# Chiral Perturbation Theory

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# Contents

# Chapter 1

# Theory

In this section we will survey some general properties of quantum field theory that is necessary for chiral perturbation theory. First, we will introduce the path integral and the 1-particle irreducible effective action, as well as the effective action. We will derive Goldstone's theorem and effective field theories, which are the basis for $\chi$PT .

The theory in this section is based on [1, 2, 3, 4]. Feynman diagrams are drawn using JaxoDraw [5].

## 1.1 QFT via path integrals

The vacuum transition amplitude is given by the path integral

$$Z = \lim_{T\to\infty} \langle \Omega, T/2 | -T/2, \Omega \rangle = \lim_{T\to\infty} \langle \Omega | e^{-iHT} | \Omega \rangle = \int \mathcal{D}\pi \mathcal{D}\varphi \, \exp\Big\{ i \int \mathrm{d}^4x \, \left(\pi\dot{\varphi} - \mathcal{H}[\pi,\varphi]\right) \Big\}, \tag{1.1}$$

where $|\Omega\rangle$ is the vacuum of the theory. By introducing a source term into the Hamiltonian density, $\mathcal{H} \to \mathcal{H} - J(x)\varphi(x)$, we get the generating functional

$$Z[J] = \int \mathcal{D}\pi \mathcal{D}\varphi \, \exp\Big\{ i \int \mathrm{d}^4x \, \left(\pi\dot{\varphi} - \mathcal{H}[\pi,\varphi] + J\varphi\right) \Big\}. \tag{1.2}$$

If $\mathcal{H}$ is quadratic in $\pi$, we can complete the square and integrate out $\pi$ to obtain

$$Z[J] = C \int \mathcal{D}\varphi \, \exp\Big\{ i \int \mathrm{d}^4x \, (\mathcal{L}[\varphi] + J\varphi) \Big\}. \tag{1.3}$$

$C$ is infinite, but constant, and will drop out of physical quantities. In scattering theory, the main objects of study are correlation functions $\langle \varphi(x_1)\varphi(x_2)...\rangle = \langle\Omega| T\{\varphi(x_1)\varphi(x_2)\dots\} |\Omega\rangle$, where $T$ is the time ordering operator. These are given by functional derivatives of $Z[J]$,

$$\langle \varphi(x_1)\varphi(x_2)...\rangle = \frac{\int \mathcal{D}\varphi(x) \, (\varphi(x_1)\varphi(x_2)...) e^{iS[\varphi]}}{\int \mathcal{D}\varphi(x) \, e^{iS[\varphi]}} = \frac{1}{Z[0]} \prod_i \left( -i \frac{\delta}{\delta J(x_i)} \right) Z[J] \Big|_{J=0}, \tag{1.4}$$

where

$$S[\varphi] = \int \mathrm{d}^4x \, \mathcal{L}[\varphi] \tag{1.5}$$

is the action of the theory. The functional derivative is described in section B.4. In a free theory, we are able to write

$$Z_0[J] = Z_0[0] \exp(iW_0[J]), \quad W_0[J] = \frac{1}{2} \int \mathrm{d}^4x \mathrm{d}^4y \, J(x) D_0(x-y) J(y), \tag{1.6}$$

where $D_0$ is the propagator of the free theory. Using this form of the generating functional, Eq. (1.4) becomes

$$\begin{aligned}
&\frac{1}{Z[0]}(-i)^n \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} Z_0[J]\Big|_{J=0} = (-i)^n \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} e^{iW_0[J]}\Big|_{J=0} \\
&= (-i)^n \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_{n-1})} \left( i \frac{\delta W_0[J]}{\delta J(x_n)} \right) e^{iW_0[J]}\Big|_{J=0} \\
&= (-i)^n \frac{\delta}{\delta J(x_2)} \cdots \frac{\delta}{\delta J(x_{n-1})} \left( i \frac{\delta^2 W_0[J]}{\delta J(x_{n-1}) \delta J(x_n)} + i^2 \frac{\delta W_0[J]}{\delta J(x_{n-1})} \frac{\delta W_0[J]}{\delta J(x_n)} \right) e^{iW_0[J]}\Big|_{J=0} = \dots \\
&= (-i)^{n/2} \sum_{(a,b)} \prod_{i=1}^{n/2} \frac{\delta^2 W_0[J]}{\delta J(x_{a(i)}) \delta J(x_{b(i)})}\Big|_{J=0}.
\end{aligned}$$

In the last line we have introduced the functions $a$, $b$ which define a way to pair up $n$ elements. The domain of the functions are the integers between 1 and $n/2$, the image a subset of the integers between 1 and $n$ of size $n/2$. A valid pairing is a set $\{(a(1), b(1)), \dots (a(n/2), b(n/2))\}$, where all elements $a(i)$ and $b(j)$ are different, such all integers up to and including $n$ are featured. A pair is not directed, so $(a(i), b(i))$ is the same pair as $(b(i), a(i))$. The sum is over the set $\{(a,b)\}$ of all possible, unique pairings. If $n$ is odd, the expression is equal to 0. This is Wick's theorem, and it can more simply be stated as *a correlation function is the sum of all possible pairings of 2-point functions*,

$$\left\langle \prod_{i=1}^{n} \varphi(x_i) \right\rangle_0 = \sum_{\{(a,b)\}} \prod_{i=1}^{n/2} \left\langle \varphi(x_{a(i)}) \varphi(x_{b(i)}) \right\rangle_0 . \tag{1.7}$$

The subscript on the expectation value indicates that it is evaluated in the free theory.

If we have an interacting theory, that is a theory with an action $S = S_0 + S_I$, where $S_0$ is a free theory, the generating functional can be written

$$Z[J] = Z_0[0] \left\langle \exp\left( iS_I + i \int \mathrm{d}^4x \, J(x)\varphi(x) \right) \right\rangle_0 . \tag{1.8}$$

We can expand the exponential in power series, which means the expectation in Eq. (1.8) becomes

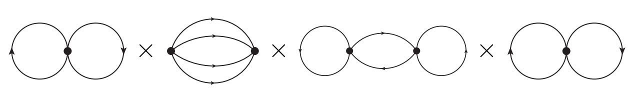
$$\sum_{n,m} \frac{1}{n!m!} \left\langle (iS_I)^n \left( i \int \mathrm{d}^4x \, J(x)\varphi(x) \right)^m \right\rangle_0 . \tag{1.9}$$

The terms in this series are represented by Feynman-diagrams, which are constructed from the Feynman-rules, and can be read from the action. We will not go into further details on how the Feynman-rules are derived, which can be found in any of the main sources for this section [1, 2, 3, 4]. The source terms gives rise to an additional vertex

 $J(x)\,.$ (1.10)

The generating functional $Z[J]$ equals $Z_0[0]$ times *the sum of all diagrams with external sources* $J(x)$.

Consider a general diagram without external legs, built up of $N$ different connected subdiagrams, where subdiagram $i$ appears $n_i$ times. As an illustration, a generic vacuum diagram in $\phi^4$-theory has the form

$V =$  $\times \dots .$ (1.11)

If sub-diagram $i$ as a stand-alone diagram equals $V_i$, then each copy of that subdiagram contribute a factor $V_i$ to the total diagram. However, due to the symmetry of permuting identical subdiagrams, one must divide by the extra symmetry factor $s = n_i!$, which is the total number of permutation of all the copies of diagram $i$. The full diagram therefore equals

$$V = \prod_{i=1}^{N} \frac{1}{n_i!} V_i^{n_i} . \tag{1.12}$$

$V$ is uniquely defined by a finite sequence of integers, $(n_1, n_2, \dots n_N, 0, 0, \dots)$, so the sum of all diagrams is the sum over the set $S$ of all finite sequences of integers. This allows us to write the sum of all diagrams as

$$\sum_{(n_1,\dots)\in S} \prod_i \frac{1}{n_i!} V_i^{n_i} = \prod_{i=1}^{\infty} \sum_{n_i=1}^{\infty} \frac{1}{n_i!} V_i^{n_i} = \exp\Big(\sum\nolimits_i V_i\Big). \tag{1.13}$$

We showed that the generating functional $Z[J]$ were the $Z_0[0]$ times the sum of all diagrams due to external sources. Using Eq. (1.13), we see that the sum of all *connected* diagrams $W[J]$ is given by

$$Z[J] = Z_0[0] \exp(iW[J]). \tag{1.14}$$

We can see that this is trivially true for the free theory, the only connected diagram is

$W_0[J] =$ 

. (1.15)

## 1.2 The 1PI effective action and the effective potential

The generating functional for connected diagrams, $W[J]$, is dependent on the external source current $J$. Analogously to what is done in thermodynamics and in Lagrangian and Hamiltonian mechanics, we can define a new quantity, with a different independent variable, using the Legendre transformation. The new independent variable is

$$\varphi_J(x) := \frac{\delta W[J]}{\delta J(x)} = \langle \varphi(x) \rangle_J \,. \tag{1.16}$$

The subscript $J$ on the expectation value indicate that it is evaluated in the presence of a source. The Legendre transformation of $W$ is then

$$\Gamma[\varphi_J] = W[J] - \int \mathrm{d}^4x\, J(x)\varphi_J(x). \tag{1.17}$$

Using the definition of $\varphi_J$, we have that

$$\frac{\delta}{\delta\varphi_J(x)}\Gamma[\varphi_J] = \int \mathrm{d}^4y\, \frac{\delta J(y)}{\delta\varphi_J(x)} \frac{\delta}{\delta J(y)} W[J] - \int \mathrm{d}^4y\, \frac{\delta J(y)}{\delta\varphi_J(x)} \varphi_J(y) - J(x) = -J(x). \tag{1.18}$$

If we compare this to the classical equations of motion of a field $\varphi$ with the action $S$,

$$\frac{\delta S[\varphi]}{\delta\varphi(x)} = -J(x), \tag{1.19}$$

we see that $\Gamma$ is an action that gives the equation of motion for the expectation value of the field, given a source current $J(x)$.

To interpret $\Gamma$ further we observe what happens if we treat $\Gamma[\varphi]$ as a classical action with a coupling $g$. The generating functional in this new theory is

$$Z[J, g] = \int \mathcal{D}\varphi \exp\Big\{ ig^{-1} \Big( \Gamma[\varphi] + \int \mathrm{d}^4x \varphi(x) J(x) \Big) \Big\} \tag{1.20}$$

The free propagator in this theory will be proportional to $g$, as it is given by the inverse of the equation of motion for the free theory. All vertices in this theory, on the other hand, will be proportional to $g^{-1}$, as they are given by the higher order terms in the action $g^{-1}\Gamma$. This means that a diagram with $V$ vertices and $I$ internal lines is proportional to $g^{I-V}$. Regardless of what the Feynman-diagrams in this theory are, the number of loops of a connected diagram is $L = I - V + 1$. [1] To see this, we first observe that one single loop must have equally many internal lines as vertices, so the formula holds for $L = 1$. If we add a new loop to a diagram with $n$ loops by joining two vertices, the formula still holds. If we attach a new vertex with one line, the formula still holds, and as the diagram is connected, any more lines connecting the new vertex to the

[1] This is a consequence of the Euler characteristic $\chi = V - E + F$.

diagram will create additional loops. This ensures that the formula holds, by induction. As a consequence of this, any diagram is proportional to $g^{L-1}$. This means that in the limit $g \to 0$, the theory is fully described at the tree-level, i.e. by only considering diagrams without loops. In this limit, we may use the stationary phase approximation, as described in section B.3, which gives

$$Z[J, g \to 0] \approx C \det\left(-\frac{\delta^2\Gamma[\varphi_J]}{\delta\varphi^2}\right) \exp\left\{ig^{-1}\left(\Gamma[\varphi_J] + \int \mathrm{d}^4x J\varphi_J\right)\right\}. \tag{1.21}$$

This means that

$$-ig\ln(Z[J,g]) = gW[J,g] = \Gamma[\varphi_J] + \int \mathrm{d}^4x\, J(x)\varphi_J(x) + \mathcal{O}(g), \tag{1.22}$$

which is exactly the Legendre transformation we started out with, modulo the factor $g$. $\Gamma$ is therefore the action which describes the full theory at the tree level. For a free theory, the classical action $S$ equals the effective action, as there are no loop diagrams.

The propagator $D(x,y)$, which is the connected 2 point function $\langle\varphi(x)\varphi(y)\rangle_J$, is given by the second functional derivative of $W[J]$, times $-i$. Using the chain rule, together with Eq. (1.18), we get

$$(-i)\int \mathrm{d}^4z \frac{\delta^2 W[J]}{\delta J(x)\delta J(z)} \frac{\delta^2\Gamma[\varphi_J]}{\delta\varphi_J(z)\varphi_J(y)} = (-i)\int \mathrm{d}^4z \frac{\delta\varphi_J[z]}{\delta J(x)} \frac{\delta^2\Gamma[\varphi_J]}{\delta\varphi_J(z)\varphi_J(y)} = \frac{\delta}{\delta J(x)} \frac{\delta\Gamma[\varphi_J]}{\delta\varphi_J(y)} = \delta(x-y). \tag{1.23}$$

This shows that the second functional derivative of the effective action is $iD^{-1}$, where $D^{-1}$ is the inverse propagator. The inverse propagator is the sum of all one-particle-irreducible (1PI) diagrams, with two external vertices. More generally, $\Gamma$ is the generating functional for 1PI diagrams, which is why it is called the 1PI effective action.

## The effective action and symmetries

The symmetries of a theory are transformations of the physical state that leaves the governing equations unchanged. A lot of physics is contained in the symmetries of a theory, such as the presence of conserved quantities and the systems low energy behavior. We distinguish between internal and external symmetries. An external symmetry is an active coordinate transformation, such as rotations or translations. They relate degrees of freedom at different space-time points, while internal symmetry transforms degrees of freedom at each space-time point independently of what happens at other points. A further distinction is between local and global symmetry transformations. Local transformations have one rule for how to transform degrees of freedom at each point, which is applied everywhere, while local transformations might themselves be functions of space-time.

In classical field theory, symmetries are encoded in how the Lagrangian changes due to a transformation of the fields. We will consider continuous transformations, which are can in general be written as

$$\varphi(x) \longrightarrow \varphi'(x) = f_t[\varphi](x), \quad t \in [0,1]. \tag{1.24}$$

Here, $f_t[\varphi]$ is a functional in $\varphi$, and a smooth function of $t$, with the constraint that $f_0[\varphi] = \varphi$. This allows us to look at "infinitesimal" transformations,

$$\varphi'(x) = f_\epsilon[\varphi] \sim \varphi(x) + \epsilon g[\varphi](x), \quad \epsilon \to 0. \tag{1.25}$$

Here, $g$ is a functional of $\varphi$. We will consider internal, global transformations in which $g$ is linear in $\varphi$. For $N$ fields, $\varphi_i$, this can be written

$$\varphi_i'(x) = \varphi_i(x) + \epsilon\, it_{ij}\varphi_j(x), \quad \epsilon \to 0. \tag{1.26}$$

$t_{ij}$ is called the generator of the transformation. A symmetry of the system is then one in which the Lagrangian is unchanged by the transformation, or at most is different by a divergence-term. That is, a transformation $\varphi \to \varphi'$ is a symmetry if

$$\mathcal{L}[\varphi'] = \mathcal{L}[\varphi] + \partial_\mu K^\mu[\varphi], \tag{1.27}$$

where $K^\mu[\varphi]$ is a functional of $\varphi$.[2] This is a requirement for a symmetry in quantum field theory too. However, as physical quantities are given by not just the action of a single state, but the path integral, the integration measure $\mathcal{D}\varphi_i$ has to be invariant as well. If a classical symmetry fails due to the integration measure, it is called an anomaly.

We want to investigate what constraints a symmetry lies on the effective action. To that end, assume

$$\mathcal{D}\varphi'(x) = \mathcal{D}\varphi(x), \quad S[\varphi'] = S[\varphi]. \tag{1.28}$$

In the generating functional, such a transformation corresponds to a change of integration variable. Using the infinitesimal version of the transformation, we may write

$$Z[J] = \int \mathcal{D}\varphi \, \exp\left\{iS[\varphi] + i\int \mathrm{d}^4x J_i(x)\varphi_i(x)\right\} = \int \mathcal{D}\varphi' \, \exp\left\{iS[\varphi'] + i\int \mathrm{d}^4x J_i(x)\varphi_i'(x)\right\} \tag{1.29}$$

$$= Z[J] - \epsilon \int \mathrm{d}^4x J_i(x) \int \mathcal{D}\varphi \, e^{iS[\varphi]}[t_{ij}\varphi_j(x)], \tag{1.30}$$

Using Eq. (1.18), we can write this as

$$\int \mathrm{d}^4x \, \frac{\delta\Gamma[\varphi_J]}{\delta\varphi_i(x)} \, t_{ij} \left\langle \varphi_j(x) \right\rangle_J = 0. \tag{1.31}$$

## Effective potential

For a constant field configuration $\varphi(x) = \varphi_0$, the effective action, which is a functional, becomes a regular function. We define the effective potential $\mathcal{V}_{\text{eff}}$ by

$$\Gamma[\varphi_0] = -VT\, \mathcal{V}_{\text{eff}}(\varphi_0), \tag{1.32}$$

$VT$ is the volume of space-time. For a constant ground state, the effective potential will equal the energy of this state. To calculate the effective potential, we can expand the action around this state to calculate the effective action, by changing variables to $\varphi(x) = \varphi_0 + \eta(x)$. $\eta(x)$ now parametrizes fluctuations around the ground state, and has by assumption a vanishing expectation value. The generating functional becomes

$$Z[J] = \int \mathcal{D}(\varphi_0 + \eta) \, \exp\left\{iS[\varphi_0 + \eta] + i\int \mathrm{d}^4x J(\varphi_0 + \eta)\right\} \tag{1.33}$$

The notation

$$\frac{\delta S[\varphi_0]}{\delta\varphi(x)} \tag{1.34}$$

indicates that the functional $S[\varphi]$ is differentiated with respect to $\varphi(x)$, then evaluated at $\varphi(x) = \varphi_0$. The functional version of a Taylor expansion is

$$S[\varphi_0 + \eta] = S[\varphi_0] + \int \mathrm{d}x \frac{\delta S[\varphi_0]}{\delta\varphi(x)}\eta(x) + \frac{1}{2}\int \mathrm{d}x\mathrm{d}y \, \frac{\delta^2 S[\varphi_0]}{\delta\varphi(x)\delta\varphi(y)}\eta(x)\eta(y) + \dots \tag{1.35}$$

We will only consider this expansion up to second order in derivatives for now. Inserting this into $Z[J]$ we get

$$Z[J] =$$
$$\int \mathcal{D}\eta \exp\left\{i\int \mathrm{d}^4x \left(\mathcal{L}[\varphi_0] + J\varphi_0\right) + i\int \mathrm{d}^4x \left(\frac{\delta S[\varphi_0]}{\delta\varphi(x)} + J(x)\right)\eta(x) + i\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4y \, \frac{\delta^2 S[\varphi_0]}{\delta\varphi(x)\delta\varphi(y)}\eta(x)\eta(y)\right\}$$

The first term is constant with respect to $\eta$, and may therefore be taken outside the path integral. The second term gives rise to tadpole diagrams, which alter the expectation value of $\eta(x)$. For $J = 0$, this

[2] Terms of the form $\partial_\mu K^\mu$ does not affect the physics, as variational principle $\delta S = 0$ which gives the equations of motion do not vary the fields at infinity.

expectation value should vanish, so this term can be ignored. Furthermore, this means that the ground state must minimize the classical potential,

$$\frac{\partial \mathcal{V}(\varphi_0)}{\partial \varphi} = 0. \tag{1.36}$$

The one loop approximation to the effective potential is given by the Taylor-expansion up to second order. This term is a Gaussian integral, and may be evaluated as described in section B.3,

$$\int \mathcal{D}\eta \, \exp\left( i\frac{1}{2} \int \mathrm{d}^4x \mathrm{d}^4y \, \frac{\delta^2 S[\varphi_0]}{\varphi(x)\varphi(y)} \eta(x)\eta(y) \right) = C \det\left( -\frac{\delta^2 S[\varphi_0]}{\delta\varphi(x)\,\delta\varphi(y)} \right)^{-1/2} \tag{1.37}$$

The generating functional for connected diagrams, as defined in Eq. (1.15), is therefore

$$W[J] = \int \mathrm{d}^4x \; (\mathcal{L}[\varphi_0] + J\varphi_0) + i\frac{1}{2} \,\mathrm{Tr}\left\{ \ln\left( -\frac{\delta^2 S[\varphi_0]}{\delta\varphi(x)\,\delta\varphi(y)} \right) \right\} + \dots, \tag{1.38}$$

where we have used the identity $\ln \det M = \mathrm{Tr} \ln M$. Using the definition of the effective action, Eq. (1.17), and Eq. (1.32) we get an explicit formula for the effective potential to 1 loop order,

$$\mathcal{V}_{\mathrm{eff}}(\varphi_0) = \mathcal{V}(\varphi_0) - \frac{i}{VT}\frac{1}{2}\,\mathrm{Tr}\left\{ \ln\left( -\frac{\delta^2 S[\varphi_0]}{\delta\varphi(x)\,\delta\varphi(y)} \right) \right\}. \tag{1.39}$$

## 1.3 Goldstone's theorem

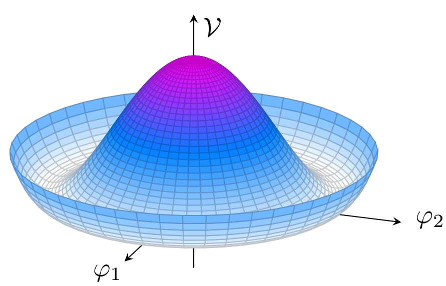



Figure 1.1: The Mexican hat potential.

The ground state of a theory is not necessarily invariant under the symmetry transformations of the theory. This is exemplified by the linear sigma model,

$$\mathcal{L}[\varphi] = \frac{1}{2}\partial_\mu\varphi_i(x)\partial^\mu\varphi_i(x) - \mathcal{V}(\varphi), \quad \mathcal{V}(\varphi) = -\frac{1}{2}\mu^2\varphi_i(x)\varphi_i(x) + \frac{1}{4}\lambda[\varphi_i(x)\varphi_i(x)]^2. \tag{1.40}$$

The linear sigma model Lagrangian is invariant under the rotation of the $N$ fields,

$$\varphi_i \longrightarrow \varphi_i' = O_{ij}\varphi_j, \quad O^{-1} = O^T. \tag{1.41}$$

This is a global, internal and continuous set of transformations, with the infinitesimal form

$$\varphi_i(x) = \varphi_i(x) + \epsilon \, i t_{ij}\varphi_j(x) \tag{1.42}$$

The group of all such transformations form the Lie group $O(N)$. (SKRIVE APPENDIX OM LIE GRUPPER?) If we assume the ground state $\varphi_0$ is translationally invariant, then it is given by minimizing the effective potential. The first approximation of this is given by the classical potential, $\mathcal{V}$. For $N = 2$, this is the famous "Mexican hat"-potential, as illustrated by Figure 1.1. The ground state is therefore given by any of the values along the brim of the potential. If we, without loss of generality, choose $\varphi = (0, v)$ as the ground state, then any symmetry transformation will change this state. We say that the symmetry has been *spontaneously broken*. We can express this mathematically as

$$t_{ij}\left\langle \varphi_i \right\rangle_0 \neq 0. \tag{1.43}$$

If we take the constraint Eq. (1.18), differentiate with respect to $\varphi_j(y)$ and evaluate in the vacuum, we get

$$\int \mathrm{d}^4x \, \frac{\delta^2\Gamma[\varphi_0]}{\delta\varphi_j(y)\,\delta\varphi_i(x)} t_{ik} \left\langle \varphi_k \right\rangle_0 = 0. \tag{1.44}$$

In Eq. (1.23), we found that the second derivative of the effective action is the inverse propagator. The momentum space propagator is therefore

$$i\delta(0)\tilde{D}^{-1}_{ji}(p) = \int \mathrm{d}^4x \, e^{-ipx} \frac{\delta^2\Gamma[\varphi_0]}{\delta\varphi_j(0)\,\delta\varphi_i(x)}. \tag{1.45}$$

(CHECK THAT) If we assume the ground state is independent of space-time, we get

$$\tilde{D}^{-1}_{ij}(p=0)\, t_{jk} \left\langle \varphi_k \right\rangle_0 = \frac{\partial^2 \mathcal{V}_{\text{eff}}}{\partial\varphi_i\,\partial\varphi_j}\, t_{jk} \left\langle \varphi_k \right\rangle_0 = 0. \tag{1.46}$$

If the symmetry transformation $\varphi_i \to \varphi_i + i\epsilon t_{ij}\varphi_j$ remains unbroken, then this is trivial, as $t_{ij}\left\langle \varphi_j \right\rangle_0 = 0$. However, if it is a broken symmetry, then by definition $t_{ij}\left\langle \varphi_j \right\rangle_0 \neq 0$. In that case, $t_{ij}\left\langle \varphi_j \right\rangle_0$ has a zero eigenvalue of the inverse propagator, at $p = 0$. In other words, the system contains a zero-mass particle, a Goldstone boson.[3]

The set of continuous symmetry transformations,

$$G = \{\, g \mid g\varphi = \varphi',\, S[\varphi'] = S[\varphi], \mathcal{D}\varphi' = \mathcal{D}\varphi \,\}, \tag{1.47}$$

form a Lie group. This is a manifold, i.e. a space that is local homeomorphic to $\mathbb{R}^N$, and thus can be parametrized by $N$ real numbers $\eta_\alpha$, while also having a group structure, composition of transformations. We will focus on connected Lie groups, in which we all elements $g \in G$ is in the same connected piece as the identity map $\mathrm{id}(\varphi) = \varphi$. This means that for each $g \in G$, one can find a continuous path $\gamma(t)$ in the manifold, such that $\gamma(0) = \mathrm{id}$ and $\gamma(1) = g$. Given such a path, we can study transformations close to the identity. As the Lie group is a smooth manifold, we can write

$$\gamma(\epsilon) = \mathrm{id} + i\epsilon V + \mathcal{O}\left(\epsilon^2\right). \tag{1.48}$$

$V$ is a generator, and is a part of the tangent space of the identity element, $T_{\mathrm{id}}$.[4] The set of generators such $T_{\mathrm{id}}$, thus forms a vector space, and the coordinates $\eta_a$ induce a coordinate basis. This can be written as a path through parameter space, $\gamma(t) = g(\eta(t))$, which gives

$$V = \frac{\mathrm{d}\gamma}{\mathrm{d}t}\Big|_{t=0} = \frac{\mathrm{d}\eta_a}{\mathrm{d}t}\Big|_{t=0} \frac{\partial g}{\partial\eta_a}\Big|_{\eta=0}. \tag{1.49}$$

The parameters therefore gives a basis for $T_{\mathrm{id}}$, and we can write

$$\gamma(\epsilon) = \mathrm{id} + i\epsilon v_\alpha T_\alpha + \mathcal{O}\left(\epsilon^2\right), \quad T_\alpha = \frac{\partial g}{\partial\eta_\alpha}. \tag{1.50}$$

The tangent space, together with the additional operation

$$[T_\alpha, T_\beta] = iC^\gamma_{\alpha\beta}T_\gamma, \tag{1.51}$$

called the Lie bracket, forms a Lie algebra. For matrix groups, which are what we deal with in this text, the Lie bracket is the commutator. $C^\gamma_{\alpha\beta}$ are called the structure constants, and are totally antisymmetric. This is equivalent with them obeying the Jacobi identity,

$$C_{\alpha\beta\gamma} + C_{\beta\gamma\alpha} + C_{\gamma\alpha\beta} = 0. \tag{1.52}$$

The exponential maps elements of the Lie algebra to the Lie group. In fact, every element of the Lie group can be written

$$g(\eta) = \exp\{i\eta_\alpha T_\alpha\}. \tag{1.53}$$

[3] The particles are bosons due to the bosonic nature of the transformations, $t$. If the generators are Grassmann numbers, the resulting particle, called a goldstinos, are fermions.

[4] The factor $i$ is a physics convention, and differs from how mathematicians define generators of a lie group.

For matrix group, the exponential is defined by its series expansion,

$$\exp(X) = \sum_n \frac{1}{n!} X^n. \tag{1.54}$$

A subset of the original Lie group, $H \subset G$, which is closed under the group action is called a subgroup. In the case of symmetry breaking, a subset of the original generators, $x_i$, where $i \in \{1, ...m\}$ may correspond to a symmetry of the ground state. That is,

$$x_i \varphi_0 = 0. \tag{1.55}$$

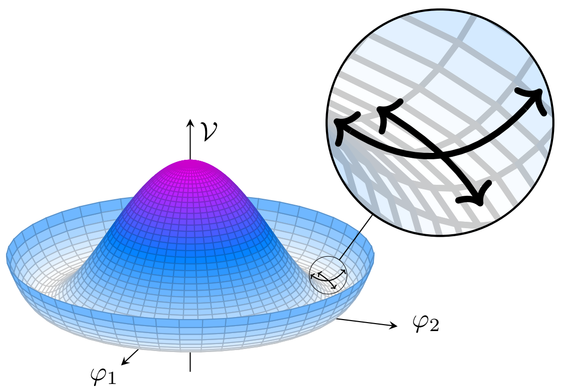
The group elements of $G$ that has the form $h = \exp\{i\xi_i x_i\}$ form a subgroup $H$, with dimension $m = \dim H$. The set of generators $t_a$, on the other hand, do not form a subgroup, as the identity transformation id is a part of $H$. These are called *Broken generators*. As the union of $x_i$ and $t_a$ make up the total set of original generators, $T_\alpha$, the number of broken generators are $n_G = \dim G - \dim H$ In Lorentz invariant systems this corresponds to one massless mode per broken generator, so there are a total of $n_G$ Goldstone bosons. As a subgroup $H$ of a Lie group $G$ is a Lie group in its own right, the generators of the subgroup forms a closed algebra. We can write the commutators as

$$[x_i, x_j] = iC^k_{ij} x_k, \tag{1.56}$$

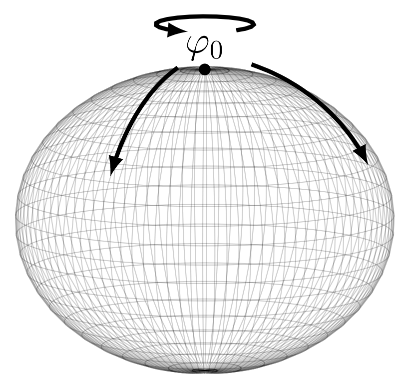
$$[x_i, t_a] = iC^b_{ia} t_b, \tag{1.57}$$

$$[t_a, t_b] = iC^k_{ab} x_k + iC^c_{ab} t_c, \tag{1.58}$$

where $ijk$ runs over the unbroken generators, and $abc$ runs over the broken. The second formula can be derived using the Jacobi identity Eq. (1.52), which implies that $C^k_{ia} = -C^a_{ik} = 0$.




(a) Excitations along the brim does not cost any energy.




(b) Excitations for the $N = 3$ sigma model. Two of the symmetries are broken, while rotations around the groundstate leaves the system unchanged.

The Mexican hat potential gives an intuition for the Goldstone mode. In the case of the two-dimensional sigma model, the symmetry of the Lagrangian are rotations in the plane. As the ground stat is along the "brim" of the hat, this rotational symmetry is broken. Any excitations in this direction, however, does not cost any energy, which is indicative of a massless mode. This is illustrated in Figure 1.2a. In this example, the original symmetry group is one dimensional, so there is no unbroken symmetries. If we instead consider the three-dimensional linear sigma model, which has a three-dimensional symmetry group, rotations of the sphere. The ground state manifold, the set of all the degenerate ground states, is then a sphere. When the system chooses one single ground state, this symmetry is broken, but only for two of the generators. The generator for rotations around the ground state leaves the that point unchanged, and is thus an unbroken symmetry. This is illustrated in Figure 1.2b.

## 1.4 CCWZ construction

As Goldstone bosons are massless, they play a crucial role in the low energy dynamics. To best describe this limit, we seek a parametrization of the fields in which they are the degrees of freedom. This can be done using the CCWZ construction, named after Callan, Coleman, Wess and Zumino. As well as the original papers [6, 7], this section is based on [3, 8, 9] and[5]. (HVORDAN SITERE EN BEAMER?)

We saw that the Goldstone bosons corresponds to excitations within the vacuum manifold. The vacuum manifold corresponds to points in field space $\varphi$ that can be reach from the vacuum $\varphi_0$ with a transformation $g \in G$. This means that we can write such excitations as

$$\varphi = \tilde{\Sigma}\varphi_0, \quad \tilde{\Sigma} = \tilde{\Sigma}(\eta) = \exp\{i\eta_\alpha T_\alpha\} \tag{1.59}$$

$\tilde{\Sigma}$ is thus a function from the parameter space, $\eta_\alpha \in \mathbb{R}^n$ to $G$,

$$\tilde{\Sigma} : \mathbb{R}^n \longmapsto G. \tag{1.60}$$

We then get space-time dependent field configurations by making the parameters dependent on spacetime. We will for now assume $\eta_\alpha$ is constant. This parametrization is highly redundant. There are $n = \dim G$ parameters $\eta$, but Goldstone's theorem says that there is one massless mode per broken generator. Two elements $\tilde{\Sigma}$ and $\tilde{\Sigma}'$, related by

$$\tilde{\Sigma}' = \tilde{\Sigma} e^{i\theta_a t_a} \tag{1.61}$$

results in the same $\varphi(x)$. This is because $e^{i\theta_a t_a} = h \in H$, and $h\varphi_0 = \varphi_0$, by assumption. The set of all equivalent $\Sigma$'s is exactly the left coset, $gH = \{\, gh \mid h \in H \,\}$. The set of cosets forms a new manifold, $G/H$, called the Goldstone manifold, and has dimension $\dim(G/H) = \dim(G) - \dim(H)$, which is exactly the number of broken generators, and thus also the number of Goldstone modes. Membership of a certain coset from an equivalence relation, $g \sim g'$ if $g' = gh$. This means that the cosets $gH$ form a partition of $G$, and that each element $g \in G$ belongs to one, and only one, coset. To remove the redundancy in the parametrization, we need to choose one representative element from each coset.

By the inverse function theorem, any mapping between manifolds $f : \mathcal{M} \mapsto \mathcal{N}$ that has a non-degenerate differential, that is an invertible Jacobian, at a point $p \in \mathcal{M}$, is invertible in a neighborhood of $p$. The map

$$\Sigma(\xi, \theta) = \exp\{i\xi_i x_i\} \exp\{i\theta_a t_a\} \tag{1.62}$$

is invertible at $p = (\xi_i = 0, \theta_a = 0)$, which is mapped to the identity, as the Jacobian is the identity matrix. This means that, in a neighborhood $U \subset G$ of the identity, each element $g$ has a unique representation $g = \Sigma$. [10] Furthermore, two elements $\Sigma'$ and $\Sigma$ related by $\Sigma' = \Sigma h$, $h \in H$ have the same $\xi$-arguments. We see that $\xi_i$ parametrize $G/H$, in the neighborhood of the identity. We therefore demand that $\Sigma$ always appear in the standard form

$$\Sigma(\xi) = \exp\{i\xi_i x_i\}. \tag{1.63}$$

The field $\varphi(x)$ can therefore be written as

$$\varphi(x) = \Sigma(x)\varphi_0 = \exp\{i\xi_i(x)x_i\}\varphi_0, \tag{1.64}$$

and $\xi_i(x)$ can be associated with the Goldstone bosons.

In the linear sigma model, the original $O(N)$ symmetry is broken down to $O(N-1)$, which transforms the remaining $N-1$ fields with vanishing expectation value into each other. However, $O(N)$ consists of two disconnected subsets, those matrices with determinant 1 and those with determinant -1. There is no continuous path that takes an element of $O(N)$ with determinant of $-1$ to an element with determinant 1.[6] The set of symmetries that are connected to the identity is

$$G = SO(N) = \{\, M \in O(N) \mid \det M = 1 \,\}. \tag{1.65}$$

If we choose $\varphi_0 = (0, 0, ..., v)$, then it is apparent that the ground state is invariant under the rotations of the $N-1$ first fields, so the unbroken symmetry is $H = SO(N-1)$. The Goldstone manifold is $G/H = SO(N)/SO(N-1)$.

Consider the case of $N = 3$, which is illustrated in Figure 1.2b. $G$ is the rotations of the sphere, while $H$ is rotations around $\varphi_0$, $SO(2)$. The Goldstone manifold consists of the rotations of $\varphi_0$ to other points of the

[5] `http://scipp.ucsc.edu/~haber/archives/physics251_17/PHYS251_Presentation_L_Morrison`

[6] A simple proof of this is the fact that the determinant is a continuous function, while any path $\det M(t)$ such that $\det M(1) = -1$, $\det m(0) = 1$ must make a discontinuous jump.

sphere, i.e. $G/H = SO(3)/S(2) = S^2$, the 2-sphere. This is not a Lie group, as translating $\varphi$ in a closed path around the sphere may result in a rotation around the z-axis.

To check that $\xi_i$ in fact are the Goldstone modes, we study the way they appear in the Lagrangian. As they are massless, no mass term of the form $\xi_i\xi_i$ should appear in the Lagrangian. The original Lagrangian $\mathcal{L}[\varphi]$ was invariant under global transformations $\varphi(x) \mapsto g\varphi(x)$. However, any terms that only depend on $\varphi(x)$, and not its derivatives, will also be invariant under a *local* transformation, $\varphi(x) \mapsto g(x)\varphi(x)$. Our parametrization of the fields, $\varphi(x) = \Sigma(x)\varphi_0$ is exactly such a transformation, which means that any such terms are independent of the Goldstone bosons. We can therefore write

$$\mathcal{L}[\varphi] = \mathcal{L}_{\text{kin}}[\varphi] + V(\varphi_0), \tag{1.66}$$

where all terms in $\mathcal{L}_{\text{kin}}$ are proportional to at least one derivative term, $\partial_\mu\varphi(x)$. Inserting the parametrization into this derivative term, we get

$$\partial_\mu\varphi(x) = \partial_\mu[\Sigma(x)\varphi_0] = \Sigma(x)[\Sigma(x)^{-1}\partial_\mu\Sigma(x)]\varphi_0. \tag{1.67}$$

The dependence of the Lagrangian on $\xi_i$ only appears through terms of the form $\Sigma(x)^{-1}\partial_\mu\Sigma(x)$. These can always be written on the form

$$i\Sigma(x)^{-1}\partial_\mu\Sigma(x) = D_\mu(x) + E_\mu(x),$$
$$D_\mu = ix_i D_{ij}(\xi)\partial_\mu\xi_j, \quad E_\mu = it_a E_{ai}(\xi)\partial_\mu\xi_i.$$

This is called the Maurer-Cartan form, and $D_{ij}$ and $E_{ai}$ are as-of-yet unknown real valued functions of $\xi$. (HVORFOR??)

### Transformation properties of Goldstone bosons

We can deduce how the Goldstone bosons transforms under $G$ from how $\varphi$ transforms. In general,

$$\varphi' = g\varphi = (g\Sigma(\xi))\varphi_0 = \Sigma(\xi')\varphi_0 \quad g \in G. \tag{1.68}$$

While $\Sigma(\xi')$ has the standard form by assumption,

$$\Sigma(\xi') = \exp\{i\xi'_i x_i\}, \tag{1.69}$$

$g\Sigma(\xi)$ does not, in general.

Figure 1.3 illustrates this in the case of $G = O(3)$. $\Sigma(\xi)$ transforms $\varphi_0$ to $\varphi$, then $g$ transforms $\Sigma(\xi)\varphi_0$ to $\varphi(\xi')$. Assuming $\varphi$ and $\varphi'$ are close enough to $\varphi_0$, we can write $\Sigma(\xi)$ and $\Sigma(\xi')$ on the standard form. However, if we follow a small neighborhood around $\varphi_0$ as it is acted on by $\Sigma(\xi)$, then $g$, it will be rotated by the time it arrives at $\varphi'$, when compared to the same neighborhood if it was acted on by $\Sigma(\xi')$

$g\Sigma(\xi)$ and $\Sigma(\xi')$ are in the same coset, as they by assumption corresponds to the same physical state. This means that we can write $g\Sigma(\xi) = \Sigma(\xi')h(g,\xi)$, where $h(g,\xi) \in H$. The transformation rule of $\xi$ under $G$ is therefore implicitly defined by

$$\Sigma(\xi') = g\Sigma(\xi)[h(g,\xi)]^{-1}. \tag{1.70}$$

This is in general not a linear representation, which is why this construction also is called a *non-linear realization.* Using the transformation rule, we can obtain the transformation rule of the Maurer-Cartan form. We use the shorthand $\Sigma = \Sigma(\xi)$, $\Sigma' = \Sigma(\xi')$, and $h = h(g,\xi)$. This gives

$$\Sigma^{-1}\partial_\mu\Sigma \to \Sigma'^{-1}\partial_\mu\Sigma' \tag{1.71}$$
$$= (g\Sigma h^{-1})^{-1}\partial_\mu(g\Sigma h^{-1}) \tag{1.72}$$
$$= (h\Sigma^{-1}g^{-1})g[(\partial_\mu\Sigma)h^{-1} + \Sigma\partial_\mu h^{-1}] \tag{1.73}$$
$$= h\Sigma^{-1}(\partial_\mu\Sigma)h^{-1} + h\partial_\mu h^{-1} \tag{1.74}$$
$$= h(\Sigma^{-1}\partial_\mu\Sigma + \partial_\mu)h^{-1}. \tag{1.75}$$

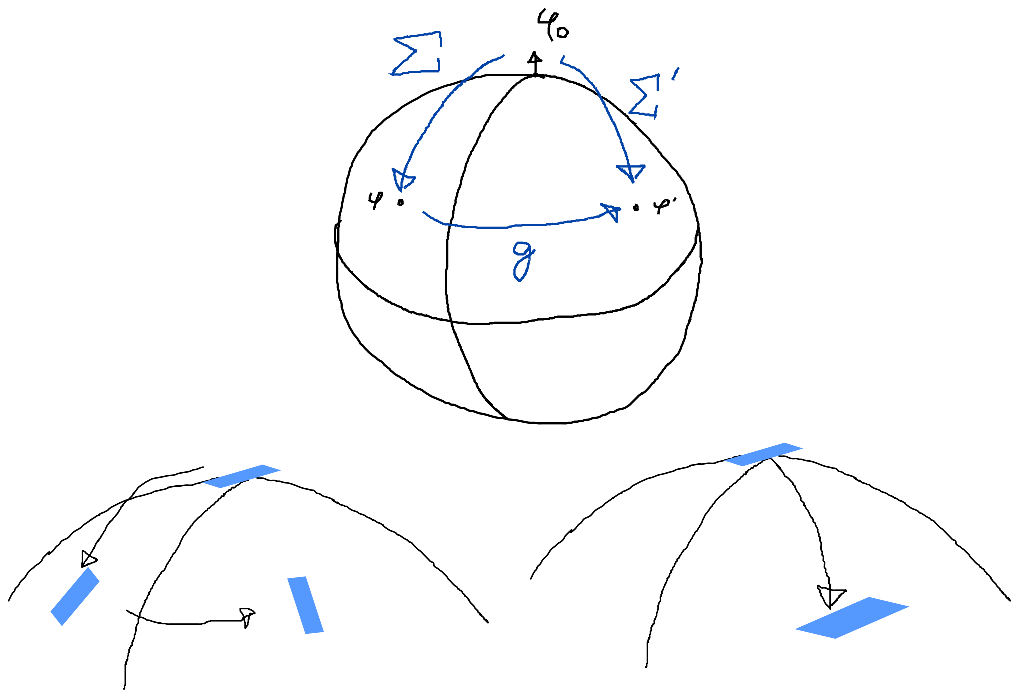



Figure 1.3: (KLADD) The top figure illustrates the transformation of $\varphi_0$ to $\varphi$ and then $\varphi$, and the alternative transformation $\varphi_0 \to \varphi'$. The bottom figure illustrates how this can rotate a neighborhood of $\varphi_0$ differently.

In terms of $D_\mu$ and $E_\mu$,

$$D_\mu \to h D_\mu h^{-1} \tag{1.76}$$

$$E_\mu \to h(E_\mu + i\partial_\mu)h^{-1}. \tag{1.77}$$

We see that $E_\mu$ transforms like a gauge field, with the gauge group $H$. These are the peicees we need to construct a Lagrangian in terms of the Goldstone bosons. (MANGLER: Inkluder massiver partikler, med $\varphi(x) = \Sigma(x)\tilde{\varphi}(x)$, og hvordan det transfoormerer. Skriver om representasjoner? (adj, r pi))

## 1.5 Effective theories

In section 1.2, we studied the effective action, and found that it gave the equation of motion for the expectation value of the field in the full quantum theory. Let $\varphi^*)x = \langle \varphi(x) \rangle$, and $\varphi(x) = \varphi^*(x) + \eta(x)$. We can write this as

$$\exp\{i\Gamma[\varphi^*]\} = \int \mathcal{D}\eta \, \exp\{iS[\varphi^* + \eta]\}. \tag{1.78}$$

As, by assumption, $\langle \eta \rangle = 0$, this only includes 1PI diagrams. We say that the degree of freedom $\eta$ has been *integrated out*. More generally, we can integrate out some of the degrees of freedom of a system, to get an effective theory for what is left. If we have two sets of fields, $\varphi$ and $\psi$, and a Lagrangian $\mathcal{L}[\varphi, \psi]$, then the effective theory of the $\varphi$ fields are defined by

$$\int \mathcal{D}\varphi \mathcal{D}\psi \, \exp\left\{ i \int \mathrm{d}x \, \mathcal{L}[\varphi, \psi] \right\} = \int \mathcal{D}\varphi \exp\{iS_{\text{eff}}[\varphi]\}. \tag{1.79}$$

(EKSEMPLER? WILSON RENORMALISERING; FERMI TEORI)

The effective action can not in general be written as a single integral over a power series in the field, it migh for example be non-local [4]. To construct the effective a effective theory of Goldstone bosons, such as chiral pertubation theory, we rely on a "theorem", as formulated by Weinberg:

> [I]f one writes down the most general possible Lagrangian, including all terms consistent with assumed symmetry principles, and then calculates matrix elements with this Lagrangian to any given order of perturbation theory, the result will simply be the most general possible S-matrix consistent with analyticity, perturbative unitary, cluster decomposition and the assumed symmetry principles. [11]

In other words, if we write down the most general Lagrange density consistent with symmetries of the underlying theory, it will result in the most general S-matrix consistent with that theory, and important physical assumptions. In last section, we found a parametrization that guarantees for a simple way the fields must appear in the Lagrangian. Thus, any combination of these building blocks $\mathcal{O}_i(\xi)$ that must appear in the Lagrangian. The effective Lagrangian is therefore

$$\mathcal{L}[\xi] = \sum_i c_i \mathcal{O}_i(\xi), \tag{1.80}$$

where $c_i$ are free parameters.

This leaves a Lagrange density with infinitely many terms, and infinitely many free parameters. To be able to use this theory for anything one must have a method for ordering the terms in order of importance. As described in [12], by rescaling the external momenta $p_\mu \to tp_\mu$ and quark masses $m_i \to t^2 m_i$, each term in the Lagrangian obtains a factor $t^D$. The Lagrangian is then expanded as $\mathcal{L} = \sum_D \mathcal{L}_D$, where $\mathcal{L}_D$ contains all terms with a factor $t^D$.

# Chapter 2

# The effective theory of pions

## 2.1 QCD

In this paper we consider an effective theory of quantum chromo dynamics, QCD, at low temperature and with two quarks, up and down. These quarks has a mass matrix

$$M = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}. \tag{2.1}$$

In the isospin limit, $m_u = m_d$, the theory is invariant under global transformations by elements of the group $G' = \mathrm{SU}\left(2\right)_L \times \mathrm{SU}\left(2\right)_R \times \mathrm{U}\left(1\right)_V$. All terms involving only pions are trivially invariant under $\mathrm{U}\left(1\right)_V$, (HVORFOR?) so we focus on the $G = \mathrm{SU}\left(2\right)_L \times \mathrm{SU}\left(2\right)_R$ subgroup.

(TODO: SKRIVE OM DE RELEVANTE EGENSKAPENE TIL QCD)

## 2.2 Chiral perturbation theory

The $\mathrm{SU}\left(2\right)_L \times \mathrm{SU}\left(2\right)_R$ symmetry of QCD is spontaneously broken if the quark field has a non-zero ground state expectation value $\langle \bar{q} q \rangle$, leaving only a subgroup $H = \mathrm{SU}\left(2\right)_V \subseteq G$ of symmetry transformations of the vacuum state. The Goldstone manifold $G/H = \mathrm{SU}\left(2\right)_A$ is a three-dimensional Lie group, and therefore results in three (pseudo) Goldstone bosons, the pions. There exists an isomorphism from a subset $S \subseteq M_1$ of the set of all Goldstone-fields

$$M_1 = \left\{ \pi_a : \mathcal{M}_4 \longrightarrow \mathbb{R}^3 | \pi_a \, \text{smooth} \right\}$$

close to the ground state, into fields taking values in the Goldstone manifold $G/H$. (BEVISE?)(HVA ER ISOMORFISME HER?). The $\chi$PT effective Lagrangian will be constructed using this map, through the parametrization

$$\begin{aligned} \Sigma : \mathcal{M}_4 &\longrightarrow \mathrm{SU}\left(2\right), \\ x &\longrightarrow \Sigma(x) = A_\alpha (U(x) \Sigma_0 U(x)) A_\alpha, \end{aligned} \tag{2.2}$$

where

$$\Sigma_0 = \mathbb{1},\, A_\alpha = \exp\left(\frac{i\alpha}{2}\tau_1\right),\, U(x) = \exp\left(i\frac{\tau_a \pi_a(x)}{2f}\right).$$

$\tau_a$ are the SU(2) generators, i.e. Pauli matrices, as described in section B.1. $\pi_a$, where $a \in \{1,2,3\}$, are the pion fields. These are real fields, meaning $\pi_a^\dagger = \pi_a$.

### 2.2.1 Leading order Lagrangian

The leading order Lagrangian in $\chi$PT is [12, 13]

$$\mathcal{L}_2 = \frac{f^2}{4}\,\mathrm{Tr}\left[\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\right] + \frac{f^2}{4}\,\mathrm{Tr}\left[\chi^\dagger\Sigma + \Sigma^\dagger\chi\right]. \tag{2.3}$$

$\chi$ and $f$ are the free parameters of the theory. $f$ is the pion decay constant, while $\chi = 2B_0M$. Here, $M$ is the mass matrix Eq. (2.1), and $B_0$ is related to the quark condensate through $f^2B_0 = -\langle\bar{u}u\rangle$. The covariant derivative is defined by

$$\nabla_\mu\Sigma = \partial_\mu\Sigma - i\left[v_\mu, \Sigma\right], \quad (\nabla_\mu\Sigma)^\dagger = \partial_\mu\Sigma^\dagger - i\left[v_\mu, \Sigma^\dagger\right], \quad v_\mu = \frac{1}{2}\mu_I\delta^0_\mu\tau_3,$$

where $\mu_I$ is the isospin chemical potential. To get the series expansion of $\Sigma$ in powers of $\pi/f$, we start by using the fact that $\tau_a^2 = \mathbb{1}$ to write

$$A_\alpha = \sum_n^\infty \frac{1}{n!}\left(\frac{i\alpha}{2}\tau_1\right)^n = \sum_n^\infty\left[\frac{\mathbb{1}}{(2n)!}\left(\frac{i\alpha}{2}\right)^{(2n)} + \frac{\tau_1}{(2n+1)!}\left(\frac{i\alpha}{2}\right)^{(2n+1)}\right] = \mathbb{1}\cos\frac{\alpha}{2} + i\tau_1\sin\frac{\alpha}{2}. \tag{2.4}$$

The series expansion of $U$ is

$$U = \exp\!\left(\frac{i\pi_a\tau_a}{2f}\right) = 1 + \frac{i\pi_a\tau_a}{2f} + \frac{1}{2}\left(\frac{i\pi_a\tau_a}{2f}\right)^2 + \frac{1}{6}\left(\frac{i\pi_a\tau_a}{2f}\right)^3 + \frac{1}{24}\left(\frac{i\pi_a\tau_a}{2f}\right)^4 + \mathcal{O}\left((\pi/f)^5\right),$$

which we use to calculate the expansion of the inner part of $\Sigma$, as given in Eq. (2.2),

$$\begin{aligned}
U\Sigma_0U = &\left(1 + \frac{i\pi_a\tau_a}{2f} + \frac{1}{2}\left(\frac{i\pi_a\tau_a}{2f}\right)^2 + \frac{1}{6}\left(\frac{i\pi_a\tau_a}{2f}\right)^3 + \frac{1}{24}\left(\frac{i\pi_a\tau_a}{2f}\right)^4\right) \\
&\times\left(1 + \frac{i\pi_a\tau_a}{2f} + \frac{1}{2}\left(\frac{i\pi_a\tau_a}{2f}\right)^2 + \frac{1}{6}\left(\frac{i\pi_a\tau_a}{2f}\right)^3 + \frac{1}{24}\left(\frac{i\pi_a\tau_a}{2f}\right)^4\right) + \mathcal{O}\left((\pi/f)^5\right) \\
&= 1 + \frac{i\pi_a\tau_a}{f} + 2\left(\frac{i\pi_a\tau_a}{2f}\right)^2 + \frac{4}{3}\left(\frac{i\pi_a\tau_a}{2f}\right)^3 + \frac{2}{3}\left(\frac{i\pi_a\tau_a}{2f}\right)^4 + \mathcal{O}\left((\pi/f)^5\right).
\end{aligned}$$

The symmetry of $\pi_a\pi_b$ means that

$$(\pi_a\tau_a)^2 = \pi_a\pi_b\frac{1}{2}\{\tau_a, \tau_b\} = \pi_a\pi_a, \quad (\pi_a\tau_a)^3 = \pi_a\pi_a\pi_b\tau_b, \quad (\pi_a\tau_a)^4 = \pi_a\pi_a\pi_b\pi_b.$$

This gives us the expression

$$U\Sigma_0U = 1 + i\frac{\pi_a\tau_a}{f} - \frac{\pi_a^2}{2f^2} - i\frac{\pi_a^2\pi_b\tau_b}{6f^3} + \frac{\pi_a^2\pi_b^2}{24f^4} + \mathcal{O}\left((\pi/f)^5\right).$$

We combine this result with Eq. (2.4) to get an expression for $\Sigma$ up to $\mathcal{O}\left((\pi/f)^5\right)$

$$\begin{aligned}
\Sigma = &\left(\cos\frac{\alpha}{2} + i\tau_1\sin\frac{\alpha}{2}\right)\left(1 + i\frac{\pi_a\tau_a}{f} - \frac{\pi_a^2}{2f^2} - i\frac{\pi_a^2\pi_b\tau_b}{6f^3} + \frac{\pi_a^2\pi_b^2}{24f^4}\right)\left(\cos\frac{\alpha}{2} + i\tau_1\sin\frac{\alpha}{2}\right) \\
= &\left(1 + i\frac{\pi_a\tau_a}{f} - \frac{\pi_a^2}{2f^2} - i\frac{\pi_a^2\pi_b\tau_b}{6f^3} + \frac{\pi_a^2\pi_b^2}{24f^4}\right)\cos^2\frac{\alpha}{2} \\
- &\left(1 + i\frac{\pi_a}{f}\tau_1\tau_a\tau_1 - \frac{\pi_a^2}{2f^2} - i\frac{\pi_a^2\pi_b}{6f^3}\tau_1\tau_b\tau_1 + \frac{\pi_a^2\pi_b^2}{24f^4}\right)\sin^2\frac{\alpha}{2} \\
+ i&\left(2\tau_1 + i\frac{\pi_a}{f}\{\tau_1, \tau_a\} - 2\tau_1\frac{\pi_a^2}{2f^2} - i\frac{\pi_a^2\pi_b}{6f^3}\{\tau_1, \tau_b\} + 2\tau_1\frac{\pi_a^2\pi_b^2}{24f^4}\right)\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}.
\end{aligned}$$

Using trigonometric identities and the commutator,

$$\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2} = \cos\alpha, \quad 2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2} = \sin\frac{\alpha}{2}, \quad \tau_1\tau_a\tau_1 = -\tau_a + 2\delta_{1a}\tau_1,$$

the final expression of $\Sigma$ to $\mathcal{O}\left((\pi/f)^5\right)$ is

$$\Sigma = \left(1 - \frac{\pi_a^2}{2f^2} + \frac{\pi_a^2\pi_b^2}{24f^4}\right)(\cos\alpha + i\tau_1\sin\alpha) + \left(\frac{\pi_a}{f} - \frac{\pi_b^2\pi_a}{6f^3}\right)\left(i\tau_a - 2i\delta_{a1}\tau_1\sin^2\frac{\alpha}{2} - \delta_{a1}\sin\alpha\right). \tag{2.5}$$

The kinetic term in the $\chi$PT Lagrangian is

$$\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger = \partial_\mu\Sigma\partial^\mu\Sigma^\dagger - i\left(\partial_\mu\Sigma\left[v^\mu,\Sigma^\dagger\right] - \text{h.c.}\right) - \left[v_\mu,\Sigma\right]\left[v_\mu,\Sigma^\dagger\right]. \tag{2.6}$$

Using Eq. (2.5) we find the expansion of the constitutive parts of the kinetic term to be

$$\begin{aligned}
\partial_\mu\Sigma &= \left(\frac{-1}{f^2} + \frac{\pi_b^2}{6f^4}\right)(\cos\alpha + i\tau_1\sin\alpha)(\pi_a\partial_\mu\pi_a) \\
&+ \left(\frac{\partial_\mu\pi_a}{f} - \frac{\pi_b^2\partial_\mu\pi_a + 2\pi_a\pi_b\partial_\mu\pi_b}{6f^3}\right)\left(i\tau_a - 2i\delta_{a1}\tau_1\sin^2\frac{\alpha}{2} - \delta_{1a}\sin\alpha\right) \\
&= \left[\left(\frac{-1}{f^2} + \frac{\pi_b^2}{6f^4}\right)(\pi_a\partial_\mu\pi_a)\cos\alpha - \left(\frac{\partial_\mu\pi_1}{f} - \frac{\pi_b^2\partial_\mu\pi_1 + 2\pi_1\pi_b\partial_\mu\pi_b}{6f^3}\right)\sin\alpha\right] \\
&- \left[\left(\frac{-1}{f^2} + \frac{\pi_b^2}{6f^4}\right)(\pi_a\partial_\mu\pi_a)\sin\alpha - \left(\frac{\partial_\mu\pi_1}{f} - \frac{\pi_b^2\partial_\mu\pi_1 + 2\pi_1\pi_b\partial_\mu\pi_b}{6f^3}\right)2\sin^2\frac{\alpha}{2}\right]i\tau_1 \\
&+ \left(\frac{\partial_\mu\pi_a}{f} - \frac{\pi_b^2\partial_\mu\pi_a + 2\pi_a\pi_b\partial_\mu\pi_b}{6f^3}\right)i\tau_a,
\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}
\left[v_\mu,\Sigma\right] &= \frac{1}{2}\mu_I\delta_\mu^0\left[\left(1 - \frac{\pi_a^2}{2f^2} + \frac{\pi_a^2\pi_b^2}{24f^4}\right)i\sin\alpha\left[\tau_3,\tau_1\right] + \left(\frac{\pi_a}{f} - \frac{\pi_b^2\pi_a}{6f^3}\right)\left(i\left[\tau_a,\tau_3\right] - 2i\delta_{a1}\sin^2\frac{\alpha}{2}\left[\tau_3,\tau_1\right]\right)\right] \\
&= -\mu_I\delta_\mu^0\left\{\left(1 - \frac{\pi_a^2}{2f^2} + \frac{\pi_a^2\pi_b^2}{24f^4}\right)\tau_2\sin\alpha + \left(\frac{\pi_a}{f} - \frac{\pi_b^2\pi_a}{6f^3}\right)\left[(\delta_{a1}\tau_2 - \delta_{a2}\tau_1) - 2\delta_{a1}\tau_2\sin^2\frac{\alpha}{2}\right]\right\} \\
&= -\mu_I\delta_\mu^0\left\{\left[\left(1 - \frac{\pi_a^2}{2f^2} + \frac{\pi_a^2\pi_b^2}{24f^4}\right)\sin\alpha + \left(\frac{\pi_1}{f} - \frac{\pi_b^2\pi_1}{6f^3}\right)\cos\alpha\right]\tau_2 - \left(\frac{\pi_2}{f} - \frac{\pi_b^2\pi_2}{6f^3}\right)\tau_1\right\}.
\end{aligned} \tag{2.8}$$

Combining Eq. (2.7) and Eq. (2.8) gives the following terms [1]

$$\begin{aligned}
\text{Tr}\left\{\partial_\mu\Sigma\partial^\mu\Sigma^\dagger\right\} &= \frac{2}{f^2}\partial_\mu\pi_a\partial^\mu\pi_a + \frac{2}{3f^4}\left[(\pi_a\partial_\mu\pi_a)(\pi_b\partial^\mu\pi_b) - (\pi_a\partial_\mu\pi_b)(\pi_b\partial^\mu\pi_a)\right], \\
-i\,\text{Tr}\left\{\partial^\mu\Sigma\left[v_\mu,\Sigma^\dagger\right] - \text{h.c.}\right\} &= 4\mu_I\frac{\partial_0\pi_2}{f} + 8\mu_I\frac{\pi_3}{3f^3}\sin\alpha(\pi_2\partial_0\pi_3 - \pi_3\partial_0\pi_2)\sin\alpha \\
&+ \left(\frac{4\mu_I}{f^2}\cos\alpha - \frac{8\mu_I\pi_1}{3f^3}\sin\alpha - \frac{4\mu_I\pi_a\pi_a}{3f^4}\cos\alpha\right)(\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1), \\
-\text{Tr}\left\{\left[v_\mu,\Sigma\right]\left[v^\mu,\Sigma^\dagger\right]\right\} &= {\mu_I}^2\left[2\sin^2\alpha + \left(\frac{2}{f} - \frac{4\pi_a\pi_a}{3f^3}\right)\pi_1\sin2\alpha + \left(\frac{2}{f^2} - \frac{2\pi_a\pi_a}{3f^4}\right)\pi_a\pi_b k_{ab}\right], \\
\text{Tr}\left\{\Sigma + \Sigma^\dagger\right\} &= 4\cos\alpha - \frac{4\pi_1}{f}\sin\alpha - \frac{2\pi_a\pi_a}{f^2}\cos\alpha + \frac{2\pi_1\pi_a\pi_a}{3f^3}\sin\alpha + \frac{(\pi_a\pi_a)^2}{6f^4}\cos\alpha,
\end{aligned}$$

where $k_{ab} = \delta_{a1}\delta_{b1}\cos2\alpha + \delta_{a2}\delta_{b2}\cos^2\alpha - \delta_{a3}\delta_{b3}\sin^2\alpha$. If we write the Lagrangian as show in Eq. (2.3) as $\mathcal{L}_2 = \mathcal{L}_2^{(0)} + \mathcal{L}_2^{(1)} + \mathcal{L}_2^{(2)} + ...$, where $\mathcal{L}_2^{(n)}$ contains all terms of order $\mathcal{O}\left((\pi/f)^n\right)$, then the result of the series

[1] The scripts used to aid the calculation of the Lagrangian is available at `https://github.com/martkjoh/prosjektopggave`

expansion is

$$
\begin{aligned}
\mathcal{L}_2^{(0)} &= f^2 \left( \bar{m}^2 \cos\alpha + \frac{1}{2}\mu^2 \sin^2\alpha \right), && (2.9)\\
\mathcal{L}_2^{(1)} &= f(\mu_I^2 \cos\alpha - \bar{m}^2)\pi_1 \sin\alpha + f\mu_I \partial_0 \pi_2 \sin\alpha, && (2.10)\\
\mathcal{L}_2^{(2)} &= \frac{1}{2}\partial_\mu \pi_a \partial^\mu \pi_a + \mu_I \cos\alpha\, (\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1) - \frac{1}{2}\bar{m}^2 \pi_a\pi_a \cos\alpha + \frac{1}{2}\mu_I^2 \pi_a\pi_b k_{ab}, && (2.11)\\
\mathcal{L}_2^{(3)} &= \frac{\pi_a\pi_a\pi_1}{6f}(\bar{m}^2\sin\alpha - 2{\mu_I}^2 \sin 2\alpha)\\
&\quad - \frac{2\mu_I}{3f}\left[\pi_1(\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1) + \pi_3(\pi_3\partial_0\pi_2 - \pi_2\partial_0\pi_3)\right]\sin\alpha, && (2.12)\\
\mathcal{L}_2^{(4)} &= \frac{1}{6f^2}\left\{ \frac{1}{4}\bar{m}^2(\pi_a\pi_a)^2\cos\alpha - \left[(\pi_a\pi_a)(\partial_\mu\pi_b\partial^\mu\pi_b) - (\pi_a\partial_\mu\pi_a)(\pi_b\partial^\mu\pi_b)\right]\right\}\\
&\quad - \frac{\mu_I\pi_a\pi_a}{3f^2}\left[(\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1)\cos\alpha + \frac{1}{2}\mu_I\pi_a\pi_b k_{ab}\right]. && (2.13)
\end{aligned}
$$

We have introduced $\bar{m}^2 = B_0(m_u + m_d)$, the bare pion mass, and assumed $2\delta m^2 = m_u - m_d = 0$.

## 2.3 Equation of motion and redundant terms

Changing the field parametrization that appear in the Lagrangian does not affect any of the physics, as it corresponds to a change of variables in the path integral [12, 14, 15]. However, a change of variables can result in new terms in the Lagrangian. As a result of this, terms that on the face of it appear independent may be redundant. These terms can be eliminated by using the classical equation of motion. In this section we show first the derivation of the equation of motion, then use this result to identify redundant terms which need not be included in the most general Lagrangian.

We derive the equation of motion for the leading order Lagrangian using the principle of least action. Choosing the parametrization $\Sigma = \exp(i\pi_a\tau_a)$, a variation $\pi_a \to \pi_a + \delta\pi_a$ results in a variation in $\Sigma$, $\delta\Sigma = i\tau_a\delta\pi_a\Sigma$. The variation of the leading order action,

$$
S_2 = \int_\Omega \mathrm{d}^4x\, \mathcal{L}_2, \tag{2.14}
$$

when varying $\pi_a$ is

$$
\delta S = \int_\Omega \mathrm{d}x\, \frac{f^2}{4}\,\mathrm{Tr}\big\{(\nabla_\mu\delta\Sigma)(\nabla^\mu\Sigma)^\dagger + (\nabla_\mu\Sigma)(\nabla^\mu\delta\Sigma)^\dagger + \chi\delta\Sigma^\dagger + \delta\Sigma\chi^\dagger\big\}.
$$

Using the properties of the covariant derivative to do partial integration, as show in section B.2, as well as $\delta(\Sigma\Sigma^\dagger) = (\delta\Sigma)\Sigma^\dagger + \Sigma(\delta\Sigma)^\dagger = 0$, the variation of the action can be written

$$
\begin{aligned}
\delta S &= \frac{f^2}{4}\int_\Omega \mathrm{d}x\ \mathrm{Tr}\big\{-\delta\Sigma\nabla^2\Sigma^\dagger + (\nabla^2\Sigma)(\Sigma^\dagger\delta\Sigma\Sigma^\dagger) - \chi(\Sigma^\dagger\delta\Sigma\Sigma^\dagger) + \delta\Sigma\chi^\dagger\big\}\\
&= \frac{f^2}{4}\int_\Omega \mathrm{d}x\ \mathrm{Tr}\big\{\delta\Sigma\Sigma^\dagger\left[(\nabla^2\Sigma)\Sigma^\dagger - \Sigma\nabla^2\Sigma^\dagger - \chi\Sigma^\dagger + \Sigma\chi^\dagger\right]\big\}\\
&= i\frac{f^2}{4}\int_\Omega \mathrm{d}x\ \mathrm{Tr}\big\{\tau_a\left[(\nabla^2\Sigma)\Sigma^\dagger - \Sigma\nabla^2\Sigma^\dagger - \chi\Sigma^\dagger + \Sigma\chi^\dagger\right]\big\}\delta\pi_a = 0.
\end{aligned}
$$

As the variation is arbitrary, the equation of motion to leading order is

$$
\mathrm{Tr}\big\{\tau_a\left[(\nabla^2\Sigma)\Sigma^\dagger - \Sigma\nabla^2\Sigma^\dagger - \chi\Sigma^\dagger + \Sigma\chi^\dagger\right]\big\} = 0. \tag{2.15}
$$

This may be rewritten as a matrix equation. Using that

$$
\mathrm{Tr}\big\{(\nabla_\mu\Sigma)\Sigma^\dagger\big\} = \mathrm{Tr}\big\{i\tau_a(\partial_\mu\pi_a)\Sigma\Sigma^\dagger\big\} - i\,\mathrm{Tr}\big\{[v_\mu,\Sigma]\Sigma^\dagger\big\} = 0,
$$

we can see that $\mathrm{Tr}\{(\nabla^2\Sigma)\Sigma^\dagger - \Sigma\nabla^2\Sigma^\dagger\} = 0$, and the equation of motion may be written as

$$\mathcal{O}^{(2)}_{\mathrm{EOM}}(\Sigma) = (\nabla^2\Sigma)\Sigma^\dagger - \Sigma\nabla^2\Sigma^\dagger - \chi\Sigma^\dagger + \Sigma\chi^\dagger - \frac{1}{2}\mathrm{Tr}\{\chi\Sigma^\dagger - \Sigma\chi^\dagger\} = 0. \tag{2.16}$$

The next step in eliminating redundant terms is to change the parametrization of $\Sigma$ by $\Sigma(x) \to \Sigma'(x)$. Here, $\Sigma(x) = e^{iS(x)}\Sigma'(x)$, $S(x) \in \mathfrak{su}(2)$. This change leads to a new Lagrange density, $\mathcal{L}[\Sigma] = \mathcal{L}[\Sigma'] + \Delta\mathcal{L}[\Sigma']$. We are free to choose $S(x)$, as long $\Sigma'$ still obeys the required transformation properties. Any terms in the Lagrangian $\Delta\mathcal{L}$ due to a reparametrization can be neglected, as argued earlier. When demanding that $\Sigma'$ obey the same symmetries as $\Sigma$, the most general transformation to second order in Weinberg's power counting scheme is [12]

$$S_2 = i\alpha_2\left[(\nabla^2\Sigma')\Sigma^\dagger - \Sigma'(\nabla^2\Sigma')^\dagger\right] + i\alpha_2\left[\chi\Sigma'^\dagger - \Sigma'\chi^\dagger - \frac{1}{2}\mathrm{Tr}\{\chi\Sigma'^\dagger - \Sigma'\chi^\dagger\}\right]. \tag{2.17}$$

$\alpha_1$ and $\alpha_2$ are arbitrary real numbers. As Eq. (2.17) is to second order, $\Delta\mathcal{L}$ is fourth order in Weinberg's power counting scheme. To leading order is given by

$$\begin{aligned}\mathcal{L}_2\left[e^{iS_2}\Sigma'\right] &= \frac{f^2}{4}\mathrm{Tr}\{[\nabla_\mu(1+iS_2)\Sigma'][\nabla^\mu\Sigma'^\dagger(1-iS_2)]\} + \frac{f^2}{4}\mathrm{Tr}\{\chi\Sigma'^\dagger(1-iS_2) + (1+iS_2)\Sigma'\chi^\dagger\}\\ &= \mathcal{L}[\Sigma'] + i\frac{f^2}{4}\mathrm{Tr}\{[\nabla_\mu(S_2\Sigma')][\nabla^\mu\Sigma']^\dagger - [\nabla_\mu\Sigma'][\nabla^\mu(\Sigma'^\dagger S_2)]\} - i\frac{f^2}{4}\mathrm{Tr}\{\chi\Sigma'^\dagger S_2 - S_2\Sigma'\chi^\dagger\}\end{aligned}$$

Using the properties of the covariant derivative, as described in section B.2, we may use the product rule and partial integration to write the difference between the two Lagrangians to fourth order as

$$\begin{aligned}\Delta\mathcal{L}[\Sigma'] &= i\frac{f^2}{4}\mathrm{Tr}\{(\nabla_\mu S_2)(\Sigma'\nabla^\mu\Sigma'^\dagger - (\nabla^\mu\Sigma')\Sigma'^\dagger)\} - i\frac{f^2}{4}\mathrm{Tr}\{\chi\Sigma'^\dagger S_2 - S_2\Sigma'\chi^\dagger\}\\ &= i\frac{f^2}{4}\mathrm{Tr}\{S_2\left[\Sigma'^\dagger\nabla^2\Sigma' - (\nabla^2\Sigma')\Sigma'^\dagger - \chi\Sigma'^\dagger + \Sigma'\chi^\dagger\right]\}.\end{aligned}$$

Using the equation of motion Eq. (2.16), and the fact that $\mathrm{Tr}\{S_2\} = 0$, this difference can be written as

$$\Delta\mathcal{L}[\Sigma'] = \frac{f^2}{4}\mathrm{Tr}\Big\{iS_2\mathcal{O}^{(2)}_{\mathrm{EOM}}(\Sigma')\Big\}. \tag{2.18}$$

Any term that can be written in the form of Eq. (2.18) for arbitrary $\alpha_1, \alpha_2 \in \mathbb{R}$ is redundant, as we argued earlier, and may therefore be discarded. $\Delta\mathcal{L}_2$ is of fourth order, and it can thus be used to remove terms from $\mathcal{L}_4$ or higher order.

## 2.4 Next to leading order Lagrangian

The next to leading order Lagrangian density is, assuming no external fields

$$\begin{aligned}\mathcal{L}_4 = &\frac{l_1}{4}\mathrm{Tr}\{\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\}^2 + \frac{l_2}{4}\mathrm{Tr}\{\nabla_\mu\Sigma(\nabla_\nu\Sigma)^\dagger\}\mathrm{Tr}\{\nabla^\mu\Sigma(\nabla^\nu\Sigma)^\dagger\} + \frac{l_3+h_1-h_3}{16}\mathrm{Tr}\{\chi\Sigma^\dagger + \Sigma\chi^\dagger\}^2\\ &+ \frac{l_4}{8}\mathrm{Tr}\{\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\}\mathrm{Tr}\{\chi\Sigma^\dagger + \Sigma\chi^\dagger\} + \frac{h_1-h_3-l_4-l_7}{16}\mathrm{Tr}\{\chi\Sigma^\dagger - \Sigma\chi^\dagger\}^2 + \frac{h_1+h_3-l_4}{4}\mathrm{Tr}\{\chi\chi^\dagger\}\\ &- \frac{h_1-h_3-l_4}{8}\mathrm{Tr}\Big\{\left(\chi\Sigma^\dagger\right)^2 + \left(\Sigma\chi^\dagger\right)^2\Big\}\end{aligned} \tag{2.19}$$

To $\mathcal{L}_4$ to $\mathcal{O}\left((\pi/f)^3\right)$, we use the result from Eq. (2.7) and Eq. (2.8), up to and including $\mathcal{O}\left((\pi/f)^2\right)$, which gives

$$\begin{aligned}\mathrm{Tr}\{\partial_\mu\Sigma\partial_\nu\Sigma^\dagger\} &= 2\frac{\partial_\mu\pi_a\partial_\nu\pi_a}{f^2}\\ -i\,\mathrm{Tr}\{\partial_\mu\Sigma\left[v_\nu,\Sigma^\dagger\right] - \mathrm{h.c.}\} &= \frac{2\mu_I\pi_2}{f}(\delta^0_\mu\partial_\nu + \delta^0_\nu\partial_\mu)\sin\alpha + \frac{2\mu_I}{f^2}[\pi_1(\delta^0_\mu\partial_\nu + \delta^0_\nu\partial_\mu)\pi_2 - \pi_2(\delta^0_\mu\partial_\nu + \delta^0_\nu\partial_\mu)\pi_1]\cos\alpha\\ -\mathrm{Tr}\{[v_\nu,\Sigma]\left[v_\nu,\Sigma^\dagger\right]\} &= 2{\mu_I}^2\delta^0_\mu\delta^0_\nu\left[\sin^2\alpha + \frac{\pi_1}{f}\sin 2\alpha + \frac{\pi_a\pi_b}{f^2}k_{ab}\right].\end{aligned}$$

Using the form of the Pauli matrices, we can write $\chi$ as

$$\chi = 2B_0 M = \bar{m}^2 \mathbb{1} + \Delta m^2 \tau_3,$$

where $\bar{m} = B_0(m_u + m_d)$, $\Delta m = B_0(m_u - m_d)/2$, which gives

$$\begin{aligned}
\chi\Sigma^\dagger + \Sigma\chi^\dagger &= 2(\bar{m}^2 + \Delta m^2 \tau_3) \left[ \left(1 - \frac{\pi_a^2}{2f^2}\right) \cos\alpha - \frac{\pi_1}{f} \sin\alpha \right] \\
&\quad + 2\Delta m^2 \left[ \left(1 - \frac{\pi_a^2}{2f^2}\right) \tau_2 \sin\alpha + \frac{\pi_a}{f} \left(\delta_{a1}\tau_2 \cos\alpha - \delta_{a2}\tau_1\right) \right], \\
\chi\Sigma^\dagger - \Sigma\chi^\dagger &= -2i\bar{m}^2 \left[ \left(1 - \frac{\pi_a^2}{2f^2}\right) \tau_1 \sin\alpha + \frac{\pi_a}{f} \left(\tau_a - 2\delta_{1a}\tau_1 \sin^2\frac{\alpha}{2}\right) \right] - 2i\Delta m^2 \frac{\pi_3}{f}.
\end{aligned}$$

Combining these results gives all the terms in $\mathcal{L}_4$, to $\mathcal{O}\left((\pi/f)^3\right)$:

$$\begin{aligned}
&\mathrm{Tr}\{\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\}^2 = \mathrm{Tr}\{\partial_\mu\Sigma\partial^\mu\Sigma^\dagger - i\left(\partial_\mu\Sigma\left[v^\mu, \Sigma^\dagger\right] - \text{h.c.}\right) - [v_\mu, \Sigma]\left[v^\mu, \Sigma^\dagger\right]\}^2 \\
&\quad = \frac{8\mu_I^2}{f^2}(\partial_\mu\pi_a\partial^\mu\pi_a + 2\partial_\mu\pi_2\partial^\mu\pi_2)\sin^2\alpha \\
&\quad + 16\mu_I^3\left[\frac{\partial_0\pi_2}{f}\sin^3\alpha + \frac{1}{f^2}\left(3\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1\right)\cos\alpha\sin^2\alpha\right] \\
&\quad + 4\mu_I^4\left\{\sin^4\alpha + 2\sin^2\alpha\left[\frac{\pi_1}{f}\sin 2\alpha + \frac{\pi_a\pi_b}{f^2}\left(k_{ab} + 2\delta_{a1}\delta_{a2}\cos^2\alpha\right)\right]\right\}, && (2.20) \\
&\mathrm{Tr}\{\nabla_\mu\Sigma(\nabla_\nu\Sigma)^\dagger\}\,\mathrm{Tr}\{\nabla^\mu\Sigma(\nabla^\nu\Sigma)^\dagger\} \\
&\quad = \frac{4{\mu_I}^2}{f^2}\left(\partial_0\pi_a\partial_0\pi_a + \partial_0\pi_2\partial_0\pi_2 + \partial_\mu\pi_2\partial^\mu\pi_2\right)\sin^2\alpha \\
&\quad + 16\mu_I^3\left[\frac{\partial_0\pi_2}{f}\sin^3\alpha + \frac{1}{f^2}\left(3\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1\right)\cos\alpha\sin^2\alpha\right] \\
&\quad + 4\mu_I^4\left\{\sin^4\alpha + 2\sin^2\alpha\left[\frac{\pi_1}{f}\sin 2\alpha + \frac{\pi_a\pi_b}{f^2}\left(k_{ab} + 2\delta_{a1}\delta_{a2}\cos^2\alpha\right)\right]\right\}, && (2.21) \\
&\mathrm{Tr}\{\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\}\,\mathrm{Tr}\{\chi\Sigma^\dagger + \Sigma\chi^\dagger\} \\
&\quad = 4\bar{m}^2\left\{2\frac{\partial_\mu\pi_a\partial^\mu\pi_a}{f^2}\cos\alpha + 4\mu_I\left[\frac{\partial_0\pi_2}{2f}\sin 2\alpha + \frac{1}{f^2}\left(\pi_1\partial_0\pi_2\cos 2\alpha - \pi_2\partial_0\pi_1\cos^2\alpha\right)\right]\right. && (2.22) \\
&\quad \left. + {\mu_I}^2\left[2\cos\alpha\sin^2\alpha - 2\frac{\pi_1}{f}\sin\alpha\left(2 - 3\sin^2\alpha\right) + \frac{1}{f^2}\left(\pi_1^2[2 - 9\sin^2\alpha] + \pi_2^2[2 - 3\sin^2\alpha] - 3\pi_3^2\sin^2\alpha\right)\cos\alpha\right]\right\}, \\
&\mathrm{Tr}\{\chi\Sigma^\dagger + \Sigma\chi^\dagger\}^2 = 16\bar{m}^4\left[\cos^2\alpha - \frac{\pi_1}{f}\sin 2\alpha + \frac{1}{f^2}\left(\pi_1^2\sin^2\alpha - \pi_a\pi_a\cos^2\alpha\right)\right], && (2.23) \\
&\mathrm{Tr}\{\chi\Sigma^\dagger - \Sigma\chi^\dagger\}^2 = -16\left(\frac{\Delta m^2\pi_3}{f}\right)^2, && (2.24) \\
&\mathrm{Tr}\left\{\left(\chi\Sigma^\dagger\right)^2 + \left(\Sigma\chi^\dagger\right)^2\right\} \\
&\qquad = 4\bar{m}^4\left(\cos 2\alpha - 2\frac{\pi_1}{f}\sin 2\alpha - 2\frac{\pi_a\pi_a}{f^2}\cos^2\alpha + 2\frac{\pi_1^2}{f^2}\sin^2\alpha\right) + 4\Delta m^4\left(1 - 2\frac{\pi_3^2}{f^2}\right), && (2.25) \\
&\mathrm{Tr}\{\chi^\dagger\chi\} = 2\bar{m}^4 + 2\Delta m^4. && (2.26)
\end{aligned}$$

The different terms of the NLO Lagrangian is

$$\mathcal{L}_4^{(0)} = (l_1+l_2)\mu_I^4 \sin^4\alpha + (l_3+l_4)\bar{m}^2\cos^2\alpha + l_4\bar{m}{\mu_I}^2\cos\alpha\sin^2\alpha + (h_1-l_4)\bar{m}^4 + h_3\Delta m^4 \tag{2.27}$$

$$\begin{aligned}\mathcal{L}_4^{(1)} &= 4\mu_I^3\frac{l_1+l_2}{f}\left(\partial_0\pi_2 + \mu_I\cos\alpha\pi_1\right)\sin^3\alpha - \frac{l_3+l_4}{f}\bar{m}^4\pi_1\sin 2\alpha \\ &+ \bar{m}^2\frac{l_4}{f}\left[\mu_I\partial_0\pi_2\sin 2\alpha - {\mu_I}^2\pi_1\sin\alpha\left(3\sin^2\alpha - 2\right)\right]\end{aligned} \tag{2.28}$$

$$\begin{aligned}\mathcal{L}_4^{(2)} &= 2\mu_I^2\frac{l_1}{f^2}(\partial_\mu\pi_a\partial^\mu\pi_a + 2\partial_0\pi_2\partial_0\pi_2)\sin^2\alpha + {\mu_I}^2\frac{l_2}{f^2}\left(\partial_\mu\pi_2\partial^\mu\pi_2 + 2\partial_0\pi_a\partial_0\pi_a + 2\partial_0\pi_2\partial_0\pi_2\right)\sin^2\alpha \\ &+ 2\frac{l_1+l_2}{f^2}\left[2\mu_I^3\left(3\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1\right)\cos\alpha + \mu_I^4\pi_a\pi_b\left(k_{ab} + 2\delta_{a1}\delta_{a2}\cos^2\alpha\right)\right]\sin^2\alpha \\ &+ \frac{l_3+l_4}{f^2}\bar{m}^2\left(\pi_1^2\sin^2\alpha - \pi_a\pi_a\cos^2\alpha\right) + \frac{l_4}{f^2}\bar{m}^2\Big[\partial_\mu\pi_a\partial^\mu\pi_a\cos\alpha + 4\mu_I\left(\pi_1\partial_0\pi_2\cos 2\alpha - \pi_2\partial_0\pi_1\cos^2\alpha\right) \\ &+ \frac{1}{2}{\mu_I}^2\left(\pi_1^2[2-9\sin^2\alpha] + \pi_2^2[2-3\sin^2\alpha] - 3\pi_3^2\sin^2\alpha\right)\cos\alpha\Big] + \frac{l_7}{f^2}\Delta m^2\pi_3^2\end{aligned} \tag{2.29}$$

## 2.5 Propagator

We may write the quadratic part of the Lagrangian Eq. (2.11) as [2]

$$\mathcal{L}_2^{(2)} = \frac{1}{2}\sum_a \partial_\mu\pi_a\partial^\mu\pi_a + \frac{1}{2}m_{12}(\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1) - \frac{1}{2}\sum_a m_a^2\pi_a^2, \tag{2.30}$$

where

$$m_1^2 = \bar{m}^2\cos\alpha - \mu_I^2\cos 2\alpha, \tag{2.31}$$

$$m_2^2 = \bar{m}^2\cos\alpha - \mu_I^2\cos^2\alpha, \tag{2.32}$$

$$m_3^2 = \bar{m}^2\cos\alpha + \mu_I^2\sin^2\alpha, \tag{2.33}$$

$$m_{12} = 2\mu_I\cos\alpha. \tag{2.34}$$

The components of the Euler-Lagrange equations of this field are

$$\frac{\partial\mathcal{L}}{\partial\pi_a} = \frac{1}{2}m_{12}(\delta_{a1}\partial_0\pi_2 - \delta_{a2}\partial_0\pi_1) - m_a^2\pi_a, \quad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\pi_a)} = \partial^\mu\pi_a - \frac{1}{2}m_{12}\delta_0^\mu(\delta_{a1}\pi_2 - \delta_{a2}\pi_1).$$

This gives the equation of motion for the field

$$\partial^\mu\partial_\mu\pi_a + m_a^2\pi_a = m_{12}(\delta_{a1}\partial_0\pi_2 - \delta_{a2}\partial_0\pi_1). \tag{2.35}$$

The propagator of the pion field is defined by

$$\left[\delta_{ab}(\partial^\mu\partial_\mu + m_a^2) - m_{12}(\delta_{a1}\delta_{b2} - \delta_{a2}\delta_{b1})\partial_0\right]D_{bc}(x,x') = -i\delta(x-x')\delta_{ac}. \tag{2.36}$$

The momentum space propagator, as defined in the section B.1, fulfills (MOTSATT FORTEGN FRA FRI ENERGI; FIKS!!!)

$$-\left[\delta_{ab}(p^2 - m_a^2) + ip_0 m_{12}(\delta_{a1}\delta_{b2} - \delta_{a2}\delta_{b1})\right]D_{bc}(p) := D^{-1}_{ab}D_{bc}(p) = -i\delta_{ac},$$

where

$$D^{-1} = -\begin{pmatrix} p^2 - m_1^2 & ip_0m_{12} & 0 \\ -ip_0m_{12} & p^2 - m_2^2 & 0 \\ 0 & 0 & p^2 - m_3^2 \end{pmatrix}.$$

The spectrum of the particles is given by solving $\det\left(D^{-1}\right) = 0$ for $p^0$. With $p = (p_0, \vec{p})$ as the four momentum, this gives

$$\det\left(D^{-1}\right) = D^{-1}_{33}\left(D^{-1}_{11}D^{-1}_{22} + (D^{-1}_{12})^2\right) = -\left(p^2 - m_3^2\right)\left[\left(p^2 - m_1^2\right)\left(p^2 - m_2^2\right) - p_0^2 m_{12}^2\right] = 0,$$

[2]Summation over isospin index $(a,b,c)$ will be explicit in this section.

This equation has the solutions

$$E_0^2 = |\vec{p}|^2 + m_3^2, \tag{2.37}$$

$$E_\pm^2 = |\vec{p}|^2 + \frac{1}{2}\left(m_1^2 + m_2^2 + m_{12}^2\right) \pm \frac{1}{2}\sqrt{4|\vec{p}|^2 m_{12}^2 + \left(m_1^2 + m_2^2 + m_{12}^2\right)^2 - 4m_1^2 m_2^2}. \tag{2.38}$$

These are the energies of three particles $\pi_0$, $\pi_+$ and $\pi_-$. $\pi_0$ is $\pi_3$, while $\pi_\pm$ are linear combinations of $\pi_1$ and $\pi_2$. The (tree-level) masses of these particles are found by setting $\vec{p} = 0$, i.e. the rest-frame energy, and are

$$m_0^2 = m_3^2, \tag{2.39}$$

$$m_\pm^2 = \frac{1}{2}\left[m_1^2 + m_2^2 + m_{12}^2\right] \pm \frac{1}{2}\sqrt{\left(m_1^2 + m_2^2 + m_{12}^2\right)^2 - 4m_1^2 m_2^2}. \tag{2.40}$$

Figure 2.1 shows the masses as functions of $\mu_I$.

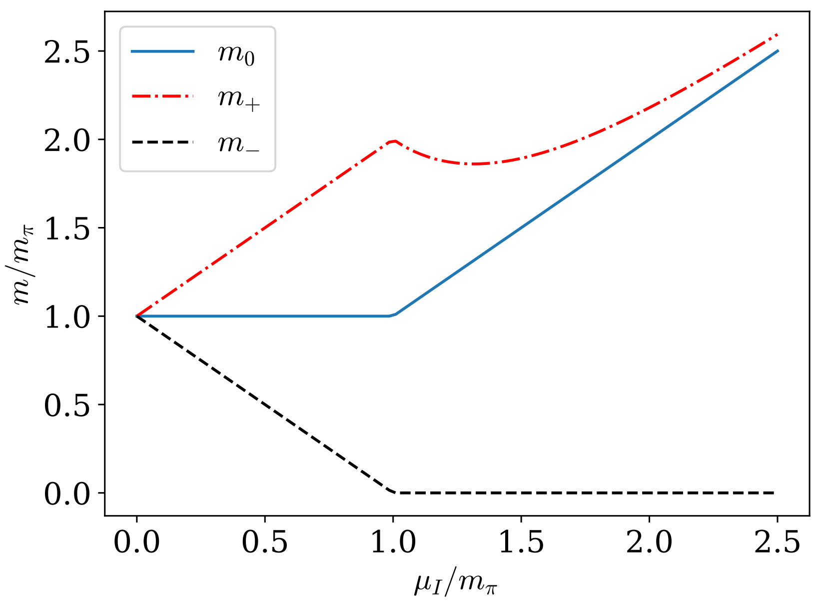



Figure 2.1: The masses of the three particles as functions of isosopin chemical potential.

With these energies we can write the determinant of the inverse propagator as

$$\det\left(D^{-1}\right) = -(p_0^2 - E_0^2)(p_0^2 - E_+^2)(p_0^2 - E_-^2). \tag{2.41}$$

The propagator may then be obtained as described in section B.1,

$$\begin{aligned} D = i(D^{-1})^{-1} &= \frac{i}{\det(D^{-1})}\begin{pmatrix} D_{22}^{-1}D_{33}^{-1} & D_{12}^{-1}D_{33}^{-1} & 0 \\ -D_{12}^{-1}D_{33}^{-1} & D_{11}^{-1}D_{33}^{-1} & 0 \\ 0 & 0 & D_{11}^{-1}D_{22}^{-1} + (D_{12}^{-1})^2 \end{pmatrix} \\ &= i\begin{pmatrix} \frac{p^2-m_2^2}{(p_0^2-E_+^2)(p_0^2-E_-^2)} & \frac{-ip_0 m_{12}}{(p_0^2-E_+^2)(p_0^2-E_-^2)} & 0 \\ \frac{ip_0 m_{12}}{(p_0^2-E_+^2)(p_0^2-E_-^2)} & \frac{p^2-m_1^2}{(p_0^2-E_+^2)(p_0^2-E_-^2)} & 0 \\ 0 & 0 & \frac{1}{p_0^2-E_0^2} \end{pmatrix}. \end{aligned} \tag{2.42}$$

# Chapter 3

# Equation of state for pions

## 3.1 Free energy at lowest order

The equation of state (EOS) relates the thermodynamic variables of as system. In this section, we will obtain the equation of state of the pions by calculating their free energy. We use the effective Lagrangian found in chapter 2 to find the leading-order contribution to one loop, and the next-to-leading order contribution at the tree-level, following the procedure used in [13, 16].[1] The free energy density of a homogenous system is

$$\mathcal{F} = -\frac{1}{V\beta} \ln Z. \tag{3.1}$$

Here, $Z$ is the partition function, and $V$ the volume of space. Using imaginary time formalism for thermal field theory, which is described in Appendix A, we find that the partition function is given by the path integral of the *Euclidean* Lagrange density, as shown in equation Eq. (A.14). In the zero temperature limit $\beta \to \infty$, the partition function is related to vacuum transition amplitude $Z_0 = Z[J = 0]$, as described in section 1.1, by a Wick rotation. The free energy density at zero temperature is therefore

$$\mathcal{F} = \frac{i}{VT} \ln Z_0, \tag{3.2}$$

where $VT$ is the volume of space-time. This equals the effective potential in the ground state, which we found an explicit formula for in section 1.2, Eq. (1.39). We write $\mathcal{F} = \mathcal{F}^{(0)} + \mathcal{F}^{(1)} + \dots$, where $\mathcal{F}^{(n)}$ refers to the $n$-loop contributions to the free energy density.

### Tree-level contribution

The tree-level contribution $\mathcal{F}^{(0)}$ is the classical potential, which is given by the static ($\pi$-independent) part of the Lagrangian. From Eq. (2.9) we have the leading order contribution,

$$\mathcal{F}_2^{(0)} = -\mathcal{L}_2^{(0)} = -f^2 \left( \bar{m}^2 \cos\alpha + \frac{1}{2}\mu^2 \sin^2\alpha \right), \tag{3.3}$$

where $\alpha$ parameterizes the ground state, which means that its value must minimize the free energy.

$$\frac{\partial}{\partial\alpha}\mathcal{F}_2^{(0)} = f^2 \left( \bar{m}^2 - \mu_I^2 \cos\alpha \right) \sin\alpha = 0.$$

This equation defines the relationship between the chemical potential $\mu_I$, and the ground state parameter $\alpha$, as illustrated in Figure 3.1. This gives the criterion

$$\alpha \in \{0, \pi\} \quad \text{or} \quad \cos\alpha = \frac{\bar{m}^2}{{\mu_I}^2}. \tag{3.4}$$

[1] Leading order and next-to-leading order, in this context, refers to Weinberg's power counting scheme.

As we see in the figure, $\alpha = \pi$ is a maximum, and thus unstable. This means that for all values $\mu_I^2 \leq \bar{m}^2$, we will have $\alpha = 0$, and the system will remain in its ground state.

In our discussion of the effective potential we also found that the ground state should minimize the classical potential, as shown by Eq. (1.36). This means that the linear part of the classical potential should vanish. The linear part of the classical potential is given by Eq. (2.10) to leading order, and reads $\mathcal{V}^{(1)} = f({\mu_I}^2 \cos\alpha - \bar{m}^2)\sin\alpha\, \pi_1$, which vanishes given Eq. (3.4).

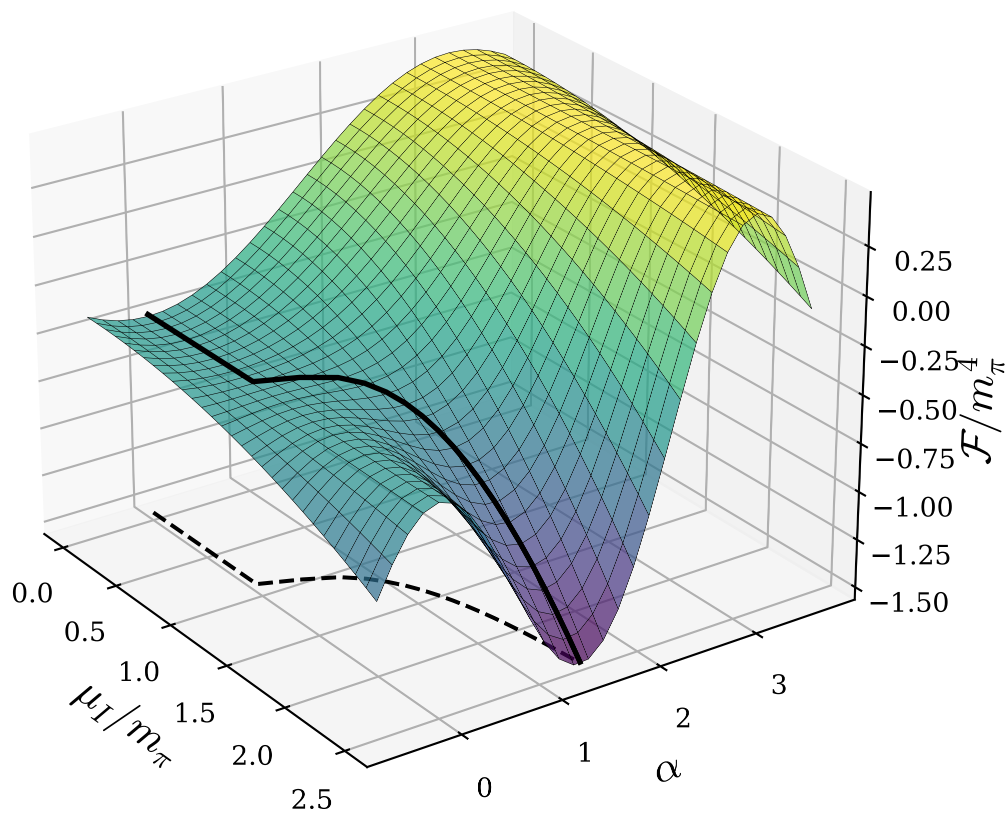



Figure 3.1: The surface gives free energy as a function of $\mu_I$ and $\alpha$. $\alpha$ is the found by minimizing $\mathcal{F}$ for a given $\mu_I$. This leads to a curve across the free energy surface, as show in the plot.

## One-loop contribution

The one loop contribution to the free energy density is

$$\mathcal{F}^{(1)} = -\frac{i}{VT}\frac{1}{2}\,\mathrm{Tr}\left\{\ln\left(-\frac{\delta^2 S[\pi = 0]}{\delta\pi_a(x)\,\delta\pi_b(y)}\right)\right\}. \tag{3.5}$$

This can be evaluated using the rules for functional differentiation given in section B.4. To leading order,

$$\frac{\delta^2 S[\pi = 0]}{\delta\pi_a(x)\,\delta\pi_b(y)} = \frac{\delta^2}{\delta\pi_a(x)\,\delta\pi_b(y)}\int \mathrm{d}^4x\,\mathcal{L}_2^{(2)} = D_x^{-1}\delta(x-y). \tag{3.6}$$

Here, $\mathcal{L}_2^{(2)}$ is the quadratic part of the Lagrangian, as given in Eq. (2.30), and $D_x^{-1}$ is the corresponding inverse propagator of the pion fields,

$$D_x^{-1} = -\left[\delta_{ab}(\partial_x^\mu\partial_{x,\mu} + m_a^2) - m_{12}(\delta_{a1}\delta_{b2} - \delta_{a2}\delta_{b1})\partial_{x,0}\right] \tag{3.7}$$

The inverse propagator is a matrix, which means that the determinant in Eq. (3.5) i both a matrix determinant, over the three pion indices, as well as a functional determinant. In section 2.5 we found the matrix part of the determinant in momentum space, which we can write using the dispersion relations of the pion fields

$$\det\big(-D^{-1}\big) = \det\big(-p_0^2 + E_0^2\big)\det\big(-p_0^2 + E_+^2\big)\det\big(-p_0^2 + E_-^2\big). \tag{3.8}$$

These dispersion relations are functions of the three-momentum $\vec{p}$, and are given in Eqs. (2.37) and (2.38). The functional determinant can therefore be evaluated as

$$\begin{aligned}
\mathrm{Tr}\Big\{\ln\Big(-\frac{\delta^2 S[\pi=0]}{\delta\pi_a(x)\,\delta\pi_b(y)}\Big)\Big\} &= \ln\det\big(-p_0^2 + E_0^2\big) + \ln\det\big(-p_0^2 + E_+^2\big) + \ln\det\big(-p_0^2 + E_-^2\big) \\
&= \mathrm{Tr}\big\{\ln\big(-p_0^2 + E_0^2\big) + \ln\big(-p_0^2 + E_+^2\big) + \ln\big(-p_0^2 + E_-^2\big)\big\} \\
&= (VT)\int \frac{\mathrm{d}^4p}{(2\pi)^4}\left[\ln\big(-p_0^2 + E_0^2\big) + \ln\big(-p_0^2 + E_+^2\big) + \ln\big(-p_0^2 + E_-^2\big)\right], 
\end{aligned} \tag{3.9}$$

where we have used the identity $\ln\det M = \mathrm{Tr}\ln M$. These terms all have the form

$$I = \int \frac{\mathrm{d}^4p}{(2\pi)^2}\ln\big(-p_0^2 + E^2\big), \tag{3.10}$$

where $E$ is some function of the 3-momentum $\vec{p}$, but not $p_0$. We use the trick

$$\frac{\partial}{\partial\alpha}\left(-p_0^2 + E^2\right)^{-\alpha}\Big|_{\alpha=0} = \frac{\partial}{\partial\alpha}\exp\left[-\alpha\ln\left(-p_0^2 + E^2\right)\right]\Big|_{\alpha=0} = \ln\left(-p_0^2 + E^2\right), \tag{3.11}$$

and then perform a Wick-rotation of the $p_0$-integral to write the integral on the form

$$I = i\frac{\partial}{\partial\alpha}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\left(p_0^2 + E^2\right)^{-\alpha}\Big|_{\alpha=0}, \tag{3.12}$$

where $p$ now is a Euclidean four-vector. The $p_0$ integral equals $\Phi_1(E, 1, \alpha)$, as defined in Eq. (A.33). The result is therefore given by Eq. (A.40),

$$\int\frac{\mathrm{d}p_0}{2\pi}(p_0^2 + E)^{-\alpha} = \frac{E^{1-2\alpha}}{\sqrt{4\pi}}\frac{\Gamma(\alpha-\frac{1}{2})}{\Gamma(\alpha)}. \tag{3.13}$$

The derivative of the Gamma function is $\Gamma'(\alpha) = \psi(\alpha)\Gamma(\alpha)$, where $\psi(\alpha)$ is the digamma function. Using

$$\frac{\partial}{\partial\alpha}\frac{\Gamma(\alpha-\frac{1}{2})}{\Gamma(\alpha)}\Big|_{\alpha=0} = \Gamma\left(\alpha-\frac{1}{2}\right)\frac{\psi(\alpha-\frac{1}{2})-\psi(\alpha)}{\Gamma(\alpha)}\Big|_{\alpha=0} = \sqrt{4\pi}, \tag{3.14}$$

$$\frac{\Gamma(\alpha-\frac{1}{2})}{\Gamma(\alpha)}\Big|_{\alpha=0} = 0, \tag{3.15}$$

we get

$$I = i\int\frac{\mathrm{d}^3p}{(2\pi)^3}E. \tag{3.16}$$

We see that the result is what we would expect physically, the total energy is the integral of the energy of each mode. This also agrees with the result from Appendix A in low temperature limit $\beta\to\infty$. This results in

$$\mathcal{F}^{(1)} = \frac{1}{2}\left[\int\frac{\mathrm{d}^3p}{(2\pi)^3}E_0 + \int\frac{\mathrm{d}^3p}{(2\pi)^3}(E_+ + E_-)\right] = \mathcal{F}^{(1)}_{\pi_0} + \mathcal{F}^{(1)}_{\pi_\pm}. \tag{3.17}$$

The first integral is identical to what we find for a free field in section A.3, in the zero temperature limit $\beta\to\infty$. These terms are all divergent, and must be regularized. We will use dimensional regularization, in which the integral is generalized to $d$ dimensions, and the $\overline{\mathrm{MS}}$-scheme, as described in section A.5. Using the result for a free field Eq. (A.47), we get

$$\mathcal{F}^{(1)}_{\pi_0} = -\frac{1}{4}\frac{m_3^4}{(4\pi)^2}\left[\frac{1}{\epsilon} + \frac{3}{2} + \ln\left(\frac{\mu^2}{m_3^2}\right)\right] + \mathcal{O}(\epsilon), \tag{3.18}$$

where $\mu$ is the renormalization scale, which is introduced ensure that the integral has the same engineering dimension for $d \neq 3$.

The contribution to the free energy from the $\pi_+$ and $\pi_-$ particles is more complicated, as the dispersion relation is given by

$$E_\pm = \sqrt{|\vec{p}|^2 + \frac{1}{2}\left(m_1^2 + m_2^2 + m_{12}^2\right) \pm \frac{1}{2}\sqrt{4|\vec{p}|^2 m_{12}^2 + \left(m_1^2 + m_2^2 + m_{12}^2\right)^2 - 4m_1^2 m_2^2}}. \tag{3.19}$$

This is not an integral we can easily do in dimensional regularization. Instead, we will seek a function $f(|\vec{p}|)$ with the same UV-behavior, that is behavior for large $\vec{p}$, as $E_+ + E_-$. If we then add $0 = f(|\vec{p}|) - f(|\vec{p}|)$ to the integrand, we can isolate the divergent behavior

$$\mathcal{F}^{(1)}_{\pi_\pm} = \frac{1}{2}\int \frac{\mathrm{d}^3 p}{(2\pi)^3}[E_+ + E_- + f(|\vec{p}|) - f(|\vec{p}|)] = \mathcal{F}^{(1)}_{\text{fin},\pi_\pm} + \mathcal{F}^{(1)}_{\text{div},\pi_\pm}. \tag{3.20}$$

This results in a finite integral,

$$\mathcal{F}^{(1)}_{\text{fin},\pi_\pm} = \frac{1}{2}\int \frac{\mathrm{d}^3 p}{(2\pi)^3}[E_+ + E_- - f(|\vec{p}|)], \tag{3.21}$$

which we can evaluate numerically, and isolate the divergence to

$$\mathcal{F}^{(1)}_{\text{div},\pi_\pm} = \frac{1}{2}\int \frac{\mathrm{d}^3 p}{(2\pi)^3} f(|\vec{p}|), \tag{3.22}$$

which we hopefully will be able to do in dimensional regularization. We can explore the UV-behavior of $E_+ + E_-$ by expanding it in powers of $1/|\vec{p}|$,

$$\begin{aligned} E_+ + E_- &= 2|\vec{p}| + \frac{m_{12} + 2(m_1^2 + m_2^2)}{4}|\vec{p}|^{-1} - \frac{m_{12}^4 + 4m_{12}^2(m_1^2 + m_2^2) + 8(m_1^4 + m_2^4)}{64}|\vec{p}|^{-3} + \mathcal{O}\left(|\vec{p}|^{-5}\right) \\ &= a_1|\vec{p}| + a_2|\vec{p}|^{-1} + a_3|\vec{p}|^{-3} + \mathcal{O}\left(|\vec{p}|^{-5}\right). \end{aligned} \tag{3.23}$$

We have defined the new constants $a_i$ for brevity of notation. As

$$\int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3}|\vec{p}|^n = C\int_0^\infty \mathrm{d}p\, p^{2+n} \tag{3.24}$$

is UV divergent for $n \geq -3$, $f$ need to match the expansion of $E_+ + E_-$ up to and including $\mathcal{O}\left(|\vec{p}|^{-3}\right)$ for $\mathcal{F}^{(1)}_{\text{fin},\pi_\pm}$ to be finite. The most obvious choice for $f$ is

$$f(|\vec{p}|) = a_1|\vec{p}| + a_2|\vec{p}|^{-1} + a_3|\vec{p}|^{-3}. \tag{3.25}$$

However, this introduces a new problem. $f$ has the same UV-behavior as $E_+ + E_-$, but the last term diverges in the IR, that is for low $|\vec{p}|$. This can be amended by introducing a mass term. Let

$$|\vec{p}|^{-3} = \left(\frac{1}{\sqrt{|\vec{p}|^2}}\right)^3 \longrightarrow \left(\frac{1}{\sqrt{|\vec{p}|^2 + m^2}}\right)^3. \tag{3.26}$$

For $|\vec{p}|^2 \to \infty$, this term is asymptotic to $|\vec{p}|^{-3}$, so it retains its UV behavior. However, for $|\vec{p}| \to 0$, it now approaches $m^{-3}$, so the IR-divergence is gone. The cost of this technique is that we have introduced an arbitrary mass parameter. Any final result must thus be independent of the value of $m$ to be acceptable.

We will instead regularize the integral by defining $E_i = \sqrt{|\vec{p}|^2 + \tilde{m}_i^2}$, and $\tilde{m}_i^2 = m_i^2 + \frac{1}{4}m_{12}^2$. Using the definition of the mass parameters, Eqs. (2.31) to (2.34), we get

$$m_3^2 = \bar{m}^2\cos\alpha + \mu_I^2\sin^2\alpha, \tag{3.27}$$

$$\tilde{m}_1^2 = m_1^2 + \mu^2\cos\alpha^2 = \bar{m}^2\cos\alpha + \mu_I^2\sin^2\alpha = m_3^2 \tag{3.28}$$

$$\tilde{m}_2^2 = m_2^2 + \mu^2\cos\alpha^2 = \bar{m}^2\cos\alpha. \tag{3.29}$$

Finally, we define $f(|\vec{p}|) = E_1 + E_2$ which differ from $E_+ + E_-$ by $\mathcal{O}\left(|\vec{p}|^{-5}\right)$, and is well-behaved in the IR. This leads to a divergent integral the same form as in the case of a free scalar. Thus, in the $\overline{\text{MS}}$-scheme,

$$\mathcal{F}^{(1)}_{\text{div},\pi\pm} = -\frac{1}{4}\frac{\tilde{m}_1^4}{(4\pi)^2}\left[\frac{1}{\epsilon} + \frac{3}{2} + \ln\left(\frac{\mu^2}{\tilde{m}_1^2}\right)\right] - \frac{1}{4}\frac{\tilde{m}_2^4}{(4\pi)^2}\left[\frac{1}{\epsilon} + \frac{3}{2} + \ln\left(\frac{\mu^2}{\tilde{m}_2^2}\right)\right] + \mathcal{O}(\epsilon). \tag{3.30}$$

We define

$$\mathcal{F}^{(1)}_{\text{fin},\pi\pm} = \frac{1}{2}\int \frac{\mathrm{d}^3 p}{(2\pi)^3}(E_+ + E_- - E_1 - E_2), \tag{3.31}$$

which is a finite integral. The total one-loop contribution is then, using Eqs. (3.28) and (3.29),

$$\mathcal{F}^{(2)} = \mathcal{F}^{(1)}_{\text{fin},\pi\pm} - \frac{1}{2}\frac{1}{(4\pi)^2}\left[\left(\frac{1}{\epsilon} + \frac{3}{2} + \ln\frac{\mu^2}{m_3^2}\right)m_3^4 + \frac{1}{2}\left(\frac{1}{\epsilon} + \frac{3}{2} + \ln\frac{\mu^2}{\tilde{m}_2^2}\right)\tilde{m}_2^4\right] \tag{3.32}$$

## 3.2 Next-to-leading order and renormalization

We have now regularized the divergences, so they can be handled in a well-defined way. However, they are still there. To get rid of them, we need to use renormalization. The power counting scheme used when constructing the Effective Lagrangians ensures that all terms in $\mathcal{L}_{2n}$ is of order $p^{2n}$ in the pion momenta.[2] The tree level free energy from $\mathcal{L}_{2n}$ is thus of order $p^{2n}$. The n-loop correction to the tree level result is then suppressed by $p^{2n}$ [17, 11]. Our one-loop calculation using $\mathcal{L}_2$ therefore contains divergences of order $p^4$. Since $\mathcal{L}_4$ is, by construction, the most general possible Lagrangian at order $p^4$, it contains coupling constants which can be renormalized to absorb all these divergences.

The renormalized coupling constants in $\mathcal{L}_2$ have been calculated for $\mu_I = 0$ [17]. They are independent of $\mu_I$, and we can therefore use them in this calculation. The renormalized coupling constants in the $\overline{\text{MS}}$-scheme are related to the bare couplings through

$$l_i = l_i^r - \frac{1}{2}\mu^{-2\epsilon}\frac{\gamma_i}{(4\pi)^2}\left(\frac{1}{\epsilon} + 1\right), \qquad i \in \{1, ...7\} \tag{3.33}$$

$$h_i = h_i^r - \frac{1}{2}\mu^{-2\epsilon}\frac{\delta_i}{(4\pi)^2}\left(\frac{1}{\epsilon} + 1\right), \qquad i \in \{1, ...3\}. \tag{3.34}$$

Here, $\gamma_i$ and $\delta_i$ are numerical constants that are used to match the divergences. The relevant terms are[3]

$$\gamma_1 = \frac{1}{3}, \quad \gamma_2 = \frac{2}{3}, \quad \gamma_3 = -\frac{1}{2}, \quad \gamma_4 = 2, \tag{3.35}$$

$$\delta_1 = 2, \quad \delta_3 = 0. \tag{3.36}$$

The bare coupling constants, though infinite, are independent of our renormalization scale $\mu$. From this we obtain the renormalization group equations for the running coupling constants,

$$\mu\frac{\mathrm{d}l_i^r}{\mathrm{d}\mu} = -\frac{\gamma_i}{(4\pi)^2} + \mathcal{O}\left(\epsilon^2\right), \quad \mu\frac{\mathrm{d}h_i^r}{\mathrm{d}\mu} = -\frac{\delta_i}{(4\pi)^2} + \mathcal{O}\left(\epsilon^2\right). \tag{3.37}$$

These have the general solutions

$$l_i^r = \frac{1}{2}\frac{\gamma_i}{(2\pi)^2}\left(\bar{l}_i - \ln\frac{\mu^2}{M^2}\right), \quad h_i^r = \frac{1}{2}\frac{\gamma_i}{(2\pi)^2}\left(\bar{h}_i - \ln\frac{\mu^2}{M^2}\right), \tag{3.38}$$

where $\bar{l}_i$ and $\bar{h}_i$ are the values of the coupling constants (times a constant) measured at the energy $M$. This only applies if the numerical constants $\gamma_i/\delta_i$ are non-zero. If they are zero, then the renormalized constant is not running, and instead equal to its measured value at all energies. The bare couplings are thus given by

$$l_i = \frac{1}{2}\frac{\gamma_i}{(4\pi)^2}\left[\bar{l}_i - \mu^{-2\epsilon}\left(1 + \frac{1}{\epsilon}\right) - \ln\frac{\mu^2}{M^2}\right], \quad h_i = \frac{1}{2}\frac{\delta_i}{(4\pi)^2}\left[\bar{h}_i - \mu^{-2\epsilon}\left(1 + \frac{1}{\epsilon}\right) - \ln\frac{\mu^2}{M^2}\right] \tag{3.39}$$

[2] Remember that the bare pion mass $\bar{m} = B_0(m_u + m_d)$ is considered to be of order $p^2$.

[3] Some authors [16, 18] instead use $h_1' = h_1 - l_4$, with a corresponding $\delta_1' = \delta_1 - \gamma_1 = 0$.

(HVOR BLE DET AV $\ln\mu$?) The free energy at tree-level, at next-to-leading order is, according to Eq. (2.27),

$$\begin{aligned}
\mathcal{F}_4^{(0)} &= -\mathcal{L}_4^{(0)} \\
&= -(l_1+l_2)\mu_I^4\sin^4\alpha - (l_3+l_4)\bar{m}^4\cos^2\alpha - l_4\bar{m}^2{\mu_I}^2\cos\alpha\sin^2\alpha - (h_1-l_4)\bar{m}^2 - h_3\Delta m^2 \\
&= -\frac{1}{2}\frac{1}{(4\pi)^2}\Bigg[\frac{1}{3}\left(\bar{l}_1+2\bar{l}_2-3\right)\mu_I^4\sin^4\alpha + \frac{1}{2}\left(-\bar{l}_3+4\bar{l}_4-3\right)\bar{m}^4\cos^2\alpha \\
&\qquad + 2\left(\bar{l}_4-1\right)\bar{m}^2\mu_I^2\cos\alpha\sin^2\alpha + 2(\bar{l}_4-\bar{h}_1)\bar{m}^2 + \bar{h}_3\Delta m^2 \\
&\qquad - \left(\frac{1}{\epsilon}+\ln\frac{\mu^2}{M^2}\right)\left(\mu_I^4\sin^4\alpha + \frac{3}{2}\bar{m}^4\cos^2\alpha + 2\bar{m}^2\mu_I^2\cos\alpha\sin^2\alpha\right)\Bigg]
\end{aligned}$$

Adding all the contribution to the free energy density, we get the next-to-leading order free energy,

$$\begin{aligned}
\mathcal{F}_{\text{NLO}} &= -f^2\left(\bar{m}^2\cos\alpha + \frac{1}{2}\mu_I^2\sin^2\alpha\right) + \mathcal{F}^{(1)}_{\text{fin},\pi\pm} - \frac{1}{2}\frac{1}{(4\pi)^2}\Bigg[\frac{1}{3}\left(\bar{l}_1+2\bar{l}_2+\frac{3}{2}+3\ln\frac{M^2}{m_3^2}\right)\mu_I^4\sin^4\alpha \\
&+ \frac{1}{2}\left(-\bar{l}_3+4\bar{l}_4+\frac{3}{2}+2\ln\frac{M^2}{m_3^2}+\ln\frac{M^2}{\tilde{m}_2^2}\right)\bar{m}^4\cos^2\alpha + 2\left(\bar{l}_4-\frac{1}{2}+\ln\frac{M^2}{m_3^2}\right)\bar{m}^2\mu_I^2\cos\alpha\sin^2\alpha\Bigg]. \quad (3.40)
\end{aligned}$$

We have dropped the terms proportional to $\bar{l}_4-\bar{h}_1$ and $\bar{h}_3$, as they only add an unobservable constant value to the free energy.

## Parameters, and keeping the expansion consistent

The coupling constants are free parameters, and can therefore not be calculated from first principles, but must be measured. The values for the pion mass and pion decay constants are (HVORFOR?)

$$m_\pi = 131\,\text{MeV}, \quad f_\pi = \frac{1}{\sqrt{2}}128\,\text{MeV}. \quad (3.41)$$

This is the physical mass, $m_\pi$, is defined as the pole of the propagator and thus the zero of the inverse propagator,

$$D^{-1}(p^2 = m_\pi^2) = 0. \quad (3.42)$$

This relates it to the which relates it to the bare mass $\bar{m}$. We found, in Eq. (2.31), that $m_\pi = m_3^2(\mu_I = 0) = \bar{m}^2$ to leading order. Similarly, $f_\pi = f$ to leading order. (HVORFOR) However, in any NLO results we need $\bar{m}^2$ and $f$ to NLO. This is given by [16] (UTLEDE?)

$$m_\pi^2 = \bar{m}^2 + \frac{\bar{l}_3}{2(4\pi)^2}\frac{\bar{m}^4}{f^2}, \quad (3.43)$$

$$f_\pi^2 = f^2 + \frac{2\bar{l}_4}{(4\pi)^2}\frac{\bar{m}^2}{f^2} \quad (3.44)$$

All results in this text are given in units of $m_\pi$.

Table 3.1: The measured values and corresponding uncertainties of the relevant coupling constants.

| | value | uncertainty | source |
|---|---|---|---|
| $\bar{l}_1$ | -0.4 | $\pm0.6$ | [19] |
| $\bar{l}_2$ | 4.3 | $\pm0.1$ | [19] |
| $\bar{l}_3$ | 2.9 | $\pm2.4$ | [17] |
| $\bar{l}_4$ | 4.4 | $\pm0.2$ | [19] |

The values for the coupling constants used in this text are given in Table 3.1. The constants $\bar{l}_1$, $\bar{l}_2$ and $\bar{l}_3$ are estimated using data from $\pi\pi$-scattering [19], while $\bar{l}_3$ is estimated using three flavor chiral perturbation theory [17]. These are the same values as those used in [16]. Together with Eqs. (3.43) and (3.44), the NLO results for the bare mass and decay constant are

$$\bar{m}/m_\pi = 1.01136, \quad (3.45)$$

$$f/m_\pi = 0.64835. \quad (3.46)$$

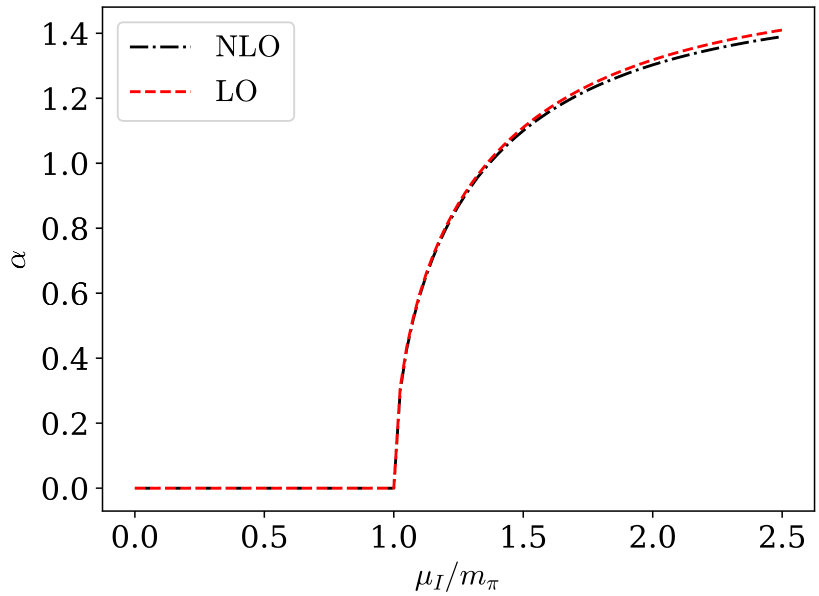



Figure 3.2: The leading order and next-to-leading order results for $\alpha$ as a function of $\mu_I$.

In section 3.1, we found a relationship between $\alpha$ and $\mu_i$, using the lowest order estimate of $\mathcal{F}$, given in Eq. (3.4). To calculate any thermodynamic quantities to leading order, we must thus use this result. If we are to calculate any other quantities to next-to-leading order, we also have to use the next-to-leading order criterion for $\alpha$ as a function of $\mu_I$, which is given by

$$\frac{\partial \mathcal{F}_{\text{NLO}}}{\partial \alpha} = 0, \tag{3.47}$$

using the result Eq. (3.40). In Figure 3.2, the NLO result is compared with the LO result, Eq. (3.4).

## 3.3 Thermodynamics

The free energy[4] is defined as

$$F = U - TS - \mu_I Q_I, \quad \mathrm{d}F = -S\mathrm{d}T - P\mathrm{d}V - Q_I \mathrm{d}\mu_I, \tag{3.48}$$

where $Q_I$ is the isospin charge, and $U$ is the energy. As we have seen earlier, our system is homogenous. This means that the free energy density is independent of volume, and thus $F = V\mathcal{F}$. From

$$P = -\left(\frac{\partial F}{\partial V}\right)_{T,\mu_I} \tag{3.49}$$

We are interested in the pressure relative to $\mu_I$,

$$P(\mu_I) = -(\mathcal{F}(\mu_I) - \mathcal{F}(\mu_I = 0)) \tag{3.50}$$

This is illustrated in Figure 3.3. Likewise, the total isospin is proportional to volume, which means that the isospin density is

$$n_I = \frac{Q_I}{V} = -\frac{1}{V}\left(\frac{\partial F}{\partial \mu_I}\right)_{T,V} = -\frac{\partial \mathcal{F}}{\partial \mu_I}. \tag{3.51}$$

The isospin density, as a function of $\mu_I$, is shown in Figure 3.4. Finally, from Eq. (3.48), we get the energy

[4] As we are in the grand canonical ensemble, this is the *grand canonical* free energy, and not Helmholtz' free energy.

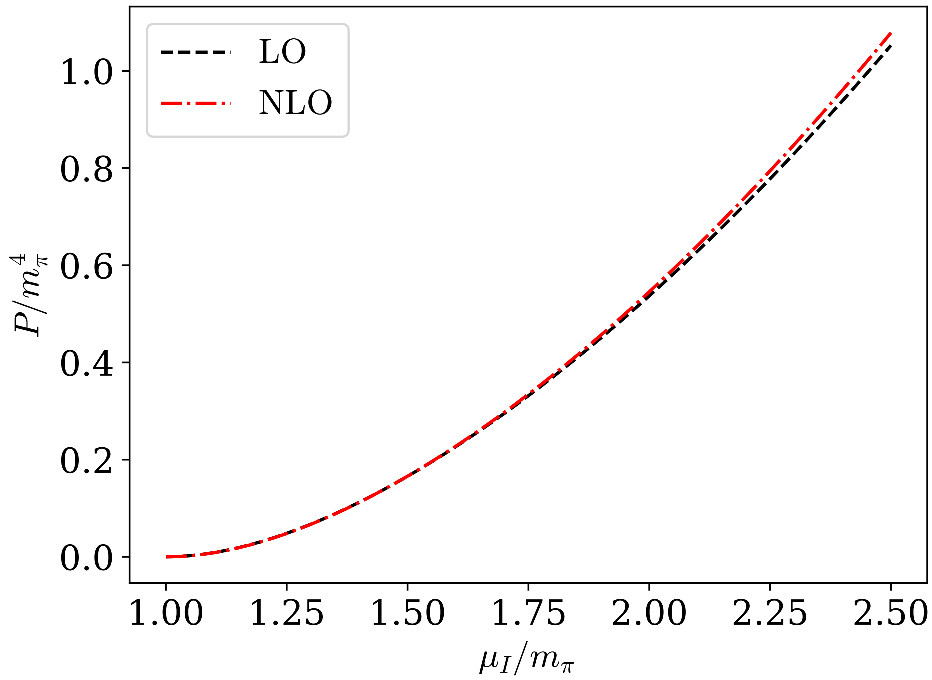



Figure 3.3: The NLO and LO result for the pressure of the pions, as a function of $\mu_I$.

density, $u = U/V$ is given by

$$u(\mu_I) = -P(\mu_I) + \mu_I n_I(\mu_I). \tag{3.52}$$

The isospin chemical potential $\mu_I$ parametrizes a line through the energy density-pressure plane, the relationship between the energy density and the pressure, which is shown in Figure 3.5. This relationship has the form

$$f(p, u) = 0, \tag{3.53}$$

and is the equation of state of the pion system.

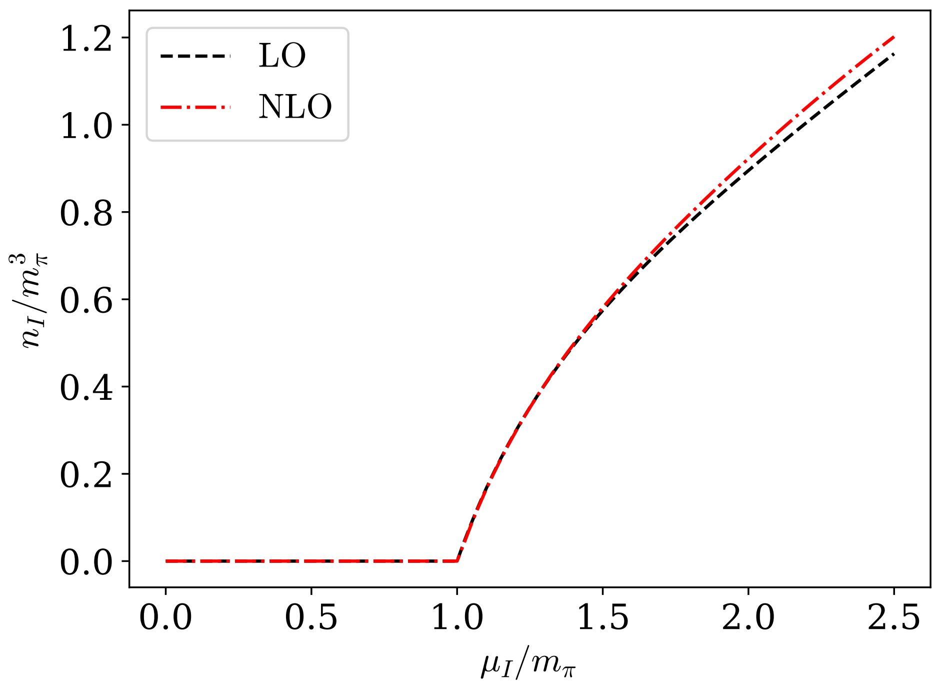



Figure 3.4: The NLO and LO result for the isosopin densisty, as a function of $\mu_I$.

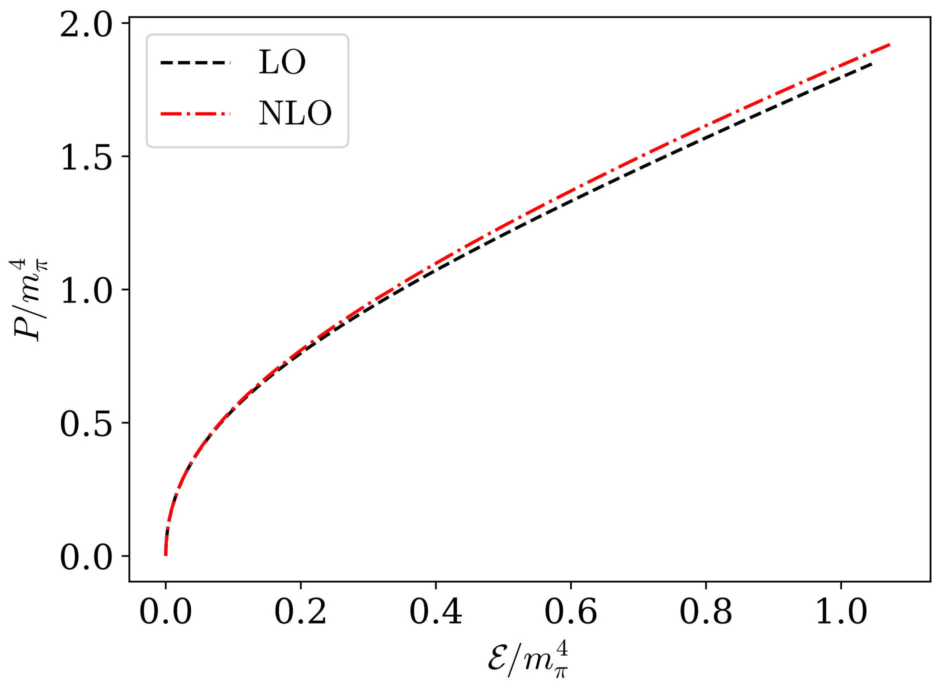



Figure 3.5: The equation of state of the pions.

# Appendix A

# Thermal Field Theory

This section is based on [20, 21].

## A.1 Statistical Mechanics

In classical mechanics, a thermal system at temperature $T = 1/\beta$ is described as an ensemble state, which have a probability $P_n$ of being in state $n$, with energy $E_n$. In the canonical ensemble, the probability is proportional to $e^{-\beta E_n}$. The expectation value of some quantity $A$, with value $A_n$ in state $n$ is

$$\langle A \rangle = \sum_n A_n P_n = \frac{1}{Z} \sum_n A_n e^{-\beta E_n}, \quad Z = \sum_n e^{-\beta A_n}.$$

$Z$ is the partition function. In quantum mechanics, an ensemble configuration is described by a non-pure density operator,

$$\hat{\rho} = C \sum_n P_n \left| n \right\rangle\!\left\langle n \right|,$$

where $|n\rangle$ is some basis for the relevant Hilbert space and $C$ is a constant. Assuming $|n\rangle$ are energy eigenvectors, i.e. $\hat{H} |n\rangle = E_n |n\rangle$, the density operator for the canonical ensemble is

$$\hat{\rho} = C \sum_n e^{-\beta E_n} \left| n \right\rangle\!\left\langle n \right| = C e^{-\beta \hat{H}} \sum_n \left| n \right\rangle\!\left\langle n \right| = C e^{-\beta \hat{H}}.$$

The expectation value in the ensemble state of a quantity corresponding to the operator $\hat{A}$ is given by

$$\langle A \rangle = \frac{\mathrm{Tr}\left\{\hat{\rho} \hat{A}\right\}}{\mathrm{Tr}\{\hat{\rho}\}} = \frac{1}{Z} \mathrm{Tr}\left\{\hat{A} e^{-\beta \hat{H}}\right\} \tag{A.1}$$

The partition function $Z$ ensures that the probabilities adds up to 1, and is defined as

$$Z = \mathrm{Tr}\left\{e^{-\beta \hat{H}}\right\}. \tag{A.2}$$

(REFFERER TIL THEROY-DELEN OM SYMMETRI; DETTE ER DOBBELT OPP) The grand canonical ensemble takes into account the conserved charges of the system. Conserved charges are a result of Nöther's theorem. Assume we have a set of fields $\varphi_\alpha$. Nöther's theorem tells us that if the Lagrangian $\mathcal{L}[\varphi_\alpha]$ has a *continuous symmetry*, then there is a corresponding conserved current [1, 22]. To define a continuous symmetry of the Lagrangian, we need a one-parameter family of transformations,

$$\varphi_\alpha(x) \longrightarrow \varphi'_\alpha(x; \epsilon) \sim \varphi_\alpha(x) + \varepsilon \eta_\alpha(x), \; \varepsilon \to 0.$$

Here, $\eta_\alpha(x)$ is some arbitrary function which define the transformation as $\varepsilon \to 0$. Applying this transformation to the Lagrangian will in general change its form,

$$\mathcal{L}[\varphi_\alpha] \to \mathcal{L}[\varphi'_\alpha] \sim \mathcal{L}[\varphi_\alpha] + \delta\mathcal{L},\, \varepsilon \to 0.$$

If the change in the Lagrangian can be written as a total derivative, i.e.

$$\delta\mathcal{L} = \varepsilon\partial_\mu K^\mu(x),$$

we say that the Lagrangian has a continuous symmetry. This is because a term of this form will result in a boundary term in the action integral, which does not contribute to the variation of the action. Nöther's theorem states more precisely that the current

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_\alpha)}\eta_\alpha - K^\mu \tag{A.3}$$

obeys the conservation law

$$\partial_\mu j^\mu = 0. \tag{A.4}$$

The flux of current through some space-like surface $V$, i.e. a surface with a time-like normal vector, defines a conserved charge. This surface is most commonly a surface of constant time in some reference frame. The conserved charge is then

$$Q = \int_V \mathrm{d}^3x\, j^0.$$

Using the divergence theorem again, and assuming the current falls of quickly enough towards infinity, we can show that the total charge is conserved,

$$\frac{\partial}{\partial t}Q = -\int_V \mathrm{d}^3x\, \nabla\cdot\vec{j} = -\int_{\partial V} \mathrm{d}^2x\, n_\mu j^\mu = 0.$$

Here, $n^\mu$ is the time-like normal vector of the surface of $V$, $\partial V$.

In the grand canonical ensemble, a system with $n$ conserved charges $Q_i$ has probability proportional to $e^{-\beta(H-\mu_i Q_i)}$. $\mu_i$ are the chemical potentials corresponding to conserved charge $Q_i$. This leads to the partition function

$$Z = \mathrm{Tr}\left\{e^{-\beta(\hat{H}-\mu_i\hat{Q}_i)}\right\}. \tag{A.5}$$

## A.2 Imaginary-time formalism

The partition function may be calculated in a similar way to the path integral approach, in what is called the imaginary-time formalism. This formalism is restricted to time independent problems, and is used to study fields in a volume $V$. This volume is taken to infinity in the thermodynamic limit. As an example, take a scalar quantum field theory with the Hamiltonian

$$\hat{H} = \int_V \mathrm{d}^3x\, \hat{\mathcal{H}}\left[\hat{\varphi}(\vec{x}), \hat{\pi}(\vec{x})\right], \tag{A.6}$$

where $\hat{\varphi}(\vec{x})$ is the field operator, and $\hat{\pi}(\vec{x})$ is the corresponding canonical momentum operator. These field operators have time independent eigenvectors, $|\varphi\rangle$ and $|\pi\rangle$, defined by

$$\hat{\varphi}(\vec{x})\,|\varphi\rangle = \varphi(\vec{x})\,|\varphi\rangle\,, \quad \hat{\pi}(\vec{x})\,|\pi\rangle = \pi(\vec{x})\,|\pi\rangle\,. \tag{A.7}$$

In analogy with regular quantum mechanics, they obey the relations [1]

$$\mathbb{1} = \int \mathcal{D}\varphi(\vec{x})\,|\varphi\rangle\langle\varphi| = \int \mathcal{D}\pi(\vec{x})\,|\pi\rangle\langle\pi|\,, \tag{A.8}$$

$$\langle\varphi|\pi\rangle = \exp\left(i\int_V \mathrm{d}^3x\,\varphi(\vec{x})\pi(\vec{x})\right), \tag{A.9}$$

$$\langle\pi_a|\pi_b\rangle = \delta(\phi_a - \phi_b), \quad \langle\varphi_a|\varphi_b\rangle = \delta(\varphi_a - \varphi_b). \tag{A.10}$$

[1] Some authors write $\mathcal{D}\pi/2\pi$. This extra factor $2\pi$ is a convention which in this text is left out for notational clarity.

The functional integral is defined by starting with $M$ degrees of freedom, $\{\varphi_m\}_{m=1}^M$ located at a finite grid $\{\vec{x}_m\}_{m=1}^M \subset V$. The integral is then the limit of the integral over all degrees of freedom, as $M \to \infty$:

$$\int \mathcal{D}\varphi(\vec{x}) = \lim_{M\to\infty} \int \left( \prod_{m=1}^{M} \mathrm{d}\varphi_m \right).$$

The functional Dirac-delta $\delta(f) = \prod_x \delta(f(x))$ is generalization of the familiar Dirac delta function. Given a functional $\mathcal{F}[f]$, it is defined by the relation

$$\int \mathcal{D}f(x)\, \mathcal{F}[f]\delta(f-g) = \mathcal{F}[g]. \tag{A.11}$$

The Hamiltonian is the limit of a sum of Hamiltonians $\hat{H}_m$ for each point $\vec{x}_m$

$$\hat{H} = \lim_{M\to\infty} \sum_{m=1}^{M} \frac{V}{M} \hat{H}_m(\{\hat{\varphi}_m\}, \{\hat{\pi}_m\}).$$

$H_m$ may depend on the local degrees of freedom $\hat{\varphi}_m$, $\hat{\pi}_m$ as well as those at neighboring points. By inserting the completeness relations Eq. (A.8) $N$ times into the definition of the partition function, it may be written as

$$Z = \int \mathcal{D}\varphi(\vec{x}) \left\langle \varphi \middle| e^{-\beta \hat{H}} \middle| \varphi \right\rangle = \prod_{n=1}^{N} \left( \int \mathcal{D}\varphi_n(\vec{x}) \int \mathcal{D}\pi_n(\vec{x}) \right) \prod_{n=1}^{N} \langle \varphi_n | \pi_n \rangle \left\langle \pi_n \middle| e^{-\epsilon \hat{H}} \middle| \varphi_{n+1} \right\rangle \langle \varphi_1 | \varphi_{N+1} \rangle,$$

where $\epsilon = \beta/N$. The last term ensures that $\varphi_1 = \varphi_{N+1}$. Strictly speaking, we only need to require $\varphi_1 = e^{i\theta}\varphi_{N+1}$, as the partition function is only defined up to a constant. As will be shown later, bosons such as the scalar field $\varphi$, follow the periodic boundary condition $\varphi(0,\vec{x}) = \varphi(\beta,\vec{x})$, i.e. $e^{i\theta} = 1$, while fermions follow the anti-periodic boundary condition $\psi(0,\vec{x}) = -\psi(\beta,\vec{x})$, i.e. $e^{i\theta} = -1$. We now want to exploit the fact that $|\pi\rangle$ and $|\varphi\rangle$ are the eigenvectors of the operators that define the Hamiltonian. In our case, as the Hamiltonian density $\mathcal{H}$ can be written as a sum of functions of $\varphi$ and $\pi$ separately, $\mathcal{H}[\varphi(\vec{x}),\pi(\vec{x})] = \mathcal{F}_1[\varphi(\vec{x})] + \mathcal{F}_2[\pi(\vec{x})]$ we may evaluate it as $\langle \pi_n | \mathcal{H}[\hat{\varphi}(\vec{x}), \hat{\pi}(\vec{x})] | \varphi_{n+1} \rangle = \mathcal{H}[\varphi_{n+1}(\vec{x}), \pi_n(\vec{x})] \langle \pi_n | \varphi_{n+1} \rangle$. This relationship does not, however, hold for more general functions of the field operators. In that case, one has to be more careful about the ordering of the operators, for example by using *Weyl ordering* [1]. By series expanding $e^{-\epsilon \hat{H}}$ and exploiting this relationship, the partition function can be written as, to second order in $\epsilon$,

$$Z = \prod_{n=1}^{N} \left( \int \mathcal{D}\varphi_n(\vec{x}) \int \mathcal{D}\pi_n(\vec{x}) \right) \exp\left[ -\epsilon \sum_{n=1}^{N} \int_V \mathrm{d}^3x \left( \mathcal{H}[\varphi_n(\vec{x}), \pi_n(\vec{x})] - i\pi_n(\vec{x}) \frac{\varphi_n(\vec{x}) - \varphi_{n+1}(\vec{x})}{\epsilon} \right) \right].$$

We denote $\varphi_n(\vec{x}) = \varphi(\tau_n, \vec{x})$, $\tau \in [0,\beta]$ and likewise with $\pi_n(\vec{x})$. In the limit $N \to \infty$, the expression for the partition function becomes

$$Z = \int_S \mathcal{D}\varphi(\tau,\vec{x}) \int \mathcal{D}\pi(\tau,\vec{x}) \exp\left\{ -\int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}\vec{x}\, \{ \mathcal{H}[\varphi(\tau,\vec{x}), \pi(\tau,\vec{x})] - i\pi(\tau,\vec{x})\dot{\varphi}(\tau,\vec{x}) \} \right\}, \tag{A.12}$$

where $S$ is the set of field configurations $\varphi$ such that $\varphi(\beta,\vec{x}) = \varphi(0,\vec{x})$. With a Hamiltonian density of the form $\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\varphi)^2 + \mathcal{V}(\varphi)$, we can evaluate the integral over the canonical momentum $\pi$ by discretizing $\pi(\tau_n, \vec{x}_m) = \pi_{n,m}$,

$$\begin{aligned}
&\int \mathcal{D}\pi \exp\left\{ -\int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}^3x \left( \frac{1}{2}\pi^2 - i\pi\dot{\varphi} \right) \right\} \\
&= \lim_{M,N\to\infty} \int \left( \prod_{m,n=1}^{M,N} \frac{\mathrm{d}\pi_{m,n}}{2\pi} \right) \exp\left\{ -\sum_{m,n} \frac{V\beta}{MN} \left[ \frac{1}{2}(\pi_{m,n} - i\dot{\varphi}_{m,n})^2 + \frac{1}{2}\dot{\varphi}_{m,n}^2 \right] \right\} \\
&= \lim_{M,N\to\infty} \left( \frac{MN}{2\pi V\beta} \right)^{MN/2} \exp\left\{ -\int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}^3x\, \frac{1}{2}\dot{\varphi}^2 \right\},
\end{aligned}$$

where $\dot{\varphi}_{m,n} = (\varphi_{m,n+1} - \varphi_{m,n})/\epsilon$. The partition function is then,

$$Z = C \int \mathcal{D}\varphi \exp\left\{-\int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}^3x \left[\frac{1}{2}\left(\dot{\varphi}^2 + \nabla\varphi^2\right) + \mathcal{V}(\varphi)\right]\right\}. \tag{A.13}$$

Here, $C$ is the divergent constant that results from the $\pi$-integral. In the last line, we exploited the fact that the variable of integration $\pi_{n,m}$ may be shifted by a constant without changing the integral, and used the Gaussian integral

$$\int_{-\infty}^{\infty} \mathrm{d}x\, e^{-ax^2/2} = \sqrt{\frac{2\pi}{a}}.$$

The partition function resulting from this procedure may also be obtained by starting with the ground state path integral

$$Z_g = \int \mathcal{D}\varphi\mathcal{D}\pi \exp\left\{i\int_{\Omega'} \mathrm{d}^4x\ (\pi\dot{\varphi} - \mathcal{H}[\varphi,\pi])\right\} = C' \int \mathcal{D}\varphi(x) \exp\left\{i\int_{\Omega'} \mathrm{d}^4x\, \mathcal{L}[\varphi, \partial_\mu\varphi]\right\},$$

and follow a formal procedure. First, the action integral is modified by performing a Wick-rotation of the time coordinate $t$. This involves changing the domain of $t$ from the real line to the imaginary line by closing the contour at infinity, and making the change of variable $it \to \tau$. The new variable is then restricted to the interval $\tau \in [0,\beta]$, and the domain of the functional integral $\int \mathcal{D}\varphi$ is restricted from *all* (smooth enough) field configurations $\varphi(t,\vec{x})$, to only those that obey $\varphi(\beta,\vec{x}) = e^{i\theta}\varphi(0,\vec{x})$, which is denoted $S$. This procedure motivates the introduction of the Euclidean Lagrange density, $\mathcal{L}_E(\tau,\vec{x}) = -\mathcal{L}(-i\tau,\vec{x})$, as well as the name "imaginary-time formalism". The result is the same partition function as before,

$$\begin{aligned} Z &= C\int_S \mathcal{D}\varphi \int \mathcal{D}\pi \exp\left\{-\int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}^3x\ [-i\dot{\varphi}\pi + \mathcal{H}(\varphi,\pi)]\right\} \\ &= C'\int_S \mathcal{D}\varphi \exp\left\{-\int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}^3x\, \mathcal{L}_E(\varphi,\pi)\right\}. \end{aligned} \tag{A.14}$$

## A.3 Free scalar field

This section uses notation as described in section B.1. The procedure for obtaining the thermal properties of an interacting scalar field is similar to that used in scattering theory. One starts with a free theory, which can be solved exactly. Then an interaction term is added, which is accounted for peturbatively by using Feynman diagrams. The Euclidean Lagrangian for a free scalar gas is, after integrating by parts,

$$\mathcal{L}_E = \frac{1}{2}\varphi(X)\left(-\partial_E^2 + m^2\right)\varphi(X) \tag{A.15}$$

Here, $X = (\tau,\vec{x})$ is the Euclidean coordinate which is a result of the Wick-rotation. We have also introduced the Euclidean Laplace operator, $\partial_E^2 = \partial_\tau^2 + \nabla^2$. Following the procedure as described in section A.2 to obtain the thermal partition function yields

$$Z = C\int_S \mathcal{D}\varphi(X) \exp\left\{-\int_\Omega \mathrm{d}X \frac{1}{2}\varphi(X)\left(-\partial_E^2 + m^2\right)\varphi(X)\right\}. \tag{A.16}$$

Here, $\Omega$ is the domain $[0,\beta] \times V$. We then insert the Fourier expansion of $\varphi$, and change the functional integration variable to the Fourier components. The integration measures are related by

$$\mathcal{D}\varphi(X) = \det\left(\frac{\delta\varphi(X)}{\delta\tilde{\varphi}(K)}\right)\mathcal{D}\tilde{\varphi}(K),$$

where $K = (\omega_n, \vec{k})$ is the Euclidean Fourier-space coordinate. The determinant factor which appears may be absorbed into the constant $C$, as the integration variables are related by a linear transform. The action becomes

$$S = -\int_\Omega \mathrm{d}X\, \mathcal{L}_e = -\frac{1}{2}V\beta \int_\Omega \mathrm{d}X \int_{\tilde{\Omega}} \mathrm{d}K \int_{\tilde{\Omega}} \mathrm{d}K'\, \tilde{\varphi}(K') \left(\omega_n^2 + \vec{k}^2 + m^2\right) \tilde{\varphi}(K) e^{iX\cdot(K-K')}$$
$$= -\frac{1}{2}V\beta^2 \int_{\tilde{\Omega}} \mathrm{d}K\, \tilde{\varphi}(K)^* \left(\omega_n^2 + \omega_k^2\right) \tilde{\varphi}(K),$$

where $\omega_k^2 = \vec{k}^2 + m^2$. $\tilde{\Omega}$ is the reciprocal space corresponding to $\Omega$, as described in section B.1. We used the fact that $\varphi$ is real, which implies that $\tilde{\varphi}(-K) = \tilde{\varphi}(K)^*$, as well as the identity Eq. (B.9). This gives the partition function

$$Z = C \int_{\tilde{S}} \mathcal{D}\tilde{\varphi}(K) \exp\left\{-\frac{1}{2}V \int_{\tilde{\Omega}} \mathrm{d}K\, \tilde{\varphi}(K)^* \left[\beta^2(\omega_n^2 + \omega_k^2)\right] \tilde{\varphi}(K)\right\}, \tag{A.17}$$

Going back to before the continuum limit, this integral can be written as a product of Gaussian integrals, and may therefore be evaluated

$$Z = C \prod_{n=-\infty}^{\infty} \prod_{k\in\tilde{V}} \left(\int \mathrm{d}\tilde{\varphi}_{n,\vec{k}}\, \exp\left\{-\frac{1}{2}\tilde{\varphi}^*_{n,\vec{k}} \left[\beta^2(\omega_n^2 + \omega_k^2)\right] \tilde{\varphi}_{n,\vec{k}}\right\}\right) = C \prod_{n=-\infty}^{\infty} \prod_{k\in\tilde{V}} \sqrt{\frac{2\pi}{\beta^2(\omega_n^2 + \omega_k^2)}}.$$

The partition function is related to Helmholtz free energy $F$ through

$$\frac{F}{TV} = -\frac{\ln(Z)}{V} = \frac{1}{2}\int_{\tilde{\Omega}} \mathrm{d}K \frac{1}{2} \ln[\beta^2(\omega_n^2 + \omega_k^2)] + \frac{F_0}{TV}, \tag{A.18}$$

where $F_0$ is a constant.

A faster and more formal way to get to this result is to compare the partition function to the multidimensional version of the Gaussian integral [20, 1]. The partition function has the form

$$I_n = \int_{\mathbb{R}^n} \mathrm{d}^n x\, \exp\left\{-\frac{1}{2}\langle x, D_0^{-1} x\rangle\right\},$$

where $D_0^{-1}$ is a linear operator, and $\langle\cdot,\cdot\rangle$ an inner product on the corresponding vector space. By diagonalizing $D_0^{-1}$, we get the result

$$I_n = \sqrt{\frac{(2\pi)^n}{\det\left(D_0^{-1}\right)}}.$$

We may then use the identity

$$\det\left(D_0^{-1}\right) = \prod_i \lambda_i = \exp\left\{\mathrm{Tr}\left[\ln\left(D_0^{-1}\right)\right]\right\}, \tag{A.19}$$

where $\lambda_i$ are the eigenvalues of $D_0^{-1}$. The trace in this context is defined by the vector space $D_0^{-1}$ acts on. For given an orthonormal basis $x_n$, such that $\langle x_n, x'_n\rangle = \delta_{nn'}$ the trace can be evaluated as $\mathrm{Tr}\left\{D_0^{-1}\right\} = \sum_n \langle x_n, D_0^{-1} x_n\rangle$. Identifying

$$\langle x, D_0^{-1} x\rangle = \int_\Omega \mathrm{d}X \varphi(X) \left(-\partial_E^2 + m^2\right) \varphi(X),$$

we get the formal result

$$Z = \det\left(-\partial_E^2 + m^2\right)^{-1/2},$$

and

$$\beta F = \frac{1}{2}\,\mathrm{Tr}\left\{\ln\left(-\partial_E^2 + m^2\right)\right\}.$$

The logarithm may then be evaluated by using the eigenvalues of the linear operator. This is found by diagonalizing the operator,

$$\langle x, D_0^{-1} x\rangle = \int_\Omega \mathrm{d}X \varphi(X) \left(-\partial_E^2 + m^2\right) \varphi(X) = V \int_{\tilde{\Omega}} \mathrm{d}K \tilde{\varphi}(K)^* [\beta^2(\omega_k^2 + \omega_n^2)] \tilde{\varphi}(K),$$

leaving us with the same result as we obtained in Eq. (A.18),

$$\beta F = \frac{1}{2}\,\mathrm{Tr}\left\{\ln\left(-\partial_E^2 + m^2\right)\right\} = \frac{1}{2}V \int_{\tilde{\Omega}} \mathrm{d}K \ln[\beta^2(\omega_n^2 + \omega_k^2)].$$

Sums similar to this show up a lot, and it is show how to evaluate them in section A.4.

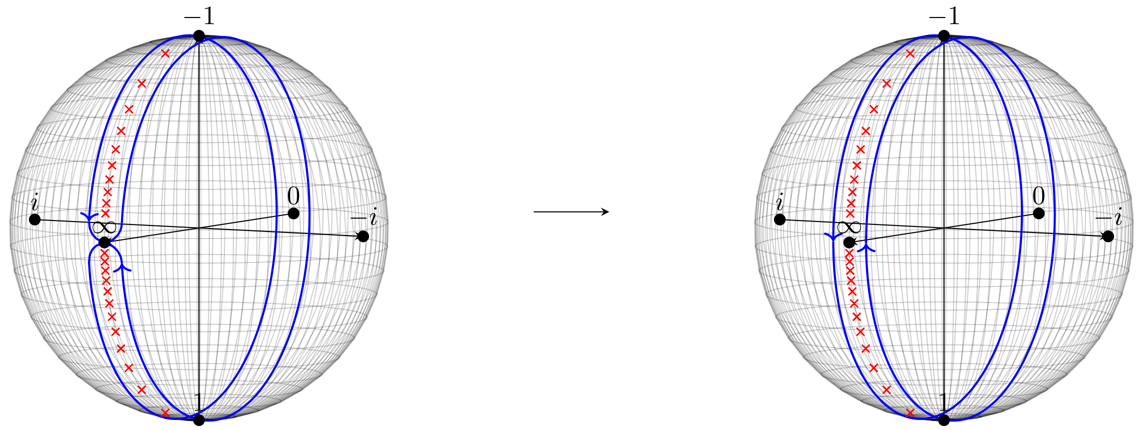



Figure A.1: The integral contour $\gamma$, and the result of deforming it into to contours close to the real line. The red crosses illustrate the poles of $n_B$.

## A.4 Thermal sum

When evaluating thermal integral, we will often encounter sums of the form

$$j(\omega, \mu) = \frac{1}{2\beta} \sum_{\omega_n} \ln\{\beta^2[(\omega_n + i\mu) + \omega^2]\} + g(\beta), \tag{A.20}$$

where the sum is over either the bosonic Matsubara frequencies $\omega_n = 2n\pi/\beta$, $n \in \mathbb{Z}$, or the fermionic ones, $\omega_n = (2n+1)\pi/\beta$, $n \in \mathbb{Z}$. $\mu \in \mathbb{R}$ is the chemical potential. $g$ may be a function of $\beta$, but we assume it is independent of $\omega$. Thus, the factor $\beta^2$ could strictly be dropped, but it is kept to make the argument within the logarithm dimensionless. We define the function

$$i(\omega, \mu) = \frac{1}{\omega} \frac{\mathrm{d}}{\mathrm{d}\omega} j(\omega, \mu) = \frac{1}{\beta} \sum_{\omega_n} \frac{1}{(\omega_n + i\mu)^2 + \omega^2}. \tag{A.21}$$

We will first work with the sum over bosonic Matsubara frequencies. Assume $f(z)$ is an analytic function, except perhaps on a set of isolated poles $\{z_i\}$ located outside the real line. By exploiting the properties of the Bose-distribution $n_B(z)$, as described in section B.1, we can rewrite the sum over Matsubara frequencies as a contour integral

$$\frac{1}{\beta} \sum_{\omega_n} f(\omega_n) = \oint_\gamma \frac{\mathrm{d}z}{2\pi i} f(z) i n_B(iz),$$

where $\gamma$ is a contour that goes from $-\infty - i\epsilon$ to $+\infty - i\epsilon$, crosses the real line at $\infty$, goes from $+\infty - i\epsilon$ to $-\infty + i\epsilon$ before closing the curve. The contour $\gamma$, and the change of integral contours is illustrated in Figure A.1 This result exploits Cauchy's integral formula, by letting the poles of $in_B(iz)$ at the Matsubara frequencies "pick out" the necessary residues. The integral over $\gamma$ is equivalent to two integrals along $\mathbb{R} \pm i\epsilon$,

$$\begin{aligned} \frac{1}{\beta} \sum_{\omega_n} f(\omega_n) &= \left( \int_{\infty + i\epsilon}^{-\infty + i\epsilon} \frac{\mathrm{d}z}{2\pi} + \int_{-\infty - i\epsilon}^{\infty - i\epsilon} \frac{\mathrm{d}z}{2\pi} \right) f(z) n_B(iz), \\ &= \int_{-\infty - i\epsilon}^{\infty - i\epsilon} \frac{\mathrm{d}z}{2\pi} \left\{ f(-z) + [f(z) + f(-z)]\, n_B(iz) \right\} \\ &= \int_{-\infty}^{\infty} \frac{\mathrm{d}z}{2\pi} f(z) + \int_{-\infty - i\epsilon}^{\infty - i\epsilon} \frac{\mathrm{d}z}{2\pi} [f(z) - f(-z)]\, n_B(iz). \end{aligned} \tag{A.22}$$

In the second line, we have changed variables $z \to -z$ in the first integral, and exploited the property $n_B(-iz) = -1 - n_B(iz)$. In the last line, we use the assumption that $f(z)$ is analytic on the real line, and therefore also in a neigbourhood of it. This allows us to shift the first integral back to the real line. As $n_B(iz)$

is analytic outside the real line, the result of the second integral is the sum of residues of $f(z)+f(-z)$ in the lower half plane. The function

$$f(z) = \frac{1}{(z+i\mu)^2+\omega^2} = \frac{i}{2\omega}\left(\frac{1}{z+i(\mu+\omega)} - \frac{1}{z+i(\mu-\omega)}\right) \tag{A.23}$$

obeys the assumed properties, as it has poles at $z=-i(\mu\pm\omega)$, with residue $1/2\omega$, so the function defined in Eq. (A.21) may be written [2]

$$i(\omega,\mu) = \frac{1}{2\omega}[1+n_B(\omega-\mu)+n_B(\omega+\mu)]. \tag{A.24}$$

Using the anti-derivative of the Bose distribution, we get the final form of Eq. (A.20)

$$j(\omega,\mu) = \int \mathrm{d}\omega'\,\omega' i(\omega',\mu) = \frac{1}{2}\omega + \frac{1}{2\beta}\left[\ln\left(1-e^{-\beta(\omega-\mu)}\right) + \ln\left(1-e^{-\beta(\omega+\mu)}\right)\right] + g'(\beta). \tag{A.25}$$

The extra $\omega$-independent term $g'(\beta)$ is an integration constant. We see there are temperature dependent terms, one due to the particle and one due to the anti-particle, and one due to the antiparticle, as they have opposite chemical potentials.

We now consider the sum over fermionic frequencies, which we for clarity denote $\tilde{\omega}_n$ in this chapter. The procedure in this case is the same, except that we have to use a function to with poles at the fermionic Matsubara frequencies. This is done by the Fermi distribution, $n_F(z)$, as described in section B.1. The result is

$$\frac{1}{\beta}\sum_{\tilde{\omega}_n} f(\tilde{\omega}_n) = -\int_{-\infty}^{\infty}\frac{\mathrm{d}z}{2\pi}f(z) + \int_{-\infty-i\epsilon}^{\infty-i\epsilon}\frac{\mathrm{d}z}{2\pi}\left[f(z)-f(-z)\right]n_F(iz), \tag{A.26}$$

and

$$i(\omega,\mu) = \frac{1}{2\omega}[-1+n_F(\omega-\mu)+n_F(\omega+\mu)]. \tag{A.27}$$

Using the antiderivative of the Fermi-distribution, we get

$$j(\omega,\mu) = -\frac{1}{2}\omega - \frac{1}{2\beta}\left[\ln\left(1+e^{-\beta(\omega-\mu)}\right) + \ln\left(1+e^{-\beta(\omega+\mu)}\right)\right]. \tag{A.28}$$

## A.5 Regulating the free energy

Using the result from section A.4 on the result for the free energy density of the free scalar field, Eq. (A.14), we get

$$\mathcal{F} = \frac{\ln(Z)}{\beta V} = \frac{1}{2}\int_{\tilde{V}}\frac{\mathrm{d}^3k}{(2\pi)^3}\left[\omega_k + \frac{2}{\beta}\ln\bigl(1-e^{-\beta\omega_k}\bigr)\right]. \tag{A.29}$$

The first part of this integral is a temperature independent vacuum energy, while the second part encodes all the termperature dependence of the free energy density This free energy has two parts, the first part is dependent on temperature, the other is temperature independent vacuum contribution. Noticing that the integral is spherically symmetric, we may write the two contributions as

$$J_0 = \frac{1}{2}\frac{1}{2\pi^2}\int_{\mathbb{R}}\mathrm{d}k\,k^2\sqrt{k^2+m^2}, \quad J_T = \frac{T^4}{2\pi^2}\int_{\mathbb{R}}\mathrm{d}x\,x^2\ln\Bigl(1-e^{-\sqrt{x^2+(m/T)^2}}\Bigr), \tag{A.30}$$

The temperature-independent part, $J_0$, is clearly divergent, and we must therefore impose a regulator, and then add counter-terms. $J_T$, however, is convergent. To see this, we use the series expansion $\ln(1+\epsilon)\sim\epsilon+\mathcal{O}\left(\epsilon^2\right)$ to find the leading part of the integrand for large $k$'s,

$$x^2\ln\Bigl(1-e^{-\sqrt{x^2+(\beta m)^2}}\Bigr)\sim -x^2e^{-x}, \tag{A.31}$$

[2] Assuming $\omega>\mu$.

which is exponentially suppressed, making the integral convergent. In the limit of $T \to \infty$, we get

$$J_\infty \sim \frac{T^4}{2\pi^2}\int_{\mathbb{R}} \mathrm{d}x\, x^2 \ln\!\left(1-e^{-x}\right) = -\frac{T^4}{2\pi^2}\sum_{n=1}\frac{1}{n}\frac{\partial^2}{\partial n^2}\int \mathrm{d}x e^{-nx} = -\frac{T^4}{2\pi^2}\sum_{n=1}\frac{2}{n^4} = -\frac{T^4}{\pi^2}\zeta(4) = -T^4\frac{\pi^2}{90}. \tag{A.32}$$

Where $\zeta$ is the Riemann-zeta function.

Returning to the temperature-independent part, we use dimensional regularization to see its singular behavior. To that end, we define

$$\Phi_n(m,d,\alpha) = \mu^{n-d}\int_{\tilde{\Omega}} \frac{\mathrm{d}^d k}{(2\pi)^d}(k^2+m^2)^{-\alpha}, \tag{A.33}$$

so that $J_0 = \Phi_3(m,3,1/2)/2$. The parameter $\mu$ has the dimensions of $k$, and is inserted to ensure that $\Phi_n$ does not change physical dimension for $d \neq n$. Furthermore, as non-rational exponents are defined through the exponential functions, this parameter is needed to make the expression well-defined. Dimensional regularization takes uses the formula for integration of spherically symmetric function in $d$-dimensions,

$$\int_{\mathbb{R}^d} \mathrm{d}^d x\, f(r) = \frac{2\pi^{d/2}}{\Gamma(d/2)}\int_{\mathbb{R}} \mathrm{d}r\, r^{d-1} f(r), \tag{A.34}$$

where $r = \sqrt{x_i x_i}$ is the radial distance, and $\Gamma$ is the Gamma function. The factor in the front of the integral comes from the solid angle. By extending this formula from integer-valued $d$ to real numbers, the function we defined becomes

$$\Phi_n = \frac{2\pi^{d/2}\mu^{n-d}}{\Gamma(d/2)}\int_{\mathbb{R}} \mathrm{d}k\, \frac{k^{d-1}}{(k^2+m^2)^\alpha} = \frac{m^{n-2\alpha}}{(4\pi)^{d/2}\Gamma(d/2)}\left(\frac{m}{\mu}\right)^{d-n} 2\int_{\mathbb{R}} \mathrm{d}z\, \frac{z^{d-1}}{(1+z)^\alpha}, \tag{A.35}$$

where we have made the change of variables $mz = k$. We make one more change of variable to the integral,

$$I = 2\int_{\mathbb{R}} \mathrm{d}z\, \frac{z^{d-1}}{(1+z)^\alpha} \tag{A.36}$$

Let

$$z^2 = \frac{1}{s} - 1 \implies 2z\mathrm{d}z = -\frac{\mathrm{d}s}{s^2} \tag{A.37}$$

Thus,

$$I = \int_0^a \mathrm{d}s\, s^{\alpha-d/2-1}(1-z)^{d/2-1}. \tag{A.38}$$

This is the beta function, which can be written in terms of Gamma functions [1]

$$I = B\left(\alpha - \frac{d}{2}, \frac{d}{2}\right) = \frac{\Gamma\left(\alpha-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}\right)}{\Gamma(\alpha)}. \tag{A.39}$$

Combining this gives

$$\Phi_n(m,d,\alpha) = \frac{m^{n-2\alpha}}{(4\pi)^{d/2}}\frac{\Gamma\left(\alpha-\frac{d}{2}\right)}{\Gamma(\alpha)}\left(\frac{m^2}{\mu^2}\right)^{(d-n)/2}. \tag{A.40}$$

Inserting $n=3$, $d = 3-2\epsilon$ and $\alpha = -1/2$, we get

$$\Phi_3(m,3-2\epsilon,-1/2) = \frac{m^4}{(4\pi)^{d/2}\Gamma(-1/2)}\Gamma(-2+\epsilon)\left(\frac{m^2}{\mu^2}\right)^{-\epsilon} = -\frac{m^4}{(4\pi)^2}\left(\frac{m^2}{4\pi\mu^2}\right)^{-\epsilon}\frac{\Gamma(\epsilon)}{(\epsilon-2)(\epsilon-1)}, \tag{A.41}$$

where we have used the defining property $\Gamma(z+1) = z\Gamma(z)$ and $\Gamma(1/2) = \sqrt{\pi}$. Expanding around $\epsilon = 0$ gives

$$\left(\frac{m^2}{4\pi\mu^2}\right)^{-\epsilon} \sim 1 + \epsilon\ln\left(4\pi\frac{\mu^2}{m^2}\right), \tag{A.42}$$

$$\Gamma(\epsilon) \sim \frac{1}{\epsilon} - \gamma, \tag{A.43}$$

$$\frac{1}{(\epsilon-2)(\epsilon-1)} \sim \frac{1}{2}\left(1+\frac{3}{2}\epsilon\right). \tag{A.44}$$

The singular behavior of the time-independent term is therefore

$$J_0 \sim -\frac{1}{4}\frac{m^4}{(4\pi)^2}\left[\frac{1}{\epsilon} - \gamma + \frac{3}{2} + \ln\left(4\pi\frac{\mu^2}{m^2}\right)\right]. \tag{A.45}$$

With this regulator, one can then add counter-terms to cancel the $\frac{1}{\epsilon}$-divergence. The exact form of the counter-term is convention. One may also cancel the finite contribution due to the regulator. The minimal subtraction, or MS, scheme, is to only subtract the divergent term, as the name suggest. We will use the modified minimal subtraction, or $\overline{\text{MS}}$, scheme. In this scheme, one also removes the $-\gamma$ and $\ln(4\pi)$ term, which can be interpreted as changing the parameter $\mu$

$$-\gamma + \ln\left(4\pi\frac{\mu^2}{m^2}\right) \to \ln\left(\frac{\mu^2}{m^2}\right), \tag{A.46}$$

which leads to the expression

$$J_0 \sim -\frac{1}{4}\frac{m^4}{(4\pi)^2}\left[\frac{1}{\epsilon} + \frac{3}{2} + \ln\left(\frac{\mu^2}{m^2}\right)\right]. \tag{A.47}$$

## A.6 Interacting scalar

We now study a scalar field with a $\lambda\varphi^4$ interaction term. We write the Lagrangian in the form

$$\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(I)}, \quad \mathcal{L}^{(0)} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - m^2\varphi^2, \quad \mathcal{L}^{(I)} = -\frac{\lambda}{4!}\varphi^4$$

$\mathcal{L}^{(I)}$ is called the interaction term, and makes it impossible to exactly solve for the partition function. Instead, we turn to perturbation theory. The canonical partition function in this theory is

$$Z = \mathrm{Tr}\left\{e^{-\beta\hat{H}}\right\} = \int_S \mathcal{D}\varphi\, \exp\left\{-\int_\Omega \mathrm{d}X\left(\mathcal{L}_E^{(0)} + \mathcal{L}_E^{(I)}\right)\right\} = \int_S \mathcal{D}\varphi\, e^{-S_0}e^{-S_I}. \tag{A.48}$$

Here, $S_0$ and $S_I$ denote the Euclidean action due to the free and interacting Lagrangian, respectively. The domain of integration $S$ is again periodic field configurations $\varphi(\beta,\vec{x}) = \varphi(0,\vec{x})$. We may write the free energy as

$$-\beta F = \ln\left[\int_S \mathcal{D}\varphi\, e^{-S_0}\sum_n \frac{1}{n!}(-S_I)^n\right] = \ln[Z_0] + \ln[Z_I],$$

where $Z_0$ is the partition function of the free theory. The correction to the partition function is thus given by

$$Z_I = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left\langle {S_I}^n\right\rangle_0, \tag{A.49}$$

where

$$\langle A\rangle_0 = \frac{\int_S \mathcal{D}\varphi\, A\, e^{-S_0}}{\int_S \mathcal{D}\varphi\, e^{-S_0}}. \tag{A.50}$$

To evaluate expectation values of the form $\langle\varphi(X_1)...\rangle_0$, we introduce the partition function with a source term

$$Z[J] = \int_S \mathcal{D}\varphi\, \exp\left\{-\frac{1}{2}\int_\Omega \mathrm{d}X\, \varphi(-\partial_E^2 + m^2)\varphi + \int_\Omega \mathrm{d}X\, J\varphi\right\}. \tag{A.51}$$

Using the thermal Greens function $D_0(X,Y)$, as defined in section B.1, we may complete the square to write

$$Z[J] = Z[0]\exp\left\{\frac{1}{2}\int_\Omega \mathrm{d}X\mathrm{d}Y\, J(X)D_0(X,Y)J(Y)\right\} = Z[0]\exp(W[J]) \tag{A.52}$$

We can now write

$$\langle \varphi(X)\varphi(Y)\rangle_0 = \frac{1}{Z[0]}\frac{\delta}{\delta J(X)}\frac{\delta}{\delta J(Y)}Z[J]\Big|_{J=0} = D_0(X,Y), \tag{A.53}$$

This generalizes to higher order expectation values,

$$\langle \varphi(X_i)\dots\varphi(X_n)\rangle_0 = \left(\prod_{i=1}^{n}\frac{\delta}{\delta J(X_i)}\right)Z[J]\Big|_{J=0}, \tag{A.54}$$

Using Wick's theorem, as described in section 1.1, the expectation values we are evaluating can be written

$$\begin{aligned}\langle {S_I}^m\rangle &= \left(-\frac{\lambda}{4!}\right)^m \int_\Omega \mathrm{d}X_1\dots\mathrm{d}X_m \left\langle \varphi^4(X_1)\dots\varphi^4(X_m)\right\rangle \\ &= \left(-\frac{\lambda}{4!}\right)^m \int_\Omega \mathrm{d}X_1\dots\mathrm{d}X_m \sum_{\{a,b\}} \left\langle \varphi(X_{a(1)})\varphi(X_{b(1)})\right\rangle\dots\left\langle \varphi(X_{a(2m)})\varphi(X_{b(2m)})\right\rangle\end{aligned}$$

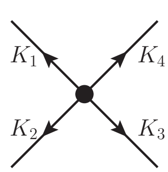
where $X_i$ for $i > m$ is defined as $X_j$, where $j = i \mod m$. This means that $X_{m+i} = X_i$. The functions $a$, $b$ represents a possible pairing, as described in section 1.1. Inserting the Fourier expansions of the field gives

$$\begin{aligned}&\langle {S_I}^m\rangle \\ &= \left(-\frac{\lambda}{4!}\right)^m \int_\Omega \mathrm{d}X_1\dots\mathrm{d}X_m (V\beta)^2 \int_{\tilde\Omega} \mathrm{d}K_1 ... \mathrm{d}K_{2m} \sum_{\{a,b\}} \\ &\qquad \left\langle \varphi(K_{a(1)})\varphi(K_{b(1)})\right\rangle\dots\left\langle \varphi(K_{a(2m)})\varphi(K_{b(2m)})\right\rangle \exp\left(i\sum\nolimits_{i=1}^{m} X_i\cdot K_i\right) \\ &= \left(-\frac{\lambda}{4!}\right)^m \frac{(V\beta)^{2m}\beta^m}{(V\beta^2)^{2m}} \int_{\tilde\Omega} \mathrm{d}K_1 ... \mathrm{d}K_{2m} \sum_{\{a,b\}} \\ &\qquad \tilde D(K_{a(1)})\delta(K_{a(1)}+K_{b(1)})\dots\tilde D(K_{a(2m)})\delta(K_{a(2m)}+K_{b(2m)}) \prod_{i=1}^{m}\delta\left(X_i\cdot\sum\nolimits_{j=0}^{3}K_{i+jm}\right) \\ &= \left(-\frac{\lambda\beta}{4!}\right)^m \prod_{i=1}^{2m}\int_{\tilde\Omega}\left(\mathrm{d}K_i\frac{1}{\beta}\tilde D(K_i)\right)\prod_{i=1}^{m}\delta\left(X_i\cdot\sum\nolimits_{j=0}^{3}K_{i+jm}\right)\sum_{\{a,b\}}\prod_{n=1}^{2m}\delta(K_{a(k)}+K_{b(k)})\end{aligned}$$

Here we have used that $V\beta^2\tilde D_0(K,P) = \tilde D_0(K)\delta(P+K)$, where $\tilde D_0(K)$ is the thermal propagator for the free field, as defined in section B.1. In this case, it is

$$\tilde D_0(K) = \tilde D_0(\omega_n,\vec k) = \frac{1}{\omega_k^2+\omega_n^2}. \tag{A.55}$$

This expectation value can be represented graphically using Feynman diagrams. The thermal $\lambda\varphi^2$-theory gets the prescription




$$= -\lambda\beta\delta\left(\sum\nolimits_i K_i\right), \tag{A.56}$$

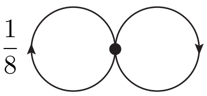
$K$ 

$$= \frac{1}{\beta}D_0(K). \tag{A.57}$$

Lastly, one has to integrate over all internal momenta, and divide by a symmetry factor of the diagram $s$, which is described in detail in [1].

Calculating $\langle {S_I}^n \rangle_0$ boils down to the sum of all possible Feynman diagrams with $n$ vertices. The first example is

$$\langle S_I \rangle = \frac{1}{8} \tag{A.58}$$



In section 1.1, we saw that the sum of all vacuum diagrams is the exponential of the sum of all *connected* diagrams, so the free energy of the interacting theory is given by

$$-\beta F = \ln(Z_0) + \Sigma(\text{all connected diagrams}) \tag{A.59}$$

## A.7 Fermions

The phase factor $e^{i\theta}$ that was introduced in section A.2 can be determined by studying the properties of the thermal Greens function. The thermal Greens function may be written

$$D(X_1, X_2) = D(\vec{x}, \vec{y}, \tau_1, \tau_2) = \left\langle e^{-\beta\hat{H}} \mathrm{T}\left\{\hat{\varphi}(X_1)\hat{\varphi}(X_2)\right\}\right\rangle.$$

$\mathrm{T}\{...\}$ is time-ordering operator, and is defined as

$$\mathrm{T}\left\{\varphi(\tau_1)\varphi(\tau_2)\right\} = \theta(\tau_1 - \tau_2)\varphi(\tau_1)\varphi(\tau_2) + \nu\theta(\tau_2 - \tau_1)\varphi(\tau_2)\varphi(\tau_1),$$

where $\nu = \pm 1$ for bosons and fermions respectively, and $\theta(\tau)$ is the Heaviside step function. In the same way that $i\hat{H}$ generates the time translation of a quantum field operator through $\hat{\varphi}(x) = \hat{\varphi}(t, \vec{x}) = e^{it\hat{H}}\hat{\varphi}(0, \vec{x})e^{-it\hat{H}}$, the imaginary-time formalism implies the relation

$$\hat{\varphi}(X) = \hat{\varphi}(\tau, \vec{x}) = e^{\tau\hat{H}}\hat{\varphi}(0, \vec{x})e^{-\tau\hat{H}}. \tag{A.60}$$

Using $\mathbb{1} = e^{\tau\hat{H}}e^{-\tau\hat{H}}$ and the cyclic property of the trace, we show that, assuming $\beta > \tau > 0$,

$$\begin{aligned}
G(\vec{x}, \vec{y}, \tau, 0) &= \left\langle e^{-\beta\hat{H}} \mathrm{T}\left\{\varphi(\tau, \vec{x})\varphi(0, \vec{y})\right\}\right\rangle \\
&= \frac{1}{Z}\,\mathrm{Tr}\Big\{e^{-\beta\hat{H}}\varphi(\tau, \vec{x})\varphi(0, \vec{y})\Big\} \\
&= \frac{1}{Z}\,\mathrm{Tr}\Big\{\varphi(0, \vec{y})e^{-\beta\hat{H}}\varphi(\tau, \vec{x})\Big\} \\
&= \frac{1}{Z}\,\mathrm{Tr}\Big\{e^{-\beta\hat{H}}e^{\beta\hat{H}}\varphi(0, \vec{y})e^{-\beta\hat{H}}\varphi(\tau, \vec{x})\Big\} \\
&= \frac{1}{Z}\,\mathrm{Tr}\Big\{e^{-\beta\hat{H}}\varphi(\vec{y}, \beta)\varphi(\tau, \vec{x})\Big\} \\
&= \nu\left\langle e^{-\beta\hat{H}} \mathrm{T}\left\{\varphi(\tau, \vec{x})\varphi(\beta, \vec{y})\right\}\right\rangle.
\end{aligned}$$

This implies that $\varphi(0, x) = \nu\varphi(\beta, \varphi)$, which shows that bosons are periodic in time, as stated earlier, while fermions are anti-periodic.

The Lagrangian density of a free fermion is

$$\mathcal{L} = \bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi. \tag{A.61}$$

This Lagrangian is invariant under the transformation $\psi \to e^{-i\alpha}\psi$, which by Nöther's theorem results in a conserved current

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta\psi = \bar{\psi}\gamma^\mu\psi. \tag{A.62}$$

The corresponding conserved charge is

$$Q = \int_V \mathrm{d}^3x\, j^0 = \int_V \mathrm{d}^3x\, \bar{\psi}\gamma^0\psi. \tag{A.63}$$

We can now use our earlier result for the thermal partition function, Eq. (A.12), only with the substitution $\mathcal{H} \to \mathcal{H} - \mu\bar{\psi}\gamma^0\psi$, and integrate over anti-periodic $\psi$'s:

$$Z = \mathrm{Tr}\Big\{e^{-\beta(\hat{H}-\mu\hat{Q})}\Big\} = \prod_{ab}\int \mathcal{D}\psi_a\mathcal{D}\pi_b \exp\bigg\{\int_\Omega \mathrm{d}X\ \Big(i\dot{\psi}\pi - \mathcal{H}(\psi,\pi) + \mu\bar{\psi}\gamma^0\psi\Big)\bigg\},$$

where $a, b$ are the spinor indices. The canonical momentum corresponding to $\psi$ is

$$\pi = \frac{\partial\mathcal{L}}{\partial(\partial_0\psi)} = i\bar{\psi}\gamma^0, \tag{A.64}$$

and the Hamiltonian density is

$$\mathcal{H} = \pi\dot{\psi} - \mathcal{L} = \bar{\psi}(-i\gamma^i\partial_i + m)\psi \tag{A.65}$$

which gives

$$\mathcal{L}_E = -i\pi\dot{\psi} + \mathcal{H}(\psi,\pi) - \mu\bar{\psi}\gamma^0\psi = \bar{\psi}[\gamma^0(\partial_\tau - \mu) - i\gamma^i\partial_i + m]\psi, \tag{A.66}$$

By using the Grassman-version of the Gaussian integral formula, the partition function can be written

$$\begin{aligned}
Z &= \prod_{ab}\int \mathcal{D}\psi_a\mathcal{D}\bar{\psi}_b \exp\bigg\{-\int_\Omega \mathrm{d}X\,\bar{\psi}\left[\tilde{\gamma}_0(\partial_\tau - \mu) - i\gamma^i\partial_i + m\right]\psi\bigg\}\\
&= C\prod_{ab}\int \mathcal{D}\tilde{\psi}_a\mathcal{D}\tilde{\bar{\psi}}_b \exp\bigg\{-\int_{\tilde{\Omega}} \mathrm{d}K\,\tilde{\bar{\psi}}\left[i\tilde{\gamma}_0(\omega_n + i\mu) + i\gamma_i p_i + m\right]\tilde{\psi}\bigg\}\\
&= C\prod_{ab}\int \mathcal{D}\tilde{\psi}_a\mathcal{D}\tilde{\bar{\psi}}_b e^{-\langle\tilde{\bar{\psi}}, D_0^{-1}\psi\rangle} = \det\big(D_0^{-1}\big).
\end{aligned}$$

In the second line, we have inserted the Fourier expansion of the field, as defined in section B.1, and changed variable of integration, as we did for the scalar field. The linear operator in this case is

$$D_0^{-1} = i\gamma^0(-i\partial_\tau + i\mu) - (-i\gamma^i)\partial_i + m = \beta[i\tilde{\gamma}_a p_a + m]. \tag{A.67}$$

This equality must be understood as an equality between linear operators, which are represented in different bases. We introduced the notation $p_{n;a} = (\omega_n + i\mu, p_i)$ and use the Euclidean gamma matrices, as defined in section B.1. We use the fact that

$$\det(i\tilde{\gamma}_a p_a + m) = \det\big(\gamma^5\gamma^5\big)\det(i\tilde{\gamma}_a p_a + m) = \det\big[\gamma^5(i\tilde{\gamma}_a p_a + m)\gamma^5\big] = \det(-i\tilde{\gamma}_a p_a + m),$$

Let $\tilde{D} = -i\tilde{\gamma}_a p_a + m$, which means we can write

$$Z = \sqrt{\det(D)\det\big(\tilde{D}\big)} = \sqrt{\det\big(D\tilde{D}\big)} = \det\big[\mathbb{1}(p_a p_a + m^2)\big]^{1/2}, \tag{A.68}$$

where we have used the anti-commutation rule for the Euclidean gamma-matrices, $\{\gamma_a, \gamma_b\} = 2\delta_{ab}$. It is important to keep in mind that the determinant here refers to linear operators on the space of spinor functions.

$$\begin{aligned}
\ln(Z) &= \ln\Big\{\det\big[\mathbb{1}(p_a p_a + m^2)\big]^{1/2}\Big\} = \frac{1}{2}\,\mathrm{Tr}\big\{\ln[\mathbb{1}(p_a p_a + m^2)]\big\}\\
&= \frac{1}{2}\int_{\tilde{\Omega}} \mathrm{d}K\,\ln[\mathbb{1}\beta^2(p_a p_a + m^2)]_{aa}
\end{aligned} \tag{A.69}$$

As the matrix within the logarithm is diagonal, the matrix-part of the trace is trivial, and the free energy may be written

$$\beta\mathcal{F} = -2\int_{\tilde{\Omega}} \mathrm{d}X\ \ln\{\beta^2[(\omega_n + i\mu)^2 + \omega_k^2]\}. \tag{A.70}$$

Using the fermionic version of the thermal sum from section A.4 gives the answer

$$\mathcal{F} = 2\int \frac{\mathrm{d}^3p}{(2\pi)^3}\left[\beta\omega_k + \frac{1}{\beta}\ln\Big(1 + e^{-\beta(\omega_k - \mu)}\Big) + \frac{1}{\beta}\ln\Big(1 + e^{-\beta(\omega_k + \mu)}\Big)\right]. \tag{A.71}$$

We see again that the temperature-independent part of the integral diverges, and must be regulated. There are two temperature-dependent terms, one from the particle and one from the anti-particle.

# Appendix B

# Appendices

## B.1 Conventions and notation

Throughout this text, natural units are employed, in which

$$\hbar = c = k_B = 1, \tag{B.1}$$

where $\hbar$ is the Planck reduced constant, $k_B$ is the Boltzmann constant and $c$ is the speed of light. The Minkowski metric convention used is the "mostly minus", $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.

The $\mathfrak{su}(2)$ basis used is the Pauli matrices,

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

They obey

$$[\tau_a, \tau_b] = 2i\varepsilon_{abc}\tau_c, \quad \{\tau_a, \tau_b\} = 2\delta_{ab}\mathbb{1}, \quad \mathrm{Tr}[\tau_a] = 0, \quad \mathrm{Tr}[\tau_a\tau_b] = 2\delta_{ab}\mathbb{1}.$$

Together with the identity matrix $\mathbb{1}$, the Pauli matrices form a basis for the vector space of all 2-by-2 matrices. An arbitrary 2-by-2 matrix $M$ may be written

$$M = M_0\mathbb{1} + M_a\tau_a, \quad M_0 = \frac{1}{2}\mathrm{Tr}\{M\}, \; M_a = \frac{1}{2}\mathrm{Tr}\{\tau_a M\}. \tag{B.2}$$

The gamma matrices $\gamma^\mu$, $\mu \in \{0, 1, 2, 3\}$, obey

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}. \tag{B.3}$$

The "fifth $\gamma$-matrix" is defined by

$$\gamma^5 = \frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = i\gamma^0\gamma^1\gamma^2\gamma^3. \tag{B.4}$$

The $\gamma^5$-matrix obey

$$\{\gamma^5, \gamma^\mu\} = 0, \quad (\gamma^5)^2 = \mathbb{1}, \quad {\gamma^0}^\dagger = \gamma^0, \; {\gamma^i}^\dagger = -\gamma^i \tag{B.5}$$

Their Euclidean counterpart obey

$$\{\tilde{\gamma}_a, \tilde{\gamma}_b\} = 2\delta_{ab}\mathbb{1}, \quad \tilde{\gamma}_a^\dagger = \tilde{\gamma}_a, \tag{B.6}$$

and they are related by $\tilde{\gamma}_0 = \gamma^0$, and $\tilde{\gamma}_j = -i\gamma^j$. The Euclidean $\tilde{\gamma}_5$ is defined as

$$\tilde{\gamma}_5 = \gamma_0\gamma_1\gamma_2\gamma_3 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5. \tag{B.7}$$

It thus also anti-commutes with the Euclidean $\gamma$-matrices.

### B.1.1 Fourier transform

The Fourier transform used in this text is defined by

$$\mathcal{F}\left\{f(x)\right\}(p) = \tilde{f}(p) = \int \mathrm{d}x\, e^{ipx} f(x), \quad \mathcal{F}^{-1}\left\{\tilde{f}(p)\right\}(x) = f(x) = \int \frac{\mathrm{d}p}{2\pi}\, e^{-ipx}\tilde{f}(p).$$

### B.1.2 Fourier series

Imaginary-time formalism is set in a Euclidean space $\Omega = [0, \beta] \times V$, where $V = L_x L_y L_z$ is a space-like volume. The possible momenta in this space are

$$\tilde{V} = \left\{ \vec{k} \in \mathbb{R}^3 \,\middle|\, \vec{k} = \left( \frac{2\pi n_x}{L_x}, \frac{2\pi n_y}{L_y}, \frac{2\pi n_z}{L_z} \right) \right\}$$

$\omega_n$ are the Matsubara-frequencies, with $\omega_n = 2n\pi/\beta$ for bosons and $\omega_n = (2n+1)\pi/\beta$ for fermions. They together form the reciprocal space $\tilde{\Omega} = \{\omega_n\} \times \tilde{V}$. The Euclidean coordinates are denoted $X = (\tau, \vec{x})$ and $K = (\omega_n, \vec{K})$, and have the dot product $X \cdot P = \omega_n \tau + \vec{k} \cdot \vec{x}$. In the limit $V \to \infty$, we follow the prescription

$$\frac{1}{V} \sum_{\vec{p} \in \tilde{V}} \to \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3}.$$

The sum over all degrees of freedom, and the corresponding integrals for the thermodynamic limit are

$$\begin{aligned} \frac{\beta V}{NM} \sum_{n=1}^{N} \sum_{\vec{x}_m \in V} &\xrightarrow{N,M \to \infty} \int_0^\beta \mathrm{d}\tau \int_{\mathbb{R}^3} \mathrm{d}^3 x = \int_\Omega \mathrm{d}X, \\ \frac{1}{V} \sum_{n=-\infty}^{\infty} \sum_{\vec{k} \in \tilde{V}} &\xrightarrow{V \to \infty} \sum_{n=-\infty}^{\infty} \int_{R^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} = \int_{\tilde{\Omega}} \mathrm{d}K. \end{aligned}$$

The convention used for the Fourier expansion of thermal fields is in accordance with [20]. The prefactor is chosen to make the Fourier components of the field dimensionless, which makes it easier to evaluate the trace correctly. For bosons, the Fourier expansion is

$$\begin{aligned} \varphi(X) =& \sqrt{V\beta} \int_{\tilde{\Omega}} \mathrm{d}K\, \tilde{\varphi}(K) e^{iX\cdot K} = \sqrt{\frac{\beta}{V}} \sum_{n=-\infty}^{\infty} \sum_{\vec{k} \in \tilde{V}} \tilde{\varphi}_n(\vec{p}) \exp\left\{ i(\omega_n \tau + \vec{x} \cdot \vec{k}) \right\}, \\ \tilde{\varphi}(K) =& \sqrt{\frac{1}{V\beta^3}} \int_{\tilde{\Omega}} \mathrm{d}X\, \tilde{\varphi}(X) e^{-iX\cdot K} \end{aligned}$$

while for Fermions it is

$$\psi(X) = \sqrt{V} \int_{\tilde{\Omega}} \mathrm{d}K\, \tilde{\psi}(K) e^{iX\cdot K} = \frac{1}{\sqrt{V}} \sum_{n=-\infty}^{\infty} \sum_{\vec{k} \in \tilde{V}} \psi(\omega_n, \vec{k}) \exp\left\{ i(\omega_n \tau + \vec{x} \cdot \vec{k}) \right\} \tag{B.8}$$

A often used identity is

$$\int_\Omega \mathrm{d}X e^{iX\cdot(K-K')} = \beta \delta_{nn'} (2\pi)^3 \delta^3(\vec{k} - \vec{k}') := \beta\delta(K - K'), \tag{B.9}$$

$$\int_{\tilde{\Omega}} \mathrm{d}K\, e^{iK(X-X')} = \beta\delta(\tau - \tau')\delta^3(\vec{x} - \vec{x}') := \beta\delta(X - X'). \tag{B.10}$$

### B.1.3 Particle distributions

The Bose distribution is defined as

$$n_B(\omega) = \frac{1}{e^{\beta\omega} - 1}. \tag{B.11}$$

This function obeys

$$n_B(-i\omega) = -1 - n_B(i\omega). \tag{B.12}$$

We can expand it around the Bose Matsubara frequencies on the imaginary line:

$$in_B[i(\omega_n + \epsilon)] = \frac{i}{e^{i\beta\epsilon + 2\pi i n} - 1} = i[i\beta\epsilon + \mathcal{O}\left(\epsilon^2\right)]^{-1} \sim \frac{1}{\epsilon\beta}. \tag{B.13}$$

This means that $in_B(i\omega)$ has a pole on all Matsubara-frequencies, with residue $1/\beta$. Furthermore, we have

$$\frac{\mathrm{d}}{\mathrm{d}\omega} \ln\left(1 - e^{-\beta\omega}\right) = \beta n_B(\omega). \tag{B.14}$$

The Fermi distribution is

$$n_F(\omega) = \frac{1}{e^{\beta\omega} + 1}. \tag{B.15}$$

It obeys

$$\frac{\mathrm{d}}{\mathrm{d}\omega} \ln\left(1 - e^{-\beta\omega}\right) = -\beta n_F(\omega), \tag{B.16}$$

$$n_F(-i\omega) = 1 - n_F(i\omega). \tag{B.17}$$

The two distributions are related by

$$2n_B(i\omega; 2\beta) - n_B(i\omega; \beta) = -n_F(i\omega; \beta). \tag{B.18}$$

### B.1.4 Propagators

If $D^{-1}[f(x)] = 0$ is the equation of motion for some field $f$, where $D^{-1}$ in general is a differential operator, then the propagator $D(x, x')$ for this field is defined by

$$D^{-1}[D(x, x')] = -i\delta(x - x')\mathbb{1}.$$

Assuming $A$ is linear and independent of space, we may redefine $D(x - x', 0) \to D(x - x')$, and the Fourier transform with respect to both $x$ and $x'$ to obtain

$$\mathcal{F}\left\{D^{-1}[D(x, x')]\right\}(p, p') = \tilde{D}^{-1}(p)\,\tilde{D}(p)\,\delta(p + p') = -i\delta(p + p'),$$

meaning the momentum space propagator $\tilde{D}(p) = \mathcal{F}\left\{D(x)\right\}(p)$ is given by $\tilde{D} = -i(\tilde{D}^{-1})^{-1}$.

For some differential operator $D^{-1}$, the thermal propagator is defined as

$$D^{-1}D(X, Y) = \beta\delta(X - Y). \tag{B.19}$$

The Fourier transformed propagator is, assuming $D(X, Y) = D(X - Y, 0)$,

$$\tilde{D}(K, K') = \frac{1}{V\beta^3}\int_\Omega \mathrm{d}X\mathrm{d}Y\, D(X, Y)\exp(-i[X \cdot K + Y \cdot K']) \tag{B.20}$$

$$= \frac{1}{V\beta^3}\int_\Omega \mathrm{d}X'\mathrm{d}Y'\, D(X', 0)\exp\left(-i[X' \cdot \frac{1}{2}(K - K') + Y \cdot (K + K')]\right) \tag{B.21}$$

$$= \frac{1}{V\beta^2}\tilde{D}(K)\delta(K + K'), \tag{B.22}$$

where

$$\tilde{D}(K) = \int \mathrm{d}X e^{iK\cdot X} D(X, 0). \tag{B.23}$$

## B.2 Covariant derivative

In $\chi$PT at finite isospin chemical potential $\mu_I$, the covariant derivative acts on functions $A(x) : \mathcal{M}_4 \to \mathrm{SU}(2)$, where $\mathcal{M}_4$ is the space-time manifold. It is defined as

$$\nabla_\mu A(x) = \partial_\mu A(x) - i\left[v_\mu, A(x)\right], \quad v_\mu = \frac{1}{2}\mu_I \delta^0_\mu \tau_3. \tag{B.24}$$

The covariant derivative obeys the product rule, as

$$\nabla_\mu(AB) = (\partial_\mu A)B + A(\partial_\mu B) - i\left[v_\mu, AB\right] = (\partial_\mu A - i\left[v_\mu, A\right])B + A(\partial_\mu B - i\left[v_\mu, B\right]) = (\nabla_\mu A)B + A(\nabla_\mu B).$$

Decomposing a 2-by-2 matrix $M$, as described in section B.1, shows that the trace of the commutator of $\tau_b$ and $M$ is zero:

$$\mathrm{Tr}\{[\tau_a, M]]\} = M_b \, \mathrm{Tr}\{[\tau_a, \tau_b]\} = 0.$$

Together with the fact that $\mathrm{Tr}\{\partial_\mu A\} = \partial_\mu \mathrm{Tr}\{A\}$, this gives the product rule for invariant traces:

$$\mathrm{Tr}\{A \nabla_\mu B\} = \partial_\mu \mathrm{Tr}\{AB\} - \mathrm{Tr}\{(\nabla_\mu A)B\}.$$

This allows for the use of the divergence theorem when doing partial integration. Let $\mathrm{Tr}\{K^\mu\}$ be a space-time vector, and $\mathrm{Tr}\{A\}$ scalar. Let $\Omega$ be the domain of integration, with coordinates $x$ and $\partial\Omega$ its boundary, with coordinates $y$. Then,

$$\int_\Omega \mathrm{d}x \, \mathrm{Tr}\{A \nabla_\mu K^\mu\} = \int_{\partial\Omega} \mathrm{d}y \, n_\mu \, \mathrm{Tr}\{A K^\mu\} - \int_\Omega \mathrm{d}x \, \mathrm{Tr}\{(\nabla_\mu A) K^\mu\},$$

where $n_\mu$ is the normal vector of $\partial\Omega$.[22] This makes it possible to do partial integration and discard surface terms in the $\chi$PT Lagrangian, given the assumption of no variation on the boundary.

## B.3 Integrals

A useful integral is the Gaussian integral,

$$\int_{\mathbb{R}} \mathrm{d}z \, \exp\left(-\frac{1}{2}az^2\right) = \sqrt{\frac{2\pi}{a}}, \tag{B.25}$$

for $a \in \mathbb{R}$. The imaginary version,

$$\int_R \mathrm{d}z \, \exp\left(i\frac{1}{2}az^2\right) \tag{B.26}$$

does not converge. However, if we change $a \to a + i\epsilon$, then the integrand is exponentially suppressed.

$$f(x) = \exp\left(i\frac{1}{2}ax^2\right) \to \exp\left(i\frac{1}{2}ax^2 - \frac{1}{2}\epsilon x^2\right), \tag{B.27}$$

As the integrand falls of exponentially for $x \to \infty$, and contains no poles in the upper right nor lower left quarter of the complex plane, we may perform a wick rotation by closing the contour as shown in Figure B.1. This gives the result

$$\int_{\mathbb{R}} \mathrm{d}x \, \exp\left(i\frac{1}{2}(a+i\epsilon)x^2\right) = \int_{\sqrt{i}\mathbb{R}} \mathrm{d}x \, \exp\left(i\frac{1}{2}ax^2\right) = \sqrt{i}\int_{\mathbb{R}} \mathrm{d}y \, \exp\left(-\frac{1}{2}(-a)y^2\right) = \sqrt{\frac{2\pi i}{(-a)}} \tag{B.28}$$

where we have made the change of variable $y = (1+i)/\sqrt{2}x = \sqrt{i}x$. In $n$ dimensions, the Gaussian integral formula generalizes to

$$\int_{\mathbb{R}^n} \mathrm{d}^n x \, \exp\left\{-\frac{1}{2}x_n A_{nm} x_m\right\} = \sqrt{\frac{(2\pi)^n}{\det(A)}}, \tag{B.29}$$

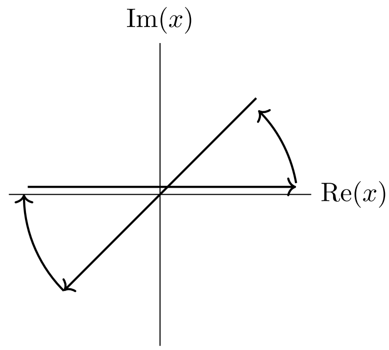



Figure B.1: Wick rotation

where $A$ is a matrix with $n$ real, positive eigenvalues. We may also generalize Eq. (B.28),

$$\int_{\mathbb{R}^n} \mathrm{d}^n x \, \exp\left\{ i\frac{1}{2} x_n (A_{nm} + i\epsilon\delta_{nm}) x_m \right\} = \sqrt{\frac{(2\pi i)^n}{\det(-A)}}. \tag{B.30}$$

The final generalization is to functional integrals,

$$\int \mathcal{D}\varphi \, \exp\left( -\frac{1}{2} \int \mathrm{d}x \, \varphi(x) A \varphi(x) \right) = C(\det(A))^{-1/2}, \int \mathcal{D}\varphi \, \exp\left( i\frac{1}{2} \int \mathrm{d}x \, \varphi(x) A \varphi(x) \right) = C(\det(-A))^{-1/2}. \tag{B.31}$$

$C$ is here a divergent constant, but will either fall away as we are only looking at the logarithm of $I_\infty$ and are able to throw away additive constants, or ratios between quantities which are both multiplied by $C$.

The Gaussian integral can be used for the stationary phase approximation. In one dimension, it is

$$\int \mathrm{d}x \, \exp(i\alpha f(x)) \approx \sqrt{\frac{2\pi}{f''(x_0)}} \exp(f(x_0)), \; f'(x) = 0, \; \alpha \to \infty \tag{B.32}$$

The functional generalization of this is

$$\int \mathcal{D}\varphi \exp\{iS[\varphi]\} \approx C \det\left( -\frac{\delta^2 S[\varphi_0]}{\delta \varphi^2} \right) \exp\{i\alpha S[\varphi_0]\}, \quad \frac{\delta S[\varphi_0]}{\delta \varphi} = 0 \tag{B.33}$$

Here, $S[\varphi]$ is a general functional of $\varphi$, and we have used the Taylor expansion, as described in section B.4, and $\varphi_0$ fulfills

$$\frac{\delta}{\delta\varphi(x)} S[\varphi_0] = 0, \tag{B.34}$$

## B.4 Functional Derivatives

Functional derivatives generalize the notion of a gradient and the directional derivative. A function $f(p)$, where $p$ is point with coordinates $x_i = x_i(p)$, has a gradient

$$\mathrm{d}f_p = \frac{\partial f(p)}{\partial x_i} \mathrm{d}x_i. \tag{B.35}$$

The derivative in a particular direction $v = v^i \partial_i$ is

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon} f(x_i + \epsilon v_i) = f(x) + \mathrm{d}f_x(v) = f(x) + \frac{\partial f}{\partial x^i} v_i. \tag{B.36}$$

This is generalized to functionals through the definition of functional derivative, and the variation of a functional. Let $F[f]$ be a functional, i.e. a machine that takes in a function, and returns a number. The obvious example in our case is the action, which takes in one or more field-configurations, and returns a single real number. We will assume here that the functions have the domain $\Omega$, with coordinates $x$. The functional derivative is defined as

$$\delta F[f] = \frac{\mathrm{d}}{\mathrm{d}\epsilon} F[f + \epsilon\eta]\Big|_{\epsilon=0} = \int_\Omega \mathrm{d}x\, \frac{\delta F[f]}{\delta f(x)} \eta(x). \tag{B.37}$$

$\eta(x)$ is here an arbitrary function, but we will make the important assumption that it as well as all its derivatives are zero at the boundary of its domain $\Omega$. This allows us to discard surface terms stemming from partial integration, which we will use frequently. We may use the definition to derive one of the fundamental relations of functional derivation. Take the functional $F[f] = f(x)$. Then,

$$\delta F[f] = \frac{\mathrm{d}}{\mathrm{d}\epsilon}[f(x) + \epsilon\eta(x)] = \eta(x) = \int \mathrm{d}y\, \delta(x-y)\eta(y) \tag{B.38}$$

This leads to the identity

$$\frac{\delta f(x)}{\delta f(y)} = \delta(x-y), \tag{B.39}$$

for any function $f$. Higher functional derivatives are defined similarly, by applying functional variation repeatedly

$$\delta^n F[f] = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\delta^{n-1} F[f + \epsilon\eta_n]\big|_{\epsilon=0} = \int \left(\prod_{i=1}^n \mathrm{d}x_i\right) \frac{\delta^n F[f]}{\delta f(x_n)\ldots\delta f(x_n)} \left(\prod_{i=1}^n \eta_i(x_i)\right). \tag{B.40}$$

A functional may be expanded in a generalization of the Fourier series, which has the form

$$F[f_0 + f] = F[f_0] + \int_\Omega \mathrm{d}x\, f(x) \frac{\delta F[f_0]}{\delta f(x)}\bigg|_{f=f_0} + \frac{1}{2!}\int_\Omega \mathrm{d}x\mathrm{d}y\, f(x) f(y) \frac{\delta^2 F[f_0]}{\delta f(x)\delta f(y)} + \ldots \tag{B.41}$$

As an example, the Klein-Gorodn action

$$S[\varphi] = -\frac{1}{2}\int_\Omega \mathrm{d}x\, \varphi(\partial^2 + m^2)\varphi(x) \tag{B.42}$$

can be evaluated quickly by using Eq. (B.38) and partial integration

$$\begin{aligned}\frac{\delta}{\delta\varphi(x)} S[\varphi] &= -\frac{1}{2}\int_\Omega \mathrm{d}y\, [\delta(x-y)(\partial_y^2 + m^2)\varphi(y) + \varphi(y)(\partial_y^2 + m^2)\delta(x-y)] \\ &= -\int_\Omega \mathrm{d}y\, \delta(x-y)(\partial_y^2 + m^2)\varphi(y) = (\partial_x^2 + m^2)\varphi(x)\end{aligned} \tag{B.43}$$

The second derivative is

$$\frac{\delta^2 S[\varphi]}{\delta\varphi(x)\delta\varphi(y)} = \frac{\delta}{\delta\varphi(x)}(\partial_y^2 + m^2)\varphi(y) = (\partial_y^2 + m^2)\delta(x-y). \tag{B.44}$$

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