The thermodynamics of pion condensation at $T = 0$

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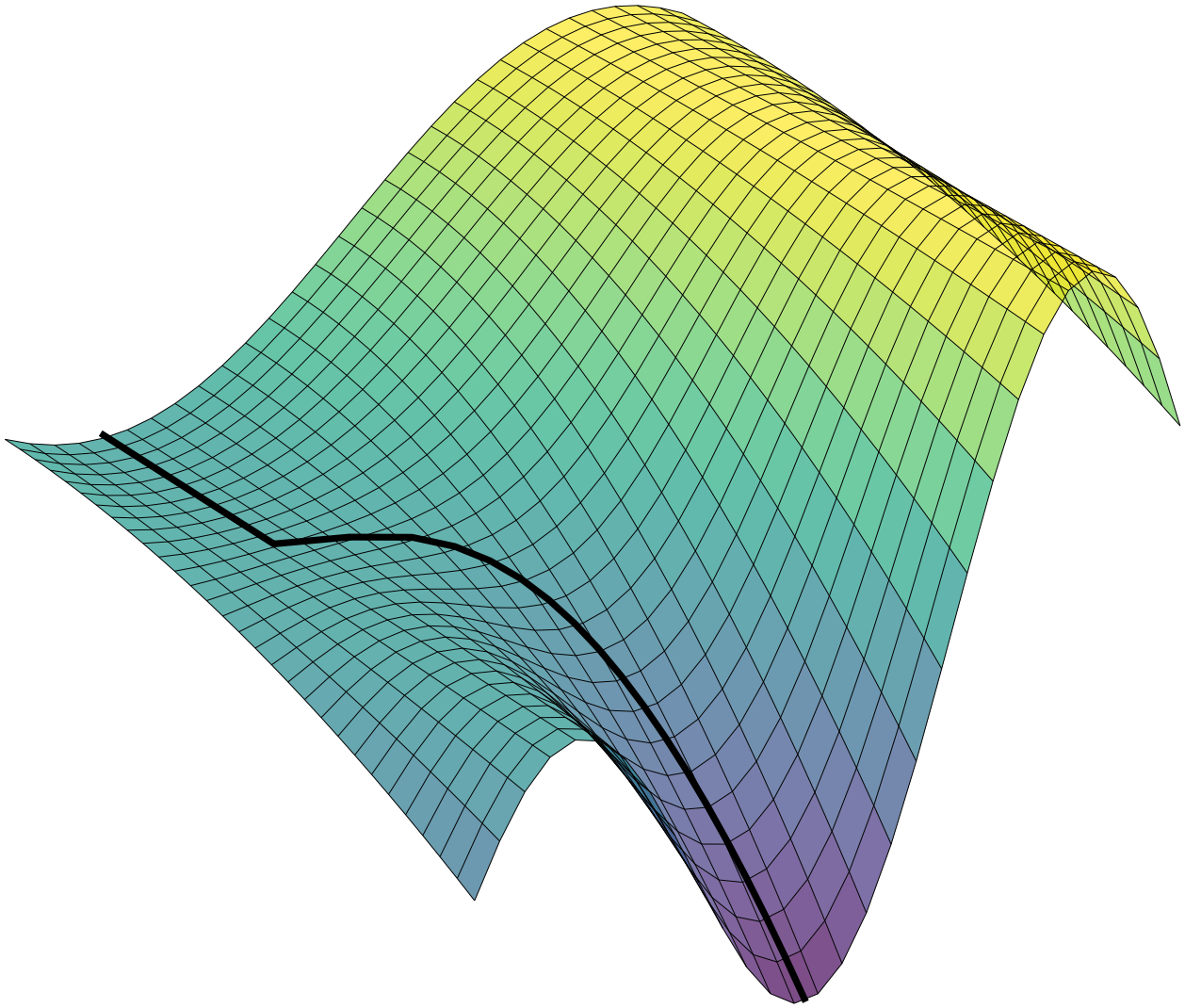


Abstract

Pions are particles that describe the dynamics of QCD at low energies, and it has recently been proposed that they can form compact stellar objects called pion stars. We use two-flavor chiral perturbation theory to calculate the grand canonical free energy density to next-to-leading order, at $T = 0$ and with non-zero isospin chemical potential μ_I. At $\mu_I = m_\pi$, a pion condensate is formed and spontaneously breaks the isospin symmetry of the QCD Lagrangian. We observe the resulting Goldstone mode. The pion condensate phase is characterized by a non-zero isospin density. We discuss the nature of the phase transition using Landua-theory of second-order phase transitions. The free energy density is used to obtain the relationship between the pressure and the energy density of the system, the equation of state. This, together with the Tolman-Oppenheimer-Volkoff equation, can be used to model pion stars, allowing for further investigation of these newly proposed objects.

Conventions and notation

Throughout this text, *natural units* are used. These units are defined so that

$$\hbar = c = k_B = 1, \tag{1}$$

where $\hbar$ is the Planck reduced constant, k_B is the Boltzmann constant, and c is the speed of light. These constants will therefore be dropped from all expressions. They can be reintroduced using dimensional analysis. In natural units, *mass dimension* is the only engineering dimension. Dimensionfull results are given in units of electronvolt (eV), or pion-masses,

$$m_\pi = 131\,\mathrm{MeV}. \tag{2}$$

The Minkowski metric convention used is the "mostly minus",

$$g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1). \tag{3}$$

The Fourier transform used in this text is defined by

$$\mathcal{F}\left\{f(x)\right\}(p) = \tilde{f}(p) = \int \mathrm{d}x\, e^{ipx} f(x), \quad \mathcal{F}^{-1}\left\{\tilde{f}(p)\right\}(x) = f(x) = \int \frac{\mathrm{d}p}{2\pi}\, e^{-ipx} \tilde{f}(p).$$

We employ the *Einstiein summation convention*, in which pairwise matching indices are summed. That is,

$$a_i b_i = \sum_i a_i b_i = a_1 b_1 + \dots. \tag{4}$$

For Minkowski-space indices, μ, ν, ρ and σ, the metric raises and lower indices, and summation should always be over one raised and one lowered index,

$$a_\mu b^\mu = g_{\mu\nu} a^\mu b^\nu = a^0 b^0 - a^1 b^1 - \dots. \tag{5}$$

Contents

Chapter 1

Introduction

The Standard Model is arguably the most successful theory in all of physics. It is the culmination of centuries, if not millennia, of scientific advancements. As it stands, it is humanity's best answer to the age-old question—what are the laws that govern the universe? The Standard Model is formulated in the language of quantum field theory (QFT) and describes the elementary particles and their interactions via the electromagnetic, weak, and strong forces. When including massive neutrinos in the standard model, it contains 26 free parameters. These are quantities, such as the masses of particles and the strength of the forces, which cannot be calculated using the Standard Model but must be measured and put in by hand. This might seem like a lot of freedom, but it is a vanishing amount when compared to the Standard Model's vast amount of precise prediction [1–3]. Figure 1.1 illustrates the particles in the Standard Model.

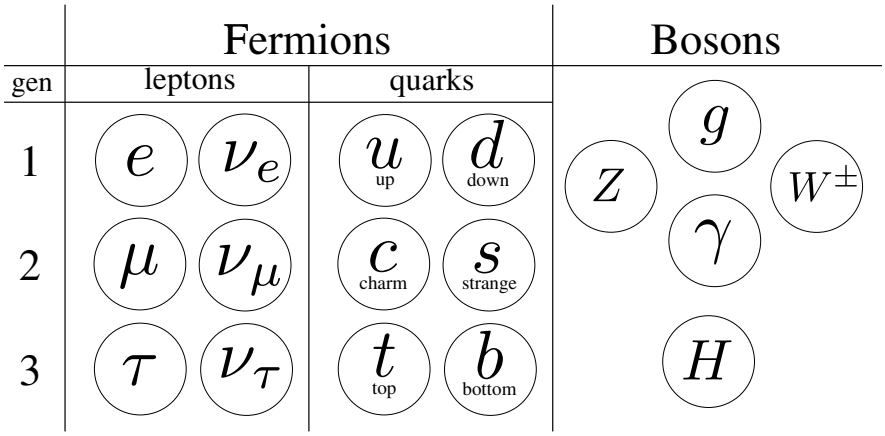


Figure 1.1: The particles of the Standard Model. The fermions are made up of two leptons and two quarks in each generation, with three generations. The photon (γ) is the particle of the electromagnetic force, the gluon (g) of the strong force, and the Z and $W^\pm$ bosons are responsible for the weak force. The final piece of the puzzle is the Higgs boson (H).

When we use quantum electrodynamics (QED), the quantum theory of electromagnetic interactions, and the theory of weak interactions to calculate observable quantities, we get highly precise predictions that agree with experiments to an astounding degree [1]. These calculations are done using perturbation theory. Perturbation theory is a calculation scheme available for weakly coupled interactions. It describes a process as the sum of all possible sequences of interactions that could give rise to that process. A Feynman diagram illustrates each such possible sequence. Together with the Feynman rules, each Feynman diagram is a recipe for calculating the contribution of that particular sequence to the total sum. Figure 1.2 illustrates the lowest order contributions to the process of electron-electron scattering. The weakness of QED is quantified in the fine structure constant, $\alpha \approx 0.007297$ [4], one of the free parameters mentioned earlier. Each vertex in a QED Feynman diagram is proportional to α. A diagram with n vertices is proportional to α^n. This tends to zero as n approaches infinity, and as a result, a complicated Feynman diagram—with many vertices and thus

also many loops—will generally give a small contribution to the overall sum. In the case of QED, calculating a few orders in perturbation theory yield precise and accurate estimates.[1]

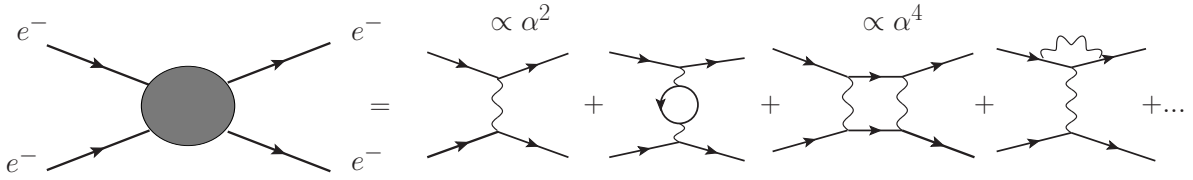


Figure 1.2: The process of electron-electron scattering is described by the summation of Feynman diagramms representing all possible ways for the electrons to interact. Higher order terms are less important, as they pick up more and more powers of α. Diagrams are drawn using [7].

Quantum chromodynamics, or QCD, is the part of the Standard Model that describes quarks, the constituents of protons and neutrons, and how they interact via the strong nuclear force. When dealing with the strong force, the fact that the strength of interaction depends on the energy scale becomes apparent. This dependence is due to what is called the *running* of the coupling constant. In high energy interactions at the energy scale of the Z-boson, $m_z \approx 91.19\,\text{GeV}$, the strong force equivalent to the fine structure constant is $\alpha_s(m_Z) \approx 0.118$ [4]. This makes it possible to do perturbative calculations using QCD. However, the strong force has its name for a reason. For scales around 1 GeV and below, the perturbation method breaks down. In this case, quarks bond together and form *hadrons*. Hadrons are classified as *mesons* or *baryons*. The most familiar of the baryons are the proton and neutrons.

None of the mesons are stable, but they play an essential role in describing the strong force in the non-perturbative regime. This is done using *effective field theories*, which we will cover in the following subsection. Mesons, of which pions are the lightest, were first proposed by Hideki Yukawa as the mechanism to hold nucleons together and form the nucleus of atoms. Though first believed to appear in the showers of particles created by cosmic rays, they were decisively discovered in 1947 by Cecil F. Powell *et al.* [2]. Pions do not show up in the standard model, as quarks do, but rather as an effective degree of freedom at low energy.

Effective field theories

A profound feature of physics is the possibility of describing a system by isolating the degrees of freedom of interest, ignoring the rest. We can describe the motion of the planets in the solar system, massive and complex systems, by only their mass, velocity, and position. In quantum field theory, this feature manifests in the power of effective field theory. An effective field theory describes a system not by the fundamental, underlying particles but by effective degrees of freedom. A theory of two interacting fields φ and ψ will be described by an action that depends on both fields, $S[\varphi, \psi]$. In the path integral formalism, predictions can then be made by integrating over all possible states of the fields. An example is the vacuum transition amplitude,

$$Z = \int \mathcal{D}\varphi \mathcal{D}\psi \, \exp\{iS[\varphi, \psi]\}. \tag{1.1}$$

We obtain an effective description of only the φ-degrees of freedom by *integrating out* the ψ-degrees of freedom, which results in an effective action $S_{\text{eff}}[\varphi]$, related to the underlying theory by [1]

$$\int D\varphi \, \exp\{iS_{\text{eff}}[\varphi]\} = \int \mathcal{D}\varphi \mathcal{D}\psi \, \exp\{iS[\varphi, \psi]\}. \tag{1.2}$$

This gives us hope for describing low-energy QCD as an effective theory of the particles we observe in experiments. In this specialization project, we will derive and explore chiral perturbation theory (χPT), a low-energy effective theory of QCD where mesons are the degrees of freedom.

[1] The magnitude of a complicated diagram is small, but there are also a lot of them. The inclusion of higher-order diagrams will eventually lead to large corrections, and the sum will diverge [5]. In QED, this is estimated to occur for diagrams of order $n \approx 1/\alpha \approx 137$, which is far more than we could hope to calculate at present. Perturbation theory must be interpreted, not as a convergent series, but an asymptotic expansion, which gives accurate estimates when truncated [6].

The action of the standard model has the form of an integral over a local Lagrangian,

$$S[\varphi] = \int \mathrm{d}^4x\, \mathcal{L}[\varphi], \tag{1.3}$$

where φ denotes all fundamental particles. The locality of $\mathcal{L}$ means that it is made up of terms like $\varphi(x)\varphi(x)$, where all interactions happen at one point in space-time, as opposed to a term such as $f(x,y)\varphi(x)\varphi(y)$. We can not a priori expect an effective action to take this form [1]. However, we have general principles we expect particles to obey, such Lorentz invariance and cluster decomposition. Cluster decomposition concerns a system of N sets of particles, α_i, that evolve into the sets β_i. That is,

$$|\alpha_1, \alpha_2, ...\alpha_N\rangle_{\text{in}} \longrightarrow |\beta_1, \beta_2, ...\beta_N\rangle_{\text{out}}\,. \tag{1.4}$$

It says that if the sets of particles α_i, β_i are located far enough apart, then the S-matrix factors as

$$\langle\beta_1, \beta_2, ...\beta_N|\alpha_1, \alpha_2, ...\alpha_N\rangle = \langle\beta_1|\alpha_1\rangle\,\langle\beta_2|\alpha_2\rangle\,...\,\langle\beta_N|\alpha_N\rangle\,. \tag{1.5}$$

This is a familiar property, as it essentially says that wildly separated experiments do not interfere, and one that we expect all good effective descriptions to have [8, 9]. These principles greatly constrain any effective action and are the basis for constructing the χPT effective action. This method was formulated by Weinberg [10] It relies on—as Weinberg himself called it—a "theorem":

> "[I]f one writes down the most general possible Lagrangian, including all terms consistent with assumed symmetry principles, and then calculates matrix elements with this Lagrangian to any given order of perturbation theory, the result will simply be the most general possible S-matrix consistent with analyticity, perturbative unitary, cluster decomposition and the assumed symmetry principles." [10]

In other words, if we write down the most general Lagrangian consistent with symmetries of the underlying theory, then we have not made any restrictions on the theory, other than some foundational assumptions. This Lagrangian will be of the form

$$\mathcal{L}_{\text{eff}}[\varphi] = \sum_i \lambda_i \mathcal{O}_i, \tag{1.6}$$

where $\mathcal{O}_i$ are local functions of the fields and their derivatives, and λ_i are coupling constants. The coupling constants are free parameters, which parametrizes the most general S-matrix consistent with foundational assumptions and the underlying theory. A Lagrangian with an infinite amount of free parameters might seem useless. However, if we can find a consistent series expansion, then only a finite number of terms are needed to calculate quantities to any given order in perturbation theory. In the case of χPT, the expansion is in the momentum of the pions. We will detail this later in the text.

Stars

Although it might seem counterintuitive, stars are one of the objects we might hope to describe using QCD at low energies. Neutron stars, one of the most extreme objects in the universe, quickly cool down to temperatures below 10^{10} K. This might be hot by almost all standards. However it corresponds to an energy of 0.862 MeV. This is well below the perturbative regime of QCD, and the stars must therefore be described by an effective model of interacting nuclear matter [11, 12]. Stars are modeled using the Tolman-Oppenheimer-Volkoff, or TOV, equations. The TOV equations are based on Einstein's general theory of relativity, and its solution gives the star's pressure as a function of its radius. The only input needed is the *equation of state*, or EOS, of the matter that makes up the star [13]. The equation of state of a system is the relationship between its energy density, u, and pressure P, i.e. a relationship of the form

$$f(P, u) = 0. \tag{1.7}$$

This is where QCD comes in. One way to compute the equation of state of QCD systems is using the numerical method called lattice QCD. Here, space-time is approximated as finite and discrete, and Monte-Carlo importance sampling is used to perform the path integral. This method, however, fails for non-zero

baryon chemical potential μ_B, in what is known as the fermion sign problem. The baryon chemical potential μ_B parametrizes the matter-antimatter asymmetry. A high value of μ_B corresponds to a high matter density. There is much more matter than antimatter in neutron stars, or more generally, all observed stars. This makes lattice QCD unsuited for simulating these systems. Recently, pions have been proposed to form a new type of compact, gravitationally bound object, i.e., a star. Pions can form states with zero baryon chemical potential, which are thus amenable to lattice QCD simulations. This offers a way to model stellar objects from first principles, as well as by analytical methods using χPT [14, 15].

Pion condensate and the QCD phase diagram

In addition to their electrical charge, pions have an *isospin* quantum number, $I_3 = -1, 0$ or 1, and a corresponding isospin chemical potential μ_I. This chemical potential parametrizes how much the system favors negative or positive isospin charges. When the isospin chemical potential reaches a critical value, $\mu_I = \mu_I^c$, the system undergoes a phase transition and form a pion condensate with a non-zero isospin density n_I. The pion condensate is an expample of Bose-Einstein condensation, in which a macroscopic number of bosons occupy a single quantum state [16–18].

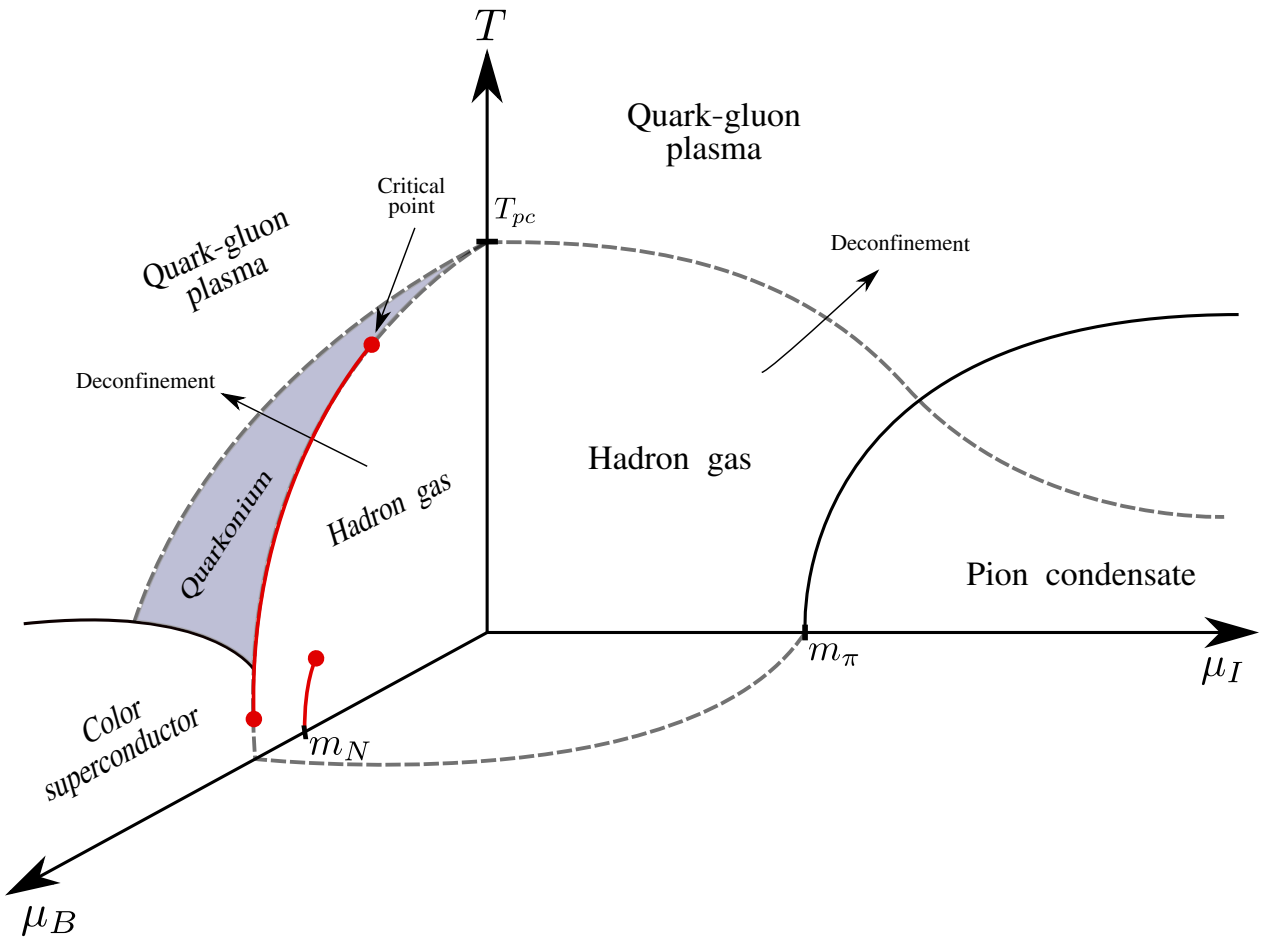


Figure 1.3: A sketch scetch of the phase diagram of QCD. See text for description. Based on [12, 16–20]

The pion condensate phase is just one part of the rich phase structure of QCD, which is an active field of research. In Figure 1.3, we show a rough sketch of this diagram. We emphasize that the understanding of this diagram is far from complete. The difficulties of exploring QCD, such as the non-perturbativity at low temperatures and the fermion sign problem mentioned in this text, means that much of the phase diagram is only conjectured. At low temperatures and chemical potentials, where the strong interaction is just that, strong, QCD-matter form a hadronic gas. This is called the normal phase or vacuum phase. Here, QCD is

best described by an effective theory of hadrons. QCD matter is in its vacuum phase in all but the most extreme situations. In the case where the hadronic description fails, under extreme temperatures or densities, *quark matter* is formed. Quark matter is conjectured to be present at the center of neutron stars [12], and there were done observations of a non-hadronic state of QCD-matter at the Relativistic Heavy Ion Collider (RHIC) in 2005 [21, 22].

At high temperatures, QCD-matter undergoes *deconfinment*, as the strong force weakens due to the running of the coupling constant. In this phase, quarks are no longer tightly bound in hadrons, but together with gluons form a soup called *quark-gluon plasma*. At $\mu_B = \mu_I = 0$, the crossover from the vacuum phase to a quark-gluon plasma is characterized by the pseudo-critical temperature, T_{pc}, which is estimated to be in the range of 150 MeV to 200 MeV [19]. At low temperatures and high baryon density, and thus high μ_B, the hadron gas undergoes a liquid-gas phase transition. This happens when $\mu_B \approx m_N = 939\,\text{MeV}$, where m_N is the neucleon mass [19]. At arbitrarily high μ_B, rigorous results are available which show QCD-matter form a color-superconducting phase. The color-superconductor is a state of matter analogous to electrical superconductors, in which electrons form Cooper pairs allowing for unimpeded electrical current. The color superconducting phase is due to Cooper pairs of quarks forming and results in effects such as the Meissner effect in which gluons can acquire mass. The nature of the transition between cold hadronic matter to a color-superconducting phase is not well understood [20].

Understanding the phase diagram of QCD is an integral part of research into the standard model and its consequences. It is essential to use all possible sound approaches to validate the techniques used. This allows for validation of methods by comparing results in the overlapping regimes, such as when comparing χPT with lattice QCD.

Outline of thesis

In this specialization project we calculate the next-to-leading order equation of state of a system at finite isospin chemical potential, using two-flavor chiral perturbation theory. We will also investigate the phase transition from the vacuum phase into a pion condensate phase. In chapter 2, we take a survey of some general theory needed for χPT. We introduce the generating functional in the path integral formalism and use this to define the one-particle-irreducible effective action and the effective potential. This allows us to prove Goldstone's theorem, a significant result that connects the symmetries of a theory to its low energy dynamics. The theorem states that spontaneous symmetry breaking gives rise to massless particles. We then present the CCWZ construction, which provides a procedure to construct an effective Lagrangian of Goldstone bosons. We also present some mathematical prerequisites, such as Lie groups and Lie algebras, and discuss the role and mathematical implementation of symmetries in physics in general and quantum field theory in particular.

In chapter 3, we take the general theory of the last chapter and apply it to QCD. We start the chapter by discussing QCD, its constituent parts, symmetries, and the corresponding conserved currents. We then use the theory from the last chapter to find the terms that make up the Lagrangian of χPT and incorporate explicit symmetry breaking, external source currents, and a finite isospin chemical potential. This section also discusses how to order these terms in a well-defined series expansion. With this, we construct the leading order and next-to-leading order Lagrangian, which is expanded in powers of the pion fields. We use our result to find properties of the pion, such as their tree-level mass and propagator.

chapter 4 is dedicated to the thermodynamic properties of χPT. We use the derived Lagrangian to calculate the free energy density to one loop using the leading order Lagrangian. Then, we use the tree level free energy density at next-to-leading order to renormalize the result. We discuss the low energy parameters we use and how to consistently evaluate observable to the same order in the series expansion. With the free energy density, we derive the equation of state at finite isospin chemical potential. We also discuss the phase transition to the pion condensate phase using the Landau theory of phase transitions.

In chapter 5, we summarize the results and discuss further work. The appendix is referenced throughout the text. In Appendix A, we review thermal field theory and the imaginary time formalism. This chapter contains details of calculations used in the main text, where they are referenced. We also discuss dimensional regularization, derive the Feynman rules for and interacting scalar, and generalize thermal field theory to

fermions. Appendix B, Appendix C and Appendix D contain additional material, and are referenced when relevant. In Appendix E, we link to an online repository containing all computer code used in this project.

Chapter 2

Theory

In this section, we survey some general properties of quantum field theory that are necessary for chiral perturbation theory. First, we introduce the path integral and the 1-particle irreducible effective action and the effective potential. We will derive Goldstone's theorem and present the CCWZ construction, which are the basis for χPT.

2.1 QFT via path integrals

This section is based on [1, 8, 9, 23]. Feynman diagrams are drawn using JaxoDraw [7].

In the path integral formalism, one evaluates quantum observable by integrating over the contributions of all possible configurations. If the system has specified initial and final states, this amounts to all possible paths the system might evolve between these, hence the name. We assume the reader has some familiarity with this formalism. However, if a refresher is needed, section A.2 contains a derivation of the closely related imaginary-time formalism and compares it with the path integral approach.

In the path integral formalism, the vacuum-to-vacuum transition amplitude, i.e., the probability that that vacuum at $t = -\infty$ evolves to the vacuum at time $t = \infty$, is given by

$$Z = \lim_{T\to\infty} \langle \Omega, T/2 | -T/2, \Omega\rangle = \lim_{T\to\infty} \langle \Omega | e^{-iHT} | \Omega\rangle = \int \mathcal{D}\pi\mathcal{D}\varphi \, \exp\left\{ i \int \mathrm{d}^4x \, (\pi\dot{\varphi} - \mathcal{H}[\pi,\varphi]) \right\}, \tag{2.1}$$

where $|\Omega\rangle$ is the vacuum state. The φ are the fields of the theory, and π their canonical momenta. We will work as if φ are a bosonic field. However, this can be readily generalized to fermions. By introducing a source term into the Hamiltonian density, $\mathcal{H} \to \mathcal{H} - J(x)\varphi(x)$, we get the generating functional

$$Z[J] = \int \mathcal{D}\pi\mathcal{D}\varphi \, \exp\left\{ i \int \mathrm{d}^4x \, (\pi\dot{\varphi} - \mathcal{H}[\pi,\varphi] + J\varphi) \right\}. \tag{2.2}$$

If $\mathcal{H}$ is quadratic in π, we can complete the square and integrate out π to obtain

$$Z[J] = C \int \mathcal{D}\varphi \, \exp\left\{ i \int \mathrm{d}^4x \, (\mathcal{L}[\varphi] + J\varphi) \right\}. \tag{2.3}$$

C is infinite, but constant, and will drop out of physical quantities. In scattering theory, the main objects of study are correlation functions $\langle\varphi(x_1)\varphi(x_2)...\rangle = \langle\Omega|T\{\varphi(x_1)\varphi(x_2)\dots\}|\Omega\rangle$, where T is the time ordering operator. These are given by functional derivatives of $Z[J]$,

$$\langle\varphi(x_1)\varphi(x_2)...\rangle = \frac{\int \mathcal{D}\varphi(x)\,[\varphi(x_1)\varphi(x_2)...]e^{iS[\varphi]}}{\int \mathcal{D}\varphi(x)\,e^{iS[\varphi]}} = \frac{1}{Z[0]} \prod_i \left(-i\frac{\delta}{\delta J(x_i)} \right) Z[J]\Big|_{J=0}, \tag{2.4}$$

where

$$S[\varphi] = \int \mathrm{d}^4x\, \mathcal{L}[\varphi] \tag{2.5}$$

is the action of the theory. The functional derivative is described in Appendix C. In a free theory, we are able to write

$$Z_0[J] = Z_0[0]\exp(iW_0[J]), \quad iW_0[J] = -\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4y\, J(x)D_0(x-y)J(y), \tag{2.6}$$

where D_0 is the propagator of the free theory. Using this form of the generating functional, Eq. (2.4) becomes

$$\begin{aligned}
&\frac{1}{Z[0]}(-i)^n\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_n)}Z_0[J]\Big|_{J=0} = (-i)^n\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_n)}e^{iW_0[J]}\Big|_{J=0}\\
&= (-i)^n\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_{n-1})}\left(i\frac{\delta W_0[J]}{\delta J(x_n)}\right)e^{iW_0[J]}\Big|_{J=0}\\
&= (-i)^n\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_{n-2})}\left(i\frac{\delta^2 W_0[J]}{\delta J(x_{n-1})\,\delta J(x_n)} + i^2\frac{\delta W_0[J]}{\delta J(x_{n-1})}\frac{\delta W_0[J]}{\delta J(x_n)}\right)e^{iW_0[J]}\Big|_{J=0} = \dots\\
&= (-i)^{\lfloor n/2\rfloor}\sum_{(a,b)}\prod_{i=1}^{\lfloor n/2\rfloor}\frac{\delta^2 W_0[J]}{\delta J(x_{a(i)})\,\delta J(x_{b(i)})}\Big|_{J=0}.
\end{aligned}$$

In the last line we have introduced the functions a, b, which define a way to pair up n elements. $\lfloor\cdot\rfloor$ is the floor function. The domain of these functions are the integers between 1 and $\lfloor n/2\rfloor$, the image a subset of the integers between 1 and n of size $\lfloor n/2\rfloor$. A valid pairing is a set $\{(a(1),b(1)),\dots(a(\lfloor n/2\rfloor),b(\lfloor n/2\rfloor))\}$, where all elements $a(i)$ and $b(j)$ are different, such that all integers up to and including n are featured. A pair is not directed, so $(a(i),b(i))$ is the same pair as $(b(i),a(i))$. The sum is over the set $\{(a,b)\}$ of all possible, unique pairings. If n is odd, the expression is equal to 0. This is Wick's theorem, and it can more simply be stated as *a correlation function is the sum of all possible pairings of 2-point functions*,

$$\left\langle\prod\nolimits_{i=1}^{n}\varphi(x_i)\right\rangle_0 = \sum_{\{(a,b)\}}\prod_{i=1}^{\lfloor n/2\rfloor}\left\langle\varphi(x_{a(i)})\varphi(x_{b(i)})\right\rangle_0. \tag{2.7}$$

The subscript on the expectation value indicates that it is evaluated in the free theory.

If we have an interacting theory, that is, a theory with an action $S = S_0 + S_I$, where S_0 is a free theory, the generating functional can be written

$$Z[J] = Z_0[0]\left\langle\exp\left(iS_I + i\int\mathrm{d}^4x\, J(x)\varphi(x)\right)\right\rangle_0. \tag{2.8}$$

We can expand the exponential in power series, which means the expectation value in Eq. (2.8) becomes

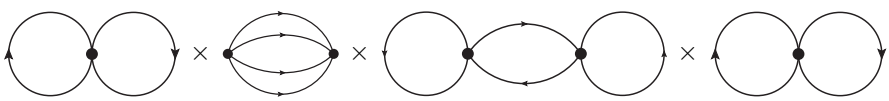
$$\sum_{n,m}\frac{1}{n!m!}\left\langle(iS_I)^n\left(i\int\mathrm{d}^4x\, J(x)\varphi(x)\right)^m\right\rangle_0. \tag{2.9}$$

The terms in this series are represented by Feynman diagrams, constructed using the Feynman rules, and can be read from the action. We will not further detail how the Feynman rules are derived. The Feynman rules for a free scalar field in thermal field theory are derived in Appendix A.4, and the general procedure is found in any of the main sources for this section [1, 8, 9, 23]. The source terms gives rise to an additional vertex

. (2.10)

The generating functional $Z[J]$ thus equals $Z_0[0]$ times *the sum of all diagrams with external sources* $J(x)$.

Consider a general diagram without external legs, built up of N different connected subdiagrams, where subdiagram i appears n_i times. As an illustration, a generic vacuum diagram in φ^4-theory has the form

$\mathcal{M} =$  $\times\dots$ (2.11)

If sub-diagram i as a stand-alone diagram equals $\mathcal{M}_i$, each copy of that subdiagram contributes a factor $\mathcal{M}_i$ to the total diagram. However, due to the symmetry of permuting identical subdiagrams, one must divide by the extra symmetry factor $s = n_i!$, the total number of permutations of all the copies of diagram i. The full diagram therefore equals

$$\mathcal{M} = \prod_{i=1}^{N} \frac{1}{n_i!} \mathcal{M}_i^{n_i}. \tag{2.12}$$

$\mathcal{M}$ is uniquely defined by a finite sequence of integers, $(n_1, n_2, \dots n_N, 0, 0, \dots)$, so the sum of all diagrams is the sum over the set S of all finite sequences of integers. This allows us to write the sum of all diagrams as

$$\sum_{(n_1,\dots)\in S} \prod_i \frac{1}{n_i!} \mathcal{M}_i^{n_i} = \prod_{i=1}^{\infty} \sum_{n_i=1}^{\infty} \frac{1}{n_i!} \mathcal{M}_i^{n_i} = \exp\Big(\sum\nolimits_i \mathcal{M}_i\Big). \tag{2.13}$$

We showed that the generating functional $Z[J]$ were the $Z_0[0]$ times the sum of all diagrams due to external sources. From Eq. (2.13), if we define

$$Z[J] = Z_0[0] \exp(iW[J]), \tag{2.14}$$

then $W[J]$ is the sum of all connected diagrams. This is trivially true for the free theory, where the only connected diagram is

$W_0[J] = J(x)$ $J(y)$. (2.15)

The two-point function in the full, interacting theory can thus be written

$$-i \frac{\delta^2 W[J]}{\delta J(x)\, \delta J(y)} = D(x-y). \tag{2.16}$$

The 1PI effective action

The generating functional for connected diagrams, $W[J]$, is dependent on the external source current J. We can define a new quantity with a different independent variable, using the Legendre transformation analogously to what is done in thermodynamics and Lagrangian mechanics. The new independent variable is

$$\varphi_J(x) := \frac{\delta W[J]}{\delta J(x)} = \langle \varphi(x) \rangle_J \,. \tag{2.17}$$

The subscript J on the expectation value indicate that it is evaluated in the presence of a source. The Legendre transformation of W is then

$$\Gamma[\varphi_J] = W[J] - \int \mathrm{d}^4x\, J(x) \varphi_J(x). \tag{2.18}$$

Using the definition of φ_J, we have that

$$\frac{\delta}{\delta \varphi_J(x)} \Gamma[\varphi_J] = \int \mathrm{d}^4y\, \frac{\delta J(y)}{\delta \varphi_J(x)} \frac{\delta}{\delta J(y)} W[J] - \int \mathrm{d}^4y\, \frac{\delta J(y)}{\delta \varphi_J(x)} \varphi_J(y) - J(x) = -J(x). \tag{2.19}$$

If we compare this to the classical equations of motion of a field φ with the action S,

$$\frac{\delta S[\varphi]}{\delta \varphi(x)} = -J(x), \tag{2.20}$$

we see that Γ is an action that gives the equation of motion for the expectation value of the field, given a source current $J(x)$.

To interpret Γ further, we observe what happens if we treat $\Gamma[\varphi]$ as a classical action with a coupling g. The generating functional in this new theory is

$$Z[J, g] = \int \mathcal{D}\varphi \exp\left\{ ig^{-1} \left(\Gamma[\varphi] + \int \mathrm{d}^4x\, \varphi(x) J(x) \right) \right\} \tag{2.21}$$

The free propagator in this theory will be proportional to g, as it is given by the inverse of the equation of motion for the free theory. All vertices in this theory, on the other hand, will be proportional to g^{-1}, as they are given by the higher-order terms in the action $g^{-1}\Gamma$. This means that a diagram with V vertices and I internal lines is proportional to g^{I-V}. Regardless of what the Feynman-diagrams in this theory are, the number of loops of a connected diagram is[1]

$$L = I - V + 1. \tag{2.22}$$

To see this, we first observe that diagrams with one single loop must have equally many internal lines as vertices, so the formula holds for $L = 1$. The formula still holds if we add a new loop to a diagram with n loops by joining two vertices. If we attach a new vertex with one line, the formula still holds, and as the diagram is connected, any more lines connecting the new vertex to the diagram will create additional loops. This ensures that the formula holds by induction. As a consequence of this, any diagram is proportional to g^{L-1}. This means that in the limit $g \to 0$, the theory is fully described at the tree-level, i.e., by only considering diagrams without loops. In this limit, we may use the stationary phase approximation, as described in Appendix C, which gives

$$Z[J, g \to 0] \approx C \det\left(-\frac{\delta^2\Gamma[\varphi_J]}{\delta\varphi(x)\,\delta\varphi(y)}\right) \exp\left\{ig^{-1}\left(\Gamma[\varphi_J] + \int \mathrm{d}^4x\, J(x)\varphi_J(x)\right)\right\}. \tag{2.23}$$

This means that

$$-ig\ln(Z[J,g]) = gW[J,g] = \Gamma[\varphi_J] + \int \mathrm{d}^4x\, J(x)\varphi_J(x) + \mathcal{O}(g), \tag{2.24}$$

which is exactly the Legendre transformation we started out with, modulo the factor g. Γ is, therefore, the action that describes the full theory at the tree level. For a free theory, the classical action S equals the effective action.

As we found in the last section, the propagator $D(x,y) = \langle\varphi(x)\varphi(y)\rangle_J$ is given by $-i$ times the second functional derivative of $W[J]$. Using the chain rule, together with Eq. (2.19), we get

$$(-i)\int \mathrm{d}^4z \frac{\delta^2 W[J]}{\delta J(x)\delta J(z)} \frac{\delta^2\Gamma[\varphi_J]}{\delta\varphi_J(z)\varphi_J(y)} = (-i)\int \mathrm{d}^4z \frac{\delta\varphi_J[z]}{\delta J(x)} \frac{\delta^2\Gamma[\varphi_J]}{\delta\varphi_J(z)\varphi_J(y)} = \frac{\delta}{\delta J(x)}\frac{\delta\Gamma[\varphi_J]}{\delta\varphi_J(y)} = i\delta(x-y). \tag{2.25}$$

This is exactly the definition of the inverse propagator,

$$\frac{\delta^2\Gamma[\varphi_J]}{\delta\varphi_J(x)\,\delta\varphi_J(y)} = D^{-1}(x,y). \tag{2.26}$$

The inverse propagator is the sum of all one-particle-irreducible (1PI) diagrams, with two external vertices. More generally, Γ is the generating functional for 1PI diagrams, which is why it is called the 1PI effective action.

Γ may be viewed as an effective action as defined in the introduction. We define η as the fluctuations around the expectation value of the field, $\varphi(x) = \varphi_J(x) + \eta(x)$, and use this to change variables of integration in the path integral. The expectation value φ_J is constant with respect to the integral, so

$$\int \mathcal{D}\varphi\, \exp\{iS[\varphi]\} = \int \mathcal{D}\eta\, \exp\{iS[\varphi_J + \eta]\}. \tag{2.27}$$

By assumption, $\langle\eta\rangle_J = 0$, which means this path integral is described by only 1PI diagrams, connected or not. We can therefore write

$$\exp\{i\Gamma[\varphi_J]\} = \int \mathcal{D}\eta\, \exp\{iS[\varphi_J + \eta]\}. \tag{2.28}$$

Comparing this to Eq. (1.2), we see that the 1PI effective potential corresponds to integrating out *all* degrees of freedom, and let the expectation value appear as a static background field,

[1] This is a consequence of the Euler characteristic $\chi = V - E + F$.

Effective potential

For a constant field configuration $\varphi(x) = \varphi_0$, the effective action, which is a functional, becomes a regular function. We define the effective potential $\mathcal{V}_{\text{eff}}$ by

$$\Gamma[\varphi_0] = -VT\,\mathcal{V}_{\text{eff}}(\varphi_0), \tag{2.29}$$

where VT is the volume of space-time. For a constant ground state, the effective potential will equal the energy of this state. To calculate the effective potential, we can expand the action around this state to calculate the effective action, by changing variables to $\varphi(x) = \varphi_0 + \eta(x)$. $\eta(x)$ now parametrizes fluctuations around the ground state, and has by assumption a vanishing expectation value. The generating functional becomes

$$Z[J] = \int \mathcal{D}(\varphi_0 + \eta)\, \exp\left\{ iS[\varphi_0 + \eta] + i \int \mathrm{d}^4x\, [\varphi_0 + \eta(x)]J(x) \right\}. \tag{2.30}$$

The functional version of a Taylor expansion, as described in Appendix C, is

$$S[\varphi_0 + \eta] = S[\varphi_0] + \int \mathrm{d}x\, \eta(x)\, \frac{\delta S[\varphi_0]}{\delta\varphi(x)} + \frac{1}{2}\int \mathrm{d}x\mathrm{d}y\, \eta(x)\eta(y)\, \frac{\delta^2 S[\varphi_0]}{\delta\varphi(x)\delta\varphi(y)} + \dots \tag{2.31}$$

The notation

$$\frac{\delta S[\varphi_0]}{\delta\varphi(x)} \tag{2.32}$$

indicates that the functional $S[\varphi]$ is differentiated with respect to $\varphi(x)$, then evaluated at $\varphi(x) = \varphi_0$. We define

$$S_0[\eta] := \int \mathrm{d}^4x\mathrm{d}^4y\, \eta(x)\eta(y)\, \frac{\delta^2 S[\varphi_0]}{\delta\varphi(x)\,\delta\varphi(y)}, \tag{2.33}$$

$$S_I[\eta] := \int \mathrm{d}^4x\mathrm{d}^4y\mathrm{d}^4z\, \eta(x)\eta(y)\eta(z)\, \frac{\delta^3 S[\varphi_0]}{\delta\varphi(x)\,\delta\varphi(y)\,\delta\varphi(z)} + \dots, \tag{2.34}$$

where the dots indicate higher derivatives. When we insert this expansion into the generating functional $Z[J]$ we get

$$Z[J] = \int \mathcal{D}\eta \exp\left\{ i\int \mathrm{d}^4x\, (\mathcal{L}[\varphi_0] + J\varphi_0) + i\int \mathrm{d}^4x\, \eta(x)\, \left(\frac{\delta S[\varphi_0]}{\delta\varphi(x)} + J(x) \right) + iS_0[\eta] + iS_I[\eta] \right\} \tag{2.35}$$

The first term is constant with respect to η and may be taken outside the path integral. The second term gives rise to tadpole diagrams, which alter the expectation value of $\eta(x)$. For $J = 0$, this expectation value should vanish, and this term can be ignored. Furthermore, this means that the ground state must minimize the classical potential,

$$\frac{\partial \mathcal{V}(\varphi_0)}{\partial\varphi} = 0. \tag{2.36}$$

This leaves us with

$$-i\ln Z[J] = W[J] = \int \mathrm{d}^4x\, (\mathcal{L}[\varphi_0] + J\varphi_0) - i\ln\left(\int \mathcal{D}\eta \exp\{iS_0[\eta] + iS_I[\eta]\} \right) \tag{2.37}$$

We can now use the definition of the 1PI effective action to obtain a formula for the effective potential,

$$\mathcal{V}_{\text{eff}}(\varphi_0) = -\frac{1}{VT}\left(W[J] - \int \mathrm{d}^4x\, J(x)\varphi_0 \right) = \mathcal{V}(\varphi_0) - i\ln\left(\int \mathcal{D}\eta \exp\{iS_0[\eta] + iS_I[\eta]\} \right). \tag{2.38}$$

In Eq. (2.18), we showed that the 1PI effective action describes the whole quantum theory of the original action at the tree-level. This was done by inspecting a theory with an action proportional to g^{-1}. In this theory, Feynman diagrams with L loops are proportional to g^{L-1}. We can use the same argument to expand the effective potential in loops. This is done by modifying the action $S[\varphi] \to g^{-1}S[\varphi]$, and then expand in

power of g. The first term in the effective potential is modified by $\mathcal{V} \to g^{-1}\mathcal{V}$, which means that it is made up of tree-level terms. This is as expected, since the tree-level result corresponds to the classical result without any quantum corrections. The second term becomes

$$\ln\left(\int \mathcal{D}\eta\, e^{iS_0+iS_I}\right) \longrightarrow \ln\left(\int \mathcal{D}\eta\, e^{ig^{-1}S_0+ig^{-1}S_I}\right) = \ln\left(\int \mathcal{D}\eta\, e^{ig^{-1}S_0}\right) + \ln\left(\frac{\int \mathcal{D}\eta\, e^{ig^{-1}S_I}\, e^{ig^{-1}S_0}}{\int \mathcal{D}\eta\, e^{ig^{-1}S_0}}\right)$$

The first term is quadratic in η, and can therefore be evaluated as a generalized Gaussian integral, as described in Appendix C,

$$\begin{aligned}
&\ln\left\{\int \mathcal{D}\eta\, \exp\left(ig^{-1}\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4y\, \eta(x)\eta(y)\, \frac{\delta^2 S[\varphi_0]}{\delta\varphi(x)\,\delta\varphi(y)}\right)\right\} \\
&= \ln\left\{\det\left(-g^{-1}\frac{\delta^2 S[\varphi_0]}{\delta\varphi(x)\,\delta\varphi(y)}\right)^{-1/2}\right\} = -\frac{1}{2}\,\mathrm{Tr}\left\{\ln\left(-\frac{\delta^2 S[\varphi_0]}{\delta\varphi(x)\,\delta\varphi(y)}\right)\right\} + \text{const.}
\end{aligned}$$

We then use the identity $\ln\det M = \mathrm{Tr}\ln M$. After we remove the constant, this term is proportional to g^0, i.e., it is made up of one-loop terms.

The last term can be evaluated by first expanding the exponential containing the S_I term, then using $\ln(1+x) = \sum_n \frac{1}{n}x^n$. Using

$$\langle A\rangle_0 = \frac{\int \mathcal{D}\varphi\, A\, e^{ig^{-1}S_0}}{\int \mathcal{D}\varphi\, e^{ig^{-1}S_0}}, \tag{2.39}$$

we can write

$$\ln\left[\frac{\int \mathcal{D}\eta\, e^{ig^{-1}S_I} e^{ig^{-1}S_0}}{\int \mathcal{D}\varphi\, e^{ig^{-1}S_0}}\right] = \ln\left(\sum_{n=0}^{\infty}\frac{1}{n!}\left\langle (ig^{-1}S_I)^n\right\rangle_0\right). \tag{2.40}$$

We recognize this as the sum of all connected Feynman diagrams, with Feynman ruels from the interaction term S_I. We know that S_I is made up of terms that are third power or higher in the fields. Each internal line is connected to two vertices, and each vertex is connected to at least three internal lines, i.e., $I \geq 3/2V$. The number of loops is therefore $L = I - V + 1 \geq (3/2-1)V + 1$. There is at leas one vertex, i.e. $L \geq 3/2$. This shows that the first logarith contains *all* one-loop contributions. The effective potential to one-loop order is therefore

$$\mathcal{V}_{\text{eff}}(\varphi_0) = \mathcal{V}(\varphi_0) - \frac{i}{VT}\frac{1}{2}\,\mathrm{Tr}\left\{\ln\left(-\frac{\delta^2 S[\varphi_0]}{\delta\varphi(x)\,\delta\varphi(y)}\right)\right\}. \tag{2.41}$$

2.2 Symmetry and Goldstone's theorem

This section is based on [8, 9, 23, 24].

Symmetry plays a prominent role in modern physics. If we can transform a physical state in such a way that the governing equations of this system are unchanged, we call that transformation a *symmetry transformation*. All such transformations are known as the symmetries of that theory. The symmetries of a theory encode a lot of physics, such as the presence of conserved quantities and the system's low energy behavior. We distinguish between internal and external symmetries. An external symmetry is an active coordinate transformation, such as rotations or translations. They relate degrees of freedom at different space-time points, while internal symmetries transform degrees of freedom at each space-time point independently. A further distinction is between global and local symmetry transformations. Global transformations have one rule for transforming degrees of freedom at each point, which is applied everywhere, while local transformations are functions of space-time.

In classical field theory, symmetries are encoded in the behavior of the Lagrangian when the fields are transformed. We will consider continuous transformations, which can in general be written as

$$\varphi(x) \longrightarrow \varphi'(x) = f_t[\varphi](x), \quad t \in [0,1]. \tag{2.42}$$

Here, $f_t[\varphi]$ is a functional in φ, and a smooth function of t, with the constraint that $f_0[\varphi] = \varphi$. This allows us to look at "infinitesimal" transformations,

$$\varphi'(x) = f_\epsilon[\varphi] = \varphi(x) + \epsilon \frac{\mathrm{d}f_t[\varphi]}{\mathrm{d}t}\bigg|_{t=0} + \mathcal{O}\left(\epsilon^2\right). \tag{2.43}$$

When considering infinitesimal transformations, we will not always write $+\mathcal{O}\left(\epsilon^2\right)$, but rather consider it implicit. We will consider internal, global transformations which act linearly on φ. For N fields, φ_i, this can be written

$$\varphi_i'(x) = \varphi_i(x) + \epsilon\, iV_{ij}\varphi_j(x). \tag{2.44}$$

V_{ij} is called the generator of the transformation. A symmetry transformation of the system is then a transformation in which the Lagrangian left is unchanged, or at most differ by a 4-divergence term. That is, a transformation $\varphi \to \varphi'$ is a symmetry if

$$\mathcal{L}[\varphi'] = \mathcal{L}[\varphi] + \partial_\mu K^\mu[\varphi], \tag{2.45}$$

where $K^\mu[\varphi]$ is a functional of φ.[2] This is a requirement for symmetry in quantum field theory too. However, as physical quantities in quantum field theory are given not just by the action of a single state but the path integral, the integration measure $\mathcal{D}\varphi$ has to be invariant as well. If a classical symmetry fails due to the non-invariance of the integration measure, it is called an *anomaly*.

To investigate the symmetry properties of a quantum theory, we explore what constraints a symmetry imposes on the effective action. To that end, assume

$$\mathcal{D}\varphi'(x) = \mathcal{D}\varphi(x), \quad S[\varphi'] = S[\varphi]. \tag{2.46}$$

In the generating functional, such a transformation corresponds to a change of integration variable. Using the infinitesimal version of the transformation, we may write

$$\begin{aligned} Z[J] = \int \mathcal{D}\varphi\, \exp\left\{iS[\varphi] + i\int \mathrm{d}^4x\, J_i(x)\varphi_i(x)\right\} &= \int \mathcal{D}\varphi'\, \exp\left\{iS[\varphi'] + i\int \mathrm{d}^4x\, J_i(x)\varphi_i'(x)\right\} \\ &= Z[J] + i\epsilon \int \mathrm{d}^4x\, J_i(x) \int \mathcal{D}\varphi\, [V_{ij}\varphi_j(x)]e^{iS[\varphi]}, \end{aligned} \tag{2.47}$$

Using Eq. (2.19), we can write this as

$$\int \mathrm{d}^4x\, \frac{\delta\Gamma[\varphi_J]}{\delta\varphi_i(x)}\, V_{ij}\, \langle\varphi_j(x)\rangle_J = 0. \tag{2.48}$$

This constraint will allow us to deduce the properties of a theory close to the ground state, only using information about the symmetries of the theory.

The archetypical example of an internal, global, and continuous symmetry is the linear sigma model, which we will use as an example throughout this section. The linear sigma model is made up of N real scalar fields φ_i, whose Lagrangian is

$$\mathcal{L}[\varphi] = \frac{1}{2}\partial_\mu\varphi_i(x)\partial^\mu\varphi_i(x) - \mathcal{V}(\varphi), \quad \mathcal{V}(\varphi) = -\frac{1}{2}\mu^2\varphi_i(x)\varphi_i(x) + \frac{1}{4}\lambda[\varphi_i(x)\varphi_i(x)]^2. \tag{2.49}$$

This system is invariant under the rotation of the N fields into each other,

$$\varphi_i \longrightarrow \varphi_i' = M_{ij}\varphi_j, \quad M^{-1} = M^T. \tag{2.50}$$

The set of all such transformations forms the Lie group $\mathrm{O}(N)$. Lie groups will be discussed in the next section. For $N = 2$, and $\mu^2, \lambda > 0$ we get the ubiquitous Mexican hat potential, as illustrated in Figure 2.1.

[2] Terms of the form $\partial_\mu K^\mu$ does not affect the physics, as variational principle $\delta S = 0$ do not vary the fields at infinity. Together with the divergence theorem, this means that such terms do not influence the equations of motion.

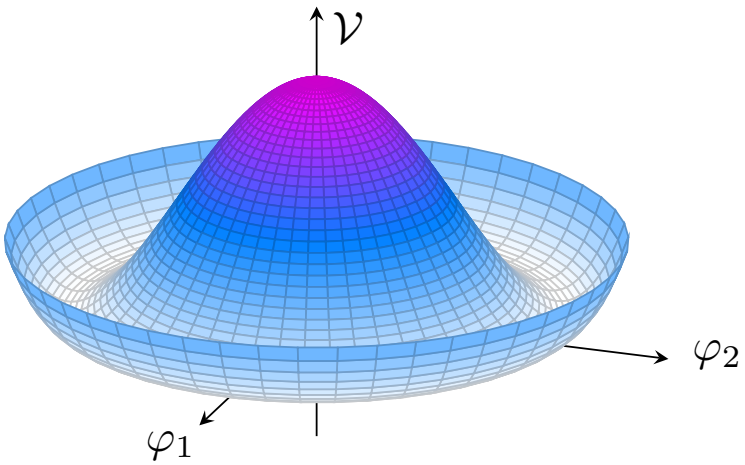


Figure 2.1: The Mexican hat potential is the classical potential $\mathcal{V}$ for the $N = 2$ linear sigma model.

Lie groups

Lie groups are a natural structure to capture the symmetries of a theory. A Lie group is a smooth manifold, i.e., a space that is locally diffeomorphic to $\mathbb{R}^N$. This means that we can locally parametrize the space by N real numbers η_α, using smooth invertible functions. A Lie group is also equipped with group structure. A group is a set, G, together with a map

$$(\cdot,\cdot) : G \times G \longmapsto G, \tag{2.51}$$
$$(g_1, g_2) \longmapsto g_3, \tag{2.52}$$

called group multiplication. This map obeys the group axioms, which are the existence of an identity element $\mathbb{1}$, associativity and the existence of an inverse element g^{-1} for all $g \in G$. These can be written as

$$\begin{array}{ll} \forall g \in G, & (g, \mathbb{1}) = g, \\ \forall g_1, g_2, g_3 \in G, & (g_1, (g_2, g_3)) = ((g_1, g_2), g_3), \\ \forall g \in G,\ \exists g^{-1} \in G, \text{ s.t.}, & (g, g^{-1}) = \mathbb{1}. \end{array}$$

In addition, we require that both the multiplication map and the inverse map, $g \mapsto g^{-1}$, are smooth. We describe the set of continuous symmetry transformations,

$$G = \{\, g \mid g\varphi = \varphi',\, S[\varphi'] = S[\varphi], \mathcal{D}\varphi' = \mathcal{D}\varphi \,\}, \tag{2.53}$$

as a Lie group. The group G might act on φ linearly, so $(g\varphi)_i = g_{ij}\varphi_j$, or in a more complicated matter. In this case, the group multiplication is composition, i.e., performing transformations in succession. This map is closed, as the composite of two symmetry transformations is another symmetry transformation. The identity map is a symmetry transformation, and composition is associative. This means that invertible symmetry transformations form a group.

We will focus on connected Lie groups, in which all elements $g \in G$ are in the same connected piece as the identity map $\mathbb{1}\varphi = \varphi$. This means that for each $g \in G$, one can find a continuous path $\gamma(t)$ in the manifold, such that $\gamma(0) = \mathbb{1}$ and $\gamma(1) = g$. Given such a path, we can study transformations close to the identity. As the Lie group is a smooth manifold, we can write[3]

$$\gamma(\epsilon) = \mathbb{1} + i\epsilon V + \mathcal{O}\left(\epsilon^2\right). \tag{2.54}$$

V is a generator, and is defined as

$$iV = \frac{\mathrm{d}\gamma}{\mathrm{d}t}\Big|_{t=0}. \tag{2.55}$$

[3]The factor i is a physics convention, and differs from how mathematicians define generators of a Lie group.

We can define a path γ in G by its path through parameter space $\mathbb{R}^n$, $\gamma(t) = g(\eta(t))$. Here, $\eta_\alpha(t)$ is a path through $\mathbb{R}^N$, the coordinates of G, such that $\eta_\alpha(0) = 0$ and $g(0) = \mathbb{1}$. We can thus write the generator as

$$V = \frac{\mathrm{d}\gamma}{\mathrm{d}t}\Big|_{t=0} = \frac{\mathrm{d}\eta_\alpha}{\mathrm{d}t}\Big|_{t=0}\frac{\partial g}{\partial \eta_\alpha}\Big|_{\eta=0} = v_\alpha T_\alpha, \quad T_\alpha = \frac{\mathrm{d}\eta_\alpha}{\mathrm{d}t}\Big|_{t=0}, \quad \frac{\partial g}{\partial \eta_\alpha}\Big|_{\eta=0}. \tag{2.56}$$

One can show that the generators form a vector space, with the basis T_α, induced by the coordinates η_α [24]. This vector space is called the tangent space of the identity element, $T_{\mathbb{1}}G$. Infinitesimal transformations can therefore be written as

$$\gamma(\epsilon) = \mathbb{1} + i\epsilon v_\alpha T_\alpha + \mathcal{O}\left(\epsilon^2\right). \tag{2.57}$$

The tangent space, together with the additional operation

$$[T_\alpha, T_\beta] = iC^\gamma_{\alpha\beta}T_\gamma, \tag{2.58}$$

called the Lie bracket, form a Lie algebra denoted $\mathfrak{g}$. $C^\gamma_{\alpha\beta}$ are called structure constants. They obey the Jacobi identity,

$$C^\gamma_{\alpha\beta} + C^\alpha_{\beta\gamma} + C^\beta_{\gamma\alpha} = 0, \tag{2.59}$$

which mean that they are totally antisymmetric. For matrix groups, which we deal with in this text, the Lie bracket is the commutator. A subset of the original Lie group, $H \subset G$, closed under the group action, is called a subgroup. H then has its own Lie algebra $\mathfrak{h}$, with a set of $m = \dim H$ generators, t_a, which is a subset of the original generators T_α. We denote the remaining set of generators x_i, such that t_a and x_i together span $\mathfrak{g}$. The commutators of t_a must be closed, which means that we can write

$$[t_a, t_b] = iC^c_{ab}t_c, \tag{2.60}$$

$$[t_a, x_i] = iC^k_{ai}x_k, \tag{2.61}$$

$$[x_i, x_j] = iC^k_{ij}x_k + iC^c_{ij}t_c, \tag{2.62}$$

where abc runs over the generators of $\mathfrak{h}$, and ijk runs over the rest. The second formula can be derived using the Jacobi identity Eq. (2.59), which implies that $C^k_{ab} = 0 = -C^b_{ak}$. This is called a Cartan decomposition.

One parameter subgroups are one special case of Lie subgroups. If a curve $\gamma(t)$ through G obey

$$\gamma(t)\gamma(s) = \gamma(t+s), \quad \gamma(0) = \mathbb{1}, \tag{2.63}$$

then all the points on this curve from a one parameter subgroup of G. This path is associated with a generator,

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t}\Big|_{t=0} = i\eta_\alpha T_\alpha. \tag{2.64}$$

This association is one-to-one, and allows us to define the exponential map,

$$\exp\{i\eta_\alpha T_\alpha\} := \gamma(1). \tag{2.65}$$

For connected and compact Lie groups, all elements of the Lie group $g \in G$ can be written as an exponential of elements in the corresponding Lie algebra $\eta_\alpha T_\alpha \in \mathfrak{g}$. For matrix groups, the exponential equals the familiar series expansion [24]

$$\exp\{X\} = \sum_n \frac{1}{n!}X^n. \tag{2.66}$$

Nöther's theorem

One of the most profound consequences of symmetry in physics is the appearance of conserved quantities. Assume we have a set of fields φ_i. Nöther's theorem tells us that if the Lagrangian $\mathcal{L}[\varphi_i]$ has a continuous symmetry, then there is a corresponding conserved current [13, 23]. Consider an infinitesimal transformation,

$$\varphi_i(x) \longrightarrow \varphi_i'(x) = \varphi_i(x) + \delta\varphi_i(x), . \tag{2.67}$$

Applying this transformation to the Lagrangian will in general change its form,

$$\mathcal{L}[\varphi] \to \mathcal{L}[\varphi'] = \mathcal{L}[\varphi] + \delta\mathcal{L}. \tag{2.68}$$

We assume this transformation is a symmetry, i.e.,

$$\delta\mathcal{L} = \partial_\mu K^\mu.$$

By considering the Lagrangian as a function of the field and its derivatives, $\mathcal{L} = \mathcal{L}(\varphi_i, \partial_\mu\varphi_i)$, we can write the difference term as a Taylor expansion around $(\varphi_i, \partial_\mu\varphi_i)$,

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\varphi_i}\delta\varphi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}\delta(\partial_\mu\varphi_i), \tag{2.69}$$

where $\delta(\partial_\mu\varphi_i) = \partial_\mu\varphi_i' - \partial_\mu\varphi_i$. By the linearlity of the derivative,

$$\delta(\partial_\mu\varphi_i) = \partial_\mu\varphi_i' - \partial_\mu\varphi_i = \partial_\mu(\varphi_i' - \varphi_i) = \partial_\mu\delta\varphi_i. \tag{2.70}$$

With this, and the Euler-Lagrange equations

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} - \frac{\partial\mathcal{L}}{\partial\varphi_i} = 0, \tag{2.71}$$

we can rewrite

$$\delta\mathcal{L} = \left(\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}\right)\delta\varphi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}(\partial_\mu\delta\varphi_i) = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}\delta\varphi_i\right) \tag{2.72}$$

If we define the current

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}\delta\varphi_i(x) - K^\mu, \tag{2.73}$$

then

$$\partial_\mu j^\mu = \delta\mathcal{L} - \delta\mathcal{L} = 0. \tag{2.74}$$

This is Nöther's theorem; a continuous symmetry implies the existence of a conserved current.

The current flux through some spacelike surface V defines a conserved charge. The surface of constant time in some reference frame has the normal vector $n_\mu = (1, 0, 0, 0)$, so the charge is

$$Q = \int_V \mathrm{d}^4x\, n_\mu j^\nu = \int_V \mathrm{d}^3x\, j^0. \tag{2.75}$$

We can then use the divergence theorem. Assume ∂V is the boundary of V, which has the space-like normal vector k_i, and that the current falls of quickly towards infinity. Then

$$\frac{\partial}{\partial t}Q = -\int_V \mathrm{d}^3x\, \partial_i j^i = -\int_{\partial V} \mathrm{d}^2x\, k_i j^i = 0, \tag{2.76}$$

proving that the charge is conserved.

Goldstone's theorem

A symmetry transformation will leave the governing equation of a theory unchanged. This, however, does not imply that physical states, such as the ground state, are invariant under this transformation. The $N = 2$ linear sigma model illustrates this. If we assume the ground state φ_0 is translationally invariant, then it is given by minimizing the effective potential, of which the classical potential, $\mathcal{V}$, is the leading order approximation. This potential is illustrated in Figure 2.1. The ground state is therefore given by any of the values along the brim of the potential. If we, without loss of generality, choose $\varphi_0 = (0, v)$ as the ground state, then any rotation will change this state. We say that the symmetry has been *spontaneously broken*.

To explore this in a general context, assume a theory of N real scalar fields φ_i are invariant under the actions of some Lie group, G. A symmetry $g \in G$ is broken if the vacuum expectation value of the original fields and the transformed fields differ. That is, if

$$\langle\varphi\rangle_0 \neq \langle\varphi'\rangle_0 = \langle g\varphi\rangle_0 \tag{2.77}$$

We can now exploit what we learned about Lie groups to write the infinitesimal transformation as

$$\langle\varphi'\rangle_0 = \langle\varphi\rangle_0 + i\epsilon\eta_\alpha T_\alpha \langle\varphi\rangle_0 . \tag{2.78}$$

Let x_i be the set of generators corresponding to broken symmetries, i.e.,

$$x_i \langle\varphi\rangle_0 \neq 0. \tag{2.79}$$

These are called the *broken generators*. The remaining set of generators t_a, which obey

$$t_a \langle\varphi\rangle_0 = 0, \tag{2.80}$$

are called unbroken, and generate a subgroup $H \subset G$ as the set of symmetry transformations of the vacuum is a group.

In Eq. (2.48) we found that, if V is the generator of some symmetry, then the effective action obeys

$$\int \mathrm{d}^4x \, \frac{\delta\Gamma[\varphi_J]}{\delta\varphi_i} V_{ij} \langle\varphi_j\rangle_0 = 0, \tag{2.81}$$

We now differentiate this expression with respect to $\varphi_k(y)$ and evaluate it in the vacuum, which gives

$$\int \mathrm{d}^4x \, \frac{\delta^2\Gamma[\varphi_0]}{\delta\varphi_k(y)\,\delta\varphi_i(x)} V_{ij} \langle\varphi_j\rangle_0 = 0. \tag{2.82}$$

With the assumption that the ground state is constant, we get

$$\frac{\partial^2\mathcal{V}_{\text{eff}}}{\partial\varphi_k\,\partial\varphi_i} V_{ij} \langle\varphi_j\rangle_0 = 0. \tag{2.83}$$

This is trival for unbroken symmetries, as $t^a_{ij} \langle\varphi_j\rangle_0 = 0$ by definition. However, in the case of a broken symmetry, the second derivative of the effective potential has an eigenvector $x^\ell_{ij} \langle\varphi_j\rangle_0$ with a zero eigenvalue for each broken generator. Here, ℓ label the set of generators, while (ij) are the indices corresponding to field-components φ_i. In Eq. (2.25), we found that the second derivative of the effective action is the inverse propagator,

$$D^{-1}_{ij}(x,y) = \frac{\delta^2\Gamma[\varphi_0]}{\delta\varphi_i(y)\,\delta\varphi_j(x)} = \int \frac{\mathrm{d}^4p}{(2\pi)^4} e^{-ip(x-y)} \tilde{D}^{-1}_{ij}(p). \tag{2.84}$$

Using this, we can write

$$\tilde{D}^{-1}_{ij}(p=0)\, x^\ell_{jk} \langle\varphi_k\rangle_0 = 0. \tag{2.85}$$

Zeros of the inverse propagator correspond to the physical mass of particles. In Lorentz-invariant systems, each zero-eigenvalue vector corresponds to a masses particle, called a Goldstone boson.[4] This means there are $n_G = \dim G - \dim H$ zero-mass modes. In general, the counting of massless modes is complicated and depends on the dispersion relation of the particles at low momenta. Systems with Goldstone bosons with quadratic dispersion relation, that is $E \propto |\vec{p}|^2$ when $\vec{p} \to 0$, often exhibit a lower number of massless modes. An example is ferromagnets, where the SU(2) rotational symmetry is broken down to U(1) when they align along one axis. This corresponds to two broken generators, yet the system exhibits only one massless mode [25].

The linear sigma model gives an intuition for the Goldstone mode. In the case of $N = 2$, the symmetry of the Lagrangian are rotations in the plane. As the ground state is a point along the "brim" of the hat, this rotational symmetry is broken. However, any excitations in the angular direction do not cost any energy,

[4]The particles are bosons due to the bosonic nature of the transformations, g. If the generators are Grassmann numbers, the resulting particles, called goldstinos, are fermions.

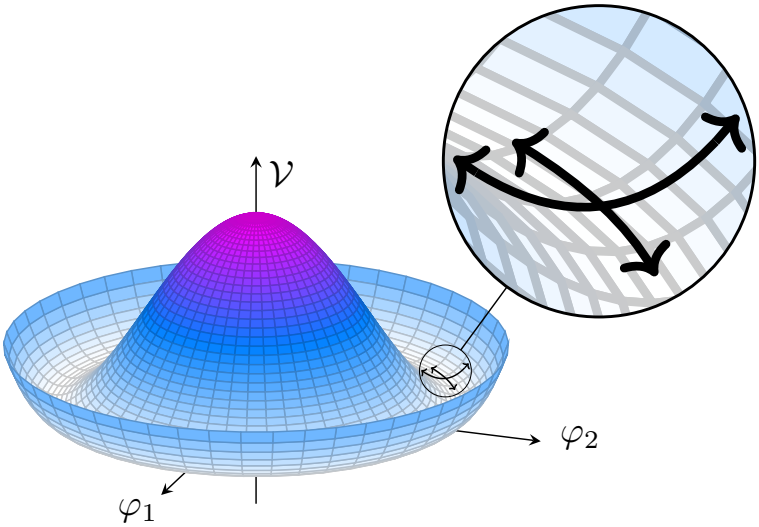


Figure 2.2: Excitations along the brim does not cost any energy, as the potential is flat, unlike excitations in the radial direction.

which is indicative of a massless mode. This is illustrated in Figure 2.2. In this example, the original symmetry group is one-dimensional, so there are no unbroken symmetries. Consider instead the $N = 3$ linear sigma model, which has the three-dimensional symmetry group SO(3), rotations of the sphere. We see that the ground state is left invariant under a subgroup of the original symmetry transformations. The ground state manifold of this system, the set of all states that minimizes the effective potential, is then a sphere. When the system chooses one single ground state, this symmetry is broken, but only for two of the generators. The generator for rotations around the ground state leaves that point unchanged and is thus an unbroken symmetry. Any excitations in the direction of the broken symmetries do not cost energy, as it is in the ground state manifold. On the other hand, the unbroken symmetry does not correspond to an excitation. This is illustrated in Figure 2.3.

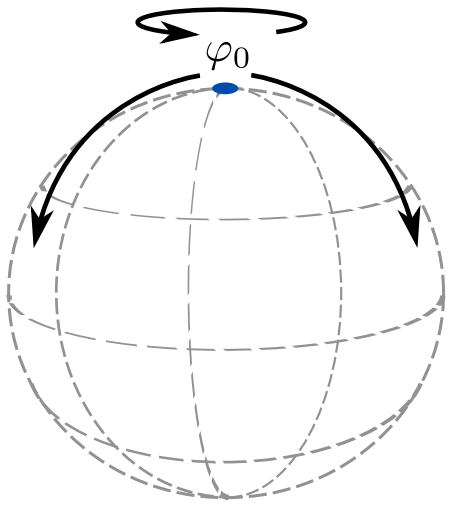


Figure 2.3: Excitations for the $N = 3$ sigma model. Two of the symmetries are broken, while rotations around the groundstate leaves the system unchanged.

2.3 CCWZ construction

As Goldstone bosons are massless, they play a crucial role in low-energy dynamics. To best describe this limit, we seek a parametrization of the theory in which they are the degrees of freedom. This can be done using the CCWZ construction, named after Callan, Coleman, Wess, and Zumino. This section is based on [9, 26–28], as well as the original papers [29, 30].

We saw that the Goldstone bosons correspond to excitations within the vacuum manifold. The vacuum manifold corresponds to points in field space φ that can be reached from the vacuum φ_0 with a transformation $g \in G$. Assume this group acts linearly on the fields. This means that we can write such excitations as

$$\varphi_i = (\tilde{\Sigma}\varphi_0)_i = \tilde{\Sigma}_{ij}(\varphi_0)_j, \quad \tilde{\Sigma} = \tilde{\Sigma}(\eta) = \exp\{i\eta_\alpha T_\alpha\} \tag{2.86}$$

We will drop the indices for the sake of compact notation. $\tilde{\Sigma}$ is thus a function from the parameter space, $\eta_\alpha \in \mathbb{R}^n$, to G,

$$\tilde{\Sigma} : \mathbb{R}^n \longmapsto G. \tag{2.87}$$

We then get space-time-dependent field configurations by making the parameters dependent on space-time. We will for now assume η_α is constant. This parametrization is highly redundant. Two elements $\tilde{\Sigma}$ and $\tilde{\Sigma}'$, related by

$$\tilde{\Sigma}' = \tilde{\Sigma}e^{i\theta_a t_a} \tag{2.88}$$

results in the same φ. This is because $e^{i\theta_a t_a} = h \in H$, and $h\varphi_0 = \varphi_0$, by assumption. The set of all equivalent $\tilde{\Sigma}$'s is exactly the left coset, $gH = \{\, gh \mid h \in H \,\}$. The set of cosets forms a new manifold, G/H, called the Goldstone manifold. This is a manifold of dimension $\dim(G/H) = \dim(G) - \dim(H)$, which is the number of broken generators and thus also the number of Goldstone modes. Membership of a certain coset form an equivalence relation, $g \sim g'$ if $g' = gh$. This means that the cosets gH form a partition of G and that each element $g \in G$ belongs to one, and only one, coset. To remove the redundancy in the parametrization, we need to choose one representative element from each coset.

By the inverse function theorem, any mapping between manifolds $f : \mathcal{M} \mapsto \mathcal{N}$ that has a non-degenerate differential, that is an invertible Jacobian, at a point $p \in \mathcal{M}$, is invertible in a neighborhood of p. If we write

$$\tilde{\Sigma}(\xi, \theta) = \exp\{i\xi_i x_i\} \exp\{i\theta_a t_a\}, \tag{2.89}$$

then the map is invertible at $p = (\xi_i = 0, \theta_a = 0)$, as the Jacobian is the identity matrix. This point is mapped to the identity element of G. This means that, in a neighborhood $U \subset G$ of the identity, each element g has a unique representation $g = \tilde{\Sigma}$ [24]. Furthermore, two elements $\tilde{\Sigma}'$ and $\tilde{\Sigma}$ related by $\tilde{\Sigma}' = \tilde{\Sigma}h$, $h \in H$ have the same ξ-arguments. We see that ξ_i parametrize G/H, in the neighborhood of the identity. We therefore demand that $\tilde{\Sigma}$ always appear in the standard form

$$\Sigma(\xi) = \tilde{\Sigma}(\xi, 0) = \exp\{i\xi_i x_i\}. \tag{2.90}$$

The field $\varphi(x)$ can therefore be written as

$$\varphi(x) = \Sigma(x)\varphi_0 = \exp\{i\xi_i(x)x_i\}\varphi_0, \tag{2.91}$$

and $\xi_i(x)$ can be associated with the Goldstone bosons.

In the linear sigma model, the original $\mathrm{O}(N)$ symmetry is broken down to $\mathrm{O}(N-1)$, which transforms the remaining $N-1$ fields with vanishing expectation values into each other. However, $\mathrm{O}(N)$ consists of two disconnected subsets, those matrices with determinant 1 and those with determinant -1. There is no continuous path that takes an element of $\mathrm{O}(N)$ with determinant of -1 to an element with determinant 1.[5] The set of symmetries that are connected to the identity is

$$G = SO(N) = \{\, M \in \mathrm{O}(N) \mid \det M = 1 \,\}. \tag{2.92}$$

[5] A simple proof of this is the fact that the determinant is a continuous function, while any path $\det M(t)$ such that $\det M(1) = -1$, $\det M(0) = 1$ must make a discontinuous jump.

If we choose $\varphi_0 = (0, 0, ..., v)$, then it is apparent that the ground state is invariant under the rotations of the $N-1$ first fields, so the unbroken symmetry is $H = \mathrm{SO}(N-1)$. The Goldstone manifold is $G/H = \mathrm{SO}(N)/\mathrm{SO}(N-1)$.

Consider the case of $N = 3$, which is illustrated in Figure 2.3. G is the rotations of the sphere, while H is rotations around φ_0, SO(2). The Goldstone manifold consists of the rotations of φ_0 to other points of the sphere, i.e. $G/H = \mathrm{SO}(3)/\mathrm{SO}(2) = S^2$, the 2-sphere. This is not a Lie group, as translating φ in a closed path around the sphere may result in a rotation around the z-axis. This is illustrated in Figure 2.4

To check that ξ_i, in fact, are the Goldstone modes, we study the way they appear in the Lagrangian. As they are massless, no mass term of the form $M_{ij}\xi_i\xi_j$ should appear. The original Lagrangian $\mathcal{L}[\varphi]$ was invariant under global transformations $\varphi(x) \mapsto g\varphi(x)$. However, any terms that only depend on $\varphi(x)$, and not its derivatives, will also be invariant under a *local* transformation, $\varphi(x) \mapsto g(x)\varphi(x)$. Our parametrization of the fields, $\varphi(x) = \Sigma(x)\varphi_0$ is exactly such a transformation, which means that any such terms are independent of the Goldstone bosons. We can therefore write

$$\mathcal{L}[\varphi] = \mathcal{L}_{\mathrm{kin}}[\varphi] + V(\varphi_0), \tag{2.93}$$

where all terms in $\mathcal{L}_{\mathrm{kin}}$ are proportional to at least one derivative term, $\partial_\mu\varphi(x)$. Inserting the parametrization into this derivative term, we get

$$\partial_\mu\varphi(x) = \partial_\mu[\Sigma(x)\varphi_0] = \Sigma(x)[\Sigma(x)^{-1}\partial_\mu\Sigma(x)]\varphi_0. \tag{2.94}$$

The Lagrangian will therefore depend on ξ_i via terms of the form $\Sigma(x)^{-1}\partial_\mu\Sigma(x)$, which is called the Mauer-Cartan form. This is a $\mathfrak{g}$-valued function, which means that it can be written as

$$i\Sigma(x)^{-1}\partial_\mu\Sigma(x) = d_\mu(x) + e_\mu(x), \tag{2.95}$$
$$d_\mu = ix_i d_{ij}(\xi)\partial_\mu\xi_j, \tag{2.96}$$
$$e_\mu = it_a e_{ai}(\xi)\partial_\mu\xi_i, \tag{2.97}$$

where d_{ij} and e_{ai} are as-of-yet unknown real valued functions of ξ [9, 31].

Transformation properties of Goldstone bosons

We can deduce how the Goldstone bosons transforms under G from how φ transforms. In general,

$$\varphi' = g\varphi = (g\Sigma(\xi))\varphi_0 = \Sigma(\xi')\varphi_0 \quad g \in G. \tag{2.98}$$

While $\Sigma(\xi')$ has the standard form by assumption,

$$\Sigma(\xi') = \exp\{i\xi'_i x_i\}, \tag{2.99}$$

$g\Sigma(\xi)$ does not, in general.

Figure 2.4 illustrates this in the case of $G = \mathrm{SO}(3)$. $\Sigma(\xi)$ transforms φ_0 to φ, then g transforms φ to $\varphi' = \Sigma(\xi')\varphi_0$. Assuming φ and φ' are close enough to φ_0, we can write $\Sigma(\xi)$ and $\Sigma(\xi')$ on the standard form. However, if we follow a small neighborhood around φ_0 as it is acted on by $\Sigma(\xi)$, then g, it will be rotated by the time it arrives at φ' when compared to the same neighborhood if it was acted on by $\Sigma(\xi')$.

$g\Sigma(\xi)$ and $\Sigma(\xi')$ are in the same coset, as they by assumption corresponds to the same physical state. This means that we can write $g\Sigma(\xi) = \Sigma(\xi')h(g, \xi)$, where $h(g, \xi) \in H$. The transformation rule of ξ under G is therefore implicitly defined by

$$\Sigma(\xi') = g\Sigma(\xi)[h(g, \xi)]^{-1}. \tag{2.100}$$

This is, in general, not a linear representation, which is why this construction also is called a *non-linear realization*. Using the transformation rule, we can obtain the transformation rule of the Maurer-Cartan form.

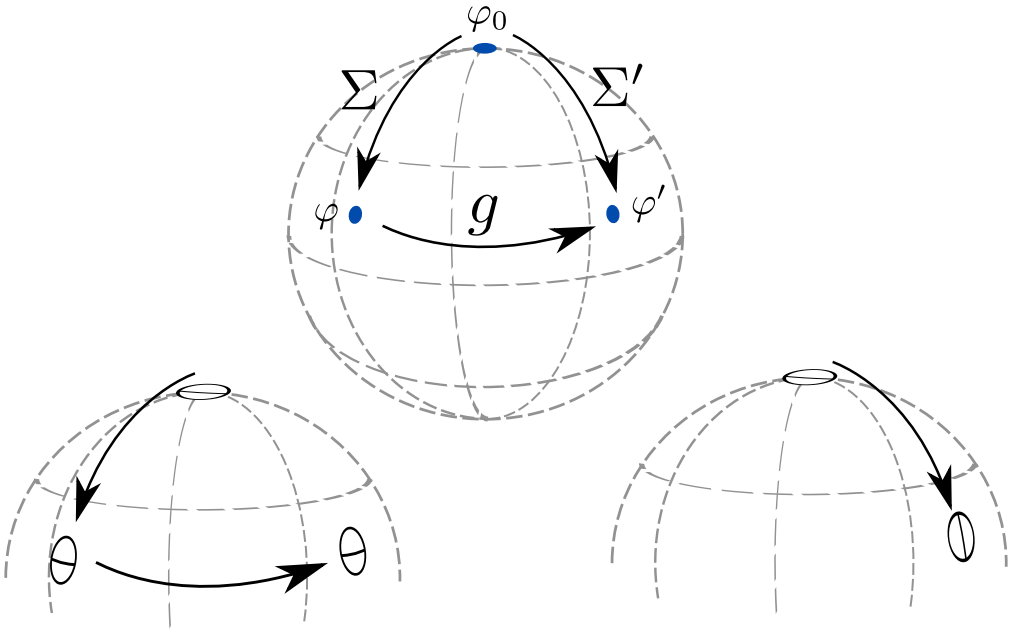


Figure 2.4: The top figure illustrates the transformation of φ_0 to φ and then φ', and the alternative, direct transformation $\varphi_0 \to \varphi'$. The bottom figure illustrates how this can rotate a neighborhood of φ_0 differently.

We use the shorthand $\Sigma = \Sigma(\xi)$, $\Sigma' = \Sigma(\xi')$, and $h = h(g, \xi)$. This gives

$$\begin{aligned}\Sigma^{-1}\partial_\mu\Sigma \to \Sigma'^{-1}\partial_\mu\Sigma' &= (g\Sigma h^{-1})^{-1}\partial_\mu(g\Sigma h^{-1}) \\ &= (h\Sigma^{-1}g^{-1})g[(\partial_\mu\Sigma)h^{-1} + \Sigma\partial_\mu h^{-1}] \\ &= h\Sigma^{-1}(\partial_\mu\Sigma)h^{-1} + h\partial_\mu h^{-1} \\ &= h(\Sigma^{-1}\partial_\mu\Sigma + \partial_\mu)h^{-1}.\end{aligned}$$

In terms of d_μ and e_μ,

$$d_\mu \to h d_\mu h^{-1} \tag{2.101}$$

$$e_\mu \to h(e_\mu + i\partial_\mu)h^{-1}. \tag{2.102}$$

These are our building blocks for constructing a general, G-invariant effective Lagrangian. The trace of a product of d_μ's are invariant under G,

$$\mathrm{Tr}\{d_\mu d_\nu \dots d_\rho\} \to \mathrm{Tr}\{h d_\mu h^{-1} h d_\nu h^{-1} h \dots d_\rho h^{-1}\} = \mathrm{Tr}\{d_\mu d_\nu \dots d_\rho\}, \tag{2.103}$$

where we have used the cyclic property of trace. However, the terms must also obey the other symmetries of the Lagrangian, such as C or P-parity and Lorentz invariance. The last criterion excludes any terms with free space-time indices. In section 3.2, we will construct an effective Lagrangian in powers of d. The lowest order terms are therefore

$$\mathrm{Tr}\{d_\mu\}\,\mathrm{Tr}\{d^\mu\}, \quad \mathrm{Tr}\{d_\mu d^\mu\}. \tag{2.104}$$

We see that e_μ transforms like a gauge field, with the gauge group H. If we include massive degrees of freedom and not only the Goldtone modes, e_μ is used to create a covariant derivative of the massive modes. We are only interested in the Goldstone modes and will therefore be satisfied with d_μ. With these tools, we can create an effective theory of quantum chromodynamics at low energies.

Chapter 3

The effective theory of pions

3.1 QCD

This section is based on [1, 23, 32]

Quantum chromodynamics, or QCD, is the theory of how quarks interact via the strong force. There are six flavors of quarks, labeled by the index f, called up, down, charm, strange, top, and bottom. The up, charm, and top quarks, collectively called up-type quarks, have electrical charge $+\frac{2}{3}$, while the down-type quarks—down, strange, and bottom—have an electrical charge of $-\frac{1}{3}$. Quarks have one additional degree of freedom called color, indexed by c, which takes on the value of red, green, and blue for quarks, and anti-red, -green, and -blue for antiquarks. The analogy to colors is meant to capture the empirical fact that quarks are only ever observed in *colorless* configurations, that is, either an equal amount of each color and its anti-color, or equal amounts of red, green, and blue. Quarks are spin-$\frac{1}{2}$ particles, so each quark q_{fc} is a spinor. Free quarks are governed by the Dirac-Lagrangian,

$$\mathcal{L}[q,\bar{q}] = \sum_{f=1}^{N_f}\sum_{c=1}^{N_c} \bar{q}_{fc}(i\gamma^\mu\partial_\mu - m_f)q_{fc} = \bar{q}(i\partial\!\!\!/ - m)q, \tag{3.1}$$

where N_f is the number of flavors, and N_c is the number of colors. In the last equality, we have hidden the indices to reduce clutter. The gamma matrices, γ^μ and γ^5 are described in Appendix B, and we have employed the Feynman slash notation $\gamma^\mu A_\mu = A\!\!\!/$. Furthermore, $\bar{q} = q^\dagger\gamma^0$. This Lagrangian is invariant under rations of the quarks in the color indices, i.e., transformations

$$q_c \to q'_c = U_{cc'}q_{c'}, \quad \bar{q}'_c = \bar{q}_{c'}U^\dagger_{c'c} \tag{3.2}$$

where $U_{cc'}$ is an $N_c \times N_c$ unitary matrix. The set of all $N \times N$ unitary matrices form the Lie group $\mathrm{U}(N) = \mathrm{U}(1) \times \mathrm{SU}(N)$. The subgroup of color transformations with determinant 1, $\mathrm{SU}(N_c)$, is the gauge group of the strong force. A theory with an $\mathrm{SU}(N)$ gauge group is called a Yang-Mills theory.

Yang-Mills theory

Given an element $U \in \mathrm{SU}(N_c)$, we can write

$$U = \exp\{i\chi_\alpha\lambda_\alpha\}, \quad \chi_\alpha\lambda_\alpha \in \mathfrak{su}(N_c), \tag{3.3}$$

where $\mathfrak{su}(N_c)$ is the Lie algebra of $\mathrm{SU}(N_c)$, with generators λ_α. We derive the Yang-Mills Lagrangian by including all normalizable and gauge invariant terms. A term is gauge invariant if it remains unchanged after a *local* $\mathrm{SU}(N_c)$ transformation, i.e.,

$$q \to \exp\{i\chi_\alpha(x)\lambda_\alpha\}q, \quad \bar{q} \to \bar{q}\exp\{-i\chi_\alpha(x)\lambda_\alpha\}. \tag{3.4}$$

The mass term $m_f \bar{q}_f q_f$ is gauge invariant, while the kinetic derivative term needs to be modified. This is done by exchanging the partial derivative with a *covariant derivative*. A covariant derivative D_μ is defined to transform the same way as the object it acts on. It must therefore obey

$$D_\mu q \to (D_\mu q)' = U(D_\mu q) = D'_\mu(Uq). \tag{3.5}$$

We assume that the covariant derivative has the form

$$D_\mu = \mathbb{1}\partial_\mu - igA_\mu, \tag{3.6}$$

where $A_\mu = A^\alpha_\mu \lambda_\alpha$, A^α_μ is a set of vector fields, and g a constant. Inserting this into Eq. (3.5) yields

$$(\mathbb{1}\partial_\mu - igA'_\mu)U = U(\mathbb{1}\partial_\mu - igA_\mu). \tag{3.7}$$

This means that if we demand that A_μ transforms as

$$A_\mu \to U\left(A_\mu + \frac{i}{g}\partial_\mu\right)U^\dagger, \tag{3.8}$$

then $D_\mu q$ transforms in the same way as the quarks q, and the kinetic term becomes gauge invariant. The second derivative,

$$D_\mu D_\nu = \partial_\mu\partial_\nu - ig(\partial_\mu A_\nu + A_\mu\partial_\nu + A_\nu\partial_\mu) - g^2 A_\mu A_\nu, \tag{3.9}$$

transforms in the same way as the first derivative. We see that the "operator-part" of this derivative is symmetric in the space-time indices, which means that the commutator is just a tensor, not an operator. We define

$$G_{\mu\nu} := \frac{i}{g}[D_\mu, D_\nu] = (\partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu + gC^\alpha_{\beta\gamma}A^\beta_\mu A^\gamma_\nu)\lambda_\alpha, \tag{3.10}$$

which is the gluon field strength tensor. This transforms as

$$G_{\mu\nu} \to UG_{\mu\nu}U^\dagger. \tag{3.11}$$

We now need to include terms governing the gauge field A_μ in the Lagrangian. The tensor $G_{\mu\nu}$ allows construct all gauge invariant renormalizable, which are

$$G^a_{\mu\nu}G^{\mu\nu}_a, \quad \varepsilon^{\mu\nu\rho\sigma}G^a_{\mu\nu}G^a_{\rho\sigma}. \tag{3.12}$$

Here, ε is the Levi-Civita symbol. The Yang-Mills Lagrangian is therefore

$$\mathcal{L} = \bar{q}(i\not{D} - m)q - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a + \theta\varepsilon^{\mu\nu\rho\sigma}G^a_{\mu\nu}G^a_{\rho\sigma}, \tag{3.13}$$

where θ is a coupling constant. In the case of QCD, $N_c = 3$, and the generators λ_α, $\alpha \in \{1, ...8\}$ are the 3×3 Gell-Mann matrices. Furthermore, experiments indicate that $\theta = 0$, or at least very close. No symmetry forbids $\theta \neq 0$, and its absence is dubbed the strong CP-problem [1].

Chiral symmetry

In addition to the color and flavor indices c and f, the quarks also have spinor indices, i, on which the γ-matrices act. We can define the projection operators,

$$P_\pm = \frac{1}{2}(\mathbb{1} \pm \gamma^5), \tag{3.14}$$

which obey $P^2_\pm = P_\pm$, $P_+P_- = P_-P_+ = 0$ and $P_+ + P_- = 1$, as good projection operators should. Furthermore, $P^\dagger_\pm = P_\pm$. These project spinors down to their chiral components, called left- and right-handed spinors,

$$P_+ q = q_R, \quad P_- q = q_L. \tag{3.15}$$

From Appendix B, we have

$$\{\gamma^\mu, \gamma^5\} = 0, \tag{3.16}$$

which means that

$$\bar{q}P_\pm = (P_\mp q)^\dagger \gamma^0. \tag{3.17}$$

Using the chiral projection operators, we can write the quark term of the QCD Lagrangian as

$$\begin{aligned}\bar{q}(i\not{D} - m)q &= \bar{q}(P_+ + P_-)(P_+ + P_-)(i\not{D} - m)q = (qP_-)\gamma^0 P_+(i\not{D} - m)q + (qP_+)\gamma^0 P_-(i\not{D} - m)q \\ &= \bar{q}_L(i\not{D})q_L + \bar{q}_R(i\not{D})q_R - \bar{q}_L m q_R - \bar{q}_R m q_L.\end{aligned}$$

The mass-term mixes the two chiral components. In the case that $m_f = 0$, called the *chiral limit*, the QCD Lagrangian is invariant under the seperate transformations

$$q_R \to e^{i\theta_R} U_R q_R, \quad q_L \to e^{i\theta_L} U_L q_L, \tag{3.18}$$

where U_L and U_R are complex, unitary $N_f \times N_f$ matrices with determinant 1, that act on the flavor indices, and θ_L and θ_R real numbers. The full set of such transformations forms the Lie group $\mathrm{U}(N_f)_L \times \mathrm{U}(N_f)_R = \mathrm{U}(1)_L \times U(1)_R \times \mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$. We can write

$$U_R = P_- + \exp\{i\eta_\alpha^R T_\alpha^R\}P_+, \quad U_L = P_+ + \exp\{i\eta_\alpha^L T_\alpha^L\}P_-, \tag{3.19}$$

where $T_\alpha^L P_-$ and $T_\alpha^R P_+$ are the generator for $\mathrm{SU}(N_f)_L$ and $\mathrm{SU}(N_f)_R$. The conserved currents corresponding to these symmetries are

$$L_\alpha^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu q)}(-iT_\alpha^L q_L) = \bar{q}\gamma^\mu T_\alpha^L q_L = \bar{q}_L \gamma^\mu T_\alpha^L q_L, \tag{3.20}$$

$$R_\alpha^\mu = \bar{q}_R \gamma^\mu T_\alpha^R q_R, \tag{3.21}$$

$$J_R^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu q)}(-iq_L) = \bar{q}_R \gamma^\mu q_R \tag{3.22}$$

$$J_L^\mu = \bar{q}_L \gamma^\mu q_L \tag{3.23}$$

We have defined the currents with a positive sign to conform to standard convention. When we introduce a non-zero mass, the symmetry of the Lagrangian is explicitly broken. For now, we assume the mass matrix has the form $m = m_q \mathbb{1}$. In this case, transforming left- and right-handed quarks, in the same way, i.e., $\eta_R = \eta_L$, leaves the Lagrangian unchanged. The diagonal subgroup of $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$, denoted $\mathrm{SU}(N_f)_V$, remains unbroken. Elements of this group have the form

$$U_V = \exp\{i\eta_\alpha^V T_\alpha^V \mathbb{1}\}. \tag{3.24}$$

This is a normal subgroup of $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$, and the quotient $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R / \mathrm{SU}(N_f)_V$ therefore forms a Lie group, $\mathrm{SU}(N_f)_A$, with elements of the form

$$U_A = \exp\{i\eta_\alpha^A T_\alpha^A \gamma^5\}. \tag{3.25}$$

Similarly, the diagonal subgroup of $\mathrm{U}(1)_L \times \mathrm{U}(1)_R$ has an unbroken, normal subgroup $\mathrm{U}(1)_V$, with elements of the form $e^{i\theta_A \mathbb{1}}$, and a quotient group, $\mathrm{U}(1)_A$, with elements of the form $e^{i\theta_V \gamma^5}$. In the chiral limit, these groups are all symmetries of the Lagrangian, and the corresponding conserved currents are

$$V_\alpha^\mu = R_\alpha^\mu + L_\alpha^\mu = \bar{q}\gamma^\mu T_\alpha^V q, \tag{3.26}$$

$$A_\alpha^\mu = R_\alpha^\mu - L_\alpha^\mu = \bar{q}\gamma^\mu \gamma^5 T_\alpha^A q, \tag{3.27}$$

$$J_V^\mu = J_R^\mu + L^\mu = \bar{q}\gamma^\mu q, \tag{3.28}$$

$$J_A^\mu = J_L^\mu - L^\mu = \bar{q}\gamma^\mu \gamma^5 q. \tag{3.29}$$

The currents corresponding to $\mathrm{U}(1)_V$ and $\mathrm{U}(1)_A$ are called the vector and axial currents, as they transform as vectors and axial-vectors under space-time transformations. The conserved charge corresponding to the vector current,

$$Q = \int \mathrm{d}^3x \, \bar{q}\gamma^0 q, \tag{3.30}$$

is the quark number, which is 3 times the *baryon number*. Up until now, we have only made classical considerations. In the full quantum theory, the baryon number remains conserved. The axial current J_A^μ however, is subject to an anomaly. As the integration measure $\mathcal{D}\bar{q}\mathcal{D}q$ is not invariant under transformations in $\mathrm{U}(1)_A$, as we saw earlier is required, the quantum correction to $\partial_\mu J_A^\mu$ is non-zero. The remaining symmetry, $\mathrm{U}(1)_V \times \mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$, and how it is broken is the foundation for chiral perturbation theory, which is the subject of the next section.

3.2 Chiral perturbation theory

This section is based on [1, 9, 32], in addition to the original work [10, 33, 34]

In this paper, we will consider the interaction of the two lightest quarks, the up and down quarks u and d, i.e., $N_f = 2$. In this case, the symmetry group of rotations in the flavor indices is $G_f = U(1) \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$. The generators of SU(2) are $T_\alpha = \tau_\alpha/2$, where τ_α are the Pauli matrices, as described in Appendix B. The mass matrix of the quarks is

$$m = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} = \frac{1}{2}(m_u + m_d)\mathbb{1} + \frac{1}{2}(m_u - m_d)\tau_3, \tag{3.31}$$

where $m_u \approx 2.16\,\mathrm{MeV}$, and $m_d \approx 4.67\,\mathrm{MeV}$ [4]. The G_f symmetry is *explicitly* broken. Even though the underlying symmetry is only approximate, we can still apply the formalism from Goldstone's theorem and the CCWZ construction by including a small mass term for the Goldstone bosons, which in this case are called *Pseudo Goldstone bosons*.

We now focus on the subgroup $G = \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$. The ground state spontaneously breaks this approximate symmetry of the two-flavor QCD Lagrangian. As quarks q are spinors, a non-zero expectation value of the quark field would break Lorentz-invariance. Instead, the spontaneous symmetry breaking is characterized by the *scalar quark condensate*, $\langle \bar{q}q \rangle$. The scalar quantity $\bar{q}q$ is invariant under isospin transformations $H = \mathrm{SU}(2)_V$, but not under $\mathrm{SU}(2)_A$. The Goldstone manifold, therefore, is $G/H = \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R/\mathrm{SU}(2)_V = \mathrm{SU}(2)_A$. To model the low-energy dynamics of QCD, we start with the massless QCD Lagrangian,

$$\mathcal{L}^0_{\mathrm{QCD}}[q, \bar{q}, A_\mu] = i\bar{q}\not{D}q - \frac{1}{4}G^\alpha_{\mu\nu}G^{\mu\nu}_\alpha.$$

Following [32, 33], we couple quarks to external currents. These can be used to model external fields or capture the symmetry-breaking mass term. As found in section 3.1, the conserved currents are

$$V_a^\mu = \frac{1}{2}\bar{q}\gamma^\mu\tau_a q, \quad A_a^\mu = \frac{1}{2}\bar{q}\gamma^\mu\gamma^5\tau_a q, \quad J^\mu = \bar{q}\gamma^\mu q. \tag{3.32}$$

In addition, external currents can couple to the scalar and pseudo-scalar quark bilinears

$$\bar{q}q, \quad \bar{q}\gamma^5 q, \quad \bar{q}\tau_a q, \quad \bar{q}\gamma^5\tau_a q. \tag{3.33}$$

The Lagrangian containing external terms is

$$\mathcal{L}_{\mathrm{ext}} = -(\bar{q}q)\, s_0 + (i\bar{q}\gamma^5 q)\, p_0 - (\bar{q}\tau_a q)\, s_a + (i\bar{q}\gamma^5\tau_a q)\, p_a + \frac{1}{3}J_\mu v^\mu_{(s)} + V^a_\mu v^\mu_a + A^a_\mu a^\mu_a \tag{3.34}$$

$$= -\bar{q}\left(s + i\gamma^5 p\right)q + \bar{q}\left(\frac{1}{3}v^\mu_{(s)} + v^\mu + a^\mu\gamma^5\right)\gamma_\mu q. \tag{3.35}$$

Here, $v^\mu_a, v^\mu_{(s)}, a^\mu_a, s_a, s_0$ and p_a are the external source currents, and we denote

$$s = s_0\mathbb{1} + s_a\tau_a, \quad p = p_0\mathbb{1} + p_a\tau_a, \quad v^\mu = \frac{1}{2}v^\mu_a\tau_a, \quad a^\mu = \frac{1}{2}a^\mu_a\tau_a. \tag{3.36}$$

Setting $v = v_{(s)} = a = s = p = 0$, we recover the chiral limit. To include the effect of the quark masses, we set $s = m$. We denote all the external currents as $j = (v, v_{(s)}, a, s, p)$. As with the conserved currents, we define

$$v_\mu = \frac{1}{2}(r_\mu + l_\mu), \quad a_\mu = \frac{1}{2}(r_\mu - l_\mu). \tag{3.37}$$

With this, as well as

$$\bar{q}(s - i\gamma^5 p)q = \bar{q}_R(s - ip)q_L + \bar{q}_L(s + ip)q_R, \tag{3.38}$$

we can write the Lagrangian as

$$\mathcal{L}_{\text{ext}} = -\bar{q}_R(s + ip)q_L - \bar{q}_L(s - ip)q_R + \frac{1}{3}(J^\mu_R + J^\mu_V)(v_{(s)})_\mu + R^a_\mu r^\mu_a + L^a_\mu l^\mu_a \tag{3.39}$$

The generating functional is then

$$Z[j] = \int \mathcal{D}\bar{q}\mathcal{D}q\mathcal{D}A_\mu \, \exp\left\{ i \int \mathrm{d}^4x \left(\mathcal{L}^0_{\text{QCD}}[q, \bar{q}, A_\mu] + \mathcal{L}_{\text{ext}}[q, \bar{q}, j] \right) . \right\} \tag{3.40}$$

We will now use the CCWZ construction and effective field theory to construct the effective Lagrangian, which obeys

$$Z[j] = \int \mathcal{D}\pi \, \exp\left\{ i \int \mathrm{d}^4x \, \mathcal{L}_{\text{eff}}[\pi] \right\}, \tag{3.41}$$

where π are the Goldstone bosons.

Weinberg's power counting scheme

We now have a relationship between the underlying Lagrangian and the effective Lagrangian, of the form Eq. (1.2). However, as discussed earlier, we cannot solve QCD at low energies and thus have no way to integrate out the high energy degrees of freedom. Instead, we have to make use of Weinberg's "theorem". Using the CCWZ construction, we can obtain the most general Lagrangian obeying the symmetries of QCD. In this case, we have not laid any constraints on the theory but the most basic assumptions. This, however, will result in a theory with an infinite number of free parameters, making it rather unwieldy. To amend this, we need a scheme to order the terms to compute observable perturbatively. We are working in the low-energy regime, so it is natural to expand in pion momenta. As we saw in section 2.3, the terms in the Lagrangian will be made up of combinations of the terms e_μ and d_μ of the Maurer-Cartan form, $i\Sigma\partial_\mu\Sigma = e_\mu + d_\mu$. Therefore, all terms in the effective Lagrangian will be proportional to a certain number of derivatives of the pion fields, $\partial_\mu\pi$, which Lorentz invariance demands to be even.

Consider the matrix element $\mathcal{M}$ for a given Feynman diagram with external pion lines with momenta q_n, where both the energies and momenta are less than or equal to some energy scale Q. If we scale $Q \to tQ$, and consequently also the external momenta $q_n \to tq_n$, momentum conservation at each vertex ensures that each internal momentum p of the diagram scales as $p \to tp$. Assume this diagram is made up of V_i copies of the vertex i, which contain d_i derivatives. Each of these vertices then scale as t^{d_i}. The propagators contribute a factor p^{-2} and will therefore scale as t^{-2}, and the integration measure d^4p scales as t^4. This means that a matrix element with L loops and I internal lines scales as

$$\mathcal{M}(q) \to \mathcal{M}(tq) = t^D \mathcal{M}(q), \tag{3.42}$$

where

$$D = \sum_i V_i d_i - 2I + 4L. \tag{3.43}$$

D is called the *chiral dimension* of $\mathcal{M}$. Using the formula Eq. (2.22) for number of loops in a Feynman diagram, we get

$$D = \sum_i V_i(d_i - 2) + 2L + 2. \tag{3.44}$$

For low energy scales Q, the larges contribution will come from the matrix elements of the smallest chiral dimension D. A general process will consist of a sum of matrix elements of different chiral dimensions. We can expand this element in powers of the pion momenta by using t as the expansion parameter. The leading order term will be those where $L = 0$ and $d_i = 2$ so that $D = 2$. This means that all tree-level contributions of the lowest chiral dimension are from terms in the Lagrangian with exactly two derivatives. Next is $D = 4$,

which contains both tree-level contributions from terms with $d_i = 4$ and a one-loop contribution from $d_i = 2$. We therefore expand the effective Lagrangian as

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_2 + \mathcal{L}_4 + ..., \tag{3.45}$$

where $\mathcal{L}_{2n}$ contains $2n$ derivatives. This is equivalent to scaling the space-time cooridnates as $x^\mu \to tx^\mu$, and expanding the Lagrangian in powers of t.

We must also allow for the fact that pions have non-zero mass and that there is a finite isospin chemical potential. This is all implemented by external currents, which we also have to assume are small. Any external pions are on-shell, so the pion mass m_π must be less than the energy scale Q. As we will see, this corresponds to scaling the quark masses as $m_q \to t^2 m_q$. Similarly, μ_I must also be less than Q, which means that we scale it as $\mu_I \to t\mu_I$. Following these rules, each term in the effective Lagrangian will have a well-defined chiral dimension D, ensuring a consistent series expansion. The term $\mathcal{L}_D$ then contain all allowed terms that scale as t^D [9, 10, 32].

Non-linear realization

To parametrize the low energy behavior of QCD, we must find a representative element of the coset space of G, $G/H = \text{SU}(2)_A$. Let $g \in G$. We write $g = (U_L, U_R)$, where $U_R \in \text{SU}(2)_R$, $U_L \in \text{SU}(2)_L$. Elements $h \in H$ are then of the form $h = (U, U)$. A general element g can be written as

$$g = (U_L, U_R) = (1, U_R U_L^\dagger)(U_L, U_L). \tag{3.46}$$

Since $(U_L, U_L) \in H$, this means that we can write the coset gH as $(1, U_R U_L^\dagger)H$, which gives a way to choose a representative element for each coset. We identify

$$\Sigma = U_R U_L^\dagger. \tag{3.47}$$

This is our standard form for elements in gH. As we saw in section 2.3, it therefore implicitly define transformation properties of the Goldstone bosons, which is given by the function $h(g, \xi)$. For $\tilde{g} \in G$, we have

$$\tilde{g}(1, \Sigma) = (\tilde{U}_L, \tilde{U}_R)(1, U_R U_L^\dagger) = (1, \tilde{U}_R (U_R U_L^\dagger)\tilde{U}_L^\dagger)(\tilde{U}_L, \tilde{U}_L) = (1, \tilde{U}_R \Sigma \tilde{U}_L)\tilde{h}. \tag{3.48}$$

This gives the transformation rule

$$\Sigma \to \Sigma' = U_R \Sigma U_L^\dagger. \tag{3.49}$$

Under transformations by $h = (U, U^\dagger) \in H$, we have

$$\Sigma \to \Sigma' = U \Sigma U^\dagger. \tag{3.50}$$

Du to how G factors into two Lie groups, the constituents of the Mauer-Cartan form are

$$d_\mu = i\Sigma(x)^\dagger \partial_\mu \Sigma(x), \quad e_\mu = 0. \tag{3.51}$$

Using $\partial_\mu[\Sigma(x)^\dagger \Sigma(x)] = 0$, we can write

$$d_\mu d^\mu = -\Sigma(x)^\dagger [\partial_\mu \Sigma(x)]\Sigma(x)^\dagger [\partial^\mu \Sigma(x)] = \Sigma(x)^\dagger [\partial_\mu \Sigma(x)][\partial^\mu \Sigma(x)^\dagger]\Sigma(x). \tag{3.52}$$

In section 2.3, we found the lowest order terms, Eq. (2.104). As $d_\mu \in \mathfrak{su}(2)$, which we represent by the traceless Pauli matrices, we have

$$\text{Tr}\{d_\mu\} = 0. \tag{3.53}$$

This leaves us with the single leading order term

$$\text{Tr}\{d_\mu d^\mu\} = \text{Tr}\big\{\partial_\mu \Sigma (\partial^\mu \Sigma)^\dagger\big\}, \tag{3.54}$$

where we have used the cyclic property of the trace.

However, constructing the effective Lagrangian out of terms invariant under G is too restrictive to get the most general effective action. This only allows for an even number of d_μ's, and observed processes such as the decay of the neutral pion through $\pi^0 \to \gamma\gamma$ would not be possible [32]. This is because we have not allowed for terms that change the Lagrangian with a divergence term, as discussed in section 2.2. Terms of this type are called Wess-Zumino-Witten (WZW) terms [9]. We will not consider these here, as they do not affect the thermodynamic quantities in question [35].

External currents and explicit symmetry breaking

Using d_μ we are able to construct any terms in the effective Lagrangian corresponding to $j = 0$. To construct the effective Lagrangian when $j \neq 0$, which is needed to capture the effects of non-zero quark masses and external currents, we treat $G = U(1)_V \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ as a gauge group, and the source currents as the corresponding gauge fields. The effective Lagrangian is then constructed as the most general Lagrangian that is invariant under *local* transformations in G. The Ward-identities corresponding to the local symmetries of the Lagrangian are equivalent with the statement that $Z[j]$ is invariant under a gauge transformation of the external fields, in the absence of anomalies [36].

Following [32], we write the gauge transformation as

$$q_L \to e^{i\theta(x)/3} U_L(x) q_L, \quad q_R \to e^{i\theta(x)/3} U_R(x) q_R. \tag{3.55}$$

First, we consider the $U(1)_V$ transformation. The massless QCD Lagrangian then transforms as

$$\mathcal{L}^0_{\mathrm{QCD}} = i\bar{q}\not{D}q \to i\bar{q}e^{-i\theta(x)/3}\not{D}e^{i\theta(x)/3}q = i\bar{q}\not{D}q - \frac{1}{3}\bar{q}\gamma^\mu q \partial_\mu \theta(x). \tag{3.56}$$

This gives us the transformation rule

$$v^\mu_{(s)} \to v^\mu_{(s)} - \partial^\mu \theta(x). \tag{3.57}$$

Then, applying the $\mathrm{SU}(2)_R$ transformation, we get

$$\bar{q}_R \not{D} q_R + (i\bar{q}_R \gamma^\mu \tau_a q_R)\, r^a_\mu \to i\bar{q}_R \not{D} q_R + (i\bar{q}_R \gamma^\mu q_R)\, U_R^\dagger \partial_\mu U_R + [\bar{q}_R \gamma^\mu (U_R^\dagger \tau_a U_R) q_R]\, r^a_\mu \tag{3.58}$$

This, and the similar expression for U_L, gives the gauge transformation rules

$$r_\mu \to U_R(r_\mu + i\mathbb{1}\partial_\mu)U_R^\dagger, \tag{3.59}$$

$$l_\mu \to U_L(l_\mu + i\mathbb{1}\partial_\mu)U_L^\dagger, \tag{3.60}$$

$$s + ip \to U_R(s + ip)U_L^\dagger, \tag{3.61}$$

$$s - ip \to U_L(s - ip)U_R^\dagger. \tag{3.62}$$

The Goldstone fields now transform as

$$\Sigma(x) \to U_R(x)\Sigma(x)U_L(x)^\dagger, \tag{3.63}$$

and the derivative as

$$\begin{aligned} \partial_\mu \Sigma \to{}& \partial_\mu (U_R \Sigma U_L^\dagger) \\ ={}& U_R(\partial_\mu \Sigma)U_L^\dagger + (\partial_\mu U_R)\Sigma U_L^\dagger + U_R \Sigma (\partial_\mu U_L^\dagger) \\ ={}& U_R \left[\partial_\mu \Sigma + U_R^\dagger (\partial_\mu U_R)\Sigma + \Sigma(\partial_\mu U_L^\dagger)U_L\right] U_L^\dagger. \end{aligned} \tag{3.64}$$

As in the case of the gauge group of the strong force, we must introduce a covariant derivative, which is defined to transform as the object it acts on does. By this definition, the form of the covariant derivative is decided by what it acts on. Assume A, B and Σ transforms as $A \to U_R A U_R^\dagger$, $B \to U_L A U_L^\dagger$, and $\Sigma \to U_R \Sigma U_L^\dagger$. In each of these cases, we define the covariant derivative as

$$\nabla_\mu A = \partial_\mu A - i[r_\mu, A], \tag{3.65}$$

$$\nabla_\mu B = \partial_\mu B - i[l_\mu, B], \tag{3.66}$$

$$\nabla_\mu \Sigma = \partial_\mu \Sigma - i r_\mu \Sigma + i \Sigma l_\mu. \tag{3.67}$$

It follows from Eq. (3.64) that $\nabla_\mu \Sigma$ transforms as Σ, and the proof for the other two is similar. For invariant quantities under $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$, the covariant derivative is the ordinary partial derivative. In our case, $a_\mu = 0$, and the covariant derivative is

$$\nabla_\mu \Sigma = \partial_\mu \Sigma - i[v_\mu, \Sigma]. \tag{3.68}$$

If A, B and $C = AB$ are all quantities with a well-defined covariant derivative, then

$$\nabla_\mu(AB) = (\nabla_\mu A)B + A(\nabla_\mu B). \tag{3.69}$$

We show this in the case where $a_\mu = 0$. Assume A, B transforms as Σ. Then,

$$\begin{aligned}\nabla_\mu(AB) &= (\partial_\mu A)B + A(\partial_\mu B) - i\,[v_\mu, AB] = (\partial_\mu A - i\,[v_\mu, A])B + A(\partial_\mu B - i\,[v_\mu, B]) \\ &= (\nabla_\mu A)B + A(\nabla_\mu B).\end{aligned}$$

The more general theorem follows by applying the definition of the various covariant derivatives [34]. We can extend this to do integration by parts. Decomposing a 2-by-2 matrix M, as described in Appendix B, shows that the trace of the commutator of τ_b and M is zero:

$$\operatorname{Tr}\{[\tau_a, M]]\} = M_b \operatorname{Tr}\{[\tau_a, \tau_b]\} = 0.$$

Together with the fact that $\operatorname{Tr}\{\partial_\mu A\} = \partial_\mu \operatorname{Tr}\{A\}$, this gives the product rule for invariant traces:

$$\operatorname{Tr}\{A\nabla_\mu B\} = \partial_\mu \operatorname{Tr}\{AB\} - \operatorname{Tr}\{(\nabla_\mu A)B\}.$$

Assume A, K^μ, and AK^μ have well-defined covariant derivatives, and that $\operatorname{Tr}\{AK^\mu\}$ is invariant under transformations by the gauge group. Let $\operatorname{Tr}\{K^\mu\}$ be a space-time vector, and $\operatorname{Tr}\{A\}$ scalar. Let Ω be the domain of integration, with coordinates x and $\partial\Omega$ its boundary, with coordinates y. Then,

$$\int_\Omega \mathrm{d}x\ \operatorname{Tr}\{A\nabla_\mu K^\mu\} = -\int_\Omega \mathrm{d}x\ \operatorname{Tr}\{(\nabla_\mu A)K^\mu\} + \int_{\partial\Omega} \mathrm{d}y\, n_\mu \operatorname{Tr}\{AK^\mu\},$$

where n_μ is the normal vector of $\partial\Omega$ [13]. By the assumption of no variation on the boundary, the last term is constant when varying the action and may therefore be discarded.

In the presence of external source-currents, the Maurer-Cartan form becomes

$$d_\mu = i\Sigma^\dagger \nabla_\mu \Sigma. \tag{3.70}$$

Any invariant terms we constructed for $j = 0$ remain invariant, however the gauge fields provide new building blocks. The leading order kinetic term now becomes

$$\operatorname{Tr}\{d_\mu d^\mu\} = \operatorname{Tr}\big\{\nabla_\mu \Sigma (\nabla^\mu \Sigma)^\dagger\big\} \tag{3.71}$$

We define the scalar current

$$\chi = 2B_0(s + ip), \tag{3.72}$$

where B_0 is defined by the up quark condensate in the chiral limit by

$$\langle \bar{u}u \rangle_{j=0} = -f^2 B_0. \tag{3.73}$$

Here, f is the bare pion decay constant. The pseudoscalar current is thus $\chi^\dagger = 2B_0(s - ip)$. As we are going to set $\chi = 2B_0 m$, both χ and $\chi^\dagger$ are of chiral dimension 2. These can be combined to give more gauge-invariant terms, such as

$$\operatorname{Tr}\big\{\chi^\dagger \Sigma\big\}, \quad \operatorname{Tr}\big\{\Sigma^\dagger \chi\big\}. \tag{3.74}$$

However, under parity transformation, $\chi \to \chi^\dagger$ and $\Sigma \to \Sigma^\dagger$ [32], so to ensure the Lagrangian is scalar, the only allowed term is[1]

$$\operatorname{Tr}\big\{\chi^\dagger \Sigma + \Sigma^\dagger \chi\big\} \tag{3.75}$$

In the grand canonical ensemble, as described in Appendix A.1, we modify the Lagrangian as

$$\mathcal{L} \to \mathcal{L} + \mu Q, \tag{3.76}$$

[1] This corresponds to the fact that pions are pseudo scalars, they transform as $\pi \to -\pi$ under parity, as they are exitations in $\mathrm{SU}(2)_A$.

where Q is a conserved charge, and μ is the corresponding chemical component. We are interested in systems with non-zero isospin,

$$Q_I = \int \mathrm{d}^3x\, V_3^0 = \int \mathrm{d}^3x\, \frac{1}{2}\bar{q}\gamma^0\tau_3 q, \tag{3.77}$$

and a corresponding non-zero isospin chemical potential μ_I. We do this by considering the system with a external current

$$v_\mu = \frac{1}{2}\mu_I \delta_\mu^0 \tau_3. \tag{3.78}$$

The covariant derivative therefore is of chiral dimension 1. As in the case of Yang-Mills theory, we may form a field strength tensor to create invariant terms,

$$f_{\mu\nu}^R = \partial_\mu r_\nu - \partial_\nu r_\mu - i[r_\mu, r_\nu], \tag{3.79}$$

$$f_{\mu\nu}^L = \partial_\mu l_\nu - \partial_\nu l_\mu - i[l_\mu, l_\nu]. \tag{3.80}$$

In our case, these vanish, and we can therefore safely ignore terms including them.

For $j = 0$, the ground state is by assumption $\Sigma = \mathbb{1}$, the vacuum, and we can use the paramterization

$$\Sigma(x) = \exp\left\{i\frac{\pi_a\tau_a}{f}\right\}, \tag{3.81}$$

where f is the bare pion decay constant, π_a are the three Goldstone bosons, a set of real functions of space-time. This ensures that $\pi = 0$ corresponds to the vacuum. If we perform an infinitesimal isospin-transformation, and assume π/f small, then

$$\Sigma \to U_V \Sigma U_V^\dagger = \left(1 + i\eta_a\frac{1}{2}\tau_a\right)\left(1 + i\frac{1}{f}\pi_b\tau_b\right)\left(1 - i\eta_c\frac{1}{2}\tau_c\right) = 1 + i\frac{1}{f}\pi_a(\delta_{ac} + i\eta_b\epsilon_{abc})\tau_c, \tag{3.82}$$

or

$$\pi_a \to (\delta_{ac} + i\eta_b\epsilon_{abc})\pi_c. \tag{3.83}$$

The generators of π_a under isospin-transformations are thus the adjoint representation of $\mathfrak{su}(2)$, and they form an isospin triplet. For $\eta_1 = \eta_2 = 0$, i.e. transformations generated by τ_3, π_3 is invariant, which means that it has quantum number $I_3 = 0$.[2] π_1 and π_2 do not have a definite value of the third component of isospin, but rather for the first and second component. They are related to the observed, charged pions π_+ and π_- by [32]

$$\pi_a\tau_a = \begin{pmatrix} \pi_3 & \pi_1 - i\pi_2 \\ \pi_1 + i\pi_2 & -\pi_3 \end{pmatrix} = \begin{pmatrix} \pi_0 & \sqrt{2}\pi_- \\ \sqrt{2}\pi_+ & -\pi_0 \end{pmatrix}, \tag{3.84}$$

where $\pi_\pm$ has a third isospin-component of $I_3 = \pm 1$. For non-zero isospin chemical potential, however, we expect that the ground state may be rotated away from the vacuum. To find what the new ground state is, we have to minimize the Hamiltonian.

3.3 Leading order Lagrangian

The leading order χPT Lagrangian is made up of the terms Eq. (3.71) and Eq. (3.75), and reads

$$\mathcal{L}_2 = \frac{1}{4}f^2\,\mathrm{Tr}\{\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\} + \frac{1}{4}f^2\,\mathrm{Tr}\{\chi^\dagger\Sigma + \Sigma^\dagger\chi\}. \tag{3.85}$$

The external source currents are

$$s = m, \quad v_\mu = \frac{1}{2}\mu_I\delta_\mu^0\tau_3, \tag{3.86}$$

where m is the quark-mass matrix Eq. (3.31), and

$$\chi = 2B_0 m = \bar{m}^2\mathbb{1} + \Delta m^2\tau_3, \tag{3.87}$$

$$\nabla_\mu\Sigma = \partial_\mu\Sigma - i[v_\mu, \Sigma]. \tag{3.88}$$

[2] Authors differe if they define $\sqrt{2}\pi_\pm = \pi_1 \pm i\pi_2$, or with opposite signs.We choose the former, so that $\pi_+|0\rangle$ is the state with the quantum numbers of the positive pion.

We have defined

$$\bar{m}^2 = B_0(m_u + m_d), \quad \Delta m^2 = B_0(m_u - m_d). \tag{3.89}$$

To incorporate a finite isospin density, we must parametrize the Goldstone manifold differently than in the vacuum. We follow the analysis in [35]. We assume the ground state is independent of space, $\pi_a(x) = \pi_a^0$, and write it as

$$\Sigma_\alpha := \exp\{i\alpha n_a \tau_a\} = \cos\alpha + i n_a \tau_a \sin\alpha, \tag{3.90}$$

where

$$\alpha = \frac{1}{f}\sqrt{\pi_a^0 \pi_a^0}, \quad n_a = \frac{\pi_a^0}{\sqrt{\pi_a^0 \pi_a^0}}. \tag{3.91}$$

The covariant derivative is then

$$\nabla_\mu \Sigma_\alpha = -i v_\mu^a [\tau_a, \Sigma_\alpha]. \tag{3.92}$$

The two terms in the first order Lagrangian then becomes

$$\begin{aligned}\operatorname{Tr}\{\nabla_\mu \Sigma_\alpha (\nabla^\mu \Sigma_\alpha)^\dagger\} &= \frac{1}{4} n_a n_b \mu_I^2 \operatorname{Tr}\{[\tau_3, \tau_a][\tau_3, \tau_b]\} \sin^2\alpha = n_a n_b \mu_I^2 \sin^2\alpha \epsilon_{3ac}\epsilon_{3bd} \operatorname{Tr}\{\tau_c \tau_d\} \\ &= 2\mu_I^2 (n_1^2 + n_2^2)\sin^2\alpha.\end{aligned} \tag{3.93}$$

and

$$\begin{aligned}\operatorname{Tr}\{\chi^\dagger \Sigma_\alpha + \Sigma_\alpha^\dagger \chi\} &= \operatorname{Tr}\{\bar{m}^2(\Sigma_\alpha + \Sigma_\alpha^\dagger) + \Delta m^2(\tau_3 \Sigma_\alpha + \Sigma_\alpha^\dagger \tau_3)\} = 4\bar{m}^2 \cos\alpha + i\Delta m^2 \operatorname{Tr}\{[\tau_3, \tau_a]\} n_a \sin\alpha \\ &= 4\bar{m}^2 \cos\alpha.\end{aligned} \tag{3.94}$$

We see that, to first order, all results are independent of Δm. To find the new ground state, we minimize the Hamiltonian density. With the assumption that the fields are constant, the first order Hamiltonian density is

$$\mathcal{H}_2 = -\mathcal{L}_2 = -f^2 \left[\bar{m}^2 \cos^2\alpha + \frac{1}{2}\mu_I^2 (n_1^2 + n_2^2)\sin^2\alpha\right] \tag{3.95}$$

For $\mu_I = 0$, this is independent of n_a, and minimized by $\alpha = 0$. Now, as $n_i n_i = 1$, we have that $n_1^2 + n_2^2 = 1 - n_3^2$. This means that, for $\mu_I \neq 0$, the energy is minimized by $n_3 = 0$. We can write $n_1 = \cos\phi$, $n_2 = \sin\phi$, for some real number ϕ, which gives the ground state

$$\Sigma_\alpha = \mathbb{1}\cos\alpha + i(\tau_1 \cos\phi + \tau_2 \sin\phi)\sin\alpha. \tag{3.96}$$

We can choose, without loss of generality, $\phi = 0$ [37]. This corresponds to a change of basis of $\mathfrak{su}(2)$, $\tau_1 \to \tilde{\tau}_1 = \tau_1 \cos\phi + \tau_2 \sin\phi$ and $\tau_2 \to \tilde{\tau}_2 = -\tau_1 \sin\phi + \tau_2 \cos\phi$. With this, the new ground state is

$$\Sigma_\alpha = \exp\{i\alpha\tau_1\} \tag{3.97}$$

Any exited state is a transformation of the ground state by $\mathrm{SU}(2)_A$. For $\mu_I = 0$, this corresponds to

$$\Sigma(x) = U_R(x)\Sigma_0 U_L^\dagger(x) = U(x)\Sigma_0 U(x). \tag{3.98}$$

where

$$U(x) = \exp\left\{i\frac{\tau_a \pi_a(x)}{2f}\right\}. \tag{3.99}$$

We see that this recovers the parametrization Eq. (3.81). For $\mu_I \neq 0$, the ground state may be shifted, and so $U(x)$ must be too. The groundstate transforms as

$$\Sigma_0 \to \Sigma_\alpha = \hat{U}_L \Sigma_0 \hat{U}_R^\dagger = A_\alpha \Sigma_0 A_\alpha. \tag{3.100}$$

where

$$A_\alpha := \exp\left\{i\frac{1}{2}\alpha\tau_1\right\}. \tag{3.101}$$

This induces the following transformations for the fluctuations,

$$U_L \to \hat{U}_L U_L \hat{U}_L^\dagger = A_\alpha U_L A_\alpha^\dagger, \tag{3.102}$$

$$U_R \to \hat{U}_R U_R \hat{U}_R^\dagger = A_\alpha^\dagger U_R A_\alpha. \tag{3.103}$$

The new parametrization is thus

$$\Sigma(x) = A_\alpha[U(x)\Sigma_0 U(x)]A_\alpha. \tag{3.104}$$

With this, we can expand the first order Lagrangian, Eq. (3.85), in powers of π/f We will use this expansion to calculate the free energy density. To get the series expansion of Σ in powers of π/f, we start by using the fact that $\tau_a^2 = \mathbb{1}$ to write

$$A_\alpha = \sum_n^\infty \frac{1}{n!}\left(\frac{i\alpha}{2}\tau_1\right)^n = \sum_n^\infty \left[\frac{\mathbb{1}}{(2n)!}\left(\frac{i\alpha}{2}\right)^{(2n)} + \frac{\tau_1}{(2n+1)!}\left(\frac{i\alpha}{2}\right)^{(2n+1)}\right] = \mathbb{1}\cos\frac{\alpha}{2} + i\tau_1\sin\frac{\alpha}{2}. \tag{3.105}$$

The symmetry of $\pi_a\pi_b$ means that

$$(\pi_a\tau_a)^2 = \pi_a\pi_b\frac{1}{2}\{\tau_a,\tau_b\} = \pi_a\pi_a, \quad (\pi_a\tau_a)^3 = \pi_a\pi_a\pi_b\tau_b, \quad (\pi_a\tau_a)^4 = \pi_a\pi_a\pi_b\pi_b.$$

This gives us the expression

$$U\Sigma_0 U = 1 + i\frac{\pi_a\tau_a}{f} - \frac{\pi_a^2}{2f^2} - i\frac{\pi_a^2\pi_b\tau_b}{6f^3} + \frac{\pi_a^2\pi_b^2}{24f^4} + \mathcal{O}\left((\pi/f)^5\right). \tag{3.106}$$

We combine this result with Eq. (3.105) to get an expression for Σ up to $\mathcal{O}\left((\pi/f)^5\right)$

$$\begin{aligned}\Sigma = &\left(1 + i\frac{\pi_a\tau_a}{f} - \frac{\pi_a^2}{2f^2} - i\frac{\pi_a^2\pi_b\tau_b}{6f^3} + \frac{\pi_a^2\pi_b^2}{24f^4}\right)\cos^2\frac{\alpha}{2}\\ &- \left(1 + i\frac{\pi_a}{f}\tau_1\tau_a\tau_1 - \frac{\pi_a^2}{2f^2} - i\frac{\pi_a^2\pi_b}{6f^3}\tau_1\tau_b\tau_1 + \frac{\pi_a^2\pi_b^2}{24f^4}\right)\sin^2\frac{\alpha}{2}\\ &+ i\left(2\tau_1 + i\frac{\pi_a}{f}\{\tau_1,\tau_a\} - 2\tau_1\frac{\pi_a^2}{2f^2} - i\frac{\pi_a^2\pi_b}{6f^3}\{\tau_1,\tau_b\} + 2\tau_1\frac{\pi_a^2\pi_b^2}{24f^4}\right)\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}.\end{aligned} \tag{3.107}$$

Using trigonometric identities and the commutator,

$$\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2} = \cos\alpha, \quad 2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2} = \sin\frac{\alpha}{2}, \quad \tau_1\tau_a\tau_1 = -\tau_a + 2\delta_{1a}\tau_1,$$

the final expression of Σ to $\mathcal{O}\left((\pi/f)^5\right)$ is

$$\Sigma = \left(1 - \frac{\pi_a^2}{2f^2} + \frac{\pi_a^2\pi_b^2}{24f^4}\right)(\cos\alpha + i\tau_1\sin\alpha) + \left(\frac{\pi_a}{f} - \frac{\pi_b^2\pi_a}{6f^3}\right)\left(i\tau_a - 2i\delta_{a1}\tau_1\sin^2\frac{\alpha}{2} - \delta_{a1}\sin\alpha\right). \tag{3.108}$$

The kinetic term in the χPT Lagrangian is

$$\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger = \partial_\mu\Sigma\partial^\mu\Sigma^\dagger - i\left(\partial_\mu\Sigma\left[v^\mu,\Sigma^\dagger\right] - \text{h.c.}\right) - \left[v_\mu,\Sigma\right]\left[v_\mu,\Sigma^\dagger\right]. \tag{3.109}$$

Using Eq. (3.108) we find the expansion of the constitutive parts of the kinetic term to be

$$\begin{aligned}\partial_\mu\Sigma = &\left[\left(\frac{-1}{f^2} + \frac{\pi_b^2}{6f^4}\right)(\pi_a\partial_\mu\pi_a)\cos\alpha - \left(\frac{\partial_\mu\pi_1}{f} - \frac{\pi_b^2\partial_\mu\pi_1 + 2\pi_1\pi_b\partial_\mu\pi_b}{6f^3}\right)\sin\alpha\right]\\ &- \left[\left(\frac{-1}{f^2} + \frac{\pi_b^2}{6f^4}\right)(\pi_a\partial_\mu\pi_a)\sin\alpha - \left(\frac{\partial_\mu\pi_1}{f} - \frac{\pi_b^2\partial_\mu\pi_1 + 2\pi_1\pi_b\partial_\mu\pi_b}{6f^3}\right)2\sin^2\frac{\alpha}{2}\right]i\tau_1\\ &+ \left(\frac{\partial_\mu\pi_a}{f} - \frac{\pi_b^2\partial_\mu\pi_a + 2\pi_a\pi_b\partial_\mu\pi_b}{6f^3}\right)i\tau_a,\end{aligned} \tag{3.110}$$

and

$$[v_\mu,\Sigma] = -\mu_I\delta_\mu^0\left\{\left[\left(1 - \frac{\pi_a^2}{2f^2} + \frac{\pi_a^2\pi_b^2}{24f^4}\right)\sin\alpha + \left(\frac{\pi_1}{f} - \frac{\pi_b^2\pi_1}{6f^3}\right)\cos\alpha\right]\tau_2 - \left(\frac{\pi_2}{f} - \frac{\pi_b^2\pi_2}{6f^3}\right)\tau_1\right\}. \tag{3.111}$$

Combining Eq. (3.110) and Eq. (3.111) gives the following terms

$$\mathrm{Tr}\{\partial_\mu\Sigma\partial^\mu\Sigma^\dagger\} = \frac{2}{f^2}\partial_\mu\pi_a\partial^\mu\pi_a + \frac{2}{3f^4}\left[(\pi_a\partial_\mu\pi_a)(\pi_b\partial^\mu\pi_b) - (\pi_a\partial_\mu\pi_b)(\pi_b\partial^\mu\pi_a)\right], \tag{3.112}$$

$$\begin{aligned}-i\,\mathrm{Tr}\{\partial^\mu\Sigma\left[v_\mu,\Sigma^\dagger\right] - \text{h.c.}\} &= 4\mu_I\frac{\partial_0\pi_2}{f} + 8\mu_I\frac{\pi_3}{3f^3}\sin\alpha(\pi_2\partial_0\pi_3 - \pi_3\partial_0\pi_2)\sin\alpha \\ &\quad + \left(\frac{4\mu_I}{f^2}\cos\alpha - \frac{8\mu_I\pi_1}{3f^3}\sin\alpha - \frac{4\mu_I\pi_a\pi_a}{3f^4}\cos\alpha\right)(\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1),\end{aligned} \tag{3.113}$$

$$-\mathrm{Tr}\{[v_\mu,\Sigma]\left[v^\mu,\Sigma^\dagger\right]\} = {\mu_I}^2\left[2\sin^2\alpha + \left(\frac{2}{f} - \frac{4\pi_a\pi_a}{3f^3}\right)\pi_1\sin 2\alpha + \left(\frac{2}{f^2} - \frac{2\pi_a\pi_a}{3f^4}\right)\pi_a\pi_b k_{ab}\right], \tag{3.114}$$

$$\mathrm{Tr}\{\chi^\dagger\Sigma + \Sigma^\dagger\chi\} = \bar{m}^2\left(4\cos\alpha - \frac{4\pi_1}{f}\sin\alpha - \frac{2\pi_a\pi_a}{f^2}\cos\alpha + \frac{2\pi_1\pi_a\pi_a}{3f^3}\sin\alpha + \frac{(\pi_a\pi_a)^2}{6f^4}\cos\alpha\right), \tag{3.115}$$

where $k_{ab} = \delta_{a1}\delta_{b1}\cos 2\alpha + \delta_{a2}\delta_{b2}\cos^2\alpha - \delta_{a3}\delta_{b3}\sin^2\alpha$. Notice that the mass term is independent of the difference in quark masses, Δm. If we write the Lagrangian as show in Eq. (3.85) as $\mathcal{L}_2 = \mathcal{L}_2^{(0)} + \mathcal{L}_2^{(1)} + \mathcal{L}_2^{(2)} + ...$, where $\mathcal{L}_2^{(n)}$ contains all terms of order $(\pi/f)^n$, then the result of the series expansion is

$$\mathcal{L}_2^{(0)} = f^2\left(\bar{m}^2\cos\alpha + \frac{1}{2}\mu^2\sin^2\alpha\right), \tag{3.116}$$

$$\mathcal{L}_2^{(1)} = f(\mu_I^2\cos\alpha - \bar{m}^2)\pi_1\sin\alpha + f\mu_I\partial_0\pi_2\sin\alpha, \tag{3.117}$$

$$\mathcal{L}_2^{(2)} = \frac{1}{2}\partial_\mu\pi_a\partial^\mu\pi_a + \mu_I\cos\alpha\,(\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1) - \frac{1}{2}\bar{m}^2\pi_a\pi_a\cos\alpha + \frac{1}{2}\mu_I^2\pi_a\pi_b k_{ab}, \tag{3.118}$$

$$\begin{aligned}\mathcal{L}_2^{(3)} &= \frac{\pi_a\pi_a\pi_1}{6f}(\bar{m}^2\sin\alpha - 2{\mu_I}^2\sin 2\alpha) \\ &\quad - \frac{2\mu_I}{3f}\left[\pi_1(\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1) + \pi_3(\pi_3\partial_0\pi_2 - \pi_2\partial_0\pi_3)\right]\sin\alpha,\end{aligned} \tag{3.119}$$

$$\begin{aligned}\mathcal{L}_2^{(4)} &= \frac{1}{6f^2}\left\{\frac{1}{4}\bar{m}^2(\pi_a\pi_a)^2\cos\alpha - \left[(\pi_a\pi_a)(\partial_\mu\pi_b\partial^\mu\pi_b) - (\pi_a\partial_\mu\pi_a)(\pi_b\partial^\mu\pi_b)\right]\right\} \\ &\quad - \frac{\mu_I\pi_a\pi_a}{3f^2}\left[(\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1)\cos\alpha + \frac{1}{2}\mu_I\pi_a\pi_b k_{ab}\right].\end{aligned} \tag{3.120}$$

3.4 Next to leading order Lagrangian

Constructing the next-to-leading order (NLO) Lagrangian is now a business of combining the building blocks we found in section 3.2. We must include all terms that obey all symmetries and that are fourth-order in Weinberg's power counting scheme and remove possible redundant terms, as discussed in Appendix D. We will quote the result from [32],

$$\begin{aligned}\mathcal{L}_4 &= \frac{l_1}{4}\,\mathrm{Tr}\{\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\}^2 + \frac{l_2}{4}\,\mathrm{Tr}\{\nabla_\mu\Sigma(\nabla_\nu\Sigma)^\dagger\}\,\mathrm{Tr}\{\nabla^\mu\Sigma(\nabla^\nu\Sigma)^\dagger\} + \frac{l_3+l_4}{16}\,\mathrm{Tr}\{\chi\Sigma^\dagger + \Sigma\chi^\dagger\}^2 \\ &\quad + \frac{l_4}{8}\,\mathrm{Tr}\{\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\}\,\mathrm{Tr}\{\chi\Sigma^\dagger + \Sigma\chi^\dagger\} - \frac{l_7}{16}\,\mathrm{Tr}\{\chi\Sigma^\dagger - \Sigma\chi^\dagger\}^2 + \frac{h_1+h_3-l_4}{4}\,\mathrm{Tr}\{\chi\chi^\dagger\} \\ &\quad + \frac{h_1-h_3-l_4}{16}\left[\mathrm{Tr}\{\chi\Sigma^\dagger + \Sigma\chi^\dagger\}^2 + \mathrm{Tr}\{\chi\Sigma^\dagger - \Sigma\chi^\dagger\}^2 - 2\,\mathrm{Tr}\left\{\left(\chi\Sigma^\dagger\right)^2 + \left(\Sigma\chi^\dagger\right)^2\right\}\right].\end{aligned} \tag{3.121}$$

We have ignored terms containing the field strength tensors for external fields, as they vanish in our case. The parameters l_i and h_i are called low energy constants, or LEO. In Appendix D, we show how to rewrite the Lagrangian to match the one used in [35, 38]. To obtain $\mathcal{L}_4$ to $\mathcal{O}\left((\pi/f)^3\right)$, we use the result from Eq. (3.110)

and Eq. (3.111), which gives

$$\begin{aligned}
\mathrm{Tr}\{\partial_\mu\Sigma\partial_\nu\Sigma^\dagger\} &= 2\frac{\partial_\mu\pi_a\partial_\nu\pi_a}{f^2}\\
-i\,\mathrm{Tr}\{\partial_\mu\Sigma\left[v_\nu,\Sigma^\dagger\right]-\text{h.c.}\} &= \frac{2\mu_I\pi_2}{f}(\delta^0_\mu\partial_\nu+\delta^0_\nu\partial_\mu)\sin\alpha+\frac{2\mu_I}{f^2}[\pi_1(\delta^0_\mu\partial_\nu+\delta^0_\nu\partial_\mu)\pi_2-\pi_2(\delta^0_\mu\partial_\nu+\delta^0_\nu\partial_\mu)\pi_1]\cos\alpha\\
-\mathrm{Tr}\{[v_\nu,\Sigma]\left[v_\nu,\Sigma^\dagger\right]\} &= 2{\mu_I}^2\delta^0_\mu\delta^0_\nu\left[\sin^2\alpha+\frac{\pi_1}{f}\sin 2\alpha+\frac{\pi_a\pi_b}{f^2}k_{ab}\right],
\end{aligned}$$

up to $\mathcal{O}\left((\pi/f)^3\right)$. Inserting $\chi=2B_0m=\bar{m}^2\mathbb{1}+\Delta m^2\tau_3$ gives

$$\begin{aligned}
\chi\Sigma^\dagger+\Sigma\chi^\dagger &= 2(\bar{m}^2+\Delta m^2\tau_3)\left[\left(1-\frac{\pi_a^2}{2f^2}\right)\cos\alpha-\frac{\pi_1}{f}\sin\alpha\right]\\
&\quad+2\Delta m^2\left[\left(1-\frac{\pi_a^2}{2f^2}\right)\tau_2\sin\alpha+\frac{\pi_a}{f}\left(\delta_{a1}\tau_2\cos\alpha-\delta_{a2}\tau_1\right)\right],\\
\chi\Sigma^\dagger-\Sigma\chi^\dagger &= -2i\bar{m}^2\left[\left(1-\frac{\pi_a^2}{2f^2}\right)\tau_1\sin\alpha+\frac{\pi_a}{f}\left(\tau_a-2\delta_{1a}\tau_1\sin^2\frac{\alpha}{2}\right)\right]-2i\Delta m^2\frac{\pi_3}{f}.
\end{aligned}$$

Combining these results gives all the terms in $\mathcal{L}_4$, to $\mathcal{O}\left((\pi/f)^3\right)$:

$$\begin{aligned}
&\mathrm{Tr}\{\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\}^2=\mathrm{Tr}\{\partial_\mu\Sigma\partial^\mu\Sigma^\dagger-i\left(\partial_\mu\Sigma\left[v^\mu,\Sigma^\dagger\right]-\text{h.c.}\right)-[v_\mu,\Sigma]\left[v^\mu,\Sigma^\dagger\right]\}^2\\
&\quad=\frac{8\mu_I^2}{f^2}(\partial_\mu\pi_a\partial^\mu\pi_a+2\partial_\mu\pi_2\partial^\mu\pi_2)\sin^2\alpha\\
&\quad+16\mu_I^3\left[\frac{\partial_0\pi_2}{f}\sin^3\alpha+\frac{1}{f^2}\left(3\pi_1\partial_0\pi_2-\pi_2\partial_0\pi_1\right)\cos\alpha\sin^2\alpha\right]\\
&\quad+4\mu_I^4\left\{\sin^4\alpha+2\sin^2\alpha\left[\frac{\pi_1}{f}\sin 2\alpha+\frac{\pi_a\pi_b}{f^2}\left(k_{ab}+2\delta_{a1}\delta_{a2}\cos^2\alpha\right)\right]\right\},\qquad(3.122)\\
&\mathrm{Tr}\{\nabla_\mu\Sigma(\nabla_\nu\Sigma)^\dagger\}\,\mathrm{Tr}\{\nabla^\mu\Sigma(\nabla^\nu\Sigma)^\dagger\}\\
&\quad=\frac{4{\mu_I}^2}{f^2}\left(\partial_0\pi_a\partial_0\pi_a+\partial_0\pi_2\partial_0\pi_2+\partial_\mu\pi_2\partial^\mu\pi_2\right)\sin^2\alpha\\
&\quad+16\mu_I^3\left[\frac{\partial_0\pi_2}{f}\ \sin^3\alpha+\frac{1}{f^2}\left(3\pi_1\partial_0\pi_2-\pi_2\partial_0\pi_1\right)\cos\alpha\sin^2\alpha\right]\\
&\quad+4\mu_I^4\left\{\sin^4\alpha+2\sin^2\alpha\left[\frac{\pi_1}{f}\sin 2\alpha+\frac{\pi_a\pi_b}{f^2}\left(k_{ab}+2\delta_{a1}\delta_{a2}\cos^2\alpha\right)\right]\right\},\qquad(3.123)\\
&\mathrm{Tr}\{\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\}\,\mathrm{Tr}\{\chi\Sigma^\dagger+\Sigma\chi^\dagger\}\\
&\quad=4\bar{m}^2\left\{2\frac{\partial_\mu\pi_a\partial^\mu\pi_a}{f^2}\cos\alpha+4\mu_I\left[\frac{\partial_0\pi_2}{2f}\sin 2\alpha+\frac{1}{f^2}\left(\pi_1\partial_0\pi_2\cos 2\alpha-\pi_2\partial_0\pi_1\cos^2\alpha\right)\right]\right.\qquad(3.124)\\
&\quad\left.+{\mu_I}^2\left[2\cos\alpha\sin^2\alpha-2\frac{\pi_1}{f}\sin\alpha\left(2-3\sin^2\alpha\right)+\frac{1}{f^2}\left(\pi_1^2[2-9\sin^2\alpha]+\pi_2^2[2-3\sin^2\alpha]-3\pi_3^2\sin^2\alpha\right)\cos\alpha\right]\right\},\\
&\mathrm{Tr}\{\chi\Sigma^\dagger+\Sigma\chi^\dagger\}^2=16\bar{m}^4\left[\cos^2\alpha-\frac{\pi_1}{f}\sin 2\alpha+\frac{1}{f^2}\left(\pi_1^2\sin^2\alpha-\pi_a\pi_a\cos^2\alpha\right)\right],\qquad(3.125)\\
&\mathrm{Tr}\{\chi\Sigma^\dagger-\Sigma\chi^\dagger\}^2=-16\left(\frac{\Delta m^2\pi_3}{f}\right)^2,\qquad(3.126)\\
&\mathrm{Tr}\left\{\left(\chi\Sigma^\dagger\right)^2+\left(\Sigma\chi^\dagger\right)^2\right\}\\
&\quad=4\bar{m}^4\left(\cos 2\alpha-2\frac{\pi_1}{f}\sin 2\alpha-2\frac{\pi_a\pi_a}{f^2}\cos^2\alpha+2\frac{\pi_1^2}{f^2}\sin^2\alpha\right)+4\Delta m^4\left(1-2\frac{\pi_3^2}{f^2}\right),\qquad(3.127)\\
&\mathrm{Tr}\{\chi^\dagger\chi\}=2\bar{m}^4+2\Delta m^4.\qquad(3.128)
\end{aligned}$$

The different terms of the NLO Lagrangian is

$$\mathcal{L}_4^{(0)} = (l_1+l_2)\mu_I^4 \sin^4\alpha + (l_3+l_4)\bar{m}^2\cos^2\alpha + l_4\bar{m}{\mu_I}^2\cos\alpha\sin^2\alpha + (h_1-l_4)\bar{m}^4 + h_3\Delta m^4, \tag{3.129}$$

$$\begin{aligned}\mathcal{L}_4^{(1)} &= 4\mu_I^3\frac{l_1+l_2}{f}\left(\partial_0\pi_2 + \mu_I\cos\alpha\pi_1\right)\sin^3\alpha - \frac{l_3+l_4}{f}\bar{m}^4\pi_1\sin 2\alpha \\ &+ \bar{m}^2\frac{l_4}{f}\left[\mu_I\partial_0\pi_2\sin 2\alpha - {\mu_I}^2\pi_1\sin\alpha\left(3\sin^2\alpha - 2\right)\right],\end{aligned} \tag{3.130}$$

$$\begin{aligned}\mathcal{L}_4^{(2)} &= 2\mu_I^2\frac{l_1}{f^2}(\partial_\mu\pi_a\partial^\mu\pi_a + 2\partial_0\pi_2\partial_0\pi_2)\sin^2\alpha + {\mu_I}^2\frac{l_2}{f^2}\left(\partial_\mu\pi_2\partial^\mu\pi_2 + 2\partial_0\pi_a\partial_0\pi_a + 2\partial_0\pi_2\partial_0\pi_2\right)\sin^2\alpha \\ &+ 2\frac{l_1+l_2}{f^2}\left[2\mu_I^3\left(3\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1\right)\cos\alpha + \mu_I^4\pi_a\pi_b\left(k_{ab} + 2\delta_{a1}\delta_{a2}\cos^2\alpha\right)\right]\sin^2\alpha \\ &+ \frac{l_3+l_4}{f^2}\bar{m}^2\left(\pi_1^2\sin^2\alpha - \pi_a\pi_a\cos^2\alpha\right) + \frac{l_4}{f^2}\bar{m}^2\Big[\partial_\mu\pi_a\partial^\mu\pi_a\cos\alpha + 4\mu_I\left(\pi_1\partial_0\pi_2\cos 2\alpha - \pi_2\partial_0\pi_1\cos^2\alpha\right) \\ &+ \frac{1}{2}{\mu_I}^2\left(\pi_1^2[2-9\sin^2\alpha] + \pi_2^2[2-3\sin^2\alpha] - 3\pi_3^2\sin^2\alpha\right)\cos\alpha\Big] + \frac{l_7}{f^2}\Delta m^2\pi_3^2.\end{aligned} \tag{3.131}$$

3.5 Propagator

We may write the quadratic part of the Lagrangian Eq. (3.118) as [3]

$$\mathcal{L}_2^{(2)} = \frac{1}{2}\sum_a \partial_\mu\pi_a\partial^\mu\pi_a + \frac{1}{2}m_{12}(\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1) - \frac{1}{2}\sum_a m_a^2\pi_a^2, \tag{3.132}$$

where

$$m_1^2 = \bar{m}^2\cos\alpha - \mu_I^2\cos 2\alpha, \tag{3.133}$$

$$m_2^2 = \bar{m}^2\cos\alpha - \mu_I^2\cos^2\alpha, \tag{3.134}$$

$$m_3^2 = \bar{m}^2\cos\alpha + \mu_I^2\sin^2\alpha, \tag{3.135}$$

$$m_{12} = 2\mu_I\cos\alpha. \tag{3.136}$$

The inverse propagator is given by the functional derivative,

$$D_{ab}^{-1}(x-y) = \frac{\delta S[\pi]}{\delta\pi_a(x)\pi_b(y)} = \left[-\delta_{ab}(\partial_x^2 + m_a^2) + m_{12}(\delta_{a1}\delta_{b2} - \delta_{a2}\delta_{b1})\partial_{x,0}\right]\delta(x-y). \tag{3.137}$$

The momentum space inverse propagator is

$$D_{ab}^{-1}(p) = \delta_{ab}(p^2 - m_a^2) + ip_0 m_{12}(\delta_{a1}\delta_{b2} - \delta_{a2}\delta_{b1}). \tag{3.138}$$

The spectrum of the particles is given by solving $\det\left(D^{-1}\right) = 0$ for p^0. With $p = (p_0, \vec{p})$ as the four momentum, this gives

$$\det\left(D^{-1}\right) = D_{33}^{-1}\left(D_{11}^{-1}D_{22}^{-1} + (D_{12}^{-1})^2\right) = \left(p^2 - m_3^2\right)\left[\left(p^2 - m_1^2\right)\left(p^2 - m_2^2\right) - p_0^2 m_{12}^2\right] = 0.$$

This equation has the solutions

$$E_0^2 = |\vec{p}|^2 + m_3^2, \tag{3.139}$$

$$E_\pm^2 = |\vec{p}|^2 + \frac{1}{2}\left(m_1^2 + m_2^2 + m_{12}^2\right) \pm \frac{1}{2}\sqrt{4|\vec{p}|^2 m_{12}^2 + \left(m_1^2 + m_2^2 + m_{12}^2\right)^2 - 4m_1^2 m_2^2}. \tag{3.140}$$

These are the energies of three particles π_0, π_+ and π_-. π_0 is π_3, while $\pi_\pm$ are linear combinations of π_1 and π_2. [4] We will show that for $\mu_I < m_\pi$, $\alpha = 0$, before it starts to increas for $\mu_I \geq m_\pi$. This result is presented

[3] Summation over isospin index (a, b, c) will be explicit in this section.

[4] An unfortunate notational convention is that E_+ is the energy of π_--particle, and E_+ for the π_+-particle. This is because the positivley charged pion, π_+, has isospin $I_3 = +1$, so that the mass will decrease as μ_I increseas, and hence the negative sign.

in chapter 4. For $\alpha = 0$, we get

$$\frac{1}{2}(m_1^2 + m_2^2 + m_{12}^2) = \bar{m}^2 + \mu_I^2, \quad m_1^2 m_2^2 = (\bar{m}^2 - \mu_I^2)^2, \quad m_3^2 = \bar{m}^2,$$
$$\implies E_\pm^2 = |\vec{p}|^2 + \bar{m}^2 + \mu_I^2 \pm 2\mu_I \sqrt{|\vec{p}|^2 + \bar{m}^2}.$$

This corresponds to a Zeeman-like splitting of the energies,

$$E_0 = \sqrt{|\vec{p}|^2 + \bar{m}^2}, \tag{3.141}$$
$$E_\pm = \pm\mu_I + \sqrt{|\vec{p}|^2 + \bar{m}^2}. \tag{3.142}$$

The (tree-level) masses of these particles are found by setting $\vec{p} = 0$ and are

$$m_0^2 = m_3^2, \tag{3.143}$$
$$m_\pm^2 = \frac{1}{2}\left[m_1^2 + m_2^2 + m_{12}^2\right] \pm \frac{1}{2}\sqrt{\left(m_1^2 + m_2^2 + m_{12}^2\right)^2 - 4m_1^2 m_2^2}. \tag{3.144}$$

Using the result for α, Figure 3.1 shows the masses as functions of μ_I. We observe that the mass of the π_--particle goes to zero at $\mu_I = m_\pi$. This is indicative of spontaneous symmetry breaking, which we will investigate in the next chapter.

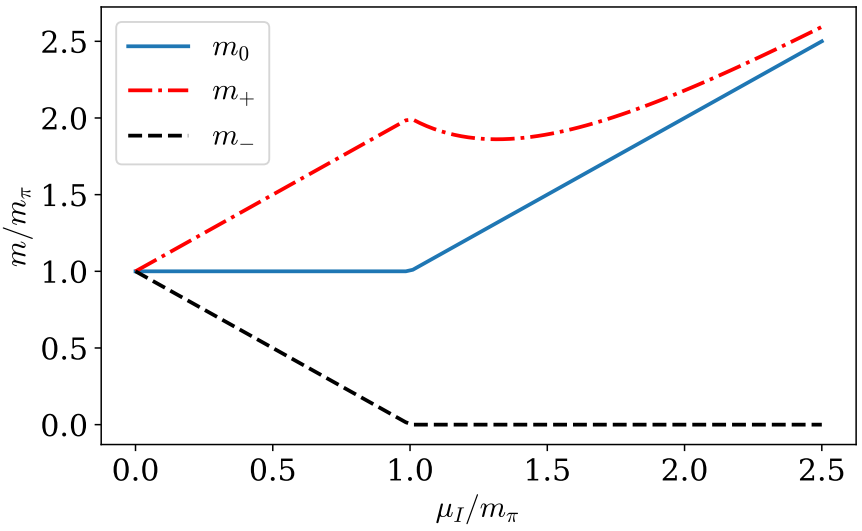


Figure 3.1: The masses of the three particles as functions of isosopin chemical potential. Rresults are given in units of the pion mass, m_π.

With the energies of the pions, we can write the determinant of the inverse propagator as

$$\det(D^{-1}) = (p_0^2 - E_0^2)(p_0^2 - E_+^2)(p_0^2 - E_-^2). \tag{3.145}$$

The propagator and the inverse propagator in momentum space obey[5]

$$\sum_c D_{ac}(p) D_{cb}^{-1}(p) = i\delta_{ab} \tag{3.146}$$

Using this, we can solve for the propagator

$$D = (-iD^{-1})^{-1} = i \begin{pmatrix} \frac{p^2 - m_2^2}{(p_0^2 - E_+^2)(p_0^2 - E_-^2)} & \frac{-ip_0 m_{12}}{(p_0^2 - E_+^2)(p_0^2 - E_-^2)} & 0 \\ \frac{ip_0 m_{12}}{(p_0^2 - E_+^2)(p_0^2 - E_-^2)} & \frac{p^2 - m_1^2}{(p_0^2 - E_+^2)(p_0^2 - E_-^2)} & 0 \\ 0 & 0 & \frac{1}{p_0^2 - E_0^2} \end{pmatrix}. \tag{3.147}$$

With these results, we can start calculating thermodynamic quantities.

[5]One has to be carefull regarding the factor i in the physicist's definition of propagators. It has the consequence that D^{-1} is not strictly the operator inverse of the propagator D.

Chapter 4

Thermodynamics

With the leading order and next-to-leading order χPT Lagrangian, we can now compute thermodynamic properties such as the free energy and the equation of state and investigate the phase transition to the pion condensate phase. We will start with the free energy, computing the leading-order contribution to one loop, and the next-to-leading order contribution at tree-level, following the procedure used in [35, 38].[1]

4.1 Free energy at lowest order

The free energy density of a homogenous system is

$$\mathcal{F} = -\frac{1}{V\beta} \ln \mathcal{Z}. \tag{4.1}$$

Here, $\mathcal{Z}$ is the partition function, V is the volume of space, and $\beta = 1/T$ is inverse temperature. Using imaginary-time formalism for thermal field theory, described in Appendix A, we find that the partition function is given by the path integral of the *Euclidean* Lagrange density, as shown in equation Eq. (A.12). In the zero temperature limit $\beta \to \infty$, the partition function is related to the generating functional $Z = Z[j]$, as described in section 2.1, by a Wick rotation. The free energy density at zero temperature is therefore

$$\mathcal{F} = \frac{i}{VT} \ln Z, \tag{4.2}$$

where VT is the volume of space-time. This equals the effective potential in the ground state. In section 2.1, we found an explicit formula for this to one-loop order, Eq. (2.41). We write $\mathcal{F} = \mathcal{F}^{(0)} + \mathcal{F}^{(1)} + \dots$, where $\mathcal{F}^{(n)}$ refers to the n-loop contributions to the free energy density.

Tree-level contribution

The tree-level contribution $\mathcal{F}^{(0)}$ is the classical potential, which is given by the static (π-independent) part of the Lagrangian. From Eq. (3.116) we have the leading order contribution,

$$\mathcal{F}_2^{(0)} = -\mathcal{L}_2^{(0)} = -f^2 \left(\bar{m}^2 \cos\alpha + \frac{1}{2}\mu_I^2 \sin^2\alpha \right), \tag{4.3}$$

where α parameterizes the ground state. This parameter must minimize the free energy, which means that

$$\frac{\partial}{\partial\alpha}\mathcal{F}_2^{(0)} = f^2 \left(\bar{m}^2 - \mu_I^2 \cos\alpha \right) \sin\alpha = 0. \tag{4.4}$$

[1] Leading order and next-to-leading order, in this context, refers to Weinberg's power counting scheme.

This equation defines the relationship between the chemical potential μ_I and α, as illustrated in Figure 4.1. This gives the criterion

$$\alpha \in \{0, \pi\} \quad \text{or} \quad \cos\alpha = \frac{\bar{m}^2}{{\mu_I}^2}. \tag{4.5}$$

As we see in the figure, $\alpha = \pi$ is a maximum and thus unstable. This means that for all values $\mu_I^2 \leq \bar{m}^2$, we will have $\alpha = 0$, and the system will remain in the vacuum.

In our discussion of the effective potential, we also found that the ground state should minimize the classical potential, as shown by Eq. (2.36). As a consequence, the linear part of the classical potential should vanish. The linear part of the classical potential is given by Eq. (3.117) to leading order. It reads $\mathcal{V}^{(1)} = f(\mu_I{}^2 \cos\alpha - \bar{m}^2)\sin\alpha\,\pi_1$, and vanishes given the minimizition criterion Eq. (4.5).

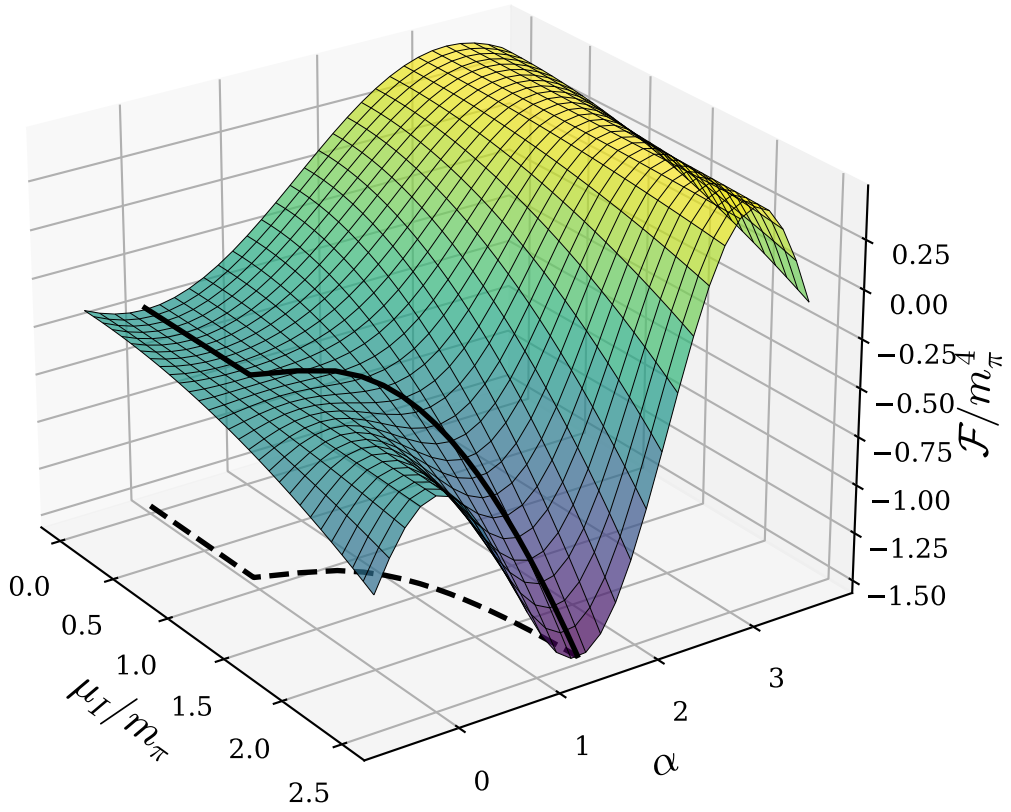


Figure 4.1: The free energy surface as a function of μ_I and α. α is the found by minimizing $\mathcal{F}$ for a given μ_I. This leads to a curve across the free energy surface, as show in the plot.

One-loop contribution

The one loop contribution to the free energy density is

$$\mathcal{F}^{(1)} = -\frac{i}{VT}\frac{1}{2}\,\mathrm{Tr}\left\{\ln\left(-\frac{\delta^2 S[\pi=0]}{\delta\pi_a(x)\,\delta\pi_b(y)}\right)\right\}. \tag{4.6}$$

This can be evaluated using the rules for functional differentiation given in Appendix C. To leading order,

$$\frac{\delta^2 S[\pi=0]}{\delta\pi_a(x)\,\delta\pi_b(y)} = \frac{\delta^2}{\delta\pi_a(x)\,\delta\pi_b(y)}\int \mathrm{d}^4x\,\mathcal{L}_2^{(2)} = D_{ab}^{-1}(x-y). \tag{4.7}$$

Here, $\mathcal{L}_2^{(2)}$ is the quadratic part of the Lagrangian, as given in Eq. (3.132), and D^{-1} is the corresponding inverse propagator of the pion fields,

$$D_{ab}^{-1}(x-y) = \left[-\delta_{ab}(\partial_x^2 + m_a^2) + m_{12}(\delta_{a1}\delta_{b2} - \delta_{a2}\delta_{b1})\partial_{x,0}\right]\delta(x-y) \tag{4.8}$$

The inverse propagator is a matrix, which means that the determinant in Eq. (4.6) is both a matrix determinant, over the three pion indices, and a functional determinant. In section 3.5 we found the matrix part of the determinant in momentum space, which we can write using the dispersion relations of the pion fields

$$\det\left(-D^{-1}\right) = \det\left(-p_0^2 + E_0^2\right)\det\left(-p_0^2 + E_+^2\right)\det\left(-p_0^2 + E_-^2\right). \tag{4.9}$$

These dispersion relations are functions of the three-momentum $\vec{p}$, and are given in Eqs. (3.139) and (3.140). The functional determinant can therefore be evaluated as

$$\begin{aligned}
\mathrm{Tr}\left\{\ln\left(-\frac{\delta^2 S[\pi=0]}{\delta\pi_a(x)\,\delta\pi_b(y)}\right)\right\} &= \ln\det\left(-p_0^2 + E_0^2\right) + \ln\det\left(-p_0^2 + E_+^2\right) + \ln\det\left(-p_0^2 + E_-^2\right) \\
&= \mathrm{Tr}\left\{\ln\left(-p_0^2 + E_0^2\right) + \ln\left(-p_0^2 + E_+^2\right) + \ln\left(-p_0^2 + E_-^2\right)\right\} \\
&= (VT)\int\frac{\mathrm{d}^4p}{(2\pi)^4}\left[\ln\left(-p_0^2 + E_0^2\right) + \ln\left(-p_0^2 + E_+^2\right) + \ln\left(-p_0^2 + E_-^2\right)\right],
\end{aligned} \tag{4.10}$$

where we have used the identity $\ln\det M = \mathrm{Tr}\ln M$. These terms all have the form

$$I = \int\frac{\mathrm{d}^4p}{(2\pi)^2}\ln\left(-p_0^2 + E^2\right), \tag{4.11}$$

where E is some function of the 3-momentum $\vec{p}$, but not p_0. We use the trick

$$\frac{\partial}{\partial\alpha}\left(-p_0^2 + E^2\right)^{-\alpha}\Big|_{\alpha=0} = \frac{\partial}{\partial\alpha}\exp\left[-\alpha\ln\left(-p_0^2 + E^2\right)\right]\Big|_{\alpha=0} = \ln\left(-p_0^2 + E^2\right), \tag{4.12}$$

and then perform a Wick-rotation of the p_0-integral to write the integral on the form

$$I = i\frac{\partial}{\partial\alpha}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\left(p_0^2 + E^2\right)^{-\alpha}\Big|_{\alpha=0}, \tag{4.13}$$

where p now is a Euclidean four-vector. The p_0 integral equals $\Phi_1(E,1,\alpha)$, as defined in Eq. (A.42). The result is therefore given by Eq. (A.49),

$$\int\frac{\mathrm{d}p_0}{2\pi}(p_0^2 + E)^{-\alpha} = \frac{E^{1-2\alpha}}{\sqrt{4\pi}}\frac{\Gamma(\alpha - \frac{1}{2})}{\Gamma(\alpha)}. \tag{4.14}$$

The derivative of the Gamma function is $\Gamma'(\alpha) = \psi(\alpha)\Gamma(\alpha)$, where $\psi(\alpha)$ is the digamma function. Using

$$\frac{\partial}{\partial\alpha}\frac{\Gamma(\alpha - \frac{1}{2})}{\Gamma(\alpha)}\Big|_{\alpha=0} = \Gamma\left(\alpha - \frac{1}{2}\right)\frac{\psi(\alpha - \frac{1}{2}) - \psi(\alpha)}{\Gamma(\alpha)}\Big|_{\alpha=0} = \sqrt{4\pi}, \tag{4.15}$$

$$\frac{\Gamma(\alpha - \frac{1}{2})}{\Gamma(\alpha)}\Big|_{\alpha=0} = 0, \tag{4.16}$$

we get

$$I = i\int\frac{\mathrm{d}^3p}{(2\pi)^3}E. \tag{4.17}$$

We see that the result is what we would expect physically; the total energy is integral of each mode's energy. This also agrees with the result from Appendix A in the zero-temperature limit $\beta\to\infty$. The one-loop contribution can therefore be written

$$\mathcal{F}^{(1)} = \frac{1}{2}\left[\int\frac{\mathrm{d}^3p}{(2\pi)^3}E_0 + \int\frac{\mathrm{d}^3p}{(2\pi)^3}(E_+ + E_-)\right] = \mathcal{F}_{\pi_0}^{(1)} + \mathcal{F}_{\pi_\pm}^{(1)}. \tag{4.18}$$

The first integral is identical to what we find for a free field in section A.3, in the zero-temperature limit. These terms are all divergent and must be regularized. We will use dimensional regularization, in which the

integral is generalized to d dimensions, and the $\overline{\text{MS}}$-scheme, as described in section A.3. Using the result for a free field Eq. (A.56), we get

$$\mathcal{F}^{(1)}_{\pi_0} = -\mu^{-2\epsilon}\frac{1}{4}\frac{m_3^4}{(4\pi)^2}\left(\frac{1}{\epsilon}+\frac{3}{2}+\ln\frac{\tilde{\mu}^2}{m_3^2}\right)+\mathcal{O}(\epsilon), \tag{4.19}$$

where μ is the renormalization scale, a parameter with mass-dimension 1, introduced to ensure the action integral remains dimensionless during dimensional regularization. $\tilde{\mu}$ is a related to μ as described in Eq. (A.55).

The contribution to the free energy from the π_+ and π_- particles is more complicated, as the dispersion relation is given by

$$E_\pm = \sqrt{|\vec{p}|^2+\frac{1}{2}\left(m_1^2+m_2^2+m_{12}^2\right)\pm\frac{1}{2}\sqrt{4|\vec{p}|^2m_{12}^2+\left(m_1^2+m_2^2+m_{12}^2\right)^2-4m_1^2m_2^2}}. \tag{4.20}$$

This is not an integral we can easily do in dimensional regularization. Instead, we will seek a function $f(|\vec{p}|)$ with the same UV-behavior, that is, behavior for large $\vec{p}$, as $E_+ + E_-$. If we then add $0 = f(|\vec{p}|) - f(|\vec{p}|)$ to the integrand, we can isolate the divergent behavior

$$\mathcal{F}^{(1)}_{\pi_\pm} = \frac{1}{2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}[E_+ + E_- + f(|\vec{p}|) - f(|\vec{p}|)] = \mathcal{F}^{(1)}_{\text{fin},\pi_\pm} + \mathcal{F}^{(1)}_{\text{div},\pi_\pm}. \tag{4.21}$$

This results in a finite integral,

$$\mathcal{F}^{(1)}_{\text{fin},\pi_\pm} = \frac{1}{2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}[E_+ + E_- - f(|\vec{p}|)], \tag{4.22}$$

which we can evaluate numerically, and a divergent integal

$$\mathcal{F}^{(1)}_{\text{div},\pi_\pm} = \frac{1}{2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}f(|\vec{p}|), \tag{4.23}$$

which we hopefully will be able to do in dimensional regularization. We can explore the UV-behavior of $E_+ + E_-$ by expanding it in powers of $1/|\vec{p}|$,

$$\begin{aligned} E_+ + E_- &= 2|\vec{p}| + \frac{m_{12}+2(m_1^2+m_2^2)}{4}|\vec{p}|^{-1} - \frac{m_{12}^4+4m_{12}^2(m_1^2+m_2^2)+8(m_1^4+m_2^4)}{64}|\vec{p}|^{-3} + \mathcal{O}\left(|\vec{p}|^{-5}\right) \\ &= a_1|\vec{p}| + a_2|\vec{p}|^{-1} + a_3|\vec{p}|^{-3} + \mathcal{O}\left(|\vec{p}|^{-5}\right). \end{aligned} \tag{4.24}$$

We have defined new constants a_i for brevity of notation. As

$$\int_{\mathbb{R}^3}\frac{\mathrm{d}^3p}{(2\pi)^3}|\vec{p}|^n = C\int_0^\infty \mathrm{d}p\, p^{2+n} \tag{4.25}$$

is UV divergent for $n \geq -3$, f need to match the expansion of $E_+ + E_-$ up to and including $\mathcal{O}\left(|\vec{p}|^{-3}\right)$ for $\mathcal{F}^{(1)}_{\text{fin},\pi_\pm}$ to be finite. The most obvious choice for f is

$$f(|\vec{p}|) = a_1|\vec{p}| + a_2|\vec{p}|^{-1} + a_3|\vec{p}|^{-3}. \tag{4.26}$$

However, this introduces a new problem. f has the same UV-behavior as $E_+ + E_-$, but the last term diverges in the IR, that is, for low $|\vec{p}|$. This can be amended by introducing a mass term. Let

$$|\vec{p}|^{-3} = \left(\frac{1}{\sqrt{|\vec{p}|^2}}\right)^3 \longrightarrow \left(\frac{1}{\sqrt{|\vec{p}|^2+m^2}}\right)^3. \tag{4.27}$$

For $|\vec{p}|^2 \to \infty$, this is asymptotic to $|\vec{p}|^{-3}$, so it retains its UV behavior. However, for $|\vec{p}| \to 0$, it now approaches m^{-3}, so the IR-divergence is gone. The cost of this technique is that we have introduced an arbitrary mass parameter. Any final result must thus be independent of the value of m to be acceptable.

We will instead regularize the integral by defining $E_i = \sqrt{|\vec{p}|^2 + \tilde{m}_i^2}$, and $\tilde{m}_i^2 = m_i^2 + \frac{1}{4}m_{12}^2$. Using the definition of the masses, Eqs. (3.133) to (3.136), we get

$$m_3^2 = \bar{m}^2 \cos\alpha + \mu_I^2 \sin^2\alpha, \tag{4.28}$$

$$\tilde{m}_1^2 = m_1^2 + \mu^2 \cos\alpha^2 = \bar{m}^2 \cos\alpha + \mu_I^2 \sin^2\alpha = m_3^2 \tag{4.29}$$

$$\tilde{m}_2^2 = m_2^2 + \mu^2 \cos\alpha^2 = \bar{m}^2 \cos\alpha. \tag{4.30}$$

Finally, we define $f(|\vec{p}|) = E_1 + E_2$, which differ from $E_+ + E_-$ by $\mathcal{O}\left(|\vec{p}|^{-5}\right)$ and is well-behaved in the IR. This leads to a divergent integral the same form as in the case of a free scalar. Thus, in the $\overline{\text{MS}}$-scheme,

$$\mathcal{F}^{(1)}_{\text{div},\pi_\pm} = -\mu^{-2\epsilon}\frac{1}{4}\frac{\tilde{m}_1^4}{(4\pi)^2}\left(\frac{1}{\epsilon} + \frac{3}{2} + \ln\frac{\tilde{\mu}^2}{\tilde{m}_1^2}\right) - \mu^{-2\epsilon}\frac{1}{4}\frac{\tilde{m}_2^4}{(4\pi)^2}\left(\frac{1}{\epsilon} + \frac{3}{2} + \ln\frac{\tilde{\mu}^2}{\tilde{m}_2^2}\right) + \mathcal{O}(\epsilon). \tag{4.31}$$

We define

$$\mathcal{F}^{(1)}_{\text{fin},\pi_\pm} = \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}(E_+ + E_- - E_1 - E_2), \tag{4.32}$$

which is a finite integral. The total one-loop contribution is then, using Eqs. (4.29) and (4.30),

$$\mathcal{F}^{(2)} = \mathcal{F}^{(1)}_{\text{fin},\pi_\pm} - \mu^{-2\epsilon}\frac{1}{2}\frac{1}{(4\pi)^2}\left[\left(\frac{1}{\epsilon} + \frac{3}{2} + \ln\frac{\tilde{\mu}^2}{m_3^2}\right)m_3^4 + \frac{1}{2}\left(\frac{1}{\epsilon} + \frac{3}{2} + \ln\frac{\tilde{\mu}^2}{\tilde{m}_2^2}\right)\tilde{m}_2^4\right] + \mathcal{O}(\epsilon). \tag{4.33}$$

4.2 Next-to-leading order and renormalization

We have now regularized the divergences, which allows them to be handled in a well-defined way. However, they are still there. To get rid of them, we need to use renormalization. As laid out in section 3.2, the power counting scheme ensures that all terms in $\mathcal{L}_{2n}$ scales as t^{2n} when the momenta p are scaled as $p \to tp$.[2] The tree-level free energy from $\mathcal{L}_{2n}$ is thus of order p^{2n}. The m-loop correction to the tree level result is then suppressed by p^{2m} [10, 33]. Our one-loop calculation using $\mathcal{L}_2$ therefore contains divergences of order p^4. Since $\mathcal{L}_4$ is, by construction, the most general possible Lagrangian at order p^4, it contains coupling constants that can be renormalized to absorb all these divergences.

The renormalized coupling constants in $\mathcal{L}_4$ have been calculated for $\mu_I = 0$ [33]. They are independent of μ_I, and we can therefore use them in this calculation. The renormalized coupling constants in the $\overline{\text{MS}}$-scheme are related to the bare couplings through

$$l_i = l_i^r - \mu^{-2\epsilon}\frac{1}{2}\frac{\gamma_i}{(4\pi)^2}\left(\frac{1}{\epsilon} + 1\right), \qquad i \in \{1, ...7\}, \tag{4.34}$$

$$h_i = h_i^r - \mu^{-2\epsilon}\frac{1}{2}\frac{\delta_i}{(4\pi)^2}\left(\frac{1}{\epsilon} + 1\right), \qquad i \in \{1, ...3\}. \tag{4.35}$$

Here, γ_i and δ_i are numerical constants which are used to match the divergences. The relevant terms are[3]

$$\gamma_1 = \frac{1}{3}, \quad \gamma_2 = \frac{2}{3}, \quad \gamma_3 = -\frac{1}{2}, \quad \gamma_4 = 2, \tag{4.36}$$

$$\delta_1 = 2, \quad \delta_3 = 0. \tag{4.37}$$

The bare coupling constants l_i and h_i, though infinite, are independent of our renormalization scale μ. From this we obtain the renormalization group equations for the running coupling constants,

$$\mu\frac{\mathrm{d}l_i^r}{\mathrm{d}\mu} = -\mu^{-2\epsilon}\frac{\gamma_i}{(4\pi)^2} + \mathcal{O}(\epsilon), \quad \mu\frac{\mathrm{d}h_i^r}{\mathrm{d}\mu} = -\mu^{-2\epsilon}\frac{\delta_i}{(4\pi)^2} + \mathcal{O}(\epsilon). \tag{4.38}$$

These have the general solutions

$$l_i^r = \frac{1}{2}\mu^{-2\epsilon}\frac{\gamma_i}{(4\pi)^2}\left(\bar{l}_i - \ln\frac{\tilde{\mu}^2}{M^2}\right) + \mathcal{O}(\epsilon), \quad h_i^r = \frac{1}{2}\mu^{-2\epsilon}\frac{\gamma_i}{(4\pi)^2}\left(\bar{h}_i - \ln\frac{\tilde{\mu}^2}{M^2}\right) + \mathcal{O}(\epsilon), \tag{4.39}$$

[2] Remember that we scale pion mass $\bar{m} = B_0(m_u + m_d)$ as t^2, and the chemical potential as t.

[3] Some authors [35, 39] instead use $h_1' = h_1 - l_4$, with a corresponding $\delta_1' = \delta_1 - \gamma_1 = 0$.

where $\bar{l}_i$ and $\bar{h}_i$ are the values of the coupling constants (times a constant) measured at the energy M. This only applies if the numerical constants γ_i/δ_i are non-zero. If they are zero, then the coupling is not running, and the measured value can be applied at all energies. The bare couplings are thus given by

$$l_i = \mu^{-2\epsilon}\frac{1}{2}\frac{\gamma_i}{(4\pi)^2}\left(\bar{l}_i - 1 - \frac{1}{\epsilon} - \ln\frac{\tilde{\mu}^2}{M^2}\right) + \mathcal{O}(\epsilon), \tag{4.40}$$

$$h_i = \mu^{-2\epsilon}\frac{1}{2}\frac{\delta_i}{(4\pi)^2}\left(\bar{h}_i - 1 - \frac{1}{\epsilon} - \ln\frac{\tilde{\mu}^2}{M^2}\right) + \mathcal{O}(\epsilon). \tag{4.41}$$

The next-to-leading contribution to the free energy at tree-level is $\mathcal{F}_4^0 = -\mathcal{L}_4^{(0)}$, which is given by Eq. (3.129). When substituting Eq. (4.40) into the bare couplings, we get

$$\begin{aligned}
\mathcal{F}_4^{(0)} &= -(l_1+l_2)\mu_I^4\sin^4\alpha - (l_3+l_4)\bar{m}^4\cos^2\alpha - l_4\bar{m}^2{\mu_I}^2\cos\alpha\sin^2\alpha - (h_1-l_4)\bar{m}^4 - h_3\Delta m^4 \\
&= -\mu^{-2\epsilon}\frac{1}{2}\frac{1}{(4\pi)^2}\Bigg[\frac{1}{3}\left(\bar{l}_1+2\bar{l}_2-3\right)\mu_I^4\sin^4\alpha + \frac{1}{2}\left(-\bar{l}_3+4\bar{l}_4-3\right)\bar{m}^4\cos^2\alpha \\
&\qquad + 2\left(\bar{l}_4-1\right)\bar{m}^2\mu_I^2\cos\alpha\sin^2\alpha + 2(\bar{l}_4-\bar{h}_1)\bar{m}^4 + \bar{h}_3\Delta m^4 \\
&\qquad - \left(\frac{1}{\epsilon}+\ln\frac{\tilde{\mu}^2}{M^2}\right)\left(\mu_I^4\sin^4\alpha + \frac{3}{2}\bar{m}^4\cos^2\alpha + 2\bar{m}^2\mu_I^2\cos\alpha\sin^2\alpha\right)\Bigg] + \mathcal{O}(\epsilon).
\end{aligned}$$

Notice that the term proportional to ϵ^{-1} cancel exactly with the divergent term from $\mathcal{F}^{(2)}$, as we expected. Adding all the contribution to the free energy density, and taking the limit $\epsilon \to 0$, we get the next-to-leading order free energy density,

$$\begin{aligned}
\mathcal{F}_{\text{NLO}} = -f^2\left(\bar{m}^2\cos\alpha + \frac{1}{2}\mu_I^2\sin^2\alpha\right) + \mathcal{F}^{(1)}_{\text{fin},\pi_\pm} - \frac{1}{2}\frac{1}{(4\pi)^2}\Bigg[\frac{1}{3}\left(\bar{l}_1 + 2\bar{l}_2 + \frac{3}{2} + 3\ln\frac{M^2}{m_3^2}\right)\mu_I^4\sin^4\alpha \\
+ \frac{1}{2}\left(-\bar{l}_3 + 4\bar{l}_4 + \frac{3}{2} + 2\ln\frac{M^2}{m_3^2} + \ln\frac{M^2}{\tilde{m}_2^2}\right)\bar{m}^4\cos^2\alpha + 2\left(\bar{l}_4 + \frac{1}{2} + \ln\frac{M^2}{m_3^2}\right)\bar{m}^2\mu_I^2\cos\alpha\sin^2\alpha\Bigg].
\end{aligned} \tag{4.42}$$

We have dropped the terms proportional to $\bar{l}_4 - \bar{h}_1$ and $\bar{h}_3$, as they only add an unobservable constant value to the free energy.

Parameters and consistent expansion

The coupling constants are free parameters of the effective theory of χPT. As we are not able to do calculations with QCD at low energies, these must be measured instead of calculated from first principles. The values for the pion mass and pion decay constants used in this text are

$$m_\pi = 131\,\text{MeV}, \quad f_\pi = \frac{1}{\sqrt{2}}128\,\text{MeV} = 90.5\,\text{MeV}. \tag{4.43}$$

These are the central vales used in [35, 38]. They were chosen to compare the result to results from lattice QCD calculations. The most up-to-date value of the pion masses from the Pardicle Data Book are [4]

$$m_0 = 139.57039 \pm 0.00018\,\text{MeV}, \quad m_\pm = 134.9768 \pm 0.0005\,\text{MeV}. \tag{4.44}$$

These are the masses of the neutral and the charged pions and includes contribution due to electromagnetic and weak interactions, which are not included in this study. The value for the pion decay constant is [4]

$$f_\pi = \frac{1}{\sqrt{2}}(130.2 \pm 1.2\,\text{MeV}) = 92.1 \pm 0.8\,\text{MeV}. \tag{4.45}$$

The physical mass, m_π, is defined as the pole of the propagator in the vacuum, and thus the zero of the inverse propagator,

$$D^{-1}(p^2 = m_\pi^2) = 0. \tag{4.46}$$

This relates it to the bare mass $\bar{m}$. We found, in Eq. (3.133), that $m_\pi^2 = \bar{m}^2$ to leading order. Similarly, $f_\pi = f$ to leading order. However, in any NLO results we need the relationship between the bare and physical constants to NLO. This is given by [33]

$$m_\pi^2 = \bar{m}^2 + \frac{\bar{l}_3}{2(4\pi)^2}\frac{\bar{m}^4}{f^2}, \tag{4.47}$$

$$f_\pi^2 = f^2 + \frac{2\bar{l}_4}{(4\pi)^2}\frac{\bar{m}^2}{f^2}. \tag{4.48}$$

Table 4.1: The measured values and corresponding uncertainties of the relevant coupling constants, measured at $M = m_\pi$.

	value	uncertainty	source
$\bar{l}_1$	-0.4	± 0.6	[40]
$\bar{l}_2$	4.3	± 0.1	[40]
$\bar{l}_3$	2.9	± 2.4	[33]
$\bar{l}_4$	4.4	± 0.2	[40]

The values for the coupling constants used in this text are given in Table 4.1. The constants $\bar{l}_1$, $\bar{l}_2$ and $\bar{l}_4$ are estimated using data from $\pi\pi$-scattering [40], while $\bar{l}_3$ is estimated using three flavor chiral perturbation theory [33]. These are the same values as those used in [35]. In this text, we use the central values of the parameters. Together with Eqs. (4.47) and (4.48), the NLO results for the central value for the bare mass and decay constant are

$$\bar{m} = 1.01136\, m_\pi = 132.5\,\text{MeV}, \tag{4.49}$$

$$f = 0.64835\, m_\pi = 85.9\,\text{MeV}. \tag{4.50}$$

In section 4.1, we found a relationship between α and μ_I, using the lowest-order result for $\mathcal{F}$, given in Eq. (4.5). To calculate any thermodynamic quantities to leading order, at tree-level, we must use this result. When using the NLO result for the free energy, we must consistently calculate this and other quantities to the same order. As we have seen earlier, when replacing the action by $S[\varphi] \to g^{-1}S[\varphi]$, the L-loop contribution is proportional to g^{L-1}. In Weinberg's power counting scheme, as laid out in section 3.2, we scale $p \to tp$ and $m_q \to t^2 m_q$. Then, the nth term in the expansion is proportional to t^{2n}. We can therefore expand the free energy as

$$\mathcal{F} = t^2 g^{-1}\mathcal{F}_2^{(0)} + t^2\mathcal{F}_2^{(1)} + t^4 g^{-1}\mathcal{F}_4^{(0)} + ... \tag{4.51}$$

We consider terms where $k = L + n$ has the same value to be of same order. This expansion can be written as

$$\mathcal{F} = \sum_{k=0}^{\infty}\sum_{n+L=k} t^{2n} g^{L-1}\mathcal{F}_{2n}^{(L)}. \tag{4.52}$$

If we now define

$$\tilde{\mathcal{F}}_k = \sum_{n+L=k} t^{2n} g^{L-1}\mathcal{F}_{2n}^{(L)}, \tag{4.53}$$

then scale $t \to \sqrt{s}\,t$ and $g \to sg$, where s is some real number, then $t^{2n}g^{L-1}$ scales as $s^{n+L-1} = s^{m-1}$. All expansions are now done in this new parameter s. The free energy expansion is

$$\mathcal{F} = s^{-1}\sum_{k=0}^{\infty}\tilde{\mathcal{F}}_k\, s^k. \tag{4.54}$$

As argued earlier, α must minimize the free energy, and therefore satisfy

$$\frac{\partial\mathcal{F}}{\partial\alpha} = 0, \tag{4.55}$$

to all orders. We expand this solution in s,

$$\alpha = \alpha_0 + \alpha_1 s + \dots. \tag{4.56}$$

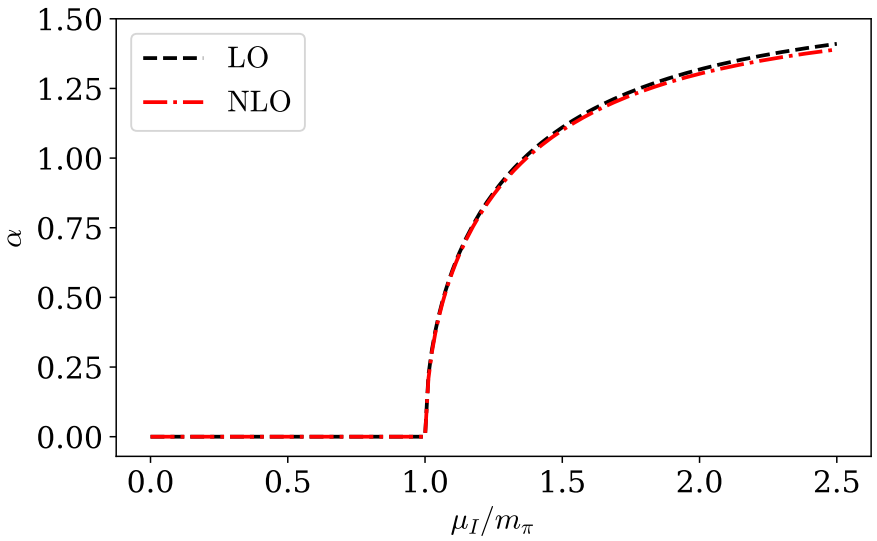


Figure 4.2: The leading order and next-to-leading order results for α as a function of μ_I, which is given in units of m_π.

Combining this, we get

$$\begin{aligned} 0 &= \frac{\partial}{\partial\alpha}\left[s^{-1}\tilde{\mathcal{F}}_0 + \tilde{\mathcal{F}}_1 + \mathcal{O}\left(s^1\right)\right]\Big|_{\alpha=\alpha_0+\alpha_1 s+\mathcal{O}(s^2)} \\ &= s^{-1}\left[\tilde{\mathcal{F}}_0'(\alpha_0) + (\alpha'-\alpha_0)\tilde{\mathcal{F}}_0''(\alpha_0) + \mathcal{O}\left(s^2\right)\right] + \tilde{\mathcal{F}}_1'(\alpha_0) + \mathcal{O}\left(s^1\right) \\ &= s^{-1}\tilde{\mathcal{F}}_1'(\alpha_0) + s^0\left[\alpha_1\tilde{\mathcal{F}}_0''(\alpha_0) + \tilde{\mathcal{F}}_1'(\alpha_0)\right] + \mathcal{O}\left(s^1\right). \end{aligned} \tag{4.57}$$

Here, the prime indicates partial derivatives with respect to α. The equality in Eq. (4.57) has to hold term for term. After setting $s = g = t = 1$, we get

$$\frac{\partial\tilde{\mathcal{F}}_0}{\partial\alpha}\bigg|_{\alpha=\alpha_0} = 0, \quad \tilde{\mathcal{F}}_0 = \mathcal{F}_2^{(0)},$$

which is what we have used as the leading-order result. The first correction is on this result is

$$\alpha_1 = -\frac{\tilde{\mathcal{F}}_1'(\alpha_0)}{\tilde{\mathcal{F}}_0''(\alpha_0)}, \quad \tilde{\mathcal{F}}_1 = \mathcal{F}_4^{(0)} + \mathcal{F}_2^{(1)}. \tag{4.58}$$

The next to leading order results for the free energy and α are

$$\mathcal{F}_{\text{NLO}} = \tilde{\mathcal{F}}_0 + \tilde{\mathcal{F}}_1, \quad \alpha_{\text{NLO}} = \alpha_0 + \alpha_1. \tag{4.59}$$

We have that

$$\mathcal{F}_{\text{NLO}}'(\alpha_{\text{NLO}}) = \tilde{\mathcal{F}}_0'(\alpha_0) + \alpha_1\tilde{\mathcal{F}}_0''(\alpha_0) + \cdots + \tilde{\mathcal{F}}_1'(\alpha_0) + \ldots, \tag{4.60}$$

where the excluded terms are beyond next-to-leading order. Using Eq. (4.58), we see that this vanishes to next-to-leading order. We therefore use the criterion

$$\frac{\partial\mathcal{F}_{\text{NLO}}}{\partial\alpha}\bigg|_{\alpha=\alpha_{\text{NLO}}} = 0 \tag{4.61}$$

to calculate α_{NLO}. The result is shown in Figure 4.2, which compares the leading order and next-to-leading order results for α.

We can use the expansion in s to consistently evaluate any observable to any power in perturbation theory. Assume that f is some obserable, and a function of α. We then expand in s,

$$f(\alpha) = f_0(\alpha) + sf_1(\alpha) + s^2 f_2(\alpha) + \ldots \tag{4.62}$$

When Taylor expanding around the leading order result for α, we get

$$f(\alpha) = f_0(\alpha_0) + s\alpha_1 f_0'(\alpha_0) + sf_1(\alpha_0) + \mathcal{O}\left(s^2\right) = f_0(\alpha_0 + s\alpha_1) + sf_1(\alpha_0) + \mathcal{O}\left(s^2\right).$$

We see that, to get a consistent expansion, we must evaluate the leading order result for the function f_0 at next-to-leading order in α, while next-to-leading order correction can be evaluated at leading order. However, as

$$f_1(\alpha_0 + s\alpha_1) = f_1(\alpha_0) + \mathcal{O}\left(s^1\right), \tag{4.63}$$

we also get a consistent expansion if we evaluate the leading-order result and its correction at next-to-leading order in α. We will do this to obtain the results in the next section.

4.3 Equation of state

The methods and analysis of this section are based on [23, 35, 41]. All results in this section are given in units of m_π.

The free energy[4] is defined as

$$F(T, V, \mu_I) = U - TS - \mu_I Q_I, \quad \mathrm{d}F = -S\mathrm{d}T - P\mathrm{d}V - Q_I \mathrm{d}\mu_I. \tag{4.64}$$

In our case, all results are for $T = 0$. As we have seen earlier, our system is homogenous. This means that the free energy density is independent of volume, and thus $F = V\mathcal{F}$. From Eq. (4.64), the pressure is given by

$$P = -\left(\frac{\partial F}{\partial V}\right)_{T,\mu_I} = -\mathcal{F}. \tag{4.65}$$

We are interested in the pressure relative to the state in which $\mu_I = 0$. We therefore normalize $P(\mu_I = 0) = 0$, which gives

$$P(\mu_I) = -[\mathcal{F}(\mu_I) - \mathcal{F}(\mu_I = 0)]. \tag{4.66}$$

This is illustrated in Figure 4.3.

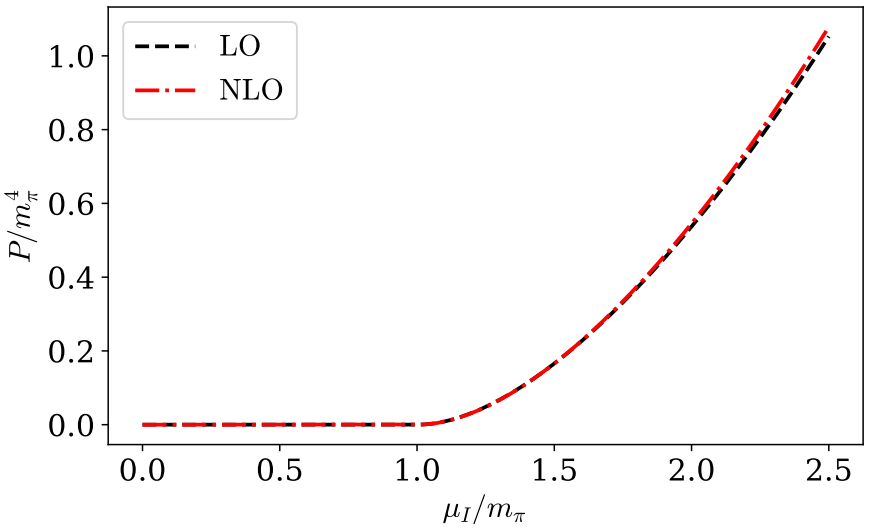


Figure 4.3: The LO and NLO result for the pressure, as a function of μ_I.

Likewise, the total isospin is proportional to volume, which means that the isospin density is

$$n_I = \frac{Q_I}{V} = -\frac{1}{V}\left(\frac{\partial F}{\partial \mu_I}\right)_{T,V} = -\frac{\partial \mathcal{F}}{\partial \mu_I}. \tag{4.67}$$

[4] As we are in the grand canonical ensemble, this is the grand canonical, or Landau, free energy, also called the grand potential, and not Helmholtz' free energy.

Using Eq. (4.42), this equals

$$n_I = f^2 \mu_I \sin^2\alpha - \frac{\partial \mathcal{F}_{\text{fin}}}{\partial \mu_I} + \frac{1}{(4\pi)^2}\left[\left(2\bar{l}_4 + \ln\frac{M^2}{m_3^2}\right)\bar{m}^2\mu_I\cos\alpha\sin^2\alpha + \frac{1}{3}\left(2\bar{l}_1 + 4\bar{l}_2 + 3\ln\frac{M^2}{m_3^2}\right)\mu_I^3\sin^4\alpha\right]. \tag{4.68}$$

The isospin density, as a function of μ_I, is shown in Figure 4.4.

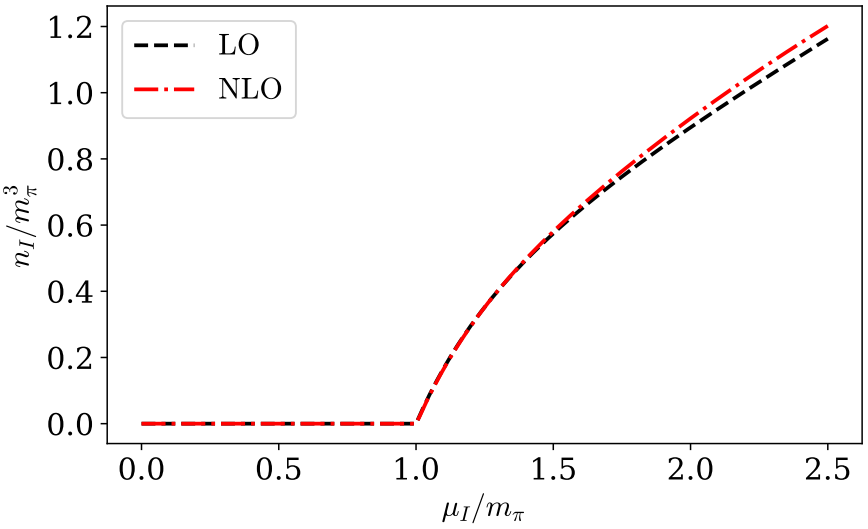


Figure 4.4: The LO and NLO result for the isospin density, as a function of μ_I.

From Eq. (4.64) we get the energy density, $u = U/V$, at $T = 0$, is given by

$$u(\mu_I) = -P(\mu_I) + \mu_I n_I(\mu_I), \tag{4.69}$$

where we again have normalized so that $u(\mu_I = 0) = 0$. Now that we have both the dependence of the pressure and the energy density on the isospin chemical potential, we can trace out the line in the pressure-energy density plane, parametrized by μ_I. This is the equation of state of the system and is shown in Figure 4.5.

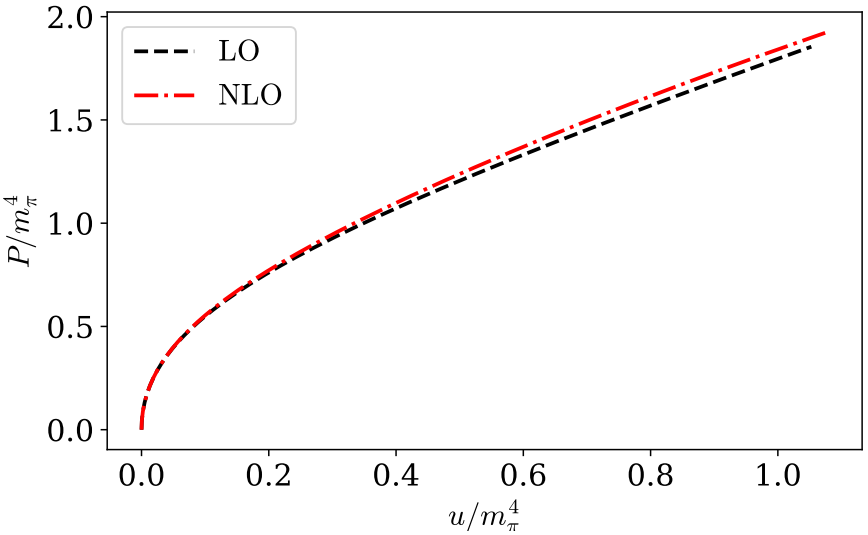


Figure 4.5: The leading and next-to-leading order equation of state. Both the pressure and energy density are given in units of m_π^4.

The next-to-leading order correction is small compared to the leading order result. The maximum correction to the energy density is $0.0717\, m_\pi^4$, while the largest correction to the pressure is $0.02590\, m_\pi^4$. We notice that

the difference steadily increases with the value of μ_I. This is expected, as the perturbation theory assumes small disturbances.

As both the pressure and isospin density are zero for $\mu_I < m_\pi$, the equation of state in the vacuum phase is trivial. It is only for $\mu_I \geq m_\pi$ that we get interesting behavior. We see that this is a recurring theme. The vacuum and the next quantum state are separated by a finite energy gap, given by the mass of the lightest particle. We see this from the energies we found for $\mu_I < 0$, Eqs. (3.141) and (3.142), which gave a Zeeman-like splitting. As the isospin chemical potential smoothly varies from zero, the free energy becomes different from the vacuum energy. A system in equilibrium minimizes the free energy. However, due to the finite gap between the vacuum and other states, we expect there to be a critical value μ_I^c below which all observable quantities are independent of μ_I. This is called the "Silver-Blaze" property [42, 43].

4.4 Phase transition

Our leading-order analysis showed that α is zero for $\mu_I \leq \bar{m}$, and then increases continuously for $\mu_I > \bar{m}$. Furthermore, $\bar{m} = m_\pi$ to leading-order. This behavior is illustrated in Figure 4.2. This is the hallmark of a phase transition, where α is the order parameter. The behavior of systems near points of phase transition is described by Landau theory [23]. Using Eq. (4.3), we can expand the leading-order free energy in α,

$$\begin{aligned}\mathcal{F} &= -f^2\bar{m}^2 + f^2\frac{1}{2}(\bar{m}^2 - \mu_I^2)\alpha^2 - \frac{1}{24}f^2(\bar{m}^2 - 4\mu_I^2)\alpha^4 + \mathcal{O}\left(\alpha^5\right) \\ &= \mathcal{F}(\alpha = 0) + a(\mu_I)\alpha^2 + \frac{1}{2}b(\mu_I)\alpha^4 + \mathcal{O}\left(\alpha^5\right). \end{aligned} \tag{4.70}$$

Notice that near $\mu_I = \bar{m}$, $b > 0$. As earlier, the equation that governs α is

$$\frac{\partial \mathcal{F}}{\partial \alpha} = 2[a(\mu_I) + b(\mu_I)\alpha^2]\alpha = 0. \tag{4.71}$$

If $a > 0$, then $\alpha = 0$ will be the only solution, which gives us the criterion for a phase transition

$$a(\mu_I) = 0. \tag{4.72}$$

As expected, this criterion is fulfilled at $\mu_I = \bar{m}$. Near $\mu_I = \bar{m}$, we can write

$$a = -a_0(\mu_I - \bar{m}), \quad b = b_0, \tag{4.73}$$

where a_0 and b_0 are positive constants, so the solution to Eq. (4.71) for $\mu_I > \bar{m}$ is

$$\alpha(\mu_I) = \sqrt{\frac{a_0}{b_0}}(\mu_I - \bar{m})^{1/2}. \tag{4.74}$$

The free energy around the phase transition is illustrated in Figure 4.6.

The order parameter α changes continuously as the system transitions between phases. This means we have a *second order* phase transition. The power-law behavior, $\alpha \propto (\mu_I - \mu_I^c)^\beta$, is typical of systems near a phase transition. The exponent β, which in this case equals $1/2$, is called a *critical exponent*. If we have a system where $b < 0$ near $\mu_I = \bar{m}$, then we must expand $\mathcal{F}$ further to show if the phase transition is continuous or not. Figure 4.7 shows the free energy surface but modified so that $b_0 < 0$, together with the corresponding value of α, which now changes discontinuously at the point of phase transition.

In the vacuum phase, $\alpha = 0$, the ground state is given by

$$\Sigma(\pi = 0) = \Sigma_0 = \mathbb{1}, \tag{4.75}$$

where we have used Eq. (3.104). Under $H = SU(2)_V$, Σ transforms as

$$\Sigma(x) \to \Sigma'(x) = V\Sigma(x)V^\dagger, \quad V = \exp\left\{i\frac{1}{2}\theta_a\tau_a\right\}, \tag{4.76}$$

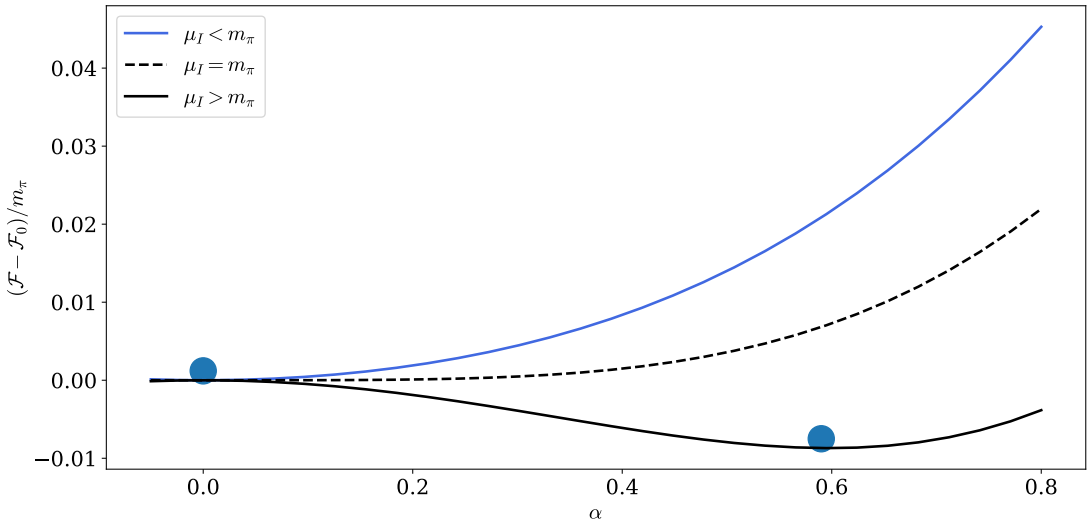


Figure 4.6: The plot shows normalized free energy density as a function of α, in the two different phases. Each line is a constant μ_I slice of the surface in Figure 4.1.

We see that the vacuum phase ground state is invariant under H. However, for $\alpha \neq 0$, the ground state is shifted to

$$\Sigma(\pi = 0) = A_\alpha \Sigma_0 A_\alpha = \exp\{i\alpha\tau_1\}. \tag{4.77}$$

This is not, in general, invariant under transformations in H. The generators τ_2 and τ_3 are broken. In Figure 3.1, we saw that the mass of the m_- particle vanishes, so we identify this particle with the corresponding Goldstone mode. There is only one Goldstone mode. However, this is not a Lorentz invariant system, in which case we cannot guarantee one massless mode per broken generator. In section 3.3, we defined

$$\alpha = \frac{1}{f}\sqrt{\pi_a^0 \pi_a^0}, \tag{4.78}$$

where π_a^0 was the ground state expectation value of the pion fields, as defined in the vacuum phase, when $\mu_I \neq 0$. The new ground state thus corresponds to a condensate of pions. The isospin symmetry $\mathrm{SU}(3)_V$ is not a perfect symmetry of the QCD Lagrangian, but is explicitly broken by the mass term

$$\bar{q}mq = \frac{1}{2}\bar{q}[(m_u + m_d)\mathbb{1} + (m_u - m_d)\tau_3]q. \tag{4.79}$$

This term, however, is invariant under the subgroup $\mathrm{U}(1)_{I_3} \subset \mathrm{SU}(2)_V$, generated by τ_3. If we write out a generic element from this subgroup,

$$U = e^{i\theta\tau_3} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \tag{4.80}$$

we see that this corresponds to rotating the phase of the up and down quark but not rotating them into each other. Thus, the pion condensate spontaneously breaks an exact symmetry of the two-flavor QCD Lagrangian, and we expect the corresponding Goldstone mode to remain massless outside the chiral limit.

To find the value of μ_I to next-to-leading order, we must expand the NLO free energy in powers of α When we expand the static, second order Lagrangian to α^2, we get

$$\begin{aligned}\mathcal{F}_4^{(0)} &= -(l_3 + l_4)\bar{m}^4 + [(l_3 + l_4)\bar{m}^4 - l_4\bar{m}^2\mu_I^2]\alpha^2 \\ &= \text{const.} + \frac{\mu^{-2\epsilon}}{(4\pi)^2}\left[\left(\bar{l}_4 - \frac{1}{4}\bar{l}_3\right)\bar{m}^4 - \bar{l}_4\bar{m}^2\mu_I^2 - \left(1 + \frac{1}{\epsilon} + \ln\frac{\tilde{\mu}^2}{M^2}\right)\left(\frac{3}{4}\bar{m}^2 - \mu_I^2\right)\bar{m}^2\right]\alpha^2 + \mathcal{O}(\epsilon),\end{aligned} \tag{4.81}$$

where const. is independent of α, and thus not of interest. From the one-loop correction, we have the contributions

$$\mathcal{F}_2^{(1)} = i\frac{1}{2}\int \frac{\mathrm{d}^4p}{(2\pi)^2}\ln(-p^2 + m_3^2) + i\frac{1}{2}\int \frac{\mathrm{d}^4p}{(2\pi)^2}\ln[(-p^2 + m_1^2)(-p^2 + m_2^2) - p_0^2 m_{12}^2]. \tag{4.82}$$

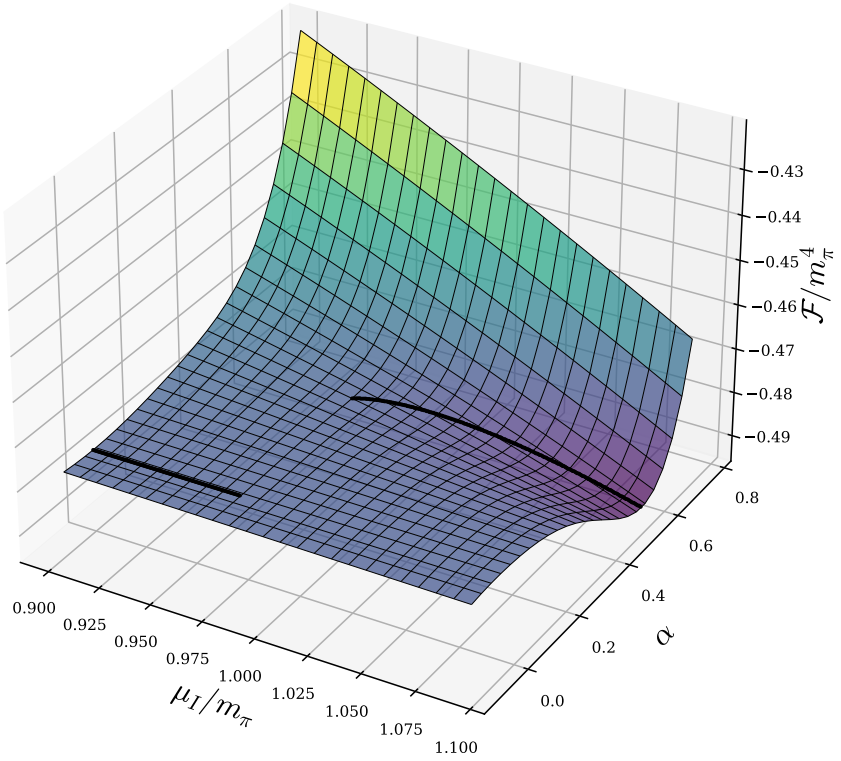


Figure 4.7: The surface is free energy density, only modified so that $b_0 < 0$. The black line traces out the minimum for each value of μ_I, which jumps discontinuously at the point of phase transition.

The first integral is the same free energy contribution from the π_0-particle as we have calculated earlier in Eq. (4.19), and it reads

$$\mathcal{F}^{(1)}_{\pi_0} = i\frac{1}{2}\int \frac{\mathrm{d}^4p}{(2\pi)^2}\ln\left(-p^2+m_3^2\right) = -\mu^{-2\epsilon}\frac{1}{4}\frac{m_3^4}{(4\pi)^2}\left(\frac{1}{\epsilon}+\frac{3}{2}+\ln\frac{\tilde{\mu}^2}{m_3^2}\right)+\mathcal{O}(\epsilon). \tag{4.83}$$

The mass m_3 is dependent on α, and has the expansion

$$\begin{aligned} m_3^4 &= \bar{m}^4+\bar{m}^2(2\mu_I^2-\bar{m}^2)\alpha^2+\mathcal{O}\left(\alpha^4\right), \\ \ln\frac{\mu^2}{m_3^2} &= \ln\frac{\mu^2}{\bar{m}_3^2}-\frac{1}{2}\frac{(2\mu_I^2-\bar{m}^2)}{\bar{m}^2}+\mathcal{O}\left(\alpha^4\right). \end{aligned}$$

In the second integral, we rewrite the argument of the logarithm as

$$(-p^2+m_1^2)(-p^2+m_2^2)-p_0^2m_{12}^2 = \left[-p^2+\frac{1}{2}(m_1^2+m_2^2)\right]^2-p_0^2m_{12}^2-\frac{1}{4}(m_1^2-m_2^2)^2. \tag{4.84}$$

When we calculate the α dependence of the last term, we get $(m_1^2-m_2^2)^2=\mu^4\sin^4\alpha=\mathcal{O}\left(\alpha^4\right)$, which means that for our purposes, we can ignore this term. We further rewrite the remaining expression by factoring it,

$$\left[-p^2+\frac{1}{2}(m_1^2+m_2^2)\right]^2-p_0^2m_{12}^2 = \left[-p^2+\frac{1}{2}(m_1^2+m_2^2)-p_0m_{12}\right]\left[-p^2+\frac{1}{2}(m_1^2+m_2^2)+p_0m_{12}\right]. \tag{4.85}$$

We then complete the square in each of the factors,

$$-p^2+\frac{1}{2}(m_1^2+m_2^2)\pm p_0m_{12} = -\left(p_0\mp\frac{1}{2}m_{12}\right)^2+|\vec{p}|^2+m_4^2, \tag{4.86}$$

where

$$m_4^2 = \frac{1}{2}\left(m_1^2 + m_2^2 + \frac{1}{2}m_{12}\right) = \bar{m}^2\cos\alpha + \frac{1}{2}\mu_I^2\sin^2\alpha, \tag{4.87}$$

$$m_4^4 = \bar{m}^4 - \bar{m}^2(m^2+\mu_I^2)\alpha^2 + \mathcal{O}\left(\alpha^4\right), \tag{4.88}$$

$$\ln\frac{\mu^2}{m_4^2} = \ln\frac{\mu_I^2}{\bar{m}^2} + \frac{1}{2}\frac{\bar{m}^2+\mu_I^2}{\bar{m}^2}\alpha^2 + \mathcal{O}\left(\alpha^4\right). \tag{4.89}$$

After a shift of variables, the integral has the same form as the logarithmic integrals we have calculated earlier, which gives us the result

$$\mathcal{F}_{\pi_\pm} = i\int\frac{\mathrm{d}^4}{(2\pi)^4}\ln\left(-p^2+m_4^2\right) = -\mu^{-2\epsilon}\frac{1}{2}\frac{m_4^4}{(4\pi)^2}\left(\frac{1}{\epsilon}+\frac{3}{2}+\ln\frac{\tilde{\mu}^2}{m_4^2}\right)+\mathcal{O}(\epsilon). \tag{4.90}$$

Combining these two contributions to the one-loop correction of the free energy gives

$$\mathcal{F}_2^{(1)} = \text{const.} + \frac{\mu^{-2\epsilon}}{(4\pi)^2}\left(1+\frac{1}{\epsilon}+\ln\frac{\tilde{\mu}^2}{m^2}\right)\left(\frac{3}{4}m^2-\mu_I^2\right)\bar{m}^2\alpha^2+\mathcal{O}(\epsilon).$$

We see that again, the ϵ^{-1} will cancel when we combine the NLO-terms. Setting $\epsilon = 0$, the NLO correction to the free energy becomes

$$\tilde{\mathcal{F}}_1 = \text{const.} + \frac{1}{(4\pi)^2}\left[\left(\bar{l}_4-\frac{1}{4}\bar{l}_3\right)\bar{m}^4 - \bar{l}_4\bar{m}^2\mu_I^2 + \ln\frac{M^2}{\bar{m}^2}\left(\frac{3}{4}\bar{m}^2-\mu_I^2\right)\bar{m}^2\right]\alpha^2+\mathcal{O}\left(\alpha^4\right). \tag{4.91}$$

All coupling constants are measured at $M = m_\pi$. Using this in the logarithm gives,

$$\ln\frac{M^2}{\bar{m}^2} = \ln\frac{m_\pi^2}{\bar{m}^2} = \ln\left[1+\mathcal{O}\left((\bar{m}/f)^2\right)\right] = \mathcal{O}\left((\bar{m}/f)^2\right). \tag{4.92}$$

The term proportional to the logarithm thus vanishes to next-to-leading order. Combining these expressions give total NLO free energy, up to second order in α, is

$$\mathcal{F}_{\text{NLO}} = \mathcal{F}_{\text{NLO}}(\alpha=0) + \frac{1}{2}f^2\bar{m}^2\left(1-\frac{1}{2}\frac{\bar{l}_3-4\bar{l}_4}{(4\pi)^2}\frac{\bar{m}^2}{f^2}\right)\alpha^2 - \frac{1}{2}f^2\mu_I^2\left(1+\frac{2\bar{l}_4}{(4\pi)^2}\frac{\bar{m}^2}{f^2}\right)\alpha^2+\mathcal{O}\left(\alpha^4\right). \tag{4.93}$$

We now insert the physical constants f_π and m_π, by using the next-to-leading order expressions Eq. (4.47) and Eq. (4.48). Notice that

$$f_\pi^2 m_\pi^2 = f^2\bar{m}^2\left[1-\frac{1}{2}\frac{\bar{l}_3-4\bar{l}_4}{(4\pi)^2}\frac{\bar{m}^2}{f^2}+\mathcal{O}\left(\frac{\bar{m}^4}{f^4}\right)\right], \tag{4.94}$$

which means that the NLO free energy has the same structure as the leading order expression,

$$\mathcal{F}_{\text{NLO}} = \mathcal{F}_{\text{NLO}}(\alpha=0) + \frac{1}{2}f_\pi^2(m_\pi^2-\mu_I^2)\alpha^2+\mathcal{O}\left(\alpha^4\right). \tag{4.95}$$

This shows that the critical isospin chemical potential is $\mu_I^c = m_\pi$, also at next-to-leading order. We expect this to hold to all orders in perturbation theory.

Chapter 5

Conclusions and outlook

The thermodynamic behavior of chiral perturbation theory at non-zero isospin chemical potential serves as a fruitful avenue for exploration of QCD at low temperatures, as it can be compared to calculations from first principles using lattice QCD. The recent proposals that pions can form compact stellar objects called pion stars [14, 15] increase the importance of better understanding of this regime. It is speculated that lepton asymetires in the early universe could result in pion condensation [14, 44–46]. In this case, pion stars might have left observable traces in the form of neutrino and photon spectra from their evaporation or in the form of gravitational waves and can have thus played a role in forming the universe we see today [14].

In this specialization project, we have discussed the theoretical foundations for chiral perturbation theory and derived the building blocks of the effective Lagrangian governing pions. Using the most general Lagrangian to next-to-leading order, we calculated the grand canonical free energy density in the case of a non-zero isospin chemical potential. From this, we calculated the equation of state (EOS). We find the EOS to remain trivial for isospin chemical potential μ_I less than some critical value μ_I^c and showed that this value equals the pion mass m_π to next-to-leading order, as expected. Furthermore, at $\mu_I = \mu_I^c$, we showed that the system undergoes a phase transition, breaking the isospin symmetry. In this new phase, it becomes energetically favorable for the system to move to an excited state, leading to a pion condensate. The pion condensate is characterized by a non-zero isospin density n_I, caused by the ground state being rotated away from the vacuum. As we expect from Goldstone's theorem, we observe a massless mode in the spectrum.

Outlook

The equation of state of a material is used in conjunction with the Tolman-Oppenheimer-Volkoff (TOV) equations to model the internal dynamics of stars. This enables calculation of the relationship between the mass and radius of stars [13]. Stellar objects, such as stars, display no electric charge neutrality on macroscopic scales. In the case of pion stars, one has to include leptons in the model to ensure electric charge neutrality. Isospin density is related to the up- and down-quark density n_u and n_d by

$$n_I = \frac{1}{TV} Q_I = \frac{1}{TV} \left(\langle \bar{u} \gamma^0 u \rangle - \langle \bar{d} \gamma^0 d \rangle \right) = n_u - n_d, \tag{5.1}$$

At zero baryon chemical potential, $\mu_B = 0$, the isospin chemical potential is related to the up- and down-quark chemical potential by $\mu_I/2 = \mu_u = -\mu_d$. We thus get the relationship $n_u = -n_d$ [14]. As the up quark has electric charge $\frac{2}{3}e$, and the down quark $-\frac{1}{3}e$, the pion condensate alone is electrically charged. A realistic stellar object thus must include a lepton-density n_l to remain neutrally charged. This gives the criterion for charge density,

$$n_Q = \frac{2}{3} n_u - \frac{1}{3} n_d - n_l = n_I - n_l = 0. \tag{5.2}$$

Together with the equation of state and the TOV-equation, this equation allows for the investigation of charge-neutral pion stars.

Comparisons between the free energy density and other thermodynamic quantities obtained from chiral perturbation theory and lattice QCD at $T = 0$ are in good agreement [35, 38]. Using thermal field thoery, χPT can be extended to finite temperature. Recent studies of χPT at non-zero temperature, find that the theory remains in good agreement with lattice QCD as well as other models for temperatures below 20 MeV [47]. A good understanding of the thermal properties of pion condensates is critical to account for the non-zero temperature of real stellar objects.

Our results can be improved by using three-flavor chiral perturbation theory, taking into account the strange quark. The strange quark s has a mass $m_s \approx 93\,\mathrm{MeV}$ [4], which is considerably larger than the up and down quark, but still within the strong-interaction regime, which suggests that it can play an important role in the equation of state. Chiral perturbation theory can be extended to include the strange quark by considering the larger group $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$, consisting of rotations of all three quarks into each other.

Appendix A

Thermal field theory

This section is based on [48, 49].

A.1 Statistical mechanics

In classical mechanics, a thermal system at temperature $T = 1/\beta$ is described as an ensemble state, which have a probability P_n of being in state n, with energy E_n. In the canonical ensemble, the probability is proportional to $e^{-\beta E_n}$. The expectation value of some quantity A, with value A_n in state n is

$$\langle A \rangle = \sum_n A_n P_n = \frac{1}{Z} \sum_n A_n e^{-\beta E_n}, \quad Z = \sum_n e^{-\beta A_n}.$$

Z is called the partition function. In quantum mechanics, an ensemble configuration is described by a non-pure density operator,

$$\hat{\rho} = \sum_n P_n \left| n \right\rangle \left\langle n \right|,$$

where $|n\rangle$ is some basis for the relevant Hilbert space. Assuming $|n\rangle$ are energy eigenvectors, i.e., $\hat{H} |n\rangle = E_n |n\rangle$, the density operator for the canonical ensemble, where $P_n = Ce^{-\beta E_n}$, is

$$\hat{\rho} = \sum_n Ce^{-\beta E_n} \left| n \right\rangle \left\langle n \right| = Ce^{-\beta \hat{H}} \sum_n \left| n \right\rangle \left\langle n \right| = Ce^{-\beta \hat{H}}.$$

The expectation value in the ensemble state of a quantity corresponding to the operator $\hat{A}$ is given by

$$\langle A \rangle = \frac{\mathrm{Tr}\left\{ \hat{\rho} \hat{A} \right\}}{\mathrm{Tr}\{\hat{\rho}\}} = \frac{1}{Z} \mathrm{Tr}\left\{ \hat{A} e^{-\beta \hat{H}} \right\}. \tag{A.1}$$

The partition function Z ensures that the probabilities adds up to 1, and is defined as

$$Z = \mathrm{Tr}\left\{ e^{-\beta \hat{H}} \right\}. \tag{A.2}$$

The grand canonical ensemble takes into account the conserved charges of the system, which are a result of Nöther's theorem, as discussed in section 2.2. In the grand canonical ensemble, a system with n conserved charges Q_i has probability proportional to $e^{-\beta(H - \mu_i Q_i)}$. Here, μ_i are the chemical potentials corresponding to conserved charge Q_i. This leads to the partition function

$$Z = \mathrm{Tr}\left\{ e^{-\beta(\hat{H} - \mu_i \hat{Q}_i)} \right\}. \tag{A.3}$$

A.2 Imaginary-time formalism

The partition function may be calculated similarly to the path integral approach, in what is called the imaginary-time formalism. This formalism is restricted to time independent problems, and is used to study fields in a volume V. This volume is taken to infinity in the thermodynamic limit. As an example, take a scalar quantum field theory with the Hamiltonian

$$\hat{H} = \int_V \mathrm{d}^3x\, \hat{\mathcal{H}}\left[\hat{\varphi}(\vec{x}), \hat{\pi}(\vec{x})\right], \tag{A.4}$$

where $\hat{\varphi}(\vec{x})$ is the field operator, and $\hat{\pi}(\vec{x})$ is the corresponding canonical momentum operator. These field operators have time independent eigenvectors, $|\varphi\rangle$ and $|\pi\rangle$, defined by

$$\hat{\varphi}(\vec{x})\,|\varphi\rangle = \varphi(\vec{x})\,|\varphi\rangle\,, \quad \hat{\pi}(\vec{x})\,|\pi\rangle = \pi(\vec{x})\,|\pi\rangle\,. \tag{A.5}$$

In analogy with regular quantum mechanics, they obey the relations [1]

$$\mathbb{1} = \int \mathcal{D}\varphi(\vec{x})\,|\varphi\rangle\langle\varphi| = \int \mathcal{D}\pi(\vec{x})\,|\pi\rangle\langle\pi|\,, \tag{A.6}$$

$$\langle\varphi|\pi\rangle = \exp\!\left(i\int_V \mathrm{d}^3x\,\varphi(\vec{x})\pi(\vec{x})\right), \tag{A.7}$$

$$\langle\pi_a|\pi_b\rangle = \delta(\phi_a - \phi_b), \quad \langle\varphi_a|\varphi_b\rangle = \delta(\varphi_a - \varphi_b). \tag{A.8}$$

The functional integral is defined by starting with M degrees of freedom, $\{\varphi_m\}_{m=1}^M$ located at a finite grid $\{\vec{x}_m\}_{m=1}^M \subset V$. The integral is then the limit of the integral over all degrees of freedom, as $M \to \infty$:

$$\int \mathcal{D}\varphi(\vec{x}) = \lim_{M\to\infty} \int \left(\prod_{m=1}^{M} \mathrm{d}\varphi_m\right).$$

The functional Dirac-delta $\delta(f) = \prod_x \delta(f(x))$ is generalization of the familiar Dirac delta function. Given a functional $\mathcal{F}[f]$, it is defined by the relation

$$\int \mathcal{D}f(x)\,\mathcal{F}[f]\delta(f-g) = \mathcal{F}[g]. \tag{A.9}$$

The Hamiltonian is the limit of a sum of Hamiltonians $\hat{H}_m$ for each point $\vec{x}_m$

$$\hat{H} = \lim_{M\to\infty} \sum_{m=1}^{M} \frac{V}{M} \hat{H}_m(\{\hat{\varphi}_m\}, \{\hat{\pi}_m\}).$$

H_m may depend on the local degrees of freedom $\hat{\varphi}_m$, $\hat{\pi}_m$ as well as those at neighboring points. By inserting the completeness relations Eq. (A.6) N times into the definition of the partition function, it may be written as

$$Z = \int \mathcal{D}\varphi(\vec{x})\,\left\langle\varphi\middle|e^{-\beta\hat{H}}\middle|\varphi\right\rangle = \prod_{n=1}^{N}\left(\int \mathcal{D}\varphi_n(\vec{x}) \int \mathcal{D}\pi_n(\vec{x})\right) \prod_{n=1}^{N} \langle\varphi_n|\pi_n\rangle \left\langle\pi_n\middle|e^{-\epsilon\hat{H}}\middle|\varphi_{n+1}\right\rangle \langle\varphi_1|\varphi_{N+1}\rangle\,,$$

where $\epsilon = \beta/N$. The last term ensures that $\varphi_1 = \varphi_{N+1}$. Bosons such as the scalar field φ, follow the periodic boundary condition $\varphi(0,\vec{x}) = \varphi(\beta,\vec{x})$. Fermions, as we will show later, follow the anti-periodic boundary condition $\psi(0,\vec{x}) = -\psi(\beta,\vec{x})$. We now want to exploit the fact that $|\pi\rangle$ and $|\varphi\rangle$ are the eigenvectors of the operators that define the Hamiltonian. In our case, as the Hamiltonian density $\mathcal{H}$ can be written as a sum of functions of φ and π separately, $\mathcal{H}[\varphi(\vec{x}),\pi(\vec{x})] = \mathcal{F}_1[\varphi(\vec{x})] + \mathcal{F}_2[\pi(\vec{x})]$ we may evaluate it as $\langle\pi_n|\mathcal{H}[\hat{\varphi}(\vec{x}),\hat{\pi}(\vec{x})]|\varphi_{n+1}\rangle = \mathcal{H}[\varphi_{n+1}(\vec{x}),\pi_n(\vec{x})]\,\langle\pi_n|\varphi_{n+1}\rangle$. This relationship does not, however, hold for more general functions of the field operators. In that case, one has to be more careful about the ordering of

[1] Some authors write $\mathcal{D}\pi/2\pi$. This extra factor 2π is a convention which in this text is left out for notational clarity.

the operators, for example, by using *Weyl ordering* [23]. By series expanding $e^{-\epsilon\hat{H}}$ and exploiting this relationship, the partition function can be written as, to second order in ϵ,

$$Z = \prod_{n=1}^{N}\left(\int \mathcal{D}\varphi_n(\vec{x})\int \mathcal{D}\pi_n(\vec{x})\right)\exp\left[-\epsilon\sum_{n=1}^{N}\int_V \mathrm{d}^3x\,\left(\mathcal{H}[\varphi_n(\vec{x}),\pi_n(\vec{x})] - i\pi_n(\vec{x})\frac{\varphi_n(\vec{x})-\varphi_{n+1}(\vec{x})}{\epsilon}\right)\right].$$

We denote $\varphi_n(\vec{x}) = \varphi(\tau_n,\vec{x})$, $\tau\in[0,\beta]$ and likewise with $\pi_n(\vec{x})$. In the limit $N\to\infty$, the expression for the partition function becomes

$$Z = \int_S \mathcal{D}\varphi(\tau,\vec{x})\int \mathcal{D}\pi(\tau,\vec{x})\exp\left\{-\int_0^\beta \mathrm{d}\tau\int_V \mathrm{d}\vec{x}\,\{\mathcal{H}[\varphi(\tau,\vec{x}),\pi(\tau,\vec{x})] - i\pi(\tau,\vec{x})\dot{\varphi}(\tau,\vec{x})\}\right\}, \tag{A.10}$$

where S is the set of field configurations φ such that $\varphi(\beta,\vec{x}) = \varphi(0,\vec{x})$. With a Hamiltonian density of the form $\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\varphi)^2 + \mathcal{V}(\varphi)$, we can evaluate the integral over the canonical momentum π by discretizing $\pi(\tau_n,\vec{x}_m) = \pi_{n,m}$,

$$\begin{aligned}
&\int \mathcal{D}\pi\exp\left\{-\int_0^\beta \mathrm{d}\tau\int_V \mathrm{d}^3x\left(\frac{1}{2}\pi^2 - i\pi\dot{\varphi}\right)\right\}\\
&= \lim_{M,N\to\infty}\int\left(\prod_{m,n=1}^{M,N}\frac{\mathrm{d}\pi_{m,n}}{2\pi}\right)\exp\left\{-\sum_{m,n}\frac{V\beta}{MN}\left[\frac{1}{2}(\pi_{m,n} - i\dot{\varphi}_{m,n})^2 + \frac{1}{2}\dot{\varphi}_{m,n}^2\right]\right\}\\
&= \lim_{M,N\to\infty}\left(\frac{MN}{2\pi V\beta}\right)^{MN/2}\exp\left\{-\int_0^\beta \mathrm{d}\tau\int_V \mathrm{d}^3x\,\frac{1}{2}\dot{\varphi}^2\right\},
\end{aligned}$$

where $\dot{\varphi}_{m,n} = (\varphi_{m,n+1} - \varphi_{m,n})/\epsilon$. The partition function is then,

$$Z = C\int \mathcal{D}\varphi\exp\left\{-\int_0^\beta \mathrm{d}\tau\int_V \mathrm{d}^3x\left[\frac{1}{2}\left(\dot{\varphi}^2 + \nabla\varphi^2\right) + \mathcal{V}(\varphi)\right]\right\}. \tag{A.11}$$

Here, C is the divergent constant that results from the π-integral. In the last line, we exploited the fact that the variable of integration $\pi_{n,m}$ may be shifted by a constant without changing the integral, and used the Gaussian integral

$$\int_{-\infty}^{\infty}\mathrm{d}x\,e^{-ax^2/2} = \sqrt{\frac{2\pi}{a}}.$$

The partition function resulting from this procedure may also be obtained by starting with the ground state path integral

$$Z_g = \int \mathcal{D}\varphi\mathcal{D}\pi\exp\left\{i\int_{\Omega'}\mathrm{d}^4x\,(\pi\dot{\varphi} - \mathcal{H}[\varphi,\pi])\right\} = C'\int\mathcal{D}\varphi(x)\exp\left\{i\int_{\Omega'}\mathrm{d}^4x\,\mathcal{L}[\varphi,\partial_\mu\varphi]\right\},$$

and follow a formal procedure. First, the action integral is modified by performing a Wick-rotation of the time coordinate t. This involves changing the domain of t from the real line to the imaginary line by closing the contour at infinity and changing variable $it\to\tau$. The new variable is then restricted to the interval $\tau\in[0,\beta]$, and the domain of the functional integral $\int\mathcal{D}\varphi$ is restricted from *all* (smooth enough) field configurations $\varphi(t,\vec{x})$, to only those that obey $\varphi(\beta,\vec{x}) = e^{i\theta}\varphi(0,\vec{x})$, which is denoted S. Here, $\theta\in\{0,\pi\}$, depending on if the particle is a boson of fermion. This procedure motivates the introduction of the Euclidean Lagrange density, $\mathcal{L}_E(\tau,\vec{x}) = -\mathcal{L}(-i\tau,\vec{x})$, as well as the name "imaginary-time formalism". The result is the same partition function as before,

$$\begin{aligned}
Z &= C\int_S\mathcal{D}\varphi\int\mathcal{D}\pi\exp\left\{-\int_0^\beta\mathrm{d}\tau\int_V\mathrm{d}^3x\,[-i\dot{\varphi}\pi + \mathcal{H}(\varphi,\pi)]\right\}\\
&= C'\int_S\mathcal{D}\varphi\exp\left\{-\int_0^\beta\mathrm{d}\tau\int_V\mathrm{d}^3x\,\mathcal{L}_E(\varphi,\pi)\right\}.
\end{aligned} \tag{A.12}$$

Fourier series

Due to the finite range of the imaginary-time coordinate $\tau \in [0, \beta]$, the momentum-space fields in imaginary-time formalism have a discrete coordinate. We define the Matsubara-frequencies as $\omega_n = 2n\pi/\beta$ for bosons and $\omega_n = (2n+1)\pi/\beta$ for fermions. They together form the reciprocal space $\tilde{\Omega} = \{\omega_n\} \times \tilde{V}$, where $\tilde{V}$ is reciprocal to V. To get a more economical notation, we denote the Euclidean real-space coordinates as $X = (\tau, \vec{x})$ and the reciprocal space coordinates as $K = (\omega_n, \vec{k})$. The dot product is $X \cdot K = \omega_n \tau + \vec{k} \cdot \vec{x}$. In the limit $V \to \infty$, we follow the prescription

$$\frac{1}{V} \sum_{\vec{p} \in \tilde{V}} \to \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3}.$$

The sum over all degrees of freedom, and the corresponding integrals for the thermodynamic limit are

$$\frac{\beta V}{NM} \sum_{n=1}^{N} \sum_{\vec{x}_m \in V} \xrightarrow{N,M \to \infty} \int_0^\beta \mathrm{d}\tau \int_{\mathbb{R}^3} \mathrm{d}^3 x = \int_\Omega \mathrm{d}X,$$
$$\frac{1}{V} \sum_{n=-\infty}^{\infty} \sum_{\vec{k} \in \tilde{V}} \xrightarrow{V \to \infty} \sum_{n=-\infty}^{\infty} \int_{R^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} = \int_{\tilde{\Omega}} \mathrm{d}K.$$

The convention used for the Fourier expansion of thermal fields is in accordance with [48]. The prefactor is chosen to make the Fourier components of the field dimensionless, which makes it easier to evaluate the trace correctly. For bosons, the Fourier expansion is

$$\varphi(X) = \sqrt{V\beta} \int_{\tilde{\Omega}} \mathrm{d}K\, \tilde{\varphi}(K) e^{iX \cdot K} = \sqrt{\frac{\beta}{V}} \sum_{n=-\infty}^{\infty} \sum_{\vec{k} \in \tilde{V}} \tilde{\varphi}_n(\vec{p}) \exp\Big\{ i(\omega_n \tau + \vec{x} \cdot \vec{k}) \Big\},$$
$$\tilde{\varphi}(K) = \sqrt{\frac{1}{V\beta^3}} \int_{\tilde{\Omega}} \mathrm{d}X\, \tilde{\varphi}(X) e^{-iX \cdot K},$$

while for Fermions it is

$$\psi(X) = \sqrt{V} \int_{\tilde{\Omega}} \mathrm{d}K\, \tilde{\psi}(K) e^{iX \cdot K} = \frac{1}{\sqrt{V}} \sum_{n=-\infty}^{\infty} \sum_{\vec{k} \in \tilde{V}} \psi(\omega_n, \vec{k}) \exp\Big\{ i(\omega_n \tau + \vec{x} \cdot \vec{k}) \Big\}. \tag{A.13}$$

Two often used identities are

$$\int_\Omega \mathrm{d}X e^{iX \cdot (K - K')} = \beta \delta_{nn'} (2\pi)^3 \delta^3(\vec{k} - \vec{k}') := \beta \delta(K - K'), \tag{A.14}$$
$$\int_{\tilde{\Omega}} \mathrm{d}K\, e^{iK(X - X')} = \beta \delta(\tau - \tau') \delta^3(\vec{x} - \vec{x}') := \beta \delta(X - X'). \tag{A.15}$$

A.3 Free scalar field

The procedure for obtaining the thermal properties of an interacting scalar field is similar to that used in scattering theory. One starts with a free theory, which can be solved exactly. Then an interaction term is added, which is accounted for perturbatively by using Feynman diagrams. The Euclidean Lagrangian for a free scalar gas is, after integrating by parts,

$$\mathcal{L}_E = \frac{1}{2} \varphi(X) \left(-\partial_E^2 + m^2 \right) \varphi(X) \tag{A.16}$$

Here, $X = (\tau, \vec{x})$ is the Euclidean coordinate resulting from the Wick-rotation as described in the last section. We have also introduced the Euclidean Laplace operator, $\partial_E^2 = \partial_\tau^2 + \nabla^2$. Following the procedure to obtain the thermal partition function yields

$$Z = C \int_S \mathcal{D}\varphi(X) \exp\Big\{ - \int_\Omega \mathrm{d}X \frac{1}{2} \varphi(X) \left(-\partial_E^2 + m^2 \right) \varphi(X) \Big\}. \tag{A.17}$$

Here, Ω is the domain $[0,\beta]\times V$. We then insert the Fourier expansion of φ and change the functional integration variable to the Fourier components. The integration measures are related by

$$\mathcal{D}\varphi(X) = \det\left(\frac{\delta\varphi(X)}{\delta\tilde{\varphi}(K)}\right)\mathcal{D}\tilde{\varphi}(K),$$

where $K=(\omega_n,\vec{k})$ is the Euclidean Fourier-space coordinate. The determinant factor which appears may be absorbed into the constant C, as the integration variables are related by a linear transform. The action becomes

$$\begin{aligned} S &= -\int_\Omega \mathrm{d}X\,\mathcal{L}_E = -\frac{1}{2}V\beta\int_\Omega \mathrm{d}X\int_{\tilde{\Omega}}\mathrm{d}K\int_{\tilde{\Omega}}\mathrm{d}K'\,\tilde{\varphi}(K')\left(\omega_n^2+\vec{k}^2+m^2\right)\tilde{\varphi}(K)e^{iX\cdot(K-K')} \\ &= -\frac{1}{2}V\beta^2\int_{\tilde{\Omega}}\mathrm{d}K\,\tilde{\varphi}(K)^*\left(\omega_n^2+\omega_k^2\right)\tilde{\varphi}(K), \end{aligned}$$

where $\omega_k^2=\vec{k}^2+m^2$. $\tilde{\Omega}$ is the reciprocal space corresponding to Ω. We used the fact that φ is real, which implies that $\tilde{\varphi}(-K)=\tilde{\varphi}(K)^*$, as well as the identity Eq. (A.14). This gives the partition function

$$Z = C\int_{\tilde{S}}\mathcal{D}\tilde{\varphi}(K)\exp\left\{-\frac{1}{2}V\int_{\tilde{\Omega}}\mathrm{d}K\,\tilde{\varphi}(K)^*\left[\beta^2(\omega_n^2+\omega_k^2)\right]\tilde{\varphi}(K)\right\}, \tag{A.18}$$

Going back to before the continuum limit, this integral can be written as a product of Gaussian integrals and may therefore be evaluated

$$Z = C\prod_{n=-\infty}^{\infty}\prod_{k\in\tilde{V}}\left(\int \mathrm{d}\tilde{\varphi}_{n,\vec{k}}\,\exp\left\{-\frac{1}{2}\tilde{\varphi}^*_{n,\vec{k}}\left[\beta^2(\omega_n^2+\omega_k^2)\right]\tilde{\varphi}_{n,\vec{k}}\right\}\right) = C\prod_{n=-\infty}^{\infty}\prod_{k\in\tilde{V}}\sqrt{\frac{2\pi}{\beta^2(\omega_n^2+\omega_k^2)}}.$$

The partition function is related to free energy F through

$$\frac{F}{TV} = -\frac{\ln(Z)}{V} = \frac{1}{2}\int_{\tilde{\Omega}}\mathrm{d}K\ln[\beta^2(\omega_n^2+\omega_k^2)] + \frac{F_0}{TV}, \tag{A.19}$$

where F_0 is a constant.

A faster and more formal way to get to this result is to compare the partition function to the multidimensional version of the Gaussian integral [23, 48]. The partition function has the form

$$I_n = \int_{\mathbb{R}^n}\mathrm{d}^n x\,\exp\left\{-\frac{1}{2}\langle x, D_0^{-1}x\rangle\right\},$$

where D_0^{-1} is a linear operator, and $\langle\cdot,\cdot\rangle$ an inner product on the corresponding vector space. By diagonalizing D_0^{-1}, we get the result

$$I_n = \sqrt{\frac{(2\pi)^n}{\det\left(D_0^{-1}\right)}}.$$

We may then use the identity

$$\det\left(D_0^{-1}\right) = \prod_i\lambda_i = \exp\left\{\mathrm{Tr}\left[\ln\left(D_0^{-1}\right)\right]\right\}, \tag{A.20}$$

where λ_i are the eigenvalues of D_0^{-1}. The trace in this context is defined by the vector space D_0^{-1} acts on. For given an orthonormal basis x_n, such that $\langle x_n, x_n'\rangle=\delta_{nn'}$ the trace can be evaluated as $\mathrm{Tr}\left\{D_0^{-1}\right\} = \sum_n\langle x_n, D_0^{-1}x_n\rangle$. Identifying

$$\langle x, D_0^{-1}x\rangle = \int_\Omega \mathrm{d}X\varphi(X)\left(-\partial_E^2+m^2\right)\varphi(X),$$

we get the formal result

$$Z = \det\left(-\partial_E^2+m^2\right)^{-1/2},$$

and

$$\beta F = \frac{1}{2}\,\mathrm{Tr}\{\ln(-\partial_E^2 + m^2)\}.$$

The logarithm may then be evaluated by using the eigenvalues of the linear operator. This is found by diagonalizing the operator,

$$\langle x, D_0^{-1} x\rangle = \int_\Omega \mathrm{d}X \varphi(X)\left(-\partial_E^2 + m^2\right)\varphi(X) = V\int_{\tilde{\Omega}} \mathrm{d}K \tilde{\varphi}(K)^*[\beta^2(\omega_k^2+\omega_n^2)]\tilde{\varphi}(K),$$

leaving us with the same result as we obtained in Eq. (A.19),

$$\beta F = \frac{1}{2}\,\mathrm{Tr}\{\ln(-\partial_E^2 + m^2)\} = \frac{1}{2}V\int_{\tilde{\Omega}} \mathrm{d}K \ln[\beta^2(\omega_n^2+\omega_k^2)].$$

Sums similar to this show up a lot, and we show how to evaluate them in the next section.

Thermal sum

When evaluating thermal integral, we will often encounter sums of the form

$$j(\omega,\mu) = \frac{1}{2\beta}\sum_{\omega_n} \ln\{\beta^2[(\omega_n + i\mu) + \omega^2]\} + g(\beta), \tag{A.21}$$

where the sum is over either the bosonic Matsubara frequencies $\omega_n = 2n\pi/\beta$, $n \in \mathbb{Z}$, or the fermionic ones, $\omega_n = (2n+1)\pi/\beta$, $n \in \mathbb{Z}$. $\mu \in \mathbb{R}$ is a chemical potential. g may be a function of β, but we assume it is independent of ω. Thus, the factor β^2 could strictly be dropped, but it is kept to make the argument within the logarithm dimensionless. We define the function

$$i(\omega,\mu) = \frac{1}{\omega}\frac{\mathrm{d}}{\mathrm{d}\omega} j(\omega,\mu) = \frac{1}{\beta}\sum_{\omega_n} \frac{1}{(\omega_n + i\mu)^2 + \omega^2}. \tag{A.22}$$

We will first work with the sum over bosonic Matsubara frequencies. Assume $f(z)$ is an analytic function, except perhaps on a set of isolated poles $\{z_i\}$ located outside the real line. We can exploit this using the properties of the Bose-distribution $n_B(z)$. The Bose distribution is defined as

$$n_B(\omega) = \frac{1}{e^{\beta\omega} - 1}. \tag{A.23}$$

This function obeys

$$n_B(-i\omega) = -1 - n_B(i\omega). \tag{A.24}$$

We can expand it around the Bose Matsubara frequencies on the imaginary line:

$$i n_B[i(\omega_n + \epsilon)] = \frac{i}{e^{i\beta\epsilon + 2\pi i n} - 1} = i[i\beta\epsilon + \mathcal{O}\left(\epsilon^2\right)]^{-1} \sim \frac{1}{\epsilon\beta}. \tag{A.25}$$

This means that $i n_B(i\omega)$ has a pole on all Matsubara-frequencies, with residue $1/\beta$. Using this, we can rewrite the sum over Matsubara frequencies as a contour integral

$$\frac{1}{\beta}\sum_{\omega_n} f(\omega_n) = \oint_\gamma \frac{\mathrm{d}z}{2\pi i} f(z) i n_B(iz),$$

where γ is a contour that goes from $-\infty - i\epsilon$ to $+\infty - i\epsilon$, crosses the real line at ∞, goes from $+\infty - i\epsilon$ to $-\infty + i\epsilon$ before closing the curve. The contour γ and the new contours are illustrated in Figure A.1. This result exploits Cauchy's integral formula by letting the poles of $i n_B(iz)$ at the Matsubara frequencies "pick out" the necessary residues. The integral over γ is equivalent to two integrals along $\mathbb{R} \pm i\epsilon$,

$$\begin{aligned}
\frac{1}{\beta}\sum_{\omega_n} f(\omega_n) &= \left(\int_{\infty+i\epsilon}^{-\infty+i\epsilon} \frac{\mathrm{d}z}{2\pi} + \int_{-\infty-i\epsilon}^{\infty-i\epsilon} \frac{\mathrm{d}z}{2\pi}\right) f(z) n_B(iz), \\
&= \int_{-\infty-i\epsilon}^{\infty-i\epsilon} \frac{\mathrm{d}z}{2\pi}\left\{f(-z) + [f(z) + f(-z)]\, n_B(iz)\right\} \\
&= \int_{-\infty}^{\infty} \frac{\mathrm{d}z}{2\pi} f(z) + \int_{-\infty-i\epsilon}^{\infty-i\epsilon} \frac{\mathrm{d}z}{2\pi}\left[f(z) - f(-z)\right] n_B(iz).
\end{aligned} \tag{A.26}$$

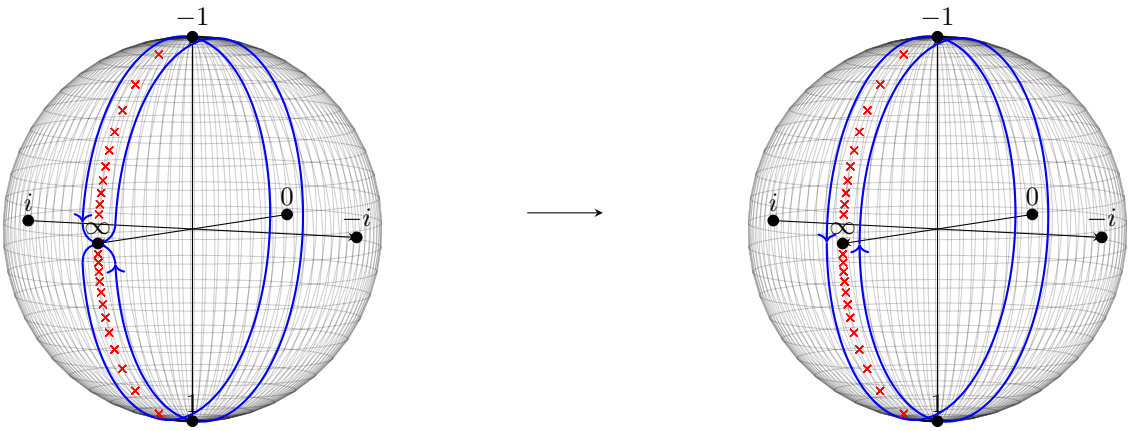


Figure A.1: The integral contour γ, and the result of deforming it into to contours close to the real line. The red crosses illustrate the poles of n_B.

In the second line, we have changed variables $z \to -z$ in the first integral, and exploited the property $n_B(-iz) = -1 - n_B(iz)$. In the last line, we use the assumption that $f(z)$ is analytic on the real line, and therefore also in a neighborhood of it. This allows us to shift the first integral back to the real line. As $n_B(iz)$ is analytic outside the real line, the result of the second integral is the sum of residues of $f(z) + f(-z)$ in the lower half-plane. The function

$$f(z) = \frac{1}{(z + i\mu)^2 + \omega^2} = \frac{i}{2\omega}\left(\frac{1}{z + i(\mu + \omega)} - \frac{1}{z + i(\mu - \omega)}\right) \tag{A.27}$$

obeys the assumed properties, as it has poles at $z = -i(\mu \pm \omega)$, with residue $1/(2\omega)$, so the function defined in Eq. (A.22) may be written

$$i(\omega, \mu) = \frac{1}{2\omega}[1 + n_B(\omega - \mu) + n_B(\omega + \mu)]. \tag{A.28}$$

Using the antiderivative of the Bose distribution,

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\ln\left(1 - e^{-\beta\omega}\right) = \beta n_B(\omega), \tag{A.29}$$

we get the final form of Eq. (A.21)

$$j(\omega, \mu) = \int \mathrm{d}\omega'\, \omega' i(\omega', \mu) = \frac{1}{2}\omega + \frac{1}{2\beta}\left[\ln\left(1 - e^{-\beta(\omega - \mu)}\right) + \ln\left(1 - e^{-\beta(\omega + \mu)}\right)\right] + g'(\beta). \tag{A.30}$$

The extra ω-independent term $g'(\beta)$ is an integration constant. We see there are temperature dependent terms, one due to the particle and one due to the anti-particle, and one due to the antiparticle, as they have opposite chemical potentials.

We now consider the sum over fermionic frequencies, which we for clarity denote $\tilde{\omega}_n$ in this chapter. The procedure, in this case, is the same, except that we have to use a function with poles at the fermionic Matsubara frequencies. This is done by the Fermi distribution, $n_F(z)$. The Fermi distribution is

$$n_F(\omega) = \frac{1}{e^{\beta\omega} + 1}. \tag{A.31}$$

It obeys

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\ln\left(1 - e^{-\beta\omega}\right) = -\beta n_F(\omega), \tag{A.32}$$

$$n_F(-i\omega) = 1 - n_F(i\omega). \tag{A.33}$$

The two distributions are related by

$$2n_B(i\omega;2\beta) - n_B(i\omega;\beta) = -n_F(i\omega;\beta). \tag{A.34}$$

With this, the sum over fermionic Matsubara frequencies gives

$$\frac{1}{\beta}\sum_{\tilde{\omega}_n} f(\tilde{\omega}_n) = -\int_{-\infty}^{\infty}\frac{\mathrm{d}z}{2\pi}f(z) + \int_{-\infty-i\epsilon}^{\infty-i\epsilon}\frac{\mathrm{d}z}{2\pi}\left[f(z)-f(-z)\right]n_F(iz), \tag{A.35}$$

and

$$i(\omega,\mu) = \frac{1}{2\omega}[-1 + n_F(\omega-\mu) + n_F(\omega+\mu)]. \tag{A.36}$$

Using the antiderivative of the Fermi-distribution, we get

$$j(\omega,\mu) = -\frac{1}{2}\omega - \frac{1}{2\beta}\left[\ln\left(1+e^{-\beta(\omega-\mu)}\right) + \ln\left(1+e^{-\beta(\omega+\mu)}\right)\right]. \tag{A.37}$$

Low-temperature limit

Using the result from section A.3 on the result for the free energy density of the free scalar field, Eq. (A.12), we get

$$\mathcal{F} = \frac{\ln(Z)}{\beta V} = \frac{1}{2}\int_{\tilde{V}}\frac{\mathrm{d}^3k}{(2\pi)^3}\left[\omega_k + \frac{2}{\beta}\ln\left(1-e^{-\beta\omega_k}\right)\right]. \tag{A.38}$$

The free energy density thus has two contributions from parts; the first part is dependent on temperature, the other is a temperature-independent vacuum contribution. Noticing that the integral is spherically symmetric, we may write the two contributions as

$$\mathcal{F}_0 = \frac{1}{2}\frac{1}{2\pi^2}\int_{\mathbb{R}}\mathrm{d}k\,k^2\sqrt{k^2+m^2}, \quad \mathcal{F}_T = \frac{T^4}{2\pi^2}\int_{\mathbb{R}}\mathrm{d}x\,x^2\ln\left(1-e^{-\sqrt{x^2+(m/T)^2}}\right), \tag{A.39}$$

The temperature-independent part, $\mathcal{F}_0$, is divergent, and we must impose a regulator and then add counter-terms. $\mathcal{F}_T$, however, is convergent. To see this, we use the series expansion $\ln(1+\epsilon)\sim\epsilon+\mathcal{O}\left(\epsilon^2\right)$ to find the leading part of the integrand for large k's,

$$x^2\ln\left(1-e^{-\sqrt{x^2+(\beta m)^2}}\right) \sim -x^2e^{-x}, \tag{A.40}$$

which is exponentially suppressed, making the integral convergent. In the limit of $T\to 0$, we get

$$\mathcal{F}_T \sim \frac{T^4}{2\pi^2}\int_{\mathbb{R}}\mathrm{d}x\,x^2\ln\left(1-e^{-x}\right) = -\frac{T^4}{2\pi^2}\sum_{n=1}\frac{1}{n}\frac{\partial^2}{\partial n^2}\int\mathrm{d}x e^{-nx} = -\frac{T^4}{2\pi^2}\sum_{n=1}\frac{2}{n^4} = -\frac{T^4}{\pi^2}\zeta(4),$$

where ζ is the Riemann-zeta function. Using $\zeta(4)=\frac{\pi^4}{90}$, we get

$$\mathcal{F}_T \sim -\frac{\pi^2}{90}T^4, \quad T\to 0. \tag{A.41}$$

Regularization

Returning to the temperature-independent part, we use dimensional regularization to control its divergent behavior. To that end, we define

$$\Phi_n(m,d,\alpha) = \int_{\tilde{\Omega}}\frac{\mathrm{d}^dk}{(2\pi)^d}(k^2+m^2)^{-\alpha}, \tag{A.42}$$

so that $\mathcal{F}_0 = \Phi_3(m,3,-1/2)/2$. We will use the formula for integration of spherically symmetric function in d-dimensions,

$$\int_{\mathbb{R}^d}\mathrm{d}^dx\,f(r) = \frac{2\pi^{d/2}}{\Gamma(d/2)}\int_{\mathbb{R}}\mathrm{d}r\,r^{d-1}f(r), \tag{A.43}$$

where $r = \sqrt{x_i x_i}$ is the radial distance, and Γ is the Gamma function. The factor in the front of the integral is the solid angle. By extending this formula from integer-valued d to real numbers, we get

$$\Phi_n = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_{\mathbb{R}} \mathrm{d}k \, \frac{k^{d-1}}{(k^2+m^2)^\alpha} = \frac{m^{n-2\alpha} m^{d-n}}{(4\pi)^{d/2}\Gamma(d/2)} 2 \int_{\mathbb{R}} \mathrm{d}z \, \frac{z^{d-1}}{(1+z)^\alpha}, \tag{A.44}$$

where we have made the change of variables $mz = k$. We make one more change of variable to the integral,

$$I = 2 \int_{\mathbb{R}} \mathrm{d}z \, \frac{z^{d-1}}{(1+z)^\alpha}. \tag{A.45}$$

Let

$$z^2 = \frac{1}{s} - 1 \implies 2z\mathrm{d}z = -\frac{\mathrm{d}s}{s^2} \tag{A.46}$$

Thus,

$$I = \int_0^a \mathrm{d}s \, s^{\alpha - d/2 - 1} (1-z)^{d/2-1}. \tag{A.47}$$

This is the beta function, which can be written in terms of Gamma functions [23]

$$I = B\left(\alpha - \frac{d}{2}, \frac{d}{2}\right) = \frac{\Gamma\left(\alpha - \frac{d}{2}\right)\Gamma\left(\frac{d}{2}\right)}{\Gamma(\alpha)}. \tag{A.48}$$

Combining this gives

$$\Phi_n(m,d,\alpha) = \mu^{n-d} \frac{m^{n-2\alpha}}{(4\pi)^{d/2}} \frac{\Gamma\left(\alpha - \frac{d}{2}\right)}{\Gamma(\alpha)} \left(\frac{m^2}{\mu^2}\right)^{(d-n)/2}. \tag{A.49}$$

In the last step, we have introduced a parameter μ with mass dimension 1, that is, $[\mu] = [m]$. This is done to be able to series expand around $d - n$ in a dimensionless variable. This parameter is arbitrary, and all physical quantities should therefore be independent of it. We will shortly justify this parameter further.

Inserting $n = 3$, $d = 3 - 2\epsilon$ and $\alpha = -1/2$, we get

$$\Phi_3(m, 3-2\epsilon, -1/2) = \frac{m^4 \mu^{-2\epsilon}}{(4\pi)^{d/2}\Gamma(-1/2)} \Gamma(-2+\epsilon) \left(\frac{m^2}{\mu^2}\right)^{-\epsilon} = -\mu^{-2\epsilon} \frac{m^4}{(4\pi)^2} \left(\frac{m^2}{4\pi\mu^2}\right)^{-\epsilon} \frac{\Gamma(\epsilon)}{(\epsilon-2)(\epsilon-1)}, \tag{A.50}$$

where we have used the defining property $\Gamma(z+1) = z\Gamma(z)$ and $\Gamma(1/2) = \sqrt{\pi}$. Expanding around $\epsilon = 0$ gives

$$\left(\frac{m^2}{4\pi\mu^2}\right)^{-\epsilon} \sim 1 + \epsilon \ln\left(4\pi \frac{\mu^2}{m^2}\right), \tag{A.51}$$

$$\Gamma(\epsilon) \sim \frac{1}{\epsilon} - \gamma, \tag{A.52}$$

$$\frac{1}{(\epsilon-2)(\epsilon-1)} \sim \frac{1}{2}\left(1 + \frac{3}{2}\epsilon\right). \tag{A.53}$$

The divergent behavior of the temperature-independent term is therefore

$$\mathcal{F}_0 \sim -\mu^{-2\epsilon} \frac{1}{4} \frac{m^4}{(4\pi)^2} \left[\frac{1}{\epsilon} - \gamma + \frac{3}{2} + \ln\left(4\pi \frac{\mu^2}{m^2}\right)\right]. \tag{A.54}$$

With this regulator, one can then add counter-terms to cancel the ϵ^{-1}-divergence. The exact form of the counter-term is convention. One may also cancel the finite contribution due to the regulator. The minimal subtraction (MS) scheme involves only subtracting the divergent term, as the name suggests. We will use the modified minimal subtraction, or $\overline{\mathrm{MS}}$, scheme. In this scheme, one also removes the $-\gamma$ and $\ln(4\pi)$ term, by defining a new mass scale $\tilde{\mu}$ by

$$-\gamma + \ln\left(4\pi \frac{\mu^2}{m^2}\right) = \ln\left(4\pi e^{-\gamma} \frac{\mu^2}{m^2}\right) = \ln\left(\frac{\tilde{\mu}^2}{m^2}\right), \tag{A.55}$$

which leads to the expression

$$\mathcal{F}_0 = -\mu^{-2\epsilon} \frac{1}{4} \frac{m^4}{(4\pi)^2} \left(\frac{1}{\epsilon} + \frac{3}{2} + \ln\frac{\tilde{\mu}^2}{m^2}\right) + \mathcal{O}\left(\epsilon^2\right). \tag{A.56}$$

Renormalization

Now that we have applied a regulator, we are able to handle the divergence in a well-defined way. When $\epsilon \neq 0$, we can subtract terms which are proportional to ϵ^{-1}, and be left with a term that is finite in the limit $\epsilon \to 0$. Consider an arbitrary Lagrangian,

$$\mathcal{L}[\varphi] = \sum_n \lambda_n \mathcal{O}_n[\varphi]. \tag{A.57}$$

Here, $\mathcal{O}_n[\varphi]$ are operators consisting of φ and $\partial_\mu\varphi$, and λ_n are coupling constants. In d dimensions, the action integral is

$$S[\varphi] = \sum_n \int \mathrm{d}^d x \, \lambda_n \mathcal{O}_n[\varphi]. \tag{A.58}$$

The action has mass dimension 0. This means that all terms $\lambda_n\mathcal{O}_n$ must have mass dimension d, as $[\mathrm{d}^d x] = -d$. We are free to choose the coupling constant corresponding to $\mathcal{O}_0 = \partial_\mu\varphi\partial^\mu\varphi$ to be of mass dimension 0, and therefore set $\lambda_0 = 1/2$ to get canonical normalization. This allows us to deduce the dimensionality of φ. As $[\partial_\mu] = 1$, we have that $[\varphi] = (d-2)/2$. Assume $\mathcal{O}_n$ consists of k_n factors of φ, and l_n factors of $\partial_\mu\varphi$. We must then have

$$[\lambda_n] + [\mathcal{O}_n] - d = [\lambda_n] + (k_n + l_n)(d-2)/2 + l_n - d = 0, \tag{A.59}$$

$$\implies D_n := [\lambda_n] = d - k_n\frac{d-2}{2} - l_n\frac{d}{2}. \tag{A.60}$$

From this formula, we recover that $[\lambda_0] = 0$, and if $\mathcal{O}_1 = \varphi^2$, then $[\lambda_1] = 2$, which we recognize as the mass term squared. The mass dimensions of these coupling constants are independent of d. However, the coupling constant for the interaction term

$$-\frac{1}{4!}\lambda_3\varphi^4 \tag{A.61}$$

has mass dimensions $[\lambda_3] = d - 4(d-2)/2 = 4 - 2d$. Our goal now is to exchange the bare coupling constants λ_n with renormalized ones, λ_n^r, and remove the divergent terms proportional to $(d-4)^{-m}$. We can always define the renormalized coupling constants as dimensionless, i.e., $[\lambda_n^r] = 0$, by measuring them in units of a mass scale. We therefore write

$$\lambda_n = \mu^{4-D_n}\left[\lambda_n^r + \sum_{m=1}\frac{a_m(\lambda_n^r)}{(d-4)^m}\right],$$

where we have introduced the dimensionfull parameter μ to ensure that λ_n has the correct mass dimension, so that the action integral stays dimensionless. The functions a_m are then determined to each order in perturbation theory by calculating Feynman diagrams. As μ again is arbitrary, λ_4' should not depend on this parameter. In this case, we chose the same renormalization scale as we did when regulating the one-loop integral. This is only for our own convenience. This means that if we change $\mu \to \mu'$, then λ_i^r and a_m must adjust to compensate and keep λ_n constant [50].

The vacuum energy term absorbs the divergence in the one loop contribution to the free energy density. It is

$$\lambda_4\mathcal{O}_4 = \lambda_4 = m^4\lambda_4'. \tag{A.62}$$

Using the expansion in terms of the renormalized coupling, we have,

$$\lambda_4' = \mu^{-2\epsilon}\left[\lambda_4^r + \frac{1}{2\epsilon}a_1(\lambda_r^4) + ...\right], \tag{A.63}$$

where $d = 4 - 2\epsilon$. After adding Eq. (A.62) to the Lagrangian of the free scalar, the temperature independent free energy density becomes

$$\mathcal{F}_0 \sim -\mu^{-2\epsilon}\frac{1}{4}\frac{m^4}{(4\pi)^2}\left[\frac{1}{\epsilon} + \frac{3}{2} + \ln\frac{\tilde{\mu}^2}{m^2} + (4(4\pi)^2)\left(\lambda_4^r + \frac{1}{2\epsilon}a_1(\lambda_4^r)\right)\right], \quad \epsilon \to 0. \tag{A.64}$$

Thus, if we choose $a_1 = -8(4\pi)^2 + \mathcal{O}(\lambda_4^r)$, and define $\lambda_4'^r = 4(4\pi)^2\lambda_4^r$, we are able to cancel the divergence, and may take the limit $\epsilon \to 0$ safely. The free energy is now

$$\mathcal{F} = -\frac{1}{4}\frac{m^4}{(4\pi)^2}\left(\frac{3}{2} + \lambda_4'^r + \ln\frac{\tilde{\mu}^2}{m^2}\right) + \frac{T^4}{2\pi^2}\int \mathrm{d}x\, x^2 \ln\Big(1 - \exp\Big\{-\sqrt{x^2+\beta^2 m^2}\Big\}\Big). \tag{A.65}$$

Notice that all choices we have made up until now, such as defining $\lambda_4 = m^4\lambda_4'$ and using the same renormalization scale μ, has no impact on this result. Different choices would force us to define λ_4^r and a_4 differently.

A.4 Interacting scalar

We now study a scalar field with a $\lambda\varphi^4$ interaction term. We write the Lagrangian in the form

$$\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(I)}, \quad \mathcal{L}^{(0)} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - m^2\varphi^2, \quad \mathcal{L}^{(I)} = -\frac{\lambda}{4!}\varphi^4$$

$\mathcal{L}^{(I)}$ is called the interaction term, and makes it impossible to exactly solve for the partition function. Instead, we turn to perturbation theory. The canonical partition function in this theory is

$$Z = \mathrm{Tr}\Big\{e^{-\beta\hat{H}}\Big\} = \int_S \mathcal{D}\varphi\, \exp\Big\{-\int_\Omega \mathrm{d}X\, \Big(\mathcal{L}_E^{(0)} + \mathcal{L}_E^{(I)}\Big)\Big\} = \int_S \mathcal{D}\varphi\, e^{-S_0}e^{-S_I}. \tag{A.66}$$

Here, S_0 and S_I denote the Euclidean action due to the free and interacting Lagrangian, respectively. The domain of integration S is again periodic field configurations $\varphi(\beta,\vec{x}) = \varphi(0,\vec{x})$. We may write the free energy as

$$-\beta F = \ln\left[\int_S \mathcal{D}\varphi\, e^{-S_0}\sum_n \frac{1}{n!}(-S_I)^n\right] = \ln Z_0 + \ln Z_I,$$

where Z_0 is the partition function of the free theory. The correction to the partition function is thus given by

$$Z_I = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left\langle {S_I}^n\right\rangle_0, \tag{A.67}$$

where

$$\langle A\rangle_0 = \frac{\int_S \mathcal{D}\varphi\, A\, e^{-S_0}}{\int_S \mathcal{D}\varphi\, e^{-S_0}}. \tag{A.68}$$

To evaluate expectation values of the form $\langle\varphi(X_1)...\rangle_0$, we introduce the partition function with a source term

$$Z[J] = \int_S \mathcal{D}\varphi\, \exp\Big\{-\frac{1}{2}\int_\Omega \mathrm{d}X\, \varphi(-\partial_E^2 + m^2)\varphi + \int_\Omega \mathrm{d}X\, J\varphi\Big\}. \tag{A.69}$$

Thermal propagators are the generalization of the time-ordered two-point functions $\langle T\{\varphi(x)\varphi(y)\}\rangle$ of the vacuum formalism. For some differential operator D^{-1}, the thermal propagator is defined as

$$D^{-1}D(X,Y) = \beta\delta(X-Y). \tag{A.70}$$

The Fourier transformed propagator is, assuming $D(X,Y) = D(X-Y,0)$,

$$\begin{aligned}\tilde{D}(K,K') &= \frac{1}{V\beta^3}\int_\Omega \mathrm{d}X\mathrm{d}Y\, D(X,Y)\exp(-i[X\cdot K + Y\cdot K']) \\ &= \frac{1}{V\beta^3}\int_\Omega \mathrm{d}X'\mathrm{d}Y'\, D(X',0)\exp\Big(-i[X'\cdot\frac{1}{2}(K-K') + Y\cdot(K+K')]\Big) \\ &= \frac{1}{V\beta^2}\tilde{D}(K)\delta(K+K'),\end{aligned} \tag{A.71}$$

where

$$\tilde{D}(K) = \int \mathrm{d}X e^{iK\cdot X} D(X,0). \tag{A.72}$$

We write the thermal propagator of the free field as $D_0(X,Y)$. With this, we may complete the square,

$$Z[J] = Z[0]\exp\left\{\frac{1}{2}\int_\Omega \mathrm{d}X\mathrm{d}Y J(X)D_0(X,Y)J(Y)\right\} = Z[0]\exp(W[J]). \tag{A.73}$$

We can now write

$$\langle\varphi(X)\varphi(Y)\rangle_0 = \frac{1}{Z[0]}\frac{\delta}{\delta J(X)}\frac{\delta}{\delta J(Y)}Z[J]\Big|_{J=0} = D_0(X,Y). \tag{A.74}$$

This generalizes to higher order expectation values,

$$\langle\varphi(X_i)\dots\varphi(X_n)\rangle_0 = \frac{1}{Z[0]}\left(\prod_{i=1}^{n}\frac{\delta}{\delta J(X_i)}\right)Z[J]\Big|_{J=0}, \tag{A.75}$$

Using Wick's theorem, as described in section 2.1, the expectation values we are evaluating can be written

$$\begin{aligned}\langle {S_I}^m\rangle_0 &= \left(-\frac{\lambda}{4!}\right)^m\int_\Omega \mathrm{d}X_1\dots\mathrm{d}X_m\left\langle\varphi^4(X_1)\dots\varphi^4(X_m)\right\rangle_0\\ &= \left(-\frac{\lambda}{4!}\right)^m\int_\Omega \mathrm{d}X_1\dots\mathrm{d}X_m\sum_{\{a,b\}}\left\langle\varphi(X_{a(1)})\varphi(X_{b(1)})\right\rangle_0\dots\left\langle\varphi(X_{a(2m)})\varphi(X_{b(2m)})\right\rangle_0,\end{aligned}$$

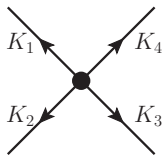
where X_i for $i > m$ is defined to equal X_j, where $j = i \mod m$. More simpliy, $X_{m+i} = X_i$. The functions a, b represents a possible pairing, as described in section 2.1. Inserting the Fourier expansions of the field gives

$$\begin{aligned}&\langle {S_I}^m\rangle_0\\ &= \left(-\frac{\lambda}{4!}\right)^m\int_\Omega \mathrm{d}X_1\dots\mathrm{d}X_m(V\beta)^2\int_{\tilde{\Omega}}\mathrm{d}K_1\dots\mathrm{d}K_{2m}\sum_{\{a,b\}}\\ &\qquad\left\langle\varphi(K_{a(1)})\varphi(K_{b(1)})\right\rangle_0\dots\left\langle\varphi(K_{a(2m)})\varphi(K_{b(2m)})\right\rangle_0\exp\Big(i\textstyle\sum_{i=1}^m X_i\cdot K_i\Big)\\ &= \left(-\frac{\lambda}{4!}\right)^m\frac{(V\beta)^{2m}\beta^m}{(V\beta^2)^{2m}}\int_{\tilde{\Omega}}\mathrm{d}K_1\dots\mathrm{d}K_{2m}\sum_{\{a,b\}}\\ &\qquad\tilde{D}(K_{a(1)})\delta(K_{a(1)}+K_{b(1)})\dots\tilde{D}(K_{a(2m)})\delta(K_{a(2m)}+K_{b(2m)})\prod_{i=1}^m\delta\left(X_i\cdot\textstyle\sum_{j=0}^3 K_{i+jm}\right)\\ &= \left(-\frac{\lambda\beta}{4!}\right)^m\prod_{i=1}^{2m}\int_{\tilde{\Omega}}\left(\mathrm{d}K_i\frac{1}{\beta}\tilde{D}(K_i)\right)\prod_{i=1}^m\delta\left(X_i\cdot\textstyle\sum_{j=0}^3 K_{i+jm}\right)\sum_{\{a,b\}}\prod_{n=1}^{2m}\delta(K_{a(k)}+K_{b(k)}).\end{aligned}$$

Here we have used that $V\beta^2\tilde{D}_0(K,P) = \tilde{D}_0(K)\delta(P+K)$, where $\tilde{D}_0(K)$ is the thermal propagator for the free field. In this case, it is

$$\tilde{D}_0(K) = \tilde{D}_0(\omega_n,\vec{k}) = \frac{1}{\omega_k^2+\omega_n^2}. \tag{A.76}$$

This expectation value can be represented graphically using Feynman diagrams. The thermal $\lambda\varphi^2$-theory gets the prescription

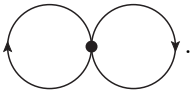


$$= -\lambda\beta\delta\left(\textstyle\sum_i K_i\right), \tag{A.77}$$

$$= \frac{1}{\beta}D_0(K). \tag{A.78}$$

Lastly, one has to integrate over internal momenta and divide by the symmetry factor of the diagram s, which is described in detail in [23].

Calculating $\langle {S_I}^n \rangle_0$ boils down to the sum of all possible Feynman diagrams with n vertices. The first example is

$$\langle S_I \rangle_0 = \frac{1}{8}$$  . (A.79)

In section 2.1, we saw that the sum of all vacuum diagrams is the exponential of the sum of all *connected* diagrams, so the free energy of the interacting theory is given by

$$-\beta F = \ln Z_0 + \Sigma(\text{all connected diagrams}). \tag{A.80}$$

A.5 Fermions

The anti-periodic nature of fermion-fields, as mentioned in section A.2, can be determined by studying the properties of the thermal Greens function. The thermal Greens function may be written

$$D(X_1, X_2) = D(\vec{x}, \vec{y}, \tau_1, \tau_2) = \langle \Omega | \, e^{-\beta \hat{H}} \mathrm{T} \left\{ \hat{\varphi}(X_1) \hat{\varphi}(X_2) \right\} | \Omega \rangle .$$

$\mathrm{T}\{...\}$ is time-ordering operator, defined as

$$\mathrm{T}\left\{ \varphi(\tau_1)\varphi(\tau_2) \right\} = \theta(\tau_1 - \tau_2)\varphi(\tau_1)\varphi(\tau_2) + \nu\theta(\tau_2 - \tau_1)\varphi(\tau_2)\varphi(\tau_1),$$

where $\nu = \pm 1$ for bosons and fermions respectively, and $\theta(\tau)$ is the Heaviside step function. In the same way that $i\hat{H}$ generates the time translation of a quantum field operator through $\hat{\varphi}(x) = \hat{\varphi}(t, \vec{x}) = e^{it\hat{H}} \hat{\varphi}(0, \vec{x}) e^{-it\hat{H}}$, the imaginary-time formalism implies the relation

$$\hat{\varphi}(X) = \hat{\varphi}(\tau, \vec{x}) = e^{\tau \hat{H}} \hat{\varphi}(0, \vec{x}) e^{-\tau \hat{H}}. \tag{A.81}$$

Using $\mathbb{1} = e^{\tau \hat{H}} e^{-\tau \hat{H}}$ and the cyclic property of the trace, we show that, assuming $\beta > \tau > 0$,

$$\begin{aligned}
D(\vec{x}, \vec{y}, \tau, 0) &= \langle \Omega | \, e^{-\beta \hat{H}} \mathrm{T} \left\{ \varphi(\tau, \vec{x}) \varphi(0, \vec{y}) \right\} | \Omega \rangle \\
&= \frac{1}{Z} \mathrm{Tr} \Big\{ e^{-\beta \hat{H}} \varphi(\tau, \vec{x}) \varphi(0, \vec{y}) \Big\} \\
&= \frac{1}{Z} \mathrm{Tr} \Big\{ \varphi(0, \vec{y}) e^{-\beta \hat{H}} \varphi(\tau, \vec{x}) \Big\} \\
&= \frac{1}{Z} \mathrm{Tr} \Big\{ e^{-\beta \hat{H}} e^{\beta \hat{H}} \varphi(0, \vec{y}) e^{-\beta \hat{H}} \varphi(\tau, \vec{x}) \Big\} \\
&= \frac{1}{Z} \mathrm{Tr} \Big\{ e^{-\beta \hat{H}} \varphi(\vec{y}, \beta) \varphi(\tau, \vec{x}) \Big\} \\
&= \nu \langle \Omega | \, e^{-\beta \hat{H}} \mathrm{T} \left\{ \varphi(\tau, \vec{x}) \varphi(\beta, \vec{y}) \right\} | \Omega \rangle .
\end{aligned}$$

This implies that $\varphi(0, x) = \nu\varphi(\beta, \varphi)$, which shows that bosons are periodic in time, as stated earlier, while fermions are anti-periodic.

The Lagrangian density of a free fermion is

$$\mathcal{L} = \bar{\psi} \left(i\partial\!\!\!/ - m \right) \psi. \tag{A.82}$$

This Lagrangian is invariant under the transformation $\psi \to e^{-i\alpha}\psi$, which by Nöther's theorem results in a conserved current

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \delta\psi = \bar{\psi} \gamma^\mu \psi. \tag{A.83}$$

The corresponding conserved charge is

$$Q = \int_V \mathrm{d}^3x\, j^0 = \int_V \mathrm{d}^3x\, \bar{\psi}\gamma^0\psi. \tag{A.84}$$

We can now use our earlier result for the thermal partition function, Eq. (A.10), only with the substitution $\mathcal{H} \to \mathcal{H} - \mu\bar{\psi}\gamma^0\psi$, and integrate over anti-periodic ψ's:

$$Z = \mathrm{Tr}\Big\{e^{-\beta(\hat{H}-\mu\hat{Q})}\Big\} = \prod_{ab}\int \mathcal{D}\psi_a\mathcal{D}\pi_b \exp\Big\{\int_\Omega \mathrm{d}X\, \Big(i\dot{\psi}\pi - \mathcal{H}(\psi,\pi) + \mu\bar{\psi}\gamma^0\psi\Big)\Big\},$$

where a, b are the spinor indices. The canonical momentum corresponding to ψ is

$$\pi = \frac{\partial\mathcal{L}}{\partial(\partial_0\psi)} = i\bar{\psi}\gamma^0, \tag{A.85}$$

and the Hamiltonian density is

$$\mathcal{H} = \pi\dot{\psi} - \mathcal{L} = \bar{\psi}(-i\gamma^i\partial_i + m)\psi \tag{A.86}$$

which gives

$$\mathcal{L}_E = -i\pi\dot{\psi} + \mathcal{H}(\psi,\pi) - \mu\bar{\psi}\gamma^0\psi = \bar{\psi}[\gamma^0(\partial_\tau - \mu) - i\gamma^i\partial_i + m]\psi, \tag{A.87}$$

By using the Grassman-version of the Gaussian integral formula, the partition function can be written

$$\begin{aligned} Z &= \prod_{ab}\int \mathcal{D}\psi_a\mathcal{D}\bar{\psi}_b \exp\Big\{-\int_\Omega \mathrm{d}X\, \bar{\psi}\left[\tilde{\gamma}_0(\partial_\tau - \mu) - i\gamma^i\partial_i + m\right]\psi\Big\} \\ &= C\prod_{ab}\int \mathcal{D}\tilde{\psi}_a\mathcal{D}\tilde{\bar{\psi}}_b \exp\Big\{-\int_{\tilde{\Omega}} \mathrm{d}K\, \tilde{\bar{\psi}}\left[i\tilde{\gamma}_0(\omega_n + i\mu) + i\gamma_i p_i + m\right]\tilde{\psi}\Big\} \\ &= C\prod_{ab}\int \mathcal{D}\tilde{\psi}_a\mathcal{D}\tilde{\bar{\psi}}_b e^{-\langle\tilde{\bar{\psi}}, D_0^{-1}\tilde{\psi}\rangle} = \det\big(D_0^{-1}\big). \end{aligned}$$

In the second line, we have inserted the Fourier expansion of the field, as defined in section A.2, and changed variable of integration, as we did for the scalar field. The linear operator in this case is

$$D_0^{-1} = i\gamma^0(-i\partial_\tau + i\mu) - (-i\gamma^i)\partial_i + m = \beta[i\tilde{\gamma}_a p_a + m]. \tag{A.88}$$

This equality must be understood as an equality between linear operators, which are represented in different bases. We introduced the notation $p_a = (\omega_n + i\mu, p_i)$ and use the Euclidean gamma matrices, as defined in Appendix B. We use the fact that

$$\det(i\tilde{\gamma}_a p_a + m) = \det\big(\gamma^5\gamma^5\big)\det(i\tilde{\gamma}_a p_a + m) = \det\big[\gamma^5(i\tilde{\gamma}_a p_a + m)\gamma^5\big] = \det(-i\tilde{\gamma}_a p_a + m),$$

Let $\tilde{D} = -i\tilde{\gamma}_a p_a + m$, which means we can write

$$Z = \sqrt{\det(D)\det\Big(\tilde{D}\Big)} = \sqrt{\det\Big(D\tilde{D}\Big)} = \det\big[\mathbb{1}(p_a p_a + m^2)\big]^{1/2}, \tag{A.89}$$

where we have used the anti-commutation rule for the Euclidean gamma-matrices, $\{\gamma_a, \gamma_b\} = 2\delta_{ab}$. It is important to keep in mind that the determinant here refers to linear operators on the space of spinor functions. Thus

$$\begin{aligned} \ln(Z) &= \ln\Big\{\det\big[\mathbb{1}(p_a p_a + m^2)\big]^{1/2}\Big\} = \frac{1}{2}\,\mathrm{Tr}\big\{\ln[\mathbb{1}(p_a p_a + m^2)]\big\} \\ &= \frac{1}{2}\int_{\tilde{\Omega}} \mathrm{d}K\, \ln[\mathbb{1}\beta^2(p_a p_a + m^2)]_{ii} \end{aligned} \tag{A.90}$$

As the matrix within the logarithm is diagonal, the matrix-part of the trace is trivial, and the free energy may be written

$$\beta\mathcal{F} = -2\int_{\tilde{\Omega}} \mathrm{d}X\, \ln\{\beta^2[(\omega_n + i\mu)^2 + \omega_k^2]\}. \tag{A.91}$$

Using the fermionic version of the thermal sum from section A.3 gives the answer

$$\mathcal{F} = 2\int \frac{\mathrm{d}^3p}{(2\pi)^3}\left[\beta\omega_k + \frac{1}{\beta}\ln\Big(1 + e^{-\beta(\omega_k - \mu)}\Big) + \frac{1}{\beta}\ln\Big(1 + e^{-\beta(\omega_k + \mu)}\Big)\right]. \tag{A.92}$$

We see again that the temperature-independent part of the integral diverges, and must be regulated. There are two temperature-dependent terms, one from the particle and one from the anti-particle.

Appendix B

Algebra bases

Pauli matrices

The $\mathfrak{su}(2)$ basis used is the Pauli matrices,

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{B.1}$$

They obey

$$[\tau_a, \tau_b] = 2i\varepsilon_{abc}\tau_c, \tag{B.2}$$
$$\{\tau_a, \tau_b\} = 2\delta_{ab}\mathbb{1}, \tag{B.3}$$
$$\mathrm{Tr}\{\tau_a\} = 0, \tag{B.4}$$
$$\mathrm{Tr}\{\tau_a\tau_b\} = 2\delta_{ab}. \tag{B.5}$$

Together with the identity matrix $\mathbb{1}$, the Pauli matrices form a basis for the vector space of all 2-by-2 matrices. An arbitrary 2-by-2 matrix M may be written

$$M = M_0\mathbb{1} + M_a\tau_a, \quad M_0 = \frac{1}{2}\mathrm{Tr}\{M\},\ M_a = \frac{1}{2}\mathrm{Tr}\{\tau_a M\}. \tag{B.6}$$

Gamma matrices

The gamma matrices γ^μ, $\mu \in \{0, 1, 2, 3\}$, obey

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}, \tag{B.7}$$
$$\gamma^{0\dagger} = \gamma^0, \quad \gamma^{i\dagger} = -\gamma^i. \tag{B.8}$$

These matrices, together with

$$\sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\mu], \tag{B.9}$$
$$\gamma^\mu_A = \gamma^\mu\gamma^5, \tag{B.10}$$
$$\gamma^5 = \frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma, \tag{B.11}$$

form the Clifford algebra $\mathrm{Cl}_{1,3}$, also known as the *space-time algebra*. The subscripts $(1, 3)$ denotes the signature of the metric. The "fifth γ-matrix", which can be expressed as $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$, obey

$$\left\{\gamma^5, \gamma^\mu\right\} = 0, \quad (\gamma^5)^2 = \mathbb{1}. \tag{B.12}$$

The Euclidian counterpart of the space-time algebra, Cl_4, is defined by the "Euclidian gamma matrices", which obey

$$\{\tilde{\gamma}_a, \tilde{\gamma}_b\} = 2\delta_{ab}\mathbb{1}. \tag{B.13}$$

These can be related to the regular Minkowski-matrices by

$$\tilde{\gamma}_0 = \gamma^0, \quad \tilde{\gamma}_j = -i\gamma^j. \tag{B.14}$$

These then obey

$$\tilde{\gamma}_a^\dagger = \tilde{\gamma}_a. \tag{B.15}$$

The Euclidean $\tilde{\gamma}_5$ is defined as

$$\tilde{\gamma}_5 = \gamma_0\gamma_1\gamma_2\gamma_3 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5. \tag{B.16}$$

It thus also anti-commutes with the Euclidean γ-matrices,

$$\{\tilde{\gamma}_5, \tilde{\gamma}_a\} = 0. \tag{B.17}$$

Appendix C

Functional derivatives

Functional derivatives generalize the notion of gradients and directional derivatives. A function $f(x)$ has a gradient

$$\mathrm{d}f_{x_0} = \frac{\partial f}{\partial x_i}\Big|_{x_0} \mathrm{d}x_i. \tag{C.1}$$

The derivative in a particular direction $v = v^i \partial_i$ is

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon} f(x_i + \epsilon v_i)\Big|_{\epsilon=0} = f(x) + \mathrm{d}f_x(v) = f(x) + \frac{\partial f}{\partial x^i} v_i. \tag{C.2}$$

This is generalized to functionals through the definition of the functional derivative and the variation of a functional. Let $F[f]$ be a functional, i.e., a machine that takes in a function and returns a number or a function. The obvious example in our case is the action, which takes in one or more field configurations, and returns a single real number. We will assume here that the functions have the domain Ω, with coordinates x. The functional derivative is defined as

$$\delta F[f] = \frac{\mathrm{d}}{\mathrm{d}\epsilon} F[f + \epsilon\eta]\Big|_{\epsilon=0} = \int_\Omega \mathrm{d}x \, \frac{\delta F[f]}{\delta f(x)} \eta(x). \tag{C.3}$$

$\eta(x)$ is here an arbitrary function, but we will make the assumption that it as well as all its derivatives are zero at the boundary of its domain Ω, i.e., $\eta(\partial\Omega) = 0$. This allows us to discard surface terms stemming from partial integration, which we will use frequently. We may use the definition to derive one of the fundamental relations of functional derivation. Take the functional $F[f] = f(x)$. Then,

$$\delta F[f] = \frac{\mathrm{d}}{\mathrm{d}\epsilon}[f(x) + \epsilon\eta(x)]\Big|_{\epsilon=0} = \eta(x) = \int \mathrm{d}y \, \delta(x-y)\eta(y) \tag{C.4}$$

This leads to the identity

$$\frac{\delta f(x)}{\delta f(y)} = \delta(x-y), \tag{C.5}$$

for any function f. Higher functional derivatives are defined similarly, by applying functional variation repeatedly

$$\delta^n F[f] = \frac{\mathrm{d}}{\mathrm{d}\epsilon} \delta^{n-1} F[f + \epsilon\eta_n]\big|_{\epsilon=0} = \int \left(\prod_{i=1}^n \mathrm{d}x_i\right) \frac{\delta^n F[f]}{\delta f(x_n)\ldots\delta f(x_1)} \left(\prod_{i=1}^n \eta_i(x_i)\right). \tag{C.6}$$

If we can write the functional $F[f]$ in terms of a new variable, $g(y)$, then the chain rule for functional derivatives is

$$\frac{\delta F[f]}{\delta f(x)} = \int \mathrm{d}y \, \frac{\delta F[f]}{\delta g(y)} \frac{\delta g(y)}{\delta f(x)}. \tag{C.7}$$

A functional may be expanded in a generalization of the Taylor series,

$$F[f_0 + f] = F[f_0] + \int_\Omega \mathrm{d}x \, f(x) \frac{\delta F[f_0]}{\delta f(x)} + \frac{1}{2!} \int_\Omega \mathrm{d}x\mathrm{d}y \, f(x) f(y) \frac{\delta^2 F[f_0]}{\delta f(x)\delta f(y)} + \ldots \tag{C.8}$$

Her, the notation

$$\frac{\delta F[f_0]}{\delta f(x)} \tag{C.9}$$

indicate that the functional $F[f]$ is first differentiated with respect to f, the evaluated at $f = f_0$. As an example, the Klein-Gorodn action

$$S[\varphi] = -\frac{1}{2}\int_\Omega \mathrm{d}x\, \varphi(x)(\partial^2 + m^2)\varphi(x). \tag{C.10}$$

Using Eq. (C.4) and partial integration,

$$\begin{aligned}\frac{\delta}{\delta\varphi(x)}S[\varphi] &= -\frac{1}{2}\int_\Omega \mathrm{d}y\, [\delta(x-y)(\partial_y^2 + m^2)\varphi(y) + \varphi(y)(\partial_y^2 + m^2)\delta(x-y)] \\ &= -\int_\Omega \mathrm{d}y\, \delta(x-y)(\partial_y^2 + m^2)\varphi(y) = (\partial_x^2 + m^2)\varphi(x)\end{aligned} \tag{C.11}$$

The second derivative is

$$\frac{\delta^2 S[\varphi]}{\delta\varphi(x)\delta\varphi(y)} = \frac{\delta}{\delta\varphi(x)}(\partial_y^2 + m^2)\varphi(y) = (\partial_y^2 + m^2)\delta(x-y). \tag{C.12}$$

Gaussian integrals

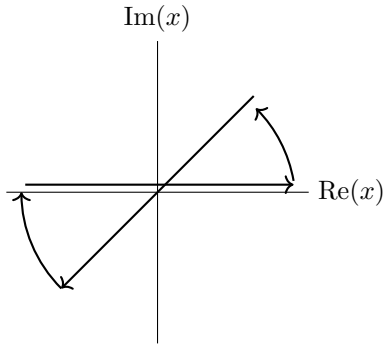


Figure C.1: Wick rotation

A useful integral is the Gaussian integral,

$$\int_{\mathbb{R}} \mathrm{d}x\, \exp\left(-\frac{1}{2}ax^2\right) = \sqrt{\frac{2\pi}{a}}, \tag{C.13}$$

for $a \in \mathbb{R}$. The imaginary version,

$$\int_R \mathrm{d}x\, \exp\left(i\frac{1}{2}ax^2\right) \tag{C.14}$$

does not converge. However, if we change $a \to a + i\epsilon$, the integrand is exponentially suppressed.

$$f(x) = \exp\left(i\frac{1}{2}ax^2\right) \to \exp\left(i\frac{1}{2}ax^2 - \frac{1}{2}\epsilon x^2\right). \tag{C.15}$$

As the integrand falls exponentially for $x \to \infty$ and contains no poles in the upper right nor lower left quarter of the complex plane, we may perform a wick rotation by closing the contour as shown in Figure C.1. This gives the result

$$\int_{\mathbb{R}} \mathrm{d}x\, \exp\left(i\frac{1}{2}(a+i\epsilon)x^2\right) = \int_{\sqrt{i}\mathbb{R}} \mathrm{d}x\, \exp\left(i\frac{1}{2}ax^2\right) = \sqrt{i}\int_{\mathbb{R}} \mathrm{d}y\, \exp\left(-\frac{1}{2}(-a)y^2\right) = \sqrt{\frac{2\pi i}{(-a)}} \tag{C.16}$$

where we have made the change of variable $y = (1+i)/\sqrt{2}x = \sqrt{i}x$. In n dimensions, the Gaussian integral formula generalizes to

$$\int_{\mathbb{R}^n} \mathrm{d}^n x \, \exp\left\{-\frac{1}{2}x_n A_{nm} x_m\right\} = \sqrt{\frac{(2\pi)^n}{\det(A)}}, \tag{C.17}$$

where A is a matrix with n real, positive eigenvalues. We may also generalize Eq. (C.16),

$$\int_{\mathbb{R}^n} \mathrm{d}^n x \, \exp\left\{i\frac{1}{2}x_n (A_{nm} + i\epsilon\delta_{nm}) x_m\right\} = \sqrt{\frac{(2\pi i)^n}{\det(-A)}}. \tag{C.18}$$

The final generalization is to functional integrals,

$$\int \mathcal{D}\varphi \, \exp\left(-\frac{1}{2}\int \mathrm{d}x \, \varphi(x) A \varphi(x)\right) = C(\det(A))^{-1/2}, \int \mathcal{D}\varphi \, \exp\left(i\frac{1}{2}\int \mathrm{d}x \, \varphi(x) A \varphi(x)\right) = C(\det(-A))^{-1/2}. \tag{C.19}$$

C is here a divergent constant, but will either fall away as we are only looking at the logarithm of I_∞ and are able to throw away additive constants, or ratios between quantities which are both multiplied by C.

The Gaussian integral can be used for the stationary phase approximation. In one dimension, it is

$$\int \mathrm{d}x \, \exp\{i\alpha f(x)\} \approx \sqrt{\frac{2\pi}{f''(x_0)}} \exp\{f(x_0)\}, \, \alpha \to \infty, \tag{C.20}$$

where the point x_0 is defined by $f'(x_0) = 0$. The functional generalization of this is

$$\int \mathcal{D}\varphi \exp\{iS[\varphi]\} \approx C \det\left(-\frac{\delta^2 S[\varphi_0]}{\delta\varphi^2}\right) \exp\{i\alpha S[\varphi_0]\}, \tag{C.21}$$

Here, $S[\varphi]$ is a general functional of φ, we have used the Taylor expansion, and φ_0 obeys

$$\frac{\delta}{\delta\varphi(x)} S[\varphi_0] = 0. \tag{C.22}$$

Appendix D

Rewriting terms

This section shows how to rewrite terms in the Lagrangian of Chiral perturbation theory. These techniques and more are used to reduce the total number of terms and to change between different conventions.

Redundant terms and the equations of motion

Changing the field parametrization that appears in the Lagrangian does not affect any of the physics, as it corresponds to a change of variables in the path integral [32, 51, 52]. However, a change of variables can result in new terms in the Lagrangian. As a result of this, terms that appear independent on their face may be redundant. These terms can be eliminated by using the classical equations of motion. In this section, we show first the derivation of the equations of motion, then use this result to identify redundant terms which need not be included in the most general Lagrangian.

We derive the equations of motion for the leading order Lagrangian using the principle of least action. Choosing the parametrization $\Sigma = \exp(i\pi_a\tau_a)$, a variation $\pi_a \to \pi_a + \delta\pi_a$ results in a variation in Σ, $\delta\Sigma = i\tau_a\delta\pi_a\Sigma$. The variation of the leading order action,

$$S_2 = \int_\Omega \mathrm{d}^4x\, \mathcal{L}_2, \tag{D.1}$$

when varying π_a is

$$\delta S = \int_\Omega \mathrm{d}x\, \frac{f^2}{4}\, \mathrm{Tr}\big\{(\nabla_\mu\delta\Sigma)(\nabla^\mu\Sigma)^\dagger + (\nabla_\mu\Sigma)(\nabla^\mu\delta\Sigma)^\dagger + \chi\delta\Sigma^\dagger + \delta\Sigma\chi^\dagger\big\}.$$

Using the properties of the covariant derivative to do partial integration, as shown in section 3.2, as well as $\delta(\Sigma\Sigma^\dagger) = (\delta\Sigma)\Sigma^\dagger + \Sigma(\delta\Sigma)^\dagger = 0$, the variation of the action can be written

$$\begin{aligned}
\delta S &= \frac{f^2}{4}\int_\Omega \mathrm{d}x\ \mathrm{Tr}\big\{-\delta\Sigma\nabla^2\Sigma^\dagger + (\nabla^2\Sigma)(\Sigma^\dagger\delta\Sigma\Sigma^\dagger) - \chi(\Sigma^\dagger\delta\Sigma\Sigma^\dagger) + \delta\Sigma\chi^\dagger\big\} \\
&= \frac{f^2}{4}\int_\Omega \mathrm{d}x\ \mathrm{Tr}\big\{\delta\Sigma\Sigma^\dagger\left[(\nabla^2\Sigma)\Sigma^\dagger - \Sigma\nabla^2\Sigma^\dagger - \chi\Sigma^\dagger + \Sigma\chi^\dagger\right]\big\} \\
&= i\frac{f^2}{4}\int_\Omega \mathrm{d}x\ \mathrm{Tr}\big\{\tau_a\left[(\nabla^2\Sigma)\Sigma^\dagger - \Sigma\nabla^2\Sigma^\dagger - \chi\Sigma^\dagger + \Sigma\chi^\dagger\right]\big\}\delta\pi_a = 0.
\end{aligned}$$

As the variation is arbitrary, the equations of motion to leading order is

$$\mathrm{Tr}\big\{\tau_a\left[(\nabla^2\Sigma)\Sigma^\dagger - \Sigma\nabla^2\Sigma^\dagger - \chi\Sigma^\dagger + \Sigma\chi^\dagger\right]\big\} = 0. \tag{D.2}$$

We define

$$\mathcal{O}^{(2)}_{\mathrm{EOM}} = (\nabla^2\Sigma)\Sigma^\dagger - \Sigma\nabla^2\Sigma^\dagger - \chi\Sigma^\dagger + \Sigma\chi^\dagger. \tag{D.3}$$

The next step in eliminating redundant terms is to change the parametrization of Σ by $\Sigma(x) \to \Sigma'(x)$. Here, $\Sigma(x) = e^{iS(x)}\Sigma'(x)$, $S(x) \in \mathfrak{su}(2)$. This change leads to a new Lagrange density, $\mathcal{L}[\Sigma] = \mathcal{L}[\Sigma'] + \Delta\mathcal{L}[\Sigma']$. We are free to choose $S(x)$, as long Σ' still obey the required transformation properties. Any terms in the Lagrangian $\Delta\mathcal{L}$ due to a reparametrization can be neglected, as argued earlier. When demanding that Σ' obey the same symmetries as Σ, the most general transformation to second order in Weinberg's power counting scheme is [32]

$$S_2 = i\alpha_2\left[(\nabla^2\Sigma')\Sigma^\dagger - \Sigma'(\nabla^2\Sigma')^\dagger\right] + i\alpha_2\left[\chi\Sigma'^\dagger - \Sigma'\chi^\dagger - \frac{1}{2}\operatorname{Tr}\{\chi\Sigma'^\dagger - \Sigma'\chi^\dagger\}\right]. \tag{D.4}$$

α_1 and α_2 are arbitrary real numbers. As Eq. (D.4) is to second order, $\Delta\mathcal{L}$ is fourth order in Weinberg's power counting scheme. Inserting this gives

$$\begin{aligned}\mathcal{L}_2\left[e^{iS_2}\Sigma'\right] &= \frac{f^2}{4}\operatorname{Tr}\{[\nabla_\mu(1+iS_2)\Sigma'][\nabla^\mu\Sigma'^\dagger(1-iS_2)]\} + \frac{f^2}{4}\operatorname{Tr}\{\chi\Sigma'^\dagger(1-iS_2) + (1+iS_2)\Sigma'\chi^\dagger\}\\ &= \mathcal{L}[\Sigma'] + i\frac{f^2}{4}\operatorname{Tr}\{[\nabla_\mu(S_2\Sigma')][\nabla^\mu\Sigma']^\dagger - [\nabla_\mu\Sigma'][\nabla^\mu(\Sigma'^\dagger S_2)]\} - i\frac{f^2}{4}\operatorname{Tr}\{\chi\Sigma'^\dagger S_2 - S_2\Sigma'\chi^\dagger\}\end{aligned}$$

Using the properties of the covariant derivative, we may use the product rule and partial integration to write the difference between the two Lagrangians to fourth-order as

$$\begin{aligned}\Delta\mathcal{L}[\Sigma'] &= i\frac{f^2}{4}\operatorname{Tr}\{(\nabla_\mu S_2)(\Sigma'\nabla^\mu\Sigma'^\dagger - (\nabla^\mu\Sigma')\Sigma'^\dagger)\} - i\frac{f^2}{4}\operatorname{Tr}\{\chi\Sigma'^\dagger S_2 - S_2\Sigma'\chi^\dagger\}\\ &= \frac{f^2}{4}\operatorname{Tr}\Big\{iS_2\mathcal{O}^{(2)}_{\mathrm{EOM}}\Big\}.\end{aligned} \tag{D.5}$$

Any term that can be written in the form of Eq. (D.5) for arbitrary $\alpha_1, \alpha_2 \in \mathbb{R}$ is redundant, as we argued earlier, and may therefore be discarded. $\Delta\mathcal{L}_2$ is of fourth order, and it can thus be used to remove terms from $\mathcal{L}_4$ or higher order.

Rewriting NLO Lagrangian

The NLO Lagrangian used in this text is given in Eq. (3.121), and is

$$\begin{aligned}\mathcal{L}_4 = {}& \frac{l_1}{4}\operatorname{Tr}\{\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\}^2 + \frac{l_2}{4}\operatorname{Tr}\{\nabla_\mu\Sigma(\nabla_\nu\Sigma)^\dagger\}\operatorname{Tr}\{\nabla^\mu\Sigma(\nabla^\nu\Sigma)^\dagger\} + \frac{l_3+l_4}{16}\operatorname{Tr}\{\chi\Sigma^\dagger + \Sigma\chi^\dagger\}^2\\ &+ \frac{l_4}{8}\operatorname{Tr}\{\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\}\operatorname{Tr}\{\chi\Sigma^\dagger + \Sigma\chi^\dagger\} - \frac{l_7}{16}\operatorname{Tr}\{\chi\Sigma^\dagger - \Sigma\chi^\dagger\}^2 + \frac{h_1+h_3-l_4}{4}\operatorname{Tr}\{\chi\chi^\dagger\}\\ &+ \frac{h_1-h_3-l_4}{16}\left[\operatorname{Tr}\{\chi\Sigma^\dagger + \Sigma\chi^\dagger\}^2 + \operatorname{Tr}\{\chi\Sigma^\dagger - \Sigma\chi^\dagger\}^2 - 2\operatorname{Tr}\Big\{\big(\chi\Sigma^\dagger\big)^2 + \big(\Sigma\chi^\dagger\big)^2\Big\}\right].\end{aligned} \tag{D.6}$$

We can rewrite it to match the one used in [35, 38], starting with

$$\begin{aligned}&\frac{h_1-h_3-l_4}{16}\left(\operatorname{Tr}\{\chi\Sigma^\dagger - \Sigma\chi^\dagger\}^2 - 2\operatorname{Tr}\{(\chi\Sigma^\dagger)^2 + (\Sigma\chi^\dagger)^2\}\right)\\ &= \frac{h_1-h_3-l_4}{16}\left(\operatorname{Tr}\{\chi\Sigma^\dagger\}^2 - 2\operatorname{Tr}\{\chi\Sigma^\dagger\}\operatorname{Tr}\{\Sigma\chi^\dagger\} + \operatorname{Tr}\{\Sigma\chi^\dagger\}^2 - 2\operatorname{Tr}\{(\chi\Sigma^\dagger)^2\} - 2\operatorname{Tr}\{(\Sigma\chi^\dagger)^2\}\right).\end{aligned}$$

Using $\operatorname{Tr}\{A^2\} = \operatorname{Tr}\{A\}^2 - \det(A)\operatorname{Tr}\{\mathbb{1}\}$, we get

$$\begin{aligned}&= -\frac{h_1-h_3-l_4}{16}\left(\operatorname{Tr}\{\chi\Sigma^\dagger\}^2 + 2\operatorname{Tr}\{\chi\Sigma^\dagger\}\operatorname{Tr}\{\Sigma\chi^\dagger\} + \operatorname{Tr}\{\Sigma\chi^\dagger\}^2 - 4\det\big(\chi\Sigma^\dagger\big) - 4\det\big(\Sigma\chi^\dagger\big)\right)\\ &= -\frac{h_1-h_3-l_4}{16}\left(\operatorname{Tr}\{\chi\Sigma^\dagger + \Sigma\chi^\dagger\}^2 - 4\det\big(\chi\Sigma^\dagger\big) - 4\det\big(\Sigma\chi^\dagger\big)\right).\end{aligned}$$

Furthermore, as $\det(\Sigma) = 1$,

$$\begin{aligned}\mathcal{L}_4 = {}& \frac{l_1}{4}\operatorname{Tr}\{\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\}^2 + \frac{l_2}{4}\operatorname{Tr}\{\nabla_\mu\Sigma(\nabla_\nu\Sigma)^\dagger\}\operatorname{Tr}\{\nabla^\mu\Sigma(\nabla^\nu\Sigma)^\dagger\} + \frac{l_3+l_4}{16}\operatorname{Tr}\{\chi\Sigma^\dagger + \Sigma\chi^\dagger\}^2\\ &+ \frac{l_4}{8}\operatorname{Tr}\{\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\}\operatorname{Tr}\{\chi\Sigma^\dagger + \Sigma\chi^\dagger\} - \frac{l_7}{16}\operatorname{Tr}\{\chi\Sigma^\dagger - \Sigma\chi^\dagger\}^2 + \frac{h_1+h_3-l_4}{4}\operatorname{Tr}\{\chi\chi^\dagger\}\\ &+ \frac{h_1-h_3-l_4}{4}(\det\{\chi\} + \det\{\chi^\dagger\}).\end{aligned} \tag{D.7}$$

For real χ, we have $\mathrm{Tr}\{\chi\chi^\dagger\} = \det(\chi) + \det(\chi^\dagger)$, and we can define $h_1' = h_1 - l_4$ to get

$$\begin{aligned}\mathcal{L}_4 = &\frac{l_1}{4}\,\mathrm{Tr}\{\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\}^2 + \frac{l_2}{4}\,\mathrm{Tr}\{\nabla_\mu\Sigma(\nabla_\nu\Sigma)^\dagger\}\,\mathrm{Tr}\{\nabla^\mu\Sigma(\nabla^\nu\Sigma)^\dagger\} + \frac{l_3+l_4}{16}\,\mathrm{Tr}\{\chi\Sigma^\dagger + \Sigma\chi^\dagger\}^2 \\ &+ \frac{l_4}{8}\,\mathrm{Tr}\{\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\}\,\mathrm{Tr}\{\chi\Sigma^\dagger + \Sigma\chi^\dagger\} - \frac{l_7}{16}\,\mathrm{Tr}\{\chi\Sigma^\dagger - \Sigma\chi^\dagger\}^2 + \frac{h_1'}{2}\,\mathrm{Tr}\{\chi\chi^\dagger\}. \qquad \text{(D.8)}\end{aligned}$$

If one assumes $\Delta m = 0$, i.e., what is called the chiral limit, then the term l_7 falls away, as $\chi = \chi^\dagger$.

Appendix E

Computer code

All computer code used in this specialization project is available in an online repository, located at `https://github.com/martkjoh/prosjektoppgave`. The code for symbolic calculation is in the folder `https://github.com/martkjoh/prosjektoppgave/tree/main/scripts`. These are Jupyter-notebooks, run with a SageMath kernel. SageMath is an open-source Python-based computer algebra system (case), which is available for download at this webpage: `https://www.sagemath.org`. The code for numerical calculation and to generate plots, as well as the generated numerical results, are available at `https://github.com/martkjoh/prosjektoppgave/tree/main/scripts/numerikk`. The code is written in Python and uses the scientific libraries NumPy, SymPy, SciPy, and Matplotlib.

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