Chiral Perturbation Theory

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October 11, 2021

1 Effective Pion Lagrangian

The technique used in χPT to obtain the effective Lagrangian of the pion relies on a "theorem", as formulated by Weinberg:

> [I]f one writes down the most general possible Lagrangian, including all terms consistent with assumed symmetry principles, and then calculates matrix elements with this Lagrangian to any given order of perturbation theory, the result will simply be the most general possible S-matrix consistent with analyticity, perturbative unitary, cluster decomposition and the assumed symmetry principles. [1]

In other words, if we write down the most general Lagrange density consistent with symmetries of the underlying theory, it will result in the most general S-matrix consistent with that theory, and important physical assumptions. This leaves a Lagrange density with infinitely many terms, and infinitely many free parameters. To be able to use this theory for anything one must have a method for ordering the terms in order of importance. As described in [2], by rescaling the external momenta $p_\mu \to t p_\mu$ and quark masses $m_i \to t^2 m_i$, each term in the Lagrangian obtains a factor t^D. The Lagrangian is then expanded as $\mathcal{L} = \sum_D \mathcal{L}_D$, where $\mathcal{L}_D$ contains all terms with a factor t^D.

In our case, the underlying theory is QCD with two quarks, up and down, with mass matrix

$$M = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}. \tag{1}$$

In the isospin limit, $m_u = m_d$, the theory is invariant under global transformations by elements of the group $G' = \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_V$. All terms involving only pions are trivially invariant under $\mathrm{U}(1)_V$, (HVORFOR?) so we focus on the $G = \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ subgroup. This symmetry is spontaneously broken if the quark field has a non-zero ground state expectation value $\langle \bar{q} q \rangle$, leaving only a subgroup $H = \mathrm{SU}(2)_V \subseteq G$ of symmetry transformations of the vacuum state. The Goldstone manifold $G/H = \mathrm{SU}(2)_A$ is a three-dimensional Lie group, and therefore results in three (pseudo) Goldstone bosons, the pions. There exists an isomorphism from a subset $S \subseteq M_1$ of the set of all Goldstone-fields

$$M_1 = \left\{ \pi_a : \mathcal{M}_4 \longrightarrow \mathbb{R}^3 | \pi_a \, \text{smooth} \right\}$$

close to the ground state, into fields taking values in the Goldstone manifold G/H. (BEVISE?)(HVA ER ISOMORFISME HER?). The χPT effective Lagrangian will be constructed using this map, through the parametrization

$$\begin{aligned} \Sigma : \mathcal{M}_4 &\longrightarrow \mathrm{SU}(2), \\ x &\longrightarrow \Sigma(x) = A_\alpha (U(x) \Sigma_0 U(x)) A_\alpha, \end{aligned} \tag{2}$$

where

$$\Sigma_0 = \mathbb{1}, \; A_\alpha = \exp\left(\frac{i\alpha}{2} \tau_1 \right), \; U(x) = \exp\left(i \frac{\tau_a \pi_a(x)}{2f} \right).$$

τ_a are the SU(2) generators, i.e. Pauli matrices, as described in Appendix A. π_a, where $a \in \{1, 2, 3\}$, are the pion fields. These are real fields, meaning $\pi_a^\dagger = \pi_a$.

1.1 Leading order Lagrangian

The leading order Lagrangian in χPT is [2, 3]

$$\mathcal{L}_2 = \frac{f^2}{4}\,\mathrm{Tr}\left[\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\right] + \frac{f^2}{4}\,\mathrm{Tr}\left[\chi^\dagger\Sigma + \Sigma^\dagger\chi\right]. \tag{3}$$

χ and f are the free parameters of the theory. f is the pion decay constant, while $\chi = 2B_0M$. Here, M is the mass matrix Eq. (1), and B_0 is related to the quark condensate through $f^2B_0 = -\langle\bar{u}u\rangle$. The covariant derivative is defined by

$$\nabla_\mu\Sigma = \partial_\mu\Sigma - i\left[v_\mu,\Sigma\right], \quad (\nabla_\mu\Sigma)^\dagger = \partial_\mu\Sigma^\dagger - i\left[v_\mu,\Sigma^\dagger\right], \quad v_\mu = \frac{1}{2}\mu_I\delta^0_\mu\tau_3,$$

where μ_I is the isospin chemical potential. To get the series expansion of Σ in powers of π/f, we start by using the fact that $\tau_a^2 = \mathbb{1}$ to write

$$A_\alpha = \sum_n^\infty \frac{1}{n!}\left(\frac{i\alpha}{2}\tau_1\right)^n = \sum_n^\infty\left[\frac{\mathbb{1}}{(2n)!}\left(\frac{i\alpha}{2}\right)^{(2n)} + \frac{\tau_1}{(2n+1)!}\left(\frac{i\alpha}{2}\right)^{(2n+1)}\right] = \mathbb{1}\cos\frac{\alpha}{2} + i\tau_1\sin\frac{\alpha}{2}. \tag{4}$$

The series expansion of U is

$$U = \exp\left(\frac{i\pi_a\tau_a}{2f}\right) = 1 + \frac{i\pi_a\tau_a}{2f} + \frac{1}{2}\left(\frac{i\pi_a\tau_a}{2f}\right)^2 + \frac{1}{6}\left(\frac{i\pi_a\tau_a}{2f}\right)^3 + \frac{1}{24}\left(\frac{i\pi_a\tau_a}{2f}\right)^4 + \mathcal{O}\left((\pi/f)^5\right),$$

which we use to calculate the expansion of the inner part of Σ, as given in Eq. (2),

$$\begin{aligned}
U\Sigma_0U &= \left(1 + \frac{i\pi_a\tau_a}{2f} + \frac{1}{2}\left(\frac{i\pi_a\tau_a}{2f}\right)^2 + \frac{1}{6}\left(\frac{i\pi_a\tau_a}{2f}\right)^3 + \frac{1}{24}\left(\frac{i\pi_a\tau_a}{2f}\right)^4\right)\\
&\quad\times\left(1 + \frac{i\pi_a\tau_a}{2f} + \frac{1}{2}\left(\frac{i\pi_a\tau_a}{2f}\right)^2 + \frac{1}{6}\left(\frac{i\pi_a\tau_a}{2f}\right)^3 + \frac{1}{24}\left(\frac{i\pi_a\tau_a}{2f}\right)^4\right) + \mathcal{O}\left((\pi/f)^5\right)\\
&= 1 + \frac{i\pi_a\tau_a}{f} + 2\left(\frac{i\pi_a\tau_a}{2f}\right)^2 + \frac{4}{3}\left(\frac{i\pi_a\tau_a}{2f}\right)^3 + \frac{2}{3}\left(\frac{i\pi_a\tau_a}{2f}\right)^4 + \mathcal{O}\left((\pi/f)^5\right).
\end{aligned}$$

The symmetry of $\pi_a\pi_b$ means that

$$(\pi_a\tau_a)^2 = \pi_a\pi_b\frac{1}{2}\left\{\tau_a,\tau_b\right\} = \pi_a\pi_a, \quad (\pi_a\tau_a)^3 = \pi_a\pi_a\pi_b\tau_b, \quad (\pi_a\tau_a)^4 = \pi_a\pi_a\pi_b\pi_b.$$

This gives us the expression

$$U\Sigma_0U = 1 + i\frac{\pi_a\tau_a}{f} - \frac{\pi_a^2}{2f^2} - i\frac{\pi_a^2\pi_b\tau_b}{6f^3} + \frac{\pi_a^2\pi_b^2}{24f^4} + \mathcal{O}\left((\pi/f)^5\right).$$

We combine this result with Eq. (4) to get an expression for Σ up to $\mathcal{O}\left((\pi/f)^5\right)$

$$\begin{aligned}
\Sigma &= \left(\cos\frac{\alpha}{2} + i\tau_1\sin\frac{\alpha}{2}\right)\left(1 + i\frac{\pi_a\tau_a}{f} - \frac{\pi_a^2}{2f^2} - i\frac{\pi_a^2\pi_b\tau_b}{6f^3} + \frac{\pi_a^2\pi_b^2}{24f^4}\right)\left(\cos\frac{\alpha}{2} + i\tau_1\sin\frac{\alpha}{2}\right)\\
&= \left(1 + i\frac{\pi_a\tau_a}{f} - \frac{\pi_a^2}{2f^2} - i\frac{\pi_a^2\pi_b\tau_b}{6f^3} + \frac{\pi_a^2\pi_b^2}{24f^4}\right)\cos^2\frac{\alpha}{2}\\
&\quad - \left(1 + i\frac{\pi_a}{f}\tau_1\tau_a\tau_1 - \frac{\pi_a^2}{2f^2} - i\frac{\pi_a^2\pi_b}{6f^3}\tau_1\tau_b\tau_1 + \frac{\pi_a^2\pi_b^2}{24f^4}\right)\sin^2\frac{\alpha}{2}\\
&\quad + i\left(2\tau_1 + i\frac{\pi_a}{f}\left\{\tau_1,\tau_a\right\} - 2\tau_1\frac{\pi_a^2}{2f^2} - i\frac{\pi_a^2\pi_b}{6f^3}\left\{\tau_1,\tau_b\right\} + 2\tau_1\frac{\pi_a^2\pi_b^2}{24f^4}\right)\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}.
\end{aligned}$$

Using trigonometric identities and the commutator,

$$\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2} = \cos\alpha, \quad 2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2} = \sin\frac{\alpha}{2}, \quad \tau_1\tau_a\tau_1 = -\tau_a + 2\delta_{1a}\tau_1,$$

the final expression of Σ to $\mathcal{O}\left((\pi/f)^5\right)$ is

$$\Sigma = \left(1 - \frac{\pi_a^2}{2f^2} + \frac{\pi_a^2\pi_b^2}{24f^4}\right)(\cos\alpha + i\tau_1\sin\alpha) + \left(\frac{\pi_a}{f} - \frac{\pi_b^2\pi_a}{6f^3}\right)\left(i\tau_a - 2i\delta_{a1}\tau_1\sin^2\frac{\alpha}{2} - \delta_{a1}\sin\alpha\right). \tag{5}$$

The kinetic term in the χPT Lagrangian is

$$\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger = \partial_\mu\Sigma\partial^\mu\Sigma^\dagger - i\left(\partial_\mu\Sigma\left[v^\mu,\Sigma^\dagger\right] - \text{h.c.}\right) - [v_\mu,\Sigma]\left[v_\mu,\Sigma^\dagger\right]. \tag{6}$$

Using Eq. (5) we find the expansion of the constitutive parts of the kinetic term to be

$$\begin{aligned}
\partial_\mu\Sigma &= \left(\frac{-1}{f^2} + \frac{\pi_b^2}{6f^4}\right)(\cos\alpha + i\tau_1\sin\alpha)(\pi_a\partial_\mu\pi_a) \\
&\quad + \left(\frac{\partial_\mu\pi_a}{f} - \frac{\pi_b^2\partial_\mu\pi_a + 2\pi_a\pi_b\partial_\mu\pi_b}{6f^3}\right)\left(i\tau_a - 2i\delta_{a1}\tau_1\sin^2\frac{\alpha}{2} - \delta_{1a}\sin\alpha\right) \\
&= \left[\left(\frac{-1}{f^2} + \frac{\pi_b^2}{6f^4}\right)(\pi_a\partial_\mu\pi_a)\cos\alpha - \left(\frac{\partial_\mu\pi_1}{f} - \frac{\pi_b^2\partial_\mu\pi_1 + 2\pi_1\pi_b\partial_\mu\pi_b}{6f^3}\right)\sin\alpha\right] \\
&\quad - \left[\left(\frac{-1}{f^2} + \frac{\pi_b^2}{6f^4}\right)(\pi_a\partial_\mu\pi_a)\sin\alpha - \left(\frac{\partial_\mu\pi_1}{f} - \frac{\pi_b^2\partial_\mu\pi_1 + 2\pi_1\pi_b\partial_\mu\pi_b}{6f^3}\right)2\sin^2\frac{\alpha}{2}\right] i\tau_1 \\
&\quad + \left(\frac{\partial_\mu\pi_a}{f} - \frac{\pi_b^2\partial_\mu\pi_a + 2\pi_a\pi_b\partial_\mu\pi_b}{6f^3}\right) i\tau_a,
\end{aligned} \tag{7}$$

and

$$\begin{aligned}
[v_\mu,\Sigma] &= \frac{1}{2}\mu_I\delta_\mu^0\left[\left(1 - \frac{\pi_a^2}{2f^2} + \frac{\pi_a^2\pi_b^2}{24f^4}\right) i\sin\alpha\,[\tau_3,\tau_1] + \left(\frac{\pi_a}{f} - \frac{\pi_b^2\pi_a}{6f^3}\right)\left(i\,[\tau_a,\tau_3] - 2i\delta_{a1}\sin^2\frac{\alpha}{2}\,[\tau_3,\tau_1]\right)\right] \\
&= -\mu_I\delta_\mu^0\left\{\left(1 - \frac{\pi_a^2}{2f^2} + \frac{\pi_a^2\pi_b^2}{24f^4}\right)\tau_2\sin\alpha + \left(\frac{\pi_a}{f} - \frac{\pi_b^2\pi_a}{6f^3}\right)\left[(\delta_{a1}\tau_2 - \delta_{a2}\tau_1) - 2\delta_{a1}\tau_2\sin^2\frac{\alpha}{2}\right]\right\} \\
&= -\mu_I\delta_\mu^0\left\{\left[\left(1 - \frac{\pi_a^2}{2f^2} + \frac{\pi_a^2\pi_b^2}{24f^4}\right)\sin\alpha + \left(\frac{\pi_1}{f} - \frac{\pi_b^2\pi_1}{6f^3}\right)\cos\alpha\right]\tau_2 - \left(\frac{\pi_2}{f} - \frac{\pi_b^2\pi_2}{6f^3}\right)\tau_1\right\}.
\end{aligned} \tag{8}$$

Combining Eq. (7) and Eq. (8) gives the following terms [1]

$$\begin{aligned}
\mathrm{Tr}\left\{\partial_\mu\Sigma\partial^\mu\Sigma^\dagger\right\} &= \frac{2}{f^2}\partial_\mu\pi_a\partial^\mu\pi_a + \frac{2}{3f^4}\left[(\pi_a\partial_\mu\pi_a)(\pi_b\partial^\mu\pi_b) - (\pi_a\partial_\mu\pi_b)(\pi_b\partial^\mu\pi_a)\right], \\
-i\,\mathrm{Tr}\left\{\partial^\mu\Sigma\left[v_\mu,\Sigma^\dagger\right] - \text{h.c.}\right\} &= 4\mu_I\frac{\partial_0\pi_2}{f} + 8\mu_I\frac{\pi_3}{3f^3}\sin\alpha(\pi_2\partial_0\pi_3 - \pi_3\partial_0\pi_2)\sin\alpha \\
&\quad + \left(\frac{4\mu_I}{f^2}\cos\alpha - \frac{8\mu_I\pi_1}{3f^3}\sin\alpha - \frac{4\mu_I\pi_a\pi_a}{3f^4}\cos\alpha\right)(\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1), \\
-\mathrm{Tr}\left\{[v_\mu,\Sigma]\left[v^\mu,\Sigma^\dagger\right]\right\} &= {\mu_I}^2\left[2\sin^2\alpha + \left(\frac{2}{f} - \frac{4\pi_a\pi_a}{3f^3}\right)\pi_1\sin 2\alpha + \left(\frac{2}{f^2} - \frac{2\pi_a\pi_a}{3f^4}\right)\pi_a\pi_b k_{ab}\right], \\
\mathrm{Tr}\left\{\Sigma + \Sigma^\dagger\right\} &= 4\cos\alpha - \frac{4\pi_1}{f}\sin\alpha - \frac{2\pi_a\pi_a}{f^2}\cos\alpha + \frac{2\pi_1\pi_a\pi_a}{3f^3}\sin\alpha + \frac{(\pi_a\pi_a)^2}{6f^4}\cos\alpha,
\end{aligned}$$

where $k_{ab} = \delta_{a1}\delta_{b1}\cos 2\alpha + \delta_{a2}\delta_{b2}\cos^2\alpha - \delta_{a3}\delta_{b3}\sin^2\alpha$. If we write the Lagrangian as show in Eq. (3) as $\mathcal{L}_2 = \mathcal{L}_2^{(0)} + \mathcal{L}_2^{(1)} + \mathcal{L}_2^{(2)} + ...$, where $\mathcal{L}_2^{(n)}$ contains all terms of order $\mathcal{O}\left((\pi/f)^n\right)$, then the result of the series

[1] The scripts used to aid the calculation of the Lagrangian is available at `https://github.com/martkjoh/prosjektopggave`

expansion is

$$\mathcal{L}_2^{(0)} = f^2\left(2B_0m\cos\alpha + \frac{1}{2}\mu^2\sin^2\alpha\right), \tag{9}$$

$$\mathcal{L}_2^{(1)} = f(\mu_I^2\cos\alpha - 2B_0m)\pi_1\sin\alpha + f\mu_I\partial_0\pi_2\sin\alpha, \tag{10}$$

$$\mathcal{L}_2^{(2)} = \frac{1}{2}\partial_\mu\pi_a\partial^\mu\pi_a + \mu_I\cos\alpha\,(\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1) - B_0m\pi_a\pi_a\cos\alpha + \frac{1}{2}\mu_I^2\pi_a\pi_b k_{ab}, \tag{11}$$

$$\begin{aligned}\mathcal{L}_2^{(3)} &= \frac{\pi_a\pi_a\pi_1}{6f}(2B_0m\sin\alpha - 2\mu_I{}^2\sin 2\alpha)\\ &\quad - \frac{2\mu_I}{3f}\left[\pi_1(\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1) + \pi_3(\pi_3\partial_0\pi_2 - \pi_2\partial_0\pi_3)\right]\sin\alpha,\end{aligned} \tag{12}$$

$$\begin{aligned}\mathcal{L}_2^{(4)} &= \frac{1}{6f^2}\left\{\frac{1}{2}B_0m(\pi_a\pi_a)^2\cos\alpha - \left[(\pi_a\pi_a)(\partial_\mu\pi_b\partial^\mu\pi_b) - (\pi_a\partial_\mu\pi_a)(\pi_b\partial^\mu\pi_b)\right]\right\}\\ &\quad - \frac{\mu_I\pi_a\pi_a}{3f^2}\left[(\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1)\cos\alpha + \frac{1}{2}\mu_I\pi_a\pi_b k_{ab}\right].\end{aligned} \tag{13}$$

1.2 Equation of motion and redundant terms

Changing the field parametrization that appear in the Lagrangian does not affect any of the physics, as it corresponds to a change of variables in the path integral [2, 4, 5]. However, a change of variables can result in new terms in the Lagrangian. As a result of this, terms that on the face of it appear independent may be redundant. These terms can be eliminated by using the classical equation of motion. In this section we show first the derivation of the equation of motion, then use this result to identify redundant terms which need not be included in the most general Lagrangian.

We derive the equation of motion for the leading order Lagrangian using the principle of least action. Choosing the parametrization $\Sigma = \exp(i\pi_a\tau_a)$, a variation $\pi_a \to \pi_a + \delta\pi_a$ results in a variation in Σ, $\delta\Sigma = i\tau_a\delta\pi_a\Sigma$. The variation of the leading order action,

$$S_2 = \int_\Omega \mathrm{d}^4x\,\mathcal{L}_2, \tag{14}$$

when varying π_a is

$$\delta S = \int_\Omega \mathrm{d}x\,\frac{f^2}{4}\,\mathrm{Tr}\big\{(\nabla_\mu\delta\Sigma)(\nabla^\mu\Sigma)^\dagger + (\nabla_\mu\Sigma)(\nabla^\mu\delta\Sigma)^\dagger + \chi\delta\Sigma^\dagger + \delta\Sigma\chi^\dagger\big\}.$$

Using the properties of the covariant derivative to do partial integration, as show in Appendix B, as well as $\delta(\Sigma\Sigma^\dagger) = (\delta\Sigma)\Sigma^\dagger + \Sigma(\delta\Sigma)^\dagger = 0$, the variation of the action can be written

$$\begin{aligned}\delta S &= \frac{f^2}{4}\int_\Omega \mathrm{d}x\;\mathrm{Tr}\big\{-\delta\Sigma\nabla^2\Sigma^\dagger + (\nabla^2\Sigma)(\Sigma^\dagger\delta\Sigma\Sigma^\dagger) - \chi(\Sigma^\dagger\delta\Sigma\Sigma^\dagger) + \delta\Sigma\chi^\dagger\big\}\\ &= \frac{f^2}{4}\int_\Omega \mathrm{d}x\;\mathrm{Tr}\big\{\delta\Sigma\Sigma^\dagger\left[(\nabla^2\Sigma)\Sigma^\dagger - \Sigma\nabla^2\Sigma^\dagger - \chi\Sigma^\dagger + \Sigma\chi^\dagger\right]\big\}\\ &= i\frac{f^2}{4}\int_\Omega \mathrm{d}x\;\mathrm{Tr}\big\{\tau_a\left[(\nabla^2\Sigma)\Sigma^\dagger - \Sigma\nabla^2\Sigma^\dagger - \chi\Sigma^\dagger + \Sigma\chi^\dagger\right]\big\}\delta\pi_a = 0.\end{aligned}$$

As the variation is arbitrary, the equation of motion to leading order is

$$\mathrm{Tr}\big\{\tau_a\left[(\nabla^2\Sigma)\Sigma^\dagger - \Sigma\nabla^2\Sigma^\dagger - \chi\Sigma^\dagger + \Sigma\chi^\dagger\right]\big\} = 0. \tag{15}$$

This may be rewritten as a matrix equation. Using that

$$\mathrm{Tr}\big\{(\nabla_\mu\Sigma)\Sigma^\dagger\big\} = \mathrm{Tr}\big\{i\tau_a(\partial_\mu\pi_a)\Sigma\Sigma^\dagger\big\} - i\,\mathrm{Tr}\big\{[v_\mu,\Sigma]\Sigma^\dagger\big\} = 0,$$

we can see that $\mathrm{Tr}\big\{(\nabla^2\Sigma)\Sigma^\dagger - \Sigma\nabla^2\Sigma^\dagger\big\} = 0$, and the equation of motion may be written as

$$\mathcal{O}^{(2)}_{\mathrm{EOM}}(\Sigma) = (\nabla^2\Sigma)\Sigma^\dagger - \Sigma\nabla^2\Sigma^\dagger - \chi\Sigma^\dagger + \Sigma\chi^\dagger - \frac{1}{2}\,\mathrm{Tr}\big\{\chi\Sigma^\dagger - \Sigma\chi^\dagger\big\} = 0. \tag{16}$$

The next step in eliminating redundant terms is to change the parametrization of Σ by $\Sigma(x) \to \Sigma'(x)$. Here, $\Sigma(x) = e^{iS(x)}\Sigma'(x)$, $S(x) \in \mathfrak{su}(2)$. This change leads to a new Lagrange density, $\mathcal{L}[\Sigma] = \mathcal{L}[\Sigma'] + \Delta\mathcal{L}[\Sigma']$. We are free to choose $S(x)$, as long Σ' still obeys the required transformation properties. Any terms in the Lagrangian $\Delta\mathcal{L}$ due to a reparametrization can be neglected, as argued earlier. When demanding that Σ' obey the same symmetries as Σ, the most general transformation to second order in Weinberg's power counting scheme is [2]

$$S_2 = i\alpha_2\left[(\nabla^2\Sigma')\Sigma^\dagger - \Sigma'(\nabla^2\Sigma')^\dagger\right] + i\alpha_2\left[\chi\Sigma'^\dagger - \Sigma'\chi^\dagger - \frac{1}{2}\operatorname{Tr}\left\{\chi\Sigma'^\dagger - \Sigma'\chi^\dagger\right\}\right]. \tag{17}$$

α_1 and α_2 are arbitrary real numbers. As Eq. (17) is to second order, ΔEll is fourth order in Weinberg's power counting scheme. To leading order is given by

$$\begin{aligned}\mathcal{L}_2\left[e^{iS_2}\Sigma'\right] &= \frac{f^2}{4}\operatorname{Tr}\left\{[\nabla_\mu(1+iS_2)\Sigma'][\nabla^\mu\Sigma'^\dagger(1-iS_2)]\right\} + \frac{f^2}{4}\operatorname{Tr}\left\{\chi\Sigma'^\dagger(1-iS_2) + (1+iS_2)\Sigma'\chi^\dagger\right\}\\ &= \mathcal{L}[\Sigma'] + i\frac{f^2}{4}\operatorname{Tr}\left\{[\nabla_\mu(S_2\Sigma')][\nabla^\mu\Sigma']^\dagger - [\nabla_\mu\Sigma'][\nabla^\mu(\Sigma'^\dagger S_2)]\right\} - i\frac{f^2}{4}\operatorname{Tr}\left\{\chi\Sigma'^\dagger S_2 - S_2\Sigma'\chi^\dagger\right\}\end{aligned}$$

Using the properties of the covariant derivative, as described in Appendix B, we may use the product rule and partial integration to write the difference between the two Lagrangians to fourth order as

$$\begin{aligned}\Delta\mathcal{L}[\Sigma'] &= i\frac{f^2}{4}\operatorname{Tr}\left\{(\nabla_\mu S_2)(\Sigma'\nabla^\mu\Sigma'^\dagger - (\nabla^\mu\Sigma')\Sigma'^\dagger)\right\} - i\frac{f^2}{4}\operatorname{Tr}\left\{\chi\Sigma'^\dagger S_2 - S_2\Sigma'\chi^\dagger\right\}\\ &= i\frac{f^2}{4}\operatorname{Tr}\left\{S_2\left[\Sigma'^\dagger\nabla^2\Sigma' - (\nabla^2\Sigma')\Sigma'^\dagger - \chi\Sigma'^\dagger + \Sigma'\chi^\dagger\right]\right\}.\end{aligned}$$

Using the equation of motion Eq. (16), and the fact that $\operatorname{Tr}\{S_2\} = 0$, this difference can be written as

$$\Delta\mathcal{L}[\Sigma'] = \frac{f^2}{4}\operatorname{Tr}\left\{iS_2\mathcal{O}^{(2)}_{\text{EOM}}(\Sigma')\right\}. \tag{18}$$

Any term that can be written in the form of Eq. (18) for arbitrary $\alpha_1, \alpha_2 \in \mathbb{R}$ is redundant, as we argued earlier, and may therefore be discarded. $\Delta\mathcal{L}_2$ is of fourth order, and it can thus be used to remove terms from $\mathcal{L}_4$ or higher order.

1.3 Next to leading order Lagrangian

The next to leading order Lagrangian density is, assuming no external fields

$$\begin{aligned}\mathcal{L}_4 = &\frac{l_1}{4}\operatorname{Tr}\left\{\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\right\}^2 + \frac{l_2}{4}\operatorname{Tr}\left\{\nabla_\mu\Sigma(\nabla_\nu\Sigma)^\dagger\right\}\operatorname{Tr}\left\{\nabla^\mu\Sigma(\nabla^\nu\Sigma)^\dagger\right\} + \frac{l_3+h_1-h_3}{16}\operatorname{Tr}\left\{\chi\Sigma^\dagger + \Sigma\chi^\dagger\right\}^2\\ &+ \frac{l_4}{4}\operatorname{Tr}\left\{\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\right\}\operatorname{Tr}\left\{\chi\Sigma^\dagger + \Sigma\chi^\dagger\right\} + \frac{h_1-h_3-l_4-l_7}{16}\operatorname{Tr}\left\{\chi\Sigma^\dagger - \Sigma\chi^\dagger\right\}^2 + \frac{h_1+h_2-l_4}{4}\operatorname{Tr}\left\{\chi\chi^\dagger\right\}\\ &- \frac{h_1-h_3-l_4}{8}\operatorname{Tr}\left\{\left(\chi\Sigma^\dagger\right)^2 + \left(\Sigma\chi^\dagger\right)^2\right\}\end{aligned} \tag{19}$$

To $\mathcal{L}_4$ to $\mathcal{O}\left((\pi/f)^3\right)$, we use the result from Eq. (7) and Eq. (8), up to and including $\mathcal{O}\left((\pi/f)^2\right)$, which gives

$$\begin{aligned}\operatorname{Tr}\left\{\partial_\mu\Sigma\partial_\nu\Sigma^\dagger\right\} &= 2\frac{\partial_\mu\pi_a\partial_\nu\pi_a}{f^2}\\ -i\operatorname{Tr}\left\{\partial_\mu\Sigma\left[v_\nu,\Sigma^\dagger\right] - \text{h.c.}\right\} &= \frac{2\mu_I\pi_2}{f}(\delta^0_\mu\partial_\nu + \delta^0_\nu\partial_\mu)\sin\alpha + \frac{2\mu_I}{f^2}[\pi_1(\delta^0_\mu\partial_\nu + \delta^0_\nu\partial_\mu)\pi_2 - \pi_2(\delta^0_\mu\partial_\nu + \delta^0_\nu\partial_\mu)\pi_1]\cos\alpha\\ -\operatorname{Tr}\left\{[v_\nu,\Sigma]\left[v_\nu,\Sigma^\dagger\right]\right\} &= 2{\mu_I}^2\delta^0_\mu\delta^0_\nu\left[\sin^2\alpha + \frac{\pi_1}{f}\sin 2\alpha + \frac{\pi_a\pi_b}{f^2}k_{ab}\right].\end{aligned}$$

Using the form of the Pauli matrices, we can write χ as

$$\chi = 2B_0M = 2B_0(\bar{m}\mathbb{1} + \Delta m\tau_3),$$

where $\bar{m} = (m_u + m_d)/2$, $\Delta m = (m_u - m_d)/2$, which gives

$$\chi\Sigma^\dagger + \Sigma\chi^\dagger = 4B_0\Bigg\{(\bar{m} + \Delta m\tau_3)\left[\left(1 - \frac{\pi_a^2}{2f^2}\right)\cos\alpha - \frac{\pi_1}{f}\sin\alpha\right] + \Delta m\left[\left(1 - \frac{\pi_a^2}{2f^2}\right)\tau_2\sin\alpha + \frac{\pi_a}{f}\left(\delta_{a1}\tau_2\cos\alpha - \delta_{a2}\tau_1\right)\right]\Bigg\},$$

$$\chi\Sigma^\dagger - \Sigma\chi^\dagger = -4iB_0\Bigg\{\bar{m}\left[\left(1 - \frac{\pi_a^2}{2f^2}\right)\tau_1\sin\alpha + \frac{\pi_a}{f}\left(\tau_a - 2\delta_{1a}\tau_1\sin^2\frac{\alpha}{2}\right)\right] + \Delta m\frac{\pi_3}{f}\Bigg\}.$$

Combining these results gives all the terms in $\mathcal{L}_4$, to $\mathcal{O}\left((\pi/f)^3\right)$:

$$\begin{aligned}
&\mathrm{Tr}\big\{\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\big\}^2 = \mathrm{Tr}\big\{\partial_\mu\Sigma\partial^\mu\Sigma^\dagger - i\left(\partial_\mu\Sigma\left[v^\mu,\Sigma^\dagger\right] - \text{h.c.}\right) - \left[v_\mu,\Sigma\right]\left[v^\mu,\Sigma^\dagger\right]\big\}^2\\
&\quad = \frac{8\mu_I^2}{f^2}(\partial_\mu\pi_a\partial^\mu\pi_a + 2\partial_\mu\pi_2\partial^\mu\pi_2)\sin^2\alpha\\
&\quad + 16\mu_I^3\left[\frac{\partial_0\pi_2}{f}\sin^3\alpha + \frac{1}{f^2}\left(3\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1\right)\cos\alpha\sin^2\alpha\right]\\
&\quad + 4\mu_I^4\left\{\sin^4\alpha + 2\sin^2\alpha\left[\frac{\pi_1}{f}\sin 2\alpha + \frac{\pi_a\pi_b}{f^2}\left(k_{ab} + 2\delta_{a1}\delta_{a2}\cos^2\alpha\right)\right]\right\},
\end{aligned} \tag{20}$$

$$\begin{aligned}
&\mathrm{Tr}\big\{\nabla_\mu\Sigma(\nabla_\nu\Sigma)^\dagger\big\}\,\mathrm{Tr}\big\{\nabla^\mu\Sigma(\nabla^\nu\Sigma)^\dagger\big\}\\
&\quad = \frac{4{\mu_I}^2}{f^2}\left(\partial_0\pi_a\partial_0\pi_a + \partial_0\pi_2\partial_0\pi_2 + \partial_\mu\pi_2\partial^\mu\pi_2\right)\sin^2\alpha\\
&\quad + 16\mu_I^3\left[\frac{\partial_0\pi_2}{f}\ \sin^3\alpha + \frac{1}{f^2}\left(3\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1\right)\cos\alpha\sin^2\alpha\right]\\
&\quad + 4\mu_I^4\left\{\sin^4\alpha + 2\sin^2\alpha\left[\frac{\pi_1}{f}\sin 2\alpha + \frac{\pi_a\pi_b}{f^2}\left(k_{ab} + 2\delta_{a1}\delta_{a2}\cos^2\alpha\right)\right]\right\},
\end{aligned} \tag{21}$$

$$\begin{aligned}
&\mathrm{Tr}\big\{\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\big\}\,\mathrm{Tr}\big\{\chi\Sigma^\dagger + \Sigma\chi^\dagger\big\}\\
&\quad = 8B_0\bar{m}\Bigg\{2\frac{\partial_\mu\pi_a\partial^\mu\pi_a}{f^2}\cos\alpha + 4\mu_I\left[\frac{\partial_0\pi_2}{2f}\sin 2\alpha + \frac{1}{f^2}\left(\pi_1\partial_0\pi_2\cos 2\alpha - \pi_2\partial_0\pi_1\cos^2\alpha\right)\right]\\
&\quad + {\mu_I}^2\left[2\cos\alpha\sin^2\alpha - 2\frac{\pi_1}{f}\sin\alpha\left(3\sin^2\alpha - 1\right) + \frac{1}{f^2}\left(\pi_1^2[2 - 9\sin^2\alpha] + \pi_2^2[2 - 3\sin^2\alpha] - 3\pi_3^2\sin^2\alpha\right)\cos\alpha\right]\Bigg\},
\end{aligned} \tag{22}$$

$$\mathrm{Tr}\big\{\chi\Sigma^\dagger + \Sigma\chi^\dagger\big\}^2 = (8B_0\bar{m})^2\left[\cos^2\alpha - \frac{\pi_1}{f}\sin 2\alpha + \frac{1}{f^2}\left(\pi_1^2\sin^2\alpha - \pi_a\pi_a\cos^2\alpha\right)\right], \tag{23}$$

$$\mathrm{Tr}\big\{\chi\Sigma^\dagger - \Sigma\chi^\dagger\big\}^2 = -16\left(\frac{2\Delta m B_0\pi_3}{f}\right)^2, \tag{24}$$

$$\begin{aligned}
&\mathrm{Tr}\Big\{\left(\chi\Sigma^\dagger\right)^2 + \left(\Sigma\chi^\dagger\right)^2\Big\}\\
&\qquad = 16B_0^2\bar{m}^2\left(\cos 2\alpha - 2\frac{\pi_1}{f}\sin 2\alpha - 2\frac{\pi_a\pi_a}{f^2}\cos^2\alpha + 2\frac{\pi_1^2}{f^2}\sin^2\alpha\right) + 16B_0^2\Delta m^2\left(1 - 2\frac{\pi_3^2}{f^2}\right),
\end{aligned} \tag{25}$$

$$\mathrm{Tr}\big\{\chi^\dagger\chi\big\} = 8B_0^2\left(\bar{m}^2 + \Delta m^2\right). \tag{26}$$

2 Effective Potential

The ground state partition function is given by

$$Z = \lim_{T\to\infty}\langle\Omega,T|-T,\Omega\rangle = \lim_{T\to\infty}\left\langle\Omega\middle|e^{-2iHT}\middle|\Omega\right\rangle = \int\mathcal{D}\pi\mathcal{D}\varphi\,\exp\Bigg\{i\int\mathrm{d}^4x\,\left(\pi\dot{\varphi} - \mathcal{H}[\pi,\varphi]\right)\Bigg\}, \tag{27}$$

where $|\Omega\rangle$ is the vacuum of the theory [6, 7]. If one introduces a source by altering the Hamiltonian, $\mathcal{H} \to \mathcal{H} - J(x)\varphi(x)$, we get the generating functional

$$Z[J] = \int \mathcal{D}\pi\mathcal{D}\varphi \, \exp\left\{ i \int \mathrm{d}^4x \, (\pi\dot{\varphi} - \mathcal{H}[\pi, \varphi] + J\varphi) \right\} \tag{28}$$

If $\mathcal{H}$ is quadratic in π, we can complete the square and integrate out π to obtain

$$Z[J] = \int \mathcal{D}\varphi \, \exp\left\{ i \int \mathrm{d}x \, (\mathcal{L}[\varphi] + J\varphi) \right\} := \exp(iW[J]). \tag{29}$$

Here we defined the *generating functional for connected diagrams*, $W[J]$. The reason for the name will become apparent later. (HUSK Å REFFERE TILBAKE) The expectation value of some function of the field-configuration, $A = A[\varphi]$, in the precesense of the source J is

$$\langle A \rangle_J = \frac{1}{Z[J]} A \left(-i \frac{\delta}{\delta J} \right) Z[J]. \tag{30}$$

(DEFINE FUNCTIONAL DERIVATIVE) The expectation value of the field defines a functional,

$$\varphi[J](x) = \langle \varphi(x) \rangle_J = \frac{1}{Z[J]} \left(-i \frac{\delta}{\delta J} \right) Z[J] = \frac{\delta}{\delta J(x)} W[J], \tag{31}$$

and is sometimes called the *classical field*. The notation $\mathcal{F}[f](x)$ means that $\mathcal{F}$ is a functional which takes in a function f, and returns the new function $(\mathcal{F}[f])(x)$. One example is the Lagrangian density, which takes in a field, and returns a function which has a value for each point in space-time.

We can use the Legendre-transformation of $W[J]$ to define the new quantity,

$$\Gamma[\varphi] = W[J] - \int \mathrm{d}x \, J(x)\varphi(x), \tag{32}$$

such that

$$\frac{\delta}{\delta\varphi(x)}\Gamma[\varphi] = \int \mathrm{d}y \, \frac{\delta J(y)}{\delta\varphi(x)} \frac{\delta}{\delta J(y)} W[J] - \int \mathrm{d}y \, \frac{\delta J(y)}{\delta\varphi(x)} \varphi(x) - J(x) = -J(x). \tag{33}$$

Here, we used the relation Eq. (31). This means that the equations of motion for the expectation value of the field in the absence of an external current is

$$\frac{\delta\Gamma}{\delta\varphi} = 0. \tag{34}$$

This is similar to the classical Euler-Lagrange equations, which are Comparing this to the classical Euler-Lagrange equations with a source,

$$\frac{\delta S}{\delta\varphi} = \frac{\partial\mathcal{L}}{\partial\pi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} = 0, \tag{35}$$

we see that the effective potential Γ and is why Γ is called the *effective potential*.

In free theory, we may write

$$W[J] = \frac{1}{2} \int \mathrm{d}^x \mathrm{d}^4 y J(x) D_0(x-y) J(y), \tag{36}$$

where D_0 is the free propagator. We may reverse the relation Eq. (31) to write the source in terms of the field,

$$J = D_0^{-1}\varphi(x) \tag{37}$$

This is the field equation for the free field with a source. For the scalar Klein-Gordon field, $D_0^{-1} = \partial^2 + m^2$ Inserting these two relation into the definition of the effective action, and assuming we can do partial integration with D_0^{-1}, we get

$$\Gamma[\varphi] = W[J] - \int \mathrm{d}^4x J(x)\varphi(x) = \int \mathrm{d}x (\frac{1}{2} \int \mathrm{d}y (D_0^{-1}\varphi) D_0 (D_0^{-1}\varphi) - (D_0^{-1}\varphi)\varphi) = -\frac{1}{2} \int \mathrm{d}^4x \varphi(x) D_0^{-1} \varphi(x) \tag{38}$$

This is the classical action. Thus, the effective action Γ and the classical action S are the same to first order in perturbation theory.

To interpret Γ further, we can define a new theory, with a coupling g,

$$Z[J,g] = \int \mathcal{D}\varphi \exp\left\{ig^{-1}\left(\Gamma[\varphi] + \int \mathrm{d}^4x \varphi(x)J(x)\right)\right\} = e^{iW[J]} \tag{39}$$

The propagator for this theory is

$$D_g(x-y) = \left(-g^{-1}\frac{\delta^2\Gamma}{\delta\varphi(x)\delta\varphi(y)}\right)^{-1} \propto g \tag{40}$$

All vertices in this theory, on the other hand, will be proportional to g^{-1}. Regardless of what the Feynman-diagrams in this theory are, the number of loops of a connected diagram is $L = V - I + 1$, where V is the number of vertices, and I the number of internal lines. To see this, one can imagine adding a new vertex to an existing diagram. This splits a line in two. Any line stemming from this vertex that connect back into the same diagram, create the same number of new loops, vertices and lines, and the formula is fulfilled by induction. This means that any diagram is proportional to g^{L+1}. In the limit $g \to 0$, the theory is fully described by only three-level diagrams. Furthermore, we may use the stationary approximation on Eq. (39). In one dimension, it is given by

$$\int \mathrm{d}x \, \exp(igf(x)) \approx \sqrt{\frac{2\pi}{f''(x_0)}} \exp(f(x_0)), \; f'(x) = 0, g \to \infty \tag{41}$$

The generalization of this approximation gives

$$Z \approx C \det\left(-\frac{\delta^2\Gamma[\varphi']}{\delta\varphi^2}\right) \exp\left\{ig^{-1}\left(\Gamma[\varphi'] + \int \mathrm{d}^4x J\varphi'\right)\right\}, \tag{42}$$

where φ' fulfills

$$\frac{\delta}{\delta\varphi}\left(\Gamma[\varphi] + \int \mathrm{d}^4x J\varphi\right) = \frac{\delta}{\delta\varphi(x)}\Gamma + J(x) = 0, \tag{43}$$

which is exactly $\varphi[J](x)$. This means that

$$gW[J] = \Gamma[\varphi] + \int \mathrm{d}^4x \, J(x)\varphi(x) + \mathcal{O}(g) \tag{44}$$

Let φ^* solve the quantum mechanical version of the equation of motion, i.e.

$$\frac{\delta\Gamma[\varphi^*]}{\delta\varphi} = 0. \tag{45}$$

We can Taylor-expand the classical action around this point, by setting $\varphi(x) = \varphi^*(x) + \eta(x)$ for some function η. The generating functional becomes

$$Z[J] = \int \mathcal{D}(\varphi^* + \eta) \, \exp\left\{iS[\varphi^* + \eta] + i\int \mathrm{d}^4x J(\varphi^* + \eta)\right\} \tag{46}$$

The functional version of a Taylor expansion is

$$S[\varphi^* + \eta] = S[\varphi^*] + \int \mathrm{d}x \frac{\delta S[\varphi^*]}{\delta\varphi(x)}\eta(x) + \frac{1}{2}\int \mathrm{d}x\mathrm{d}y \, \frac{\delta^2 S[\varphi^*]}{\delta\varphi(x)\delta\varphi(y)}\eta(x)\eta(y) + \dots \tag{47}$$

Inserting this into $Z[J]$, with S_I to denote the derivatives of higher order than 2, we get

$$Z[J] =$$
$$\int \mathcal{D}\eta \exp\left\{i\int \mathrm{d}^4x \left(\mathcal{L}[\varphi^*] + J\varphi^*\right) + i\int \mathrm{d}x \left(\frac{\delta S[\varphi^*]}{\delta\varphi(x)} + J(x)\right)\eta(x) + i\frac{1}{2}\int \mathrm{d}x\mathrm{d}y \, \frac{\delta^2 S[\varphi^*]}{\delta\varphi(x)\delta\varphi(y)}\eta(x)\eta(y) + iS_I[\eta]\right\}$$

In the first term we used the definition of the classical action. This term is constant with respect to η, and may therefore be taken outside the path integral. The next term is the calssical eqation of motion with a source,

$$\frac{\delta S[\varphi]}{\delta\varphi(x)} = -J(x), \tag{48}$$

evaluated at φ^*.

$$\frac{\delta S[\varphi^*]}{\delta\varphi(x)} + J(x) = \frac{\delta\Gamma[\varphi^*]}{\delta\varphi(x)} + J(x) + \left(\frac{\delta S[\varphi^*]}{\delta\varphi(x)} - \frac{\delta\Gamma[\varphi^*]}{\delta\varphi(x)}\right) = \left(\frac{\delta S[\varphi^*]}{\delta\varphi(x)} - \frac{\delta\Gamma[\varphi^*]}{\delta\varphi(x)}\right) := \delta J \tag{49}$$

The second to last term is a Gaussian integral, and may be evaluated as described in subsection C.1,

$$\int \mathcal{D}\eta\, \exp\left(i\frac{1}{2}\int \mathrm{d}x\mathrm{d}y\, \frac{\delta^2 S[\varphi^*]}{\varphi(x)\varphi(y)}\eta(x)\eta(y)\right) = C\det\left(\frac{\delta^2 S[\varphi^*]}{\delta\varphi^2}\right)^{-1/2} \tag{50}$$

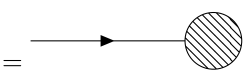
This leaves us with

$$W[J] = -i\ln(Z) \tag{51}$$

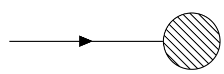
$$= \int \mathrm{d}^4x\, \left(\mathcal{L}[\varphi^*] + J\varphi^*\right) - \frac{1}{2}\mathrm{Tr}\left\{\ln\left(-\frac{\delta^2 S[\varphi^*]}{\delta\varphi^2}\right)\right\} + \int \mathcal{D}\eta\, \exp\left\{i\int \mathrm{d}^4x\delta J(x)\eta\right\} + \int \mathcal{D}\eta\, e^{iS_I} \tag{52}$$

δJ is ultimately dependent on our choice of J to define φ. It contributes to the expectation value of η, through tadpole diagrams

$$\langle\eta\rangle_{j=0} = \tag{53}$$



This can be removed by using the renormalization condition



$$= 0. \tag{54}$$

2.1 Minimizing energy

The value of α is found by minimizing the free energy. The first approximation to the free energy in the ground state is the static part of the Hamiltonian density $\mathcal{H}^{(0)}$, which we get from Eq. (9) through

$$\mathcal{H}_2^{(0)} = -\mathcal{L}_2^{(0)} = -f^2\left(2B_0 m\cos\alpha + \frac{1}{2}\mu^2\sin^2\alpha\right), \tag{55}$$

The minimum of this function is achieved when

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\mathcal{H}_2^{(0)} = f^2\left(2B_0 m - \mu_I^2\cos\alpha\right)\sin\alpha = 0.$$

This gives the solution set and minimization criterion

$$\alpha = \pi n,\, n\in\mathbb{Z} \quad \text{or} \quad \cos\alpha = \frac{2B_0 m}{{\mu_I}^2}. \tag{56}$$

We see that the linear part of the potential from Eq. (10), $\mathcal{V}^{(1)} = f({\mu_I}^2\cos\alpha - 2B_0 m)\pi_1\sin\alpha = 0$ if and only if the criterion for minimization is fulfilled.

2.2 Propagator

We may write the quadratic part of the Lagrangian Eq. (11) as [2]

$$\mathcal{L}_2^{(2)} = \frac{1}{2}\sum_a \partial_\mu\pi_a\partial^\mu\pi_a + \frac{1}{2}m_{12}(\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1) - \frac{1}{2}\sum_a m_a^2\pi_a^2, \tag{57}$$

[2]Summation over isospin index (a, b, c) will be explicit in this section.

where

$$
\begin{aligned}
m_1^2 &= 2B_0 m\cos\alpha - \mu_I^2\cos 2\alpha, && (58)\\
m_2^2 &= 2B_0 m\cos\alpha - \mu_I^2\cos^2\alpha, && (59)\\
m_3^2 &= 2B_0 m\cos\alpha + \mu_I^2\sin^2\alpha, && (60)\\
m_{12} &= 2\mu_I\cos\alpha. && (61)
\end{aligned}
$$

The components of the Euler-Lagrange equations of this field are

$$\frac{\partial\mathcal{L}}{\partial\pi_a} = \frac{1}{2}m_{12}(\delta_{a1}\partial_0\pi_2 - \delta_{a2}\partial_0\pi_1) - m_a^2\pi_a, \quad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\pi_a)} = \partial^\mu\pi_a - \frac{1}{2}m_{12}\delta_0^\mu(\delta_{a1}\pi_2 - \delta_{a2}\pi_1).$$

This gives the equation of motion for the field

$$\partial^\mu\partial_\mu\pi_a + m_a^2\pi_a = m_{12}(\delta_{a1}\partial_0\pi_2 - \delta_{a2}\partial_0\pi_1). \tag{62}$$

The propagator of the pion field is defined by

$$\left[\delta_{ab}(\partial^\mu\partial_\mu + m_a^2) - m_{12}(\delta_{a1}\delta_{b2} - \delta_{a2}\delta_{b1})\partial_0\right] D_{bc}(x,x') = -i\delta(x-x')\delta_{ac}. \tag{63}$$

The momentum space propagator, as defined in the Appendix A, fulfills

$$-\left[\delta_{ab}(p^2 - m_a^2) + ip_0 m_{12}(\delta_{a1}\delta_{b2} - \delta_{a2}\delta_{b1})\right]\tilde{D}_{bc}(p) := A_{ab}\tilde{D}_{bc}(p) = -i\delta_{ac},$$

where

$$A = -\begin{pmatrix} p^2 - m_1^2 & ip_0 m_{12} & 0 \\ -ip_0 m_{12} & p^2 - m_2^2 & 0 \\ 0 & 0 & p^2 - m_3^2 \end{pmatrix}.$$

The spectrum of the particles is given by solving $\det(A) = 0$ for p^0. With $p = (p_0, P)$ as the four momentum, this gives

$$\det(A) = A_{33}\left(A_{11}A_{22} + A_{12}^2\right) = -\left(p^2 - m_3^2\right)\left[\left(p^2 - m_1^2\right)\left(p^2 - m_2^2\right) - p_0^2 m_{12}^2\right] = 0,$$

This equation has the solutions

$$
\begin{aligned}
E_0^2 &= P^2 + m_2^2, && (64)\\
E_\pm^2 &= P^2 + \frac{1}{2}\left(m_1^2 + m_2^2 + m_{12}^2\right) \pm \frac{1}{2}\sqrt{4P^2 m_{12}^2 + \left(m_1^2 + m_2^2 + m_{12}^2\right)^2 - 4m_1^2 m_2^2}. && (65)
\end{aligned}
$$

This gives the effective masses

$$
\begin{aligned}
m_0^2 &= m_2^2, && (66)\\
m_\pm^2 &= \frac{1}{2}\left[m_1^2 + m_2^2 + m_{12}^2\right] \pm \frac{1}{2}\sqrt{\left(m_1^2 + m_2^2 + m_{12}^2\right)^2 - 4m_1^2 m_2^2}. && (67)
\end{aligned}
$$

The propagator may then be obtained as described in Appendix A,

$$
\begin{aligned}
D_0 = iA^{-1} &= \frac{i}{\det(A)}\begin{pmatrix} A_{22}A_{33} & A_{12}A_{33} & 0 \\ -A_{12}A_{33} & A_{11}A_{33} & 0 \\ 0 & 0 & A_{11}A_{22} + A_{12}^2 \end{pmatrix} \\
&= i\begin{pmatrix} \frac{p^2 - m_2^2}{(p_0^2 - E_+^2)(p_0^2 - E_-^2)} & \frac{-ip_0 m_{12}}{(p_0^2 - E_+^2)(p_0^2 - E_-^2)} & 0 \\ \frac{ip_0 m_{12}}{(p_0^2 - E_+^2)(p_0^2 - E_-^2)} & \frac{p^2 - m_1^2}{(p_0^2 - E_+^2)(p_0^2 - E_-^2)} & 0 \\ 0 & 0 & \frac{1}{p_0^2 - E_0^2} \end{pmatrix}. && (68)
\end{aligned}
$$

2.3 Free energy of the pions

The action of χPT is

$$S[\pi_a] = \int \mathrm{d}x \mathcal{L}[\pi_a] = \int \mathrm{d}x \left(\mathcal{L}_2^{(0)}[\pi_a] + \mathcal{L}_2^{(1)}[\pi_a] + \mathcal{L}_2^{(2)}[\pi_a] + \mathcal{L}_4^{(2)}[\pi_a] + \mathcal{L}_I[\pi_a]\right), \tag{69}$$

where $\mathcal{L}_I[\pi_a]$ is the higher order terms. The action may be expanded around $\pi_a = 0$,

$$S[\varphi] = S[\pi_a = 0] + \int \mathrm{d}x\, \pi_a \frac{\delta S}{\delta \pi_a(x)}\Big|_{\pi_a=0} + \int \mathrm{d}x\mathrm{d}y\, \pi_a(x)\pi_b(y) \frac{\delta^2 S}{\delta \pi_a(x)\delta\pi_a(y)}\Big|_{\pi_a=0} + \mathcal{O}\left((\pi/f)^3\right). \tag{70}$$

We showed that, when minimizing α, $\frac{\delta S_2}{\delta \pi_a(x)}\Big|_{\pi_a=0} = 0$. Furthermore, $\mathcal{L}_2[\pi_a = 0] = \mathcal{L}_2^{(0)}$. In the limit $\beta = \infty$, the free energy density of the system is given by $\beta\mathcal{F} = \frac{i}{TV}\ln(Z)$ (HVORFOR?). Using the expansion as described in section 2, the free energy density to and including second order is given by

$$\Omega = \beta\mathcal{F} = \frac{i}{TV}\ln(Z) = S[\pi_a = 0] + \ln\det\left(-\frac{\delta^2 S[\pi_a = 0]}{\delta\pi^2}\right)^{-1/2}. \tag{71}$$

As the ground sate is $\pi_a = 0$, the only terms that will remain in the trace-terms are those that are second order in the pion-fields. We can therefore evaluate the functional derivative with the rules given in Appendix D,

$$\frac{\delta^2 S}{\delta\varphi(x)\delta\varphi(y)} = \frac{\delta^2}{\delta\varphi(x)\delta\varphi(y)}\int \mathrm{d}x\, \mathcal{L}_2^{(2)} = D_x^{-1}\delta(x-y), \tag{72}$$

where $\mathcal{L}_2^{(2)}$ is the quadratic part of the Lagrangian, as given in Eq. (57), and D_x^{-1} is the corresponding inverse propagator of the pion-fields,

$$D_x^{-1} = -\left[\delta_{ab}(\partial_x^\mu \partial_{x,\mu} + m_a^2) - m_{12}(\delta_{a1}\delta_{b2} - \delta_{a2}\delta_{b1})\partial_{x,0}\right] \tag{73}$$

The first order contribution is

$$\Omega_0 = \frac{1}{VT}\int \mathrm{d}^4x\, \mathcal{L}_2^{(0)} = \mathcal{L}_2^{(0)}. \tag{74}$$

The leading order correction, Ω_2, is given by the functional determinant. The functional determinant in Eq. (71) has a matrix part, due to the 3 pion-indices, as well as a functional part. In subsection 2.2 we found the matrix part of the determinant in momentum space, which we can write using the dispersion relations of the pion fields as

$$\det\bigl(-D^{-1}\delta(x-y)\bigr) = \det\bigl(p_0^2 - E_0^2\bigr)\det\bigl(p_0^2 - E_+^2\bigr)\det\bigl(p_0^2 - E_-^2\bigr). \tag{75}$$

The Dirac-delta is removed by using partial integration. The functional determinant can therefore be evaluated as

$$-\frac{1}{2}\ln\det\left(-\frac{\delta^2 S[\pi_a = 0]}{\delta\pi^2}\right) = \ln\det\bigl(p_0^2 - E_0^2\bigr) + \ln\det\bigl(p_0^2 - E_+^2\bigr) + \ln\det\bigl(p_0^2 - E_-^2\bigr) \tag{76}$$

$$= \mathrm{Tr}\bigl\{\ln\bigl(p_0^2 - E_0^2\bigr) + \ln\bigl(p_0^2 - E_+^2\bigr) + \ln\bigl(p_0^2 - E_-^2\bigr)\bigr\} \tag{77}$$

$$= (VT)\int \frac{\mathrm{d}^4p}{(2\pi)^4}\left[\ln\bigl(p_0^2 - E_0^2\bigr) + \ln\bigl(p_0^2 - E_+^2\bigr) + \ln\bigl(p_0^2 - E_-^2\bigr)\right]. \tag{78}$$

These all have the form

$$I = \int \frac{\mathrm{d}^4p}{(2\pi)^2}\ln\bigl(-p^2 + \omega(p)^2\bigr). \tag{79}$$

We will here pereform a Wick-rotation, and use the trick

$$\frac{\partial}{\partial\alpha}\bigl(p^2 + \omega(p)^2\bigr)^{-\alpha}\Big|_{\alpha=0} = \frac{\partial}{\partial\alpha}\exp\bigl\{-\alpha\ln\bigl(p^2 + \omega(p)^2\bigr)\bigr\}\Big|_{\alpha=0} = \ln\bigl(p^2 + \omega(p)^2\bigr), \tag{80}$$

to write the integral on the form

$$I = \frac{\partial}{\partial\alpha}\int \frac{\mathrm{d}p_E^2}{(2\pi)^4}(p_E^2 + \omega(p)^2)^{-\alpha}\Big|_{\alpha=0} \tag{81}$$

3 Thermal Field Theory

3.1 Statistichal Mechanics

In classical mechanics, a thermal system at temperature $T = 1/\beta$ is described as an ensemble state, which have a probability P_n of being in state n, with energy E_n. In the canonical ensemble, the probability is proportional to $e^{-\beta E_n}$. The expectation value of some quantity A, with value A_n in state n is

$$\langle A \rangle = \sum_n A_n P_n = \frac{1}{Z} \sum_n A_n e^{-\beta E_n}, \quad Z = \sum_n e^{-\beta A_n}.$$

Z is the partition function. In quantum mechanics, an ensemble configuration is described by a non-pure density operator,

$$\hat{\rho} = C \sum_n P_n \left| n \rangle\langle n \right|,$$

where $|n\rangle$ is some basis for the relevant Hilbert space and C is a constant. Assuming $|n\rangle$ are energy eigenvectors, i.e. $\hat{H}\,|n\rangle = E_n\,|n\rangle$, the density operator for the canonical ensemble is

$$\hat{\rho} = C \sum_n e^{-\beta E_n} \left| n \rangle\langle n \right| = C e^{-\beta \hat{H}} \sum_n \left| n \rangle\langle n \right| = C e^{-\beta \hat{H}}.$$

The expectation value in the ensemble state of a quantity corresponding to the operator $\hat{A}$ is given by

$$\langle A \rangle = \frac{\mathrm{Tr}\left\{\hat{\rho}\hat{A}\right\}}{\mathrm{Tr}\{\hat{\rho}\}} = \frac{1}{Z}\,\mathrm{Tr}\left\{\hat{A} e^{-\beta \hat{H}}\right\} \tag{82}$$

The partition function Z ensures that the probabilities adds up to 1, and is defined as

$$Z = \mathrm{Tr}\left\{e^{-\beta \hat{H}}\right\}. \tag{83}$$

The grand canonical ensemble takes into account the conserved charges of the system. Conserved charges are a result of Nöther's theorem. Assume we have a set of fields φ_α. Nöther's theorem tells us that if the Lagrangian $\mathcal{L}[\varphi_\alpha]$ has a *continuous symmetry*, then there is a corresponding conserved current [6, 8]. To define a continuous symmetry of the Lagrangian, we need a one-parameter family of transformations,

$$\varphi_\alpha(x) \longrightarrow \varphi'_\alpha(x;\epsilon) \sim \varphi_\alpha(x) + \varepsilon \eta_\alpha(x),\ \varepsilon \to 0.$$

Here, $\eta_\alpha(x)$ is some arbitrary function which define the transformation as $\varepsilon \to 0$. Applying this transformation to the Lagrangian will in general change its form,

$$\mathcal{L}[\varphi_\alpha] \to \mathcal{L}[\varphi'_\alpha] \sim \mathcal{L}[\varphi_\alpha] + \varepsilon \delta \mathcal{L},\ \varepsilon \to 0.$$

If the change in the Lagrangian can be written as a total derivative, i.e.

$$\delta \mathcal{L} = \partial_\mu K^\mu(x),$$

we say that the Lagrangian has a continuous symmetry. This is because a term of this form will result in a boundary term in the action integral, which does not contribute to the variation of the action. Nöther's theorem states more precisely that the current

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_\alpha)} \eta_\alpha - K^\mu \tag{84}$$

obeys the conservation law

$$\partial_\mu j^\mu = 0. \tag{85}$$

The flux of current through some space-like surface V, i.e. a surface with a time-like normal vector, defines a conserved charge. This surface is most commonly a surface of constant time in some reference frame. The charge is defined as

$$Q = \int_V \mathrm{d}^3x \, n_\mu j^\mu, \tag{86}$$

where n^μ is the normal vector of V. Using the divergence theorem again, and assuming the current falls of quickly enough, we can show that the total charge is conserved,

$$\frac{\partial}{\partial t} Q = \int_V \mathrm{d}^3x \, \nabla \cdot \vec{j} = \int_{\partial V} \mathrm{d}^2x \, n'_\mu j^\mu = 0.$$

For a surface of constant time we may choose $n^\mu = (1, 0, 0, 0)$, and the conserved charged is

$$Q = \int_V \mathrm{d}x \, j^0.$$

In the grand canonical ensemble, a system with n conserved charges Q_i has probability proportional to $e^{-\beta(H - \mu_i Q_i)}$. μ_i are the chemical potentials corresponding to conserved charge Q_i. This leads to the partition function

$$Z = \mathrm{Tr}\Big\{ e^{-\beta(\hat{H} - \mu_i \hat{Q}_i)} \Big\}. \tag{87}$$

3.2 Imaginary-time formalism

The partition function may be calculated in a similar way to the path integral approach, in what is called the imaginary-time formalism. This formalism is restricted to time independent problems, and is used to study fields in a volume V. This volume is taken to infinity in the thermodynamic limit. As an example, take a scalar quantum field theory with the Hamiltonian

$$\hat{H} = \int_V \mathrm{d}^3x \, \hat{\mathcal{H}} \left[\hat{\varphi}(\vec{x}), \hat{\pi}(\vec{x}) \right], \tag{88}$$

where $\hat{\varphi}(\vec{x})$ is the field operator, and $\hat{\pi}(\vec{x})$ is the corresponding canonical momentum operator. These field operators have time independent eigenvectors, $|\varphi\rangle$ and $|\pi\rangle$, defined by

$$\hat{\varphi}(\vec{x}) \, |\varphi\rangle = \varphi(\vec{x}) \, |\varphi\rangle \, , \quad \hat{\pi}(\vec{x}) \, |\pi\rangle = \pi(\vec{x}) \, |\pi\rangle \, . \tag{89}$$

In analogy with regular quantum mechanics, they obey the relations [9] [3]

$$\mathbb{1} = \int \mathcal{D}\varphi(\vec{x}) \, |\varphi\rangle\langle\varphi| = \int \mathcal{D}\pi(\vec{x}) \, |\pi\rangle\langle\pi| \, , \tag{90}$$

$$\langle \varphi | \pi \rangle = \exp\!\left(i \int_V \mathrm{d}x \, \varphi(\vec{x}) \pi(\vec{x}) \right), \tag{91}$$

$$\langle \pi_a | \pi_b \rangle = \delta(\phi_a - \phi_b), \quad \langle \varphi_a | \varphi_b \rangle = \delta(\varphi_a - \varphi_b). \tag{92}$$

The functional integral is defined by starting with M degrees of freedom, $\{\varphi_m\}_{m=1}^M$ located at a finite grid $\{\vec{x}_m\}_{m=1}^M \subset V$. The integral is then the limit of the integral over all degrees of freedom, as $M \to \infty$:

$$\int \mathcal{D}\varphi(\vec{x}) = \lim_{M \to \infty} \int \left(\prod_{m=1}^M \mathrm{d}\varphi_m \right).$$

The functional Dirac-delta $\delta(f) = \prod_x \delta(f(x))$ is generalization of the familiar Dirac delta function. Given a functional $\mathcal{F}[f]$, it is defined by the relation

$$\int \mathcal{D}f(x) \, \mathcal{F}[f] \delta(f - g) = \mathcal{F}[g]. \tag{93}$$

[3] Some authors write $\mathcal{D}\pi/2\pi$. This extra factor 2π is a convention which in this text is left out for notational clarity.

The Hamiltonian is the limit of a sum of Hamiltonians $\hat{H}_m$ for each point $\vec{x}_m$

$$\hat{H} = \lim_{M\to\infty} \sum_{m=1}^{M} \frac{V}{M} \hat{H}_m(\{\hat{\varphi}_m\}, \{\hat{\pi}_m\}).$$

H_m may depend on the local degrees of freedom $\hat{\varphi}_m$, $\hat{\pi}_m$ as well as those at neighboring points. By inserting the completeness relations Eq. (90) N times into the definition of the partition function, it may be written as

$$Z = \int \mathcal{D}\varphi(\vec{x}) \left\langle \varphi \middle| e^{-\beta \hat{H}} \middle| \varphi \right\rangle = \prod_{n=1}^{N} \left(\int \mathcal{D}\varphi_n(\vec{x}) \int \mathcal{D}\pi_n(\vec{x}) \right) \prod_{n=1}^{N} \langle \varphi_n | \pi_n \rangle \left\langle \pi_n \middle| e^{-\epsilon \hat{H}} \middle| \varphi_{n+1} \right\rangle \langle \varphi_1 | \varphi_{n+1} \rangle ,$$

where $\epsilon = \beta/N$. The last term ensures that $\varphi_1 = \varphi_{N+1}$. Strictly speaking, we only need to require $\varphi_1 = e^{i\theta}\varphi_{N+1}$, as the partition function is only defined up to a constant. As will be shown later, bosons such as the scalar field φ, follow the periodic boundary condition $\varphi(0, \vec{x}) = \varphi(\beta, \vec{x})$, i.e. $e^{i\theta} = 1$, while fermions follow the anti-periodic boundary condition $\psi(0, \vec{x}) = -\psi(\beta, \vec{x})$, i.e. $e^{i\theta} = -1$. We now want to exploit the fact that $|\pi\rangle$ and $|\varphi\rangle$ are the eigenvectors of the operators that define the Hamiltonian. In our case, as the Hamiltonian density $\mathcal{H}$ can be written as a sum of functions of φ and π separately, $\mathcal{H}[\varphi(\vec{x}), \pi(\vec{x})] = \mathcal{F}_1[\varphi(\vec{x})] + \mathcal{F}_2[\pi(\vec{x})]$ we may evaluate it as $\langle \pi_n | \mathcal{H}[\hat{\varphi}(x), \hat{\pi}(x)] | \varphi_{n+1} \rangle = \mathcal{H}[\varphi_{n+1}(x), \pi_n(x)] \langle \pi_n | \varphi_{n+1} \rangle$. This relationship does not, however, hold for more general functions of the field operators. In that case, one has to be more careful about the ordering of the operators, for example by using *Weyl ordering* [6]. By series expanding $e^{-\epsilon \hat{H}}$ and exploiting this relationship, the partition function can be written as, to second order in ϵ,

$$Z = \prod_{n=1}^{N} \left(\int \mathcal{D}\varphi_n(\vec{x}) \int \mathcal{D}\pi_n(\vec{x}) \right) \exp\left[-\epsilon \sum_{n=1}^{N} \int_V \mathrm{d}^3x \, \left(\mathcal{H}(\varphi_n(\vec{x}), \pi_n(\vec{x})) - i\pi_n(\vec{x}) \frac{\varphi_n(\vec{x}) - \varphi_{n+1}(\vec{x})}{\epsilon} \right) \right].$$

We denote $\varphi_n(\vec{x}) = \varphi(\tau_n, \vec{x})$, $\tau \in [0, \beta]$ and likewise with $\pi_n(\vec{x})$. In the limit $N \to \infty$, the expression for the partition function becomes

$$Z = \int_S \mathcal{D}\varphi(\tau, \vec{x}) \int \mathcal{D}\pi(\tau, \vec{x}) \exp\left\{ -\int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}\vec{x} \, \{ \mathcal{H}[\varphi(\tau, \vec{x}), \pi(\tau, \vec{x})] - i\pi(\tau, \vec{x})\dot{\varphi}(\tau, \vec{x}) \} \right\}, \tag{94}$$

where S is the set of field configurations φ such that $\varphi(\beta, \vec{x}) = \varphi(0, \vec{x})$. With a Hamiltonian density of the form $\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\varphi)^2 + \mathcal{V}(\varphi)$, we can evaluate the integral over the canonical momentum π by discretizing $\pi(\tau_n, \vec{x}_m) = \pi_{n,m}$,

$$\begin{aligned}
&\int \mathcal{D}\pi \exp\left\{ -\int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}^3x \left(\frac{1}{2}\pi^2 - i\pi\dot{\varphi} \right) \right\} \\
&= \lim_{M,N\to\infty} \int \left(\prod_{m,n=1}^{M,N} \frac{\mathrm{d}\pi_{m,n}}{2\pi} \right) \exp\left\{ -\sum_{m,n} \frac{V\beta}{MN} \left[\frac{1}{2}(\pi_{m,n} - i\dot{\varphi}_{m,n})^2 + \frac{1}{2}\dot{\varphi}_{m,n}^2 \right] \right\} \\
&= \lim_{M,N\to\infty} \left(\frac{MN}{2\pi V\beta} \right)^{MN/2} \exp\left\{ -\int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}^3x \, \frac{1}{2}\dot{\varphi}^2 \right\},
\end{aligned}$$

where $\dot{\varphi}_{m,n} = (\varphi_{m,n+1} - \varphi_{m,n})/\epsilon$. The partition function is then,

$$Z = C \int \mathcal{D}\varphi \exp\left\{ -\int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}^3x \left[\frac{1}{2}\left(\dot{\varphi}^2 + \nabla\varphi^2 \right) + \mathcal{V}(\varphi) \right] \right\}. \tag{95}$$

Here, C is the divergent constant that results from the π-integral. In the last line, we exploited the fact that the variable of integration $\pi_{n,m}$ may be shifted by a constant without changing the integral, and used the Gaussian integral

$$\int_{-\infty}^{\infty} \mathrm{d}x \, e^{-ax^2/2} = \sqrt{\frac{2\pi}{a}}.$$

The partition function resulting from this procedure may also be obtained by starting with the ground state path integral

$$Z_g = \int \mathcal{D}\varphi \mathcal{D}\pi \exp\biggl\{ i \int_{\Omega'} \mathrm{d}^4x \; (\pi\dot{\varphi} - \mathcal{H}[\varphi,\pi]) \biggr\} = C' \int \mathcal{D}\varphi(x) \exp\biggl\{ i \int_{\Omega'} \mathrm{d}^4x \, \mathcal{L}[\varphi, \partial_\mu \varphi] \biggr\},$$

and follow a formal procedure. First, the action integral is modified by performing a Wick-rotation of the time coordinate t. This involves changing the domain of t from the real line to the imaginary line by closing the contour at infinity, and making the change of variable $it \to \tau$. The new variable is then restricted to the interval $\tau \in [0, \beta]$, and the domain of the functional integral $\int \mathcal{D}\varphi$ is restricted from *all* (smooth enough) field configurations $\varphi(t, \vec{x})$, to only those that obey $\varphi(\beta, \vec{x}) = e^{i\theta}\varphi(0, \vec{x})$, which is denoted S. This procedure motivates the introduction of the Euclidean Lagrange density, $\mathcal{L}_E(\tau, \vec{x}) = -\mathcal{L}(-i\tau, \vec{x})$, as well as the name “imaginary-time formalism”. The result is the same partition function as before,

$$\begin{aligned} Z &= C \int_S \mathcal{D}\varphi \int \mathcal{D}\pi \exp\biggl\{ - \int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}^3x \; [-i\dot{\varphi}\pi + \mathcal{H}(\varphi, \pi)] \biggr\} \\ &= C' \int_S \mathcal{D}\varphi \exp\biggl\{ - \int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}^3x \, \mathcal{L}_E(\varphi, \pi) \biggr\}. \end{aligned} \tag{96}$$

3.3 Free scalar field

This section uses notation as described in Appendix A. The procedure for obtaining the thermal properties of an interacting scalar field is similar to that used in scattering theory. One starts with a free theory, which can be solved exactly. Then an interaction term is added, which is accounted for peturbatively by using Feynman diagrams. The Euclidean Lagrangian for a free scalar gas is, after integrating by parts,

$$\mathcal{L}_E = \frac{1}{2}\varphi(X)\left(-\partial_E^2 + m^2\right)\varphi(X) \tag{97}$$

Here, $X = (\tau, \vec{x})$ is the Euclidean coordinate which is a result of the Wick-rotation. We have also introduced the Euclidean Laplace operator, $\partial_E^2 = \partial_\tau^2 + \nabla^2$. Following the procedure as described in subsection 3.2 to obtain the thermal partition function yields

$$Z = C \int_S \mathcal{D}\varphi(X) \exp\biggl\{ - \int_\Omega \mathrm{d}X \frac{1}{2}\varphi(X)\left(-\partial_E^2 + m^2\right)\varphi(X) \biggr\}. \tag{98}$$

Here, Ω is the domain $[0, \beta] \times V$. We then insert the Fourier expansion of φ, and change the functional integration variable to the Fourier components. The integration measures are related by

$$\mathcal{D}\varphi(X) = \det\biggl(\frac{\delta\varphi(X)}{\delta\tilde{\varphi}(K)}\biggr)\mathcal{D}\tilde{\varphi}(K),$$

where $K = (\omega_n, \vec{k})$ is the Euclidean Fourier-space coordinate. The determinant factor which appears may be absorbed into the constant C, as the integration variables are related by a linear transform. The action becomes

$$\begin{aligned} S = - \int_\Omega \mathrm{d}X \, \mathcal{L}_e &= -\frac{1}{2}V\beta \int_\Omega \mathrm{d}X \int_{\tilde{\Omega}} \mathrm{d}K \int_{\tilde{\Omega}} \mathrm{d}K' \, \tilde{\varphi}(K') \left(\omega_n^2 + \vec{k}^2 + m^2\right) \tilde{\varphi}(K) e^{iX\cdot(K-K')} \\ &= -\frac{1}{2}V\beta^2 \int_{\tilde{\Omega}} \mathrm{d}K \, \tilde{\varphi}(K)^* \left(\omega_n^2 + \omega_k^2\right) \tilde{\varphi}(K), \end{aligned}$$

where $\omega_k^2 = \vec{k}^2 + m^2$. $\tilde{\Omega}$ is the reciprocal space corresponding to Ω, as described in Appendix A. We used the fact that φ is real, which implies that $\tilde{\varphi}(-K) = \tilde{\varphi}(K)^*$, as well as the identity Eq. (161). This gives the partition function

$$Z = C \int_{\tilde{S}} \mathcal{D}\tilde{\varphi}(K) \exp\biggl\{ -\frac{1}{2}V \int_{\tilde{\Omega}} \mathrm{d}K \, \tilde{\varphi}(K)^* \left[\beta^2(\omega_n^2 + \omega_k^2)\right] \tilde{\varphi}(K) \biggr\}, \tag{99}$$

Going back to before the continuum limit, this integral can be written as a product of Gaussian integrals, and may therefore be evaluated

$$Z = C\prod_{n=-\infty}^{\infty}\prod_{k\in\tilde{V}}\left(\int \mathrm{d}\tilde{\varphi}_{n,\vec{k}}\,\exp\left\{-\frac{1}{2}\tilde{\varphi}^{*}_{n,\vec{k}}\left[\beta^2(\omega_n^2+\omega_k^2)\right]\tilde{\varphi}_{n,\vec{k}}\right\}\right) = C\prod_{n=-\infty}^{\infty}\prod_{k\in\tilde{V}}\sqrt{\frac{2\pi}{\beta^2(\omega_n^2+\omega_k^2)}}.$$

The partition function is related to Helmholtz free energy F through

$$\frac{F}{TV} = -\frac{\ln(Z)}{V} = \frac{1}{2}\int_{\tilde{\Omega}}\mathrm{d}K\frac{1}{2}\ln[\beta^2(\omega_n^2+\omega_k^2)] + \frac{F_0}{TV}, \tag{100}$$

where F_0 is a constant.

A faster and more formal way to get to this result is to compare the partition function to the multidimensional version of the Gaussian integral [9, 6]. The partition function is on the form

$$I_n = \int_{\mathbb{R}^n}\mathrm{d}^n x\,\exp\left\{-\frac{1}{2}\langle x, D_0^{-1}x\rangle\right\},$$

where D_0^{-1} is a linear operator, and $\langle\cdot,\cdot\rangle$ an inner product on the corresponding vector space. By diagonalizing D_0^{-1}, we get the result

$$I_n = \sqrt{\frac{(2\pi)^n}{\det\left(D_0^{-1}\right)}}.$$

We may then use the identity

$$\det(D) = \prod_i \lambda_i = \exp\left\{\mathrm{Tr}\left[\ln\left(D_0^{-1}\right)\right]\right\}, \tag{101}$$

where λ_i are the eigenvalues of D_0^{-1}. The trace in this context is defined by the vector space D_0^{-1} acts on. For given an orthonormal basis x_n, such that $\langle x_n, x_n'\rangle = \delta_{nn'}$ the trace can be evaluated as $\mathrm{Tr}\left\{D_0^{-1}\right\} = \sum_n\langle x_n, D_0^{-1}x_n\rangle$. Identifying

$$\langle x, D_0^{-1}x\rangle = \int_{\Omega}\mathrm{d}X\varphi(X)\left(-\partial_E^2+m^2\right)\varphi(X),$$

we get the formal result

$$Z = \det\left(-\partial_E^2+m^2\right)^{-1/2},$$

and

$$\beta F = \frac{1}{2}\,\mathrm{Tr}\left\{\ln\left(-\partial_E^2+m^2\right)\right\}.$$

The logarithm may then be evaluated by using the eigenvalues of the linear operator. This is found by diagonalizing the operator,

$$\langle x, D_0^{-1}x\rangle = \int_{\Omega}\mathrm{d}X\varphi(X)\left(-\partial_E^2+m^2\right)\varphi(X) = V\int_{\tilde{\Omega}}\mathrm{d}K\tilde{\varphi}(K)^*[\beta^2(\omega_k^2+\omega_n^2)]\tilde{\varphi}(K),$$

leaving us with the same result as we obtained in Eq. (100),

$$\beta F = \frac{1}{2}\,\mathrm{Tr}\left\{\ln\left(-\partial_E^2+m^2\right)\right\} = \frac{1}{2}V\int_{\tilde{\Omega}}\mathrm{d}K\ln[\beta^2(\omega_n^2+\omega_k^2)].$$

Sums similar to this show up a lot, and it is show how to evaluate them in subsection 3.4. Comparing with the definitions of the thermal propagator in Appendix A, we can write the free energy compactly as

$$\beta F = \frac{1}{2}\,\mathrm{Tr}\left\{\ln[D_0^{-1}(K,K'))]\right\} = \frac{1}{2}\,\mathrm{Tr}\left\{\ln[\beta^2 D_0^{-1}(K)]\right\}. \tag{102}$$

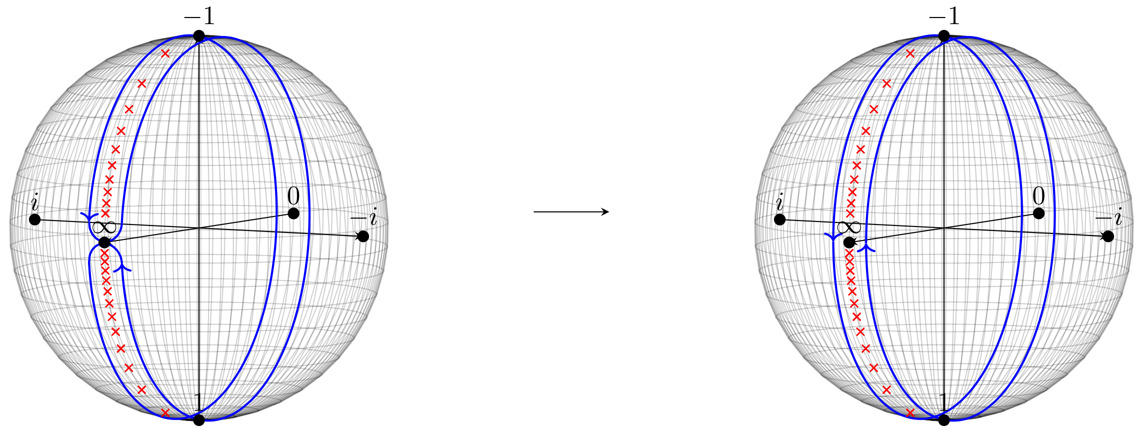


Figure 1: The integral contour γ, and the result of deforming it into to contours close to the real line. The red crosses illustrate the poles of n_B.

3.4 Thermal sum

When evaluating thermal integral, we will often encounter sums of the form

$$j(\omega, \mu) = \frac{1}{2\beta} \sum_{\omega_n} \ln\{\beta^2[(\omega_n + i\mu) + \omega^2]\} + g(\beta), \tag{103}$$

where the sum is over either the bosonic Matsubara frequencies $\omega_n = 2n\pi/\beta$, $n \in \mathbb{Z}$, or the fermionic ones, $\omega_n = (2n+1)\pi/\beta$, $n \in \mathbb{Z}$. $\mu \in \mathbb{R}$ is the chemical potential. g may be a function of β, but we assume it is independent of ω. Thus, the factor β^2 could strictly be dropped, but it is kept to make the argument within the logarithm dimensionless. We define the function

$$i(\omega, \mu) = \frac{1}{\omega} \frac{\mathrm{d}}{\mathrm{d}\omega} j(\omega, \mu) = \frac{1}{\beta} \sum_{\omega_n} \frac{1}{(\omega_n + i\mu)^2 + \omega^2}. \tag{104}$$

Assume $f(z)$ is an analytic function, except perhaps on a set of isolated poles $\{z_i\}$ located outside the real line. By exploiting the properties of $in_B(iz)$, as described in Appendix A, we can rewrite the sum over Matsubara frequencies as a contour integral

$$\frac{1}{\beta} \sum_{\omega_n} f(\omega_n) = \oint_\gamma \frac{\mathrm{d}z}{2\pi i} f(z)(in_B(iz)),$$

where γ is a contour that goes from $-\infty - i\epsilon$ to $+\infty - i\epsilon$, crosses the real line at ∞, goes from $+\infty - i\epsilon$ to $-\infty + i\epsilon$ before closing the curve. The contour γ, and the change of integral contours is illustrated in Figure 1 This result exploits Cauchy's integral formula, by letting the poles of $in_B(iz)$ at the Matsubara frequencies "pick out" the necessary residues. The integral over γ is equivalent to two integrals along $\mathbb{R} \pm i\epsilon$,

$$\begin{aligned} \frac{1}{\beta} \sum_{\omega_n} f(\omega_n) &= \left(\int_{\infty + i\epsilon}^{-\infty + i\epsilon} \frac{\mathrm{d}z}{2\pi} + \int_{-\infty - i\epsilon}^{\infty - i\epsilon} \frac{\mathrm{d}z}{2\pi} \right) f(z) n_B(iz), \\ &= \int_{-\infty - i\epsilon}^{\infty - i\epsilon} \frac{\mathrm{d}z}{2\pi} \left[f(z) + (f(z) + f(-z))\, n_B(iz) \right]. \end{aligned}$$

In the last line, we have changed variables $z \to -z$ in the first integral, and exploited the property $n_B(-iz) = -1 - n_B(iz)$. As $n_B(iz)$ is analytic on the real line, the result is the sum of residues of f in the lower half plane. The function

$$f(z) = \frac{1}{(z + i\mu)^2 + \omega^2} = \frac{i}{2\omega} \left(\frac{1}{z + i(\mu + \omega)} - \frac{1}{z + i(\mu - \omega)} \right) \tag{105}$$

obeys the assumed properties, as it has poles at $z = -i(\mu \pm \omega)$, with residue $1/2\omega$, so the function defined in Eq. (104) may be written [4]

$$i(\omega, \mu) = \frac{1}{2\omega}[1 + n_B(\omega + \mu) + n_B(\omega - \mu)]. \tag{106}$$

Using the anti-derivative of the Bose distribution, we get the final form of Eq. (103)

$$j(\omega, \mu) = \int \mathrm{d}\omega'\, \omega' i(\omega', \mu) = \frac{1}{2}\omega + \frac{1}{2\beta}\left[\ln\left(1 - e^{-\beta(\omega-\mu)}\right) + \ln\left(1 - e^{-\beta(\omega+\mu)}\right)\right] + g'(\beta). \tag{107}$$

The extra ω-independent term $g'(\beta)$ is an integration constant. We see there are temperature dependent terms, one due to the particle and one due to the anti-particle, and one due to the antiparticle, as they have opposite chemical potentials.

3.5 Regulating the free energy

Using the result from subsection 3.4 on the result for the free energy of the free scalar field, Eq. (96), we get

$$\beta\mathcal{F} = \frac{F}{VT} = \frac{1}{2}\int_{\tilde{V}} \frac{\mathrm{d}^3k}{(2\pi)^3}\left[\beta\omega_k + 2\ln\big(1 - e^{-\beta\omega_k}\big)\right]. \tag{108}$$

This free energy has two parts, the first part is dependent on temperature, the other is independent of temperature. Noticing that the integral is spherically symmetric, we may write

$$J_0 = \frac{1}{2}\frac{1}{2\pi^2}\int_{\mathbb{R}} \mathrm{d}k\, k^2\sqrt{k^2 + m^2}, \quad J_T = \frac{T^4}{2\pi^2}\int_{\mathbb{R}} \mathrm{d}x\, x^2 \ln\Big(1 - e^{-\sqrt{x^2 + (m/T)^2}}\Big), \tag{109}$$

The temperature-independent part, J_0, is clearly divergent, and we must therefore impose a regulator, and then add counter-terms. The second part of the integral is convergent. To see this, we use the series expansion $\ln(1+\epsilon) \sim \epsilon + \mathcal{O}\left(\epsilon^2\right)$ to find the leading part of the integrand for large k's,

$$x^2 \ln\Big(1 - e^{-\sqrt{x^2 + (\beta m)^2}}\Big) \sim -x^2 e^{-x}, \tag{110}$$

which is exponentially suppressed, making the integral convergent. In the limit of $T \to \infty$, we get

$$J_\infty \sim \frac{T^4}{2\pi^2}\int_{\mathbb{R}} \mathrm{d}x\, x^2 \ln\big(1 - e^{-x}\big) = -T^4\frac{\pi^2}{90} = -\frac{3}{2}\sigma T^4, \tag{111}$$

where σ is the Stefan-Boltzmann constant.

Returning to the temperature-independent part, we use dimensional regularization to see its singular behavior. To that end, we define

$$\Phi(m, d, A) = \mu^{3-d}\int_{\tilde{\Omega}} \frac{\mathrm{d}^d k}{(2\pi)^d}(k^2 + m^2)^{-A}, \tag{112}$$

so that $J_0 = \Phi(m, 3, 1/2)/2$. The parameter μ has the dimensions of k, and is inserted to ensure that Φ does not change physical dimension for $d \neq 3$. Furthermore, as non-rational exponents are defined through the exponential functions, this parameter is needed to make the expression well-defined. Dimensional regularization takes uses the formula for integration of spherically symmetric function in d-dimensions,

$$\int_{\mathbb{R}^d} \mathrm{d}^d x\, f(r) = \frac{2\pi^{d/2}}{\Gamma(d/2)}\int_{\mathbb{R}} \mathrm{d}r\, r^{d-1} f(r), \tag{113}$$

where $r = \sqrt{x_i x_i}$ is the radial distance, and Γ is the Gamma-function. The factor in the front of the integral comes from the solid angle. By extending this formula from integer-valued d to real numbers, the function we defined becomes

$$\Phi = \frac{2\pi^{d/2}\mu^{3-d}}{\Gamma(d/2)}\int_{\mathbb{R}} \mathrm{d}k\, \frac{k^{d-1}}{(k^2 + m^2)^A} = \frac{m^{3-2A}}{(4\pi)^{d/2}\Gamma(d/2)}\left(\frac{m}{\mu}\right)^{d-3} 2\int_{\mathbb{R}} \mathrm{d}z\, \frac{z^{d-1}}{(1+z)^A}, \tag{114}$$

[4] Assuming $\omega > \mu$.

where we have made the change of variables $mz = k$. We make one more change of variable to the integral,

$$I = 2\int_{\mathbb{R}} \mathrm{d}z\, \frac{z^{d-1}}{(1+z)^A} \tag{115}$$

Let

$$z^2 = \frac{1}{s} - 1 \implies 2z\mathrm{d}z = -\frac{\mathrm{d}s}{s^2} \tag{116}$$

Thus,

$$I = \int_0^a \mathrm{d}s\, s^{A-d/2-1}(1-z)^{d/2-1}. \tag{117}$$

This is the beta-function, which can be written in terms of Gamma Functions [6]

$$I = B\left(A - \frac{d}{2}, \frac{d}{2}\right) = \frac{\Gamma\left(A - \frac{d}{2}\right)\Gamma\left(\frac{d}{2}\right)}{\Gamma(A)}. \tag{118}$$

Combining this gives

$$\Phi = m^{3-2A}\left(\frac{m^2}{\mu^2}\right)^{(d-3)/2} \frac{\Gamma\left(A - \frac{d}{2}\right)}{(4\pi)^{d/2}\Gamma(A)}. \tag{119}$$

Inserting $d = 3 - 2\epsilon$ and $A = -1/2$, we get

$$\Phi(m, 3-2\epsilon, -1/2) = \frac{m^4}{(4\pi)^{d/2}\Gamma(-1/2)}\Gamma(-2+\epsilon)\left(\frac{m^2}{\mu^2}\right)^{-\epsilon} = -\frac{m^4}{(4\pi)^2}\left(\frac{m^2}{4\pi\mu^2}\right)^{-\epsilon}\frac{\Gamma(\epsilon)}{(\epsilon-2)(\epsilon-1)}, \tag{120}$$

where we have used the defining property $\Gamma(z+1) = z\Gamma(z)$, and inserted a parameter μ with the dimensions of m. Expanding around $\epsilon = 0$ gives

$$\left(\frac{m^2}{4\pi\mu^2}\right)^{-\epsilon} \sim 1 + \epsilon \ln\left(4\pi\frac{\mu^2}{m^2}\right), \tag{121}$$

$$\Gamma(\epsilon) \sim \frac{1}{\epsilon} - \gamma, \tag{122}$$

$$\frac{1}{(\epsilon-2)(\epsilon-1)} \sim \frac{1}{2}\left(1 + \frac{3}{2}\epsilon\right). \tag{123}$$

The singular behavior of the time-independent term is therefore

$$J_0 \sim -\frac{1}{4}\frac{m^4}{(4\pi)^2}\left[\frac{1}{\epsilon} - \gamma + \frac{3}{2} + \ln\left(4\pi\frac{\mu^2}{m^2}\right)\right]. \tag{124}$$

With this regulator, one can then add counter-terms to cancel the $\frac{1}{\epsilon}$-divergence. The exact form of the counter-term is convention. One may also cancel the finite contribution due to the regulator. The minimal subtraction, or MS, scheme, is to only subtract the divergent term, as the name suggest. We will use the modified minimal subtraction, or $\overline{\text{MS}}$, scheme. In this scheme, one also removes the $-\gamma$ and $\ln(4\pi)$ term, which can be interpreted as changing the parameter μ such that

$$-\gamma + \ln 4\pi\frac{\mu^2}{m^2} \to \ln\left(\frac{\mu^2}{m^2}\right), \tag{125}$$

which leads to the expression

$$J_0 \sim -\frac{1}{4}\frac{m^4}{(4\pi)^2}\left[\frac{1}{\epsilon} + \frac{3}{2} + \ln\left(\frac{\mu^2}{m^2}\right)\right]. \tag{126}$$

3.6 Interacting scalar

We now study a scalar field with a $\lambda\varphi^4$ interaction term. We write the Lagrangian in the form

$$\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(I)}, \quad \mathcal{L}^{(0)} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - m^2\varphi^2, \quad \mathcal{L}^{(I)} = -\frac{\lambda}{4!}\varphi^4$$

$\mathcal{L}^{(I)}$ is called the interaction term, and makes it impossible to exactly solve for the partition function. Instead, we turn to perturbation theory. The canonical partition function in this theory

$$Z = \mathrm{Tr}\Big\{e^{-\beta\hat{H}}\Big\} = \int_S \mathcal{D}\varphi \, \exp\Big\{-\int_\Omega \mathrm{d}X \left(\mathcal{L}_E^{(0)} + \mathcal{L}_E^{(I)}\right)\Big\} = \int_S \mathcal{D}\varphi \, e^{S_0} e^{S_I}. \tag{127}$$

Here, S_0 and S_I denote the Euclidean action due to the free and interacting Lagrangian, respectively. The domain of integration S is again periodic field configurations $\varphi(\beta, \vec{x}) = \varphi(0, \vec{x})$. We may write the free energy as

$$-\beta F = \ln\left[\int_S \mathcal{D}\varphi \, e^{S_0} \sum_n \frac{1}{n!} {S_I}^n\right] = \ln[Z_0] + \ln\left[Z_I\right],$$

where Z_0 is the partition function of the free theory. The correction to the partition function is thus given by

$$Z_I = \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle {S_I}^n \right\rangle_0 , \tag{128}$$

where

$$\langle A\rangle_0 = \frac{\int_S \mathcal{D}\varphi \, A \, e^{S_0}}{\int_S \mathcal{D}\varphi \, e^{S_0}}. \tag{129}$$

To evaluate expectation values of the form $\langle\varphi(X_1)...\rangle_0$, we introduce the partition function with a source term

$$Z[J] = \int_S \mathcal{D}\varphi \, \exp\Big\{-\frac{1}{2}\int_\Omega \mathrm{d}X \, \varphi(-\partial_E^2 + m^2)\varphi + \int_\Omega \mathrm{d}X \, J\varphi\Big\}. \tag{130}$$

Using the thermal Greens function $D_0(X, Y)$, as defined in Appendix A, we may complete the square to write

$$Z[J] = Z[0]\exp\Big\{\frac{1}{2}\int_\Omega \mathrm{d}X\mathrm{d}Y \, J(X)D_0(X,Y)J(Y)\Big\} = Z[0]\exp(W[J]) \tag{131}$$

We can now write

$$\langle\varphi(X)\varphi(Y)\rangle_0 = \frac{1}{Z[0]}\frac{\delta}{\delta J(X)}\frac{\delta}{\delta J(Y)} Z[J]\Big|_{J=0} = D_0(X,Y), \tag{132}$$

This generalizes to higher order expectation values,

$$\langle\varphi(X_i)\dots\varphi(X_n)\rangle_0 = \left(\prod_{i=1}^n \frac{\delta}{\delta J(X_i)}\right) Z[J]\Big|_{J=0}, \tag{133}$$

The exponential form of $Z[J]$ leads straight forwardly to Wick's theorem, which states that an expectation value of $2n$ fields is a sum of *all possible, distinct* combination of n propagators. To write this in a formal way, we define the functions a and b, which define a way to pair up $2m$ elements. The domain of the functions are the integers between 1 and m, the image a subset of the integers between 1 and $2m$ of size m. A valid pairing is a set $\{(a(1), b(1)), \dots (a(m), b(m))\}$, where all elements $a(i)$ and $b(j)$ are different, such all integers up to and including $2m$ are featured. A pair is not directed, so $(a(i), b(i))$ is the same pair as $(b(i), a(i))$. Wick theorem states that,

$$\left\langle \prod_{i=1}^{2m} \varphi(X_i) \right\rangle_0 = \sum_{\{(a,b)\}} \left\langle \varphi(X_{a(i)})\varphi(X_{b(i)}) \right\rangle, \tag{134}$$

where the sum is over all tuples (a, b) that define a valid and unique pairing. Using Wick's theorem, the expectation values we are evaluating can be written

$$\begin{aligned}
\left\langle {S_I}^m \right\rangle &= \left(-\frac{\lambda}{4!}\right)^m \int_\Omega \mathrm{d}X_1 \dots \mathrm{d}X_m \left\langle \varphi^4(X_1)\dots\varphi^4(X_m)\right\rangle \\
&= \left(-\frac{\lambda}{4!}\right)^m \int_\Omega \mathrm{d}X_1 \dots \mathrm{d}X_m \sum_{\{a,b\}} \left\langle \varphi(X_{a(1)})\varphi(X_{b(1)})\right\rangle \dots \left\langle \varphi(X_{a(2m)})\varphi(X_{b(2m)})\right\rangle
\end{aligned}$$

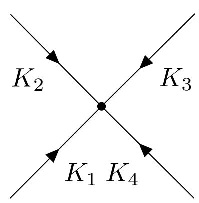
where X_i for $i > m$ is defined as X_j, where $j = i \mod m$. Inserting the Fourier expansions of the field gives

$$
\begin{aligned}
&\langle {S_I}^m \rangle \\
&= \left(-\frac{\lambda}{4!}\right)^m \int_\Omega \mathrm{d}X_1 \ldots \mathrm{d}X_m (V\beta)^2 \int_{\tilde{\Omega}} \mathrm{d}K_1 ... \mathrm{d}K_{2m} \sum_{\{a,b\}} \\
&\qquad \big\langle \varphi(K_{a(1)})\varphi(K_{b(1)}) \big\rangle \ldots \big\langle \varphi(K_{a(2m)})\varphi(K_{b(2m)}) \big\rangle \exp\Big(i \sum\nolimits_{i=1}^m X_i \cdot K_i \Big) \\
&= \left(-\frac{\lambda}{4!}\right)^m \frac{(V\beta)^{2m}\beta^m}{(V\beta^2)^{2m}} \int_{\tilde{\Omega}} \mathrm{d}K_1 ... \mathrm{d}K_{2m} \sum_{\{a,b\}} \\
&\qquad \tilde{D}(K_{a(1)})\delta(K_{a(1)} + K_{b(1)}) \ldots \tilde{D}(K_{a(2m)})\delta(K_{a(2m)} + K_{b(2m)}) \prod_{i=1}^m \delta\left(X_i \cdot \sum\nolimits_{j=0}^3 K_{i+jm} \right) \\
&= \left(-\frac{\lambda\beta}{4!}\right)^m \prod_{i=1}^{2m} \int_{\tilde{\Omega}} \left(\mathrm{d}K_i \frac{1}{\beta} \tilde{D}(K_i) \right) \prod_{i=1}^m \delta\left(X_i \cdot \sum\nolimits_{j=0}^3 K_{i+jm} \right) \sum_{\{a,b\}} \prod_{n=1}^{2m} \delta(K_{a(k)} + K_{b(k)})
\end{aligned}
$$

Here we have used that $V\beta^2 \tilde{D}_0(K,P) = \tilde{D}_0(K)\delta(P+K)$, and $\tilde{D}_0(K)$ is the thermal propagator for the free field, as defined in Appendix A. In this case, it is

$$
\tilde{D}_0(K) = \tilde{D}_0(\omega_n, \vec{k}) = \frac{1}{\omega_k^2 + \omega_n^2}. \tag{135}
$$

This expectation value can be represented graphically using Feynman diagrams. The thermal $\lambda\varphi^2$-theory gets the prescription

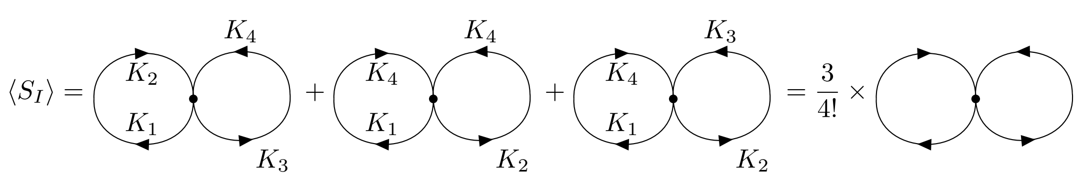


$$
= -\lambda\beta\delta\left(\sum\nolimits_i K_i\right), \tag{136}
$$

$$
= \frac{1}{\beta} D_0(K). \tag{137}
$$

The expression is the integrated over all *internal* momenta. The factor $1/4!$ is removed as a general Feynman diagram represent all diagrams with the same form, but different pairing of the momenta. Some diagrams are more symmetric, such that an exchange of momenta still gives *the same pairing.* This is dealt with by dividing with a symmetry factor s, which is described in detail in [6].

Calculating $\langle {S_I}^n \rangle_0$ boils down to the sum of all possible Feynman diagrams with n vertices. The first example is



(138)

For higher order, one gets both connected and disconnected diagrams. Let $\langle {S_I}^n \rangle_{0,c}$ be only the connected diagrams, i.e. those diagrams in which it is possible to move between all vertices along a series of edges. A general diagram contain n_i copies of a connected diagram with the value V_i. The value of the total diagram is then the product of the value of all its disconnected pieces, but with the caveat that each diagram has a symmetry factor of $n_i!$. The sum of all diagrams is thus

$$
Z_I = \sum_n \frac{1}{n!} \langle {S_I}^n \rangle = \sum_{\textit{all sets } \{n_i\}} \prod_i \frac{1}{n_i!} V_i^{n_i} = \prod_i \sum_{n_i} \frac{1}{n_i!} V_i^{n_i} = \exp\Big(\sum\nolimits_i V_i \Big). \tag{139}
$$

Thus, the correction to the free energy is given by the sum of all connected diagrams,

$$-\beta F = \ln(Z_0) + \sum_n \langle {S_I}^n \rangle_{0,c} \,. \tag{140}$$

3.7 Fermions

The phase factor $e^{i\theta}$ that was introduced in subsection 3.2 can be determined by studying the properties of the thermal Greens function. The thermal Greens function may be written

$$D(X_1, X_2) = D(\vec{x}, \vec{y}, \tau_1, \tau_2) = \left\langle e^{-\beta \hat{H}} \mathrm{T}\left\{\hat{\varphi}(X_1)\hat{\varphi}(X_2)\right\} \right\rangle .$$

$\mathrm{T}\{...\}$ is time-ordering operator, and is defined as

$$\mathrm{T}\left\{\varphi(\tau_1)\varphi(\tau_2)\right\} = \theta(\tau_1 - \tau_2)\varphi(\tau_1)\varphi(\tau_2) + \nu\theta(\tau_2 - \tau_1)\varphi(\tau_2)\varphi(\tau_1),$$

where $\nu = \pm 1$ for bosons and fermions respectively, and $\theta(\tau)$ is the Heaviside step function. In the same way that $i\hat{H}$ generates the time translation of a quantum field operator through $\hat{\varphi}(x) = \hat{\varphi}(t, \vec{x}) = e^{it\hat{H}}\hat{\varphi}(0, \vec{x})e^{-it\hat{H}}$, the imaginary-time formalism implies the relation

$$\hat{\varphi}(X) = \hat{\varphi}(\tau, \vec{x}) = e^{\tau\hat{H}}\hat{\varphi}(0, \vec{x})e^{-\tau\hat{H}}. \tag{141}$$

Using $\mathbb{1} = e^{\tau\hat{H}}e^{-\tau\hat{H}}$ and the cyclic property of the trace, we show that, assuming $\beta > \tau > 0$,

$$\begin{aligned}
G(\vec{x}, \vec{y}, \tau, 0) &= \left\langle e^{-\beta\hat{H}} \mathrm{T}\left\{\varphi(\tau, \vec{x})\varphi(0, \vec{y})\right\} \right\rangle \\
&= \frac{1}{Z} \,\mathrm{Tr}\Big\{ e^{-\beta\hat{H}} \varphi(\tau, \vec{x})\varphi(0, \vec{y}) \Big\} \\
&= \frac{1}{Z} \,\mathrm{Tr}\Big\{ \varphi(0, \vec{y}) e^{-\beta\hat{H}} \varphi(\tau, \vec{x}) \Big\} \\
&= \frac{1}{Z} \,\mathrm{Tr}\Big\{ e^{-\beta\hat{H}} e^{\beta\hat{H}} \varphi(0, \vec{y}) e^{-\beta\hat{H}} \varphi(\tau, \vec{x}) \Big\} \\
&= \frac{1}{Z} \,\mathrm{Tr}\Big\{ e^{-\beta\hat{H}} \varphi(\vec{y}, \beta) \varphi(\tau, \vec{x}) \Big\} \\
&= \nu \left\langle e^{-\beta\hat{H}} \mathrm{T}\left\{\varphi(\tau, \vec{x})\varphi(\beta, \vec{y})\right\} \right\rangle .
\end{aligned}$$

This implies that $\varphi(0, x) = \nu\varphi(\beta, \varphi)$, which shows that bosons are periodic in time, as stated earlier, while fermions are anti-periodic.

The Lagrangian density of a free fermion is

$$\mathcal{L} = \bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi. \tag{142}$$

This Lagrangian is invariant under the transformation $\psi \to e^{-i\alpha}\psi$, which by Nöther's theorem results in a conserved current

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\delta\psi = \bar{\psi}\gamma^\mu\psi. \tag{143}$$

The corresponding conserved charge is

$$Q = \int_V \mathrm{d}^3x\, j^0 = \int_V \mathrm{d}^3x\, \bar{\psi}\gamma^0\psi. \tag{144}$$

We can now use our earlier result for the thermal partition function, Eq. (94), only with the substitution $\mathcal{H} \to \mathcal{H} - \mu\bar{\psi}\gamma^0\psi$, and integrate over anti-periodic ψ's:

$$Z = \mathrm{Tr}\Big\{ e^{-\beta(\hat{H} - \mu\hat{Q})} \Big\} = \prod_{ab} \int \mathcal{D}\psi_a \mathcal{D}\pi_b \exp\Big\{ \int_\Omega \mathrm{d}X\, \Big(i\dot{\psi}\pi - \mathcal{H}(\psi, \pi) + \mu\bar{\psi}\gamma^0\psi \Big) \Big\},$$

where a, b are the spinor indices. The canonical momentum corresponding to ψ is

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} = i\bar{\psi}\gamma^0, \tag{145}$$

and the Hamiltonian density is

$$\mathcal{H} = \pi\dot{\psi} - \mathcal{L} = \bar{\psi}(-i\gamma^i\partial_i + m)\psi \tag{146}$$

which gives

$$\mathcal{L}_E = -i\pi\dot{\psi} + \mathcal{H}(\psi, \pi) - \mu\bar{\psi}\gamma^0\psi = \bar{\psi}[\gamma^0(\partial_\tau - \mu) - i\gamma^i\partial_i - m]\psi, \tag{147}$$

By using the Grassman-version of the Gaussian integral formula, the partition function can be written

$$\begin{aligned}
Z &= \prod_{ab} \int \mathcal{D}\psi_a \mathcal{D}\bar{\psi}_b \exp\Big\{ - \int_\Omega \mathrm{d}X \, \bar{\psi} \left[\tilde{\gamma}_0(\partial_\tau - \mu) - i\gamma^i\partial_i - m\right] \psi \Big\} \\
&= C \prod_{ab} \int \mathcal{D}\tilde{\psi}_a \mathcal{D}\tilde{\bar{\psi}}_b \exp\Big\{ - \int_{\tilde{\Omega}} \mathrm{d}K \, \tilde{\bar{\psi}} \left[i\tilde{\gamma}_0(\omega_n + i\mu) + i\gamma_i p_i - m\right] \tilde{\psi} \Big\} \\
&= C \prod_{ab} \int \mathcal{D}\tilde{\psi}_a \mathcal{D}\tilde{\bar{\psi}}_b e^{-\langle \tilde{\bar{\psi}}, D_0^{-1} \psi \rangle} = \det\big(D_0^{-1}\big).
\end{aligned}$$

In the second line, we have inserted the Fourier expansion of the field, as defined in Appendix A, and changed variable of integration, as we did for the scalar field. The linear operator in this case is

$$D_0^{-1} = i\gamma^0(-i\partial_\tau + i\mu) - (-i\gamma^i)\partial_i - m = \beta[i\tilde{\gamma}_a p_a - m]. \tag{148}$$

This equality must be understood as an equality between linear operators, which are represented in different bases. We introduced the notation $p_{n;a} = (\omega_n + i\mu, p_i)$ and use the Euclidean gamma matrices, as defined in Appendix A. We use the fact that

$$\det(i\tilde{\gamma}_a p_a - m) = \det\big(\gamma^5\gamma^5\big) \det(i\tilde{\gamma}_a p_a - m) = \det\big[\gamma^5(i\tilde{\gamma}_a p_a - m)\gamma^5\big] = \det(-i\tilde{\gamma}_a p_a - m),$$

Let $\tilde{D} = -i\tilde{\gamma}_a p_a - m$, which means we can write

$$Z = \sqrt{\det(D)\det\big(\tilde{D}\big)} = \sqrt{\det\big(D\tilde{D}\big)} = \det\big[\mathbb{1}(p_a p_a + m^2)\big]^{1/2}, \tag{149}$$

where we have used the anti-commutation rule for the Euclidean gamma-matrices, $\{\gamma_a, \gamma_b\} = 2\delta_{ab}$. It is important to keep in mind that the determinant here refers to linear operators on the space of spinor functions.

$$\begin{aligned}
\ln(Z) &= \ln\Big\{\det\big[\mathbb{1}(p_a p_a + m^2)\big]^{1/2}\Big\} = \frac{1}{2}\operatorname{Tr}\big\{\ln[\mathbb{1}(p_a p_a + m^2)]\big\} \\
&= \frac{1}{2}\int_{\tilde{\Omega}} \mathrm{d}K \ln[\mathbb{1}\beta^2(p_a p_a + m^2)]_{aa}
\end{aligned} \tag{150}$$

As the matrix within the logarithm is diagonal, the matrix-part of the trace is trivial, and the free energy may be written

$$\beta\mathcal{F} = -2\int_{\tilde{\Omega}} \mathrm{d}X \, \ln\{\beta^2[(\omega_n + i\mu)^2 + \omega_k^2]\}. \tag{151}$$

Using the fermionic version of the thermal sum (TODO: UTLED DENNE) gives the answer

$$\beta\mathcal{F} = -2\int \frac{\mathrm{d}^3 p}{(2\pi)^3} \, \Big[\beta\omega_k + \ln\Big(1 + e^{-\beta(\omega_k - \mu)}\Big) + \ln\Big(1 + e^{-\beta(\omega_k + \mu)}\Big)\Big]. \tag{152}$$

We see again that the temperature-independent part of the integral diverges, and must be regulated. There are two temperature-dependent terms, one from the particle and one from the anti-particle.

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Appendix A Conventions and notation

Throughout this text, natural units are employed, in which

$$\hbar = c = k_B = 1, \tag{153}$$

where $\hbar$ is the Planck reduced constant, k_B is the Boltzmann constant and c is the speed of light. The Minkowski metric convention used is the "mostly minus", $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.

The $\mathfrak{su}\,(2)$ basis used is the Pauli matrices,

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

They obey

$$[\tau_a, \tau_b] = 2i\varepsilon_{abc}\tau_c, \quad \{\tau_a, \tau_b\} = 2\delta_{ab}\mathbb{1}, \quad \mathrm{Tr}[\tau_a] = 0, \quad \mathrm{Tr}[\tau_a\tau_b] = 2\delta_{ab}\mathbb{1}.$$

Together with the identity matrix $\mathbb{1}$, the Pauli matrices form a basis for the vector space of all 2-by-2 matrices. An arbitrary 2-by-2 matrix M may be written

$$M = M_0\mathbb{1} + M_a\tau_a, \quad M_0 = \frac{1}{2}\,\mathrm{Tr}\{M\},\ M_a = \frac{1}{2}\,\mathrm{Tr}\{\tau_a M\}. \tag{154}$$

The gamma matrices γ^μ, $\mu \in \{0, 1, 2, 3\}$, obey

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}. \tag{155}$$

The "fifth γ-matrix" is defined by

$$\gamma^5 = \frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = i\gamma^0\gamma^1\gamma^2\gamma^3. \tag{156}$$

The γ^5-matrix obey

$$\left\{\gamma^5, \gamma^\mu\right\} = 0, \quad (\gamma^5)^2 = \mathbb{1}, \quad {\gamma^0}^\dagger = \gamma^0,\ {\gamma^i}^\dagger = -\gamma^i \tag{157}$$

Their Euclidean counterpart obey

$$\{\tilde{\gamma}_a, \tilde{\gamma}_b\} = 2\delta_{ab}\mathbb{1}, \quad \tilde{\gamma}_a^\dagger = \tilde{\gamma}_a, \tag{158}$$

and they are related by $\tilde{\gamma}_0 = \gamma^0$, and $\tilde{\gamma}_j = -i\gamma^j$. The Euclidean $\tilde{\gamma}_5$ is defined as

$$\tilde{\gamma}_5 = \gamma_0\gamma_1\gamma_2\gamma_3 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5. \tag{159}$$

It thus also anti-commutes with the Euclidean γ-matrices.

Fourier transform

The Fourier transform used in this text is defined by

$$\mathcal{F}\left\{f(x)\right\}(p) = \tilde{f}(p) = \int \mathrm{d}x\, e^{ipx} f(x), \quad \mathcal{F}^{-1}\left\{\tilde{f}(p)\right\}(x) = f(x) = \int \frac{\mathrm{d}p}{2\pi}\, e^{-ipx}\tilde{f}(p).$$

Fourier series

Imaginary-time formalism is set in a Euclidean space $\Omega = [0, \beta] \times V$, where $V = L_x L_y L_z$ is a space-like volume. The possible momenta in this space are

$$\tilde{V} = \left\{\vec{k} \in \mathbb{R}^3 \,\middle|\, \vec{k} = \left(\frac{2\pi n_x}{L_x}, \frac{2\pi n_y}{L_y}, \frac{2\pi n_z}{L_z}\right)\right\}$$

ω_n are the Matsubara-frequencies, with $\omega_n = 2n\pi/\beta$ for bosons and $\omega_n = (2n+1)\pi/\beta$ for fermions. They together form the reciprocal space $\tilde{\Omega} = \{\omega_n\} \times \tilde{V}$. The Euclidean coordinates are denoted $X = (\tau, \vec{x})$ and $K = (\omega_n, \vec{K})$, and have the dot product $X \cdot P = \omega_n \tau + \vec{k} \cdot \vec{x}$. In the limit $V \to \infty$, we follow the prescription

$$\frac{1}{V} \sum_{\vec{p} \in \tilde{V}} \to \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3}.$$

The sum over all degrees of freedom, and the corresponding integrals for the thermodynamic limit are

$$\frac{\beta V}{NM} \sum_{n=1}^{N} \sum_{\vec{x}_m \in V} \xrightarrow{N,M\to\infty} \int_0^\beta \mathrm{d}\tau \int_{\mathbb{R}^3} \mathrm{d}^3 x = \int_\Omega \mathrm{d}X,$$

$$\frac{1}{V} \sum_{n=-\infty}^{\infty} \sum_{\vec{k} \in \tilde{V}} \xrightarrow{V\to\infty} \sum_{n=-\infty}^{\infty} \int_{R^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} = \int_{\tilde{\Omega}} \mathrm{d}K.$$

The convention used for the Fourier expansion of thermal fields is in accordance with [9]. The prefactor is chosen to make the Fourier components of the field dimensionless, which makes it easier to evaluate the trace correctly. For bosons, the Fourier expansion is

$$\varphi(X) = \sqrt{V\beta} \int_{\tilde{\Omega}} \mathrm{d}K \, \tilde{\varphi}(K) e^{iX\cdot K} = \sqrt{\frac{\beta}{V}} \sum_{n=-\infty}^{\infty} \sum_{\vec{k} \in \tilde{V}} \tilde{\varphi}_n(\vec{p}) \exp\Big\{ i(\omega_n \tau + \vec{x} \cdot \vec{k}) \Big\},$$

$$\tilde{\varphi}(K) = \sqrt{\frac{1}{V\beta^3}} \int_{\tilde{\Omega}} \mathrm{d}X \, \tilde{\varphi}(X) e^{-iX\cdot K}$$

while for Fermions it is

$$\psi(X) = \sqrt{V} \int_{\tilde{\Omega}} \mathrm{d}K \, \tilde{\psi}(K) e^{iX\cdot K} = \frac{1}{\sqrt{V}} \sum_{n=-\infty}^{\infty} \sum_{\vec{k} \in \tilde{V}} \psi(\omega_n, \vec{k}) \exp\Big\{ i(\omega_n \tau + \vec{x} \cdot \vec{k}) \Big\} \tag{160}$$

A often used identity is

$$\int_\Omega \mathrm{d}X e^{iX\cdot(K-K')} = \beta \delta_{nn'} (2\pi)^3 \delta^3(\vec{k} - \vec{k}') := \beta \delta(K - K'), \tag{161}$$

$$\int_{\tilde{\Omega}} \mathrm{d}K \, e^{iK(X-X')} = \beta \delta(\tau - \tau') \delta^3(\vec{x} - \vec{x}') := \beta \delta(X - X'). \tag{162}$$

Bose distribution

The Bose distribution is defined as

$$n_B(\omega) = \frac{1}{e^{\beta\omega} - 1}. \tag{163}$$

This function obeys

$$n_B(-i\omega) = -1 - n_B(i\omega). \tag{164}$$

We can expand it around the Bose Matsubara frequencies on the imaginary line:

$$i n_B[i(\omega_n + \epsilon)] = \frac{i}{e^{i\beta\epsilon + 2\pi i n} - 1} = i[i\beta\epsilon + \mathcal{O}\left(\epsilon^2\right)]^{-1} \sim \frac{1}{\epsilon\beta}. \tag{165}$$

This means that $in_B(i\omega)$ has a pole on all Matsubara-frequencies, with residue $1/\beta$. Furthermore, we have

$$\frac{\mathrm{d}}{\mathrm{d}\omega} \ln\!\big(1 - e^{-\beta\omega}\big) = \beta n_B(\omega). \tag{166}$$

Propagators

If $D^{-1}[f(x)] = 0$ is the equation of motion for some field f, where D^{-1} in general is a differential operator, then the propagator $D(x, x')$ for this field is defined by

$$D^{-1}[D(x, x')] = -i\delta(x - x')\mathbb{1}.$$

Assuming A is linear and independent of space, we may redefine $D(x - x', 0) \to D(x - x')$, and the Fourier transform with respect to both x and x' to obtain

$$\mathcal{F}\left\{D^{-1}[D(x, x')]\right\}(p, p') = \tilde{D}^{-1}(p)\,\tilde{D}(p)\,\delta(p + p') = -i\delta(p + p'),$$

meaning the momentum space propagator $\tilde{D}(p) = \mathcal{F}\{D(x)\}(p)$ is given by $\tilde{D} = -i(\tilde{D}^{-1})^{-1}$.

For some differential operator D^{-1}, the thermal propagator is defined as

$$D^{-1}D(X, Y) = \beta\delta(X - Y). \tag{167}$$

The Fourier transformed propagator is, assuming $D(X, Y) = D(X - Y, 0)$,

$$\begin{aligned}
\tilde{D}(K, K') &= \frac{1}{V\beta^3}\int_\Omega \mathrm{d}X\mathrm{d}Y\, D(X, Y)\exp(-i[X \cdot K + Y \cdot K']) && (168)\\
&= \frac{1}{V\beta^3}\int_\Omega \mathrm{d}X'\mathrm{d}Y'\, D(X', 0)\exp\Big(-i[X' \cdot \frac{1}{2}(K - K') + Y \cdot (K + K')]\Big) && (169)\\
&= \frac{1}{V\beta^2}\tilde{D}(K)\delta(K + K'), && (170)
\end{aligned}$$

where

$$\tilde{D}(K) = \int \mathrm{d}X e^{iK\cdot X} D(X, 0). \tag{171}$$

Appendix B Covariant derivative

In χPT at finite isospin chemical potential μ_I, the covariant derivative acts on functions $A(x) : \mathcal{M}_4 \to \mathrm{SU}\,(2)$, where $\mathcal{M}_4$ is the space-time manifold. It is defined as

$$\nabla_\mu A(x) = \partial_\mu A(x) - i\,[v_\mu, A(x)]\,, \quad v_\mu = \frac{1}{2}\mu_I\delta^0_\mu\tau_3. \tag{172}$$

The covariant derivative obeys the product rule, as

$$\nabla_\mu(AB) = (\partial_\mu A)B + A(\partial_\mu B) - i\,[v_\mu, AB] = (\partial_\mu A - i\,[v_\mu, A])B + A(\partial_\mu B - i\,[v_\mu, B]) = (\nabla_\mu A)B + A(\nabla_\mu B).$$

Decomposing a 2-by-2 matrix M, as described in Appendix A, shows that the trace of the commutator of τ_b and M is zero:

$$\mathrm{Tr}\{[\tau_a, M]]\} = M_b\,\mathrm{Tr}\{[\tau_a, \tau_b]\} = 0.$$

Together with the fact that $\mathrm{Tr}\{\partial_\mu A\} = \partial_\mu\,\mathrm{Tr}\{A\}$, this gives the product rule for invariant traces:

$$\mathrm{Tr}\{A\nabla_\mu B\} = \partial_\mu\,\mathrm{Tr}\{AB\} - \mathrm{Tr}\{(\nabla_\mu A)B\}.$$

This allows for the use of the divergence theorem when doing partial integration. Let $\mathrm{Tr}\{K^\mu\}$ be a space-time vector, and $\mathrm{Tr}\{A\}$ scalar. Let Ω be the domain of integration, with coordinates x and $\partial\Omega$ its boundary, with coordinates y. Then,

$$\int_\Omega \mathrm{d}x\,\mathrm{Tr}\{A\nabla_\mu K^\mu\} = \int_{\partial\Omega}\mathrm{d}y\,n_\mu\,\mathrm{Tr}\{AK^\mu\} - \int_\Omega \mathrm{d}x\,\mathrm{Tr}\{(\nabla_\mu A)K^\mu\},$$

where n_μ is the normal vector of $\partial\Omega$.[8] This makes it possible to do partial integration and discard surface terms in the χPT Lagrangian, given the assumption of no variation on the boundary.

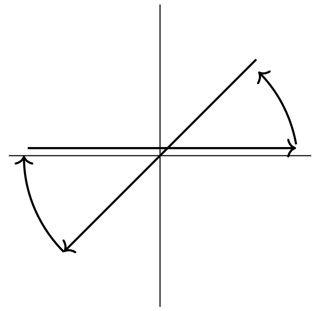

Figure 2: Wick rotation

Appendix C Integrals

C.1 Gaussian integrals

A useful integral is the Gaussian integral,

$$\int_{\mathbb{R}} \mathrm{d}z \, \exp\left(-\frac{1}{2}az^2\right) = \sqrt{\frac{2\pi}{a}}, \tag{173}$$

for $a \in \mathbb{R}$. The imaginary version,

$$\int_R \mathrm{d}z \, \exp\left(i\frac{1}{2}az^2\right) \tag{174}$$

does not converge. However, if we change $a \to a + i\epsilon$, then the integrand is exponentially supressed.

$$f(x) = \exp\left(i\frac{1}{2}ax^2\right) \to \exp\left(i\frac{1}{2}ax^2 - \frac{1}{2}\epsilon x^2\right), \tag{175}$$

As the integrand falls of exponentially for $x \to \infty$, and contains no poles in the upper right nor lower left quarter of the complex plane, we may perform a wick rotation by closing the contour as shown in Figure 2. This gives the result

$$\int_{\mathbb{R}} \mathrm{d}x \, \exp\left(i\frac{1}{2}(a+i\epsilon)x^2\right) = \int_{\sqrt{i}\mathbb{R}} \mathrm{d}x \, \exp\left(i\frac{1}{2}ax^2\right) = \sqrt{i}\int_{\mathbb{R}} \mathrm{d}y \, \exp\left(-\frac{1}{2}ay^2\right) = \sqrt{\frac{2\pi i}{a}} \tag{176}$$

where we have made the change of variable $y = (1+i)/\sqrt{2}x = \sqrt{i}x$.

In n dimensions, the Gaussian integral formula generalizes to

$$I_n = \int_{\mathbb{R}^n} \mathrm{d}^n x \, \exp\left\{-\frac{1}{2}x_n A_{nm} x_m\right\} = \sqrt{\frac{(2\pi)^n}{\det(A)}}, \tag{177}$$

where A is a matrix with n real, positive eigenvalues. We may also generalize Eq. (176),

$$I'_n = \int_{\mathbb{R}^n} \mathrm{d}^n x \, \exp\left\{i\frac{1}{2}x_n (A_{nm} + i\epsilon\delta_{nm}) x_m\right\} = \sqrt{\frac{(2\pi i)^n}{\det(A)}}. \tag{178}$$

The final generalization is to functional integrals. The bilinear becomes

$$x_n A_{nm} x_m \to \int \mathrm{d}x \, \varphi(x) A \varphi(x), \tag{179}$$

where A is some operator. The first Gaussian integral becomes

$$I_\infty = \int \mathcal{D}\varphi \, \exp\left(-\frac{1}{2}\int \mathrm{d}x \, \varphi(x) A \varphi(x)\right) = C(\det(A))^{-1/2}. \tag{180}$$

C is here a divergent constant, but will either fall away as we are only looking at the logarithm of I_∞ and are able to throw away additive constants, or ratios between quantities which are both multiplied by C.

Appendix D Functional Derivatives

Functional derivatives generalize the notion of a gradient and the directional derivative. A function $f(p)$, where p is point with coordinates $x_i = x_i(p)$, has a gradient

$$\mathrm{d}f_p = \frac{\partial f(p)}{\partial x_i}\mathrm{d}x_i. \tag{181}$$

The derivative in a particular direction $v = v^i\partial_i$ is

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}f(x_i + \epsilon v_i) = f(x) + \mathrm{d}f_x(v) = f(x) + \frac{\partial f}{\partial x^i}v_i. \tag{182}$$

This is generalized to functionals through the definition of functional derivative, and the variation of a functional. Let $F[f]$ be a functional, i.e. a machine that takes in a function, and returns a number. The obvious example in our case is the action, which takes in one or more field-configurations, and returns a single real number. We will assume here that the functions have the domain Ω, with coordinates x. The functional derivative is defined as

$$\delta F[f] = \frac{\mathrm{d}}{\mathrm{d}\epsilon}F[f + \epsilon\eta]\Big|_{\epsilon=0} = \int_\Omega \mathrm{d}x\, \frac{\delta F[f]}{\delta f(x)}\eta(x). \tag{183}$$

$\eta(x)$ is here an arbitrary function, but we will make the important assumption that it as well as all its derivatives are zero at the boundary of its domain Ω. This allows us to discard surface terms stemming from partial integration, which we will use frequently. We may use the definition to derive one of the fundamental relations of functional derivation. Take the functional $F[f] = f(x)$. Then,

$$\delta F[f] = \frac{\mathrm{d}}{\mathrm{d}\epsilon}[f(x) + \epsilon\eta(x)] = \eta(x) = \int \mathrm{d}y\, \delta(x-y)\eta(y) \tag{184}$$

This leads to the identity

$$\frac{\delta f(x)}{\delta f(y)} = \delta(x-y), \tag{185}$$

for any function f. Higher functional derivatives are defined similarly, by applying functional variation repeatedly

$$\delta^n F[f] = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\delta^{n-1}F[f + \epsilon\eta_n]\big|_{\epsilon=0} = \int \left(\prod_{i=1}^n \mathrm{d}x_i\right) \frac{\delta^n F[f]}{\delta f(x_n)\dots\delta f(x_n)} \left(\prod_{i=1}^n \eta_i(x_i)\right). \tag{186}$$

A functional may be expanded in a generalization of the Fourier series, which has the form

$$F[f_0 + f] = F[f_0] + \int_\Omega \mathrm{d}x\, f(x)\frac{\delta F[f_0]}{\delta f(x)}\bigg|_{f=f_0} + \frac{1}{2!}\int_\Omega \mathrm{d}x\mathrm{d}y\, f(x)f(y)\frac{\delta^2 F[f_0]}{\delta f(x)\delta f(y)} + \dots \tag{187}$$

As an example, the Klein-Gorodn action,

$$S[\varphi] = -\frac{1}{2}\int_\Omega \mathrm{d}x\, \varphi(\partial^2 + m^2)\varphi(x). \tag{188}$$

It can be evaluated quickly by using Eq. (184) and partial integration

$$\frac{\delta}{\delta\varphi(x)}S[\varphi] = -\frac{1}{2}\int_\Omega \mathrm{d}y\, [\delta(x-y)(\partial_y^2+m^2)\varphi(y)+\varphi(y)(\partial_y^2+m^2)\delta(x-y)] = -\int_\Omega \mathrm{d}y\, \delta(x-y)(\partial_y^2+m^2)\varphi(y) = (\partial_y^2+m^2)\varphi(y) \tag{189}$$

The second derivative is

$$\frac{\delta^2 S[\varphi]}{\delta\varphi(x)\delta\varphi(y)} = \frac{\delta}{\delta f(x)}(\partial_y^2 + m^2)\varphi(y) = (\partial_y^2 + m^2)\delta(x-y). \tag{190}$$