Utledninger

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Chapter 1

CHPT

1.1 Leading order Lagrangian

1.1.1 EM contribution only

Subs $\pi_a/f \to \pi_a$,

$$\Sigma = \exp\{i\pi_a\tau_a\} = 1 + i\pi_a\tau_a - \frac{1}{2}\pi_a\pi_a \tag{1.1}$$
$$Q = \frac{1}{6} + \frac{1}{2}\tau_3 \tag{1.2}$$

$$\begin{aligned}
Q\Sigma &= \frac{1}{2}\left[\frac{1}{3}\left(1 + i\pi_a\tau_a - \frac{1}{2}\pi_a\pi_a\right) + \tau_3\left(1 + i\pi_a\tau_a - \frac{1}{2}\pi_a\pi_a\right)\right] &(1.3)\\
&= \frac{1}{2}\left[\frac{1}{3} - \frac{1}{6}\pi_a\pi_a + i\pi_a\tau_3\tau_a + \frac{i}{3}\pi_a\tau_a + \tau_3 - \frac{1}{2}\pi_a\pi_a\tau_3\right] &(1.4)\\
Q\Sigma^\dagger &= \frac{1}{2}\left[\frac{1}{3} - \frac{1}{6}\pi_a\pi_a - i\pi_a\tau_3\tau_a - \frac{i}{3}\pi_a\tau_a + \tau_3 - \frac{1}{2}\pi_a\pi_a\tau_3\right] &(1.5)
\end{aligned}$$

Using $\mathrm{Tr}\{\tau_a\tau_b\tau_c\tau_d\} = 2(\delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd}\delta_{ad}\delta_{cb})$, and defining $\delta^i_{ab} = \delta_{ai}\delta_{bi}$,

$$\begin{aligned}
\mathrm{Tr}\left\{Q\Sigma Q\Sigma^\dagger\right\} &= \frac{1}{2^2}\mathrm{Tr}\left\{\frac{1}{9} - 2\frac{1}{2\cdot 3^2}\pi_a\pi_a + \pi_a\pi_a\tau_3\tau_a\tau_3\tau_a + \frac{1}{8}\pi_a\pi_a + 1 - \pi_a\pi_a\right\} &(1.6)\\
&= \frac{1}{2}\left(\frac{1}{9} + 1 - \frac{1}{3^2}\pi_a\pi_a - \pi_a\pi_a + \frac{1}{9}\pi_a\pi_a + \pi_a\pi_a(2\delta^3_{ab} - \delta_{ab})\right) &(1.7)\\
&= \frac{5}{9} - \pi_1^2 - \pi_2^2. &(1.8)
\end{aligned}$$

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1.1.2 Free energy EM contribution

$$\mathcal{F}/u_0 = -\left(\cos\alpha + \frac{1}{2}\frac{\mu_I^2}{\bar{m}^2}\sin^2\alpha + \frac{1}{2}\Delta\cos^2\alpha\right) \tag{1.9}$$

Intrdouce $y^2 = \mu_I^2/\bar{m}^2 = x^{-2}$.

$$-\frac{1}{u_0}\frac{d\mathcal{F}}{d\alpha} = \big(1 + [y^2 - \Delta]\cos\alpha\big)\sin\alpha = 0. \tag{1.10}$$

New phase at $y = 1 + \Delta$, where

$$\cos\alpha = \frac{1}{y^2 - \Delta} \implies \sin^2\alpha = 1 - \frac{1}{(y^2 - \Delta)^2} \tag{1.11}$$

Pressure

$$\tilde{p}' = -\mathcal{F}/u_0 = \frac{1}{y^2 - \Delta} + \frac{1}{2}y^2\left(1 - \frac{1}{(y^2 - \Delta)^2}\right) + \frac{1}{2}\Delta\frac{1}{(y^2 - \Delta)^2} \tag{1.12}$$

$$= \frac{1}{2}y^2 + \frac{1}{y^2 - \Delta} - \frac{1}{2}\frac{y^2 + \Delta}{(y^2 - \Delta)^2} \tag{1.13}$$

$$\frac{1}{2}\left(y^2 + \frac{1}{y^2 - \Delta}\right). \tag{1.14}$$

Normalize

$$p = p' - p'|_{y^2=(1-\Delta)} = \frac{1}{2}\left(y^2 + \frac{1}{y^2 + \Delta} - 2 - \Delta\right). \tag{1.15}$$

Isospin density

$$\frac{\mu_I}{u_0}n_I = -\frac{\mu_I}{u_0}\frac{d\mathcal{F}}{d\mu_I} = y^2\sin^2\alpha. \tag{1.16}$$

Energy density

$$\tilde{u} = -\tilde{p} + \frac{1}{u_0}\mu_I n_I = \frac{1}{2}\left(y^2 + \frac{1}{y^2 - \Delta} - 2 - \Delta + 2y^2\left[1 - \frac{1}{(y^2 - \Delta)^2}\right]\right) \tag{1.17}$$

$$= \frac{1}{2}\left(y^2 - \frac{3y^2 - \Delta}{(y^2 - \Delta)^2} + 2 + \Delta\right). \tag{1.18}$$

1.2 Newtonian stars

$$\mathrm{d}\left(\frac{u}{n}\right) = -p\mathrm{d}\left(\frac{1}{n}\right). \tag{1.19}$$

Polytrope: $p = Ku^\gamma$, internal energy (assuming $\gamma \neq 1$): $u' = u - mn$, where m is particle mass, n particle number density, $p = K(mn)^\gamma(1 + u'/(mn))^\gamma$.

$$\mathrm{d}\frac{u}{n} = \mathrm{d}\frac{u'}{n} = \frac{1}{n}\mathrm{d}u' + u'\mathrm{d}\frac{1}{n} = -k(mn)^\gamma\left(1 + \frac{u'}{mn}\right)^\gamma\mathrm{d}\frac{1}{n} \tag{1.20}$$

$$\implies \mathrm{d}u' = \left(\frac{u'}{mn} + k(mn)^{\gamma-1}[1 + u'/(mn)]^\gamma\right)m\mathrm{d}n. \tag{1.21}$$

non-relativistic limit, $u' \ll mn$, we get

$$u' = u - mn \sim \frac{k(mn)^\gamma}{\gamma - 1} \sim \frac{p}{\gamma - 1}, \tag{1.22}$$

as $p \sim k(mn)^\gamma$.

1.2.1 Energy

$$\Phi = -\frac{Gmu}{r}, \quad \frac{dp}{dr} = -\frac{Gmu}{r^2}. \tag{1.23}$$

Total kinetic energy is T, potential V.

$$T = 4\pi \int_0^R \mathrm{d}r\, r^2 u' = \frac{4\pi}{\gamma - 1} \int r^2 p, \tag{1.24}$$

$$V = -4\pi \int_0^R \mathrm{d}r\, r^2 \frac{Gmu}{r} = \int \mathrm{d}r\, r^3 \frac{dp}{dr} = -3 \cdot 4\pi \int \mathrm{d}r\, r^2 p \tag{1.25}$$

$$\implies T = -3(\gamma - 1)V. \tag{1.26}$$

With $\mathrm{d}m = 4\pi r^2 u \mathrm{d}r$, we get

$$I = 4\pi \int \mathrm{d}r^2 r^2 p = \int \mathrm{d}m \frac{p}{u} = -\int \mathrm{d}\left(\frac{p}{u}\right) m, \tag{1.27}$$

where we integrated by parts and used $m(0) = p(R)/u(R) = 0$. (assum $\gamma > 1$).

$$\mathrm{d}\frac{p}{u} = \frac{\gamma - 1}{\gamma} \frac{\mathrm{d}p}{u} = -\frac{\gamma - 1}{\gamma} \frac{Gm}{r^2} \mathrm{d}r, \tag{1.28}$$

as $\gamma p \mathrm{d}u = \mathrm{d}pu$. With this, we integrate by parts to obtain

$$I = \frac{\gamma - 1}{\gamma} \int \mathrm{d}r \frac{Gm^2}{r} = \frac{\gamma - 1}{\gamma} \left[-\int \mathrm{d}\left(\frac{1}{r}\right) Gm^2 \right] = -\frac{\gamma - 1}{\gamma} \left[\frac{GM^2}{R} - 2 \int \mathrm{d}m \frac{Gm}{r} \right] \tag{1.29}$$

using $\mathrm{d}m = 4\pi r^2 u$, we get

$$\int \mathrm{d}m \frac{Gm}{r} = 4\pi \int \mathrm{d}r\, r G m u = -3I, \implies I = \frac{5\gamma - 6}{\gamma - 1} \frac{GM^2}{R} \tag{1.30}$$

Combining $E = T + V$, we get

$$E = -\frac{3\gamma - 4}{5\gamma - 6} \frac{GM}{R^2} \tag{1.31}$$

1.3 Ward identities

(TODO: Gjør ward-identity utledningen med scalarfelt)

Consider, for defenders, a massless scalar field with the Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_\mu \varphi_i \partial^\mu \varphi_i - \mathcal{V}[\varphi]. \tag{1.32}$$

Assume this theory has a global $\mathrm{O}(N)$ symmetry,

$$\varphi_i \to \varphi_i' = M_{ij} \varphi_J = (\delta_{ij} + i\eta_\alpha T^\alpha_{ij}) \varphi_j. \tag{1.33}$$

so $\mathcal{V}[\varphi'] = \mathcal{V}[\varphi]$, where φ' is related to φ via a symmetry transformation. The system then has the conserved current

$$J^\mu_\alpha = (\partial^\mu \varphi_i) T^\alpha_{ij} \varphi_j. \tag{1.34}$$

The action, with scalar sources j and vector sources $v^\mu_{i,\alpha}$ is then

$$S[\varphi, j, v] = \int \mathrm{d}^4x \left(\frac{1}{2} \partial_\mu \varphi_i \partial^\mu \varphi_i - \mathcal{V}[\varphi] + j_i \varphi_i + v^\alpha_\mu J^\mu_\alpha \right). \tag{1.35}$$

As with the derivation of the Dyson-Schwinger equations, we now perform a *local* $\mathrm{O}(N)$ transformation but without setting the external sources to zero. The action then becomes

$$S[\varphi', j, v] = S[\varphi, j, v] + \int \mathrm{d}^4x\, [j_i M_{ij} \varphi_j + J\alpha, i^\mu()] \tag{1.36}$$

If a theory is globally invariant under some transformation, $\varphi(x) \to \varphi(x) + i\epsilon V \varphi(x)$, and assuming the measure is as well, then a *local* transformation $\varphi(x) \to \varphi(x) + i\epsilon V(x) \varphi(x)$ will give...

$$aaa \tag{1.37}$$