Chiral Perturbation Theory

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1 Effective Pion Lagrangian

The technique used in χPT to obtain the effective Lagrangian of the pion relies on a "theorem", as formulated by Weinberg:

> [I]f one writes down the most general possible Lagrangian, including all terms consistent with assumed symmetry principles, and then calculates matrix elements with this Lagrangian to any given order of perturbation theory, the result will simply be the most general possible S-matrix consistent with analyticity, perturbative unitary, cluster decomposition and the assumed symmetry principles. [1]

In other words, if we write down the most general Lagrange density consistent with symmetries of the underlying theory, it will result in the most general S-matrix consistent with that theory, and important physical assumptions. This leaves a Lagrange density with infinitely many terms, and infinitely many free parameters. To be able to use this theory for anything one must have a method for ordering the terms in order of importance. As described in [2], by rescaling the external momenta $p_\mu \to t p_\mu$ and quark masses $m_i \to t^2 m_i$, each term in the Lagrangian obtains a factor t^D. The Lagrangian is then expanded as $\mathcal{L} = \sum_D \mathcal{L}_D$, where $\mathcal{L}_D$ contains all terms with a factor t^D.

In our case, the underlying theory is QCD with two quarks, up and down, with mass matrix

$$M = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}. \tag{1}$$

In the isospin limit, $m_u = m_d$, the theory is invariant under global transformations by elements of the group $G' = \mathrm{SU}\left(2\right)_L \times \mathrm{SU}\left(2\right)_R \times \mathrm{U}\left(1\right)_V$. All terms involving only pions are trivially invariant under $\mathrm{U}\left(1\right)_V$, (HVORFOR?) so we focus on the $G = \mathrm{SU}\left(2\right)_L \times \mathrm{SU}\left(2\right)_R$ subgroup. This symmetry is spontaneously broken if the quark field has a non-zero ground state expectation value $\langle \bar{q} q \rangle$, leaving only a subgroup $H = \mathrm{SU}\left(2\right)_V \subseteq G$ of symmetry transformations of the vacuum state. The Goldstone manifold $G/H = \mathrm{SU}\left(2\right)_A$ is a three-dimensional Lie group, and therefore results in three (pseudo) Goldstone bosons, the pions. There exists an isomorphism from a subset $S \subseteq M_1$ of the set of all Goldstone-fields

$$M_1 = \left\{ \pi_a : \mathcal{M}_4 \longrightarrow \mathbb{R}^3 | \pi_a \, \mathrm{smooth} \right\}$$

close to the ground state, into fields taking values in the Goldstone manifold G/H. (BEVISE?)(HVA ER ISOMORFISME HER?). The χPT effective Lagrangian will be constructed using this map, through the parametrization

$$\begin{aligned} \Sigma : \mathcal{M}_4 &\longrightarrow \mathrm{SU}\left(2\right), \\ x &\longrightarrow \Sigma(x) = A_\alpha (U(x) \Sigma_0 U(x)) A_\alpha, \end{aligned} \tag{2}$$

where

$$\Sigma_0 = \mathbb{1}, \; A_\alpha = \exp\left(\frac{i\alpha}{2}\tau_1\right), \; U(x) = \exp\left(i\frac{\tau_a \pi_a(x)}{2f}\right).$$

τ_a are the SU(2) generators, i.e. Pauli matrices, as described in Appendix A. π_a, where $a \in \{1, 2, 3\}$, are the pion fields. These are real fields, meaning $\pi_a^\dagger = \pi_a$.

1.1 Leading order Lagrangian

The leading order Lagrangian in χPT is [2, 3]

$$\mathcal{L}_2 = \frac{f^2}{4}\,\mathrm{Tr}\left[\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\right] + \frac{f^2}{4}\,\mathrm{Tr}\left[\chi^\dagger\Sigma + \Sigma^\dagger\chi\right]. \tag{3}$$

χ and f are the free parameters of the theory. f is the pion decay constant, while $\chi = 2B_0M$. Here, M is the mass matrix Eq. (1), and B_0 is related to the quark condensate through $f^2B_0 = -\langle\bar{u}u\rangle$. The covariant derivative is defined by

$$\nabla_\mu\Sigma = \partial_\mu\Sigma - i\left[v_\mu,\Sigma\right], \quad (\nabla_\mu\Sigma)^\dagger = \partial_\mu\Sigma^\dagger - i\left[v_\mu,\Sigma^\dagger\right], \quad v_\mu = \frac{1}{2}\mu_I\delta^0_\mu\tau_3,$$

where μ_I is the isospin chemical potential. To get the series expansion of Σ in powers of π/f, we start by using the fact that $\tau_a^2 = \mathbb{1}$ to write

$$A_\alpha = \sum_n^\infty \frac{1}{n!}\left(\frac{i\alpha}{2}\tau_1\right)^n = \sum_n^\infty\left[\frac{\mathbb{1}}{(2n)!}\left(\frac{i\alpha}{2}\right)^{(2n)} + \frac{\tau_1}{(2n+1)!}\left(\frac{i\alpha}{2}\right)^{(2n+1)}\right] = \mathbb{1}\cos\frac{\alpha}{2} + i\tau_1\sin\frac{\alpha}{2}. \tag{4}$$

The series expansion of U is

$$U = \exp\left(\frac{i\pi_a\tau_a}{2f}\right) = 1 + \frac{i\pi_a\tau_a}{2f} + \frac{1}{2}\left(\frac{i\pi_a\tau_a}{2f}\right)^2 + \frac{1}{6}\left(\frac{i\pi_a\tau_a}{2f}\right)^3 + \frac{1}{24}\left(\frac{i\pi_a\tau_a}{2f}\right)^4 + \mathcal{O}\left((\pi/f)^5\right),$$

which we use to calculate the expansion of the inner part of Σ, as given in Eq. (2),

$$\begin{aligned}
U\Sigma_0U &= \left(1 + \frac{i\pi_a\tau_a}{2f} + \frac{1}{2}\left(\frac{i\pi_a\tau_a}{2f}\right)^2 + \frac{1}{6}\left(\frac{i\pi_a\tau_a}{2f}\right)^3 + \frac{1}{24}\left(\frac{i\pi_a\tau_a}{2f}\right)^4\right)\\
&\quad\times\left(1 + \frac{i\pi_a\tau_a}{2f} + \frac{1}{2}\left(\frac{i\pi_a\tau_a}{2f}\right)^2 + \frac{1}{6}\left(\frac{i\pi_a\tau_a}{2f}\right)^3 + \frac{1}{24}\left(\frac{i\pi_a\tau_a}{2f}\right)^4\right) + \mathcal{O}\left((\pi/f)^5\right)\\
&= 1 + \frac{i\pi_a\tau_a}{f} + 2\left(\frac{i\pi_a\tau_a}{2f}\right)^2 + \frac{4}{3}\left(\frac{i\pi_a\tau_a}{2f}\right)^3 + \frac{2}{3}\left(\frac{i\pi_a\tau_a}{2f}\right)^4 + \mathcal{O}\left((\pi/f)^5\right).
\end{aligned}$$

The symmetry of $\pi_a\pi_b$ means that

$$(\pi_a\tau_a)^2 = \pi_a\pi_b\frac{1}{2}\left\{\tau_a,\tau_b\right\} = \pi_a\pi_a, \quad (\pi_a\tau_a)^3 = \pi_a\pi_a\pi_b\tau_b, \quad (\pi_a\tau_a)^4 = \pi_a\pi_a\pi_b\pi_b.$$

This gives us the expression

$$U\Sigma_0U = 1 + i\frac{\pi_a\tau_a}{f} - \frac{\pi_a^2}{2f^2} - i\frac{\pi_a^2\pi_b\tau_b}{6f^3} + \frac{\pi_a^2\pi_b^2}{24f^4} + \mathcal{O}\left((\pi/f)^5\right).$$

We combine this result with Eq. (4) to get an expression for Σ up to $\mathcal{O}\left((\pi/f)^5\right)$

$$\begin{aligned}
\Sigma &= \left(\cos\frac{\alpha}{2} + i\tau_1\sin\frac{\alpha}{2}\right)\left(1 + i\frac{\pi_a\tau_a}{f} - \frac{\pi_a^2}{2f^2} - i\frac{\pi_a^2\pi_b\tau_b}{6f^3} + \frac{\pi_a^2\pi_b^2}{24f^4}\right)\left(\cos\frac{\alpha}{2} + i\tau_1\sin\frac{\alpha}{2}\right)\\
&= \left(1 + i\frac{\pi_a\tau_a}{f} - \frac{\pi_a^2}{2f^2} - i\frac{\pi_a^2\pi_b\tau_b}{6f^3} + \frac{\pi_a^2\pi_b^2}{24f^4}\right)\cos^2\frac{\alpha}{2}\\
&- \left(1 + i\frac{\pi_a}{f}\tau_1\tau_a\tau_1 - \frac{\pi_a^2}{2f^2} - i\frac{\pi_a^2\pi_b}{6f^3}\tau_1\tau_b\tau_1 + \frac{\pi_a^2\pi_b^2}{24f^4}\right)\sin^2\frac{\alpha}{2}\\
&+ i\left(2\tau_1 + i\frac{\pi_a}{f}\left\{\tau_1,\tau_a\right\} - 2\tau_1\frac{\pi_a^2}{2f^2} - i\frac{\pi_a^2\pi_b}{6f^3}\left\{\tau_1,\tau_b\right\} + 2\tau_1\frac{\pi_a^2\pi_b^2}{24f^4}\right)\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}.
\end{aligned}$$

Using trigonometric identities and the commutator,

$$\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2} = \cos\alpha, \quad 2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2} = \sin\frac{\alpha}{2}, \quad \tau_1\tau_a\tau_1 = -\tau_a + 2\delta_{1a}\tau_1,$$

the final expression of Σ to $\mathcal{O}\left((\pi/f)^5\right)$ is

$$\Sigma = \left(1 - \frac{\pi_a^2}{2f^2} + \frac{\pi_a^2\pi_b^2}{24f^4}\right)(\cos\alpha + i\tau_1\sin\alpha) + \left(\frac{\pi_a}{f} - \frac{\pi_b^2\pi_a}{6f^3}\right)\left(i\tau_a - 2i\delta_{a1}\tau_1\sin^2\frac{\alpha}{2} - \delta_{a1}\sin\alpha\right). \tag{5}$$

The kinetic term in the χPT Lagrangian is

$$\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger = \partial_\mu\Sigma\partial^\mu\Sigma^\dagger - i\left(\partial_\mu\Sigma\left[v^\mu,\Sigma^\dagger\right] - \text{h.c.}\right) - [v_\mu,\Sigma]\left[v_\mu,\Sigma^\dagger\right]. \tag{6}$$

Using Eq. (5) we find the expansion of the constitutive parts of the kinetic term to be

$$\begin{aligned}
\partial_\mu\Sigma &= \left(\frac{-1}{f^2} + \frac{\pi_b^2}{6f^4}\right)(\cos\alpha + i\tau_1\sin\alpha)(\pi_a\partial_\mu\pi_a) \\
&\quad + \left(\frac{\partial_\mu\pi_a}{f} - \frac{\pi_b^2\partial_\mu\pi_a + 2\pi_a\pi_b\partial_\mu\pi_b}{6f^3}\right)\left(i\tau_a - 2i\delta_{a1}\tau_1\sin^2\frac{\alpha}{2} - \delta_{1a}\sin\alpha\right) \\
&= \left[\left(\frac{-1}{f^2} + \frac{\pi_b^2}{6f^4}\right)(\pi_a\partial_\mu\pi_a)\cos\alpha - \left(\frac{\partial_\mu\pi_1}{f} - \frac{\pi_b^2\partial_\mu\pi_1 + 2\pi_1\pi_b\partial_\mu\pi_b}{6f^3}\right)\sin\alpha\right] \\
&\quad - \left[\left(\frac{-1}{f^2} + \frac{\pi_b^2}{6f^4}\right)(\pi_a\partial_\mu\pi_a)\sin\alpha - \left(\frac{\partial_\mu\pi_1}{f} - \frac{\pi_b^2\partial_\mu\pi_1 + 2\pi_1\pi_b\partial_\mu\pi_b}{6f^3}\right)2\sin^2\frac{\alpha}{2}\right] i\tau_1 \\
&\quad + \left(\frac{\partial_\mu\pi_a}{f} - \frac{\pi_b^2\partial_\mu\pi_a + 2\pi_a\pi_b\partial_\mu\pi_b}{6f^3}\right) i\tau_a,
\end{aligned} \tag{7}$$

and

$$\begin{aligned}
[v_\mu,\Sigma] &= \frac{1}{2}\mu_I\delta_\mu^0\left[\left(1 - \frac{\pi_a^2}{2f^2} + \frac{\pi_a^2\pi_b^2}{24f^4}\right) i\sin\alpha\,[\tau_3,\tau_1] + \left(\frac{\pi_a}{f} - \frac{\pi_b^2\pi_a}{6f^3}\right)\left(i\,[\tau_a,\tau_3] - 2i\delta_{a1}\sin^2\frac{\alpha}{2}\,[\tau_3,\tau_1]\right)\right] \\
&= -\mu_I\delta_\mu^0\left\{\left(1 - \frac{\pi_a^2}{2f^2} + \frac{\pi_a^2\pi_b^2}{24f^4}\right)\tau_2\sin\alpha + \left(\frac{\pi_a}{f} - \frac{\pi_b^2\pi_a}{6f^3}\right)\left[(\delta_{a1}\tau_2 - \delta_{a2}\tau_1) - 2\delta_{a1}\tau_2\sin^2\frac{\alpha}{2}\right]\right\} \\
&= -\mu_I\delta_\mu^0\left\{\left[\left(1 - \frac{\pi_a^2}{2f^2} + \frac{\pi_a^2\pi_b^2}{24f^4}\right)\sin\alpha + \left(\frac{\pi_1}{f} - \frac{\pi_b^2\pi_1}{6f^3}\right)\cos\alpha\right]\tau_2 - \left(\frac{\pi_2}{f} - \frac{\pi_b^2\pi_2}{6f^3}\right)\tau_1\right\}.
\end{aligned} \tag{8}$$

Combining Eq. (7) and Eq. (8) gives the following terms [1]

$$\begin{aligned}
\mathrm{Tr}\left\{\partial_\mu\Sigma\partial^\mu\Sigma^\dagger\right\} &= \frac{2}{f^2}\partial_\mu\pi_a\partial^\mu\pi_a + \frac{2}{3f^4}\left[(\pi_a\partial_\mu\pi_a)(\pi_b\partial^\mu\pi_b) - (\pi_a\partial_\mu\pi_b)(\pi_b\partial^\mu\pi_a)\right], \\
-i\,\mathrm{Tr}\left\{\partial^\mu\Sigma\left[v_\mu,\Sigma^\dagger\right] - \text{h.c.}\right\} &= 4\mu_I\frac{\partial_0\pi_2}{f} + 8\mu_I\frac{\pi_3}{3f^3}\sin\alpha(\pi_2\partial_0\pi_3 - \pi_3\partial_0\pi_2)\sin\alpha \\
&\quad + \left(\frac{4\mu_I}{f^2}\cos\alpha - \frac{8\mu_I\pi_1}{3f^3}\sin\alpha - \frac{4\mu_I\pi_a\pi_a}{3f^4}\cos\alpha\right)(\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1), \\
-\mathrm{Tr}\left\{[v_\mu,\Sigma]\left[v^\mu,\Sigma^\dagger\right]\right\} &= {\mu_I}^2\left[2\sin^2\alpha + \left(\frac{2}{f} - \frac{4\pi_a\pi_a}{3f^3}\right)\pi_1\sin 2\alpha + \left(\frac{2}{f^2} - \frac{2\pi_a\pi_a}{3f^4}\right)\pi_a\pi_b k_{ab}\right], \\
\mathrm{Tr}\left\{\Sigma + \Sigma^\dagger\right\} &= 4\cos\alpha - \frac{4\pi_1}{f}\sin\alpha - \frac{2\pi_a\pi_a}{f^2}\cos\alpha + \frac{2\pi_1\pi_a\pi_a}{3f^3}\sin\alpha + \frac{(\pi_a\pi_a)^2}{6f^4}\cos\alpha,
\end{aligned}$$

where $k_{ab} = \delta_{a1}\delta_{b1}\cos 2\alpha + \delta_{a2}\delta_{b2}\cos^2\alpha - \delta_{a3}\delta_{b3}\sin^2\alpha$. If we write the Lagrangian as show in Eq. (3) as $\mathcal{L}_2 = \mathcal{L}_2^{(0)} + \mathcal{L}_2^{(1)} + \mathcal{L}_2^{(2)} + ...$, where $\mathcal{L}_2^{(n)}$ contains all terms of order $\mathcal{O}\left((\pi/f)^n\right)$, then the result of the series

[1] The scripts used to aid the calculation of the Lagrangian is available at `https://github.com/martkjoh/prosjektopggave`

expansion is

$$\mathcal{L}_2^{(0)} = f^2 \left(2B_0 m \cos\alpha + \frac{1}{2}\mu^2 \sin^2\alpha \right), \tag{9}$$

$$\mathcal{L}_2^{(1)} = f(\mu_I^2 \cos\alpha - 2B_0 m)\pi_1 \sin\alpha + f\mu_I \partial_0 \pi_2 \sin\alpha, \tag{10}$$

$$\mathcal{L}_2^{(2)} = \frac{1}{2}\partial_\mu \pi_a \partial^\mu \pi_a + \mu_I \cos\alpha\,(\pi_1 \partial_0 \pi_2 - \pi_2 \partial_0 \pi_1) - B_0 m \pi_a \pi_a \cos\alpha + \frac{1}{2}\mu_I^2 \pi_a \pi_b k_{ab}, \tag{11}$$

$$\begin{aligned}\mathcal{L}_2^{(3)} &= \frac{\pi_a \pi_a \pi_1}{6f}(2B_0 m \sin\alpha - 2{\mu_I}^2 \sin 2\alpha) \\ &- \frac{2\mu_I}{3f}\left[\pi_1(\pi_1 \partial_0 \pi_2 - \pi_2 \partial_0 \pi_1) + \pi_3(\pi_3 \partial_0 \pi_2 - \pi_2 \partial_0 \pi_3)\right] \sin\alpha,\end{aligned} \tag{12}$$

$$\begin{aligned}\mathcal{L}_2^{(4)} &= \frac{1}{6f^2}\left\{ \frac{1}{2}B_0 m (\pi_a \pi_a)^2 \cos\alpha - \left[(\pi_a \pi_a)(\partial_\mu \pi_b \partial^\mu \pi_b) - (\pi_a \partial_\mu \pi_a)(\pi_b \partial^\mu \pi_b)\right] \right\} \\ &- \frac{\mu_I \pi_a \pi_a}{3f^2}\left[(\pi_1 \partial_0 \pi_2 - \pi_2 \partial_0 \pi_1)\cos\alpha + \frac{1}{2}\mu_I \pi_a \pi_b k_{ab} \right].\end{aligned} \tag{13}$$

2 Effective Potential

2.1 Minimizing energy

The value of α is found by minimizing the free energy. The first approximation to the free energy in the ground state is the static part of the Hamiltonian density $\mathcal{H}^{(0)}$, which we get from Eq. (9) through

$$\mathcal{H}_2^{(0)} = -\mathcal{L}_2^{(0)} = -f^2 \left(2B_0 m \cos\alpha + \frac{1}{2}\mu^2 \sin^2\alpha \right), \tag{14}$$

The minimum of this function is achieved when

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\mathcal{H}_2^{(0)} = f^2 \left(2B_0 m - \mu_I^2 \cos\alpha \right) \sin\alpha = 0.$$

This gives the solution set and minimization criterion

$$\alpha = \pi n,\, n \in \mathbb{Z} \quad \text{or} \quad \cos\alpha = \frac{2B_0 m}{{\mu_I}^2}. \tag{15}$$

We see that the linear part of the potential from Eq. (10), $\mathcal{V}^{(1)} = f({\mu_I}^2 \cos\alpha - 2B_0 m)\pi_1 \sin\alpha = 0$ if and only if the criterion for minimization is fulfilled.

2.2 Propagator

We may write the quadratic part of the Lagrangian Eq. (11) as [2]

$$\mathcal{L}^{(2)} = \frac{1}{2}\sum_a \partial_\mu \pi_a \partial^\mu \pi_a + \frac{1}{2}m_{12}(\pi_1 \partial_0 \pi_2 - \pi_2 \partial_0 \pi_1) - \frac{1}{2}\sum_a m_a^2 \pi_a^2, \tag{16}$$

where

$$m_1^2 = 2B_0 m \cos\alpha - \mu_I^2 \cos 2\alpha, \tag{17}$$

$$m_2^2 = 2B_0 m \cos\alpha - \mu_I^2 \cos^2\alpha, \tag{18}$$

$$m_3^2 = 2B_0 m \cos\alpha + \mu_I^2 \sin^2\alpha, \tag{19}$$

$$m_{12} = 2\mu_I \cos\alpha. \tag{20}$$

[2] Summation over isospin index (a, b, c) will be explicit in this section.

The components of the Euler-Lagrange equations of this field are

$$\frac{\partial \mathcal{L}}{\partial \pi_a} = \frac{1}{2} m_{12}(\delta_{a1}\partial_0\pi_2 - \delta_{a2}\partial_0\pi_1) - m_a^2\pi_a, \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \pi_a)} = \partial^\mu \pi_a - \frac{1}{2} m_{12}\delta_0^\mu(\delta_{a1}\pi_2 - \delta_{a2}\pi_1).$$

This gives the equation of motion for the field

$$\partial^\mu\partial_\mu\pi_a + m_a^2\pi_a = m_{12}(\delta_{a1}\partial_0\pi_2 - \delta_{a2}\partial_0\pi_1). \tag{21}$$

The propagator of the pion field is defined by

$$\left[\delta_{ab}(\partial^\mu\partial_\mu + m_a^2) - m_{12}(\delta_{a1}\delta_{b2} - \delta_{a2}\delta_{b1})\partial_0\right] D_{bc}(x, x') = -i\delta(x - x')\delta_{ac}. \tag{22}$$

The momentum space propagator, as defined in the Appendix A, fulfills

$$-\left[\delta_{ab}(p^2 - m_a^2) + ip_0 m_{12}(\delta_{a1}\delta_{b2} - \delta_{a2}\delta_{b1})\right] \tilde{D}_{bc}(p) := A_{ab}\tilde{D}_{bc}(p) = -i\delta_{ac},$$

where

$$A = -\begin{pmatrix} p^2 - m_1^2 & ip_0 m_{12} & 0 \\ -ip_0 m_{12} & p^2 - m_2^2 & 0 \\ 0 & 0 & p^2 - m_3^2 \end{pmatrix}.$$

The spectrum of the particles is given by solving $\det(A) = 0$ for p^0. With $p = (p_0, P)$ as the four momentum, this gives

$$\det(A) = A_{33}\left(A_{11}A_{22} + A_{12}^2\right) = -\left(p^2 - m_3^2\right)\left[\left(p^2 - m_1^2\right)\left(p^2 - m_2^2\right) - p_0^2 m_{12}^2\right] = 0,$$

This equation has the solutions

$$E_0^2 = P^2 + m_2^2, \tag{23}$$

$$E_\pm^2 = P^2 + \frac{1}{2}\left(m_1^2 + m_2^2 + m_{12}^2\right) \pm \frac{1}{2}\sqrt{4P^2 m_{12}^2 + \left(m_1^2 + m_2^2 + m_{12}^2\right)^2 - 4m_1^2 m_2^2}. \tag{24}$$

This gives the effective masses

$$m_0^2 = m_2^2, \tag{25}$$

$$m_\pm^2 = \frac{1}{2}\left[m_1^2 + m_2^2 + m_{12}^2\right] \pm \frac{1}{2}\sqrt{\left(m_1^2 + m_2^2 + m_{12}^2\right)^2 - 4m_1^2 m_2^2}. \tag{26}$$

The propagator may then be obtained as described in Appendix A,

$$\begin{aligned} D_0 = iA^{-1} &= \frac{i}{\det(A)} \begin{pmatrix} A_{22}A_{33} & A_{12}A_{33} & 0 \\ -A_{12}A_{33} & A_{11}A_{33} & 0 \\ 0 & 0 & A_{11}A_{22} + A_{12}^2 \end{pmatrix} \\ &= i \begin{pmatrix} \frac{p^2 - m_2^2}{(p_0^2 - E_+^2)(p_0^2 - E_-^2)} & \frac{-ip_0 m_{12}}{(p_0^2 - E_+^2)(p_0^2 - E_-^2)} & 0 \\ \frac{ip_0 m_{12}}{(p_0^2 - E_+^2)(p_0^2 - E_-^2)} & \frac{p^2 - m_1^2}{(p_0^2 - E_+^2)(p_0^2 - E_-^2)} & 0 \\ 0 & 0 & \frac{1}{p_0^2 - E_0^2} \end{pmatrix}. \end{aligned} \tag{27}$$

2.3 Free energy of the pions

The partition function of χPT is

$$Z = \int \mathcal{D}\pi_a \, \exp\left\{ i \int \mathrm{d}x \mathcal{L}[\pi_a] \right\} = \int \mathcal{D}\pi_a \, \exp\left\{ i \int \mathrm{d}x \left(\mathcal{L}_2 + \mathcal{L}_4 + ...\right) \right\}. \tag{28}$$

Considering only the lowest order Lagrangian, it may be written expanded around $\pi_a = 0$

$$\mathcal{L}_2[\pi_a] = \mathcal{L}_2[\pi_a = 0] + \pi_a \frac{\partial \mathcal{L}_2}{\partial \pi_a}\Big|_{\pi_a = 0} + \pi_a\pi_b \frac{\partial^2 \mathcal{L}_2}{\partial \pi_a \partial \pi_b}\Big|_{\pi_a = 0} + \mathcal{O}\left((\pi/f)^3\right). \tag{29}$$

We showed that, when minimizing α, $\frac{\partial \mathcal{L}_2}{\partial \pi_a}\Big|_{\pi_a=0} = 0$. Furthermore, $\mathcal{L}_2[\pi_a = 0] = \mathcal{L}_2^{(0)}$ so the partition function may be written

$$Z = \int \mathcal{D}\varphi_a \, \exp\Big\{ i \int \Big(\mathcal{L}_a^{(0)} + \Big) \Big\} \tag{30}$$

The lowest order contribution to the pion is given by the free propagator, Eq. (27), by the formula **??**. This is, however, evaluated in the imaginary-time formalism. In the zero-temperature limit, This means that the time coordinate is replaced by $t \to -\tau$, and is restricted to $\tau \in [0, \beta]$. This result in a discrete set of energies, the Matsubara frequencies $\omega = 2\pi n/\beta$. The result is

$$\beta \mathcal{F} = \frac{1}{2V} \operatorname{Tr}\big\{ \ln[\beta^2 D_0^{-1}] \big\} = \frac{1}{2} \int_{\tilde{\Omega}} \mathrm{d}K \, \sum_a \ln[\beta^2 D_0^{-1}]_{aa} \tag{31}$$

At $T = 0$, this is given by (HVORFOR?)

$$\mathcal{F} = \frac{1}{2} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \sum_a \ln[D_0^{-1}] = \frac{1}{2} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \left[\ln\big(p_0^2 - E_+^2\big) + \ln\big(p_0^2 - E_-^2\big) + \ln\big(p_0^2 - E_0^2\big) \right] \tag{32}$$

References

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Appendix A Conventions and notation

Throughout this text, natural units are employed, in which

$$\hbar = c = k_B = 1, \tag{33}$$

where $\hbar$ is the Planck reduced constant, k_B is the Boltzmann constant and c is the speed of light. The Minkowski metric convention used is the "mostly minus", $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.

The $\mathfrak{su}\,(2)$ basis used is the Pauli matrices,

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

They obey

$$[\tau_a, \tau_b] = 2i\varepsilon_{abc}\tau_c, \quad \{\tau_a, \tau_b\} = 2\delta_{ab}\mathbb{1}, \quad \mathrm{Tr}[\tau_a] = 0, \quad \mathrm{Tr}[\tau_a\tau_b] = 2\delta_{ab}\mathbb{1}.$$

Together with the identity matrix $\mathbb{1}$, the Pauli matrices form a basis for the vector space of all 2-by-2 matrices. An arbitrary 2-by-2 matrix M may be written

$$M = M_0\mathbb{1} + M_a\tau_a, \quad M_0 = \frac{1}{2}\,\mathrm{Tr}\{M\},\ M_a = \frac{1}{2}\,\mathrm{Tr}\{\tau_a M\}. \tag{34}$$

The gamma matrices γ^μ, $\mu \in \{0, 1, 2, 3\}$, obey

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}. \tag{35}$$

The "fifth γ-matrix" is defined by

$$\gamma^5 = \frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = i\gamma^0\gamma^1\gamma^2\gamma^3. \tag{36}$$

The γ^5-matrix obey

$$\left\{\gamma^5, \gamma^\mu\right\} = 0, \quad (\gamma^5)^2 = \mathbb{1}, \quad {\gamma^0}^\dagger = \gamma^0,\ {\gamma^i}^\dagger = -\gamma^i \tag{37}$$

Their Euclidean counterpart obey

$$\{\tilde{\gamma}_a, \tilde{\gamma}_b\} = 2\delta_{ab}\mathbb{1}, \quad \tilde{\gamma}_a^\dagger = \tilde{\gamma}_a, \tag{38}$$

and they are related by $\tilde{\gamma}_0 = \gamma^0$, and $\tilde{\gamma}_j = -i\gamma^j$. The Euclidean $\tilde{\gamma}_5$ is defined as

$$\tilde{\gamma}_5 = \gamma_0\gamma_1\gamma_2\gamma_3 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5. \tag{39}$$

It thus also anti-commutes with the Euclidean γ-matrices.

Fourier transform

The Fourier transform used in this text is defined by

$$\mathcal{F}\left\{f(x)\right\}(p) = \tilde{f}(p) = \int \mathrm{d}x\, e^{ipx} f(x), \quad \mathcal{F}^{-1}\left\{\tilde{f}(p)\right\}(x) = f(x) = \int \frac{\mathrm{d}p}{2\pi}\, e^{-ipx}\tilde{f}(p).$$

Fourier series

Imaginary-time formalism is set in a Euclidean space $\Omega = [0, \beta] \times V$, where $V = L_x L_y L_z$ is a space-like volume. The possible momenta in this space are

$$\tilde{V} = \left\{\vec{k} \in \mathbb{R}^3 \,\middle|\, \vec{k} = \left(\frac{2\pi n_x}{L_x}, \frac{2\pi n_y}{L_y}, \frac{2\pi n_z}{L_z}\right)\right\}$$

ω_n are the Matsubara-frequencies, with $\omega_n = 2n\pi/\beta$ for bosons and $\omega_n = (2n+1)\pi/\beta$ for fermions. They together form the reciprocal space $\tilde{\Omega} = \{\omega_n\} \times \tilde{V}$. The Euclidean coordinates are denoted $X = (\tau, \vec{x})$ and $K = (\omega_n, \vec{K})$, and have the dot product $X \cdot P = \omega_n \tau + \vec{k} \cdot \vec{x}$. In the limit $V \to \infty$, we follow the prescription

$$\frac{1}{V} \sum_{\vec{p} \in \tilde{V}} \to \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3}.$$

The sum over all degrees of freedom, and the corresponding integrals for the thermodynamic limit are

$$\begin{aligned} \frac{\beta V}{NM} \sum_{n=1}^{N} \sum_{\vec{x}_m \in V} &\xrightarrow{N,M\to\infty} \int_0^\beta \mathrm{d}\tau \int_{\mathbb{R}^3} \mathrm{d}^3 x = \int_\Omega \mathrm{d}X, \\ \frac{1}{V} \sum_{n=-\infty}^{\infty} \sum_{\vec{k} \in \tilde{V}} &\xrightarrow{V\to\infty} \sum_{n=-\infty}^{\infty} \int_{R^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} = \int_{\tilde{\Omega}} \mathrm{d}K. \end{aligned}$$

The convention used for the Fourier expansion of thermal fields is in accordance with [4]. The prefactor is chosen to make the Fourier components of the field dimensionless, which makes it easier to evaluate the trace correctly. For bosons, the Fourier expansion is

$$\begin{aligned} \varphi(X) =& \sqrt{V\beta} \int_{\tilde{\Omega}} \mathrm{d}K\, \tilde{\varphi}(K) e^{iX\cdot K} = \sqrt{\frac{\beta}{V}} \sum_{n=-\infty}^{\infty} \sum_{\vec{k} \in \tilde{V}} \tilde{\varphi}_n(\vec{p}) \exp\Big\{ i(\omega_n \tau + \vec{x} \cdot \vec{k}) \Big\}, \\ \tilde{\varphi}(K) =& \sqrt{\frac{1}{V\beta^3}} \int_{\tilde{\Omega}} \mathrm{d}X\, \tilde{\varphi}(X) e^{-iX\cdot K} \end{aligned}$$

while for Fermions it is

$$\psi(X) = \sqrt{V} \int_{\tilde{\Omega}} \mathrm{d}K\, \tilde{\psi}(K) e^{iX\cdot K} = \frac{1}{\sqrt{V}} \sum_{n=-\infty}^{\infty} \sum_{\vec{k} \in \tilde{V}} \psi(\omega_n, \vec{k}) \exp\Big\{ i(\omega_n \tau + \vec{x} \cdot \vec{k}) \Big\} \tag{40}$$

A often used identity is

$$\int_\Omega \mathrm{d}X e^{iX\cdot(K-K')} = \beta \delta_{nn'} (2\pi)^3 \delta^3(\vec{k} - \vec{k}') := \beta\delta(K - K'), \tag{41}$$

$$\int_{\tilde{\Omega}} \mathrm{d}K\, e^{iK(X-X')} = \beta\delta(\tau - \tau')\delta^3(\vec{x} - \vec{x}') := \beta\delta(X - X'). \tag{42}$$

Bose distribution

The Bose distribution is defined as

$$n_B(\omega) = \frac{1}{e^{\beta\omega} - 1}. \tag{43}$$

This function obeys

$$n_B(-i\omega) = -1 - n_B(i\omega). \tag{44}$$

We can expand it around the Bose Matsubara frequencies on the imaginary line:

$$in_B[i(\omega_n + \epsilon)] = \frac{i}{e^{i\beta\epsilon + 2\pi i n} - 1} = i[i\beta\epsilon + \mathcal{O}\left(\epsilon^2\right)]^{-1} \sim \frac{1}{\epsilon\beta}. \tag{45}$$

This means that $in_B(i\omega)$ has a pole on all Matsubara-frequencies, with residue $1/\beta$. Furthermore, we have

$$\frac{\mathrm{d}}{\mathrm{d}\omega} \ln\!\big(1 - e^{-\beta\omega}\big) = \beta n_B(\omega). \tag{46}$$

Propagators

If $D^{-1}[f(x)] = 0$ is the equation of motion for some field f, where D^{-1} in general is a differential operator, then the propagator $D(x, x')$ for this field is defined by

$$D^{-1}[D(x, x')] = -i\delta(x - x')\mathbb{1}.$$

Assuming A is linear and independent of space, we may redefine $D(x - x', 0) \to D(x - x')$, and the Fourier transform with respect to both x and x' to obtain

$$\mathcal{F}\left\{D^{-1}[D(x, x')]\right\}(p, p') = \tilde{D}^{-1}(p)\,\tilde{D}(p)\,\delta(p + p') = -i\delta(p + p'),$$

meaning the momentum space propagator $\tilde{D}(p) = \mathcal{F}\left\{D(x)\right\}(p)$ is given by $\tilde{D} = -i(\tilde{D}^{-1})^{-1}$.

For some differential operator D^{-1}, the thermal propagator is defined as

$$D^{-1}D(X, Y) = \beta\delta(X - Y). \tag{47}$$

The Fourier transformed propagator is, assuming $D(X, Y) = D(X - Y, 0)$,

$$\begin{aligned}
\tilde{D}(K, K') &= \frac{1}{V\beta^3}\int_\Omega \mathrm{d}X\mathrm{d}Y\, D(X, Y)\exp(-i[X \cdot K + Y \cdot K']) && (48)\\
&= \frac{1}{V\beta^3}\int_\Omega \mathrm{d}X'\mathrm{d}Y'\, D(X', 0)\exp\Big(-i[X' \cdot \frac{1}{2}(K - K') + Y \cdot (K + K')]\Big) && (49)\\
&= \frac{1}{V\beta^2}\tilde{D}(K)\delta(K + K'), && (50)
\end{aligned}$$

where

$$\tilde{D}(K) = \int \mathrm{d}X e^{iK\cdot X} D(X, 0). \tag{51}$$

Appendix B Covariant derivative

In χPT at finite isospin chemical potential μ_I, the covariant derivative acts on functions $A(x) : \mathcal{M}_4 \to \mathrm{SU}\,(2)$, where $\mathcal{M}_4$ is the space-time manifold. It is defined as

$$\nabla_\mu A(x) = \partial_\mu A(x) - i\,[v_\mu, A(x)]\,, \quad v_\mu = \frac{1}{2}\mu_I\delta^0_\mu\tau_3. \tag{52}$$

The covariant derivative obeys the product rule, as

$$\nabla_\mu(AB) = (\partial_\mu A)B + A(\partial_\mu B) - i\,[v_\mu, AB] = (\partial_\mu A - i\,[v_\mu, A])B + A(\partial_\mu B - i\,[v_\mu, B]) = (\nabla_\mu A)B + A(\nabla_\mu B).$$

Decomposing a 2-by-2 matrix M, as described in Appendix A, shows that the trace of the commutator of τ_b and M is zero:

$$\mathrm{Tr}\{[\tau_a, M]]\} = M_b\,\mathrm{Tr}\{[\tau_a, \tau_b]\} = 0.$$

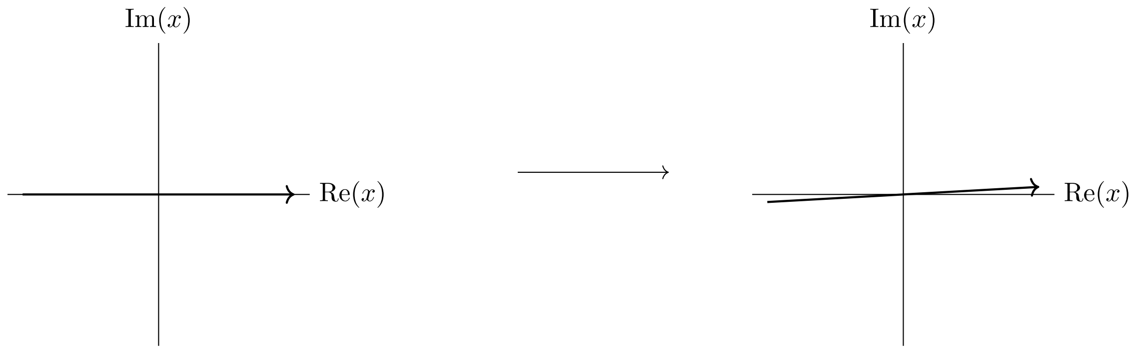
Together with the fact that $\mathrm{Tr}\{\partial_\mu A\} = \partial_\mu\,\mathrm{Tr}\{A\}$, this gives the product rule for invariant traces:

$$\mathrm{Tr}\{A\nabla_\mu B\} = \partial_\mu\,\mathrm{Tr}\{AB\} - \mathrm{Tr}\{(\nabla_\mu A)B\}.$$

This allows for the use of the divergence theorem when doing partial integration. Let $\mathrm{Tr}\{K^\mu\}$ be a space-time vector, and $\mathrm{Tr}\{A\}$ scalar. Let Ω be the domain of integration, with coordinates x and $\partial\Omega$ its boundary, with coordinates y. Then,

$$\int_\Omega \mathrm{d}x\ \mathrm{Tr}\{A\nabla_\mu K^\mu\} = \int_{\partial\Omega}\mathrm{d}y\, n_\mu\,\mathrm{Tr}\{AK^\mu\} - \int_\Omega \mathrm{d}x\ \mathrm{Tr}\{(\nabla_\mu A)K^\mu\},$$

where n_μ is the normal vector of $\partial\Omega$.[5] This makes it possible to do partial integration and discard surface terms in the χPT Lagrangian, given the assumption of no variation on the boundary.



Appendix C Integrals

C.1 Gaussian integrals

A useful integral is the Gaussian integral,

$$\int_{\mathbb{R}} \mathrm{d}x \, \exp\left(-\frac{1}{2}ax^2\right) = \sqrt{\frac{2\pi}{a}}, \tag{53}$$

for $a \in \mathbb{R}$. The imaginary version,

$$\int_{R} \mathrm{d}x \, \exp\left(i\frac{1}{2}ax^2\right) \tag{54}$$

does not converge. However, if we change the contour of integration slightly, by rotating it clockwise to $C = \mathbb{R}(1 + i\epsilon)$,