Master

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Todo list

Contents

Chapter 1

Introduction

1.1 Units

In this thesis, we employ *natural units*, defined by

$$\hbar = c = k_B = 1, \tag{1.1}$$

where $\hbar$ is Planck's reduced constant, c is the speed of light, and k_B is Boltzmann's constant. Dimensionfull results are often given in MeV. To obtain results in the SI-system, we use the following conversion factors, as given by [1]

$$c = 2.998 \cdot 10^{8}\,\mathrm{m\,s^{-1}}, \tag{1.2}$$
$$\hbar = 1.055 \cdot 10^{-34}\,\mathrm{J\,s}, \tag{1.3}$$
$$k_B = 1.380 \cdot 10^{-23}\,\mathrm{J\,K^{-1}}, \tag{1.4}$$
$$G = 6.674 \cdot 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}, \tag{1.5}$$

where G is Newton's gravitational constant. The conversion factor between MeV and SI-units is

$$1\,\mathrm{MeV} = 1.60218\ \cdot 10^{-19}\,\mathrm{J}. \tag{1.6}$$

In the calculation in section 4.3, the value for the neutron mass is [1]

$$m_N = 939.57\,\mathrm{MeV} = 1.674 \cdot 10^{-27}\,\mathrm{kg}. \tag{1.7}$$

In astronomical calculation, the solar mass is used, which is

$$M_\odot = 1.988 \cdot 10^{30}\,\mathrm{kg}. \tag{1.8}$$

Chapter 2

Mathematics

General relativity describes how matter and energy curve the fabric of space and time. Einstein first wrote down the theory more than a century ago, and it is still our most accurate theory of gravitational effects. It makes accurate and counterintuitive predictions, which experiments have borne out. This chapter surveys the basics of general relativity and some mathematical prerequisites. We will then use this to derive the Tolman-Oppenheimer-Volkoff (TOV) equation, a differential equation used to model stars. This chapter is based on [2, 3].

2.1 Differential geometry

2.1.1 Manifolds and coordinates

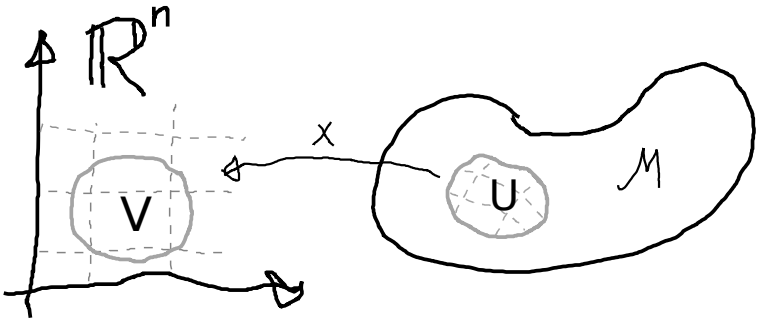

Figure 2.1: (Kladd) The coordinate function x maps a neighborhood U in the manifold $\mathcal{M}$ to a negiborhood V in $\mathbb{R}^n$.

General relativity is formulated in the language of *differential geometry*, which generalizes multivariable calculus to more general spaces than $\mathbb{R}^n$. These spaces are *smooth manifolds*. An n-dimensional manifold, $\mathcal{M}$, is a set of points, locally homeomorphic to $\mathbb{R}^n$. That is, for all points $p \in \mathcal{M}$, there exists a neighborhood U around p, together with a corresponding set of continuous, bijective functions,

$$x : U \subseteq \mathcal{M} \longmapsto V \subseteq \mathbb{R}^n, \tag{2.1}$$

$$p \longmapsto x^\mu(p). \tag{2.2}$$

We call $x(p) = (x^0(p), \ldots, x^{n-1}(p)) = x^\mu(p)$ a coordinate function of $\mathcal{M}$. The inverse of x, x^{-1}, obeys $x^{-1}(x(p)) = p$, for all $p \in U$. A smooth manifold is one in which the coordinate functions are infinitely differentiable. To define differentiability on manifolds, consider two coordinate functions, x, and x'. The corresponding domains U and U' may or may not overlap. We then define the transition function, a function

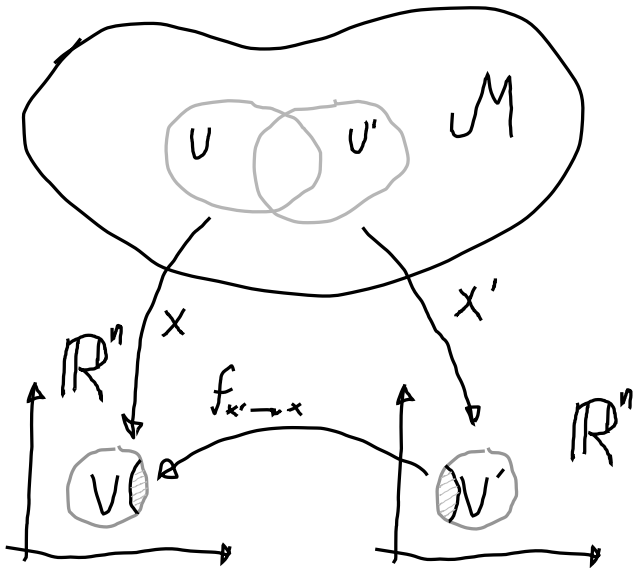

Figure 2.2: (Kladd) The transition map $f_{x'\to x}$ between two coordinate functions, x and x', maps between the images of these function, via the manifold $\mathcal{M}$. The function's domain and image are restricted to a (possibly empty) subset of the images of x and x'. This is illustrated by the shaded regions in V and V'.

between subsets of $\mathbb{R}^n$ by mapping via $\mathcal{M}$, as

$$f_{x\to x'} = x' \circ x^{-1} : \mathbb{R}^n \mapsto \mathbb{R}^n. \tag{2.3}$$

The map is illustrated in Figure 2.2. [1] A set of coordinate functions $\mathcal{A} = \{x_i\}$ whose domain cover $\mathcal{M}$ is called an *atlas* of $\mathcal{M}$. If the transition function between all pairings of coordinate functions in the atlas is smooth—that is, infinitely differentiable—we call the atlas smooth. We then define a smooth manifold as the topological manifold $\mathcal{M}$ together with a *maximal* smooth atlas $\mathcal{A}$. A smooth atlas is maximal if no coordinate function can be added while the atlas remains smooth.[2]

Consider two m- and n-dimensional smooth manifolds $\mathcal{M}$ and $\mathcal{N}$. Let x denoted the coordinates on $\mathcal{M}$, while y denotes the coordinates on $\mathcal{N}$. We can define smooth functions between these manifolds similarly to how we define smooth coordinates. Consider the function

$$F : \mathcal{M} \longmapsto \mathcal{N}. \tag{2.4}$$

It is said to be smooth if, for all points $p \in M$, there is a set of local coordinates x around p and y around $F(p)$ such that the map $\tilde{F} = y \circ F \circ x^{-1}$ is smooth. This map may be illustrated by a diagram,

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{F} & \mathcal{N} \\ \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle y} \\ \mathbb{R}^m & \xrightarrow{\tilde{F}} & \mathbb{R}^n \end{array} \tag{2.5}$$

We will not be careful with the distinction between F, the function between the abstract manifolds, and $\tilde{F}$, the function of their coordinates, but rather denote both by $F(x)$. We may take the partial derivative of such a function with respect to the coordinates x, $\partial F/\partial x^\mu$. However, this is dependent on our choice of coordinates, as a set of local coordinates can always be scaled arbitrarily. Any physical theory must be independent of our choice of coordinates, so our next task is to define the properties of a smooth manifold in a coordinate independent way.

[1] To be rigorous, one has to restrict the domains and image of the coordinate function when combining them. This is illustrated in Figure 2.2.

[2] The maximal condition ensres that two equivalent atlases correspond to the same differentiable manifold. A single manifold can be combined with different maximal atlases of smooth coordinates or differentiable structures. A set of examples are *exotic spheres*, smooth manifolds which are *homeomorphic* to S^n, but not *diffeomorphic*.

2.1.2 Vectors and tensors

A curve γ through $\mathcal{M}$ is a function from $\mathbb{R}$ to $\mathcal{M}$,

$$\gamma : \mathbb{R} \longmapsto \mathcal{M} \tag{2.6}$$
$$\lambda \longmapsto \gamma(\lambda). \tag{2.7}$$

Such curves are often denoted only by their coordinates and the parameter λ, $x^\mu(\lambda) = (x^\mu \circ \gamma)(\lambda)$. With this curve, we can take the directional derivative of a real-valued function on the manifold, $f : \mathcal{M} \mapsto \mathbb{R}$. Assume $\gamma(\lambda = 0) = p$. As we are always taking the derivative of functions between $\mathbb{R}^n$, for different n, we can use the chain rule. The directional derivative of f at p, given by this curve γ, is then

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} f(x(\lambda))\Big|_p = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\Big|_{\lambda=0} \frac{\partial}{\partial x^\mu} f(x)\Big|_p . \tag{2.8}$$

The set of all such directional derivatives, $\mathrm{d}/\mathrm{d}\lambda$ at p, form a vector space, $T_p\mathcal{M}$, called the *tangent space*. The tangent space is illustrated in Figure 2.3. The coordinates x^μ induce a basis of this vector space, namely partial derivatives with respect to the coordinate functions at p

$$e_\mu = \frac{\partial}{\partial x^\mu}\Big|_p = \partial_\mu|_p, \quad \mu \in \{0, ...n-1\}. \tag{2.9}$$

Any element $v \in T_p\mathcal{M}$ can therefore be written

$$v = v^\mu \partial_\mu|_p = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\Big|_{\lambda=0} \frac{\partial}{\partial x^\mu}\Big|_p . \tag{2.10}$$

Here, λ is the parameter of the curve corresponding to the directional derivative v.[3] The evaluation at $\lambda = 0$ and p will often be implicit for ease of notation. This directional derivative acts on functions $f : \mathcal{M} \mapsto \mathbb{R}$ as

$$v(f) = v^\mu \partial_\mu f. \tag{2.11}$$

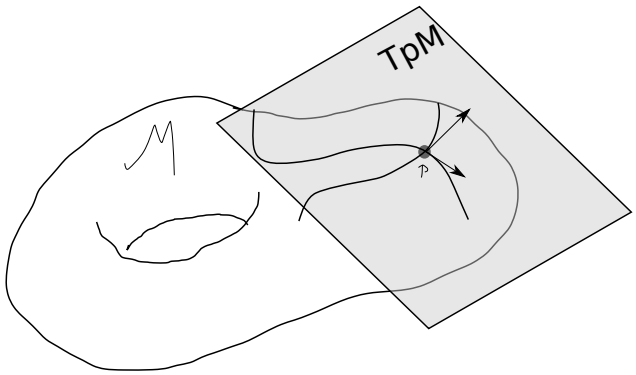

Figure 2.3: (Kladd) The tangent space $T_p\mathcal{M}$, the shaded rectangle, is the sett of all directional derivatives at $p \in \mathcal{M}$. A directional derivative is defined in terms of a curve that passes through p.

A map F between two manifolds $\mathcal{M}$ and $\mathcal{N}$ also induces a map between the tangent spaces of these manifolds. This is the *differential* of F at p,

$$\mathrm{d}F_p : T_p\mathcal{M} \longmapsto T_p\mathcal{N}, \tag{2.12}$$
$$v \longmapsto \mathrm{d}F_p(v). \tag{2.13}$$

As $\mathrm{d}F_p(v)$ is an element of $T_p\mathcal{N}$, directional derivative on $\mathcal{N}$, defined as

$$\mathrm{d}F_p(v)(g) = v(g \circ F), \tag{2.14}$$

[3] There is not only one curve corresponding to any directional derivative but rather an equivalence class. We will gloss over this technicality, as it does not affect our work.

for functions $g : \mathcal{N} \mapsto \mathbb{R}$. It thus acts on functions on $\mathcal{N}$ by "extending" the derivative v. This is a linear map between vector spaces and may be written in component form by considering the differentials of the coordinate functions. Denote the coordinates of $\mathcal{N}$ by y^μ, and $y^\mu \circ F = F^\mu$. Then,

$$\mathrm{d}F_p(\partial_\mu)(g) = \partial_\mu(g \circ F)|_p = \frac{\partial F^\nu}{\partial x^\mu}\Big|_p \frac{\partial g}{\partial y^\nu}\Big|_{F(p)}, \tag{2.15}$$

or more suggestively

$$\mathrm{d}F\left(\frac{\partial}{\partial x^\mu}\right) = \frac{\partial F^\nu}{\partial x^\mu}\frac{\partial}{\partial y_\nu}. \tag{2.16}$$

This is a linear map of vectors between two vectors by the matrix $A_\mu{}^\nu = \partial_\mu F^\nu$. The differential is thus a generalization of the Jacobian. In the case of a real valued function, $f : \mathcal{M} \mapsto \mathbb{R}$, and $g : \mathbb{R} \mapsto \mathbb{R}$, we get

$$\mathrm{d}f(v)(g) = v(g \circ f) = (v^\mu \partial_\mu f)\frac{\mathrm{d}g}{\mathrm{d}y}. \tag{2.17}$$

$\mathrm{d}f$ is thus a map from $T_p\mathcal{M}$ to $T_{f(p)}\mathbb{R}$, which is isomorphic to $\mathbb{R}$. The g be the identity function, so that $\mathrm{d}g/\mathrm{d}y = 1$. Then, the differential of a scalar function, also called a 1-form, is a map from vectors v to real numbers,

$$\mathrm{d}f(v) := v^\mu \partial_\mu f. \tag{2.18}$$

The set of all linear maps from a vector space V to the real numbers is called the *dual space* of V, denoted V^*. This is a new vector space with the same dimensionality as V. We denote the dual of $T_p\mathcal{M}$ as $T_p^*\mathcal{M}$. We can regard each coordinate function as a real-valued function with a corresponding differential. This differential obeys

$$\mathrm{d}x^\mu(\partial_\nu) = \frac{\partial x^\mu}{\partial x_\nu} = \delta^\mu_\nu. \tag{2.19}$$

The differentials of the coordinate functions thus form a basis for $T_p^*\mathcal{M}$, called the dual basis. Any differential $\mathrm{d}f$ can thus be written as $\mathrm{d}f = \omega_\mu \mathrm{d}x^\nu$ for some components ω_μ. We finde the components by applying the differential to the coordinate basis, $\mathrm{d}f(\partial_\mu) = \partial_\mu f = \omega_\mu$. In other words, we recover the classical expression

$$\mathrm{d}f = \frac{\partial f}{\partial x^\mu}\mathrm{d}x^\mu, \tag{2.20}$$

however we now interpret it as a covector-field instead of an "infinitesimal displacement".

Linear maps from vectors to real numbers is generalized by *tensors*. Given a vector space V, a general (n, m) tensor T is a multilinear map, which associates n elements from V and m from its dual V^* to the real numbers, i.e.,

$$T : V \times V \times \cdots \times V^* \times \ldots \longmapsto \mathbb{R}, \tag{2.21}$$

$$(v, u \ldots; \omega, \ldots) \longmapsto T(v, u, \ldots; \omega, \ldots). \tag{2.22}$$

Multilinear means that T is linear in each argument. The set of all such maps is the tensor product space $V \otimes V \otimes \cdots \otimes V^* \otimes \ldots$, a $\dim(V)^{n+m}$-dimensional vector space. If $\{e_\mu\}$ and $\{e^\mu\}$ are the basis for V and V^*, then we can write the basis of this of the tensor product space as $\{e_\mu \otimes \cdots \otimes e^\nu \otimes \ldots\}$. The tensor can thus be written

$$T = T^{\mu\nu\ldots}{}_{\rho\ldots}\, e_\mu \otimes e_\nu \otimes \ldots e^\rho \otimes \ldots, \tag{2.23}$$

where

$$T^{\mu\nu\ldots}{}_{\rho\ldots} = T(e^\mu e^\nu, \ldots; e_\rho, \ldots). \tag{2.24}$$

2.1.3 Geometries and the metric

The metric is a symmetric, non-degenerate $(0, 2)$ tensor

$$\mathrm{d}s^2 = g_{\mu\nu}\, \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu. \tag{2.25}$$

It defines the geometry of the manifold $\mathcal{M}$, and is the main object of study in general relativity. As it is invertible, we can define $g^{\mu\nu} = (g^{-1})_{\mu\nu}$, which is the components of a $(2,0)$ tensor. We use this to raise and lower indices, as is done with the Minkowski metric $\eta_{\mu\nu}$ in special relativity.

Up until now, we have only considered the tangent space $T_p\mathcal{M}$ at a point p and the corresponding tensor-product spaces. We are, however, more interested in *fields* of vectors, covectors, or tensors. For each point $p \in \mathcal{M}$, a tensor field T "picks out" a tensor $T(p)$ from each tensor product space corresponding to the tangent space at p, $T_p\mathcal{M}$. We will use a vector field to illustrate. This vector field can be written as

$$v(p) = v^\mu(p)\partial_\mu|_p. \tag{2.26}$$

We will mostly be working with the components v^μ, which are functions of $\mathcal{M}$. For ease of notation, we write the vector as a function of the coordinates x. The vector field $v(x)$ is unchanged by a coordinate-transformation $x^\mu \to x'^\mu$; the coordinates are only a tool for our convenience. However, with a new set of coordinates, we get a new set of basis vectors, ∂'_μ:

$$v = v^\mu\partial_\mu = v^\mu\frac{\partial x'^\nu}{\partial x^\mu}\partial'_\nu = v'^\mu\partial'_\mu, \tag{2.27}$$

This gives us the transformation rules for the components of vectors,

$$v'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}v^\nu. \tag{2.28}$$

Tangent vectors are also called *contravariant* vectors, as their components transform contra to basis vectors. For covectors, it is

$$\omega'_\mu = \frac{\partial x^\nu}{\partial x'^\mu}\omega_\nu, \tag{2.29}$$

which is why covectors also are called *covariant* vectors.

The gradient of a scalar function f, $\mathrm{d}f = \partial_\mu f \mathrm{d}x^\mu$, is a coordinate-independent derivative, as $\partial_\mu f$ follows the transformation law for covectors. We define the covariant derivative, ∇, as a map from (n,m) tensor fields to $(n,m+1)$ tensor fields. When considering a scalar as a $(0,0)$ tensor, we see that this generalizes the scalar derivative. The components of a covariant derivative, $\nabla_\rho T^{\mu_1\cdots}{}_{\nu_1,\ldots}$, must follow the tensor transformation law. However, this is not strong enough to uniquely define ∇. We further assume

- Linearity: $\nabla(T+S) = \nabla T + \nabla S$.
- The product rule: $\nabla(T\otimes S) = (\nabla T)\otimes S + T\otimes(\nabla S)$.
- Reduces to partial derivative for scalars: $\nabla_\mu f = \partial_\mu f$.
- Kronecker delta gives zero: $\nabla_\mu \delta^\rho_\nu = 0$.

With this, we can, in general, write the covariant derivative as [2]

$$\nabla_\mu v^\nu = \partial_\mu v^\mu + \Gamma^\mu_{\nu\rho}v^\rho, \tag{2.30}$$

$$\nabla_\mu \omega_\nu = \partial_\mu\omega_\nu - \Gamma^\rho_{\mu\nu}\omega_\rho, \tag{2.31}$$

for vectors and covectors. $\Gamma^\mu_{\nu\rho}$ are called *Christoffel symbols*. The generalization for higher-order tensors is straightforward,

$$\nabla_\mu T^{\nu\cdots}{}_{\rho\ldots} = \partial_\mu T^{\nu\cdots}{}_{\rho\ldots} + \Gamma^\mu_{\nu\lambda}T^{\lambda\cdots}{}_{\rho\ldots} + \cdots - \Gamma^\lambda_{\mu\rho}T^{\mu\cdots}{}_{\lambda\ldots} - \ldots. \tag{2.32}$$

This is still not enough to uniquely determine the covariant derivative. We will furthermore assume $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$ and $\nabla_\mu g_{\nu\rho} = 0$. With these, we can find an explicit formula of the Christoffel symbols in terms of the metric,

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} - \partial_\sigma g_{\mu\nu} + \partial_\nu g_{\sigma\mu}). \tag{2.33}$$

The curvature of a manifold $\mathcal{M}$, with the metic $g_{\mu\nu}$, is encoded in the Riemann tensor. It is defined by

$$[\nabla_\mu, \nabla_\nu]v^\rho = R^\rho{}_{\sigma\mu\nu}v^\sigma, \tag{2.34}$$

which in our case gives the explicit formula

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}. \tag{2.35}$$

Although the Christoffel symbols are not tensors, the Riemann tensor is due to its definition using covariant derivatives. We can therefore contract some of its indices to get other tensor quantities. We define the Ricci tensor and Ricci scalar as

$$R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}, \tag{2.36}$$

$$R = R^{\mu}{}_{\mu} = g^{\mu\nu} R_{\mu\nu}. \tag{2.37}$$

These are the quantities we need to start working with general relativity.

2.1.4 Integration on manifolds

The integral of a scalar function on a manifold is not a coordinate-independent notion, and we must introduce the notion of n-forms. A n-form is a antisymmetric $(0, n)$ tensor. To ease notation, we introduce the symmetrization of a tensor T,

$$T_{(\mu_1 \dots \mu_n)} = \frac{1}{n!} \sum_{\sigma \in S_n} T_{\mu_{\sigma(1)} \dots \mu_{\sigma(n)}}, \tag{2.38}$$

where S_n is the set of all permutations of n objects. The antisymmetrization of a tensor is defined as

$$T_{[\mu_1 \dots \mu_n]} = \frac{1}{n!} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) T_{\mu_{\sigma(1)} \dots \mu_{\sigma(n)}}. \tag{2.39}$$

The function $\sigma = \pm 1$, depending on if σ is a even or odd permutation.

We are interested in a coordinated independent quantity that we can integrate over. To that end, we define

$$\mathrm{d}^n x := \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{n-1}. \tag{2.40}$$

Here, $\wedge$ is the wedge product, defined as

$$(A \wedge B)_{\mu_1 \dots \mu_{n+m}} = \frac{(n+m)!}{n!m!} A_{[\mu_1 \dots \mu_n} B_{\mu_{n+1} \dots \mu_{n+m}]}, \tag{2.41}$$

Mer om wedge product

and $\mathrm{d}x^\mu$ is the one-form corresponding to the x^μ-coordinate function. Given a different set of coordinates, x'^μ, these are related by

$$\mathrm{d}^n x = \det\left(\frac{\partial x}{\partial x'}\right) \mathrm{d}^n x', \tag{2.42}$$

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by the properties of the wedge product. We define $|g| = |\det(g_{\mu\nu})|$, which, by the transformation properties of tensors, transforms as

$$\sqrt{|g'|} = \left|\det\left(\frac{\partial x'}{\partial x}\right)\right| \sqrt{|g'|}, \tag{2.43}$$

This means that we can use this to compensate for the transformation of $\mathrm{d}^n x$, and get a volume form with a coordinate independent expression,

$$\mathrm{d}V = \sqrt{|g|}\, \mathrm{d}^n x = \sqrt{|g'|}\, \mathrm{d}^n x'. \tag{2.44}$$

With this, we can integrate scalars in a well-defined way by mapping them to a corresponding n-form, $f \to f \mathrm{d}V$. We define the integral of a scalar function f on a manifold $\mathcal{M}$ with a metric g as

$$I = \int_{\mathcal{M}} \mathrm{d}V\, f = \int_{\mathcal{M}} \mathrm{d}^n x \sqrt{|g(x)|}\, f(x). \tag{2.45}$$

Stoke's theorem generalizes the fundamental theorem of calculus and the divergence theorem to manifolds. The most general statement of the theorem uses the exterior derivative, a map from n-forms to $n+1$-forms, defined by

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$$(\mathrm{d}T)_{\mu_1\ldots\mu_{n+1}} = (n+1)\partial_{[\mu_1}T_{\mu_2\ldots\mu_{n+1}]}. \tag{2.46}$$

Let $\mathcal{M}$ be a differential manifold of dimension n, with the boundary $\partial\mathcal{M}$. Stoke's theorem says that, for an $n-1$-form ω,

$$\int_{\mathcal{M}} \mathrm{d}\omega = \int_{\partial\mathcal{M}} \omega. \tag{2.47}$$

Stoke's theorem then implies a generalized divergence theorem. The boundary of $\mathcal{M}$ is a $n-1$ manifold dimensional, and a metric g on $\mathcal{M}$ will induce a new metric γ on $\partial\mathcal{M}$. This metric corresponds to the restriction of g to $\partial\mathcal{M}$. Furthermore, there will be a vector field n^μ of normalized vectors orthogonal to all elements of $T\partial\mathcal{M}$. This theorem states that for a vector field V^μ on $\mathcal{M}$,

$$\int_{\mathcal{M}} \mathrm{d}^n x \sqrt{|g|}\, \nabla_\mu V^\mu = \int_{\partial\mathcal{M}} \mathrm{d}^{n-1} y \sqrt{|\gamma|}\, n_\mu V^\mu. \tag{2.48}$$

2.2 Lie groups

Skriv om Lie grupper

Chapter 3

Quantum field theory and the equation of state

Skrive alt dette

3.1 QFT and the path integral formalism

3.2 Symmetries and Goldstones theorem

3.3 The CCWZ construction

Chapter 4

General relativity and the TOV equation

4.1 General relativity

(Newtonian gravity??)

Short about newtonian gravity/motivation

4.1.1 Einstein's field equations

General relativity describes spacetime as a smooth manifold $\mathcal{M}$, with a (pseudo-Riemannian) metric, $g_{\mu\nu}$. This metric is treated as a dynamical field, which is affected by the presence of matter and energy. The matter and energy contents of spacetime are encoded in the stress-energy tensor $T_{\mu\nu}$, while the behavior of $g^{\mu\nu}$ is encoded in a scalar Lagrangian density. Some of the mathematics used in this section, such as functional derivatives, are covered in section A.1.

The most obvious—and correct—choice as the Lagrangian for $g^{\mu\nu}$ is the Ricci scalar, which results in the Einstein-Hilbert action,

$$S_{\mathrm{EH}} = \frac{1}{2\kappa} \int_{\mathcal{M}} \mathrm{d}^n x \, \sqrt{|g|} \, R. \tag{4.1}$$

The $\sqrt{|g|}$-factor is included for the integral to be coordinate-independent, as discussed in subsection 2.1.4.[1] The κ is a constant and decides how strong the coupling of gravity to matter and energy is. This constant can then be related to Newton's constant of gravitation G by $\kappa = 8\pi G$. When including the contributions from other fields with an action S_{m}, the total action becomes

$$S = S_{\mathrm{EH}} + S_{\mathrm{m}}. \tag{4.2}$$

The equations of motion of the dynamical field, which in this case is the metric, are given by Hamilton's principle of stationary action. Using functional derivatives, as defined in subsection A.1.3, this is stated as

$$\frac{\delta S}{\delta g^{\mu\nu}} = 0, \tag{4.3}$$

We define the stress-energy tensor as

$$T_{\mu\nu} = -\frac{2}{\sqrt{|g|}} \frac{\delta S_{\mathrm{m}}}{\delta g^{\mu\nu}}. \tag{4.4}$$

[1] The gravitational action can also include a cosmological constant, modifying the Lagrangian to $R + 2\Lambda$. This constant does not affect the subject of this thesis and is therefore not included here.

The functional derivative of the Einstein-Hilbert action is evaluated in subsection A.1.4, and with the result, Eq. (A.26), we get the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu}, \tag{4.5}$$

The left-hand side of the Einstein field equations is called the Einstein tensor, $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$. This tensor obeys the identity

$$\nabla^\mu G_{\mu\nu} = 0, \tag{4.6}$$

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as a consequence of the more general Bianchi identity.

4.1.2 Spherically symmetric spacetime

To model stars, we will assume that the metric is spherically symmetric and time-independent. In this case, the most general metric can be written, at least locally, as [2]

$$\mathrm{d}s^2 = e^{2\alpha(r)}\mathrm{d}t^2 - e^{2\beta(r)}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2), \tag{4.7}$$

where α and β are general functions of the radial coordinate r. In matrix form, this corresponds to

$$g_{\mu\nu} = \begin{pmatrix} e^{2\alpha(r)} & 0 & 0 & 0 \\ 0 & -e^{2\beta(r)} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2(\theta) \end{pmatrix}. \tag{4.8}$$

Using Eq. (2.33), we can now compute the Christoffel symbols in terms of the unknown functions. These computations in this subsection are done using computer code, which is shown in Appendix B. The results are

$$\Gamma^t_{\mu\nu} = \begin{pmatrix} 0 & \frac{d}{dr}\alpha(r) & 0 & 0 \\ \frac{d}{dr}\alpha(r) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.9}$$

$$\Gamma^r_{\mu\nu} = \begin{pmatrix} e^{2\alpha(r)}e^{-2\beta(r)}\frac{d}{dr}\alpha(r) & 0 & 0 & 0 \\ 0 & \frac{d}{dr}\beta(r) & 0 & 0 \\ 0 & 0 & -re^{-2\beta(r)} & 0 \\ 0 & 0 & 0 & -re^{-2\beta(r)}\sin^2(\theta) \end{pmatrix}, \tag{4.10}$$

$$\Gamma^\theta_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\sin(\theta)\cos(\theta) \end{pmatrix}, \tag{4.11}$$

$$\Gamma^\phi_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \frac{\cos(\theta)}{\sin(\theta)} \\ 0 & \frac{1}{r} & \frac{\cos(\theta)}{\sin(\theta)} & 0 \end{pmatrix}. \tag{4.12}$$

The symbols not included are zero. Substituting these results into Eq. (2.35) gives the Riemann tensor curvature tensor. We can then obtain the Ricci tensor by taking the trace, as shown in Eq. (2.36). The results are

$$R_{tt} = \left(r\left(\frac{d}{dr}\alpha(r)\right)^2 - r\frac{d}{dr}\alpha(r)\frac{d}{dr}\beta(r) + r\frac{d^2}{dr^2}\alpha(r) + 2\frac{d}{dr}\alpha(r)\right)\frac{e^{2\alpha(r)}e^{-2\beta(r)}}{r}, \tag{4.13}$$

$$R_{rr} = -\frac{1}{r}\left(r\left(\frac{d}{dr}\alpha(r)\right)^2 - r\frac{d}{dr}\alpha(r)\frac{d}{dr}\beta(r) + r\frac{d^2}{dr^2}\alpha(r) - 2\frac{d}{dr}\beta(r)\right), \tag{4.14}$$

$$R_{\theta\theta} = -\left(r\frac{d}{dr}\alpha(r) - r\frac{d}{dr}\beta(r) - e^{2\beta(r)} + 1\right)e^{-2\beta(r)}, \tag{4.15}$$

$$R_{\varphi\varphi} = R_{\theta\theta}\sin^2(\theta). \tag{4.16}$$

All other components are zero. Finally, the trace of the Ricci tensor gives the Ricci scalar,

$$R = \frac{2e^{-2\beta(r)}}{r^2}\left[r^2\left(\frac{d}{dr}\alpha(r)\right)^2 - r^2\frac{d}{dr}\alpha(r)\frac{d}{dr}\beta(r) + r^2\frac{d^2}{dr^2}\alpha(r) + 2r\frac{d}{dr}\alpha(r) - 2r\frac{d}{dr}\beta(r) - e^{2\beta(r)} + 1\right]. \tag{4.17}$$

The unknown functions α and β are now determined by the matter and energy content of the universe, which is encoded in $T_{\mu\nu}$, through Einstein's field equation, Eq. (4.5).

Skriv om Schwarzchild-løsningen

4.2 The TOV equation

We will model a star as being made up of a *perfect fulid*, which is entirely described by its energy density u and pressure p. The relationship between the pressure and energy density of a substance is called the *equation of state*, or EOS, and has the form

$$f(p, u, \{\xi_i\}) = 0, \tag{4.18}$$

where $\{\xi_i\}$ are possible other thermodynamic variables. We will be working at zero temperature, in which case there are no other free thermodynamic variables. This allows us to, at least locally, express the energy density as a function of the pressure, $u = u(p)$. The stress-energy tensor of a perfect fluid is

Forklar

$$T_{\mu\nu} = (u + p)u_\mu u_\nu - p g_{\mu\nu}, \tag{4.19}$$

where u_μ is the 4-velocity of the fluid. In the rest frame of the fluid, we may write

$$v_\mu = (v_0, 0, 0, 0)\,. \tag{4.20}$$

This, together with the normalization condition of 4-velocities, $v_\mu v^\mu = 1$, allows us to calculate that

$$v_\mu v^\mu = g^{\mu\nu}v_\mu v_\nu = g^{00}(v_0)^2 = 1. \tag{4.21}$$

Using Eq. (4.8), we see that

$$v_0 = e^{\alpha(r)}. \tag{4.22}$$

This gives us the stress-energy tensor of the perfect fluid in its rest frame,

$$T_{\mu\nu} = \begin{pmatrix} u(r)e^{2\alpha(r)} & 0 & 0 & 0 \\ 0 & p(r)e^{2\beta(r)} & 0 & 0 \\ 0 & 0 & p(r)r^2 & 0 \\ 0 & 0 & 0 & p(r)r^2\sin^2(\theta) \end{pmatrix}. \tag{4.23}$$

We will use the tt and rr components of the Einstein field equations, which are

$$8\pi G r^2 u(r) e^{2\beta(r)} = 2r\frac{d}{dr}\beta(r) + e^{2\beta(r)} - 1 \tag{4.24}$$

$$8\pi G r^2 p(r) e^{2\beta(r)} = 2r\frac{d}{dr}\alpha(r) - e^{2\beta(r)} + 1. \tag{4.25}$$

In analogy with the Schwarzschild metric, we define the function $m(r)$ by

$$e^{2\beta(r)} = \left(1 - \frac{2Gm(r)}{r}\right)^{-1}. \tag{4.26}$$

Substituting this into Eq. (4.24) yields

$$\frac{\mathrm{d}m(r)}{\mathrm{d}r} = 4\pi r^2\, u(r). \tag{4.27}$$

The solution is simply

$$m(r) = 4\pi\int_0^r \mathrm{d}r'\, r'^2 u(r'). \tag{4.28}$$

We see that $m(r)$ is the matter content contained within a radius r. If $u = 0$ for $r > R$ and $m(r > R) = M$, then the metric on a constant-time surface, i.e. $\mathrm{d}t = 0$, is

Forlar forskjell på gravitasjons masse/ incl. bindingsenergi

$$\mathrm{d}s^2 = \left(1 - \frac{2GM}{r^2}\right)^{-1} \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2). \tag{4.29}$$

This is the same as for the Schwarzschild solution.

Using the Bianchi identity, Eq. (4.6), together with Einstein's equation, we find

$$\nabla^\mu G_{\mu\nu} = \nabla^\mu T_{\mu\nu} = 0. \tag{4.30}$$

The r-component of this equation is

$$\begin{aligned}\nabla_\mu T^{\mu r} &= \partial_r T^{rr} + \Gamma^\mu_{\mu\nu} T^{\nu r} + \Gamma^r_{\mu\nu} T^{\mu\nu} \\ &= \partial_r \left(p e^{-2\beta}\right) + (2\Gamma^r_{rr} + \Gamma^t_{tr}) T^{rr} + \Gamma^r_{tt} T^{tt} \\ &= e^{-2\beta}\left(\partial_r p + p\partial_r\alpha + u\partial_r\alpha\right) = 0.\end{aligned}$$

This allows us to relate α to p and u, via

$$\partial_r \alpha = -\frac{\partial_r p}{p + u} \tag{4.31}$$

When we substitute this, together with the definition of $m(r)$, into Eq. (4.25), we obtain

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{G}{r^2}\left(4\pi r^3 p + m\right)(p + u)\left(1 - \frac{2Gm}{r}\right)^{-1}, \tag{4.32}$$

the Tolman-Oppenheimer-Volkoff (TOV) equation. This equation was first obtained by Oppenheimer and Volkoff in 1939 [4] and was based on earlier work by Tolman [5]. In their paper, Oppenheimer and Volkoff studied the properties of a star made up of cold, degenerate fermions.

To summarize, we have three unknown functions, $u(r)$, $p(r)$, and $m(r)$. The equation of state, Eq. (4.18), determines $u = u(p)$, eliminating one unknown. The two differential equations Eq. (4.28) and Eq. (4.32), together with the boundary conditions $p(0) = p_c$ and $m(0) = 0$, then yield $p(r)$ and $m(r)$ when integrated. Given this, we can solve for all the unknown functions, either analytically or numerically. However, we can gain some insight into the system without solving these equations by expressing the problem in terms of dimensionless variables. We define

$$u = u_0\tilde{u}, \quad p = p_0\tilde{p}, \quad m = m_0\tilde{m}, \quad r = r_0\tilde{r}. \tag{4.33}$$

Here, quantities with subscript 0 are dimensionful constants, which may be chosen as the characteristic quantities of the problem, while the tilde indicates a dimensionless variable. By substituting this into Eq. (4.27) and Eq. (4.32), we can collect the dimensionful constants into a smaller number of dimensionless constants, k_i. These constants will decide the nature of the solution. Any change in the dimensionful constants that leave the k_i's invariant is a scaling of the problem — it corresponds to the same solution with different units. The new differential equations are

$$\frac{\mathrm{d}\tilde{m}}{\mathrm{d}\tilde{r}} = 3k_2\,\tilde{r}^2\tilde{u} \tag{4.34}$$

$$\frac{\mathrm{d}\tilde{p}}{\mathrm{d}\tilde{r}} = -\frac{k_1}{k_3}\frac{1}{\tilde{r}^2}\left(k_3\tilde{p} + \tilde{u}\right)\left(3k_2k_3\tilde{r}^3\tilde{p} + \tilde{m}\right)\left(1 - \frac{2k_1\tilde{m}}{\tilde{r}}\right)^{-1}, \tag{4.35}$$

Hvor er det best å plassere faktor 3?

where the dimensionless constants are defined as

$$k_1 = G\frac{m_0}{r_0}, \quad k_2 = \frac{4\pi}{3}\frac{r_0^3 u_0}{m_0}, \quad k_3 = \frac{p_0}{u_0}. \tag{4.36}$$

The energy density and pressure are of comparable magnitude in the relativistic regime. We will therefore often choose $k_3 = 1$, defining $p_0 = u_0$. If we have a non-complete set of characteristic quantities, the dimensionless constants k_i tell us something about the magnitude we should expect the solution to have. After defining the remaining dimensionful constants by setting $k_i = 1$, we expect that the dimensionless sizes of a typical solution will be of order 1. In other words, the dimensionful constants defined by $k_i = 1$ are new, characteristic quantities given to us by the form of the governing equation only.

4.2.1 Newtonian limit

In the Newtonian limit, the rest energy, i.e., mass, is the dominant contribution to the gravitational field. This means that the characteristic pressure, p_0, is far smaller than the characteristic energy density u_0, and we can use the approximation $k_3 \ll 1$. Furthermore, the radius of the star should be much larger than the Schwarzschild radius, $R_s = 2GM$. If we choose $r_0 = R$, then $k_1 \ll 1$. In this limit, the lowest-order contribution to the TOV equation is

$$\frac{\mathrm{d}\tilde{p}}{\mathrm{d}\tilde{r}} = -\frac{k}{\tilde{r}^2}\tilde{u}\tilde{m}, \quad k = \frac{k_1}{k_3} = G\frac{u_0 m_0}{p_0 r_0}. \tag{4.37}$$

Using the mass equation Eq. (4.34), we can write this as

$$4\pi\tilde{r}^2\frac{\mathrm{d}\tilde{p}}{\mathrm{d}\tilde{m}} = -k'\frac{\tilde{m}}{\tilde{r}^2}, \quad k' = \frac{4\pi}{3}\frac{k_1}{k_2 k_3} = G\frac{m_0^2}{r_0^4 p_0}. \tag{4.38}$$

This equation has a simple interpretation; the force due to the pressure gradient over a thin, spherical shell, $F_p = 4\pi r^2 \mathrm{d}p$, must be counteracted by the gravitational force on the same shell, $F_g = -Gm\mathrm{d}m/r^2$. This is illustrated in Figure 4.1. Both the Newtonian limit and the TOV equation are equations of *hydrostatic equilibrium*, where the forces on a small volume of the fluid cancel out. In the case of the TOV equation, we tacitly assumed hydrostatic equilibrium when we gave the fluid a rest frame where we could write $v_\mu = (v_0, 0, 0, 0)$ globaly.

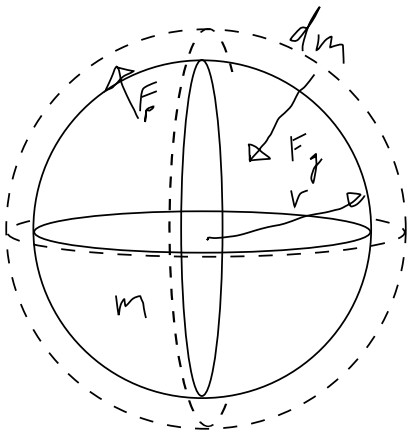

Figure 4.1: Kladd: The forces acting on a thin shell dm.

We can eliminate the equation for mass by differentiating Eq. (4.37) with respect to $\tilde{r}$. This gives us a single equation for hydrostatic equilibrium in the Newtonian limit,

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{r}}\left(\frac{\tilde{r}^2}{\tilde{u}}\frac{\mathrm{d}\tilde{p}}{\mathrm{d}\tilde{r}}\right) = -k''\tilde{r}^2\tilde{u}, \quad k'' = 3\frac{k_2 k_1}{k_3} = 4\pi G\frac{u_0^2 r_0^2}{p_0}. \tag{4.39}$$

4.2.2 Incompressible fluid

The simplest model for a star is one made up of an incompressible fluid, where the energy density is independent of the pressure. In this case, the energy density of the star will be constant for a radius $r < R$, before it drops to zero,

$$u(r) = u_0\,\theta(R - r), \tag{4.40}$$

where u_0 is a constant and $\theta(x)$ the Heaviside step function. We choose $r_0 = R$. Inserting this into the differential equation of the mass function, Eq. (4.34), together with the boundary condition $m(0) = 0$, yields

$$\tilde{m}(\tilde{r}) = k_2\tilde{r}^3, \tag{4.41}$$

when $r < R$. For $r \geq R$, or $\tilde{r} \geq 1$, this relationship is simply constant $\tilde{m}(\tilde{r}) = \tilde{m}(1) = k_2$. We choose m_0 to be the gravitational mass of the star, $M = \frac{4\pi}{3} R^3 u_0$, which is equivalent to setting $k_2 = 1$. Lastly, we choose $u_0 = p_0$, so that $k_3 = 1$. With this the TOV equation, Eq. (4.35), becomes

$$\frac{\mathrm{d}\tilde{p}}{\mathrm{d}\tilde{r}} = -k_1 \tilde{r} \frac{(1+\tilde{p})(1+3\tilde{p})}{(1-2k_1\tilde{r}^2)}. \tag{4.42}$$

This is a separable ODE, and each variable may be integrated separately. Using

$$\int \frac{\mathrm{d}x}{(1+x)(1+3x)} = \frac{1}{2} \ln \frac{3x+1}{x+1} + \text{const.}, \quad \int \mathrm{d}x \frac{x}{1-2x^2} = \frac{1}{4} \ln\left(1-2x^2\right) + \text{const.}, \tag{4.43}$$

together with the boundary condition $p(r = R) = 0$, we get

$$\tilde{p}(\tilde{r}) = -\frac{\sqrt{1-2k_1} - \sqrt{1-2k_1\tilde{r}^2}}{3\sqrt{1-2k_1} - \sqrt{1-2k_1\tilde{r}^2}}. \tag{4.44}$$

We see that the star is entirely characterized by k_1. In Figure 4.2, we have plotted the pressure as a function of radius for some values of k_1. As k_1 approaches $0.\bar{4} = 4/9$, the pressure at the center of the star increases rapidly. From the denominator of Eq. (4.44) at $r \to 0$, we find the limit

$$k_1 = G\frac{M}{R} < \frac{4}{9} \tag{4.45}$$

for the pressure to remain finite. This is an absolute limit of the mass of an object given its radius or vice versa. Although this limit is derived for a particular, unrealistic case, the more general statement can be shown to hold. General relativity does not allow for a static solution with energy densities greater than this; any such configuration would collapse [2].

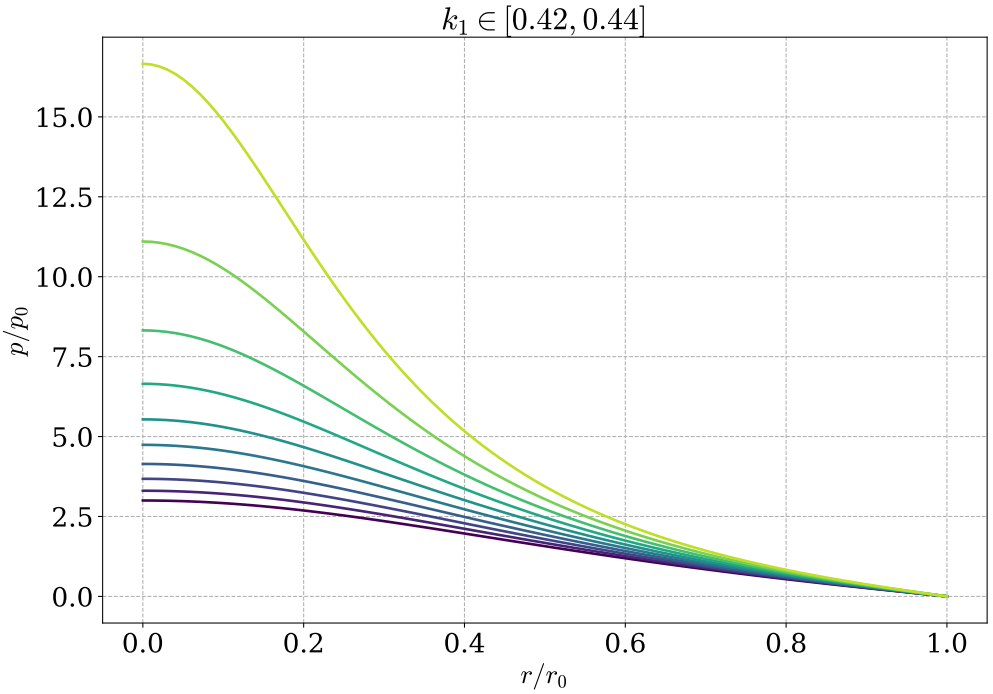

Figure 4.2: The pressure in units of p_0, as a function of the radius, in units of r_0. The graphs with lighter color and higher pressure at $r = 0$ corresponds to higher values of k_1. The values of k_1 are linearly spaced.

If we expand the solution Eq. (4.44) in powers of k_1, then the leading order contribution is

$$\tilde{p}(r) = \frac{1}{2} k_1 (1 - \tilde{r}^2). \tag{4.46}$$

This is the Newtonian limit. As a cross-check, we see that this solution obeys the equation of hydrostatic equillibrium in this limit, Eq. (4.37), as $\tilde{u} = 1$ and $k_2 = k_1 = 1$. This is the general solution for an incompressible fluid in Newtonian gravity. This solution does not have any upper limit for k_1; the limit $M/R < 4/9$ is purely relativistic phenomenon. In Figure 4.3, the Newtonian approximation is compared to the full, relativistic solution. We see that the Newtonian approximation is highly accurate for k_1 less than around 0.01.

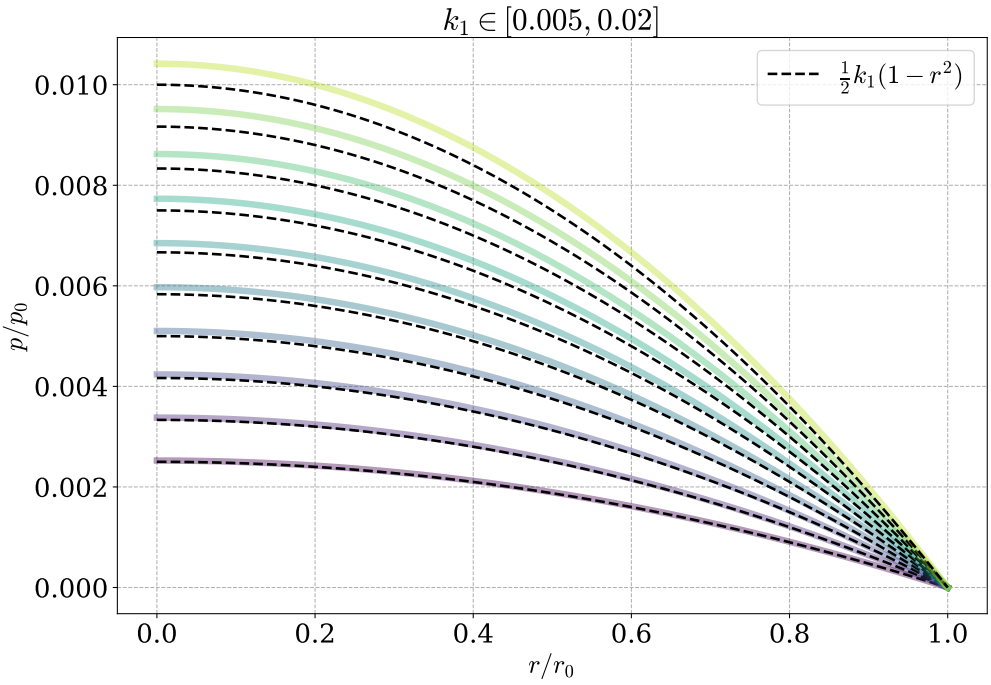

Figure 4.3: The pressure in units of p_0, as a function of the radius, in units of r_0. The wide, colored lines correspond to the full relativistic solution, while the dashed lines is the Newtonian approximation, for the same value of k_1. The values of k_1 are linearly spaced.

4.3 A star of cold, non-interacting fermions

Ekstra kilder: [6, 7]

Ha kilder på riktig sted

4.3.1 Thermodynamics and free energy

A non-interacting Fermi gas is governed by the Dirac Lagrangian

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi, \tag{4.47}$$

as described in (APPENDIX thermal field theory). In the grand canoncial ensamble, the density of a conserved charge is regulated by a chemical potential μ. The conserved current corresponding to the U(1) phase symmetry of the Dirac Lagrangian, i.e., the transformation $\psi \to e^{i\alpha}\psi \approx 1 + \alpha\delta\psi$, is

add appendix

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta\psi = \bar{\psi}\gamma^\mu\psi. \tag{4.48}$$

The conserved charge is

$$Q = \int \mathrm{d}^3x\, j^0 = \int \mathrm{d}^3x\, \bar{\psi}\gamma^0\psi, \tag{4.49}$$

the number of particles minus antiparticles. The total energy U is related to the grand canonical free energy F by a Legendre transformation,

$$F(T,V,\mu) = U - TS - \mu Q, \quad \mathrm{d}F = -S\mathrm{d}T - p\mathrm{d}V - Q\mathrm{d}\mu. \tag{4.50}$$

Here, p is pressure, $T = 1/\beta$ is temperature, and S entropy. These thermodynamic variables are related to the free energy by

$$S = -\frac{\partial F}{\partial T} = \beta^2 \frac{\partial F}{\partial \beta}, \quad Q = -\frac{\partial F}{\partial \mu}, \quad p = -\frac{\partial F}{\partial V}. \tag{4.51}$$

When the free energy can be written as $F = V\mathcal{F}$, where the free energy density $\mathcal{F}$ is independent of the volume V, then $\mathcal{F} = -p$ and

$$\mathrm{d}(V\mathcal{F}) = V\mathrm{d}\mathcal{F} - p\mathrm{d}V, \tag{4.52}$$

allowing us to write

$$\mathcal{F}(T,\mu) = u - Ts - \mu n, \quad \mathrm{d}\mathcal{F} = -s\mathrm{d}T - n\mathrm{d}\mu, \tag{4.53}$$

where s and n are entropy and charge density, defined by

$$s = -\frac{\partial \mathcal{F}}{\partial T} = \beta^2 \frac{\partial \mathcal{F}}{\partial \beta}, \quad n = -\frac{\partial \mathcal{F}}{\partial \mu}. \tag{4.54}$$

With this, we can write the energy density as

$$u = \frac{\partial}{\partial \beta}(\beta\mathcal{F}) + \mu n. \tag{4.55}$$

4.3.2 Equation of state

inkluder appendix om termisk felt-teori

The free energy density of the Fermion gas is (REF: appendix felt teori)

$$\mathcal{F} = -\frac{2}{\beta}\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\left[\beta\omega + \ln\left(1 + e^{-\beta(\omega-\mu)}\right) + \ln\left(1 + e^{-\beta(\omega+\mu)}\right)\right], \tag{4.56}$$

where $\omega = \sqrt{p^2 + m^2}$. The first term in the integral is the divergent vacuum energy, which must be renormalized. We can drop this term; it does not have any observable effects on our results, as we are interested in relative pressure and energy density. With this, we find the charge density

$$n = \frac{1}{\pi^2}\int \frac{\mathrm{d}^3 p}{(2\pi)^3}[n_f(\omega-\mu) - n_f(\omega+\mu)], \tag{4.57}$$

where

$$n_f(\omega) = \frac{1}{e^{\beta\omega} + 1}. \tag{4.58}$$

is the Fermi-Dirac distribution. The energy density is

$$u = \frac{1}{\pi^2}\int_0^\infty \mathrm{d}p\, p^2\, \omega\, [n_f(\omega-\mu) + n_f(\omega+\mu)]. \tag{4.59}$$

As expected, this is the energy per mode times the density of states, integrated over all modes. To write the pressure, $p = -\mathcal{F}$ in terms of an integral over the Fermi-Dirac distribution, we integrate by parts. We have

$$\int_0^\infty \mathrm{d}p\, p^2 \ln\left[1 + e^{-\beta(\omega\pm m)}\right] = \frac{1}{3}p^3 \ln\left[1 + e^{-\beta(\omega\pm m)}\right]\Big|_0^\infty + \frac{1}{3}\int_0^\infty \mathrm{d}p\, \frac{\beta p^4}{\omega} n_f(\omega \pm \mu), \tag{4.60}$$

where the boundary term vanish. This allows us to write the pressure as

$$p = \frac{1}{3}\int_0^\infty \mathrm{d}p\, \frac{p^4}{\omega}[n_f(\omega-\mu) + n_f(\omega+\mu)] \tag{4.61}$$

We are interested in the $T = 0$ limit. In this case, the Fermi distribution becomes a step function, $n_f(\omega) = \theta(-\omega)$. Without loss of generality, we assume that $\mu > 0$, i.e., we are dealing with an abundance of matter compared to anti-matter. The dispersion relation $\omega = \sqrt{p^2 + m^2}$ is always positivive. This means that the contribution to thermodynamic quantities from anti-particles vanish, as the integral is multiplied with $n_f(\omega + \mu) = \theta(-\omega - \mu)$, where the argument $-\omega - \mu$ is strictly negative on the domain of integration. At zero temperature, the only dynamics are due to the degeneracy pressure of the fermions, that is, due to the Pauli exclusion principle. There are no thermal fluctuations that can create a particle-antiparticle pair. Thus, if the system has a positive chemical potential, it will contain no antiparticles. Furthermore, if $\mu < m$, then integrand multiplied with $n_f(\omega - \mu)$ is also zero in the whole domain of integration. It is only when $\mu \geq m$ that it is energetically favorable for the system to be in a state with particles. We define the Fermi momentum p_f by $\mu = \sqrt{{p_f}^2 + m^2}$. In the zero-temperature limit, we can then rewrite any integral over the Fermi distribution as

$$\int_0^\infty \mathrm{d}p\, [f(p) n_f(\omega - \mu) + g(p) n_f(\omega - \mu)] = \int_0^{p_f} \mathrm{d}p\, f(p). \tag{4.62}$$

The charge density is thus,

$$n = \frac{1}{\pi^2} \int_0^{p_f} \mathrm{d}p\, p^2 = \frac{p_f^3}{3\pi^2}. \tag{4.63}$$

At $T = 0$, this is the particle number density, as there are no antiparticles. This density is given by the chemical potential and vanishes when $\mu \leq m$, i.e. when the free energy cost of creating a particle is positive. We can write the energy density and pressure integrals, Eq. (4.59) and Eq. (4.61), as

$$u = \frac{1}{\pi^2} \int_0^{p_f} \mathrm{d}p\, p^2 \sqrt{p^2 + m^2} = \frac{m^4}{\pi^2} \int_0^{x_f} \mathrm{d}x\, x^2 \sqrt{x^2 + 1}, \tag{4.64}$$

$$p = \frac{1}{3\pi^2} \int_0^{p_f} \mathrm{d}p\, \frac{p^4}{\sqrt{p^2 + m^2}} = \frac{m^4}{3\pi^2} \int_0^{x_f} \frac{\mathrm{d}x\, x^4}{\sqrt{x^2 + 1}}. \tag{4.65}$$

We have defined $x = p/m$ and $x_f = p_f/m$. These integrals can be evaluated exactly as

$$\int_0^a \mathrm{d}x\, x^2 \sqrt{x^2 + 1} = \frac{1}{8} \left[\sqrt{a^4 + 1}(2a^3 + a) - \operatorname{arcsinh}(a) \right], \tag{4.66}$$

$$\int_0^a \mathrm{d}x\, \frac{x^4}{\sqrt{x^2 + 1}} = \frac{1}{8} \left[\sqrt{a^2 + 1}(2a^3 - 3a) + 3 \operatorname{arcsinh}(a) \right]. \tag{4.67}$$

We introduce the characteristic energy and number density,

$$u_0 = \frac{m^4}{8\pi^2}, \quad n_0 = \frac{u_0}{m}, \tag{4.68}$$

which allows us to write the thermodynamic variables as

$$n = \frac{8}{3} n_0\, x_f^3 \tag{4.69}$$

$$u = u_0 \left[(2x_f^3 + x_f) \sqrt{1 + x_f^2} - \operatorname{arcsinh}(x_f) \right], \tag{4.70}$$

$$p = \frac{u_0}{3} \left[(2x_f^3 - 3x_f) \sqrt{1 + x_f^2} + 3 \operatorname{arcsinh}(x_f) \right]. \tag{4.71}$$

We have thus chosen $u_0 = p_0$, or equivalently set $k_3 = 1$. This is natural in the case of a relativistic fluid.

4.3.3 Units

The equation of state has given us the characteristic energy density and pressure, u_0 and p_0. If we demand

$$G \frac{m_0}{r_0} = \frac{4\pi}{3} \frac{r_0^3 u_0}{m_0} = 1, \tag{4.72}$$

we have two equations and two unknowns, m_0 and r_0. This thus defines a complete set of units. We are using the cold Fermi-gas as a model for a neutron star, an the mass of the fermion m is therefore the neutron mass, Eq. (1.7), $m_N = 1.674 \cdot 10^{-27}\,\mathrm{kg}$. After reinstating $\hbar$ and c in metric units, we get

$$u_0 = p_0 = \frac{m^4}{8\pi^2}\frac{c^5}{\hbar^3} = 2.032 \cdot 10^{35}\,\mathrm{J\,m^{-3}}, \tag{4.73}$$

$$m_0 = \frac{c^4}{\sqrt{\frac{4\pi}{3}u_0 G^3}} = 1.608 \cdot 10^{31}\,\mathrm{kg} = 8.082\,M_\odot \tag{4.74}$$

$$r_0 = \frac{Gm_0}{c^2} = 11.93\,\mathrm{km}. \tag{4.75}$$

From this, we expect our star to have a mass of the order of a solar mass, $M_\odot = 1.98841 \cdot 10^{30}\,\mathrm{kg}$ [1], and a radius of the order of kilometers, without solving the TOV equation.

4.3.4 Limits

In the non-relativistic limit, as the chemical potential approaches m and thus $p_f \ll m$, the lowest order contributions to the energy density and pressure are given by the Taylor series around $x_f = 0$,

$$\tilde{u}(x_f) = \frac{8}{3}x_f^3 + \frac{4}{5}x_f^5 + \mathcal{O}(x_f^7), \tag{4.76}$$

$$\tilde{p}(x_f) = \frac{8}{5}x_f^5 + \mathcal{O}(x_f^7). \tag{4.77}$$

By neglecting terms of order x_f^7 and higher, we can write this as

$$\tilde{u} = \tilde{n} + \frac{4}{5}\left(\frac{8}{3}\tilde{n}\right)^{5/3}, \quad \tilde{p} = \frac{8}{5}\left(\frac{8}{3}\tilde{n}\right)^{5/3}. \tag{4.78}$$

The leading order contribution to the energy density is the rest mass of the particles. This term does not contribute to the pressure. As discussed earlier, the non-relativistic limit corresponds to $k_3 \ll 1$, if we chose units so that $\tilde{u} \approx \tilde{p}$, or $\tilde{u} \gg \tilde{p}$ if we demand that $k_3 = 1$. We see that $x_f \to 0$ corresponds to the latter case. By including only the leading order term, we can eliminate the Fermi momentum and write the equation of state as $u = kp^{3/5}$ where $k = 8/3 \cdot (5/8)^{3/5}$. Equations of state where $u \propto p^\gamma$ are called polytropes. In the ultrarelativistic limit, where $p_f \gg m$, the leading order contributions to the pressure and energy density are

$$\tilde{u} = 2x_f^4, \quad \tilde{p} = \frac{2}{3}x_f^4, \tag{4.79}$$

and we get the particularly simple equation of state $u = 3p$, which we recognize as the formula for radiation pressure.

4.3.5 Numerical results

With the energy density, Eq. (4.70), and pressure, Eq. (4.71), we can numerically solve the TOV equation given a central pressure p_c. This is done using an adaptive Runge-Kutta method, with the stop criterion $p(r) = 0$. Description of the code and where to find it is given in Appendix B.

As $r \to 0$, parts of the TOV equation Eq. (4.35) diverge, and we must take use an approximation for numeric evaluation. The Taylor-expansion of the mass function around $\tilde{r} = 0$ is

$$\tilde{m}(r) = \tilde{m}(0) + \tilde{m}'(0)\,\tilde{r} + \frac{1}{2!}\tilde{m}''(0)\tilde{r}^2 + \frac{1}{3!}\tilde{m}'''(0)\tilde{r}^3 + \mathcal{O}\left(\tilde{r}^4\right). \tag{4.80}$$

One of the boundary conditions is $\tilde{m}(0) = 0$. We then use the differntial equation for $\tilde{m}$, Eq. (4.27), to find

$$\tilde{m}'(0) = 0, \quad \tilde{m}''(0) = 0, \quad \tilde{m}'''(0) = 6k_2\tilde{u}_0, \tag{4.81}$$

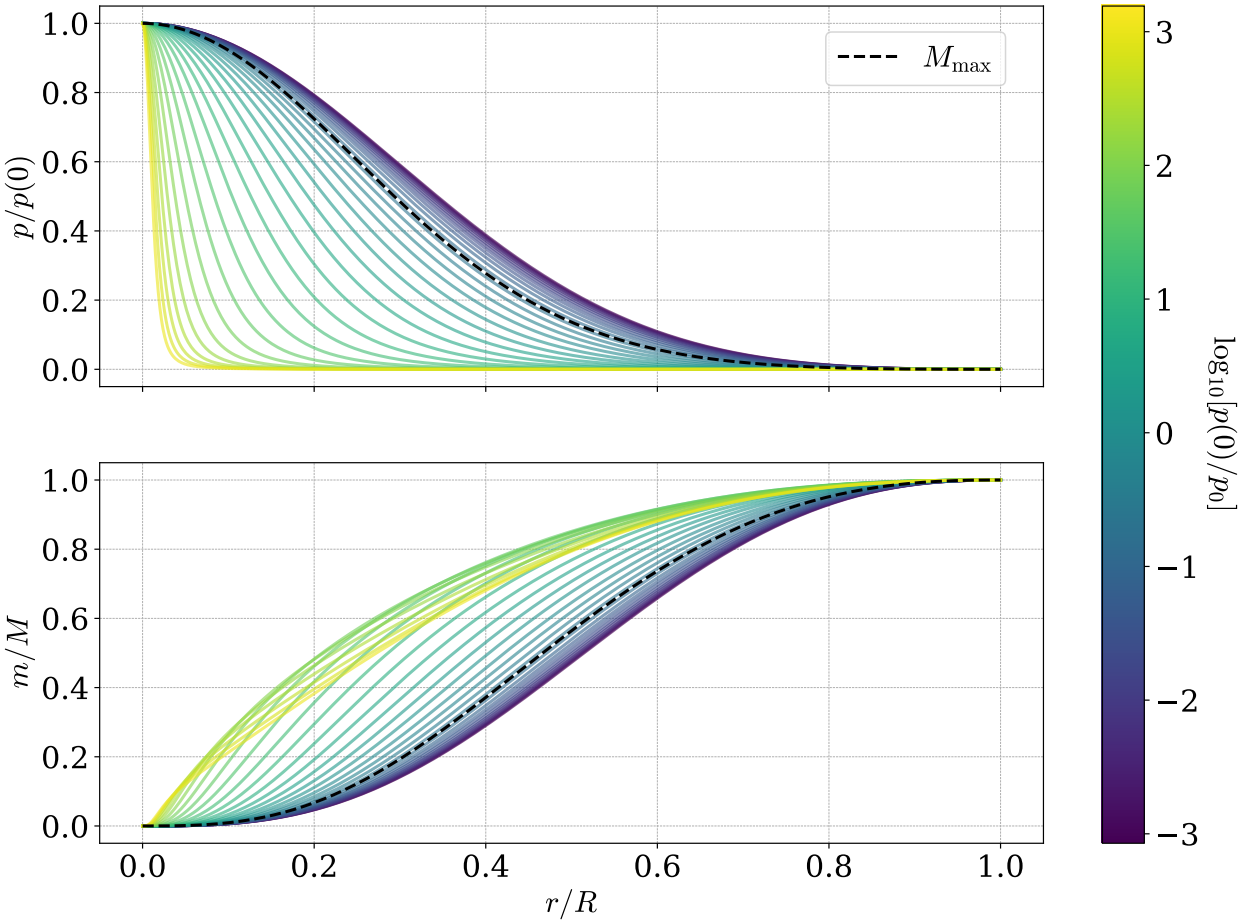

Figure 4.4: The pressure noramlized to central value (top) and the mass normalized to total mass (bottom), as a function of radius, nomralized to total radius. This is plotted for several different values of central pressure, which is indicated by the color shceme.

where $\tilde{u}_0 = \tilde{u}(r = 0)$. We get an approximation of the TOV equation for $\tilde{r} \ll 1$ by substituting the $\tilde{m}$ for its Taylor expansion and including only the leading-order term, which gives

$$\frac{\mathrm{d}\tilde{p}}{\mathrm{d}\tilde{r}} = -k_1 k_2 \tilde{r}\ (\tilde{p} + \tilde{u})\,(3\tilde{p} + \tilde{u}_0)\left(1 - 2k_1\tilde{u}_0\tilde{r}^2\right)^{-1}. \tag{4.82}$$

With this, we can find the pressure as a function of radius r for any given central pressure, $p(0) = p_0$. The stellar radius, R, is defined as the point where the pressure drops to zero, i.e., $p(R) = 0$. The top graph in Figure 4.4 shows the pressure, normalized to the central pressure p_c, as a function of radius, normalized to the corresponding stellar radius R. The boundary conditions are logarithmically spaced. The lower graph in Figure 4.4 shows the mass, normalized to the total mass $M = m(R)$, as a function of the radius, again normalized to the stellar radius. As in the case of the incompressible fluid, the pressure follows a half bell-shaped curve, with a peak that becomes sharper as the central pressure increases. The black dashed line corresponds to the solution with the maximum mass. We see that the shape changes most drastically for central pressures higher than that which corresponds to the most massive star.

In Figure 4.5, we see the relationship between the mass and radius. This line is parameterized by the base-10 logarithm of the central pressure, $p(0)$, normalized by $p_0 = u_0$. The cross marks the maximum mass, $M_{\max} = 0.711\,M_\odot$, which corresponds to a radius of $R = 9.20\,\mathrm{km}$. This matches the results obtained by Oppenheimer and Volkoff [4], $M_{\max} = 0.71$. In their 1939 paper, Oppenheimer and Volkoff computed five data-points in the mass-radius space. The results are marked by blue circles in Figure 4.5. We find good agreement between the three results closest to the maximum value and our result for the mass-radius relationship. However, the two results further away differr significantly from our results. The black dashed

Hvorfor er to punkter så langt unna?

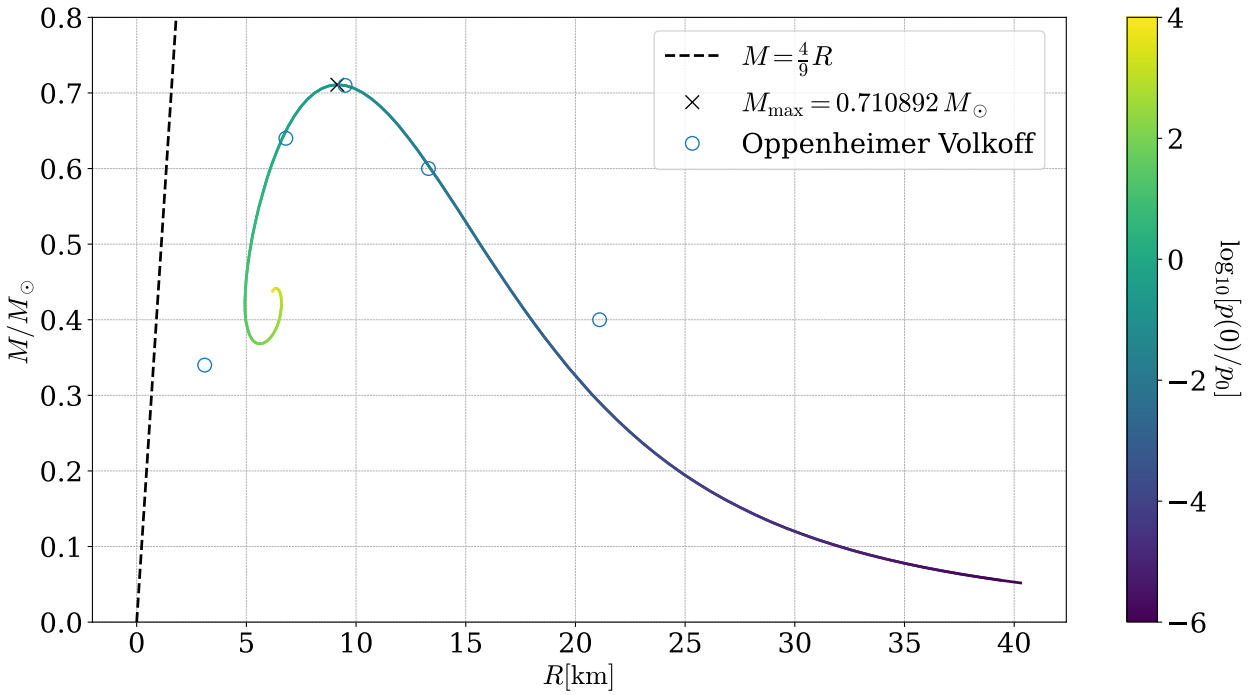

Figure 4.5: The mass-radius relationship of a star made of a cold gas of neutrons. The line is parametrized by the boundary condition $p(0)$. The corss indicate the maximum mass solution. The blue circles are results form the 1939 paper of Oppenheimer and Volkof [4].

line is the absolute mass-radius constraint, Eq. (4.45), and any stable configuration must be on the right side of this line.

Appendix A

A.1 Functionals

(TODO: INKLUDER KILER!!!!)

The principle of stationary action and the path integral method relies on functional calculus, where ordinary, n-dimensional calculus is generalized to an infinite-dimensional calculus on a space of functions. A functional, S, takes in a function $\varphi(x)$, and returns a real number $S[\varphi]$. We will be often be dealing with functionals of the form

$$S[\varphi] = \int_{\mathcal{M}} \mathrm{d}^n x \, \mathcal{L}[\varphi](x), \tag{A.1}$$

Here, $\mathcal{L}[\varphi](x)$, the Lagrangian density, is a functional which takes in a function φ, and returns a real number $\mathcal{L}[\varphi](x)$ *for each point* $x \in \mathcal{M}$. Thus, $\mathcal{L}$ does, in fact, return a real-valued function, not just a number. $\mathcal{M}$ is the manifold, in our case space-time, of which both $\varphi(x)$ and $\mathcal{L}[\varphi](x)$ are functions. The function φ can, in general, might take on the value of a scalar, complex number, spinor, vector, etc…, while $\mathcal{L}[\varphi](x)$ must be a scalar-valued function. This lays a strong constraint on the form of any Lagrangian and is an essential tool in constructing quantum field theories. Although this section is written with a single scalar-valued function φ, this can easily be generalized by adding an index, $\varphi \to \varphi_\alpha$, enumerating all the degrees of freedom, then restating the arguments.

A.1.1 Functional derivative

The functional derivative is base on an arbitrary *variation* η of the function φ. The variation η, often written $\delta\varphi$ is an arbitrary function only constrained to vanish *quickly enough* at the boundary $\partial\mathcal{M}$.[1] The variation of the functional S is defined as

$$\delta_\eta S[\varphi] = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(S[\varphi + \epsilon\eta] - S[\varphi] \right) = \frac{\mathrm{d}}{\mathrm{d}\epsilon} S[\varphi + \epsilon\eta]|_{\epsilon=0}. \tag{A.2}$$

We can regard the variation of a functional as the generalization of the differential of a function, Eq. (2.18), as the best linear approximation around a point. In regular differential geometry, a function f can be approximated around a point x by

$$f(x + \epsilon v) = f(x) + \epsilon \mathrm{d}f(v), \tag{A.3}$$

where v is a vector in the tangent space at x. In functional calculus, the functional S is analogous to f, φ to x, and η to v. We can more clearly see the resemblance by writing

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon} f(x + \epsilon v) = \mathrm{d}f(v) = \frac{\partial f}{\partial x^\mu} v^\mu. \tag{A.4}$$

In the last line we expanded the differential using the basis-representation, $v = v^\mu \partial_\mu$. To generalize this to functional, we define the *functional derivative*, by

$$\delta_\eta S[\varphi] = \int_{\mathcal{M}} \mathrm{d}^n x \, \frac{\delta S[\varphi]}{\delta \eta(x)} \eta(x). \tag{A.5}$$

[1] The condition of "q1uickly enough" is to ensure that we can integrate by parts and ignore the boundary condition, which we will do without remorse.

If we let $S[\varphi] = \varphi(x)$, for some fixed x, the variation becomes

$$\delta_\eta[\varphi] = \eta(x) = \int \mathrm{d}^n y\, \delta(x-y)\eta(y), \tag{A.6}$$

which leads to the identity

$$\frac{\delta\varphi(x)}{\delta\varphi(y)} = \delta(x-y). \tag{A.7}$$

There is also a generalized chain rule for functional derivatives. Let ψ be some function, which may be written in terms of φ. Then chain rule is

$$\frac{\delta S[\varphi]}{\delta\varphi(x)} = \int_{\mathcal{M}} \mathrm{d}^n y\, \frac{\delta S[\varphi]}{\delta\psi(y)} \frac{\delta\psi(y)}{\delta\varphi(x)}. \tag{A.8}$$

Higher functional derivatives are defined in terms of higher-order variations,

$$\delta_\eta^m S[\varphi] = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\delta_\eta^{m-1} S[\varphi+\epsilon\eta]|_{\epsilon=0} = \int_{\mathcal{M}} \left(\prod_{i=1}^m \mathrm{d}^n x_i \eta(x_i)\right) \frac{\delta^m S[\varphi]}{\delta\varphi(x_1)...\delta\varphi(x_m)}. \tag{A.9}$$

With this, we can write the functional Taylor expansion,

$$S[\varphi_0+\varphi] = S[\varphi_0] + \int_{\mathcal{M}} \mathrm{d}^n x\, \varphi(x) \frac{\delta S[\varphi_0]}{\delta\varphi(x)} + \frac{1}{2}\int_{\mathcal{M}} \mathrm{d}^n x \mathrm{d}^n y\, \varphi(x)\varphi(y) \frac{\delta^2 S[\varphi_0]}{\delta\varphi(x)\,\delta\varphi(y)} + \dots \tag{A.10}$$

Here, the notation $\delta S[\varphi_0]/\delta\varphi$ indicate that $S[\varphi]$ is first differentiated with respect to φ, then evaluated at $\varphi = \varphi_0$.

A.1.2 The Euler-Lagrange equation

The Lagrangian may also be written as a scalar function of the field-values at x, $\varphi(x)$, as well as its derivatives, $\partial_\mu\varphi(x)$, for example

$$\mathcal{L}(\varphi, \partial_\mu\varphi) = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{1}{4!}\lambda\varphi^4 + \dots \tag{A.11}$$

We have omitted the evaluation at x for the brevity of notation. We use this to evaluate the variation of a functional in the of Eq. (A.1),

$$\delta_\eta S[\varphi] = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\int_{\mathcal{M}} \mathrm{d}^n x\, \mathcal{L}[\varphi+\epsilon\eta](x), \tag{A.12}$$

by Taylor expanding the Lagrangian density as a function of φ and its derivatives,

$$\mathcal{L}[\varphi+\epsilon\eta] = \mathcal{L}\left[\varphi+\epsilon\eta, \partial_\mu\{\varphi+\epsilon\eta\}\right] = \epsilon\left(\frac{\partial\mathcal{L}}{\partial\varphi}\eta + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\partial_\mu\eta\right) + \mathcal{O}(\epsilon^2). \tag{A.13}$$

Inserting this into Eq. (A.12) and partially integrating the last term allows us to write the variation in the form Eq. (A.5), and the functional derivative is

$$\frac{\delta S}{\delta\varphi} = \frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}. \tag{A.14}$$

The principle of stationary action says that the equation of motion of a field obey $\delta_\eta S = 0$. As η is arbitrary, this is equivalent to setting the functional derivative of S equal zero. The result is the Euler-Lagrange equations of motion,

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} - \frac{\partial\mathcal{L}}{\partial\varphi} = 0. \tag{A.15}$$

A.1.3 Functional calculus on a curved manifold

(NEEDS SOME CLEANUP)

As discussed in subsection 2.1.4, when integrating a scalar on a curved manifold, we must include the $\sqrt{|g|}$-factor to get a coordinate-independent result. The action in curved space-time is therefore

$$S[g,\varphi] = \int_{\mathcal{M}} \mathrm{d}^n x \, \sqrt{|g|}\mathcal{L}[g,\varphi], \tag{A.16}$$

where the action and the Lagrangian now is a functional of both the matter-field φ and the metric $g_{\mu\nu}$. A general Lagrangian will now take the form

$$\mathcal{L}(g_{\mu\nu},\varphi,\nabla_\mu\varphi) = \frac{1}{2}g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi - \frac{1}{2}m^2\varphi^2 + \dots \tag{A.17}$$

We define the functional derivative as

$$\delta_\eta S = \int_{\mathcal{M}} \mathrm{d}^n x \sqrt{|g|}\frac{\delta S}{\delta\eta(x)}\eta(x). \tag{A.18}$$

If this is a variation in φ only, this gives the same result as before. However, if we consider the variation $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$, assuming $\mathcal{L}$ only depends on g and not its derivatives, we get

$$\delta_{\delta g} S = \int_{\mathcal{M}} \mathrm{d}^n x \, \left[\left(\delta\sqrt{|g|}\right)\mathcal{L}[g] + \sqrt{|g|}\delta\mathcal{L}[g]\right] = \int_{\mathcal{M}} \mathrm{d}^n x \, \sqrt{|g|}\left(\frac{\partial\mathcal{L}}{\partial g^{\mu\nu}} - \frac{1}{2}g_{\mu\nu}\mathcal{L}\right)\delta g^{\mu\nu}. \tag{A.19}$$

We have used

$$\delta\sqrt{|g|} = -\frac{1}{2}\sqrt{|g|}g_{\mu\nu}\delta g^{\mu\nu} \tag{A.20}$$

We find this using, for arbitrary $\eta_{\mu\nu}$, clean up

$$\begin{aligned}
\delta\sqrt{g} &= \lim_{\epsilon\to 0}\sqrt{\det(g^{\mu\nu}+\epsilon\eta^{\mu\nu})} - \sqrt{g} = \frac{1}{2}\sqrt{g}\delta\ln(g) \\
\delta\ln(g) &= \lim_{\epsilon\to 0}\ln(\det(g_{\mu\rho}[g^\rho{}_\nu - \varepsilon g^{\rho\sigma}\eta_{\sigma\nu}])) - \ln(g) = \lim_{\epsilon\to 0}\ln(\det(g^\rho{}_\nu - \varepsilon g^{\rho\sigma}\eta_{\sigma\nu})) \\
\det(g+\varepsilon M) &= \frac{1}{n!}\epsilon^{\mu_1\cdots}\epsilon_{\nu_1\dots}(g+\varepsilon M)^{\nu_1}{}_{\mu_1}\dots = \frac{1}{n!}\varepsilon^{\mu_1\cdots}\varepsilon_{\nu_1\dots}(g^{\nu_1}{}_{\mu_1}\dots + \epsilon M^{\nu_1}{}_{\mu_1}g^{\nu_2}{}_{\mu_2}\dots + \dots) \\
&= 1 + \frac{1}{n!}\epsilon\varepsilon^{\mu\mu_2\cdots}\varepsilon_{\nu\mu_2\dots}M^\nu{}_\mu + \mathcal{O}(\epsilon^2) = 1 + \epsilon M^\mu{}_\mu + \mathcal{O}(\epsilon^2)
\end{aligned}$$

Using $\ln(1+\epsilon x) \sim x$, and $\delta(g^{\mu\nu}g_{\mu\nu}) = 0$, we obtain the desired result. Wit the Lagrangian in Eq. (A.17), we get

$$\frac{\delta S}{\delta g^{\mu\nu}} = \frac{\partial\mathcal{L}}{\partial g^{\mu\nu}} - \frac{1}{2}g_{\mu\nu}\mathcal{L} = -\frac{1}{2}\left(\frac{1}{2}\nabla_\mu\varphi\nabla_\nu\varphi + \frac{1}{2}m^2\varphi^2 + \dots\right). \tag{A.21}$$

We recognize the $(\mu,\nu) = (0,0)$-component as negative half the Hamiltonian density, which supports the definition of the definition of the stress-energy tensor Eq. (4.4).

A.1.4 Functional derivative of the Einstein-Hilbert action

(NEEDS MORE CLEANUP)

In the Einstein-Hilbert action, Eq. (4.1), the Lagrangian density is $\mathcal{L} = kR = kg^{\mu\nu}R_{\mu\nu}$, where k is a constant and $R_{\mu\nu}$ the Ricci tensor, Eq. (2.36). As the Ricci tensor is dependent on both the derivative and second derivative of the metric, we can not use Eq. (A.21) directly. Instead, we use the variation

$$\delta S_{\mathrm{EH}} = k\int_{\mathcal{M}} \mathrm{d}^n x \, \sqrt{|g|}\left(\delta R - \frac{1}{2}g_{\mu\nu}R\delta g^{\mu\nu}\right). \tag{A.22}$$

The variation of the Ricci scalar is

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}, \tag{A.23}$$

We can write the variation of the Ricci scalar, and thus the Riemann curvature tensor, in terms of variations in Christoffel symbols, $\delta\Gamma^{\rho}_{\mu\nu}$ using the explicit formula for a symmetric, metric-compatible covariant derivative, Eq. (2.35). As $\delta\Gamma = \Gamma - \Gamma'$, it is a tensor, and we may write clean up

$$\begin{aligned}
\delta R^{\rho}{}_{\sigma\mu\nu} &= \delta(\partial_{[\mu}\Gamma^{\rho}_{\nu]\sigma} + \Gamma^{\rho}_{\lambda[\mu}\Gamma^{\lambda}_{\nu]\sigma}) = \partial_{[\mu}\delta\Gamma^{\rho}_{\nu]\sigma} + (\delta\Gamma^{\rho}_{\lambda[\mu})\Gamma^{\lambda}_{\nu]\sigma} + \Gamma^{\rho}_{\lambda[\mu}(\delta\Gamma^{\lambda}_{\nu]\sigma}) \\
&= \partial_{\mu}\delta\Gamma^{\rho}_{\nu\sigma} + \Gamma^{\rho}_{\lambda\mu}(\delta\Gamma^{\lambda}_{\nu\sigma}) - \Gamma^{\lambda}_{\mu\sigma}(\delta\Gamma^{\rho}_{\lambda\nu}) - \left(\partial_{\nu}\delta\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\lambda\nu}(\delta\Gamma^{\lambda}_{\mu\sigma}) - \Gamma^{\lambda}_{\nu\sigma}(\delta\Gamma^{\rho}_{\lambda\mu})\right) + (\Gamma^{\lambda}_{\mu\nu}\delta\Gamma^{\rho}_{\lambda\sigma} - \Gamma^{\lambda}_{\mu\nu}\delta\Gamma^{\rho}_{\lambda\sigma}) \\
&= \nabla_{\mu}\delta\Gamma^{\rho}_{\nu\sigma} - \nabla_{\nu}\delta\Gamma^{\rho}_{\mu\sigma} = \nabla_{\eta}\left(g^{\eta}{}_{\mu}\delta\Gamma^{\rho}_{\nu\sigma} - g^{\eta}{}_{\nu}\delta\Gamma^{\rho}_{\mu\sigma}\right) = \nabla_{\eta}(K^{\rho}{}_{\sigma\mu\nu})^{\eta},
\end{aligned}$$

where K is a tensorial quantity, which vanish at the boundary of our spacetime. Using the generalized divergence theorem, Eq. (2.48), we see that the contribution to the action from this quantity vansih. The contribution comes from an integral over $g^{\mu\nu}\delta R_{\mu\nu} = g^{\mu\nu}\delta R^{\rho}_{\mu\rho\nu} = g^{\mu\nu}\nabla_{\eta}(K^{\rho}{}_{\mu\rho\nu})^{\eta}$ Using metric compatibility, we can exchange the covariant derivative and the metric, and we have $g^{\mu\nu}\delta R_{\mu\nu} = \nabla_{\eta}[g^{\mu\nu}K^{\eta\rho}{}_{\mu\rho\nu}]$. The contribution to the action therefore becomes

$$\int_{\mathcal{M}} \mathrm{d}^4x\,\sqrt{|g|}g^{\mu\nu}\delta R_{\mu\nu} = \int_{\mathcal{M}} \mathrm{d}^4x\,\sqrt{|g|}\nabla_{\eta}[g^{\mu\nu}K^{\eta\rho}{}_{\mu\rho\nu}] = \int_{\partial\mathcal{M}} \mathrm{d}^3y\,\sqrt{|\gamma|}n_{\eta}[g^{\mu\nu}K^{\eta\rho}{}_{\mu\rho\nu}] = 0, \tag{A.24}$$

where we used the fact that $\delta g_{\mu\nu}$, and thus K, vanish at $\partial\mathcal{M}$. The variation of the action is therefore

$$\delta S_{\mathrm{EH}} = k\int_{\mathcal{M}} \mathrm{d}^nx\sqrt{|g|}\left[R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right]\delta g^{\mu\nu}, \tag{A.25}$$

and by the definition of the functional derivative,

$$\frac{\delta S_{\mathrm{EH}}}{\delta g^{\mu\nu}} = k(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}). \tag{A.26}$$

Appendix B

Code

All code is available at: `https://github.com/martkjoh/master`.

B.1 Integrating the TOV equations

Describe the numerical code for the TOV equations

B.2 Spherically symmetric metric

The calculations in chapter 4 were done using a CAS system. The code is written in Python in a Jupyter notebook. The full `.ipynb` file with executable code is available in the repository, at `https://github.com/martkjoh/master/blob/main/scripts/TOV.ipynb` Below is some of the code, which illustrates the main functions and the outputs.

1 Metric $g_{\mu\nu}$ for spherically symmetric spacetime

[8]:
```
t, r, th, ph = symbols("t, r, \\theta, \\phi")
x1 = r * cos(ph) * sin(th)
x2 = r * sin(ph) * sin(th)
x3 = r * cos(th)

one = Rational(1)
eta = sp.diag(one, -one, -one, -one)
var = (t, r, th, ph)
J = Matrix([t, x1, x2, x3]).jacobian(var)
g = np.array(simplify(J.T *eta* J))

a = sp.Function("\\alpha", real=True)(r)
b = sp.Function("\\beta", real=True)(r)
g[0, 0] *= exp(2 * a)
g[1, 1] *= exp(2 * b)
g_inv = get_g_inv(g)

print_matrix(g)
print_matrix(g_inv)
```

$$\begin{bmatrix} e^{2\alpha(r)} & 0 & 0 & 0 \\ 0 & -e^{2\beta(r)} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2(\theta) \end{bmatrix}$$

$$\begin{bmatrix} e^{-2\alpha(r)} & 0 & 0 & 0 \\ 0 & -e^{-2\beta(r)} & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{r^2 \sin^2(\theta)} \end{bmatrix}$$

[9]:
```
C = Christoffel(g, g_inv, var)
c = print_christoffel(C, var)
```

$$\Gamma^{t}_{\mu\nu} = \begin{bmatrix} 0 & \frac{d}{dr}\alpha(r) & 0 & 0 \\ \frac{d}{dr}\alpha(r) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Gamma^{r}_{\mu\nu} = \begin{bmatrix} e^{2\alpha(r)}e^{-2\beta(r)}\frac{d}{dr}\alpha(r) & 0 & 0 & 0 \\ 0 & \frac{d}{dr}\beta(r) & 0 & 0 \\ 0 & 0 & -re^{-2\beta(r)} & 0 \\ 0 & 0 & 0 & -re^{-2\beta(r)}\sin^2(\theta) \end{bmatrix}$$

$$\Gamma^{\theta}_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\sin(\theta)\cos(\theta) \end{bmatrix}$$

$$\Gamma^{\phi}_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \frac{\cos(\theta)}{\sin(\theta)} \\ 0 & \frac{1}{r} & \frac{\cos(\theta)}{\sin(\theta)} & 0 \end{bmatrix}$$

[10]:
```
Rie = Riemann_tensor(C, var)
Ricci = contract(Rie, num_indx=4, upper=1, indx=(0, 2))

for i in range(4):
    print_scalar(Ricci[i, i].factor())
```

$$\frac{\left(r\left(\frac{d}{dr}\alpha(r)\right)^2 - r\frac{d}{dr}\alpha(r)\frac{d}{dr}\beta(r) + r\frac{d^2}{dr^2}\alpha(r) + 2\frac{d}{dr}\alpha(r)\right)e^{2\alpha(r)}e^{-2\beta(r)}}{r}$$

$$-\frac{r\left(\frac{d}{dr}\alpha(r)\right)^2 - r\frac{d}{dr}\alpha(r)\frac{d}{dr}\beta(r) + r\frac{d^2}{dr^2}\alpha(r) - 2\frac{d}{dr}\beta(r)}{r}$$

$$-\left(r\frac{d}{dr}\alpha(r) - r\frac{d}{dr}\beta(r) - e^{2\beta(r)} + 1\right)e^{-2\beta(r)}$$

$$-\left(r\frac{d}{dr}\alpha(r) - r\frac{d}{dr}\beta(r) - e^{2\beta(r)} + 1\right)e^{-2\beta(r)}\sin^2(\theta)$$

[11]:
```
R = contract(Ricci, g_inv=g_inv, upper=0).simplify()
print_scalar(R)
```

$$\frac{2\left(r^2\left(\frac{d}{dr}\alpha(r)\right)^2 - r^2\frac{d}{dr}\alpha(r)\frac{d}{dr}\beta(r) + r^2\frac{d^2}{dr^2}\alpha(r) + 2r\frac{d}{dr}\alpha(r) - 2r\frac{d}{dr}\beta(r) - e^{2\beta(r)} + 1\right)e^{-2\beta(r)}}{r^2}$$

[12]:
```
G = Ricci - Rational(1, 2) * R * g
for i in range(4):
    G[i, i] = G[i, i].simplify().factor()
    print_scalar(G[i, i])
```

$$\frac{\left(2r\frac{d}{dr}\beta(r) + e^{2\beta(r)} - 1\right)e^{2\alpha(r)}e^{-2\beta(r)}}{r^2}$$

$$\frac{2r\frac{d}{dr}\alpha(r) - e^{2\beta(r)} + 1}{r^2}$$

$$r\left(r\left(\frac{d}{dr}\alpha(r)\right)^2 - r\frac{d}{dr}\alpha(r)\frac{d}{dr}\beta(r) + r\frac{d^2}{dr^2}\alpha(r) + \frac{d}{dr}\alpha(r) - \frac{d}{dr}\beta(r)\right)e^{-2\beta(r)}$$

$$r\left(r\left(\frac{d}{dr}\alpha(r)\right)^2 - r\frac{d}{dr}\alpha(r)\frac{d}{dr}\beta(r) + r\frac{d^2}{dr^2}\alpha(r) + \frac{d}{dr}\alpha(r) - \frac{d}{dr}\beta(r)\right)e^{-2\beta(r)}\sin^2(\theta)$$

1.0.1 Stress-energy tensor $T_{\mu\nu}$ for perfect fluid

[13]:
```
p = sp.Function("p")(r)
u = sp.Function("u")(r)

UU = np.zeros((4, 4), dtype=sp.Rational)
UU[0, 0] = exp(2 * a)

T = (p + u) * UU - p * g
for i in range(4):
    T[i, i] = T[i, i].simplify()
print_matrix(T)
```

$$\begin{bmatrix} u(r)e^{2\alpha(r)} & 0 & 0 & 0 \\ 0 & p(r)e^{2\beta(r)} & 0 & 0 \\ 0 & 0 & r^2p(r) & 0 \\ 0 & 0 & 0 & r^2p(r)\sin^2(\theta) \end{bmatrix}$$

2 Einstin's field equations

$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}$

[14]:
```
G_newton = sp.Symbol("G")

eq = []
for i in range(len(G)):
    eq.append((G[i, i] - 8 * pi * G_newton * T[i, i]).simplify())

# Some manual simplification
Rtt = sp.Symbol("R_{\\theta \\theta}")
eq[0] = eq[0] * r**2 / exp(2 * a)/exp(-2*b ) * (-1 )
eq[1] = eq[1] * r**2 * (-1)
eq[2] = eq[2] / r / exp(-2*b)
eq[3] = eq[3].subs(eq[2], Rtt)
for i in range(len(G)):
    print_eq(eq[i].simplify())
```

$$8\pi G r^2 u(r) e^{2\beta(r)} - 2r\frac{d}{dr}\beta(r) - e^{2\beta(r)} + 1 = 0$$

$$8\pi G r^2 p(r) e^{2\beta(r)} - 2r\frac{d}{dr}\alpha(r) + e^{2\beta(r)} - 1 = 0$$

$$-8\pi G r p(r) e^{2\beta(r)} + r\left(\frac{d}{dr}\alpha(r)\right)^2 - r\frac{d}{dr}\alpha(r)\frac{d}{dr}\beta(r) + r\frac{d^2}{dr^2}\alpha(r) + \frac{d}{dr}\alpha(r) - \frac{d}{dr}\beta(r) = 0$$

$$R_{\theta\theta} r e^{-2\beta(r)}\sin^2(\theta) = 0$$

Define $e^{2\beta} = [1 - 2Gm(r)/r]^{-1}$

[15]:
```
m = sp.Function("m", Real=True)(r)
f = (1 - 2 * G_newton * m / r)**(-1)
eq1 = (eq[0] * exp(- 2 *a)).simplify().subs(b, Rational(1, 2) * log(f)).
 ↪simplify().expand().simplify()
s = sp.solve(eq1, m.diff(r))
eq1 = m.diff(r) - s[0]
```

Use $\nabla_\mu T^{\mu r} = 0 \implies (p+\rho)\partial_r \alpha = -\partial_r p$.

[16]:
```
eq2 = (eq[1] * r**2).subs(exp(2 * b), f).simplify()
s = sp.solve(eq2, a.diff(r))
eq2 = a.diff(r) - s[0]
eq2 = ((a.diff(r) - s[0]).subs(a.diff(r), - p.diff(r) / (p + u))*(p + u)).
 ↪simplify()
s = sp.solve(eq2, p.diff())
eq2 = p.diff(r) - s[0].factor()
```

The TOV-equation and equation for $m(r)$, both expressions are equal to 0.

[17]:
```
print_eq(eq1)
print_eq(eq2)
```

$$-4\pi r^2 u(r) + \frac{d}{dr} m(r) = 0$$

$$\frac{G\left(4\pi r^3 p(r) + m(r)\right)\left(p(r) + u(r)\right)}{r\left(-2Gm(r) + r\right)} + \frac{d}{dr} p(r) = 0$$

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