f : \mathcal{M} \mapsto \mathbb{R}$ as

$$v(f) = v^\mu \partial_\mu f. \tag{1.11}$$

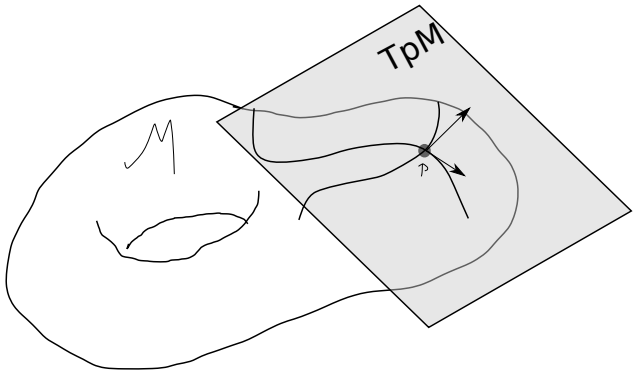

Figure 1.3: (Kladd) The tangent space $T_p\mathcal{M}$, the shaded rectangle, is the sett of all directional derivatives at $p \in \mathcal{M}$. A directional derivative is defined in terms of a curve that passes through p.

A map F between two manifolds $\mathcal{M}$ and $\mathcal{N}$ also induces a map between the tangent spaces of these manifolds. This is the *differential* of F at p,

$$\mathrm{d}F_p : T_p\mathcal{M} \longmapsto T_p\mathcal{N}, \tag{1.12}$$
$$v \longmapsto \mathrm{d}F_p(v). \tag{1.13}$$

As $\mathrm{d}F_p(v)$ is an element of $T_p\mathcal{N}$, directional derivative on $\mathcal{N}$, defined as

$$\mathrm{d}F_p(v)(g) = v(g \circ F), \tag{1.14}$$

[3]There is not only one curve corresponding to any directional derivative but rather an equivalence class.

for functions $g : \mathcal{N} \mapsto \mathbb{R}$. It thus acts on functions on $\mathcal{N}$ by "extending" the derivative v. This is a linear map between vector spaces and may be written in component form by considering the differentials of the coordinate functions. Denote the coordinates of $\mathcal{N}$ by y^μ, and $y^\mu \circ F = F^\mu$. Then,

$$\mathrm{d}F_p(\partial_\mu)(g) = \partial_\mu(g \circ F)|_p = \frac{\partial F^\nu}{\partial x^\mu}\Big|_p \frac{\partial g}{\partial y^\nu}\Big|_{F(p)}, \tag{1.15}$$

or more suggestively

$$\mathrm{d}F\left(\frac{\partial}{\partial x^\mu}\right) = \frac{\partial F^\nu}{\partial x^\mu}\frac{\partial}{\partial y_\nu}. \tag{1.16}$$

The differential is thus a generalization of the Jacobian of a function. In the case of a real valued function, $f : \mathcal{M} \mapsto \mathbb{R}$, and $g : \mathbb{R} \mapsto \mathbb{R}$, we get

$$\mathrm{d}f(v)(g) = v(g \circ f) = (v^\mu \partial_\mu f)\frac{\mathrm{d}g}{\mathrm{d}y}. \tag{1.17}$$

$\mathrm{d}f$ is thus a map from $T_p\mathcal{M}$ to $\_f(p)\mathbb{R}$, which is isomorphic to $\mathbb{R}$, i.e., it is a map from vectors v to real numbers,

$$\mathrm{d}f(v) = \mathrm{d}f(v)(y) = v^\mu \partial_\mu f. \tag{1.18}$$

The set of all linear maps from a vector space V to the real numbers is called the *dual space* of V, and is denoted V^*. This is a new vector space with the same dimensionality of V. We denote the dual of $T_p\mathcal{M}$ as $T_p^*\mathcal{M}$. We can regard each of the coordinate functions as real-valued functions, with a corresponding differential. This differential obeys

$$\mathrm{d}x^\mu(\partial_\nu) = \frac{\partial x^\mu}{\partial x_\nu} = \delta^\mu_\nu. \tag{1.19}$$

The differentials of the coordinate functions thus form a basis for $T_p^*\mathcal{M}$, called the dual basis. Using this, we can assume $\mathrm{d}f = \omega_\mu \mathrm{d}x^\nu$ for some components ω_μ, and find that $\omega_\mu = \partial_\mu f$. Or, in other words, we recover a rigorous justification for the classical expression

$$\mathrm{d}f = \frac{\partial f}{\partial x^\mu}\mathrm{d}x^\mu, \tag{1.20}$$

however we now interpret it as a covector-field instead of an "infinitesimal displacement".

Linear map from vectors to real numbers is generalized by *tensors*. Given a vector space V, a general (n, m) tensor T is a multilinear map, which associates n elements from V and m from its dual V^* to the real numbers, i.e.,

$$T : V \times V \times \cdots \times V^* \times \ldots \longmapsto \mathbb{R}, \tag{1.21}$$

$$(v, u \ldots; \omega, \ldots) \longmapsto T(v, u, \ldots; \omega, \ldots). \tag{1.22}$$

Multilinear means that T is linear in each argument. The set of all such maps is the tensor product space $V \otimes V \otimes \cdots \otimes V^* \otimes \ldots$, a $\dim(V)^{n+m}$-dimensional vector space. If $\{e_\mu\}$ and $\{e^\mu\}$ are the basis for V and V^*, then we can write the basis of this of the tensor product space as $\{e_\mu \otimes \cdots \otimes e^\nu \otimes \ldots\}$. The tensor can thus be written

$$T = T^{\mu\nu\ldots}{}_{\rho\ldots}\, e_\mu \otimes e_\nu \otimes \ldots e^\rho \otimes \ldots, \tag{1.23}$$

where

$$T^{\mu\nu\ldots}{}_{\rho\ldots} = T(e^\mu e^\nu, \ldots; e_\rho, \ldots). \tag{1.24}$$

Geometries and the metric

The metric is a symmetric, non-degenerate $(0, 2)$ tensor

$$\mathrm{d}s^2 = g_{\mu\nu}\, \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu. \tag{1.25}$$

It defines the geometry of the manifold $\mathcal{M}$, and is the main object of study in general relativity. As it is invertible, we can define $g^{\mu\nu} = (g^{-1})_{\mu\nu}$, which is the components of a $(2, 0)$ tensor. We use this to raise and lower indices, as is done with the Minkowski metric $\eta_{\mu\nu}$ in special relativity.

Up until now, we have only considered the tangent space $T_p\mathcal{M}$ at a point p. We are, however, more interested in fields of vectors, covectors, or tensors. For each point $p \in \mathcal{M}$, a tensor field T "picks out" a tensor $T(p)$ from each tensor product space corresponding to the tangent space at p, $T_p\mathcal{M}$. We will use a vector field to illustrate. This vector field can be written as

$$v(p) = v^\mu(p)\partial_\mu|_p. \tag{1.26}$$

We will mostly be working with the components v^μ, which are functions of $\mathcal{M}$. For ease of notation, we write the vector as a function of the coordinates x. The vector field $v(x)$ is unchanged by a coordinate-transformation $x^\mu \to x'^\mu$; the coordinate is only for our convenience. However, with a new set of coordinates, we get a new set of basis vectors, ∂'_μ:

$$v = v^\mu\partial_\mu = v^\mu\frac{\partial x'^\nu}{\partial x^\mu}\partial'_\nu = v'^\mu\partial'_\mu, \tag{1.27}$$

This gives us the transformation rules for the components of vectors,

$$v'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}v^\nu. \tag{1.28}$$

Tangent vectors are also called *contravariant* vectors, as their components transform contra to basis vectors. For covectors, it is

$$\omega'_\mu = \frac{\partial x^\nu}{\partial x'^\mu}\omega_\nu, \tag{1.29}$$

which is why covectors also are called *covariant* vectors.

The gradient of a scalar function f, $\mathrm{d}f = \partial_\mu f \mathrm{d}x^\mu$, is a coordinate-independent derivative, as $\partial_\mu f$ follows the transformation law for covectors. We define the covariant derivative, ∇, as a map from (n, m) tensor fields to $(n, m+1)$ tensor fields. When considering a scalar as a $(0, 0)$ tensor, we see that this generalizes the scalar derivative. Of the covariant derivative, we assume

- Linearity: $\nabla(T + S) = \nabla T + \nabla S$.
- The product rule: $\nabla(T \otimes S) = (\nabla T) \otimes S + T \otimes (\nabla S)$.
- Reduces to partial derivative for scalars: $\nabla_\mu f = \partial_\mu f$.
- Kronecker delta gives zero: $\nabla_\mu \delta^\rho_\nu = 0$.

With this, we can, in general, write the covariant derivative as [1]

$$\nabla_\mu v^\nu = \partial_\mu v^\mu + \Gamma^\mu_{\nu\rho}v^\rho, \tag{1.30}$$

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\rho_{\mu\nu}\omega_\rho, \tag{1.31}$$

for vectors and covectors. $\Gamma^\mu_{\nu\rho}$ are called *Christoffel symbols*. The generalization for higher-order tensors is straightforward,

$$\nabla_\mu T^{\nu\ldots}{}_{\rho\ldots} = \partial_\mu T^{\nu\ldots}{}_{\rho\ldots} + \Gamma^\mu_{\nu\lambda}T^{\lambda\ldots}{}_{\rho\ldots} + \cdots - \Gamma^\lambda_{\mu\rho}T^{\mu\ldots}{}_{\lambda\ldots} - \ldots. \tag{1.32}$$

This is still not enough to uniquely determine the covariant derivative. We will furthermore assume $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$ and $\nabla_\mu g_{\nu\rho} = 0$. With these, we can find an explicit formula of the Christoffel symbols in terms of the metric,

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} - \partial_\sigma g_{\mu\nu} + \partial_\nu g_{\sigma\mu}). \tag{1.33}$$

The curvature of a manifold $\mathcal{M}$, with the metic $g_{\mu\nu}$, is encoded in the Riemann tensor. It is defined by

$$[\nabla_\mu, \nabla_\nu]v^\rho = R^\rho{}_{\sigma\mu\nu}v^\sigma, \tag{1.34}$$

which in our case gives the explicit formula

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}. \tag{1.35}$$

Although the Christoffel symbols are not tensors, the Riemann tensor is due to its definition using covariant derivatives. We can therefore contract some of its indices to get other tensor quantities. We define the Ricci tensor and Ricci scalar as

$$R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}, \tag{1.36}$$
$$R = R^{\mu}{}_{\mu} = g^{\mu\nu}R_{\mu\nu}. \tag{1.37}$$

These are the quantities we need to start working with general relativity.

Integration on manifolds

The integral of a scalar function on a manifold is not a coordinate-independent notion, and we must introduce the notion of n-forms. A n-form is a antisymmetric $(0, n)$ tensor. To ease notation, we introduce the symmetrization of a tensor T,

$$T_{(\mu_1\ldots\mu_n)} = \frac{1}{n!}\sum_{\sigma\in S_n} T_{\mu_{\sigma(1)}\ldots\mu_{\sigma(n)}}, \tag{1.38}$$

where S_n is the set of all permutations of n objects. The antisymmetrization of a tensor is defined as

$$T_{[\mu_1\ldots\mu_n]} = \frac{1}{n!}\sum_{\sigma\in S_n} \mathrm{sgn}(\sigma) T_{\mu_{\sigma(1)}\ldots\mu_{\sigma(n)}}. \tag{1.39}$$

The function $\sigma = \pm 1$, depending on if σ is a even or odd permutation.

We are interested in a coordinated independent quantity that we can integrate over. To that end, we define

$$\mathrm{d}^n x := \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{n-1}. \tag{1.40}$$

Here, $\wedge$ is the wedge product, defined as

$$(A \wedge B)_{\mu_1\ldots\mu_{n+m}} = \frac{(n+m)!}{n!m!} A_{[\mu_1\ldots\mu_n} B_{\mu_{n+1}\ldots\mu_{n+m}]}, \tag{1.41}$$

Mer om wedge product

and $\mathrm{d}x^\mu$ is the one-form corresponding to the x^μ-coordinate function. Given a different set of coordinates, x'^μ, these are related by

$$\mathrm{d}^n x = \det\left(\frac{\partial x}{\partial x'}\right) \mathrm{d}^n x', \tag{1.42}$$

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by the properties of the wedge product. We define $|g| = |\det(g_{\mu\nu})|$, which, by the transformation properties of tensors, transforms as

$$\sqrt{|g'|} = \left|\det\left(\frac{\partial x'}{\partial x}\right)\right| \sqrt{|g'|}, \tag{1.43}$$

This means that we can use this to compensate for the transformation of $\mathrm{d}^n x$, and get a volume form with a coordinate independent expression,

$$\mathrm{d}V = \sqrt{|g|}\,\mathrm{d}^n x = \sqrt{|g'|}\,\mathrm{d}^n x'. \tag{1.44}$$

With this, we can integrate scalars in a well-defined way by mapping them to a corresponding n-form, $f \to f\mathrm{d}V$. We define the integral of a scalar function f on a manifold $\mathcal{M}$ with a metric g as

$$I = \int_{\mathcal{M}} \mathrm{d}V\, f = \int_{\mathcal{M}} \mathrm{d}^n x \sqrt{|g(x)|}\, f(x). \tag{1.45}$$

Stoke's theorem generalizes the fundamental theorem of calculus and the divergence theorem to manifolds. The most general statement of the theorem uses the exterior derivative, a map from n-forms to $n+1$-forms, defined by

$$(\mathrm{d}T)_{\mu_1\ldots\mu_{n+1}} = (n+1)\partial_{[\mu_1} T_{\mu_2\ldots\mu_{n+1}]}. \tag{1.46}$$

Let $\mathcal{M}$ be a differential manifold of dimension n, with boundary $\partial\mathcal{M}$. Stoke's theorem is then that, for a $n-1$-form ω,

$$\int_{\mathcal{M}} \mathrm{d}\omega = \int_{\partial\mathcal{M}} \omega. \tag{1.47}$$

Stoke's theorem then implies a generalized divergence theorem. The boundary of $\mathcal{M}$ is then $n-1$ dimensional, and a metric g on $\mathcal{M}$ will induce a new metric γ on $\partial\mathcal{M}$. This metric corresponds to the restriction of g to $\partial\mathcal{M}$. Furthermore, there will be a vector field n^μ of normalized vectors orthogonal to all elements of $T\partial\mathcal{M}$. This theorem states that for a vector field V^μ on $\mathcal{M}$,

$$\int_{\mathcal{M}} \mathrm{d}^n x \sqrt{|g|}\, \nabla_\mu V^\mu = \int_{\partial\mathcal{M}} \mathrm{d}^{n-1} y \sqrt{|\gamma|}\, n_\mu V^\mu. \tag{1.48}$$

1.2 General relativity

General relativity describes spacetime as a smooth manifold $\mathcal{M}$, with a (pseudo-Riemannian) metric, $g_{\mu\nu}$. This metric is treated as a dynamical field, which is affected by the presence of matter and energy. The matter and energy contents of spacetime are encoded in the stress-energy tensor $T_{\mu\nu}$, while the behavior of $g^{\mu\nu}$ is encoded in a scalar Lagrangian density. The most obvious—and correct—choice is to use the Ricci scalar, which results in the Einstein-Hilbert action,

$$S_{\mathrm{EH}} = k \int_{\mathcal{M}} \mathrm{d}^n x \sqrt{|g|}\, R, \tag{1.49}$$

where k is a constant. This constant can then be related to Newton's constant of gravitation by

$$k = \frac{1}{16\pi G}. \tag{1.50}$$

The total action will include contributions from other fields with an action S_{m}, so that the total action is

$$S = S_{\mathrm{EH}} + S_{\mathrm{m}}. \tag{1.51}$$

The equations of motion of the dynamical field, which in this case is the metric, are given by Hamilton's principle of stationary action. Using functional derivatives, as defined in (REF:APPENDIX), this is stated as

Ha appendix på functional derivatives

$$\frac{1}{\sqrt{|g|}} \frac{\delta S}{\delta g^{\mu\nu}} = 0, \tag{1.52}$$

We define

$$T_{\mu\nu} = -\frac{2}{\sqrt{|g|}} \frac{\delta S_{\mathrm{m}}}{\delta g^{\mu\nu}}. \tag{1.53}$$

This results in the equations of motion for the metric, the Einstein field equations

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$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu}, \tag{1.54}$$

where $\kappa = 8\pi G$. The left-hand side of the Einstein field equations is called the Einstein tensor,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}. \tag{1.55}$$

This tensor obeys the identity

$$\nabla^\mu G_{\mu\nu} = 0, \tag{1.56}$$

as a consequence of the more general Bianchi identity.

Si hva bianchi identity er

Spherically symmetric spacetime

To model stars, we will assume that the metric is spherically symmetric and time-independent. In this case, the most general metric can be written, at least locally, as [1]

$$\mathrm{d}s^2 = e^{2\alpha(r)}\mathrm{d}t^2 - e^{2\beta(r)}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2), \tag{1.57}$$

where α and β are general functions of the radial coordinate r. In matrix form, this corresponds to

$$g_{\mu\nu} = \begin{pmatrix} e^{2\alpha(r)} & 0 & 0 & 0 \\ 0 & -e^{2\beta(r)} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2(\theta) \end{pmatrix}. \tag{1.58}$$

Using Eq. (1.33), we can now compute the Christoffel symbols in terms of the unknown functions. These computations in this subsection are done using computer code, which is shown in Appendix A. The results are

$$\Gamma^t_{\mu\nu} = \begin{pmatrix} 0 & \frac{d}{dr}\alpha(r) & 0 & 0 \\ \frac{d}{dr}\alpha(r) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{1.59}$$

$$\Gamma^r_{\mu\nu} = \begin{pmatrix} e^{2\alpha(r)}e^{-2\beta(r)}\frac{d}{dr}\alpha(r) & 0 & 0 & 0 \\ 0 & \frac{d}{dr}\beta(r) & 0 & 0 \\ 0 & 0 & -re^{-2\beta(r)} & 0 \\ 0 & 0 & 0 & -re^{-2\beta(r)}\sin^2(\theta) \end{pmatrix}, \tag{1.60}$$

$$\Gamma^\theta_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\sin(\theta)\cos(\theta) \end{pmatrix}, \tag{1.61}$$

$$\Gamma^\phi_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \frac{\cos(\theta)}{\sin(\theta)} \\ 0 & \frac{1}{r} & \frac{\cos(\theta)}{\sin(\theta)} & 0 \end{pmatrix}. \tag{1.62}$$

Substituting these results into Eq. (1.35) gives the Riemann tensor curvature tensor. We can then obtain the Ricci tensor by taking the trace, as shown in Eq. (1.36). The results are

$$R_{tt} = \left(r\left(\frac{d}{dr}\alpha(r)\right)^2 - r\frac{d}{dr}\alpha(r)\frac{d}{dr}\beta(r) + r\frac{d^2}{dr^2}\alpha(r) + 2\frac{d}{dr}\alpha(r)\right)\frac{e^{2\alpha(r)}e^{-2\beta(r)}}{r}, \tag{1.63}$$

$$R_{rr} = -\frac{1}{r}\left(r\left(\frac{d}{dr}\alpha(r)\right)^2 - r\frac{d}{dr}\alpha(r)\frac{d}{dr}\beta(r) + r\frac{d^2}{dr^2}\alpha(r) - 2\frac{d}{dr}\beta(r)\right), \tag{1.64}$$

$$R_{\theta\theta} = -\left(r\frac{d}{dr}\alpha(r) - r\frac{d}{dr}\beta(r) - e^{2\beta(r)} + 1\right)e^{-2\beta(r)}, \tag{1.65}$$

$$R_{\varphi\varphi} = R_{\theta\theta}\sin^2(\theta). \tag{1.66}$$

All other components are zero. Finally, the trace of the Ricci tensor gives the Ricci scalar,

$$\begin{aligned} R = \frac{2e^{-2\beta(r)}}{r^2}\Bigg[&r^2\left(\frac{d}{dr}\alpha(r)\right)^2 - r^2\frac{d}{dr}\alpha(r)\frac{d}{dr}\beta(r) \\ &+ r^2\frac{d^2}{dr^2}\alpha(r) + 2r\frac{d}{dr}\alpha(r) - 2r\frac{d}{dr}\beta(r) - e^{2\beta(r)} + 1\Bigg]. \end{aligned} \tag{1.67}$$

The unknown functions α and β are now determined by the matter and energy content of the universe, which is encoded in $T_{\mu\nu}$, through Einstein's field equation, Eq. (1.54).

1.3 The TOV equation

We will model a star as being made up of a *perfect fulid*, with energy density u and pressure p. The relationship between the pressure and energy density of a substance is called the *equation of state*, or EOS, and has the form

$$f(p, u, \{\xi_i\}) = 0, \tag{1.68}$$

where $\{\xi_i\}$ are possible other thermodynamic variables. We will be working at zero temperature, in which case there are no other free thermodynamic variables. This allows us to, at least locally, express the energy density as a function of the pressure, $u = u(p)$. The stress-energy tensor of a perfect fluid is

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$$T_{\mu\nu} = (u + p)u_\mu u_\nu - p g_{\mu\nu}, \tag{1.69}$$

where u_μ is the 4-velocity of the fluid. In the rest frame of the fluid, we may write

$$v_\mu = (v_0, 0, 0, 0)\,. \tag{1.70}$$

This, together with the normalization condition of 4-velocities, $v_\mu v^\mu = 1$, allows us to calculate that

$$v_\mu v^\mu = g^{\mu\nu} v_\mu v_\nu = g^{00}(v_0)^2 = 1. \tag{1.71}$$

Using Eq. (1.58), we see that

$$v_0 = e^{\alpha(r)}. \tag{1.72}$$

This gives us the stress-energy tensor of the perfect fluid in its rest frame,

$$T_{\mu\nu} = \begin{pmatrix} u(r)e^{2\alpha(r)} & 0 & 0 & 0 \\ 0 & p(r)e^{2\beta(r)} & 0 & 0 \\ 0 & 0 & p(r)r^2 & 0 \\ 0 & 0 & 0 & p(r)r^2\sin^2(\theta) \end{pmatrix}. \tag{1.73}$$

We will use the tt and rr components of the Einstein field equations, which are

$$8\pi G r^2 u(r) e^{2\beta(r)} = 2r\frac{d}{dr}\beta(r) + e^{2\beta(r)} - 1 \tag{1.74}$$

$$8\pi G r^2 p(r) e^{2\beta(r)} = 2r\frac{d}{dr}\alpha(r) - e^{2\beta(r)} + 1. \tag{1.75}$$

In analogy with the Schwarzschild metric, we define the function $m(r)$ by

$$e^{2\beta(r)} = \left(1 - \frac{2Gm(r)}{r}\right)^{-1}. \tag{1.76}$$

Substituting this into Eq. (1.74) yields

$$\frac{\mathrm{d}m(r)}{\mathrm{d}r} = 4\pi r^2\, u(r). \tag{1.77}$$

The solution is simply

$$m(r) = 4\pi \int_0^r \mathrm{d}r'\, r'^2 u(r'). \tag{1.78}$$

We see that $m(r)$ is the matter content contained within a radius r. If $u = 0$ for $r > R$ and $m(r > R) = M$, then the metric on a constant-time surface, i.e. $\mathrm{d}t = 0$, is

Forlar forskjell på gravitasjons masse/ incl. bindingsenergi

$$\mathrm{d}s^2 = \left(1 - \frac{2GM}{r^2}\right)^{-1} \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2). \tag{1.79}$$

This is the same as for the Schwarzschild solution.

Using the Bianchi identity, Eq. (1.56), together with Einstein's equation, we find

$$\nabla^\mu G_{\mu\nu} = \nabla^\mu T_{\mu\nu} = 0. \tag{1.80}$$

The r-component of this equation is

$$\begin{aligned}\nabla_\mu T^{\mu r} &= \partial_r T^{rr} + \Gamma^\mu_{\mu\nu} T^{\nu r} + \Gamma^r_{\mu\nu} T^{\mu\nu} \\ &= \partial_r \left(p e^{-2\beta}\right) + (2\Gamma^r_{rr} + \Gamma^t_{tr}) T^{rr} + \Gamma^r_{tt} T^{tt} \\ &= e^{-2\beta} \left(\partial_r p + p \partial_r \alpha + u \partial_r \alpha\right) = 0.\end{aligned}$$

This allows us to relate α to p and u, via

$$\partial_r \alpha = -\frac{\partial_r p}{p+u} \tag{1.81}$$

When we substitute this, together with the definition of $m(r)$, into Eq. (1.75), we obtain

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{G}{r^2}\left(4\pi r^3 p + m\right)(p+u)\left(1 - \frac{2Gm}{r}\right)^{-1}, \tag{1.82}$$

the Tolman-Oppenheimer-Volkoff (TOV) equation. This equation was first obtained by Oppenheimer and Volkoff in 1939 [3] and was based on earlier work by Tolman [4]. In their paper, Oppenheimer and Volkoff studied the properties of a star made up of cold, degenerate fermions.

To summarize, we have three unknown functions, $u(r)$, $p(r)$, and $m(r)$. The equation of state, Eq. (1.68), determines $u = u(p)$, eliminating one unknown. The two differential equations Eq. (1.78) and Eq. (1.82), together with the boundary conditions $p(0) = p_0$ and $m(0) = 0$, then yield $p(r)$ and $m(r)$ when integrated. Given this, we can solve for all the unknown functions, either analytically or numerically. However, we can gain some insight into the system without solving these equations, by expressing the problem in terms of dimensionless variables, and corresponding characteristic quantities. We define

$$u = u_0 \tilde{u}, \quad p = p_0 \tilde{p}, \quad m = m_0 \tilde{m}, \quad r = r_0 \tilde{r}. \tag{1.83}$$

Here, quantities with subscript 0 are dimensionful constants, the characteristic quantities, while the tilde indicates a dimensionless variable. By substituting this into Eq. (1.77) and Eq. (1.82), we can collect the dimensionful constants into a smaller number of dimensionless constants, k_i. These constants will decide the nature of the solution. Any change in the dimensionful constants that leave the k_i's invariant is a scaling of the problem — it corresponds to the same solution with different units. The new differential equations are

$$\frac{\mathrm{d}\tilde{m}}{\mathrm{d}\tilde{r}} = 3k_2 \tilde{r}^2 \tilde{u} \tag{1.84}$$

$$\frac{\mathrm{d}\tilde{p}}{\mathrm{d}\tilde{r}} = -\frac{k_1}{k_3}\frac{1}{\tilde{r}^2}\left(k_3\tilde{p} + \tilde{u}\right)\left(3k_2 k_3 \tilde{r}^3 \tilde{p} + \tilde{m}\right)\left(1 - \frac{2k_1\tilde{m}}{\tilde{r}}\right)^{-1}, \tag{1.85}$$

where the dimensionless constants are defined as

$$k_1 = G\frac{m_0}{r_0}, \quad k_2 = \frac{4\pi}{3}\frac{r_0^3 u_0}{m_0}, \quad k_3 = \frac{p_0}{u_0}. \tag{1.86}$$

Given non-complete set of characteristic quantities, these constants gives us an expectation for the order of magnitude of the remaining dimensionful constants. By setting $k_i = 1$, we define the remaining, free constants to take a value, which we expect to be the natural units for this system.

Newtonian limit

In the Newtonian limit, the rest energy, i.e., mass, is the dominant contribution to the gravitational field. This means that the characteristic pressure, p_0, is far smaller than the characteristic energy density u_0, and we can use the approximation $k_3 \ll 1$. Furthermore, the radius of the star should be much larger than the Schwarzschild radius, $R_s = 2GM$, and thus $k_1 \ll 1$. In this limit, the lowest-order contribution to the TOV equation is

$$\frac{\mathrm{d}\tilde{p}}{\mathrm{d}\tilde{r}} = -k\frac{\tilde{u}\tilde{m}}{\tilde{r}^2}, \quad k = G\frac{u_0 m_0}{p_0 r_0}. \tag{1.87}$$

Using the mass equation Eq. (1.84), we can write this as

$$\tilde{r}^2 \frac{\mathrm{d}\tilde{p}}{\mathrm{d}\tilde{m}} = -k' \frac{\tilde{m}}{\tilde{r}^2}, \quad k' = \frac{G}{4\pi} \frac{m_0^2}{r_0^4 p_0}. \tag{1.88}$$

This equation has a simple interpretation; the force due to the pressure gradient over a thin, spherical shell, $F_p = 4\pi r^2 \mathrm{d}p$, must be counteracted by the gravitational force on the same shell, $F_g = -Gm\mathrm{d}m/r^2$. This is illustrated in Figure 1.4. Both the Newtonian limit and the TOV equation is an equation of *hydrostatic equilibrium*, where the forces on a small volume of the fluid cancel out. The TOV equation is an equation of hydrostatic equilibrium because we assumed tacitly that the stress-energy tensor $T_{\mu\nu}$ has a common rest frame everywhere.

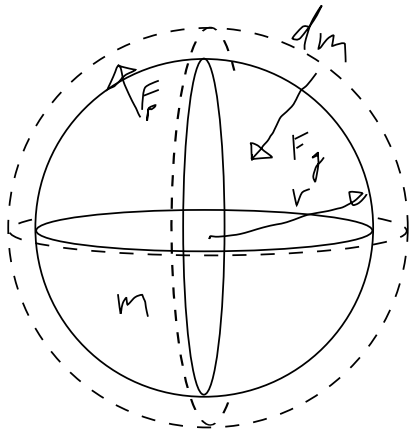

Figure 1.4: Kladd: The forces acting on a thin shell dm.

We can eliminate the equation for mass by differentiating Eq. (1.87) with respect to $\tilde{r}$, which leaves

Make sure constant is right

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{r}} \left(\frac{\tilde{r}^2}{\tilde{u}} \frac{\mathrm{d}\tilde{p}}{\mathrm{d}\tilde{r}} \right) = -k\tilde{r}^2 \tilde{u}, \quad k = 4\pi G \frac{u_0^2 r_0^2}{p_0}. \tag{1.89}$$

Incompressible fluid

The simplest model for a star is one made up of an incompressible fluid, where the energy density is independent of the pressure. In this case, the energy density of the star will be constant for a radius $r < r_0$, before it drops to zero,

$$u(r) = u_0 \, \theta(r_0 - r), \tag{1.90}$$

where u_0 is a constant and $\theta(x)$ the Heaviside step function. Inserting this into the differential equation of the mass function, Eq. (1.84), together with the boundary condition $m(0) = 0$, yields

$$\tilde{m}(\tilde{r}) = k_2 \tilde{r}^3, \tag{1.91}$$

when $r < r_0$. For $r \geq r_0$, or $\tilde{r} \geq 1$, this relationship is simply constant $\tilde{m}(\tilde{r}) = \tilde{m}(1) = k_2$. We choose m_0 to be the gravitational mass of the star, i.e. $m_0 = \frac{4\pi}{3} r_0^3 u_0$, which is equivalent to setting $k_2 = 1$. With this the TOV equation, Eq. (1.85), becomes

$$k_3 \frac{\mathrm{d}\tilde{p}}{\mathrm{d}\tilde{r}} = -k_1 \tilde{r} \frac{(1 + k_3\tilde{p})(1 + 3k_3\tilde{p})}{(1 - 2k_1\tilde{r}^2)}. \tag{1.92}$$

We see that in this case, k_3 corresponds to a scaling of the pressure, and we can therefore set $k_3 = 1$. This is a separable ODE, and each variable may be integrated separately. Using

$$\int \frac{\mathrm{d}x}{(1+x)(1+3x)} = \frac{1}{2} \ln \frac{3x+1}{x+1} + \text{const.}, \quad \int \mathrm{d}x \frac{x}{1-2x^2} = \frac{1}{4} \ln \left(1 - 2x^2\right) + \text{const.}, \tag{1.93}$$

together with the boundary condition $p(r = r_0) = 0$, we get

$$\tilde{p}(\tilde{r}) = -\frac{\sqrt{1-2k_1} - \sqrt{1-2k_1\tilde{r}^2}}{3\sqrt{1-2k_1} - \sqrt{1-2k_1\tilde{r}^2}}. \tag{1.94}$$

We see that the star is completely characterized by k_1, while k_3 corresponds to scaling the pressure. In Figure 1.5, we have plotted the pressure as a function of radius for some values of k_1. As k_1 approaches $0.\bar{4} = 4/9$, the pressure at the center of the star increases rapidly. From the denominator of Eq. (1.94) at $r \to 0$, we find the limit

$$k_1 = G\frac{m_0}{r_0} < \frac{4}{9} \tag{1.95}$$

for the pressure to remain finite. This is an absolute limit of the mass of an object given its radius or vice versa. General relativity does not allow for a static solution with energy densities greater than this; any such configuration would collapse [1].

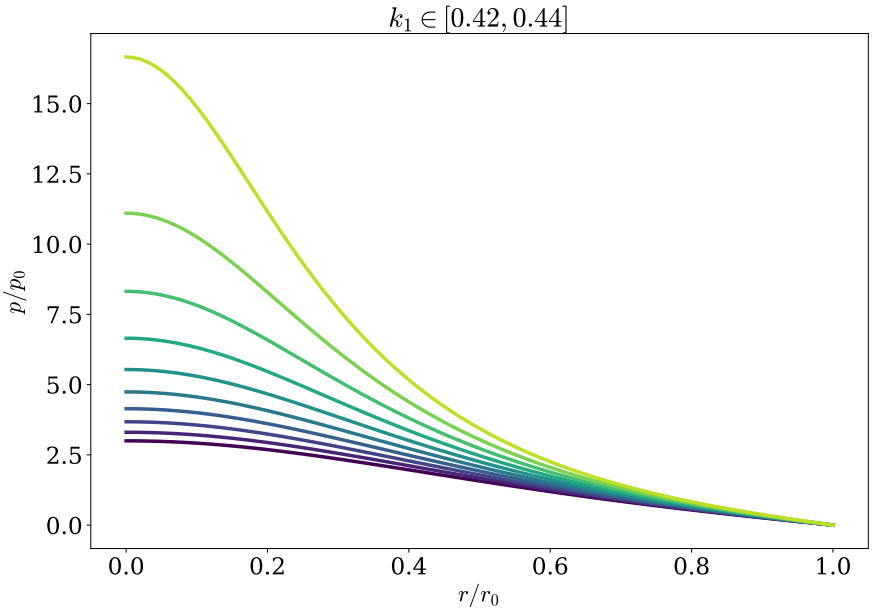

Figure 1.5: The pressure in units of p_0, as a function of the radius, in units of r_0. The graphs with lighter color and higher pressure at $r = 0$ corresponds to a higher value for k_1. The values of k_1 are linearly spaced.

If we expand the solution Eq. (1.94) in powers of k_1, then the leading order contribution is

$$\tilde{p}(r) = \frac{1}{2}k_1(1 - \tilde{r}^2). \tag{1.96}$$

As a cross-check, we see that this function obeys the Newtonian limit we derived earlier, Eq. (1.87), as $\tilde{u} = 1$, and the boundary condition $p(\tilde{r} = 1) = 0$. This is the general solution for an incompressible fluid in Newtonian gravity. This solution does not have any upper limit for k_1, showing that this phenomenon is a purely relativistic phenomenon. The Newtonian approximation is compared to the full, relativistic solution. We see that the Newtonian approximation gives highly accurate for k_1 less than around 0.01.

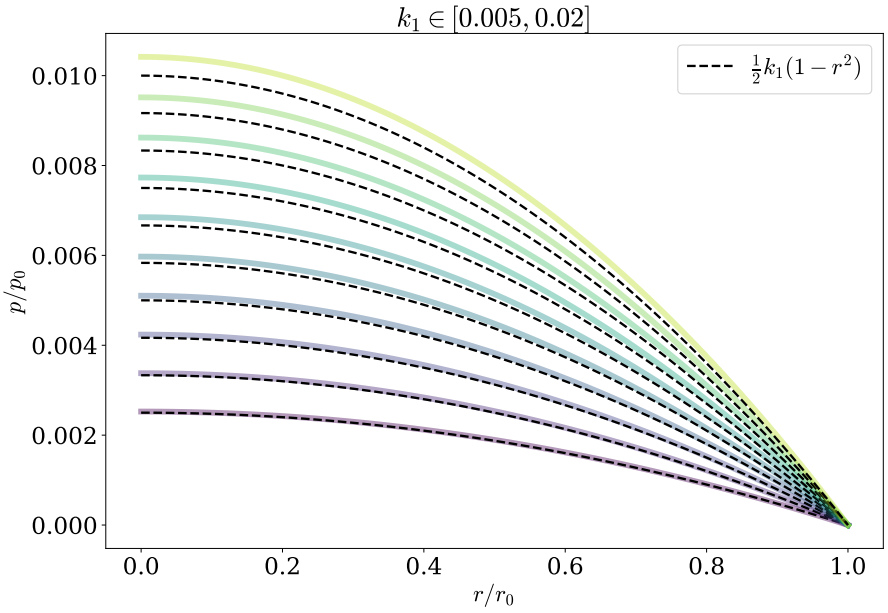

Figure 1.6: The pressure in units of p_0, as a function of the radius, in units of r_0. The wide, colored lines correspond to the full relativistic solution, while the dashed lines is the Newtonian approximation, for the same value of k_1. The values of k_1 are linearly spaced.

1.4 A star of cold, non-interacting fermions

Ekstra kilder: [5, 6]

Ha kilder på riktig sted

Skrive om white dwarfs/neutron stars?

A non-interacting Fermi gas is governed by the Dirac Lagrangian

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi, \tag{1.97}$$

as described in (APPENDIX thermal field theory). In the grand canoncial ensamble, the density of a conserved charge is regulated by a chemical potential μ. The conserved current corresponding to the U(1) phase symmetry of the Dirac Lagrangian, i.e., the transformation $\psi \to e^{i\alpha}\psi \approx 1 + \alpha\delta\psi$, is

add appendix

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\delta\psi = \bar{\psi}\gamma^\mu\psi. \tag{1.98}$$

The conserved charge is

$$Q = \int \mathrm{d}^3\, j^0 = \int \mathrm{d}^3 x\, \bar{\psi}\gamma^0\psi, \tag{1.99}$$

the number of particles minus antiparticles. The energy density u is related to the grand canonical free energy F by a Legendre transformation,

$$F(T, V, \mu) = U - TS - \mu Q, \quad \mathrm{d}F = -S\mathrm{d}T - p\mathrm{d}V - Q\mathrm{d}\mu. \tag{1.100}$$

Here, p is pressure, $T = 1/\beta$ is temprature, and S entropy. These thermodynamic variables are related to the free energy by

$$S = -\frac{\partial F}{\partial T} = \beta^2 \frac{\partial F}{\partial \beta}, \quad Q = -\frac{\mathrm{d}F}{\mathrm{d}\mu}, \quad p = -\frac{\partial F}{\partial V}. \tag{1.101}$$

When the free energy can be written as $F = V\mathcal{F}$, where the free energy density $\mathcal{F}$ is independent of the volume V, then $\mathcal{F} = -p$ and

$$\mathrm{d}(V\mathcal{F}) = V\mathrm{d}\mathcal{F} - p\mathrm{d}V, \tag{1.102}$$

allowing us to write

$$\mathcal{F}(T,\mu) = u - Ts - \mu n, \quad \mathrm{d}\mathcal{F} = -s\mathrm{d}T - n\mathrm{d}\mu, \tag{1.103}$$

where s and n are entropy and charge density, defined by

$$s = -\frac{\partial \mathcal{F}}{\partial T} = \beta^2 \frac{\partial \mathcal{F}}{\partial \beta}, \quad n = -\frac{\partial \mathcal{F}}{\partial \mu}, \tag{1.104}$$

With this, we can write the energy density as

$$u = \frac{\partial}{\partial \beta}\left(\beta \mathcal{F}\right) + \mu n. \tag{1.105}$$

The free energy density of the Fermion gas is (REF: appendix felt teori)

$$\mathcal{F} = -\frac{2}{\beta}\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\left[\beta\omega + \ln(1+e^{-\beta(\omega-\mu)}) + \ln(1+e^{-\beta(\omega+\mu)})\right], \tag{1.106}$$

where $\omega = \sqrt{p^2+m^2}$. The first term in the integral is the divergent vacuum energy, which must be renormalized. We can drop this term; it does not have any observable effects on our results, as we are interested in relative pressure and energy density. With this, we find the charge density

$$n = \frac{1}{\pi^8}\int \frac{\mathrm{d}^3 p}{(2\pi)^3}[n_f(\omega-\mu) - n_f(\omega+\mu)], \tag{1.107}$$

where

$$n_f(\omega) = \frac{1}{e^{\beta\omega}+1}. \tag{1.108}$$

is the Fermi distribution. The energy density is

$$u = \frac{1}{\pi^2}\int_0^\infty \mathrm{d}p\, p^2\, \omega[n_f(\omega-\mu) + n_f(\omega+\mu)]. \tag{1.109}$$

As expected, this is the energy per mode times the density of states, integrated over all modes. To write the pressure, $p = -\mathcal{F}$ in terms of an integral over the Fermi distribution, we integrate by parts. We have

$$\int_0^\infty \mathrm{d}p\, p^2 \ln\left[1+e^{-\beta(\omega\pm m)}\right] = \frac{1}{3}p^3 \ln\left[1+e^{-\beta(\omega\pm m)}\right]\Big|_0^\infty + \frac{1}{3}\int_0^\infty \mathrm{d}p\, \frac{\beta p^4}{\omega} n_f(\omega\pm\mu), \tag{1.110}$$

where the boundary term vanish. This allows us to write the pressure as

$$p = \frac{2}{3}\int_0^\infty \mathrm{d}p\, \frac{p^4}{\omega}[n_f(\omega-\mu) + n_f(\omega+\mu)] \tag{1.111}$$

We are interested in the $T = 0$ limit. In this case, the Fermi distribution becomes a step function, $n_f(\omega) = \theta(-\omega)$. Without loss of generality, we assume that $\mu > 0$, i.e., we are dealing with an abundance of matter compared to anti-matter. The dispersion relation $\omega = \sqrt{p^2+m^2}$ is always positivive. This means that the contribution to thermodynamic quantities from anti-particles vanish, as the integral is multiplied with $n_f(\omega+\mu) = \theta(-\omega-\mu)$, where the argument $-\omega-\mu$ is strictly negative on the domain of integration. At zero temperature, the only dynamics are due to the degeneracy pressure of the fermions, that is, due to the Pauli exclusion principle. There are no thermal fluctuations that can create a particle-antiparticle pair. Thus, if the system has a positive chemical potential, it will contain no antiparticles. In the zero-temperature limit, we can then rewrite any integral over the Fermi distribution as

$$\int_0^\infty \mathrm{d}p\,[f(p)n_f(\omega-\mu) + g(p)n_f(\omega-\mu)] = \int_0^{p_f} \mathrm{d}p\, f(p), \tag{1.112}$$

where Fermi momentum p_f is defined by the chemical potential as $\mu = \sqrt{p_f^2+m^2}$. We thus assume the chemical potential is equal or greater than the mass m. This allows us to evaluate the charge density exactly,

$$n = \frac{1}{\pi^2}\int_0^{p_f} \mathrm{d}p\, p^2 = \frac{p_f^3}{3\pi^2}. \tag{1.113}$$

At $T = 0$, this is the particle number density, as there are no antiparticles. This density is given by the chemical potential and vanishes when $\mu \leq m$, i.e. when the free energy cost of creating a particle is positive. We can write the energy density and pressure integrals, Eq. (1.109) and Eq. (1.111), as

$$u = \frac{1}{\pi^2}\int_0^{p_f} \mathrm{d}p\, p^2\sqrt{p^2+m^2} = \frac{m^4}{\pi^2}\int_0^{x_f} \mathrm{d}x\, x^2\sqrt{x^2+1}, \tag{1.114}$$

$$p = \frac{1}{3\pi^2}\int_0^{p_f} \mathrm{d}p\, \frac{p^4}{\sqrt{p^2+m^2}} = \frac{m^4}{3\pi^2}\int_0^{x_f} \frac{\mathrm{d}x\, x^4}{\sqrt{x^2+1}}. \tag{1.115}$$

We have defined $x = p/m$ and $x_f = p_f/m$. These integrals can be evaluated exactly as

$$\int_0^a \mathrm{d}x\, x^2\sqrt{x^2+1} = \frac{1}{8}\left[\sqrt{a^4+1}(2a^3+a) - \operatorname{arcsinh}(a)\right], \tag{1.116}$$

$$\int_0^a \mathrm{d}x\, \frac{x^4}{\sqrt{x^2+1}} = \frac{1}{8}\left[\sqrt{a^2+1}(2a^3-3a) + 3\operatorname{arcsinh}(a)\right]. \tag{1.117}$$

We introduce the characteristic energy and number density,

$$u_0 = \frac{m^4}{8\pi^2}, \quad n_0 = \frac{u_0}{m}, \tag{1.118}$$

which allows us to write the thermodynamic variables as

$$n = n_0\frac{8}{3}x_f^3 \tag{1.119}$$

$$u = u_0\left[(2x_f^3+x_f)\sqrt{1+x_f^2} - \operatorname{arcsinh}(x_f)\right], \tag{1.120}$$

$$p = \frac{u_0}{3}\left[(2x_f^3-3x_f)\sqrt{1+x_f^2} + 3\operatorname{arcsinh}(x_f)\right]. \tag{1.121}$$

Units

If we demand

$$G\frac{m_0}{r_0} = \frac{4\pi}{3}\frac{r_0^3 u_0}{m_0} = 1, \tag{1.122}$$

we have defined a complete set of units. Inserting $\hbar$ and c gives

$$u_0 = \frac{m^4}{8\pi^2}\frac{c^5}{\hbar^3} = 2.032\cdot 10^{35}\,\mathrm{J\,m^{-3}}, \tag{1.123}$$

$$m_0 = \frac{c^4}{\sqrt{\frac{4\pi}{3}u_0 G^3}} = 1.608\cdot 10^{31}\,\mathrm{kg} = 8.082\,M_\odot \tag{1.124}$$

$$r_0 = \frac{Gm_0}{c^2} = 11.93\,\mathrm{km}. \tag{1.125}$$

From this, we can expect our star to have a mass of the order of a solar mass, $M_\odot = 1.98841\cdot 10^{30}\,\mathrm{kg}$ [7], and a radius of the order of kilometers, without solving the TOV equation.

Limits

In the non-relativistic limit, as the chemical potential approaches m and thus $p_f \ll m$, the lowest order contributions to the energy density and pressure are given by the Taylor series around $x_f = 0$,

$$\tilde{u}(x_f) = \frac{8}{3}x_f^3 + \frac{4}{5}x_f^5 + \mathcal{O}(x_f^7), \tag{1.126}$$

$$\tilde{p}(x_f) = \frac{8}{5}x_f^5 + \mathcal{O}(x_f^7). \tag{1.127}$$

By neglecting terms of order x_f^7 and higher, we can write this as

$$\tilde{u} = \tilde{n} + \frac{4}{5}\left(\frac{8}{3}\tilde{n}\right)^{5/3}, \quad \tilde{p} = \frac{8}{5}\left(\frac{8}{3}\tilde{n}\right)^{5/3}. \tag{1.128}$$

The leading order contribution to the energy density is the rest mass of the particles. This term does not contribute to the pressure. By including only the leading order term, we can eliminate the Fermi momentum and write the equation of state as $u = kp^{3/5}$ for some constant k. Equations of state where $u \propto p^\gamma$ are called polytropes. In the ultrarelativistic limit, where $p_f \gg m$, the leading order contributions to the pressure and energy density are

$$\tilde{u} = 2x_f^4, \quad \tilde{p} = \frac{2}{3}x_f^4, \tag{1.129}$$

and we get the particularly simple equation of state $u = 3p$, which we recognize as the formula for radiation pressure.

Numerical results

With the energy density, Eq. (1.120), and pressure, Eq. (1.121), we can numerically solve the TOV equation given a central pressure p_0. This is done using a adaptive Runge-Kutta method, with the stop criterion $p(r) = 0$. Description of the code and where to find it is given in Appendix A.

Describe the code in the appendix

As $r \to 0$, parts of the TOV equation Eq. (1.85) diverges, we must take use an approximation for numeric evaluation. The Taylor-expansion of the mass function around $\tilde{r} = 0$ is

$$\tilde{m}(r) = \tilde{m}(0) + \tilde{m}'(0)\,\tilde{r} + \frac{1}{2!}\tilde{m}''(0)\tilde{r}^2 + \frac{1}{3!}\tilde{m}'''(0)\tilde{r}^3. \tag{1.130}$$

One of the boundary conditions are $\tilde{m}(0) = 0$. We then use the differntial equation for $\tilde{m}$, Eq. (1.77), to find

$$\tilde{m}'(0) = 0, \quad \tilde{m}''(0) = 0, \quad \tilde{m}'''(0) = 6k_2\tilde{u}(0). \tag{1.131}$$

For $\tilde{r} \ll 1$, we therefore approximate the TOV-equation as

$$\frac{\mathrm{d}\tilde{p}}{\mathrm{d}\tilde{r}} = -k_1k_2\tilde{r}\,\left[\tilde{p} + \tilde{u}(\tilde{p})\right]\left[3\tilde{p}(\tilde{r}) + \tilde{u}(\tilde{r}=0)\right]\left[1 - 2k_1\tilde{u}(\tilde{r}=0)r^2\right]^{-1}. \tag{1.132}$$

Appendix A

Code

All code is available at: `https://github.com/martkjoh/master`.

The code for calculations in chapter 1 is included below. The code is written in Python in a Jupyter notebook, and the `.ipynb` file with executable code is available in the repository linked above.

[17]:
```
from sympy import MatrixSymbol, Matrix, Array, pprint
from sympy import symbols, diff, exp, log, cos, sin, simplify, Rational
from sympy.core.symbol import Symbol
from sympy import pi

import numpy as np
import sympy as sp
from IPython.display import display, Latex
```

Tensor operations

[18]:
```
def INDX(i, place, num_indx):
    """
    Acceses an index at 'place' for 'num_indx' order tensor
    T_(a0 ... âp ... an-1) = T[INDX(i, place=p, num_indx=n)] = T[:,...<-p-> ,␣
 ↪i, :,...<-(n-p-1)->]
    """
    indx = []
    assert place<num_indx
    for j in range(num_indx):
        if place==j: indx.append(i)
        else: indx.append(slice(None))
    return tuple(indx)
```

[19]:
```
def contract(T, g=None, g_inv=None, num_indx=2, upper=1, indx=(0, 1)):
    """
    contracts indecies indx=(a_p, a_q) on tensor T with 'num_indx',
    'upper' of whom are upper. If upper=0, all indecies are assumed lower.
    With indx=(a_k, a_l), upper=n, num_indx=n+m, this gives
    T^(a_0...a_n-1)_(a_n...a_n+m-1) -> T^(a_0...a_k=a...a_n-1)_(a_n...a_k...
 ↪a_n+m-1),
    with the necesarry metric. If wrong metric is given, this wil throw error.
    """
    assert indx[0] < indx[1]  # we have to know if the index to the left␣
 ↪dissapears
    dim = np.shape(T)[0]
    a = (indx[0] < upper) + (indx[1] < upper) # number of upper indecies to be␣
 ↪contracted
    if a==2: g0 = g # two upper
    elif a==0: g0 = g_inv # two lower
    else: g0 = np.identity(dim, dtype=Rational)

    Tc = Rational(0) * np.ones((T.shape)[:-2], dtype=Rational)
    for i in range(dim):
        for j in range(dim):
            Tc += g0[i, j] * (T[INDX(i, indx[0], num_indx)])[INDX(j, indx[1] -␣
 ↪1, num_indx - 1)]
```

```
    return Tc

def raise_indx(T, g_inv, indx, num_indx):
    """
    Raise index 'indx' of a tensor T with 'num_indx' indices.
    """
    dim = np.shape(T)[0]
    Tu = np.zeros_like(T)
    for i in range(dim):
        I = INDX(i, indx, num_indx)
        for j in range(dim):
            J = INDX(j, indx, num_indx)
            Tu[I] += g_inv[i, j] * T[J]
    return Tu

def lower_indx(T, g, indx, num_indx):
    return raise_indx(T, g, indx, num_indx)

def get_g_inv(g):
    return np.array(Matrix(g)**(-1))
```

Calculate Christoffel symbols and Riemann curvature tensor

[20]:
```
def Christoffel(g, g_inv, var):
    """
    Work out the christoffel symbols, given a metric an its variables
    Γ^i_jk = C[i, j, k]
    """
    dim = len(var)
    C = np.zeros((dim, dim, dim), dtype=Symbol)
    for i in range(dim):
        for j in range(dim):
            for k in range(dim):
                for m in range(dim):
                    C[i, j, k] += Rational(1, 2) * (g_inv)[i, m] * (
                        diff(g[m, k], var[j])
                        + diff(g[m, j], var[k])
                        - diff(g[k, j], var[m])
                    )

    return C
```

[21]:
```
def Riemann_tensor(C, var):
    """
    Riemann_tensor(Christoffel_symbols, (x_1, ...)) = R[i, j, k, l] = R^i_jkl
    Compute the Riemann tensor from the Christoffel symbols
```

```
    """
    dim = len(var)
    R = np.zeros([dim] * 4, dtype=Symbol)
    indx = [(i, j, k, l)
        for i in range(dim)
        for j in range(dim)
        for k in range(dim)
        for l in range(dim)
    ]

    for (a, b, r, s) in indx:
        R[a, b, r, s] += diff(C[a, b, s], var[r]) - diff(C[a, b, r], var[s])
        for k in range(dim):
            R[a, b, r, s] += C[a, k, r] * C[k, b, s] - C[a, k, s] * C[k, b, r]
    return R
```

Printing functions

[22]:
```
print_latex = False

def print_christoffel(C, var):
    """ A function for dsiplaying christoffels symbols """
    output = []
    for i in range(len(var)):
        txt = "$$"
        txt += "\\Gamma^" + sp.latex(var[i]) + "_{\\mu \\nu} ="
        txt += sp.latex(Matrix(C[i]))
        txt += "$$"
        print(txt) if print_latex else print()
        output.append(display(Latex(txt)))

    return output

def print_matrix(T):
    txt = "$$" + sp.latex(Matrix(T)) +"$$"
    print(txt) if print_latex else print()
    return display(Latex(txt))

def print_scalar(T):
    txt = "$$" + sp.latex(T) +"$$"
    print(txt) if print_latex else print()
    return display(Latex(txt))

def print_eq(eq):
    txt = "$$" + sp.latex(eq) +"=0" + "$$"
    print(txt) if print_latex else print()
    return display(Latex(txt))
```

Metric $g_{\mu\nu}$ for spherically symmetric spacetime

[23]:
```
t, r, th, ph = symbols("t, r, \\theta, \\phi")
x1 = r * cos(ph) * sin(th)
x2 = r * sin(ph) * sin(th)
x3 = r * cos(th)

one = Rational(1)
eta = sp.diag(one, -one, -one, -one)
var = (t, r, th, ph)
J = Matrix([t, x1, x2, x3]).jacobian(var)
g = np.array(simplify(J.T *eta* J))

a = sp.Function("\\alpha", real=True)(r)
b = sp.Function("\\beta", real=True)(r)
g[0, 0] *= exp(2 * a)
g[1, 1] *= exp(2 * b)
g_inv = get_g_inv(g)

print_matrix(g)
print_matrix(g_inv)
```

$$\begin{bmatrix} e^{2\alpha(r)} & 0 & 0 & 0 \\ 0 & -e^{2\beta(r)} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2(\theta) \end{bmatrix}$$

$$\begin{bmatrix} e^{-2\alpha(r)} & 0 & 0 & 0 \\ 0 & -e^{-2\beta(r)} & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{r^2 \sin^2(\theta)} \end{bmatrix}$$

[24]:
```
C = Christoffel(g, g_inv, var)
c = print_christoffel(C, var)
```

$$\Gamma^{t}_{\mu\nu} = \begin{bmatrix} 0 & \frac{d}{dr}\alpha(r) & 0 & 0 \\ \frac{d}{dr}\alpha(r) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Gamma^{r}_{\mu\nu} = \begin{bmatrix} e^{2\alpha(r)}e^{-2\beta(r)}\frac{d}{dr}\alpha(r) & 0 & 0 & 0 \\ 0 & \frac{d}{dr}\beta(r) & 0 & 0 \\ 0 & 0 & -re^{-2\beta(r)} & 0 \\ 0 & 0 & 0 & -re^{-2\beta(r)}\sin^2(\theta) \end{bmatrix}$$

$$\Gamma^{\theta}_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\sin(\theta)\cos(\theta) \end{bmatrix}$$

$$\Gamma^{\phi}_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \frac{\cos(\theta)}{\sin(\theta)} \\ 0 & \frac{1}{r} & \frac{\cos(\theta)}{\sin(\theta)} & 0 \end{bmatrix}$$

[25]:
```
Rie = Riemann_tensor(C, var)
Ricci = contract(Rie, num_indx=4, upper=1, indx=(0, 2))

for i in range(4):
    print_scalar(Ricci[i, i].factor())
```

$$\frac{\left(r\left(\frac{d}{dr}\alpha(r)\right)^2 - r\frac{d}{dr}\alpha(r)\frac{d}{dr}\beta(r) + r\frac{d^2}{dr^2}\alpha(r) + 2\frac{d}{dr}\alpha(r)\right)e^{2\alpha(r)}e^{-2\beta(r)}}{r}$$

$$-\frac{r\left(\frac{d}{dr}\alpha(r)\right)^2 - r\frac{d}{dr}\alpha(r)\frac{d}{dr}\beta(r) + r\frac{d^2}{dr^2}\alpha(r) - 2\frac{d}{dr}\beta(r)}{r}$$

$$-\left(r\frac{d}{dr}\alpha(r) - r\frac{d}{dr}\beta(r) - e^{2\beta(r)} + 1\right)e^{-2\beta(r)}$$

$$-\left(r\frac{d}{dr}\alpha(r) - r\frac{d}{dr}\beta(r) - e^{2\beta(r)} + 1\right)e^{-2\beta(r)}\sin^2(\theta)$$

[26]:
```
R = contract(Ricci, g_inv=g_inv, upper=0).simplify()
print_scalar(R)
```

$$\frac{2\left(r^2\left(\frac{d}{dr}\alpha(r)\right)^2 - r^2\frac{d}{dr}\alpha(r)\frac{d}{dr}\beta(r) + r^2\frac{d^2}{dr^2}\alpha(r) + 2r\frac{d}{dr}\alpha(r) - 2r\frac{d}{dr}\beta(r) - e^{2\beta(r)} + 1\right)e^{-2\beta(r)}}{r^2}$$

[27]:
```
G = Ricci - Rational(1, 2) * R * g
for i in range(4):
    G[i, i] = G[i, i].simplify().factor()
    print_scalar(G[i, i])
```

$$\frac{\left(2r\frac{d}{dr}\beta(r) + e^{2\beta(r)} - 1\right)e^{2\alpha(r)}e^{-2\beta(r)}}{r^2}$$

$$\frac{2r\frac{d}{dr}\alpha(r) - e^{2\beta(r)} + 1}{r^2}$$

$$r\left(r\left(\frac{d}{dr}\alpha(r)\right)^2 - r\frac{d}{dr}\alpha(r)\frac{d}{dr}\beta(r) + r\frac{d^2}{dr^2}\alpha(r) + \frac{d}{dr}\alpha(r) - \frac{d}{dr}\beta(r)\right)e^{-2\beta(r)}$$

$$r\left(r\left(\frac{d}{dr}\alpha(r)\right)^2 - r\frac{d}{dr}\alpha(r)\frac{d}{dr}\beta(r) + r\frac{d^2}{dr^2}\alpha(r) + \frac{d}{dr}\alpha(r) - \frac{d}{dr}\beta(r)\right)e^{-2\beta(r)}\sin^2(\theta)$$

0.0.1 Stress-energy tensor $T_{\mu\nu}$ for perfect fluid

[28]:
```
p = sp.Function("p")(r)
rho = sp.Function("\\rho")(r)

UU = np.zeros((4, 4), dtype=sp.Rational)
UU[0, 0] = exp(2 * a)

T = (p + rho) * UU - p * g
for i in range(4):
    T[i, i] = T[i, i].simplify()
print_matrix(T)
```

$$\begin{bmatrix} \rho(r)e^{2\alpha(r)} & 0 & 0 & 0 \\ 0 & p(r)e^{2\beta(r)} & 0 & 0 \\ 0 & 0 & r^2p(r) & 0 \\ 0 & 0 & 0 & r^2p(r)\sin^2(\theta) \end{bmatrix}$$

0.0.2 Einstin equation

$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}$

[29]:
```
G_newton = sp.Symbol("G")

eq = []
for i in range(len(G)):
    eq.append((G[i, i] - 8 * pi * G_newton * T[i, i]).simplify())

# Some manual simplification
Rtt = sp.Symbol("R_{\\theta \\theta}")
eq[0] = eq[0] * r**2 / exp(2 * a)/exp(-2*b ) * (-1 )
eq[1] = eq[1] * r**2 * (-1)
eq[2] = eq[2] / r / exp(-2*b)
eq[3] = eq[3].subs(eq[2], Rtt)
for i in range(len(G)):
    print_eq(eq[i].simplify())
```

$$8\pi G r^2 \rho(r) e^{2\beta(r)} - 2r\frac{d}{dr}\beta(r) - e^{2\beta(r)} + 1 = 0$$

$$8\pi G r^2 p(r) e^{2\beta(r)} - 2r\frac{d}{dr}\alpha(r) + e^{2\beta(r)} - 1 = 0$$

$$-8\pi G r p(r) e^{2\beta(r)} + r\left(\frac{d}{dr}\alpha(r)\right)^2 - r\frac{d}{dr}\alpha(r)\frac{d}{dr}\beta(r) + r\frac{d^2}{dr^2}\alpha(r) + \frac{d}{dr}\alpha(r) - \frac{d}{dr}\beta(r) = 0$$

$$R_{\theta\theta} r e^{-2\beta(r)} \sin^2(\theta) = 0$$

Define $e^{2\beta} = [1 - 2Gm(r)/r]^{-1}$

[30]:
```
m = sp.Function("m", Real=True)(r)
f = (1 - 2 * G_newton * m / r)**(-1)
eq1 = (eq[0] * exp(- 2 *a)).simplify().subs(b, Rational(1, 2) * log(f)).
 ↪simplify().expand().simplify()
s = sp.solve(eq1, m.diff(r))
eq1 = m.diff(r) - s[0]
```

Use $\nabla_\mu T^{\mu r} = 0 \implies (p+\rho)\partial_r \alpha = -\partial_r p$.

[31]:
```
eq2 = (eq[1] * r**2).subs(exp(2 * b), f).simplify()
s = sp.solve(eq2, a.diff(r))
eq2 = a.diff(r) - s[0]
eq2 = ((a.diff(r) - s[0]).subs(a.diff(r), - p.diff(r) / (p + rho))*(p + rho)).
 ↪simplify()
s = sp.solve(eq2, p.diff())
eq2 = p.diff(r) - s[0].factor()
```

The TOV-equation and equation for $m(r)$, both expressions are equal to 0.

[32]:
```
print_eq(eq1)
print_eq(eq2)
```

$$-4\pi r^2 \rho(r) + \frac{d}{dr} m(r) = 0$$

$$\frac{G\left(4\pi r^3 p(r) + m(r)\right)\left(\rho(r) + p(r)\right)}{r\left(-2Gm(r) + r\right)} + \frac{d}{dr} p(r) = 0$$

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