Master

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Todo list

Contents

Chapter 1

Chrial perturbation theory

In this chapter, we will take the general knowledge from the general theory in **??** and apply it to the specific case of quantum chromodynamics, which results in *chiral perturbation theory*, or χPT.

1.1 QCD

This section is based on [1–3].

1.1.1 Yang-Mills theory and Gauge symmetry

In our discussion on global symmetries, we considered the global transformation of fields by some group G. In gauge theories, we will consider local transformations. That is, the transformations are themselves functions of spacetime, $U = U(x)$, and take on some value in G for all points in space. With this, however, we encounter a problem with comparing the value of a field at different points. As the symmetry is local, a gauge transformation will generally affect the field at two points differently. We must find a way to compare fields at different points independent of gauge transformations. This is similar to a problem we have encountered before. In differential geometry, as described in **??**, we needed a connection $\Gamma^\rho_{\mu\nu}$ to compare vectors in different tangent spaces in a coordinate independent way. In gauge theories, we generalize this by defining a connection, A_μ, to compare field values at different points in a gauge-independent way.

Consider a set of N_c fields ψ_c, which the symmetry group $\mathrm{SU}(N)$ acts linearly on as $\psi_c \to U_{cc'}\psi_{c'}$. We can write $U = \exp\{i\eta_\alpha T_\alpha\}$, where T_α are the generators of $\mathfrak{su}(N_c)_c$, and can therefore be written $A_\mu = A^\alpha_\mu T_\alpha$. The transformation is then made local by letting the coordinates of $\mathrm{SU}(N)$ be functions of spacetime, $\eta_\alpha = \eta_\alpha(x)$. As we did in **??**, we define the covariant derivative D_μ to transform as the thing on which it acts. It has the form

$$D^{cc'}_\mu \psi_{c'} = (\delta_{cc'}\partial_\mu - igA^{cc'}_\mu)\psi_{c'}, \tag{1.1}$$

where $A^{cc'}_\mu$ is a new, dynamic field, the gauge field. This field takes values in the Lie algebra of the gauge group, $\mathfrak{su}(N)$. We will suppress the c-indices for cleaner notation. This field also transform under the gauge group. By enforcing the transformation rule $D_\mu A_\nu \to U D_\mu A_\nu$, we can deduce the transformation properties of the gauge field,

Why? Mention it is adjoint rep., and explain.

$$A_\mu \to U\left(A_\mu + \frac{i}{g}\partial_\mu\right)U^\dagger \tag{1.2}$$

With the covariant derivative, we can create gauge-invariant terms, such as $\bar\psi D_\mu\psi$. In **??** we introduced the Riemann tensor as the commutator of covaraint derivatives, **??**. This ensures that it transforms as a tensor and gives us the interpretation as a quantity that measures the amount vectors curved when parallel transported in a small loop. In analogy, we define the *field strength tensor*,

$$G_{\mu\nu} := \frac{i}{g}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]. \tag{1.3}$$

A_μ is an element of a Lie algebra, so the commutator is given by the structure constants of that algebra, **??**. The field strength tensor transforms as $G_\mu \to U G_{\mu\nu} U^\dagger$. This allows us to create gauge-invariant terms of only this tensor, which, as with the Ricci scalar in general relativity, are the building blocks of the Lagrangian of the gauge field. The lowest order terms are

$$G_\alpha^{\mu\nu} G_{\mu\nu}^\alpha, \quad \epsilon^{\mu\nu\rho\sigma} G_{\mu\nu}^\alpha G_{\rho\sigma}^\alpha. \tag{1.4}$$

Here, α is the index in $\mathfrak{su}(N)$-space.

1.1.2 The QCD Lagrangian

Quantum chromodynamics, or QCD, is the specific gauge theory of quarks q_{fc}, spin-$\frac{1}{2}$ particles, interacting via the strong force, a $\mathrm{SU}(3)_c$ gauge field denoted A_μ. There are six quarks q, called flavors and indexed by f, and the quantum number corresponding to the strong force is called color indexed by c. The quarks, labeled u, d, s, c, t, and b, have different masses. This thesis will include the two or three lightest quarks at different times. Therefore, we denote the number of flavors by N_f. The Lagrangian of QCD, including only the strong force, is

$$\mathcal{L}_{\mathrm{QCD}} = \bar{q}(i\not{D} - m)q - \frac{1}{4} G_{\mu\nu}^\alpha G_\alpha^{\mu\nu}. \tag{1.5}$$

We have suppressed color and flavor indices. $\not{D}q = \gamma^\mu(\partial_\mu - igA_\mu)q$ is the covariant derivative associated with the $\mathrm{SU}(3)_c$ gauge group with coupling constant g, and γ^μ are the Dirac matrices, as described in section A.1. The quark mass matrix, m, acts on the flavor indexes as the flavor states are mass eigenstates. There are no known symmetries that forbid a $\epsilon^{\mu\nu\rho\sigma} G_{\mu\nu}^\alpha G_{\rho\sigma}^\alpha$-term, and its absence is dubbed the strong CP problem [3].

1.1.3 Chiral symmetry

If we consider the massless QCD Lagrangian, $m = 0$, it has an additional symmetry of rotation in its flavour indices. We can project the quarks down to their *chiral* components by introducing projection operators

$$P_R = \frac{1}{2}\left(1 + \gamma^5\right), \quad P_L = \frac{1}{2}\left(1 - \gamma^5\right). \tag{1.6}$$

Here, γ^5 is the "fifth gamma-matrix", as described in section A.1. As good projection operators, they obey

$$P_R + P_L = 0, \quad P_R P_L = P_L P_R = 0, \quad P_I^2 = P_I,\, I = R, L. \tag{1.7}$$

By the properties of γ^5 and $\bar{q} = q^\dagger \gamma^0$, these operators project out the opposite chirality of q and q,

$$P_I q = q_I, \quad \bar{q} P_I = q_{\bar{I}}, \quad I = R, L, \quad \bar{I} = L, R. \tag{1.8}$$

With this, we can write the quark-sector of massless QCD as

$$i\bar{q}\not{D}q = i\bar{q}\not{D}(P_R + P_L)^2 q = i\bar{q}_L \not{D} q_L + i\bar{q}_R \not{D} q_R. \tag{1.9}$$

This operator is invariant under the transformations

$$q_R \to U_R q_R, \quad q_L \to U_L q_L, \tag{1.10}$$

where U_L and U_R are Hermitian matrices that act on the flavor indices. These transformations form the Lie group $\mathrm{U}(N_f)_R \times \mathrm{U}(N_f)_L = \mathrm{U}(1)_R \times \mathrm{SU}(N_f)_R \times \mathrm{U}(1)_L \times \mathrm{SU}(N_f)_L$. This transformation can also be described in terms of the diagonal subgroup. This subgroup is made up of transformations where $U_R = U_L$, called vector transformations, and the remaining subgroup of transformations where $U_L = U_R^\dagger$, called axial transformations. These together form $\mathrm{U}(N_f)_A \times \mathrm{U}(N_f)_V = \mathrm{U}(1)_V \times \mathrm{SU}(N_f)_V \times \mathrm{U}(1)_A \times \mathrm{SU}(N_f)_A$. The currents corresponding to these transformations are

$$J_V^\mu = \bar{q}_R \gamma^\mu q_R, \quad V_\alpha^\mu = \bar{q} T_\alpha \gamma^\mu q, \quad J_A^\mu = \bar{q}_L \gamma^\mu \gamma^5 q_L, \quad A_\alpha^\mu = \bar{q} T_\alpha \gamma^\mu \gamma^5 q. \tag{1.11}$$

Here, T_α and $T_\alpha \gamma^5$ are the generators of $\mathrm{SU}(N_f)_V$ and $\mathrm{SU}(N_f)_A$. This symmetry, though, is broken in several ways. Firstly, transformations of the form $e^{i\alpha\gamma^5} \in \mathrm{U}(1)_A$ are subject to the *axial anomaly*. As mentioned in **??**, in a quantum theory not only the action has to be invariant but the integration measure as well, and $\mathcal{D}q\mathcal{D}\bar{q}$ is not. This is encoded in the Schwinger-Dyson equation [3]

write more about SD-eqs, ward identities

$$\partial_\mu \langle J_A^\mu \rangle = -\frac{e^2}{(4\pi)^2} \langle \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \rangle, \tag{1.12}$$

whose right side would vanish if the quantum theory was invariant under $\mathrm{U}(1)_A$. The remaining symmetry is $G = \mathrm{U}(1)_V \times \mathrm{SU}(N_f)_V \times \mathrm{SU}(N_f)_A$. Next, the mass term explicitly breaks this symmetry. If we consider all quarks to have the same mass m_q, so that $m = m_q \mathbb{1}$, only $\mathrm{U}(N_f)_A$ is broken. This is called the *chiral limit*. However, when we include the fact that the masses of the quarks are different, we break the symmetry further. Other external currents, chemical potentials, and the electromagnetic interaction also break the symmetry. We will discuss how to incorporate this in the next chapter. Lastly, the G-symmetry is broken spontaneously by the ground state quark condensate,

$$\langle \bar{q}_f q_f \rangle = -f^2 B_0 \neq 0, \; f \in \{u, d, s\}. \tag{1.13}$$

The scalar quark operator is not invariant under $\mathrm{SU}(N_f)_A$, and as discussed in **??**, this leads to the spontaneous symmetry breaking pattern.

$$\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \longrightarrow \mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R / \mathrm{SU}(N_f)_A = \mathrm{SU}(N_f)_V. \tag{1.14}$$

This pattern enables us to construct an effective low energy theory for QCD physics. We will take this symmetry breaking as an axiom and use it to construct χPT.

1.2 Chiral perturbation theory

The systematics of chiral perturbation theory, or χPT, was laid out by Gasser and Leutwyler [4, 5] and is based on Weinberg's idea that quantum field theories on their own does not contain more information than the bare minimum [6]. In addition to these paper, this section is base on [2, 7, 8].

1.2.1 *Non-linear realization

To construct the Lagrangian of chiral perturbation theory we start with the Lagrangian of massless QCD,

$$\mathcal{L}^0_{\mathrm{QCD}} = i\bar{q} \not{D} q - \frac{1}{4} G^\alpha_{\mu\nu} G^{\mu\nu}_\alpha \tag{1.15}$$

As discussed in last section, this Lagrangian is invariant under the full symmetry group $G = \mathrm{SU}(N_f)_R \times \mathrm{SU}(N_f)_L$, but the system undergoes spontaneous symmetry breaking to the smaller group $H = \mathrm{SU}(N_f)_V$. As we found in **??**, the low energy dynamics will therefore be described by a $G/H = \mathrm{SU}(N_f)_A$-valued field Σ. Let $g \in G$. We write $g = (U_L, U_R)$, where $U_R \in \mathrm{SU}(N_f)_R$, $U_L \in \mathrm{SU}(N_f)_L$. Elements $h \in H$ are then of the form $h = (U, U)$. A general element g can be written as

$$g = (U_L, U_R) = (1, U_R U_L^\dagger)(U_L, U_L). \tag{1.16}$$

Since $(U_L, U_L) \in H$, this means that we can write the coset gH as $(1, U_R U_L^\dagger)H$, which gives a way to choose a representative element for each coset. We identify

$$\Sigma = U_R U_L^\dagger. \tag{1.17}$$

This is our standard form for elements in gH. As we saw in **??**, it therefore implicitly define transformation properties of the Goldstone bosons, which is given by the function $h(g,\xi)$. For $\tilde{g}\in G$, we have

$$\tilde{g}(1,\Sigma) = (\tilde{U}_L,\tilde{U}_R)(1,U_R U_L^\dagger) = (1,\tilde{U}_R(U_R U_L^\dagger)\tilde{U}_L^\dagger)(\tilde{U}_L,\tilde{U}_L) = (1,\tilde{U}_R\Sigma\tilde{U}_L)\tilde{h}. \tag{1.18}$$

This gives the transformation rule

$$\Sigma\to\Sigma' = U_R\Sigma U_L^\dagger. \tag{1.19}$$

Under transformations by $h=(U,U^\dagger)\in H$, we have

$$\Sigma\to\Sigma' = U\Sigma U^\dagger. \tag{1.20}$$

hvorfor?

Due to how G factors into two Lie groups, the constituents of the Mauer-Cartan form are

$$d_\mu = i\Sigma(x)^\dagger\partial_\mu\Sigma(x), \quad e_\mu = 0. \tag{1.21}$$

Using $\partial_\mu[\Sigma(x)^\dagger\Sigma(x)]=0$, we can write

$$d_\mu d^\mu = -\Sigma(x)^\dagger[\partial_\mu\Sigma(x)]\Sigma(x)^\dagger[\partial^\mu\Sigma(x)] = \Sigma(x)^\dagger[\partial_\mu\Sigma(x)][\partial^\mu\Sigma(x)^\dagger]\Sigma(x). \tag{1.22}$$

In **??**, we found the lowest order terms, **??**. As $d_\mu\in\mathfrak{su}(N_f)$, which we represent by the traceless matrices, we have

$$\mathrm{Tr}\left\{d_\mu\right\}=0. \tag{1.23}$$

This leaves us with the single leading order term

$$\mathrm{Tr}\left\{d_\mu d^\mu\right\} = \mathrm{Tr}\left\{\partial_\mu\Sigma(\partial^\mu\Sigma)^\dagger\right\}, \tag{1.24}$$

where we have used the cyclic property of the trace.

However, constructing the effective Lagrangian out of terms invariant under G is too restrictive to get the most general effective action. This only allows for an even number of d_μ's, and observed processes such as the decay of the neutral pion through $\pi^0\to\gamma\gamma$ would not be possible [2]. This is because we have not allowed for terms that change the Lagrangian with a divergence term, as discussed in **??**. Terms of this type are called Wess-Zumino-Witten (WZW) terms [9]. We will not consider these here, as they do not affect the thermodynamic quantities in question [10].

1.2.2 External currents

To incorporate other fields or terms that break G, such as the quark masses, we add a Lagrangian containing external currents. These can include either couple to the conserved currents, Eq. (1.11), or the other bilinears we can create out of quarks, $\bar{q}q$, $\bar{q}\gamma^5 q$, $\bar{q}T_\alpha q$, and $\bar{q}T_\alpha\gamma^5 q$. The Lagrangian is

$$\mathcal{L}_{\mathrm{ext}} = -\bar{q}\left(s-i\gamma^5 p\right)q + \bar{q}\gamma^\mu\left(v_\mu+\gamma^5 a_\mu\right)q. \tag{1.25}$$

Hva er grunnen til valgene av fortegn?

Can we allow for them to have a trace?

Here, s, p, v_μ and a_μ are all $N_f\times N_f$ matrices acting on the flavor indices. They are, respectively, the scalar, pseudo-scalar, vector, and pseudo-vector currents. We denote these currents collectively as $j=(s,p,v^\mu,a^\mu)$. The masses of the quarks are accounted for by setting the scalar current $s=m+\tilde{s}$. Here, m is the mass matrix of the quarks, while $\tilde{s}$ are possible other scalar currents. Other examples of external currents are chemical potentials, such as the isospin chemical potential, which regulate conserved charges in the system.

We define the right handed and left handed currents as

$$r_\mu = v_\mu + a_\mu, \quad l_\mu = v_\mu - a_\mu \tag{1.26}$$

Write about gaugeing the symmetry

Including dynamical fields, such as the photon field $\mathcal{A}_\mu$, is slightly more complicated. The electromagnetic interactions is a gauge theory with gauge group $\mathrm{U}(1)_{\mathrm{EM}}$, a subgroup of G. The electromagnetic covariant derivative acing on quarks is

$$i\bar{q}\not{D}'q = i\bar{q}\gamma^\mu\left(\mathbb{1}\partial_\mu - ieQ\mathcal{A}_\mu\right)q = i\bar{q}\not{\partial}q - e\mathcal{A}_\mu J^\mu, \tag{1.27}$$

where $\mathcal{A}_\mu$ is the photon field corresponding to the gauge group, $e = |e|$ is the elementary charge as given in **??**, $J^\mu = -\bar{q}Q\gamma^\mu q$ is the electromagnetic charge current, and Q is the quark charge matrix. This matrix is the generator of $\mathrm{U}(1)_{\mathrm{EM}}$, and thus is a part of Lie algebra corresponding to G. In the case of $N_f = 3$, $Q = \mathrm{diag}(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$, while the last term is not included for $N_f = 2$. From Eq. (1.27), we see that $eQ\mathcal{A}_\mu$ is a vector current. We therefore include it by setting $v^\mu = eQ\mathcal{A}^\mu + \tilde{v}^\mu$, where again $\tilde{v}^\mu$ are possible other vector currents.

skriv om innføringen av Q_I, $I = R, L$

Lastly, we must include terms from quantum electrodynamics involving only the photon field, which are

$$\mathcal{L}^0_{\mathrm{QED}}[\mathcal{A}] = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \quad F_{\mu\nu} = 2\partial_{[\mu}\mathcal{A}_{\nu]}. \tag{1.28}$$

The full Lagrangian is then

$$\mathcal{L}_{\mathrm{QCD}}[q, \bar{q}, A, \mathcal{A}, j] = \mathcal{L}^0_{\mathrm{QCD}}[q, \bar{q}, A] + \mathcal{L}^0_{\mathrm{QED}}[\mathcal{A}] + \mathcal{L}_{\mathrm{ext}}[\mathcal{A}, j]. \tag{1.29}$$

We now define the effective Lagrangian of χPT, $\mathcal{L}_{\mathrm{eff}}$ as

$$Z[j] = \int \mathcal{D}q\mathcal{D}\bar{q}\mathcal{D}A\mathcal{D}\mathcal{A} \exp\left\{i\int \mathrm{d}^4x \mathcal{L}_{\mathrm{QCD}}[q, \bar{q}, A, \mathcal{A}, j]\right\} = \int \mathcal{D}\pi\mathcal{D}\mathcal{A} \exp\left\{i\int \mathrm{d}^4x \mathcal{L}_{\mathrm{eff}}[\pi, \mathcal{A}, j]\right\}. \tag{1.30}$$

1.2.3 Winberg's power counting scheme

Skriv/kopier tekst om Weinberg's power counting scheme

1.2.4 Building blocks

Write summary of included

Covariant derivative

$$\nabla_\mu\Sigma = \partial_\mu\Sigma - ir_\mu\Sigma + i\Sigma l_\mu. \tag{1.31}$$

Scalar

$$\chi = 2B_0(s + ip), \quad \chi^\dagger = 2B_0(s - ip) \tag{1.32}$$

Field strength tensor

$$f^{(r)}_{\mu\nu} = \partial_\mu r_\nu - \partial_\nu r_\mu - i[r_\mu, r_\nu], \quad f^{(l)}_{\mu\nu} = \partial_\mu l_\nu - \partial_\nu l_\mu - i[l_\mu, l_\nu]. \tag{1.33}$$

Transformations under $\mathrm{U}(N_f)_R \times \mathrm{U}(N_f)_L$, where

$$U_I = e^{-i\theta}\exp\{-i\eta_\alpha T_\alpha\}, \quad I = R, L. \tag{1.34}$$

$$\Sigma \to U_R\Sigma U_L^\dagger, \tag{1.35}$$

$$r_\mu \to U_R(r_\mu + i\partial_\mu)U_R^\dagger, \quad r, R \to l, L. \tag{1.36}$$

$$\chi \to U_R\chi U_L^\dagger \tag{1.37}$$

$$Q_I \to U_I Q_I U_I^\dagger,\ I = R, L. \tag{1.38}$$

We count χ as order 2, e as order 2 and $\nabla_\mu\Sigma$ as order 1. Notice that e and Q must always appear as eQ, as the orignal Lagrangian Eq. (1.30) is invariant under the transformation $e \to e/\lambda$ and $Q \to \lambda Q$ [11].

1.3 *Two-flavour χPT to leading order

1.3.1 Paramterization

In this section, we will assume $N_f = 2$, which means the generators are $T_a = \frac{1}{2}\tau_a$, where τ_a are the Pauli Matrices, as described in section A.1. The quark mass matrix is

$$m = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}. \tag{1.39}$$

We define $\bar{m}^2 = B_0(m_u + m_d)$ and $\Delta m^2 = B_0(m_u - m_d)$, so that when we set the scalar current s equal to the quark masses, and the pseudoscalar current to zero, we get

$$\chi = \bar{m}^2 \mathbb{1} + \Delta m^2 \tau_3, \tag{1.40}$$

For $j = 0$, the ground state is by assumption $\Sigma = \mathbb{1}$, the vacuum, and we can use the paramterization

$$\Sigma(x) = \exp\left\{ i\frac{\pi_a \tau_a}{f} \right\}, \tag{1.41}$$

where f is the bare pion decay constant, π_a are the three Goldstone bosons, a set of real functions of space-time. This ensures that $\pi = 0$ corresponds to the vacuum. If we perform an infinitesimal isospin transformation, and assume π/f small, then

$$\Sigma \to U_V \Sigma U_V^\dagger = \left(1 + i\eta_a \frac{1}{2}\tau_a\right)\left(1 + i\frac{1}{f}\pi_b\tau_b\right)\left(1 - i\eta_c\frac{1}{2}\tau_c\right) = 1 + i\frac{1}{f}\pi_a(\delta_{ac} + i\eta_b\epsilon_{abc})\tau_c, \tag{1.42}$$

or

$$\pi_a \to (\delta_{ac} + i\eta_b\epsilon_{abc})\pi_c. \tag{1.43}$$

The generators of π_a under isospin-transformations are thus the adjoint representation of $\mathfrak{su}(2)$, and they form an isospin triplet. For $\eta_1 = \eta_2 = 0$, i.e. transformations generated by τ_3, π_3 is invariant, which means that it has quantum number $I_3 = 0$.[1] π_1 and π_2 do not have a definite value of the third component of isospin, but rather for the first and second component. They are related to the observed, charged pions π^+ and π^- by [2]

$$\pi_a\tau_a = \begin{pmatrix} \pi_3 & \pi_1 - i\pi_2 \\ \pi_1 + i\pi_2 & -\pi_3 \end{pmatrix} = \begin{pmatrix} \pi^0 & \sqrt{2}\pi^- \\ \sqrt{2}\pi^+ & -\pi^0 \end{pmatrix}, \tag{1.44}$$

where $\pi^\pm$ has a third isospin-component of $I_3 = \pm 1$. For non-zero isospin chemical potential, however, we expect that the ground state may be rotated away from the vacuum. To find what the new ground state is, we have to minimize the Hamiltonian.

1.3.2 Leading order Lagrangian

The leading order Lagrangian in Winberg's power counting scheme, with $e = 0$, is

$$\mathcal{L}_2 = \frac{1}{4}f^2 \mathrm{Tr}\left\{\nabla_\mu \Sigma (\nabla^\mu \Sigma)^\dagger\right\} + \frac{1}{4}f^2 \mathrm{Tr}\left\{\chi^\dagger \Sigma + \Sigma^\dagger \chi\right\}. \tag{1.45}$$

The external source currents are

$$\nabla_\mu \Sigma = \partial_\mu \Sigma - i[v_\mu, \Sigma], \quad v_\mu = \frac{1}{2}\mu_I \delta_\mu^0 \tau_3. \tag{1.46}$$

To incorporate a finite isospin density, we must parametrize the Goldstone manifold differently than in the vacuum. We follow the analysis in [10]. We assume the ground state is independent of space, $\pi_a(x) = \pi_a^0$, and write it as

$$\Sigma_\alpha := \exp\{i\alpha n_a \tau_a\} = \cos\alpha + i n_a \tau_a \sin\alpha, \tag{1.47}$$

[1] Authors differe if they define $\sqrt{2}\pi^\pm = \pi_1 \pm i\pi_2$, or with opposite signs. We choose the former, so that $\pi_+ |0\rangle$ is the state with the quantum numbers of the positive pion.

where

$$\alpha = \frac{1}{f}\sqrt{\pi_a^0\pi_a^0}, \quad n_a = \frac{\pi_a^0}{\sqrt{\pi_a^0\pi_a^0}}. \tag{1.48}$$

With this, the covariant derivative is $\nabla_\mu\Sigma_\alpha = -iv_\mu^a[\tau_a, \Sigma_\alpha]$, and the two terms in the first order Lagrangian are

$$\mathrm{Tr}\left\{\nabla_\mu\Sigma_\alpha(\nabla^\mu\Sigma_\alpha)^\dagger\right\} = 2\mu_I^2(n_1^2+n_2^2)\sin^2\alpha, \quad \mathrm{Tr}\left\{\chi^\dagger\Sigma_\alpha + \Sigma_\alpha^\dagger\chi\right\} = 4\bar{m}^2\cos\alpha. \tag{1.49}$$

We see that, to first order, all results are independent of Δm. To find the new ground state, we minimize the Hamiltonian density. With the assumption that the fields are constant, the first order Hamiltonian density is

$$\mathcal{H}_2 = -\mathcal{L}_2 = -f^2\left[\bar{m}^2\cos\alpha + \frac{1}{2}\mu_I^2(n_1^2+n_2^2)\sin^2\alpha\right] \tag{1.50}$$

For $\mu_I = 0$, this is independent of n_a, and minimized by $\alpha = 0$. Now, as $n_in_i = 1$, we have that $n_1^2 + n_2^2 = 1 - n_3^2$. This means that, for $\mu_I \neq 0$, the energy is minimized by $n_3 = 0$. We can write $n_1 = \cos\phi$, $n_2 = \sin\phi$, for some real number ϕ, which gives the ground state

$$\Sigma_\alpha = \mathbb{1}\cos\alpha + i(\tau_1\cos\phi + \tau_2\sin\phi)\sin\alpha. \tag{1.51}$$

We can choose, without loss of generality, $\phi = 0$ [12]. This corresponds to a change of basis of $\mathfrak{su}(2)$, $\tau_1 \to \tilde{\tau}_1 = \tau_1\cos\phi + \tau_2\sin\phi$ and $\tau_2 \to \tilde{\tau}_2 = -\tau_1\sin\phi + \tau_2\cos\phi$. With this, the new ground state is

$$\Sigma_\alpha = \exp\{i\alpha\tau_1\} \tag{1.52}$$

Any exited state is a transformation of the ground state by $\mathrm{SU}(2)_A$. For $\mu_I = 0$, this corresponds to

$$\Sigma(x) = U_R(x)\Sigma_0U_L^\dagger(x) = U(x)\Sigma_0U(x). \tag{1.53}$$

where

$$U(x) = \exp\left\{i\frac{\tau_a\pi_a(x)}{2f}\right\}. \tag{1.54}$$

We see that this recovers the parametrization Eq. (1.41). For $\mu_I \neq 0$, the ground state may be shifted, and so $U(x)$ must be too. The groundstate transforms as

$$\Sigma_0 \to \Sigma_\alpha = \hat{U}_L\Sigma_0\hat{U}_R^\dagger = A_\alpha\Sigma_0A_\alpha. \tag{1.55}$$

where

$$A_\alpha := \exp\left\{i\frac{1}{2}\alpha\tau_1\right\} = \cos\frac{\alpha}{2} + i\tau_1\sin\frac{\alpha}{2}. \tag{1.56}$$

This induces the following transformations for the fluctuations,

$$U_L \to \hat{U}_LU_L\hat{U}_L^\dagger = A_\alpha U_LA_\alpha^\dagger, \tag{1.57}$$
$$U_R \to \hat{U}_RU_R\hat{U}_R^\dagger = A_\alpha^\dagger U_RA_\alpha. \tag{1.58}$$

The new parametrization is thus

$$\Sigma(x) = A_\alpha[U(x)\Sigma_0U(x)]A_\alpha. \tag{1.59}$$

With this, we can expand the first order Lagrangian, Eq. (1.45), in powers of π/f We will use this expansion to calculate the free energy density. Expanding Σ to $\mathcal{O}\left((\pi/f)^5\right)$, we get

$$\Sigma = \left(1 - \frac{\pi_a^2}{2f^2} + \frac{\pi_a^2\pi_b^2}{24f^4}\right)(\cos\alpha + i\tau_1\sin\alpha) + \left(\frac{\pi_a}{f} - \frac{\pi_b^2\pi_a}{6f^3}\right)\left(i\tau_a - 2i\delta_{a1}\tau_1\sin^2\frac{\alpha}{2} - \delta_{a1}\sin\alpha\right). \tag{1.60}$$

The kinetic term in the χPT Lagrangian is

$$\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger = \partial_\mu\Sigma\partial^\mu\Sigma^\dagger - i\left(\partial_\mu\Sigma[v^\mu, \Sigma^\dagger] - \text{h.c.}\right) - [v_\mu, \Sigma][v_\mu, \Sigma^\dagger]. \tag{1.61}$$

Using Eq. (1.60) we find the expansion of the constitutive parts of the kinetic term to be

$$\begin{aligned}\partial_\mu\Sigma = &\left[\left(\frac{-1}{f^2}+\frac{\pi_b^2}{6f^4}\right)(\pi_a\partial_\mu\pi_a)\cos\alpha - \left(\frac{\partial_\mu\pi_1}{f}-\frac{\pi_b^2\partial_\mu\pi_1+2\pi_1\pi_b\partial_\mu\pi_b}{6f^3}\right)\sin\alpha\right]\\ &-\left[\left(\frac{-1}{f^2}+\frac{\pi_b^2}{6f^4}\right)(\pi_a\partial_\mu\pi_a)\sin\alpha - \left(\frac{\partial_\mu\pi_1}{f}-\frac{\pi_b^2\partial_\mu\pi_1+2\pi_1\pi_b\partial_\mu\pi_b}{6f^3}\right)2\sin^2\frac{\alpha}{2}\right]i\tau_1\\ &+\left(\frac{\partial_\mu\pi_a}{f}-\frac{\pi_b^2\partial_\mu\pi_a+2\pi_a\pi_b\partial_\mu\pi_b}{6f^3}\right)i\tau_a,\end{aligned} \tag{1.62}$$

and

$$[v_\mu,\Sigma] = -\mu_I\delta_\mu^0\left\{\left[\left(1-\frac{\pi_a^2}{2f^2}+\frac{\pi_a^2\pi_b^2}{24f^4}\right)\sin\alpha+\left(\frac{\pi_1}{f}-\frac{\pi_b^2\pi_1}{6f^3}\right)\cos\alpha\right]\tau_2-\left(\frac{\pi_2}{f}-\frac{\pi_b^2\pi_2}{6f^3}\right)\tau_1\right\}. \tag{1.63}$$

Combining Eq. (1.62) and Eq. (1.63) gives the following terms

$$\mathrm{Tr}\left\{\partial_\mu\Sigma\partial^\mu\Sigma^\dagger\right\} = \frac{2}{f^2}\partial_\mu\pi_a\partial^\mu\pi_a+\frac{2}{3f^4}\left[(\pi_a\partial_\mu\pi_a)(\pi_b\partial^\mu\pi_b)-(\pi_a\partial_\mu\pi_b)(\pi_b\partial^\mu\pi_a)\right], \tag{1.64}$$

$$\begin{aligned}-i\mathrm{Tr}\left\{\partial^\mu\Sigma[v_\mu,\Sigma^\dagger]-\text{h.c.}\right\} = &\,4\mu_I\frac{\partial_0\pi_2}{f}+8\mu_I\frac{\pi_3}{3f^3}\sin\alpha(\pi_2\partial_0\pi_3-\pi_3\partial_0\pi_2)\sin\alpha\\ &+\left(\frac{4\mu_I}{f^2}\cos\alpha-\frac{8\mu_I\pi_1}{3f^3}\sin\alpha-\frac{4\mu_I\pi_a\pi_a}{3f^4}\cos\alpha\right)(\pi_1\partial_0\pi_2-\pi_2\partial_0\pi_1),\end{aligned} \tag{1.65}$$

$$-\mathrm{Tr}\left\{[v_\mu,\Sigma][v^\mu,\Sigma^\dagger]\right\} = {\mu_I}^2\left[2\sin^2\alpha+\left(\frac{2}{f}-\frac{4\pi_a\pi_a}{3f^3}\right)\pi_1\sin 2\alpha+\left(\frac{2}{f^2}-\frac{2\pi_a\pi_a}{3f^4}\right)\pi_a\pi_b k_{ab}\right], \tag{1.66}$$

$$\mathrm{Tr}\left\{\chi^\dagger\Sigma+\Sigma^\dagger\chi\right\} = \bar{m}^2\left(4\cos\alpha-\frac{4\pi_1}{f}\sin\alpha-\frac{2\pi_a\pi_a}{f^2}\cos\alpha+\frac{2\pi_1\pi_a\pi_a}{3f^3}\sin\alpha+\frac{(\pi_a\pi_a)^2}{6f^4}\cos\alpha\right), \tag{1.67}$$

where $k_{ab}=\delta_{a1}\delta_{b1}\cos 2\alpha+\delta_{a2}\delta_{b2}\cos^2\alpha-\delta_{a3}\delta_{b3}\sin^2\alpha$. Notice that the mass term is independent of the difference in quark masses, Δm. If we write the Lagrangian Eq. (1.45) as $\mathcal{L}_2=\mathcal{L}_2^{(0)}+\mathcal{L}_2^{(1)}+\mathcal{L}_2^{(2)}+...$, where $\mathcal{L}_2^{(n)}$ contains all terms of order $(\pi/f)^n$, then the result of the series expansion is

$$\mathcal{L}_2^{(0)} = f^2\left(\bar{m}^2\cos\alpha+\frac{1}{2}\mu^2\sin^2\alpha\right), \tag{1.68}$$

$$\mathcal{L}_2^{(1)} = f(\mu_I^2\cos\alpha-\bar{m}^2)\pi_1\sin\alpha+f\mu_I\partial_0\pi_2\sin\alpha, \tag{1.69}$$

$$\mathcal{L}_2^{(2)} = \frac{1}{2}\partial_\mu\pi_a\partial^\mu\pi_a+\mu_I\cos\alpha\,(\pi_1\partial_0\pi_2-\pi_2\partial_0\pi_1)-\frac{1}{2}\bar{m}^2\pi_a\pi_a\cos\alpha+\frac{1}{2}\mu_I^2\pi_a\pi_b k_{ab}, \tag{1.70}$$

$$\begin{aligned}\mathcal{L}_2^{(3)} = &\,\frac{\pi_a\pi_a\pi_1}{6f}(\bar{m}^2\sin\alpha-2{\mu_I}^2\sin 2\alpha)\\ &-\frac{2\mu_I}{3f}\left[\pi_1(\pi_1\partial_0\pi_2-\pi_2\partial_0\pi_1)+\pi_3(\pi_3\partial_0\pi_2-\pi_2\partial_0\pi_3)\right]\sin\alpha,\end{aligned} \tag{1.71}$$

$$\begin{aligned}\mathcal{L}_2^{(4)} = &\,\frac{1}{6f^2}\left\{\frac{1}{4}\bar{m}^2(\pi_a\pi_a)^2\cos\alpha-\left[(\pi_a\pi_a)(\partial_\mu\pi_b\partial^\mu\pi_b)-(\pi_a\partial_\mu\pi_a)(\pi_b\partial^\mu\pi_b)\right]\right\}\\ &-\frac{\mu_I\pi_a\pi_a}{3f^2}\left[(\pi_1\partial_0\pi_2-\pi_2\partial_0\pi_1)\cos\alpha+\frac{1}{2}\mu_I\pi_a\pi_b k_{ab}\right].\end{aligned} \tag{1.72}$$

1.3.3 Propagator

We may write the quadratic part of the Lagrangian Eq. (1.70) as[2]

$$\mathcal{L}_2^{(2)} = \frac{1}{2}\sum_a\partial_\mu\pi_a\partial^\mu\pi_a+\frac{1}{2}m_{12}(\pi_1\partial_0\pi_2-\pi_2\partial_0\pi_1)-\frac{1}{2}\sum_a m_a^2\pi_a^2, \tag{1.73}$$

[2]Summation over isospin index (a,b,c) will be explicit in this section.

where

$$m_1^2 = \bar{m}^2\cos\alpha - \mu_I^2\cos 2\alpha, \tag{1.74}$$
$$m_2^2 = \bar{m}^2\cos\alpha - \mu_I^2\cos^2\alpha, \tag{1.75}$$
$$m_3^2 = \bar{m}^2\cos\alpha + \mu_I^2\sin^2\alpha, \tag{1.76}$$
$$m_{12} = 2\mu_I\cos\alpha. \tag{1.77}$$

The inverse propagator is given by the functional derivative,

$$D^{-1}_{ab}(x-y) = \frac{\delta S[\pi]}{\delta\pi_a(x)\pi_b(y)} = \left[-\delta_{ab}(\partial_x^2 + m_a^2) + m_{12}(\delta_{a1}\delta_{b2} - \delta_{a2}\delta_{b1})\partial_{x,0}\right]\delta(x-y). \tag{1.78}$$

The momentum space inverse propagator is

$$D^{-1}_{ab}(p) = \delta_{ab}(p^2 - m_a^2) + ip_0 m_{12}(\delta_{a1}\delta_{b2} - \delta_{a2}\delta_{b1}). \tag{1.79}$$

The spectrum of the particles is given by solving $\det(D^{-1}) = 0$ for p^0. With $p = (p_0, \vec{p})$ as the four momentum, this gives

$$\det(D^{-1}) = D^{-1}_{33}\left(D^{-1}_{11}D^{-1}_{22} + (D^{-1}_{12})^2\right) = \left(p^2 - m_3^2\right)\left[\left(p^2 - m_1^2\right)\left(p^2 - m_2^2\right) - p_0^2 m_{12}^2\right] = 0.$$

This equation has the solutions

$$E_0^2 = |\vec{p}|^2 + m_3^2, \tag{1.80}$$
$$E_\pm^2 = |\vec{p}|^2 + \frac{1}{2}\left(m_1^2 + m_2^2 + m_{12}^2\right) \pm \frac{1}{2}\sqrt{4|\vec{p}|^2 m_{12}^2 + \left(m_1^2 + m_2^2 + m_{12}^2\right)^2 - 4m_1^2 m_2^2}. \tag{1.81}$$

These are the energies of three particles π_0, π_+ and π_-. π_0 is π_3, while $\pi_\pm$ are linear combinations of π_1 and π_2. [3] We will show that for $\mu_I < m_\pi$, $\alpha = 0$, before it starts to increase for $\mu_I \geq m_\pi$. This result is presented in chapter 2. For $\alpha = 0$, we get

$$\frac{1}{2}(m_1^2 + m_2^2 + m_{12}^2) = \bar{m}^2 + \mu_I^2, \quad m_1^2 m_2^2 = (\bar{m}^2 - \mu_I^2)^2, \quad m_3^2 = \bar{m}^2,$$
$$\implies E_\pm^2 = |\vec{p}|^2 + \bar{m}^2 + \mu_I^2 \pm 2\mu_I\sqrt{|\vec{p}|^2 + \bar{m}^2}.$$

This corresponds to a Zeeman-like splitting of the energies,

$$E_0 = \sqrt{|\vec{p}|^2 + \bar{m}^2}, \tag{1.82}$$
$$E_\pm = \pm\mu_I + \sqrt{|\vec{p}|^2 + \bar{m}^2}. \tag{1.83}$$

The (tree-level) masses of these particles are found by setting $\vec{p} = 0$ and are

$$m_0^2 = m_3^2, \tag{1.84}$$
$$m_\pm^2 = \frac{1}{2}\left[m_1^2 + m_2^2 + m_{12}^2\right] \pm \frac{1}{2}\sqrt{\left(m_1^2 + m_2^2 + m_{12}^2\right)^2 - 4m_1^2 m_2^2}. \tag{1.85}$$

Using the result for α, Figure 1.1 shows the masses as functions of μ_I. We observe that the mass of the π_--particle goes to zero at $\mu_I = m_\pi$. This is indicative of spontaneous symmetry breaking, which we will investigate in the next chapter.

With the energies of the pions, we can write the determinant of the inverse propagator as

$$\det(D^{-1}) = (p_0^2 - E_0^2)(p_0^2 - E_+^2)(p_0^2 - E_-^2). \tag{1.86}$$

The propagator and the inverse propagator in momentum space obey[4]

$$\sum_c D_{ac}(p)D^{-1}_{cb}(p) = i\delta_{ab} \tag{1.87}$$

[3] An unfortunate notational convention is that E_+ is the energy of π^--particle, and E_+ for the π^+-particle. This is because the positivley charged pion, π^+, has isospin $I_3 = +1$, so that the mass will decrease as μ_I increseas, and hence the negative sign.

[4] One has to be careful regarding the factor i in the physicist's definition of propagators. It has the consequence that D^{-1} is not strictly the operator inverse of the propagator D.

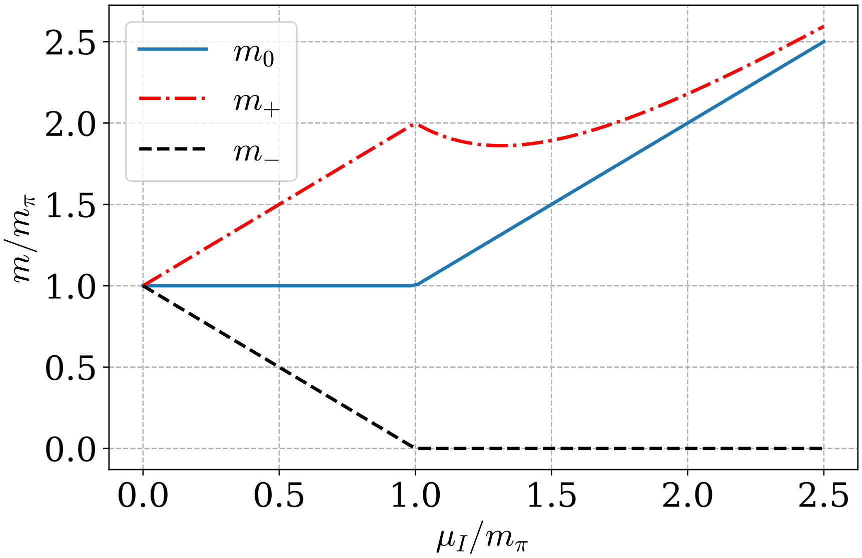

Figure 1.1: The masses of the three particles as functions of isosopin chemical potential. Rresults are given in units of the pion mass, m_π.

Using this, we can solve for the propagator

$$D = (-iD^{-1})^{-1} = i \begin{pmatrix} \frac{p^2-m_2^2}{(p_0^2-E_+^2)(p_0^2-E_-^2)} & \frac{-ip_0m_{12}}{(p_0^2-E_+^2)(p_0^2-E_-^2)} & 0 \\ \frac{ip_0m_{12}}{(p_0^2-E_+^2)(p_0^2-E_-^2)} & \frac{p^2-m_1^2}{(p_0^2-E_+^2)(p_0^2-E_-^2)} & 0 \\ 0 & 0 & \frac{1}{p_0^2-E_0^2} \end{pmatrix}. \tag{1.88}$$

1.4 Electromagnetic effects

When including contribution from a dynamical photon field, the leading order Lagrangian is [13, 14]

$$\mathcal{L}_2^{\text{EM}} = \frac{1}{4}f^2\text{Tr}\left\{\nabla_\mu\Sigma\nabla^\mu\Sigma^\dagger\right\} + \frac{1}{4}f^2\text{Tr}\left\{\chi\Sigma^\dagger + \Sigma\chi^\dagger\right\} + e^2C\text{Tr}\left\{Q\Sigma Q\Sigma^\dagger\right\} \tag{1.89}$$

Q is the quark charge matrix, which for $N_f = 2$ is

$$Q = \frac{1}{3}\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{2}\mathbb{1} + \frac{1}{6}\tau_3. \tag{1.90}$$

Can we include trace?

C and dimensionfull constant, and $\chi = 2B_0 m$, where m is the quark mass matrix Eq. (1.39). To find the electromagnetic effect on the pion mass, we assume $\mu_I = 0$. We use the parametrization $\Sigma = \exp\{i\pi_a\tau_a/f\}$, and the covariant derivative is in this case

$$\nabla_\mu\Sigma = \partial_\mu\Sigma - ie\mathcal{A}_\mu[Q,\Sigma]. \tag{1.91}$$

We expand to second order in π_a/f, which gives

$$\frac{1}{4}f^2\text{Tr}\left\{\nabla_\mu\Sigma\nabla^\mu\Sigma\right\} = \frac{1}{2}\partial_\mu\pi_a\partial^\mu\pi_a + e\mathcal{A}^\mu(\pi_1\partial_\mu\pi_2 - \pi_2\partial_\mu\pi_1) + e^2\mathcal{A}^2(\pi_1^2+\pi_2^2), \tag{1.92}$$

$$\frac{1}{4}f^2\text{Tr}\left\{\chi\Sigma^\dagger + \Sigma\chi^\dagger\right\} = \bar{m}^2\left(f^2 - \frac{1}{2}\pi_a\pi_a\right), \tag{1.93}$$

$$\text{Tr}\left\{Q\Sigma Q\Sigma^\dagger\right\} = \frac{5}{9} - \frac{\pi_1^2+\pi_2^2}{f^2}. \tag{1.94}$$

Inserting this into Eq. (1.89), we get

$$\mathcal{L}_2^{\mathrm{EM}} = \bar{m}^2 f^2 + \frac{5}{9}e^2 C + \frac{1}{2}\partial_\mu \pi_a \partial^\mu \pi_a - \frac{1}{2}\bar{m}_\pm^2(\pi_1^2+\pi_2^2) - \frac{1}{2}\bar{m}^2\pi_3^2 + e\mathcal{A}^\mu(\pi_1\partial_\mu\pi_2 - \pi_2\partial_\mu\pi_1) + e^2\mathcal{A}^2(\pi_1^2+\pi_2^2). \tag{1.95}$$

where

$$\bar{m}_\pm^2 = \bar{m}^2 + 2\frac{e^2}{f^2}C. \tag{1.96}$$

This is the leading order electromagnetic contribution to the mass. It only affects the π_1, π_2 pions, as they are linear combinations of the charged pions $\pi^\pm$, while $\pi_3 = \pi^0$, the neutral pion. To leading order, $\bar{m} = m_\pi$, the neutral pion mass, and $\bar{m}_\pm = m_{\pi^\pm}$ From the values listed in **??**, we find

$$\Delta m_\pm := \frac{e}{f}\sqrt{2C} = \sqrt{m_{\pi^\pm}^2 - m_\pi^2} = 35.50\,\mathrm{MeV}. \tag{1.97}$$

This corresponds to $C = 0.3771\,u_0 = 5.824 \cdot 10^{-5}\,\mathrm{GeV}^4$. We now no longer assume $\mu_I = 0$. The zeroth-order expansion in π/f is

Compare/use Urec's results: C = 61.1 × 10 −6 (GeV)^4

$$\Sigma = e^{i\alpha\tau_1} = \sin\alpha + i\tau_1\cos\alpha. \tag{1.98}$$

This gives the contributions

$$\mathrm{Tr}\left\{\nabla_\mu\Sigma\nabla^\mu\Sigma^\dagger\right\} = 2\sin^2\alpha\left(\mu_I^2 + 2e\mu\mathcal{A}_0 + e^2\mathcal{A}^2\right), \tag{1.99}$$

$$\mathrm{Tr}\left\{\chi\Sigma^\dagger + \Sigma\chi^\dagger\right\} = 4\bar{m}^2\cos\alpha, \tag{1.100}$$

$$\mathrm{Tr}\left\{Q\Sigma Q\Sigma^\dagger\right\} = \cos^2\alpha - \frac{4}{9}. \tag{1.101}$$

We are interested in the static Lagrangian, the Lagrangian for $\pi_a = \mathcal{A}_\mu = 0$. Inserting these terms into Eq. (1.89), we get

$$\mathcal{L}_2^{\mathrm{EM},0} = f^2\left[\frac{1}{2}\mu_I^2\sin^2\alpha + \bar{m}^2\cos\alpha + \frac{1}{2}\Delta m_{\pi^\pm}^2\left(\cos^2\alpha - \frac{4}{9}\right)\right]. \tag{1.102}$$

1.5 *Next-to-leading order Lagrangian

Constructing the next-to-leading order (NLO) Lagrangian is now a business of combining the building blocks we found in section 1.2. We must include all terms that obey all symmetries and that are fourth-order in Weinberg's power counting scheme and remove possible redundant terms, as discussed in section A.3. We will quote the result from [2],

$$\begin{aligned}
\mathcal{L}_4 = {} & \frac{l_1}{4}\mathrm{Tr}\left\{\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\right\}^2 + \frac{l_2}{4}\mathrm{Tr}\left\{\nabla_\mu\Sigma(\nabla_\nu\Sigma)^\dagger\right\}\mathrm{Tr}\left\{\nabla^\mu\Sigma(\nabla^\nu\Sigma)^\dagger\right\} + \frac{l_3+l_4}{16}\mathrm{Tr}\left\{\chi\Sigma^\dagger + \Sigma\chi^\dagger\right\}^2 \\
& + \frac{l_4}{8}\mathrm{Tr}\left\{\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\right\}\mathrm{Tr}\left\{\chi\Sigma^\dagger + \Sigma\chi^\dagger\right\} - \frac{l_7}{16}\mathrm{Tr}\left\{\chi\Sigma^\dagger - \Sigma\chi^\dagger\right\}^2 + \frac{h_1+h_3-l_4}{4}\mathrm{Tr}\left\{\chi\chi^\dagger\right\} \\
& + \frac{h_1-h_3-l_4}{16}\left[\mathrm{Tr}\left\{\chi\Sigma^\dagger + \Sigma\chi^\dagger\right\}^2 + \mathrm{Tr}\left\{\chi\Sigma^\dagger - \Sigma\chi^\dagger\right\}^2 - 2\mathrm{Tr}\left\{\left(\chi\Sigma^\dagger\right)^2 + \left(\Sigma\chi^\dagger\right)^2\right\}\right].
\end{aligned} \tag{1.103}$$

We have ignored terms containing the field strength tensors for external fields, as they vanish in our case. The parameters l_i and h_i are called low energy constants, or LEO. In section A.3, we show how to rewrite the Lagrangian to match the one used in [10, 15]. To obtain $\mathcal{L}_4$ to $\mathcal{O}\big((\pi/f)^3\big)$, we use the result from Eq. (1.62) and Eq. (1.63), which gives

$$\begin{aligned}
\mathrm{Tr}\left\{\partial_\mu\Sigma\partial_\nu\Sigma^\dagger\right\} &= 2\frac{\partial_\mu\pi_a\partial_\nu\pi_a}{f^2} \\
-i\mathrm{Tr}\left\{\partial_\mu\Sigma[v_\nu,\Sigma^\dagger] - \mathrm{h.c.}\right\} &= \frac{2\mu_I\pi_2}{f}(\delta_\mu^0\partial_\nu + \delta_\nu^0\partial_\mu)\sin\alpha + \frac{2\mu_I}{f^2}[\pi_1(\delta_\mu^0\partial_\nu + \delta_\nu^0\partial_\mu)\pi_2 - \pi_2(\delta_\mu^0\partial_\nu + \delta_\nu^0\partial_\mu)\pi_1]\cos\alpha \\
-\mathrm{Tr}\left\{[v_\nu,\Sigma][v_\nu,\Sigma^\dagger]\right\} &= 2{\mu_I}^2\delta_\mu^0\delta_\nu^0\left[\sin^2\alpha + \frac{\pi_1}{f}\sin 2\alpha + \frac{\pi_a\pi_b}{f^2}k_{ab}\right],
\end{aligned}$$

up to $\mathcal{O}\big((\pi/f)^3\big)$. Inserting $\chi = 2B_0 m = \bar{m}^2\mathbb{1} + \Delta m^2\tau_3$ gives

$$\begin{aligned}
\chi\Sigma^\dagger + \Sigma\chi^\dagger &= 2(\bar{m}^2 + \Delta m^2\tau_3)\left[\left(1 - \frac{\pi_a^2}{2f^2}\right)\cos\alpha - \frac{\pi_1}{f}\sin\alpha\right] \\
&\quad + 2\Delta m^2\left[\left(1 - \frac{\pi_a^2}{2f^2}\right)\tau_2\sin\alpha + \frac{\pi_a}{f}\left(\delta_{a1}\tau_2\cos\alpha - \delta_{a2}\tau_1\right)\right], \\
\chi\Sigma^\dagger - \Sigma\chi^\dagger &= -2i\bar{m}^2\left[\left(1 - \frac{\pi_a^2}{2f^2}\right)\tau_1\sin\alpha + \frac{\pi_a}{f}\left(\tau_a - 2\delta_{1a}\tau_1\sin^2\frac{\alpha}{2}\right)\right] - 2i\Delta m^2\frac{\pi_3}{f}.
\end{aligned}$$

Combining these results gives all the terms in $\mathcal{L}_4$, to $\mathcal{O}\big((\pi/f)^3\big)$:

$$\begin{aligned}
&\mathrm{Tr}\left\{\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\right\}^2 = \mathrm{Tr}\left\{\partial_\mu\Sigma\partial^\mu\Sigma^\dagger - i\left(\partial_\mu\Sigma[v^\mu,\Sigma^\dagger] - \mathrm{h.c.}\right) - [v_\mu,\Sigma][v^\mu,\Sigma^\dagger]\right\}^2 \\
&\quad = \frac{8\mu_I^2}{f^2}(\partial_\mu\pi_a\partial^\mu\pi_a + 2\partial_\mu\pi_2\partial^\mu\pi_2)\sin^2\alpha \\
&\quad + 16\mu_I^3\left[\frac{\partial_0\pi_2}{f}\sin^3\alpha + \frac{1}{f^2}\left(3\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1\right)\cos\alpha\sin^2\alpha\right] \\
&\quad + 4\mu_I^4\left\{\sin^4\alpha + 2\sin^2\alpha\left[\frac{\pi_1}{f}\sin 2\alpha + \frac{\pi_a\pi_b}{f^2}\left(k_{ab} + 2\delta_{a1}\delta_{a2}\cos^2\alpha\right)\right]\right\}, \qquad (1.104) \\
&\mathrm{Tr}\left\{\nabla_\mu\Sigma(\nabla_\nu\Sigma)^\dagger\right\}\mathrm{Tr}\left\{\nabla^\mu\Sigma(\nabla^\nu\Sigma)^\dagger\right\} \\
&\quad = \frac{4{\mu_I}^2}{f^2}\left(\partial_0\pi_a\partial_0\pi_a + \partial_0\pi_2\partial_0\pi_2 + \partial_\mu\pi_2\partial^\mu\pi_2\right)\sin^2\alpha \\
&\quad + 16\mu_I^3\left[\frac{\partial_0\pi_2}{f}\ \sin^3\alpha + \frac{1}{f^2}\left(3\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1\right)\cos\alpha\sin^2\alpha\right] \\
&\quad + 4\mu_I^4\left\{\sin^4\alpha + 2\sin^2\alpha\left[\frac{\pi_1}{f}\sin 2\alpha + \frac{\pi_a\pi_b}{f^2}\left(k_{ab} + 2\delta_{a1}\delta_{a2}\cos^2\alpha\right)\right]\right\}, \qquad (1.105) \\
&\mathrm{Tr}\left\{\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\right\}\mathrm{Tr}\left\{\chi\Sigma^\dagger + \Sigma\chi^\dagger\right\} \\
&\quad = 4\bar{m}^2\left\{2\frac{\partial_\mu\pi_a\partial^\mu\pi_a}{f^2}\cos\alpha + 4\mu_I\left[\frac{\partial_0\pi_2}{2f}\sin 2\alpha + \frac{1}{f^2}\left(\pi_1\partial_0\pi_2\cos 2\alpha - \pi_2\partial_0\pi_1\cos^2\alpha\right)\right]\right. \qquad (1.106) \\
&\quad \left. + {\mu_I}^2\left[2\cos\alpha\sin^2\alpha - 2\frac{\pi_1}{f}\sin\alpha\left(2 - 3\sin^2\alpha\right) + \frac{1}{f^2}\left(\pi_1^2[2 - 9\sin^2\alpha] + \pi_2^2[2 - 3\sin^2\alpha] - 3\pi_3^2\sin^2\alpha\right)\cos\alpha\right]\right\}, \\
&\mathrm{Tr}\left\{\chi\Sigma^\dagger + \Sigma\chi^\dagger\right\}^2 = 16\bar{m}^4\left[\cos^2\alpha - \frac{\pi_1}{f}\sin 2\alpha + \frac{1}{f^2}\left(\pi_1^2\sin^2\alpha - \pi_a\pi_a\cos^2\alpha\right)\right], \qquad (1.107) \\
&\mathrm{Tr}\left\{\chi\Sigma^\dagger - \Sigma\chi^\dagger\right\}^2 = -16\left(\frac{\Delta m^2\pi_3}{f}\right)^2, \qquad (1.108) \\
&\mathrm{Tr}\left\{\left(\chi\Sigma^\dagger\right)^2 + \left(\Sigma\chi^\dagger\right)^2\right\} \\
&\qquad = 4\bar{m}^4\left(\cos 2\alpha - 2\frac{\pi_1}{f}\sin 2\alpha - 2\frac{\pi_a\pi_a}{f^2}\cos^2\alpha + 2\frac{\pi_1^2}{f^2}\sin^2\alpha\right) + 4\Delta m^4\left(1 - 2\frac{\pi_3^2}{f^2}\right), \qquad (1.109) \\
&\mathrm{Tr}\left\{\chi^\dagger\chi\right\} = 2\bar{m}^4 + 2\Delta m^4. \qquad (1.110)
\end{aligned}$$

The different terms of the NLO Lagrangian is

$$\begin{aligned}
\mathcal{L}_4^{(0)} &= (l_1 + l_2)\mu_I^4\sin^4\alpha + (l_3 + l_4)\bar{m}^2\cos^2\alpha + l_4\bar{m}{\mu_I}^2\cos\alpha\sin^2\alpha + (h_1 - l_4)\bar{m}^4 + h_3\Delta m^4, \qquad (1.111) \\
\mathcal{L}_4^{(1)} &= 4\mu_I^3\frac{l_1 + l_2}{f}\left(\partial_0\pi_2 + \mu_I\cos\alpha\pi_1\right)\sin^3\alpha - \frac{l_3 + l_4}{f}\bar{m}^4\pi_1\sin 2\alpha \\
&\quad + \bar{m}^2\frac{l_4}{f}\left[\mu_I\partial_0\pi_2\sin 2\alpha - {\mu_I}^2\pi_1\sin\alpha\left(3\sin^2\alpha - 2\right)\right], \qquad (1.112) \\
\mathcal{L}_4^{(2)} &= 2\mu_I^2\frac{l_1}{f^2}(\partial_\mu\pi_a\partial^\mu\pi_a + 2\partial_0\pi_2\partial_0\pi_2)\sin^2\alpha + {\mu_I}^2\frac{l_2}{f^2}\left(\partial_\mu\pi_2\partial^\mu\pi_2 + 2\partial_0\pi_a\partial_0\pi_a + 2\partial_0\pi_2\partial_0\pi_2\right)\sin^2\alpha
\end{aligned}$$

$$
\begin{aligned}
&+2\frac{l_1+l_2}{f^2}\left[2\mu_I^3\left(3\pi_1\partial_0\pi_2-\pi_2\partial_0\pi_1\right)\cos\alpha+\mu_I^4\pi_a\pi_b\left(k_{ab}+2\delta_{a1}\delta_{a2}\cos^2\alpha\right)\right]\sin^2\alpha\\
&+\frac{l_3+l_4}{f^2}\bar{m}^2\left(\pi_1^2\sin^2\alpha-\pi_a\pi_a\cos^2\alpha\right)+\frac{l_4}{f^2}\bar{m}^2\Big[\partial_\mu\pi_a\partial^\mu\pi_a\cos\alpha+4\mu_I\left(\pi_1\partial_0\pi_2\cos2\alpha-\pi_2\partial_0\pi_1\cos^2\alpha\right)\\
&+\frac{1}{2}{\mu_I}^2\left(\pi_1^2[2-9\sin^2\alpha]+\pi_2^2[2-3\sin^2\alpha]-3\pi_3^2\sin^2\alpha\right)\cos\alpha\Big]+\frac{l_7}{f^2}\Delta m^2\pi_3^2. \qquad (1.113)
\end{aligned}
$$

1.6 Three-flavor χPT to leading order

1.6.1 Ground state

For $N_f=3$, the generators are $T_\alpha=\frac{1}{2}\lambda_\alpha$, where λ_α are the Gell-Mann matrices, as shown in section A.1. The mass matrix is now

$$
m=\begin{pmatrix} m_u & 0 & 0\\ 0 & m_d & 0\\ 0 & 0 & m_s\end{pmatrix}, \qquad (1.114)
$$

and the charge matrix is

$$
Q=\frac{1}{3}\begin{pmatrix} 2 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}=\frac{1}{2}\left(\lambda_3+\frac{1}{\sqrt{3}}\lambda_8\right). \qquad (1.115)
$$

The leading order Lagrangian still has the same form,

$$
\mathcal{L}_2=\frac{1}{4}f^2\mathrm{Tr}\left\{\nabla_\mu\Sigma\nabla^\mu\Sigma^\dagger\right\}+\frac{1}{4}f^2\mathrm{Tr}\left\{\chi\Sigma^\dagger+\Sigma\chi^\dagger\right\}+e^2C\mathrm{Tr}\left\{\Sigma Q\Sigma^\dagger Q\right\}, \qquad (1.116)
$$

where

$$
\chi=2B_0m=\frac{2}{3}M_1^2\mathbb{1}+M_2^2\lambda_3+M_3^2\lambda_8, \qquad (1.117)
$$

and

$$
M_1^2=B_0(m_u+m_d+m_s),\quad M_2^2=B_0(m_u-m_d),\quad M_3^2=\frac{1}{\sqrt{3}}B_0(m_u+m_d-2m_S). \qquad (1.118)
$$

To find the ground state, we start with $e=0$. The covariant derivative is then

$$
\nabla_\mu\Sigma=\partial_\mu\Sigma-i[v_\mu,\Sigma],\quad v_\mu=\mu\delta_\mu^0, \qquad (1.119)
$$

Here, μ is the chemical potential matrix,

$$
\mu=\begin{pmatrix} \mu_u & 0 & 0\\ 0 & \mu_d & 0\\ 0 & 0 & \mu_s\end{pmatrix}=\begin{pmatrix} \frac{1}{3}\mu_B+\frac{1}{2}\mu_I & 0 & 0\\ 0 & \frac{1}{3}\mu_B-\frac{1}{2}\mu_I & 0\\ 0 & 0 & \frac{1}{3}\mu_B-\frac{1}{2}\mu_S\end{pmatrix}=\frac{1}{3}(\mu_B-\mu_S)\mathbb{1}+\frac{1}{2}\mu_I\lambda_3+\frac{1}{\sqrt{3}}\mu_S\lambda_8, \qquad (1.120)
$$

where $\mu_B=\frac{3}{2}(\mu_u+\mu_d)$, $\mu_I=\mu_u-\mu_d$ and $\mu_S=\frac{1}{2}(\mu_u+\mu_d)-\mu_s$. Here, μ_u, μ_d, and μ_s are the up, down, and strange quark chemical potentials, while μ_B, μ_I, and μ_S are the baryon, isospin, and strangeness chemical potentials. The baryon number of all mesons, the π_a's, is zero, and Σ transforms as $\Sigma\to\Sigma$ under $U(1)_V$, the corresponding symmetry group. As a consequence, any result will be independent of μ_B. We can also see this from the fact that μ_B only appears with the identity matrix $\mathbb{1}$ in μ. As a consequence, any dependence on μ_B in $\nabla_\mu\Sigma$ will vanish as $\mathbb{1}$ commutes with everything. We will assume $\mu_I,\mu_S>0$.

1.6.2 Parametrization

From the structure constants of $\mathfrak{su}(3)$, Eq. (A.14), we see that we can create three independent $\mathfrak{su}(2)$ sub-algebras. We introduce the matrices

$$
\lambda_Q=\lambda_3+\frac{1}{\sqrt{3}}\lambda_8,\quad \lambda_K=\lambda_3-\frac{1}{\sqrt{3}}\lambda_8, \qquad (1.121)
$$

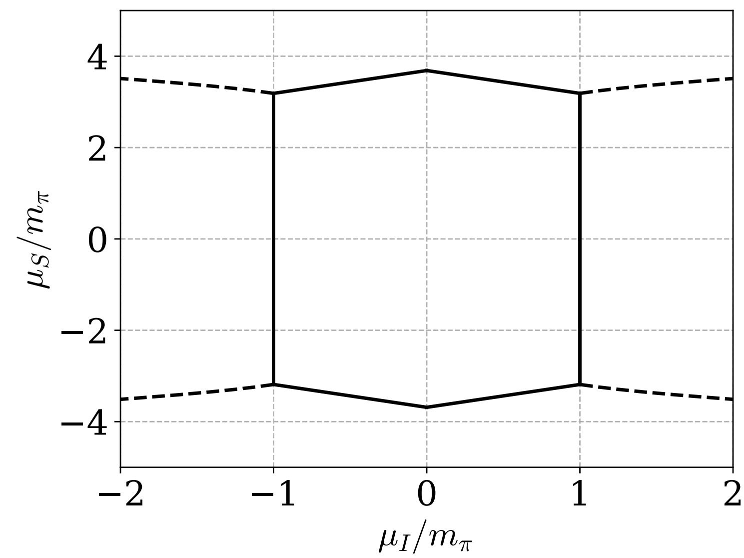

Figure 1.2: Kladd

which commute, i.e., $[\lambda_Q, \lambda_K] = 0$. With this, we get the commutation relations

$$[\lambda_i, \lambda_j] = 2i\epsilon_{ijk}\lambda_k, \quad ijk \in \begin{cases} \{1,2,3\} \\ \{4,5,Q\} \\ \{6,7,K\}. \end{cases} \tag{1.122}$$

We here define the Levi-Civita symbol by $\epsilon_{123} = \epsilon_{34Q} = \epsilon_{76K} = 1$. To find the ground state, we define

$$\Sigma_\alpha = \exp\{i\alpha n_a \lambda_a\} = \cos\alpha + i n_a \lambda_a \sin\alpha, \quad \alpha = \frac{1}{f}\sqrt{\pi_a^0 \pi_a^0}, \quad n_a = \frac{\pi_a^0}{\sqrt{\pi_b^0 \pi_b^0}}. \tag{1.123}$$

For $\mu_S = 0$, we expect to recover the results from the two-flavor case, which corresponds to $n_1^2 + n_2^2 = 1$, $n_a = 0$ for $i > 2$. As argued earlier, we may choose $n_1 = 0$ without loss of generality, in which case the ground state becomes

$$\Sigma_\alpha^{\pi^\pm} = \exp\{i\alpha\lambda_2\} = (\mathbb{1} - \lambda_2^2) + \lambda_2^2 \cos\alpha + i\lambda_2 \sin\alpha. \tag{1.124}$$

If we define $\mu_{K^\pm} = \frac{1}{2}(\frac{1}{2}\mu_I + \mu_S)$ and $\mu_{K^0} = \frac{1}{2}(\frac{1}{2}\mu_I - \mu_S)$, then we can write the external currents corresponding to μ_I and μ_S as

$$\mu = \frac{1}{2}\mu_I Q_I + \mu_S Q_8 = \mu_{K^\pm} Q_{K^\pm} + \mu_{K^0} Q_{K^0}. \tag{1.125}$$

We assume $\mu_I, \mu_S > 0$, so $\mu_{K^\pm} > \mu_{K^0}$. Analogously to how turning up μ_I leads to a condensate in the first $\mathfrak{su}(2)$ subalgebra, we can expect these chemical potentials to lead to a condensation in their respective subalgebra. As we work with $\mu_I, \mu_S > 0$, we can assume μ_{K^0}, in which case we would expect the form

$$\Sigma_\alpha^{K^\pm} = \exp\{i\alpha\lambda_5\} = (\mathbb{1} - \lambda_5^2) + \lambda_5^2 \cos\alpha + i\lambda_5 \sin\alpha. \tag{1.126}$$

Similarly, for $\mu_S < 0$, we could set $\mu_{K^\pm} = 0$ and expect a ground state of the form $e^{i\alpha\lambda_7}$. In [16], Kogut and Toublan show that exactly this happens. At $\mu_{K^\pm}^2 = m_K^2 = \frac{1}{2}B(m_u + m_d + 2m_s)$, we get a charged pion condensate, and a neutral kaon condensate at $\mu_{K^0}^2 > m_K^2$. However, these domains are overlapping, so there is a first-order phase transition between the different condensates. Figure **??** show the phase diagram in the μ_I, μ_S-plane.

1.6.3 Leading order Lagrangian

Excitations in these ground states are parametrized as

$$\Sigma(x) = A_\alpha^i U(x) \Sigma_0 U(x) A_\alpha^i, \quad U(x) = \exp\left\{i\frac{\pi_a \lambda_a}{2f}\right\}, \quad A_\alpha^i = \exp\left\{i\frac{\alpha\lambda_i}{2}\right\}, \tag{1.127}$$

where $i = 2, 5, 7$ depending on which ground state we are in. We start working in the pion condensate phase, so $i = 2$, and assume $\mu_I > 0$ and $e = 0$. Inserting this into Eq. (1.116), and expending up to and including $\mathcal{O}\left((\pi/f)^2\right)$, we get

$$\mathcal{L}_2^{(0)} = \frac{1}{2}f^2\left(\mu_I^2\sin^2\alpha + 2\bar{m}\cos\alpha + m_S\right), \tag{1.128}$$

$$\mathcal{L}_2^{(1)} = -f\mu_I\partial_0\pi_1\sin\alpha + f\sin\alpha\left(\mu_I^2\cos\alpha - \bar{m}^2\right)\pi_2, \tag{1.129}$$

$$\mathcal{L}_2^{(2)} = \frac{1}{2}\partial_\mu\pi_a\partial^\mu\pi_a + \frac{1}{2}m_{ab}\pi_a\partial_0\pi_b - \frac{1}{2}m_a^2\pi_a^2 - \frac{1}{\sqrt{3}}\Delta m^2\pi_3\pi_8, \tag{1.130}$$

where

$$m_1^2 = \bar{m}^2\cos\alpha - \mu_I^2\cos^2\alpha, \tag{1.131}$$

$$m_2^2 = \bar{m}^2\cos\alpha - \mu_I^2\cos 2\alpha, \tag{1.132}$$

$$m_3^2 = \bar{m}^2\cos\alpha + \mu_I^2\sin^2\alpha, \tag{1.133}$$

$$m_4^2 = m_5^2 = m_+^2 - m_{\mu+}^2, \tag{1.134}$$

$$m_6^2 = m_7^2 = m_-^2 - m_{\mu-}^2, \tag{1.135}$$

$$m_8^2 = \frac{1}{3}(\bar{m}\cos\alpha + 2m_S^2), \tag{1.136}$$

$$m_{12} = 2\mu_I\cos\alpha, \tag{1.137}$$

$$m_{45} = \mu_I\cos\alpha + \mu_S, \tag{1.138}$$

$$m_{76} = \mu_I\cos\alpha - \mu_S, \tag{1.139}$$

and

$$\bar{m}^2 = B_0(m_u + m_d), \quad \Delta m^2 = B_0(m_u - m_d), \quad m_S^2 = 2B_0m_s, \tag{1.140}$$

$$m_\pm^2 = \frac{1}{2}(\bar{m}^2\cos\alpha \pm \Delta m^2 + m_S^2), \quad m_{\mu\pm}^2 = \frac{1}{4}\mu_I^2\cos 2\alpha \pm \mu_I\mu_S\cos\alpha + \mu_S^2. \tag{1.141}$$

Here, $m_{ab} = -m_{ba}$, and terms not defined above are zero. At $\mu_S = \mu_I = 0$ and $\alpha = 0$, the off-diagonal terms m_{ab} vanish, and m_a^2 thus corresponds to the leading order masses,

$$m_\pi^2 = m_1^2 = m_2^2 = m_3^2 = \bar{m}^2 = B_0(m_u + m_d), \tag{1.142}$$

$$m_{K^\pm}^2 = m_4^2 = m_5^2 = \frac{1}{2}(\bar{m}^2 + \Delta m^2 + m_S^2) = B_0(m_u + m_s), \tag{1.143}$$

$$m_{K^0}^2 = m_6^2 = m_7^2 = \frac{1}{2}(\bar{m}^2 - \Delta m^2 + m_S^2) = B_0(m_d + m_s), \tag{1.144}$$

$$m_\eta^2 = m_8^2 = \frac{1}{3}(\bar{m}^2 + 2m_S^2) = \frac{1}{3}B_0(m_u + m_d + 4m_s). \tag{1.145}$$

In the $K^\pm$-condensate, we get

$$\mathcal{L}_2^{(0)} = \frac{1}{2}f^2\left(\mu_{K^\pm}^2\sin^2\alpha + 2m_{K^\pm}^2\cos\alpha + \bar{m}^2 - \Delta m^2\right), \tag{1.146}$$

$$\mathcal{L}_2^{(1)} = -\frac{1}{2}f\mu_{K^\pm}\partial_0\pi_4\,\sin\alpha + f\left(\mu_{K^\pm}^2\cos\alpha - m_{K^\pm}^2\right)\pi_5\sin\alpha \tag{1.147}$$

$$\mathcal{L}_2^{(1)} = \frac{1}{2}\partial_\mu\pi_a\partial^\mu\pi_a + \frac{1}{2}m'_{ab}\pi_a\partial_0\pi_a - \frac{1}{2}m_a'^2\pi_a^2 - \frac{1}{2}\Delta_{\pi\eta}\pi_3\pi_8 \tag{1.148}$$

where

$$m'^2_1 = m_2^2 = m'^2_- + m'^2_{\mu -} \tag{1.149}$$

$$m'^2_3 = \frac{1}{4}\left(\mu^2_{K^\pm}\sin^2\alpha + \bar{m}^2(\cos\alpha+3) + \Delta m^2(\cos\alpha-1) + m_S^2(\cos\alpha-1)\right) \tag{1.150}$$

$$m'^2_4 = m^2_{K^\pm}\cos\alpha - \mu_{K^\pm}\cos^2\alpha \tag{1.151}$$

$$m'^2_5 = m^2_{K^\pm}\cos\alpha - \mu_{K^\pm}\cos 2\alpha \tag{1.152}$$

$$m'^2_6 = m_7^2 = m'^2_+ + m'^2_{\mu +} \tag{1.153}$$

$$m'^2_8 = \frac{1}{12}\left[\frac{9}{4}\mu^2_{K^\pm}\sin^2\alpha + \bar{m}^2(5\cos\alpha-1) + 5\Delta m^2(\cos\alpha-1) + m_S^2(5\cos\alpha+3))\right] \tag{1.154}$$

$$\Delta_{\eta\pi} = \frac{\sqrt{3}}{2}\left[\mu^2_{K^\pm}\sin^2\alpha + \frac{1}{3}\bar{m}^2(\cos\alpha-1) + \frac{1}{3}\Delta m^2(\cos\alpha+3) + \frac{1}{3}m_S^2(\cos\alpha-1)\right] \tag{1.155}$$

$$m'^2_\pm = \frac{1}{4}\bar{m}^2(\cos\alpha \mp 1 + 2) + \frac{1}{4}\Delta m^2(\cos\alpha \mp 1 - 2) + \frac{1}{4}m_S^2(\cos\alpha \pm 1), \tag{1.156}$$

$$m'^2_{\mu\pm} = \frac{1}{2}(\sin^2\alpha \pm 3\cos\alpha - 5)\mu_I^2 + (\sin^2\alpha \pm \cos\alpha + 1)\mu_I\mu_s + (\sin^2\alpha \mp \cos\alpha - 1)\mu_s^2 \tag{1.157}$$

$$m'_{12} = \frac{1}{2}(\cos\alpha+3)\mu_I + (\cos\alpha-1)\mu_S \tag{1.158}$$

$$m'_{45} = 2\mu_{K^\pm}\cos\alpha \tag{1.159}$$

$$m'_{76} = \frac{1}{2}(3-\cos\alpha)\mu_I - (1+\cos\alpha)\mu_S. \tag{1.160}$$

Chapter 2

Pion stars

2.1 Leading order, two flavor pion stars

Flytt diskusjon om free energy til Thermodynamics seksjon

2.1.1 Equation of state

The free energy density of two-flavor chiral perturbation theory, to leading-order and at $T = 0$, is

$$\mathcal{F} = -f^2 \left(\bar{m}^2 \cos\alpha + \frac{1}{2}\mu_I^2 \sin^2\alpha \right). \tag{2.1}$$

The parameter α is determined by minimizing $\mathcal{F}$ for a given value of μ_I,

$$\frac{\partial \mathcal{F}}{\partial \alpha} = f^2 \left(\bar{m}^2 - \mu_I^2 \cos\alpha \right) \sin\alpha = 0. \tag{2.2}$$

This gives an explicit formula for α in terms of μ_I. As long as the chemical potential is lower than the critical value $\mu_I^c = \bar{m}$, the only solution to this equation is $\alpha = 0$. As the chemical potential reaches this critical value, the system undergoes a phase transition from the vacuum phase to the *pion condensate* phase. In this new phase, the solution is

$$\cos\alpha = \frac{\bar{m}^2}{\mu_I^2}. \tag{2.3}$$

We introduce a dimensionless variable $x^2 = \cos\alpha = \bar{m}^2/\mu_I^2$. This variable has the domain $[0, 1]$, and $\cos\alpha = x^2$ implies that $\sin^2\alpha = 1 - x^4$. Substituting the dimensionless variable into the free energy density, we get

$$\mathcal{F} = -\frac{u_0}{2} \left(x^2 + \frac{1}{x^2} \right). \tag{2.4}$$

We have introduced the characteristic energy density $u_0 = \bar{m}^2 f^2$. As we found in **??**, the pressure is given by negative the free energy density, normalized to $\mu_I = \bar{m}$, or $x = 1$. We choose $p_0 = u_0$, so the dimensionless pressure is

$$\tilde{p} = -\frac{1}{p_0} \left(\mathcal{F} - \mathcal{F}_{x=1} \right) = \frac{1}{2} \left(x^2 + \frac{1}{x^2} - 2 \right). \tag{2.5}$$

The charge density corresponding to a chemical potential is given by minus the derivative of the free energy with respect to that chemical potential. We must, however, not assume any dependence of α on μ_I when taking this derivative. The isospin density therefore is

$$n_I = -\frac{\partial \mathcal{F}}{\partial \mu_I} = \mu_I^2 \sin^2\alpha = \frac{u_0}{\mu_I} \left(\frac{1}{x^2} - x^2 \right). \tag{2.6}$$

With this, the dimensionless energy density at $T = 0$ is

$$\tilde{u} = -\tilde{p} + \frac{\mu_I n_I}{u_0} = \frac{1}{2}\left(2 + \frac{1}{x^2} - 3x^2\right). \tag{2.7}$$

The ratio of pressure to energy density is [17]

$$\frac{p}{u} = \frac{1 - x^2}{1 + 3x^2}. \tag{2.8}$$

In the ultrarelativistic limit, where $\mu_I \to \infty$ and thus $x \to 0$, we get $p/u = 1$, or $u_{\mathrm{ur}} = p$. The non-relativistic limit is $\mu_I^2 = m_\pi^2(1+\epsilon)$ and thus $x^{-2} = 1 + \epsilon$, $\epsilon \ll 1$. With this we get $\tilde{p} = \epsilon^2/2$, and $\tilde{u} = 2\epsilon$, so the equation of state is $\tilde{u}_{\mathrm{nr}} = \sqrt{8}\sqrt{\tilde{p}}$. The isospin density, and thus the pion number density, is $n_I = 2\frac{u_0}{\bar{m}}\epsilon$, and we can therefore write the energy density in this limit as $u = \bar{m}n_I + \mathcal{O}(\epsilon^2)$. The energy density is thus dominated by the rest mass as $\epsilon \to 0$, as we expect from the non-relativistic limit. Figure 2.1 shows the equation of state in two different regimes and compares it with the ultrarelativistic and non-relativistic limit.

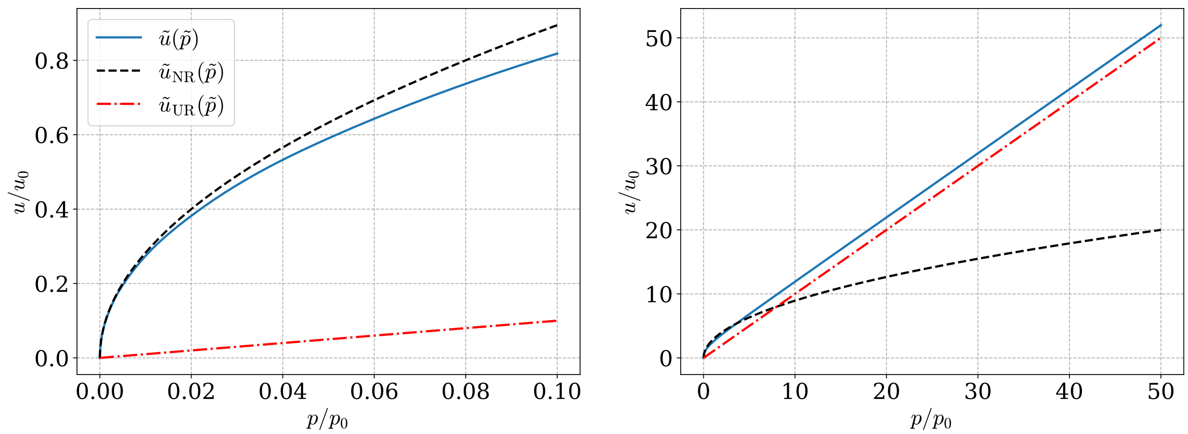

Figure 2.1: The leading order equation of state from two-flavor chiral perturbation theory, in two different regimes. The full equation is shown as a solid line, and is compared to the ultrarelativistic and non-relativistic limit, shown as dashed lines. The y-axis shows the energy density normalized to u_0, x-axis shows the pressure normalized to p_0. We have chosen $p_0 = u_0$.

2.1.2 Units

The characteristic mass and length, as discussed in **??**, are found by setting $k_1 = k_2 = k_3 = 1$. These are the dimensionless constants of the TOV equation, **??**. At leading order, the bare constants f and $\bar{m}$ are related to physical constants by $f = f_\pi$ and $m = m_\pi$, the pion decay constant and the pion mass. Using the values for f_π and m_π as given in **??** and reinstating c and $\hbar$, these quantities are give in by

$$u_0 = m_\pi^2 f_\pi^2 \frac{c}{\hbar^3} = 3.216 \cdot 10^{33}\,\mathrm{J\,m^{-3}}, \tag{2.9}$$

$$m_0 = \frac{c^4}{\sqrt{\frac{4\pi}{3}u_0 G^3}} = 64.21\,M_\odot, \tag{2.10}$$

$$r_0 = \frac{G}{c^2}m_0 = 94.79\,\mathrm{km}. \tag{2.11}$$

We, therefore, expect both the radius and mass of the pion star to be around one order of magnitude larger than the star made up of cold neutrons.

2.1.3 Limiting radius

We found that the non-relativisitc limit of the equation of state is $\tilde{p} = 8^{-1}\tilde{u}^2$, i.e., it is a polytrope with $\gamma = 2$. As discussed in **??**, this corresponds to a situation where the radius of the star is independent of the central pressure, at least in the Newtonian limit of gravity. When simulating the Newtonian, non-relativistic limit of the pion star, we should expect the radius to be constant. From **??**, the radius is $R = C\xi_1$, where

$$C = \frac{1}{\sqrt{4(4\pi)Gu_0}} = \frac{1}{\sqrt{12}} r_0, \tag{2.12}$$

and ξ_1 is the root of the Lane-Emden function $\theta(\xi)$ for polytrope index $n = 1$, the solution to

$$\theta'' + \frac{2}{\xi}\theta' + \theta = 0. \tag{2.13}$$

By substituting θ for its power series expansion, $\theta = \sum_n a_n \xi^n$, we get

$$\sum_n \left[(n+2)(n+1)a_{n+2} + 2(n+1)a_{n+1}\xi^{-1} + a_n\right] \xi^n = 0. \tag{2.14}$$

This must be obeyed for arbitrary ξ. We therefore get the recursion relation $a_{n+2} = -a_n/(n+1)(n+2)$. With our boundary condition, the solution is

$$\theta(\xi) = \frac{\sin(\xi)}{\xi}, \tag{2.15}$$

and the first root is therefore $\xi_1 = \pi$. With this, we get a closed-form expression for the stellar radius of this non-relativistic and Newtonian limit—which we expect the full theory to approach as the central pressure decreases—namely

$$R = \frac{\pi}{\sqrt{12}} r_0 = 85.97\,\mathrm{km}. \tag{2.16}$$

2.1.4 Results

The code used for obtaining numerical results is discussed in **??**.

Figure 2.2 show the pressure and mass as a function of radius for varying values of central pressure. The quantities are normalized to the stellar radius, stellar mass, and central pressure, respectively. The black dashed line corresponds to the configuration with the maximum mass. We see that both the pressure and mass distribution are very similar for stars with a mass less than the maximum. As the central pressure increase beyond that of the star with maximum mass, the pressure gradient close to the center grows sharply. This is similar to what we saw in the case of an incompressible fluid, **??**.

Figure 2.3 shows the mass-radius relation for the pion star. As in the case of the neutron star, it has a maximum mass, in this case of $M_{\max} = 10.47\,M_\odot$. However, in contrast to the case of the neutron star, the stellar radius approaches a maximum radius as the central pressure decreases. This matches our expectation from the non-relativistic, Newtonian limit. We see that the largest radius in our results, corresponding to $p_c = 10^{-6}\,p_0$, is $R = 85.82\,\mathrm{km}$, which is in good agreement with our earlier analysis, Eq. (2.16).

Figure 2.4 compares the mass-radius relation from the full equation of state and TOV equation with various limits. In the non-relativistic, Newtonian limit, the stellar radius is independent of the mass, as we found in our earlier analysis.

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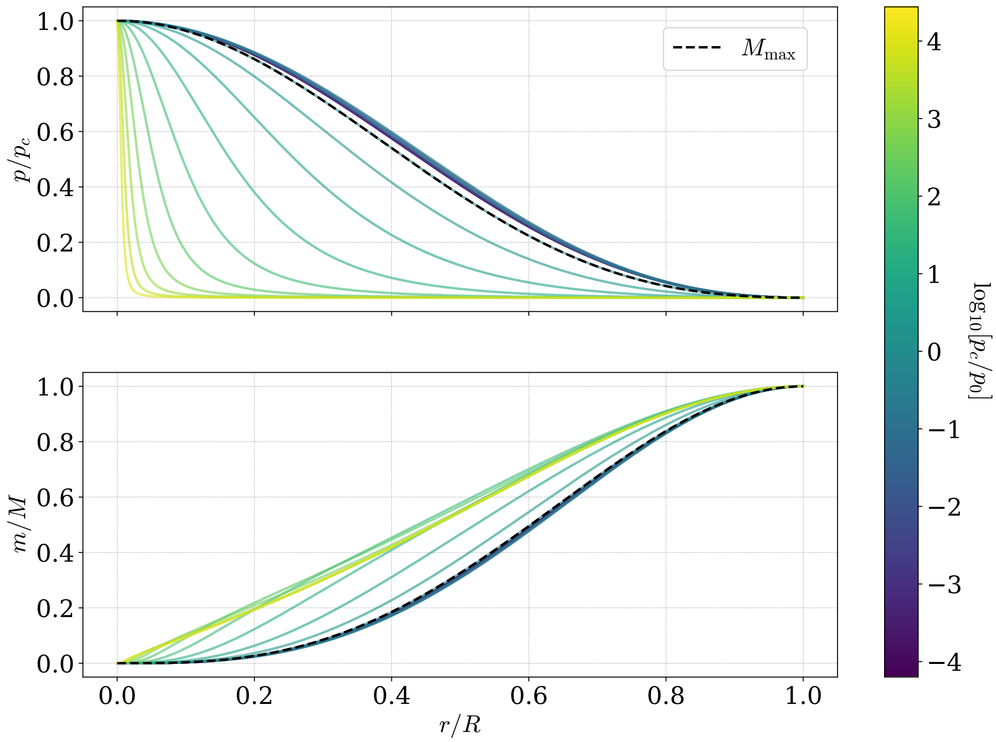

Figure 2.2: Top: The pressure normalized to the central pressure, as a function of radius, normalized to the stellar radius. Bottom: The mass, normalized to the stellar mass, within a radius r, normalized to the stellar radius. Both plots show a range of stars with different central pressures, indicated by the color. The black dashed line corresponds to the star with the largest mass.

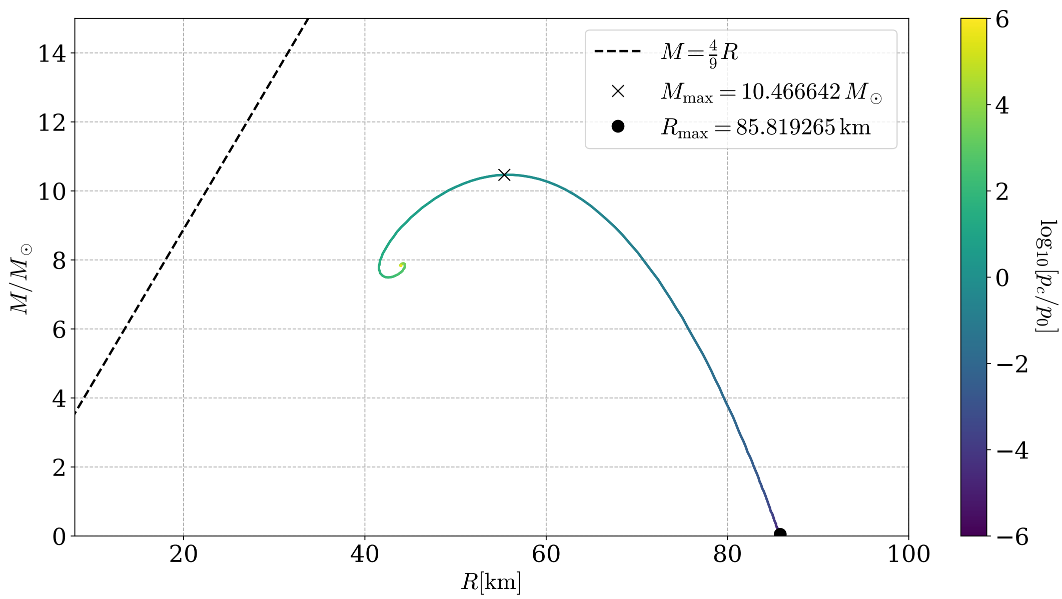

Figure 2.3: The lowest order mass-radius relation a pion star using two-flavor chiral perturbation theory. The mass is given in units of solar masses, while the radius is measured in kilometers. This line is parameterized by the central pressure p_c of the star, as indicated by the color gradient. The dashed black line indicates the theoretical maximum mass for a given radius, and any configuration above it will collapse to a black hole.

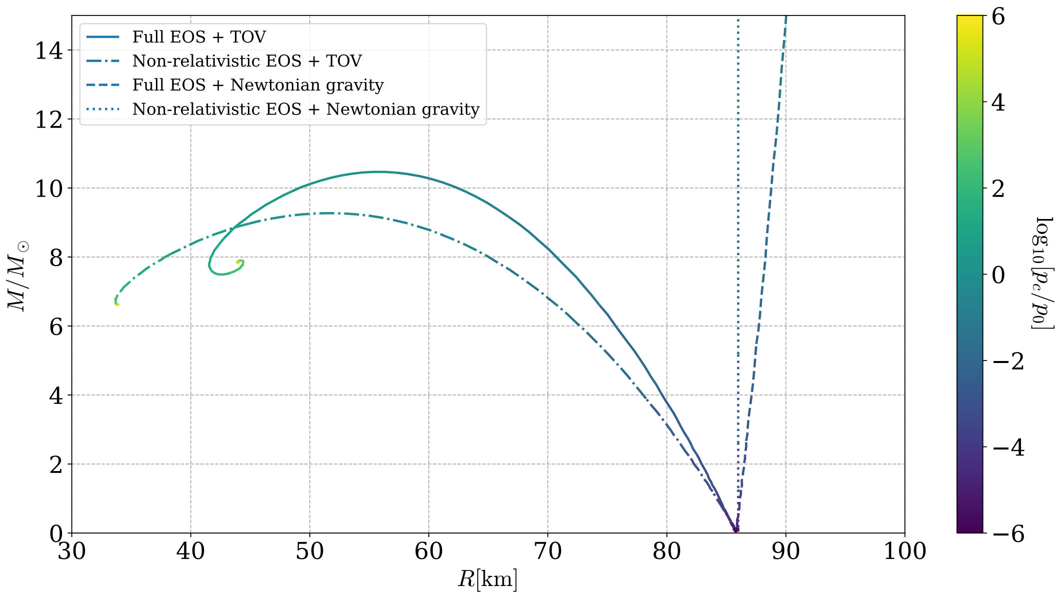

Figure 2.4: The mass-radius relationship of the pion star from the full, leading-order equation of state from two-flavor chiral perturbation and the TOV equation, compared with results in various limits.

2.1.5 Including electromagnetic contributions

From Eq. (1.102), the free energy density, including electromagnetic interactions, is

$$\mathcal{F} = -f^2 \left[\bar{m}^2 \cos\alpha + \frac{1}{2}\mu_I^2 \sin^2\alpha + \frac{1}{2}\Delta m_\pm^2 \left(\cos^2\alpha - \frac{4}{9} \right) \right]. \tag{2.17}$$

Free energy minimization now gives

$$\frac{1}{u_0}\frac{\partial \mathcal{F}}{\partial \alpha} = \left[\left(\frac{1}{x^2} - \Delta \right) \cos\alpha - 1 \right] \sin\alpha = 0. \tag{2.18}$$

Here, x is defined as before, and we introduced the new quantity $\Delta = \Delta m_\pm^2/\bar{m}^2 = 0.06916$. We see that the phase transition is raised, the critical chemical potential is now $\mu_I^c = \bar{m}\sqrt{1+\Delta}$, the mass of the charged pions. Below this value, $\alpha = 0$ remains the only solution. In the pion condensate phase, the solution is

$$\cos\alpha = \frac{x^2}{1 - x^2\Delta}. \tag{2.19}$$

This reduces to our old solution for $\Delta = 0$, as it should. With the same procedure as in the last section, we get the pressure and energy density

$$\tilde{p}_{\text{EM}} = \frac{1}{2}\left[\frac{1}{x^2} + \frac{x^2}{1-x^2\Delta} - 2 - \Delta \right], \tag{2.20}$$

$$\tilde{u}_{\text{EM}} = \frac{1}{2}\left[\frac{1}{x^2} - x^2\frac{3 - x^2\Delta}{(1-x^2\Delta)^2} + 2 + \Delta \right]. \tag{2.21}$$

In the limit $\Delta = 0$, these reduce to Eq. (2.5) and Eq. (2.7). In the ultra-relativistic limit, that is, for $x \ll 1$, the behavior is the same as before, and we again approach $p = u$. We find the non-relativistic limit by substituting $x^{-2} = 1 + \Delta + \epsilon$. To first order in ϵ we get $\tilde{p} = \epsilon/2$, which is the same as before. However, the limit of the energy density is slightly perturbed by the inclusion of electromagnetism and is now $\tilde{u} = 2(1+\Delta)\epsilon$. The non-relativistic equation of state is thus still a polytrope of the form $p = Ku^2$, however the constant

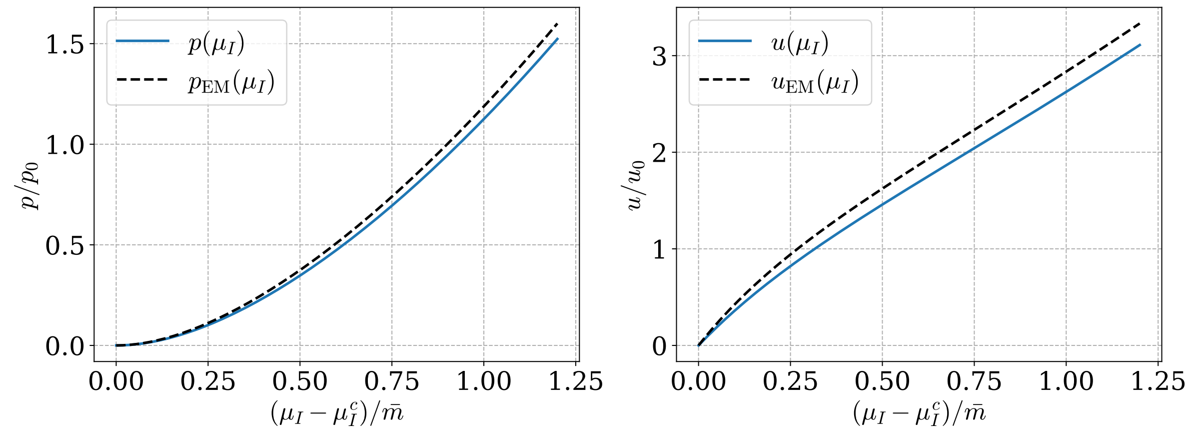

Figure 2.5: Left: The pressure, normalized to p_0, as a function of the chemical potential above the critical value, normalized to $\bar{m}$. Right: The energy density, normalized to u_0, also as a function of the chemical potential. Results with electromagnetic interaction are shown as dashed lines. The y-axis corresponds to different absolute values of isospin-chemical potential, as the critical value of the chemical is changed by the inclusion of electromagnetic interactions, see main text for details.

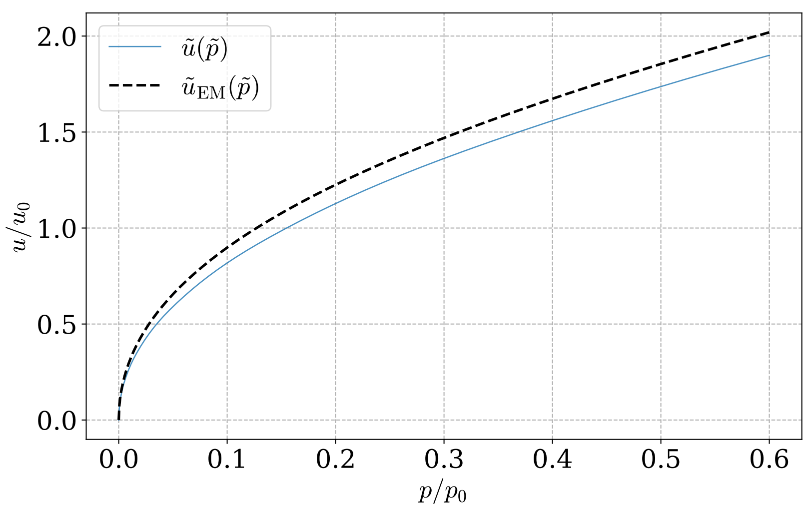

Figure 2.6: The equation of state in the pion condensate phase. Results with electromagnetic interactions are shown as dashed lines. The energy density and pressure is normalized to u_0 and $p_0 = u_0$.

is now $K^{-1} = 8(1+\Delta)^2$. With this, the radius of the polytrope and the limiting radius of the full system changes and is now

$$R = \frac{\pi}{\sqrt{12}(1+\Delta)} r_0 = 80.40\,\mathrm{km}. \tag{2.22}$$

Figure 2.5 shows the pressure and energy density, normalized to their characteristic quantities, as a function of chemical potential above the critical value, normalized to $\bar{m}$. Figure 2.6 shows the equation of state. The results with and without electromagnetic results are compared. We see that the inclusion of electromagnetic contributions results in a less stiff equation of state; a given pressure correspond to a higher energy density when including electromagnetic interactions.

Figure 2.7 shows the mass-radius reaction of the pion star when the electromagnetic interaction is taken into account. We see that the shape of the curve has not changed much from our earlier result. Both the maximum mass and radius are slightly smaller. The new result for maximum radius, $R_{\rm max} = 80.35\,\mathrm{km}$, is in excellent agreement with our expectation, Eq. (2.22). The result with and without electromagnetic interaction is compared in Figure 2.8. As discussed in **??**, we expect a stiffer equation of state to correspond to a more massive star, as happens in this case.

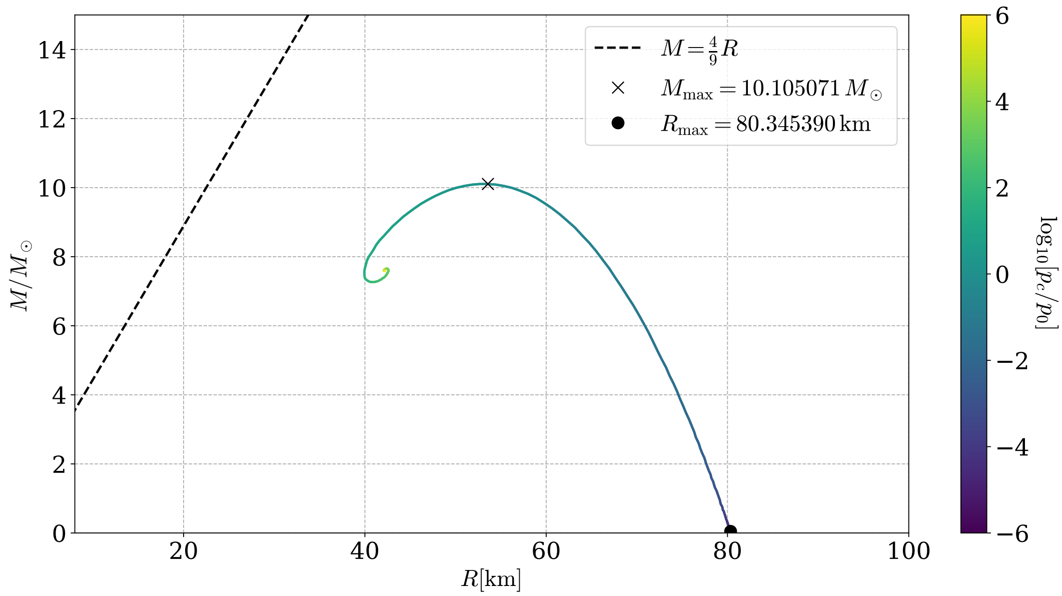

Figure 2.7: The mass-radius relation of a pion star including electromagnetic interactions, parameterized by the logarithm of the central pressure. The dashed line shows the absolute limiting mass for a given radius. The cross indicates the maximum mass configuration, and the dot the maximum radius configuration. The mass is given in units of solar masses, while the radius is in kilometers.

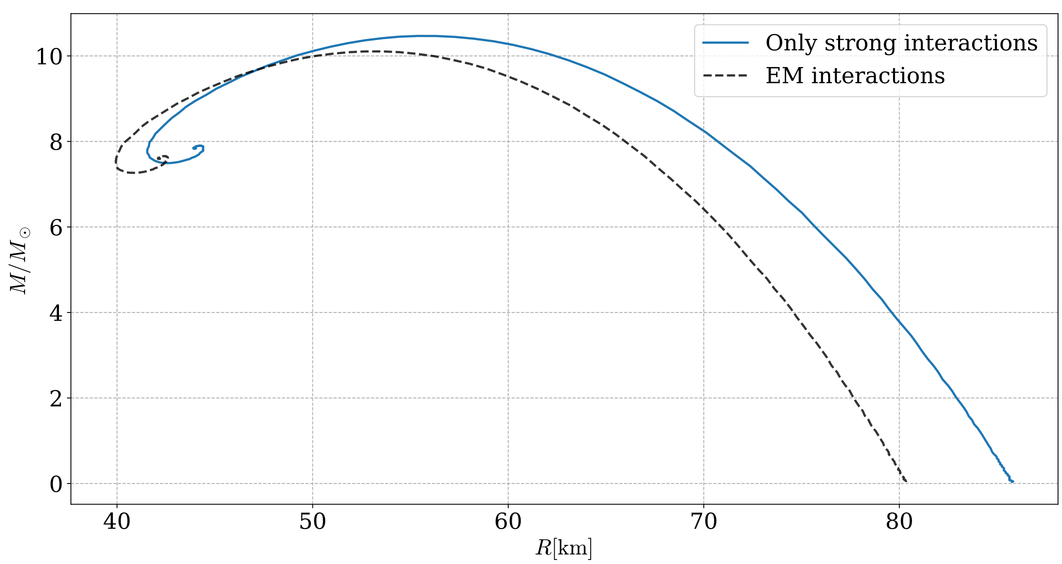

Figure 2.8: This plot compares the mass-radius relation of pion stars with and without the inclusion of electromagnetism.

Appendix A

A.1 Algebra bases

A.1.1 Pauli matrices

The Pauli matrices are

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.1}$$

They obey

$$[\tau_a, \tau_b] = 2i\varepsilon_{abc}\tau_c, \tag{A.2}$$
$$\{\tau_a, \tau_b\} = 2\delta_{ab}\mathbb{1}, \tag{A.3}$$
$$\mathrm{Tr}\,\{\tau_a\} = 0, \tag{A.4}$$
$$\mathrm{Tr}\,\{\tau_a\tau_b\} = 2\delta_{ab}, \tag{A.5}$$
$$\mathrm{Tr}\,\{\tau_a\tau_b\tau_c\tau_d\} = 2(\delta_{ab}\delta_{cd} - \delta_{ac}\delta_{cb} + \delta_{ad}\delta_{cb}). \tag{A.6}$$

Together with the identity matrix $\mathbb{1}$, the Pauli matrices form a basis for the vector space of all 2-by-2 matrices. An arbitrary 2-by-2 matrix M may be written

$$M = M_0\mathbb{1} + M_a\tau_a, \quad M_0 = \frac{1}{2}\mathrm{Tr}\,\{M\}\,,\; M_a = \frac{1}{2}\mathrm{Tr}\,\{\tau_a M\}\,. \tag{A.7}$$

A.1.2 Gell-Mann matrices

The Gell-Mann matrices are

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$
$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

They obey

$$[\lambda_a, \lambda_b] = 2if^{abc}\lambda_c, \tag{A.8}$$
$$\{\lambda_a, \lambda_b\} = \frac{4}{3}\mathbb{1}\delta_{ab} + 2d_{abc}\lambda_c, \tag{A.9}$$
$$\mathrm{Tr}\,\{\lambda_a\} = 0, \tag{A.10}$$
$$\mathrm{Tr}\,\{\lambda_a\lambda_b\} = 2\delta_{ab}, \tag{A.11}$$
$$\mathrm{Tr}\,\{\lambda_a\lambda_b\lambda_c\lambda_d\} = \frac{4}{3}\delta_{ab}\delta_{cd} + 2(d_{abe} + if_{abe})(d_{cde} + if_{cde}). \tag{A.12}$$

where

$$f_{abc} = -\frac{i}{4}\mathrm{Tr}\left\{\lambda_a[\lambda_b, \lambda_c]\right\}, \quad d_{abc} = -\frac{i}{4}\mathrm{Tr}\left\{\lambda_a\{\lambda_b, \lambda_c\}\right\}. \tag{A.13}$$

where the non-zero elements of f_{abc} and d_{abc} are

$$f_{123} = 1, \quad f_{147} = f_{246} = f_{257} = f_{345} = -f_{156} = f_{367} = \frac{1}{2}, \quad f_{458} = f_{678} = \frac{\sqrt{3}}{2}, \tag{A.14}$$

$$d_{146} = d_{157} = d_{256} = -d_{247} = d_{355} = -d_{366} = -d_{377} = \frac{1}{2}$$

$$d_{118} = d_{228} = d_{338} = -d_{888} = \frac{1}{\sqrt{3}}, \quad d_{448} = d_{558} = d_{668} = d_{778} = -\frac{1}{2\sqrt{3}}, \tag{A.15}$$

or a permutation of the indices. The indices of f are totally antisymmetric, while those of d are totally symmetric [18].

A.1.3 Gamma matrices

The gamma matrices γ^μ, $\mu \in \{0, 1, 2, 3\}$, obey

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}, \tag{A.16}$$

$$\gamma^{0\dagger} = \gamma^0, \quad \gamma^{i\dagger} = -\gamma^i. \tag{A.17}$$

These matrices, together with

$$\sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\mu], \tag{A.18}$$

$$\gamma_A^\mu = \gamma^\mu\gamma^5, \tag{A.19}$$

$$\gamma^5 = \frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma, \tag{A.20}$$

form the Clifford algebra $\mathrm{Cl}_{1,3}$, also known as the *space-time algebra*. The subscripts $(1, 3)$ denotes the signature of the metric. The “fifth γ-matrix”, which can be expressed as $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$, obey

$$\{\gamma^5, \gamma^\mu\} = 0, \quad (\gamma^5)^2 = \mathbb{1}. \tag{A.21}$$

The Euclidian counterpart of the space-time algebra, Cl_4, is defined by the “Euclidian gamma matrices”, which obey

$$\{\tilde{\gamma}_a, \tilde{\gamma}_b\} = 2\delta_{ab}\mathbb{1}. \tag{A.22}$$

These can be related to the regular Minkowski-matrices by

$$\tilde{\gamma}_0 = \gamma^0, \quad \tilde{\gamma}_j = -i\gamma^j. \tag{A.23}$$

These then obey

$$\tilde{\gamma}_a^\dagger = \tilde{\gamma}_a. \tag{A.24}$$

The Euclidean $\tilde{\gamma}_5$ is defined as

$$\tilde{\gamma}_5 = \gamma_0\gamma_1\gamma_2\gamma_3 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5. \tag{A.25}$$

It thus also anti-commutes with the Euclidean γ-matrices,

$$\{\tilde{\gamma}_5, \tilde{\gamma}_a\} = 0. \tag{A.26}$$

A.2 Functionals

(TODO: INKLUDER KILER!!!!)

The principle of stationary action and the path integral method relies on functional calculus, where ordinary, n-dimensional calculus is generalized to an infinite-dimensional calculus on a space of functions. A functional, S, takes in a function $\varphi(x)$, and returns a real number $S[\varphi]$. We will be often be dealing with functionals of the form

$$S[\varphi] = \int_{\mathcal{M}} \mathrm{d}^n x \, \mathcal{L}[\varphi](x), \tag{A.27}$$

Here, $\mathcal{L}[\varphi](x)$, the Lagrangian density, is a functional which takes in a function φ, and returns a real number $\mathcal{L}[\varphi](x)$ *for each point* $x \in \mathcal{M}$. Thus, $\mathcal{L}$ does, in fact, return a real-valued function, not just a number. $\mathcal{M}$ is the manifold, in our case space-time, of which both $\varphi(x)$ and $\mathcal{L}[\varphi](x)$ are functions. The function φ can, in general, take on the value of a scalar, complex number, spinor, vector, etc..., while $\mathcal{L}[\varphi](x)$ must be a scalar-valued function. This strongly constraints the form of any Lagrangian and is an essential tool in constructing quantum field theories. Although this section is written with a single scalar-valued function φ, this can easily be generalized by adding an index, $\varphi \to \varphi_\alpha$, enumerating all the degrees of freedom, then restating the arguments [3, 19].

A.2.1 Functional derivative

The functional derivative is base on an arbitrary *variation* η of the function φ. The variation η, often written $\delta\varphi$ is an arbitrary function only constrained to vanish *quickly enough* at the boundary $\partial\mathcal{M}$.[1] The variation of the functional S is defined as

$$\delta_\eta S[\varphi] = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(S[\varphi + \epsilon\eta] - S[\varphi] \right) = \frac{\mathrm{d}}{\mathrm{d}\epsilon} S[\varphi + \epsilon\eta]|_{\epsilon=0}. \tag{A.28}$$

We can regard the variation of a functional as the generalization of the differential of a function, **??**, as the best linear approximation around a point. In regular differential geometry, a function f can be approximated around a point x by

$$f(x + \epsilon v) = f(x) + \epsilon \mathrm{d}f(v), \tag{A.29}$$

where v is a vector in the tangent space at x. In functional calculus, the functional S is analogous to f, φ to x, and η to v. We can more clearly see the resemblance by writing

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon} f(x + \epsilon v) = \mathrm{d}f(v) = \frac{\partial f}{\partial x^\mu} v^\mu. \tag{A.30}$$

In the last line we expanded the differential using the basis-representation, $v = v^\mu \partial_\mu$. To generalize this to functional, we define the *functional derivative*, by

$$\delta_\eta S[\varphi] = \int_{\mathcal{M}} \mathrm{d}^n x \, \frac{\delta S[\varphi]}{\delta\eta(x)} \eta(x). \tag{A.31}$$

If we let $S[\varphi] = \varphi(x)$, for some fixed x, the variation becomes

$$\delta_\eta S[\varphi] = \eta(x) = \int \mathrm{d}^n y \, \delta(x - y) \eta(y), \tag{A.32}$$

which leads to the identity

$$\frac{\delta\varphi(x)}{\delta\varphi(y)} = \delta(x - y). \tag{A.33}$$

There is also a generalized chain rule for functional derivatives. If ψ is some new functional variable, then

$$\frac{\delta S[\varphi]}{\delta\varphi(x)} = \int_{\mathcal{M}} \mathrm{d}^n y \, \frac{\delta S[\varphi]}{\delta\psi(y)} \frac{\delta\psi(y)}{\delta\varphi(x)}. \tag{A.34}$$

[1] The condition of "quickly enough" is to ensure that we can integrate by parts and ignore the boundary condition, which we will do without remorse.

Higher functional derivatives are defined in terms of higher-order variations,

$$\delta_\eta^m S[\varphi] = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\delta_\eta^{m-1} S[\varphi+\epsilon\eta]|_{\epsilon=0} = \int_{\mathcal{M}} \left(\prod_{i=1}^{m} \mathrm{d}^n x_i \eta(x_i)\right) \frac{\delta^m S[\varphi]}{\delta\varphi(x_1)...\delta\varphi(x_m)}. \tag{A.35}$$

With this, we can write the functional Taylor expansion,

$$S[\varphi_0+\varphi] = S[\varphi_0] + \int_{\mathcal{M}} \mathrm{d}^n x\, \varphi(x) \frac{\delta S[\varphi_0]}{\delta\varphi(x)} + \frac{1}{2}\int_{\mathcal{M}} \mathrm{d}^n x \mathrm{d}^n y\, \varphi(x)\varphi(y) \frac{\delta S[\varphi_0]}{\delta\varphi(x),\varphi(y)} + \dots \tag{A.36}$$

Here, the notation $\delta S[\varphi_0]/\delta\varphi$ indicate that $S[\varphi]$ is first differentiated with respect to φ, then evaluated at $\varphi = \varphi_0$ [1].

A.2.2 The Euler-Lagrange equation

The Lagrangian may also be written as a scalar function of the field-values at x, $\varphi(x)$, as well as its derivatives, $\partial_\mu\varphi(x)$, for example

$$\mathcal{L}(\varphi, \partial_\mu\varphi) = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{1}{4!}\lambda\varphi^4 + \dots \tag{A.37}$$

We have omitted the evaluation at x for the brevity of notation. We use this to evaluate the variation of a functional in the of Eq. (A.27),

$$\delta_\eta S[\varphi] = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\int_{\mathcal{M}} \mathrm{d}^n x\, \mathcal{L}[\varphi+\epsilon\eta](x), \tag{A.38}$$

by Taylor expanding the Lagrangian density as a function of φ and its derivatives,

$$\mathcal{L}[\varphi+\epsilon\eta] = \mathcal{L}\left(\varphi+\epsilon\eta, \partial_\mu\{\varphi+\epsilon\eta\}\right) = \mathcal{L}[\varphi] + \epsilon\left(\frac{\partial\mathcal{L}}{\partial\varphi}\eta + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\partial_\mu\eta\right) + \mathcal{O}(\epsilon^2). \tag{A.39}$$

Inserting this into Eq. (A.38) and partially integrating the last term allows us to write the variation in the form Eq. (A.31), and the functional derivative is

$$\frac{\delta S}{\delta\varphi} = \frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}. \tag{A.40}$$

The principle of stationary action says that the equation of motion of a field obeys $\delta_\eta S = 0$. As η is arbitrary, this is equivalent to setting the functional derivative of S equal to zero. The result is the Euler-Lagrange equations of motion [3],

$$\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} = 0. \tag{A.41}$$

A.2.3 Functional calculus on a curved manifold

As discussed in **??**, when integrating a scalar on a curved manifold, we must include the $\sqrt{|g|}$-factor to get a coordinate-independent result. The action in curved spacetime is therefore [19]

$$S[g,\varphi] = \int_{\mathcal{M}} \mathrm{d}^n x\, \sqrt{|g|}\mathcal{L}[g,\varphi], \tag{A.42}$$

where the action and the Lagrangian now is a functional of both the matter-field φ and the metric $g_{\mu\nu}$. Our example Lagrangian from last section now takes the form

$$\mathcal{L}(g_{\mu\nu}, \varphi, \nabla_\mu\varphi) = \frac{1}{2}g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{1}{4!}\lambda\varphi^4 \dots, \tag{A.43}$$

where partial derivatives are substituted with covariant derivatives. We define the functional derivative as

$$\delta_\eta S = \int_{\mathcal{M}} \mathrm{d}^n x \sqrt{|g|}\frac{\delta S}{\delta\eta(x)}\eta(x). \tag{A.44}$$

If this is a variation in φ only, this gives the same result as before. However, in general relativity, the metric itself is a dynamic field, and we may thereofre vary it. Consider $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$. The variation of the action is then assuming $\mathcal{L}$ only depends on g and not its derivatives, we get

$$\delta_g S = \int_{\mathcal{M}} \mathrm{d}^n x \, \left[\left(\delta \sqrt{|g|} \right) \mathcal{L}[g] + \sqrt{|g|} \delta \mathcal{L}[g] \right] \tag{A.45}$$

We have used

$$\delta \sqrt{|g|} = -\frac{1}{2} \sqrt{|g|} g_{\mu\nu} \delta g^{\mu\nu} \tag{A.46}$$

The variation of the $\sqrt{|g|}$-factor can be evaluated using Using the Levi-Civita symbol $\varepsilon_{\mu_1 \ldots \mu_n}$, a determinant of a $n \times n$-matrix may be written as

$$\det(A) = \frac{1}{n!} \varepsilon_{\mu_1 \ldots \mu_n} \varepsilon^{\nu_1 \ldots \nu_n} A^{\mu_1}{}_{\nu_1} \ldots A^{\mu_n}{}_{\nu_n}. \tag{A.47}$$

Using this, we can write for a matrix M

$$\begin{aligned}
\det(\mathbb{1} + \varepsilon M) &= \frac{1}{n!} \varepsilon_{\mu_1 \ldots} \varepsilon^{\nu_1 \cdots} (\mathbb{1} + \epsilon M)^{\mu_1}{}_{\nu_1} (\mathbb{1} + \epsilon M)^{\mu_2}{}_{\nu_2} \ldots && \text{(A.48)} \\
&= \frac{1}{n!} \varepsilon_{\mu_1 \ldots} \varepsilon^{\nu_1 \cdots} [\delta^{\mu_1}_{\nu_1} \cdots + \epsilon (M^{\mu_1}{}_{\nu_1} \delta^{\mu_2}_{\nu_1} \cdots + M^{\mu_2}{}_{\nu_2} \delta^{\mu_1}_{\nu_1} \cdots + \ldots) + \ldots] && \text{(A.49)} \\
&= 1 + M^{\mu}{}_{\mu} + \mathcal{O}(\epsilon^2) && \text{(A.50)}
\end{aligned}$$

Thus,

$$\delta \sqrt{|g|} = \sqrt{|\det[g_{\mu\nu}(\delta^{\nu}_{\rho} + g^{\nu\sigma} \delta g_{\sigma\rho})]|} - \sqrt{|g|} = \sqrt{|g|} \left(\sqrt{|1 + g^{\mu\nu} \delta g_{\mu\nu}|} - 1 \right) = -\frac{1}{2} \sqrt{|g|} g^{\mu\nu} \delta g_{\mu\nu}. \tag{A.51}$$

The minus sign is included as the determinant of a Lorentzian metric is negative. Assuming the Lagrangian only depneds on the metric directly, and not its derivatives, the variation of the action is

$$\delta_g S = \int_{\mathcal{M}} \mathrm{d}^n x \, \sqrt{|g|} \left(\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} - \frac{1}{2} g_{\mu\nu} \mathcal{L} \right) \delta g^{\mu\nu}. \tag{A.52}$$

With the Lagrangian in Eq. (A.43), we get

$$\frac{\delta S}{\delta g^{\mu\nu}} = \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} - \frac{1}{2} g_{\mu\nu} \mathcal{L} = -\frac{1}{2} \left(\frac{1}{2} \nabla_\mu \varphi \nabla_\nu \varphi + \frac{1}{2} m^2 \varphi^2 + \ldots \right). \tag{A.53}$$

We recognize the $(\mu, \nu) = (0, 0)$-component as negative half the Hamiltonian density, which supports the definition of the definition of the stress-energy tensor **??**.

A.2.4 Functional derivative of the Einstein-Hilbert action

(NEEDS MORE CLEANUP)

In the Einstein-Hilbert action, **??**, the Lagrangian density is $\mathcal{L} = kR = kg^{\mu\nu} R_{\mu\nu}$, where k is a constant and $R_{\mu\nu}$ the Ricci tensor, **??**. As the Ricci tensor is dependent on both the derivative and second derivative of the metric, we can not use Eq. (A.53) directly. Instead, we use the variation

$$\delta S_{\mathrm{EH}} = k \int_{\mathcal{M}} \mathrm{d}^n x \, \sqrt{|g|} \left(\delta R - \frac{1}{2} g_{\mu\nu} R \delta g^{\mu\nu} \right). \tag{A.54}$$

The variation of the Ricci scalar is

$$\delta R = R_{\mu\nu} \delta g^{\mu\nu} + g^{\mu\nu} \delta R_{\mu\nu}, \tag{A.55}$$

We can write the variation of the Ricci scalar, and thus the Riemann curvature tensor, in terms of variations in Christoffel symbols, $\delta \Gamma^{\rho}_{\mu\nu}$ using the explicit formula for a symmetric, metric-compatible covariant derivative, **??**. As $\delta\Gamma = \Gamma - \Gamma'$, it is a tensor, and we may write clean up

$$
\begin{aligned}
\delta R^{\rho}{}_{\sigma\mu\nu} &= \delta(\partial_{[\mu}\Gamma^{\rho}_{\nu]\sigma} + \Gamma^{\rho}_{\lambda[\mu}\Gamma^{\lambda}_{\nu]\sigma}) = \partial_{[\mu}\delta\Gamma^{\rho}_{\nu]\sigma} + (\delta\Gamma^{\rho}_{\lambda[\mu})\Gamma^{\lambda}_{\nu]\sigma} + \Gamma^{\rho}_{\lambda[\mu}(\delta\Gamma^{\lambda}_{\nu]\sigma}) \\
&= \partial_{\mu}\delta\Gamma^{\rho}_{\nu\sigma} + \Gamma^{\rho}_{\lambda\mu}(\delta\Gamma^{\lambda}_{\nu\sigma}) - \Gamma^{\lambda}_{\mu\sigma}(\delta\Gamma^{\rho}_{\lambda\nu}) - \left(\partial_{\nu}\delta\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\lambda\nu}(\delta\Gamma^{\lambda}_{\mu\sigma}) - \Gamma^{\lambda}_{\nu\sigma}(\delta\Gamma^{\rho}_{\lambda\mu})\right) + (\Gamma^{\lambda}_{\mu\nu}\delta\Gamma^{\rho}_{\lambda\sigma} - \Gamma^{\lambda}_{\mu\nu}\delta\Gamma^{\rho}_{\lambda\sigma}) \\
&= \nabla_{\mu}\delta\Gamma^{\rho}_{\nu\sigma} - \nabla_{\nu}\delta\Gamma^{\rho}_{\mu\sigma} = \nabla_{\eta}\left(g^{\eta}{}_{\mu}\delta\Gamma^{\rho}_{\nu\sigma} - g^{\eta}{}_{\nu}\delta\Gamma^{\rho}_{\mu\sigma}\right) = \nabla_{\eta}(K^{\rho}{}_{\sigma\mu\nu})^{\eta},
\end{aligned}
$$

where K is a tensorial quantity, which vanish at the boundary of our spacetime. Using the generalized divergence theorem, **??**, we see that the contribution to the action from this quantity vansih. The contribution comes from an integral over $g^{\mu\nu}\delta R_{\mu\nu} = g^{\mu\nu}\delta R^{\rho}_{\mu\rho\nu} = g^{\mu\nu}\nabla_{\eta}(K^{\rho}{}_{\mu\rho\nu})^{\eta}$ Using metric compatibility, we can exchange the covariant derivative and the metric, and we have $g^{\mu\nu}\delta R_{\mu\nu} = \nabla_{\eta}[g^{\mu\nu}K^{\eta\rho}{}_{\mu\rho\nu}]$. The contribution to the action therefore becomes

$$
\int_{\mathcal{M}} \mathrm{d}^4x\,\sqrt{|g|}g^{\mu\nu}\delta R_{\mu\nu} = \int_{\mathcal{M}} \mathrm{d}^4x\,\sqrt{|g|}\nabla_{\eta}[g^{\mu\nu}K^{\eta\rho}{}_{\mu\rho\nu}] = \int_{\partial\mathcal{M}} \mathrm{d}^3y\,\sqrt{|\gamma|}n_{\eta}[g^{\mu\nu}K^{\eta\rho}{}_{\mu\rho\nu}] = 0, \tag{A.56}
$$

where we used the fact that $\delta g_{\mu\nu}$, and thus K, vanish at $\partial\mathcal{M}$. The variation of the action is therefore

$$
\delta S_{\mathrm{EH}} = k\int_{\mathcal{M}} \mathrm{d}^n x\sqrt{|g|}\left[R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right]\delta g^{\mu\nu}, \tag{A.57}
$$

and by the definition of the functional derivative,

$$
\frac{\delta S_{\mathrm{EH}}}{\delta g^{\mu\nu}} = k(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}). \tag{A.58}
$$

A.3 *Rewriting terms

This section shows how to rewrite terms in the Lagrangian of Chiral perturbation theory. These techniques and more are used to reduce the total number of terms and to change between different conventions. Changing the field parametrization that appears in the Lagrangian does not affect any of the physics, as it corresponds to a change of variables in the path integral [2, 20, 21]. However, a change of variables can result in new terms in the Lagrangian. As a result of this, terms that appear independent on their face may be redundant. These terms can be eliminated by using the classical equations of motion. In this section, we show first the derivation of the equations of motion, then use this result to identify redundant terms which need not be included in the most general Lagrangian.

We derive the equations of motion for the leading order Lagrangian using the principle of least action. Choosing the parametrization $\Sigma = \exp\{()\, i\pi_a\tau_a)$, a variation $\pi_a \to \pi_a + \delta\pi_a$ results in a variation in Σ, $\delta\Sigma = i\tau_a\delta\pi_a\Sigma$. The variation of the leading order action,

$$
S_2 = \int_{\Omega} \mathrm{d}^4x\,\mathcal{L}_2, \tag{A.59}
$$

when varying π_a is

$$
\delta S = \int_{\Omega} \mathrm{d}x\,\frac{f^2}{4}\mathrm{Tr}\left\{(\nabla_{\mu}\delta\Sigma)(\nabla^{\mu}\Sigma)^{\dagger} + (\nabla_{\mu}\Sigma)(\nabla^{\mu}\delta\Sigma)^{\dagger} + \chi\delta\Sigma^{\dagger} + \delta\Sigma\chi^{\dagger}\right\}.
$$

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Using the properties of the covariant derivative to do partial integration, as shown in (), as well as $\delta(\Sigma\Sigma^{\dagger}) = (\delta\Sigma)\Sigma^{\dagger} + \Sigma(\delta\Sigma)^{\dagger} = 0$, the variation of the action can be written

$$
\begin{aligned}
\delta S &= \frac{f^2}{4}\int_{\Omega} \mathrm{d}x\,\mathrm{Tr}\left\{-\delta\Sigma\nabla^2\Sigma^{\dagger} + (\nabla^2\Sigma)(\Sigma^{\dagger}\delta\Sigma\Sigma^{\dagger}) - \chi(\Sigma^{\dagger}\delta\Sigma\Sigma^{\dagger}) + \delta\Sigma\chi^{\dagger}\right\} \\
&= \frac{f^2}{4}\int_{\Omega} \mathrm{d}x\,\mathrm{Tr}\left\{\delta\Sigma\Sigma^{\dagger}\left[(\nabla^2\Sigma)\Sigma^{\dagger} - \Sigma\nabla^2\Sigma^{\dagger} - \chi\Sigma^{\dagger} + \Sigma\chi^{\dagger}\right]\right\} \\
&= i\frac{f^2}{4}\int_{\Omega} \mathrm{d}x\,\mathrm{Tr}\left\{\tau_a\left[(\nabla^2\Sigma)\Sigma^{\dagger} - \Sigma\nabla^2\Sigma^{\dagger} - \chi\Sigma^{\dagger} + \Sigma\chi^{\dagger}\right]\right\}\delta\pi_a = 0.
\end{aligned}
$$

As the variation is arbitrary, the equations of motion to leading order is

$$\mathrm{Tr}\left\{\tau_a\left[(\nabla^2\Sigma)\Sigma^\dagger - \Sigma\nabla^2\Sigma^\dagger - \chi\Sigma^\dagger + \Sigma\chi^\dagger\right]\right\} = 0. \tag{A.60}$$

We define

$$\mathcal{O}^{(2)}_{\mathrm{EOM}} = (\nabla^2\Sigma)\Sigma^\dagger - \Sigma\nabla^2\Sigma^\dagger - \chi\Sigma^\dagger + \Sigma\chi^\dagger. \tag{A.61}$$

The next step in eliminating redundant terms is to change the parametrization of Σ by $\Sigma(x) \to \Sigma'(x)$. Here, $\Sigma(x) = e^{iS(x)}\Sigma'(x)$, $S(x) \in \mathfrak{su}(2)$. This change leads to a new Lagrange density, $\mathcal{L}[\Sigma] = \mathcal{L}[\Sigma'] + \Delta\mathcal{L}[\Sigma']$. We are free to choose $S(x)$, as long Σ' still obey the required transformation properties. Any terms in the Lagrangian $\Delta\mathcal{L}$ due to a reparametrization can be neglected, as argued earlier. When demanding that Σ' obey the same symmetries as Σ, the most general transformation to second order in Weinberg's power counting scheme is [2]

$$S_2 = i\alpha_2\left[(\nabla^2\Sigma')\Sigma^\dagger - \Sigma'(\nabla^2\Sigma')^\dagger\right] + i\alpha_2\left[\chi\Sigma'^\dagger - \Sigma'\chi^\dagger - \frac{1}{2}\mathrm{Tr}\left\{\chi\Sigma'^\dagger - \Sigma'\chi^\dagger\right\}\right]. \tag{A.62}$$

α_1 and α_2 are arbitrary real numbers. As Eq. (A.62) is to second order, $\Delta\mathcal{L}$ is fourth order in Weinberg's power counting scheme. Inserting this gives

$$\begin{aligned}
\mathcal{L}_2\left[e^{iS_2}\Sigma'\right] &= \frac{f^2}{4}\mathrm{Tr}\left\{[\nabla_\mu(1+iS_2)\Sigma'][\nabla^\mu\Sigma'^\dagger(1-iS_2)]\right\} + \frac{f^2}{4}\mathrm{Tr}\left\{\chi\Sigma'^\dagger(1-iS_2) + (1+iS_2)\Sigma'\chi^\dagger\right\} \\
&= \mathcal{L}[\Sigma'] + i\frac{f^2}{4}\mathrm{Tr}\left\{[\nabla_\mu(S_2\Sigma')][\nabla^\mu\Sigma']^\dagger - [\nabla_\mu\Sigma'][\nabla^\mu(\Sigma'^\dagger S_2)]\right\} - i\frac{f^2}{4}\mathrm{Tr}\left\{\chi\Sigma'^\dagger S_2 - S_2\Sigma'\chi^\dagger\right\}
\end{aligned}$$

Using the properties of the covariant derivative, we may use the product rule and partial integration to write the difference between the two Lagrangians to fourth-order as

$$\begin{aligned}
\Delta\mathcal{L}[\Sigma'] &= i\frac{f^2}{4}\mathrm{Tr}\left\{(\nabla_\mu S_2)(\Sigma'\nabla^\mu\Sigma'^\dagger - (\nabla^\mu\Sigma')\Sigma'^\dagger)\right\} - i\frac{f^2}{4}\mathrm{Tr}\left\{\chi\Sigma'^\dagger S_2 - S_2\Sigma'\chi^\dagger\right\} \\
&= \frac{f^2}{4}\mathrm{Tr}\left\{iS_2\mathcal{O}^{(2)}_{\mathrm{EOM}}\right\}.
\end{aligned} \tag{A.63}$$

Any term that can be written in the form of Eq. (A.63) for arbitrary $\alpha_1, \alpha_2 \in \mathbb{R}$ is redundant, as we argued earlier, and may therefore be discarded. $\Delta\mathcal{L}_2$ is of fourth order, and it can thus be used to remove terms from $\mathcal{L}_4$ or higher order.

A.3.1 Rewriting NLO Lagrangian

The NLO Lagrangian used in this text is given in Eq. (1.103), and is

$$\begin{aligned}
\mathcal{L}_4 = &\frac{l_1}{4}\mathrm{Tr}\left\{\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\right\}^2 + \frac{l_2}{4}\mathrm{Tr}\left\{\nabla_\mu\Sigma(\nabla_\nu\Sigma)^\dagger\right\}\mathrm{Tr}\left\{\nabla^\mu\Sigma(\nabla^\nu\Sigma)^\dagger\right\} + \frac{l_3+l_4}{16}\mathrm{Tr}\left\{\chi\Sigma^\dagger + \Sigma\chi^\dagger\right\}^2 \\
&+ \frac{l_4}{8}\mathrm{Tr}\left\{\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\right\}\mathrm{Tr}\left\{\chi\Sigma^\dagger + \Sigma\chi^\dagger\right\} - \frac{l_7}{16}\mathrm{Tr}\left\{\chi\Sigma^\dagger - \Sigma\chi^\dagger\right\}^2 + \frac{h_1+h_3-l_4}{4}\mathrm{Tr}\left\{\chi\chi^\dagger\right\} \\
&+ \frac{h_1-h_3-l_4}{16}\left[\mathrm{Tr}\left\{\chi\Sigma^\dagger + \Sigma\chi^\dagger\right\}^2 + \mathrm{Tr}\left\{\chi\Sigma^\dagger - \Sigma\chi^\dagger\right\}^2 - 2\mathrm{Tr}\left\{\left(\chi\Sigma^\dagger\right)^2 + \left(\Sigma\chi^\dagger\right)^2\right\}\right].
\end{aligned} \tag{A.64}$$

We can rewrite it to match the one used in [10, 15], starting with

$$\begin{aligned}
&\frac{h_1-h_3-l_4}{16}\left(\mathrm{Tr}\left\{\chi\Sigma^\dagger - \Sigma\chi^\dagger\right\}^2 - 2\mathrm{Tr}\left\{(\chi\Sigma^\dagger)^2 + (\Sigma\chi^\dagger)^2\right\}\right) \\
&= \frac{h_1-h_3-l_4}{16}\left(\mathrm{Tr}\left\{\chi\Sigma^\dagger\right\}^2 - 2\mathrm{Tr}\left\{\chi\Sigma^\dagger\right\}\mathrm{Tr}\left\{\Sigma\chi^\dagger\right\} + \mathrm{Tr}\left\{\Sigma\chi^\dagger\right\}^2 - 2\mathrm{Tr}\left\{(\chi\Sigma^\dagger)^2\right\} - 2\mathrm{Tr}\left\{(\Sigma\chi^\dagger)^2\right\}\right).
\end{aligned}$$

Using $\mathrm{Tr}\left\{A^2\right\} = \mathrm{Tr}\left\{A\right\}^2 - \det(A)\mathrm{Tr}\left\{\mathbb{1}\right\}$, we get

$$\begin{aligned}
&= -\frac{h_1-h_3-l_4}{16}\left(\mathrm{Tr}\left\{\chi\Sigma^\dagger\right\}^2 + 2\mathrm{Tr}\left\{\chi\Sigma^\dagger\right\}\mathrm{Tr}\left\{\Sigma\chi^\dagger\right\} + \mathrm{Tr}\left\{\Sigma\chi^\dagger\right\}^2 - 4\det(\chi\Sigma^\dagger) - 4\det(\Sigma\chi^\dagger)\right) \\
&= -\frac{h_1-h_3-l_4}{16}\left(\mathrm{Tr}\left\{\chi\Sigma^\dagger + \Sigma\chi^\dagger\right\}^2 - 4\det(\chi\Sigma^\dagger) - 4\det(\Sigma\chi^\dagger)\right).
\end{aligned}$$

Furthermore, as $\det(\Sigma) = 1$,

$$\begin{aligned}\mathcal{L}_4 = &\frac{l_1}{4}\mathrm{Tr}\left\{\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\right\}^2 + \frac{l_2}{4}\mathrm{Tr}\left\{\nabla_\mu\Sigma(\nabla_\nu\Sigma)^\dagger\right\}\mathrm{Tr}\left\{\nabla^\mu\Sigma(\nabla^\nu\Sigma)^\dagger\right\} + \frac{l_3+l_4}{16}\mathrm{Tr}\left\{\chi\Sigma^\dagger + \Sigma\chi^\dagger\right\}^2 \\ &+ \frac{l_4}{8}\mathrm{Tr}\left\{\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\right\}\mathrm{Tr}\left\{\chi\Sigma^\dagger + \Sigma\chi^\dagger\right\} - \frac{l_7}{16}\mathrm{Tr}\left\{\chi\Sigma^\dagger - \Sigma\chi^\dagger\right\}^2 + \frac{h_1+h_3-l_4}{4}\mathrm{Tr}\left\{\chi\chi^\dagger\right\} \\ &+ \frac{h_1-h_3-l_4}{4}(\det\chi + \det\chi^\dagger).\end{aligned} \tag{A.65}$$

For real χ, we have $\mathrm{Tr}\left\{\chi\chi^\dagger\right\} = \det(\chi) + \det(\chi^\dagger)$, and we can define $h_1' = h_1 - l_4$ to get

$$\begin{aligned}\mathcal{L}_4 = &\frac{l_1}{4}\mathrm{Tr}\left\{\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\right\}^2 + \frac{l_2}{4}\mathrm{Tr}\left\{\nabla_\mu\Sigma(\nabla_\nu\Sigma)^\dagger\right\}\mathrm{Tr}\left\{\nabla^\mu\Sigma(\nabla^\nu\Sigma)^\dagger\right\} + \frac{l_3+l_4}{16}\mathrm{Tr}\left\{\chi\Sigma^\dagger + \Sigma\chi^\dagger\right\}^2 \\ &+ \frac{l_4}{8}\mathrm{Tr}\left\{\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\right\}\mathrm{Tr}\left\{\chi\Sigma^\dagger + \Sigma\chi^\dagger\right\} - \frac{l_7}{16}\mathrm{Tr}\left\{\chi\Sigma^\dagger - \Sigma\chi^\dagger\right\}^2 + \frac{h_1'}{2}\mathrm{Tr}\left\{\chi\chi^\dagger\right\}.\end{aligned} \tag{A.66}$$

If one assumes $\Delta m = 0$, i.e., what is called the chiral limit, then the term l_7 falls away, as $\chi = \chi^\dagger$.

A.4 *Covariant derivative

In χPT at finite isospin chemical potential μ_I, the covariant derivative acts on functions $A(x) : \mathcal{M}_4 \to \mathrm{SU}(2)$, where $\mathcal{M}_4$ is the space-time manifold. It is defined as

$$\nabla_\mu A(x) = \partial_\mu A(x) - i[v_\mu, A(x)], \quad v_\mu = \frac{1}{2}\mu_I\delta^0_\mu\tau_3. \tag{A.67}$$

The covariant derivative obeys the product rule, as

$$\nabla_\mu(AB) = (\partial_\mu A)B + A(\partial_\mu B) - i[v_\mu, AB] = (\partial_\mu A - i[v_\mu, A])B + A(\partial_\mu B - i[v_\mu, B]) = (\nabla_\mu A)B + A(\nabla_\mu B).$$

Decomposing a 2-by-2 matrix M, as described in section A.1, shows that the trace of the commutator of τ_b and M is zero:

$$\mathrm{Tr}\left\{[\tau_a, M]\right\} = M_b\mathrm{Tr}\left\{[\tau_a, \tau_b]\right\} = 0.$$

Together with the fact that $\mathrm{Tr}\left\{\partial_\mu A\right\} = \partial_\mu\mathrm{Tr}\left\{A\right\}$, this gives the product rule for invariant traces:

$$\mathrm{Tr}\left\{A\nabla_\mu B\right\} = \partial_\mu\mathrm{Tr}\left\{AB\right\} - \mathrm{Tr}\left\{(\nabla_\mu A)B\right\}.$$

This allows for the use of the divergence theorem when doing partial integration. Let $\mathrm{Tr}\left\{K^\mu\right\}$ be a space-time vector, and $\mathrm{Tr}\left\{A\right\}$ scalar. Let Ω be the domain of integration, with coordinates x and $\partial\Omega$ its boundary, with coordinates y. Then,

$$\int_\Omega \mathrm{d}x\,\mathrm{Tr}\left\{A\nabla_\mu K^\mu\right\} = \int_{\partial\Omega}\mathrm{d}y\, n_\mu\mathrm{Tr}\left\{AK^\mu\right\} - \int_\Omega \mathrm{d}x\,\mathrm{Tr}\left\{(\nabla_\mu A)K^\mu\right\},$$

where n_μ is the normal vector of $\partial\Omega$ [19]. This makes it possible to do partial integration and discard surface terms in the χPT Lagrangian, given the assumption of no variation on the boundary.

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