Chiral Perturbation Theory

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Contents

Chapter 1

Theory

In this section we will survey some general properties of quantum field theory that is necessary for chiral perturbation theory. First, we will introduce the path integral and the 1-particle irreducible effective action, as well as the effective action. We will derive Goldstone's theorem and effective field theories, which are the basis for χPT .

1.1 QFT via path integrals

The theory in this section is based on [1, 2, 3, 4]. Feynman diagrams are drawn using JaxoDraw [5].

The vacuum transition amplitude is given by the path integral

$$Z = \lim_{T\to\infty} \langle \Omega, T/2 | -T/2, \Omega \rangle = \lim_{T\to\infty} \langle \Omega | e^{-iHT} | \Omega \rangle = \int \mathcal{D}\pi\mathcal{D}\varphi \, \exp\left\{ i \int \mathrm{d}^4x \, \left(\pi\dot{\varphi} - \mathcal{H}[\pi, \varphi] \right) \right\}, \tag{1.1}$$

where $|\Omega\rangle$ is the vacuum of the theory. By introducing a source term into the Hamiltonian density, $\mathcal{H} \to \mathcal{H} - J(x)\varphi(x)$, we get the generating functional

$$Z[J] = \int \mathcal{D}\pi\mathcal{D}\varphi \, \exp\left\{ i \int \mathrm{d}^4x \, \left(\pi\dot{\varphi} - \mathcal{H}[\pi, \varphi] + J\varphi \right) \right\}. \tag{1.2}$$

If $\mathcal{H}$ is quadratic in π, we can complete the square and integrate out π to obtain

$$Z[J] = C \int \mathcal{D}\varphi \, \exp\left\{ i \int \mathrm{d}^4x \, \left(\mathcal{L}[\varphi] + J\varphi \right) \right\}. \tag{1.3}$$

C is infinite, but constant, and will drop out of physical quantities. In scattering theory, the main objects of study are correlation functions $\langle \varphi(x_1)\varphi(x_2)... \rangle = \langle \Omega | T \{ \varphi(x_1)\varphi(x_2) \dots \} | \Omega \rangle$, where T is the time ordering operator. These are given by functional derivatives of $Z[J]$,

$$\langle \varphi(x_1)\varphi(x_2)... \rangle = \frac{\int \mathcal{D}\varphi(x) \, (\varphi(x_1)\varphi(x_2)...) e^{iS[\varphi]}}{\int \mathcal{D}\varphi(x) \, e^{iS[\varphi]}} = \frac{1}{Z[0]} \prod_i \left(-i \frac{\delta}{\delta J(x_i)} \right) Z[J] \Big|_{J=0}, \tag{1.4}$$

where

$$S[\varphi] = \int \mathrm{d}^4x \, \mathcal{L}[\varphi] \tag{1.5}$$

is the action of the theory. The functional derivative is described in section B.4. In a free theory, we are able to write

$$Z_0[J] = Z_0[0] \exp(iW_0[J]), \quad W_0[J] = \frac{1}{2} \int \mathrm{d}^4x \mathrm{d}^4y \, J(x) D_0(x-y) J(y), \tag{1.6}$$

where D_0 is the propagator of the free theory. Using this form of the generating functional, Eq. (1.4) becomes

$$\begin{aligned}
&\frac{1}{Z[0]}(-i)^n \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} Z_0[J]\Big|_{J=0} = (-i)^n \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} e^{iW_0[J]}\Big|_{J=0} \\
&= (-i)^n \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_{n-1})} \left(i \frac{\delta W_0[J]}{\delta J(x_n)} \right) e^{iW_0[J]}\Big|_{J=0} \\
&= (-i)^n \frac{\delta}{\delta J(x_2)} \cdots \frac{\delta}{\delta J(x_{n-1})} \left(i \frac{\delta^2 W_0[J]}{\delta J(x_{n-1})\, \delta J(x_n)} + i^2 \frac{\delta W_0[J]}{\delta J(x_{n-1})} \frac{\delta W_0[J]}{\delta J(x_n)} \right) e^{iW_0[J]}\Big|_{J=0} = \dots \\
&= (-i)^{n/2} \sum_{(a,b)} \prod_{i=1}^{n/2} \frac{\delta^2 W_0[J]}{\delta J(x_{a(i)})\, \delta J(x_{b(i)})}\Big|_{J=0}.
\end{aligned}$$

In the last line we have introduced the functions a, b which define a way to pair up n elements. The domain of the functions are the integers between 1 and $n/2$, the image a subset of the integers between 1 and n of size $n/2$. A valid pairing is a set $\{(a(1), b(1)), \dots (a(n/2), b(n/2))\}$, where all elements $a(i)$ and $b(j)$ are different, such all integers up to and including n are featured. A pair is not directed, so $(a(i), b(i))$ is the same pair as $(b(i), a(i))$. The sum is over the set $\{(a, b)\}$ of all possible, unique pairings. If n is odd, the expression is equal to 0. This is Wick's theorem, and it can more simply be stated as *a correlation function is the sum of all possible pairings of 2-point functions*,

$$\left\langle \prod_{i=1}^{n} \varphi(x_i) \right\rangle_0 = \sum_{\{(a,b)\}} \prod_{i=1}^{n/2} \left\langle \varphi(x_{a(i)}) \varphi(x_{b(i)}) \right\rangle_0 . \tag{1.7}$$

The subscript on the expectation value indicates that it is evaluated in the free theory.

If we have an interacting theory, that is a theory with an action $S = S_0 + S_I$, where S_0 is a free theory, the generating functional can be written

$$Z[J] = Z_0[0] \left\langle \exp\left(iS_I + i \int \mathrm{d}^4x\, J(x)\varphi(x) \right) \right\rangle_0 . \tag{1.8}$$

We can expand the exponential in power series, which means the expectation in Eq. (1.8) becomes

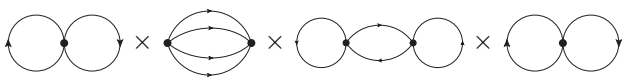
$$\sum_{n,m} \frac{1}{n!m!} \left\langle (iS_I)^n \left(i \int \mathrm{d}^4x\, J(x)\varphi(x) \right)^m \right\rangle_0 . \tag{1.9}$$

The terms in this series are represented by Feynman-diagrams, which are constructed from the Feynman-rules, and can be read from the action. We will not go into further details on how the Feynman-rules are derived, which can be found in any of the main sources for this section [1, 2, 3, 4]. The source terms gives rise to an additional vertex

(1.10)

The generating functional $Z[J]$ equals $Z_0[0]$ times *the sum of all diagrams with external sources* $J(x)$.

Consider a general diagram without external legs, built up of N different connected subdiagrams, where subdiagram i appears n_i times. As an illustration, a generic vacuum diagram in ϕ^4-theory has the form

$V =$  $\times \dots$ (1.11)

If sub-diagram i as a stand-alone diagram equals V_i, then each copy of that subdiagram contribute a factor V_i to the total diagram. However, due to the symmetry of permuting identical subdiagrams, one must divide by the extra symmetry factor $s = n_i!$, which is the total number of permutation of all the copies of diagram i. The full diagram therefore equals

$$V = \prod_{i=1}^{N} \frac{1}{n_i!} V_i^{n_i}. \tag{1.12}$$

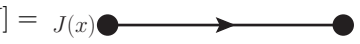
V is uniquely defined by a finite sequence of integers, $(n_1, n_2, \dots n_N, 0, 0, \dots)$, so the sum of all diagrams is the sum over the set S of all finite sequences of integers. This allows us to write the sum of all diagrams as

$$\sum_{(n_1,\dots)\in S} \prod_i \frac{1}{n_i!} V_i^{n_i} = \prod_{i=1}^{\infty} \sum_{n_i=1}^{\infty} \frac{1}{n_i!} V_i^{n_i} = \exp\Big(\sum\nolimits_i V_i\Big). \tag{1.13}$$

We showed that the generating functional $Z[J]$ were the $Z_0[0]$ times the sum of all diagrams due to external sources. Using Eq. (1.13), we see that the sum of all *connected* diagrams $W[J]$ is given by

$$Z[J] = Z_0[0] \exp(iW[J]). \tag{1.14}$$

We can see that this is trivially true for the free theory, the only connected diagram is

$W_0[J] =$  . (1.15)

The 1PI effective action

The generating functional for connected diagrams, $W[J]$, is dependent on the external source current J. Analogously to what is done in thermodynamics and in Lagrangian and Hamiltonian mechanics, we can define a new quantity, with a different independent variable, using the Legendre transformation. The new independent variable is

$$\varphi_J(x) := \frac{\delta W[J]}{\delta J(x)} = \langle \varphi(x) \rangle_J \,. \tag{1.16}$$

The subscript J on the expectation value indicate that it is evaluated in the presence of a source. The Legendre transformation of W is then

$$\Gamma[\varphi_J] = W[J] - \int \mathrm{d}^4x \, J(x)\varphi_J(x). \tag{1.17}$$

Using the definition of φ_J, we have that

$$\frac{\delta}{\delta\varphi_J(x)}\Gamma[\varphi_J] = \int \mathrm{d}^4y \, \frac{\delta J(y)}{\delta\varphi_J(x)} \frac{\delta}{\delta J(y)} W[J] - \int \mathrm{d}^4y \, \frac{\delta J(y)}{\delta\varphi_J(x)}\varphi_J(y) - J(x) = -J(x). \tag{1.18}$$

If we compare this to the classical equations of motion of a field φ with the action S,

$$\frac{\delta S[\varphi]}{\delta\varphi(x)} = -J(x), \tag{1.19}$$

we see that Γ is an action that gives the equation of motion for the expectation value of the field, given a source current $J(x)$.

To interpret Γ further we observe what happens if we treat $\Gamma[\varphi]$ as a classical action with a coupling g. The generating functional in this new theory is

$$Z[J, g] = \int \mathcal{D}\varphi \exp\left\{ ig^{-1} \left(\Gamma[\varphi] + \int \mathrm{d}^4x \varphi(x) J(x) \right) \right\} \tag{1.20}$$

The free propagator in this theory will be proportional to g, as it is given by the inverse of the equation of motion for the free theory. All vertices in this theory, on the other hand, will be proportional to g^{-1}, as they are given by the higher order terms in the action $g^{-1}\Gamma$. This means that a diagram with V vertices and I internal lines is proportional to g^{I-V}. Regardless of what the Feynman-diagrams in this theory are, the number of loops of a connected diagram is $L = I - V + 1$. [1] To see this, we first observe that one single loop must have equally many internal lines as vertices, so the formula holds for $L = 1$. If we add a new loop to a diagram with n loops by joining two vertices, the formula still holds. If we attach a new vertex with one line, the formula still holds, and as the diagram is connected, any more lines connecting the new vertex to the diagram will create additional loops. This ensures that the formula holds, by induction. As a consequence of

[1] This is a consequence of the Euler characteristic $\chi = V - E + F$.

this, any diagram is proportional to g^{L-1}. This means that in the limit $g \to 0$, the theory is fully described at the tree-level, i.e. by only considering diagrams without loops. In this limit, we may use the stationary phase approximation, as described in section B.3, which gives

$$Z[J, g \to 0] \approx C \det\left(-\frac{\delta^2\Gamma[\varphi_J]}{\delta\varphi^2}\right) \exp\left\{ig^{-1}\left(\Gamma[\varphi_J] + \int \mathrm{d}^4x J\varphi_J\right)\right\}. \tag{1.21}$$

This means that

$$-ig\ln(Z[J,g]) = gW[J,g] = \Gamma[\varphi_J] + \int \mathrm{d}^4x\, J(x)\varphi_J(x) + \mathcal{O}(g), \tag{1.22}$$

which is exactly the Legendre transformation we started out with, modulo the factor g. Γ is therefore the action which describes the full theory at the tree level. For a free theory, the classical action S equals the effective action, as there are no loop diagrams.

The propagator $D(x, y)$, which is the connected 2 point function $\langle\varphi(x)\varphi(y)\rangle_J$, is given by the second functional derivative of $W[J]$, times $-i$. Using the chain rule, together with Eq. (1.18), we get

$$(-i)\int \mathrm{d}^4z \frac{\delta^2 W[J]}{\delta J(x)\delta J(z)} \frac{\delta^2\Gamma[\varphi_J]}{\delta\varphi_J(z)\varphi_J(y)} = (-i)\int \mathrm{d}^4z \frac{\delta\varphi_J[z]}{\delta J(x)} \frac{\delta^2\Gamma[\varphi_J]}{\delta\varphi_J(z)\varphi_J(y)} = \frac{\delta}{\delta J(x)}\frac{\delta\Gamma[\varphi_J]}{\delta\varphi_J(y)} = \delta(x-y). \tag{1.23}$$

This shows that the second functional derivative of the effective action is iD^{-1}, where D^{-1} is the inverse propagator. The inverse propagator is the sum of all one-particle-irreducible (1PI) diagrams, with two external vertices. More generally, Γ is the generating functional for 1PI diagrams, which is why it is called the 1PI effective action.

Effective potential

For a constant field configuration $\varphi(x) = \varphi_0$, the effective action, which is a functional, becomes a regular function. We define the effective potential $\mathcal{V}_{\text{eff}}$ by

$$\Gamma[\varphi_0] = -VT\,\mathcal{V}_{\text{eff}}(\varphi_0), \tag{1.24}$$

VT is the volume of space-time. For a constant ground state, the effective potential will equal the energy of this state. To calculate the effective potential, we can expand the action around this state to calculate the effective action, by changing variables to $\varphi(x) = \varphi_0 + \eta(x)$. $\eta(x)$ now parametrizes fluctuations around the ground state, and has by assumption a vanishing expectation value. The generating functional becomes

$$Z[J] = \int \mathcal{D}(\varphi_0 + \eta)\, \exp\left\{iS[\varphi_0+\eta] + i\int \mathrm{d}^4x J(\varphi_0+\eta)\right\} \tag{1.25}$$

The functional version of a Taylor expansion, as described in section B.4, is

$$S[\varphi_0+\eta] = S[\varphi_0] + \int \mathrm{d}x\,\eta(x)\frac{\delta S[\varphi_0]}{\delta\varphi(x)} + \frac{1}{2}\int \mathrm{d}x\mathrm{d}y\,\eta(x)\eta(y)\frac{\delta^2 S[\varphi_0]}{\delta\varphi(x)\delta\varphi(y)} + \ldots \tag{1.26}$$

The notation

$$\frac{\delta S[\varphi_0]}{\delta\varphi(x)} \tag{1.27}$$

indicates that the functional $S[\varphi]$ is differentiated with respect to $\varphi(x)$, then evaluated at $\varphi(x) = \varphi_0$. Denote

$$S_0[\eta] := \int \mathrm{d}^4x\mathrm{d}^4y\,\eta(x)\eta(y)\frac{\delta^2 S[\varphi_0]}{\delta\eta(x)\,\delta\eta(y)}, \tag{1.28}$$

$$S_i[\eta] := \int \mathrm{d}^4x\mathrm{d}^4y\mathrm{d}^4z\,\eta(x)\eta(y)\eta(z)\frac{\delta^3 S[\varphi_0]}{\delta\varphi(x)\,\delta\varphi(y)\,\delta\varphi(z)} + \ldots, \tag{1.29}$$

where the dots indicate higher derivatives. When we insert this expansion into the generating functional $Z[J]$ we get

$$Z[J] = \int \mathcal{D}\eta \exp\left\{i\int \mathrm{d}^4x\,(\mathcal{L}[\varphi_0] + J\varphi_0) + i\int \mathrm{d}^4x\,\eta(x)\left(\frac{\delta S[\varphi_0]}{\delta\varphi(x)} + J(x)\right) + iS_0[\eta] + iS_i[\eta]\right\}$$

The first term is constant with respect to η, and may therefore be taken outside the path integral. The second term gives rise to tadpole diagrams, which alter the expectation value of $\eta(x)$. For $J = 0$, this expectation value should vanish, so this term can be ignored. Furthermore, this means that the ground state must minimize the classical potential,

$$\frac{\partial \mathcal{V}(\varphi_0)}{\partial \varphi} = 0. \tag{1.30}$$

This leaves us with

$$-i \ln Z[J] = W[J] = \int \mathrm{d}^4x \left(\mathcal{L}[\varphi_0] + J\varphi_0\right) - i \ln \left(\int \mathcal{D}\eta \exp\{iS_0[\eta] + iS_i[\eta]\} \right) \tag{1.31}$$

We can now use the definition of the 1PI effective action to obtain a formula for the effective potential,

$$\mathcal{V}_{\text{eff}}(\varphi_0) = -\frac{1}{VT}\left(W[J] - \int \mathrm{d}^4x J\varphi_0 \right) = \mathcal{V}(\varphi_0) - i \ln \left(\int \mathcal{D}\eta \exp\{iS_0[\eta] + iS_i[\eta]\} \right). \tag{1.32}$$

In Eq. (1.17), we showed that the 1PI effective action had the property that it describes the whole quantum theory of the original action at the tree-level. This was done by inspecting a theory with the action proportional to g^{-1}, and argued that Feynman diagrams with L loops are proportional to g^{L-1}. We can use the same argument to expand the effective potential in loops. This is done by modifying the action $S[\varphi] \to g^{-1}S[\varphi]$, and then expand in power of g. The first term in the effective potential is simply modified by $\mathcal{V} \to g^{-1}\mathcal{V}$, which means that it is made up of tree-level, i.e. zero loop terms. This is as expected, since the tree-level result corresponds to the classical result without any quantum corrections. The second term becomes

$$\ln\left(\int \mathcal{D}\eta e^{iS_0 + iS_i} \right) \longrightarrow \ln\left(\int \mathcal{D}\eta e^{ig^{-1}S_0 + ig^{-1}S_i} \right) = \ln\left(\int \mathcal{D}\eta\, e^{ig^{-1}S_0} \right) + \ln\left(\frac{\int \mathcal{D}\eta\, e^{ig^{-1}S_i}\, e^{ig^{-1}S_0}}{\int \mathcal{D}\eta\, e^{ig^{-1}S_0}} \right)$$

The first term is quadratic in η, and can therefore be evaluated as a generalized Gaussian integral, as described in section B.3,

$$\begin{aligned} &\ln\left\{ \int \mathcal{D}\eta\, \exp\left(ig^{-1}\frac{1}{2}\int \mathrm{d}^4x \mathrm{d}^4y\, \eta(x)\eta(y)\, \frac{\delta^2 S[\varphi_0]}{\varphi(x)\varphi(y)} \right) \right\} \\ &= \ln\left\{ \det\left(-g^{-1}\frac{\delta^2 S[\varphi_0]}{\delta\varphi(x)\,\delta\varphi(y)} \right)^{-1/2} \right\} = -\frac{1}{2}\,\mathrm{Tr}\left\{ \ln\left(-\frac{\delta^2 S[\varphi_0]}{\delta\varphi(x)\,\delta\varphi(y)} \right) \right\} \end{aligned}$$

We then use the identity $\ln\det M = \mathrm{Tr}\ln M$. After we remove the constant term, this term proportional to g^0, i.e. it is made up of one-loop terms.

The last term can be evaluated by first expanding the exponential containing the S_i term, then using $\ln(1+x) = \sum_n \frac{1}{n}x^n$. Notice that

$$\langle A \rangle_i = \frac{\int \mathcal{D}\varphi\, A\, e^{ig^{-1}S_0}}{\int \mathcal{D}\varphi\, e^{ig^{-1}S_0}} \tag{1.33}$$

with this, we can write

$$\ln\left[\frac{\int \mathcal{D}\eta\, e^{ig^{-1}S_i} e^{ig^{-1}S_0}}{\int \mathcal{D}\varphi\, e^{ig^{-1}S_0}} \right] = \ln\left(\sum_n \frac{1}{n!} \left\langle (ig^{-1}S_i)^n \right\rangle \right) \tag{1.34}$$

Expectation values of functionals of the field can be written as

$$\langle f[\varphi] \rangle = f\left[-i\frac{\delta}{\delta J} \right] Z[J]\big|_{J=0}, \tag{1.35}$$

where the functional of a functional derivative is evaluated by its series expansion. From Wick's theorem, each term in this series is given by a product of functional derivatives of W. As argued earlier, the propagator of the interacting theory is proportional to g, and thus so is W. We know that S_i is made up of terms that

is third power or higher in the fields. Furthermore, as the expectation value of an *odd* number of fields are proportional to tadpole-diagrams, those can be ignored. This means that $\langle S_i^n \rangle \propto g^k$, where k is more or equal to $3n$, if n is even, $3n+1$, if n is odd. This shows that $Z_i = \mathcal{O}\left(g^2\right)$, and thus that Z_1 contains *all* one loop contributions to $\ln Z$. The effective potential to 1 loop order is thus

$$\mathcal{V}_{\text{eff}}(\varphi_0) = \mathcal{V}(\varphi_0) - \frac{i}{VT}\frac{1}{2}\,\text{Tr}\left\{\ln\left(-\frac{\delta^2 S[\varphi_0]}{\delta\varphi(x)\,\delta\varphi(y)}\right)\right\}. \tag{1.36}$$

1.2 Symmetry and Goldstone's theorem

This section is based on [1, 2, 3, 6].

The symmetries of a theory are transformations of the physical state that leave the governing equations unchanged. A lot of physics is contained in the symmetries of a theory, such as the presence of conserved quantities and the systems low energy behavior. We distinguish between internal and external symmetries. An external symmetry is an active coordinate transformation, such as rotations or translations. They relate degrees of freedom at different space-time points, while internal symmetry transforms degrees of freedom at each space-time point independently of what happens at other points. A further distinction is between local and global symmetry transformations. Local transformations have one rule for how to transform degrees of freedom at each point, which is applied everywhere, while local transformations might themselves be functions of space-time.

In classical field theory, symmetries are encoded in how the Lagrangian changes due to a transformation of the fields. We will consider continuous transformations, which are can in general be written as

$$\varphi(x) \longrightarrow \varphi'(x) = f_t[\varphi](x), \quad t \in [0,1]. \tag{1.37}$$

Here, $f_t[\varphi]$ is a functional in φ, and a smooth function of t, with the constraint that $f_0[\varphi] = \varphi$. This allows us to look at "infinitesimal" transformations,

$$\varphi'(x) = f_\epsilon[\varphi] \sim \varphi(x) + \epsilon \frac{\mathrm{d}f[\varphi]}{\mathrm{d}t}\bigg|_{t=0}, \quad \epsilon \to 0. \tag{1.38}$$

We will consider internal, global transformations which act as linearly on φ. For N fields, φ_i, this can be written

$$\varphi_i'(x) = \varphi_i(x) + \epsilon\, i t_{ij}\varphi_j(x), \quad \epsilon \to 0. \tag{1.39}$$

t_{ij} is called the generator of the transformation. A symmetry of the system is then one in which the Lagrangian is unchanged by the transformation, or at most is different by a divergence-term. That is, a transformation $\varphi \to \varphi'$ is a symmetry if

$$\mathcal{L}[\varphi'] = \mathcal{L}[\varphi] + \partial_\mu K^\mu[\varphi], \tag{1.40}$$

where $K^\mu[\varphi]$ is a functional of φ.[2] This is a requirement for a symmetry in quantum field theory too. However, as physical quantities in quantum field theory are given not just by the action of a single state but the path integral, the integration measure $\mathcal{D}\varphi_i$ has to be invariant as well. If a classical symmetry fails due to the integration measure, it is called an anomaly.

To investigate the symmetry properties of a quantum theory, we explore what constraints a symmetry lies on the effective action. To that end, assume

$$\mathcal{D}\varphi'(x) = \mathcal{D}\varphi(x), \quad S[\varphi'] = S[\varphi]. \tag{1.41}$$

In the generating functional, such a transformation corresponds to a change of integration variable. Using the infinitesimal version of the transformation, we may write

$$Z[J] = \int \mathcal{D}\varphi\, \exp\left\{iS[\varphi] + i\int \mathrm{d}^4x J_i(x)\varphi_i(x)\right\} = \int \mathcal{D}\varphi'\, \exp\left\{iS[\varphi'] + i\int \mathrm{d}^4x J_i(x)\varphi_i'(x)\right\} \tag{1.42}$$

$$= Z[J] - \epsilon \int \mathrm{d}^4x J_i(x) \int \mathcal{D}\varphi\, e^{iS[\varphi]}[t_{ij}\varphi_j(x)], \tag{1.43}$$

[2] Terms of the form $\partial_\mu K^\mu$ does not affect the physics, as variational principle $\delta S = 0$ do not vary the fields at infinity.

Using Eq. (1.18), we can write this as

$$\int \mathrm{d}^4x \, \frac{\delta\Gamma[\varphi_J]}{\delta\varphi_i(x)} \, t_{ij} \, \langle\varphi_j(x)\rangle_J = 0. \tag{1.44}$$

This constraint will allow us to deduce the properties of a theory close to the ground state, only using information about the symmetries of the theory.

The arch typical example of an internal, global and continuous symmetry is the linear sigma model, which we will use as an example throughout this section. The linear sigma model is made up of N real scalar fields φ_i, which are governed by the Lagrangian

$$\mathcal{L}[\varphi] = \frac{1}{2}\partial_\mu\varphi_i(x)\partial^\mu\varphi_i(x) - \mathcal{V}(\varphi), \quad \mathcal{V}(\varphi) = -\frac{1}{2}\mu^2\varphi_i(x)\varphi_i(x) + \frac{1}{4}\lambda[\varphi_i(x)\varphi_i(x)]^2. \tag{1.45}$$

This system is invariant under the rotation of the N fields into each other,

$$\varphi_i \longrightarrow \varphi_i' = O_{ij}\varphi_j, \quad O^{-1} = O^T. \tag{1.46}$$

The set of all such transformations form the Lie group $O(N)$. Lie groups will be discussed in the next section. For $N = 2$, $\mu^2, \lambda > 0$ we get the famous Mexican hat potential, as illustrated in Figure 1.1.

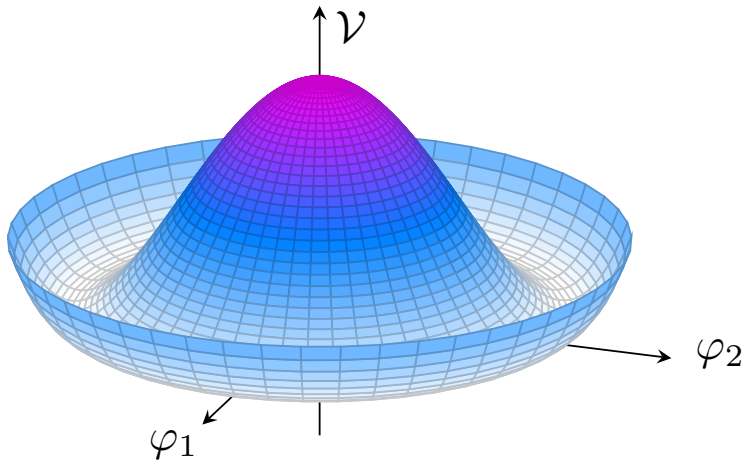


Figure 1.1: The Mexican hat potential is the classical potential $\mathcal{V}$ for the $N = 3$ linear sigma model.

Lie groups

Lie groups are a natural structure to capture the symmetries of a theory. A Lie group is a smooth manifold, i.e. a space that is locally diffeomorphic to $\mathbb{R}^N$. This means that we can locally parametrize the space by N real numbers η_α, using smooth invertible functions. A Lie group is also equipped with group structure. A group is a set, G, together with a map

$$(\cdot,\cdot) : G \times G \longmapsto G, \tag{1.47}$$
$$(g_1, g_2) \longmapsto g_3, \tag{1.48}$$

called group multiplication. This map obeys the group axioms, which are the existence of an identity element $\mathbb{1}$, associativity and the existence of an inverse element g^{-1} for all $g \in G$. These can be written as

$$\begin{aligned} (g, \mathbb{1}) &= g, \\ (g_1, (g_2, g_3)) &= ((g_1, g_2), g_3), \\ (g, g^{-1}) &= \mathbb{1}. \end{aligned}$$

In addition, we require that both the multiplication map and the inverse map, $g \mapsto g^{-1}$ are smooth. We describe the set of continuous symmetry transformations,

$$G = \{\, g \mid g\varphi = \varphi',\, S[\varphi'] = S[\varphi], \mathcal{D}\varphi' = \mathcal{D}\varphi \,\}, \tag{1.49}$$

as a Lie group, where the group multiplication is composition, i.e. performing transformations in succession. This map is closed, as two symmetry transformations are another transformation. The identity map is a symmetry transformation, and composition is associative. This means that invertible symmetry transformations form a group.

We will focus on connected Lie groups, in which we all elements $g \in G$ is in the same connected piece as the identity map $\mathbb{1}\varphi = \varphi$. This means that for each $g \in G$, one can find a continuous path $\gamma(t)$ in the manifold, such that $\gamma(0) = \mathbb{1}$ and $\gamma(1) = g$. Given such a path, we can study transformations close to the identity. As the Lie group is a smooth manifold, we can write[3]

$$\gamma(\epsilon) = \mathbb{1} + i\epsilon V + \mathcal{O}\left(\epsilon^2\right) \tag{1.50}$$

V is a generator, and can be defined as

$$iV = \frac{\mathrm{d}\gamma}{\mathrm{d}t}\Big|_{t=0}. \tag{1.51}$$

We can define a path γ by its path through parameter space, $\gamma(t) = g(\eta(t))$, which means that we can write the generator as

$$V = \frac{\mathrm{d}\gamma}{\mathrm{d}t}\Big|_{t=0} = \frac{\mathrm{d}\eta_a}{\mathrm{d}t}\Big|_{t=0} \frac{\partial g}{\partial \eta_a}\Big|_{\eta=0} = \frac{\partial g}{\partial \eta_a}\Big|_{\eta=0} T_\alpha, \quad T_\alpha = \frac{\partial g}{\partial \eta_\alpha}. \tag{1.52}$$

We see that the generators form a vector space, with the basis T_α, induced by the coordinates η_a. This vector space is called the tangent space of the identity element, $T_{\mathbb{1}}G$. Infinitesimal transformations can therefore be written as

$$\gamma(\epsilon) = \mathbb{1} + i\epsilon v_\alpha T_\alpha, \quad \epsilon \to 0. \tag{1.53}$$

The tangent space, together with the additional operation

$$[T_\alpha, T_\beta] = iC^\gamma_{\alpha\beta} T_\gamma, \tag{1.54}$$

called the Lie bracket, forms a Lie algebra denoted $\mathfrak{g}$. For matrix groups, which are what we deal with in this text, the Lie bracket is the commutator. $C^\gamma_{\alpha\beta}$ are called structure constants. They obey the Jacobi identity,

$$C^\gamma_{\alpha\beta} + C^\alpha_{\beta\gamma} + C^\beta_{\gamma\alpha} = 0, \tag{1.55}$$

which mean that they are totally antisymmetric. A subset of the original Lie group, $H \subset G$, which is closed under the group action is called a subgroup. H then has its own Lie algebra $\mathfrak{h}$, with a set of $m = \dim H$ generators, x_i, which is a subset of the original generators T_α We denote the remaining set of generators t_a, such that x_i and t_a together span $\mathfrak{g}$.

Of special importance are one parameter subgroups. If a curve $\gamma(t)$ through G obey

$$\gamma(t)\gamma(s) = \gamma(t+s), \quad \gamma(0) = \mathbb{1}, \tag{1.56}$$

then all the points on this curve from a one parameter subgroup of G. This path is associated with a generator,

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t}\Big|_{t=0} = i\eta_\alpha T_\alpha. \tag{1.57}$$

This allows us to define the exponential map,

$$\exp\{i\eta_\alpha T_\alpha\} := \gamma(1). \tag{1.58}$$

For connected and compact Lie groups, all elements of the Lie group $g \in G$ can be written as an exponential of elements in the corresponding Lie algebra $\eta_\alpha T_\alpha \in \mathfrak{g}$. For matrix groups, the exponential equals the familiar series expansion

$$\exp\{X\} = \sum_n \frac{1}{n!} X^n. \tag{1.59}$$

[3]The factor i is a physics convention, and differs from how mathematicians define generators of a lie group.

A subgroup $H \in G$ has its own Lie algebra $\mathfrak{h}$, with a set of $m = \dim H$ generators, x_i, which is a subset of the original generators T_α. We denote the remaining set of generators t_a, such that x_i and t_a together span $\mathfrak{g}$. The commutators of x_i must be closed, which means that we can write

$$[x_i, x_j] = iC_{ij}^k x_k, \tag{1.60}$$

$$[x_i, t_a] = iC_{ia}^b t_b, \tag{1.61}$$

$$[t_a, t_b] = iC_{ab}^k x_k + iC_{ab}^c t_c, \tag{1.62}$$

where ijk runs over the generators of $\mathfrak{h}$, and abc runs over the rest. The second formula can be derived using the Jacobi identity Eq. (1.55), which implies that $C_{ia}^k = -C_{ik}^a = 0$. This is called a Cartan decomposition.

Goldstone's theorem

The fact that a theory is left invariant under some symmetry transformation does not imply that the ground state is invariant under this transformation. The $N = 2$ linear sigma model illustrates this. If we assume the ground state φ_0 is translationally invariant, then it is given by minimizing the effective potential, of which the classical potential, $\mathcal{V}$, is the leading order approximation. This potential is illustrated in Figure 1.1. The ground state is therefore given by any of the values along the brim of the potential. If we, without loss of generality, choose $\varphi_0 = (0, v)$ as the ground state, then any symmetry transformation will change this state. We say that the symmetry has been *spontaneously broken*.

To explore this in a general context, assume a theory of φ_i real scalar fields are invariant under the actions of some Lie group, G. A symmetry $g \in G$ is broken if the vacuum expectation value of the original fields and the transformed fields differ. That is, if

$$\langle\varphi\rangle_0 \neq \langle\varphi'\rangle_0 = \langle g\varphi\rangle_0 \tag{1.63}$$

We can now exploit what we learned about Lie groups to write

$$\langle\varphi'\rangle_0 = \langle\varphi\rangle_0 + i\epsilon\eta_\alpha T_\alpha \langle\varphi\rangle_0 . \tag{1.64}$$

Let t_a be the set of generators corresponding to broken symmetries, i.e.

$$t_\alpha \langle\varphi\rangle_0 \neq 0. \tag{1.65}$$

These are called the *broken generators*. The remaining set of generators x_i, which obey

$$x_i \langle\varphi\rangle_0 = 0, \tag{1.66}$$

are called unbroken, and generate a subgroup $H \subset G$ as the set of symmetry transformations of the vacuum is a group.

In Eq. (1.18) we found that the effective action obey

$$\int \mathrm{d}^4x \frac{\delta\Gamma[\varphi_J]}{\delta\varphi_i} t_{ij} \langle\varphi_j\rangle_0 = 0. \tag{1.67}$$

We now differentiate this expression with respect to $\varphi_k(y)$ and evaluate it in the vacuum, which gives

$$\int \mathrm{d}^4x \frac{\delta^2\Gamma[\varphi_0]}{\delta\varphi_k(y)\,\delta\varphi_i(x)} t_{ij} \langle\varphi_j\rangle_0 = 0. \tag{1.68}$$

With the assumption that the ground state is constant, we get

$$\frac{\partial^2\mathcal{V}_{\text{eff}}}{\partial\varphi_k\,\partial\varphi_i} t_{ij} \langle\varphi_j\rangle_0 = 0. \tag{1.69}$$

This is trival for unbroken symmetries, as $t_{ij} \langle\varphi_j\rangle_0 = 0$ by definition. However, in the case of a broken symmetry, the second derivative of the effective potentail has an eigenvector $t_{ij}^\alpha \langle\varphi_j\rangle$ with a zero eigenvalue

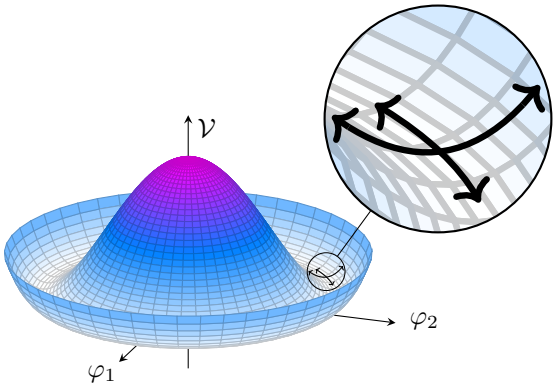
for each broken generator. In Eq. (1.23), we found that the second derivative of the effective action is the inverse propagator,

$$D_{ij}^{-1}(x,y) = -i\frac{\delta^2\Gamma[\varphi_0]}{\delta\varphi_i(y)\,\delta\varphi_j(x)} = \int \frac{\mathrm{d}^4p}{(2\pi)^4} e^{-ip(x-y)}\,\tilde{D}_{ij}^{-1}(p). \tag{1.70}$$

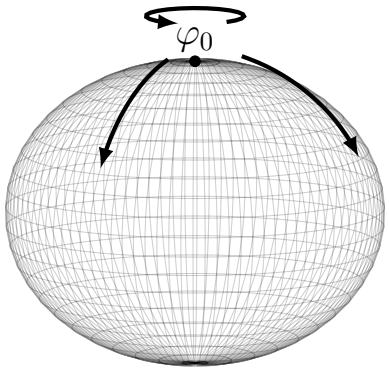
Using this, we can write

$$\tilde{D}_{ij}^{-1}(p=0)\,t_{jk}^{\alpha}\,\langle\varphi_k\rangle_0 = 0. \tag{1.71}$$

Zeros of the inverse propagator corresponds to the physical mass of particles. In Lorentz-invariant systems, each zero-mass vector corresponds to a masses particles, called a Goldstone boson.[4] This means there are $n_G = \dim G - \dim H$ zero mass modes. In general, counting of massless modes is more complicated, and depends on the dispersion relation of the particles at low momenta. Systems with Goldstone bosons with quadratic dispersion relation, that is $E = |\vec{p}|^2$ when $\vec{p} \to 0$, often exhibit a lower number of massless modes. An example is ferromagnets, where the SU(2) rotational symmetry is broken down to U(1) when they align along one axis. This corresponds to two broken generators, yet the system only exhibits one massless mode [7].



(a) Excitations along the brim does not cost any energy, as the potential is flat, unlike excitations in the radial direction.



(b) Excitations for the $N = 3$ sigma model. Two of the symmetries are broken, while rotations around the groundstate leaves the system unchanged.

The linear sigma model gives an intuition for the Goldstone mode. In the case of $N = 2$, the symmetry of the Lagrangian are rotations in the plane. As the ground state is a point along the "brim" of the hat, this rotational symmetry is broken. Any excitations in the rotational direction, however, does not cost any energy, which is indicative of a massless mode. This is illustrated in Figure 1.2a. In this example, the original symmetry group is one dimensional, so there are no unbroken symmetries. If we instead consider the $N = 3$ linear sigma model, which has a three-dimensional symmetry group, rotations of the sphere, we see that the ground state is left invariant under subgroup of the original symmetry transformations. The ground state manifold of this system, the set of all states that minimizes the effective potential, is then a sphere. When the system chooses one single ground state, this symmetry is broken, but only for two of the generators. The generator for rotations around the ground state leaves the that point unchanged, and is thus an unbroken symmetry. Any excitations in the direction of the broken symmetries does not cost energy, as it is in the ground state manifold. The unbroken symmetry, on the other hand, does not correspond to an excitation. This is illustrated in Figure 1.2b.

1.3 CCWZ construction

As Goldstone bosons are massless, they play a crucial role in the low energy dynamics. To best describe this limit, we seek a parametrization of the fields in which they are the degrees of freedom. This can be done using the CCWZ construction, named after Callan, Coleman, Wess and Zumino. As well as the original papers [8, 9], this section is based on [3, 10, 11] and[5]. (REFERER ORDENTLIG)

[4] The particles are bosons due to the bosonic nature of the transformations, t. If the generators are Grassmann numbers, the resulting particle, called a goldstinos, are fermions.

[5] `http://scipp.ucsc.edu/~haber/archives/physics251_17/PHYS251_Presentation_L_Morrison`

We saw that the Goldstone bosons corresponds to excitations within the vacuum manifold. The vacuum manifold corresponds to points in field space φ that can be reach from the vacuum φ_0 with a transformation $g \in G$. This means that we can write such excitations as

$$\varphi = \tilde{\Sigma}\varphi_0, \quad \tilde{\Sigma} = \tilde{\Sigma}(\eta) = \exp\{i\eta_\alpha T_\alpha\} \tag{1.72}$$

$\tilde{\Sigma}$ is thus a function from the parameter space, $\eta_\alpha \in \mathbb{R}^n$ to G,

$$\tilde{\Sigma} : \mathbb{R}^n \longmapsto G. \tag{1.73}$$

We then get space-time dependent field configurations by making the parameters dependent on spacetime. We will for now assume η_α is constant. This parametrization is highly redundant. There are $n = \dim G$ parameters η, but Goldstone's theorem says that there is one massless mode per broken generator. Two elements $\tilde{\Sigma}$ and $\tilde{\Sigma}'$, related by

$$\tilde{\Sigma}' = \tilde{\Sigma}e^{i\theta_a t_a} \tag{1.74}$$

results in the same φ. This is because $e^{i\theta_a t_a} = h \in H$, and $h\varphi_0 = \varphi_0$, by assumption. The set of all equivalent $\tilde{\Sigma}$'s is exactly the left coset, $gH = \{\, gh \mid h \in H \,\}$. The set of cosets forms a new manifold, G/H, called the Goldstone manifold. This is a manifold of dimension $\dim(G/H) = \dim(G) - \dim(H)$, which is exactly the number of broken generators, and thus also the number of Goldstone modes. Membership of a certain coset from an equivalence relation, $g \sim g'$ if $g' = gh$. This means that the cosets gH form a partition of G, and that each element $g \in G$ belongs to one, and only one, coset. To remove the redundancy in the parametrization, we need to choose one representative element from each coset.

By the inverse function theorem, any mapping between manifolds $f : \mathcal{M} \mapsto \mathcal{N}$ that has a non-degenerate differential, that is an invertible Jacobian, at a point $p \in \mathcal{M}$, is invertible in a neighborhood of p. The map

$$\tilde{\Sigma}(\xi, \theta) = \exp\{i\xi_i x_i\} \exp\{i\theta_a t_a\} \tag{1.75}$$

is invertible at $p = (\xi_i = 0, \theta_a = 0)$, which is mapped to the identity, as the Jacobian is the identity matrix. This means that, in a neighborhood $U \subset G$ of the identity, each element g has a unique representation $g = \Sigma$ [6]. Furthermore, two elements $\tilde{\Sigma}'$ and $\tilde{\Sigma}$ related by $\tilde{\Sigma}' = \tilde{\Sigma}h$, $h \in H$ have the same ξ-arguments. We see that ξ_i parametrize G/H, in the neighborhood of the identity. We therefore demand that $\tilde{\Sigma}$ always appear in the standard form

$$\Sigma(\xi) = \tilde{\Sigma}(\xi, 0) = \exp\{i\xi_i x_i\}. \tag{1.76}$$

The field $\varphi(x)$ can therefore be written as

$$\varphi(x) = \Sigma(x)\varphi_0 = \exp\{i\xi_i(x)x_i\}\varphi_0, \tag{1.77}$$

and $\xi_i(x)$ can be associated with the Goldstone bosons.

In the linear sigma model, the original $O(N)$ symmetry is broken down to $O(N-1)$, which transforms the remaining $N-1$ fields with vanishing expectation value into each other. However, $O(N)$ consists of two disconnected subsets, those matrices with determinant 1 and those with determinant -1. There is no continuous path that takes an element of $O(N)$ with determinant of -1 to an element with determinant 1.[6] The set of symmetries that are connected to the identity is

$$G = SO(N) = \{\, M \in O(N) \mid \det M = 1 \,\}. \tag{1.78}$$

If we choose $\varphi_0 = (0, 0, ..., v)$, then it is apparent that the ground state is invariant under the rotations of the $N-1$ first fields, so the unbroken symmetry is $H = SO(N-1)$. The Goldstone manifold is $G/H = SO(N)/SO(N-1)$.

Consider the case of $N = 3$, which is illustrated in Figure 1.2b. G is the rotations of the sphere, while H is rotations around φ_0, $SO(2)$. The Goldstone manifold consists of the rotations of φ_0 to other points of the sphere, i.e. $G/H = SO(3)/S(2) = S^2$, the 2-sphere. This is not a Lie group, as translating φ in a closed path around the sphere may result in a rotation around the z-axis.

[6] A simple proof of this is the fact that the determinant is a continuous function, while any path $\det M(t)$ such that $\det M(1) = -1$, $\det m(0) = 1$ must make a discontinuous jump.

To check that ξ_i in fact are the Goldstone modes, we study the way they appear in the Lagrangian. As they are massless, no mass term of the form $M_{ij}\xi_i\xi_j$ should appear in the Lagrangian. The original Lagrangian $\mathcal{L}[\varphi]$ was invariant under global transformations $\varphi(x) \mapsto g\varphi(x)$. However, any terms that only depend on $\varphi(x)$, and not its derivatives, will also be invariant under a *local* transformation, $\varphi(x) \mapsto g(x)\varphi(x)$. Our parametrization of the fields, $\varphi(x) = \Sigma(x)\varphi_0$ is exactly such a transformation, which means that any such terms are independent of the Goldstone bosons. We can therefore write

$$\mathcal{L}[\varphi] = \mathcal{L}_{\text{kin}}[\varphi] + V(\varphi_0), \tag{1.79}$$

where all terms in $\mathcal{L}_{\text{kin}}$ are proportional to at least one derivative term, $\partial_\mu\varphi(x)$. Inserting the parametrization into this derivative term, we get

$$\partial_\mu\varphi(x) = \partial_\mu[\Sigma(x)\varphi_0] = \Sigma(x)[\Sigma(x)^{-1}\partial_\mu\Sigma(x)]\varphi_0. \tag{1.80}$$

The dependence of the Lagrangian on ξ_i only appears through terms of the form $\Sigma(x)^{-1}\partial_\mu\Sigma(x)$. These can always be written on the form

$$\begin{aligned} &i\Sigma(x)^{-1}\partial_\mu\Sigma(x) = D_\mu(x) + E_\mu(x), \\ &D_\mu = ix_iD_{ij}(\xi)\partial_\mu\xi_j, \\ &E_\mu = it_aE_{ai}(\xi)\partial_\mu\xi_i. \end{aligned}$$

This is called the Maurer-Cartan form, and D_{ij} and E_{ai} are as-of-yet unknown real valued functions of ξ. (HVORFOR??)

Transformation properties of Goldstone bosons

We can deduce how the Goldstone bosons transforms under G from how φ transforms. In general,

$$\varphi' = g\varphi = (g\Sigma(\xi))\varphi_0 = \Sigma(\xi')\varphi_0 \quad g \in G. \tag{1.81}$$

While $\Sigma(\xi')$ has the standard form by assumption,

$$\Sigma(\xi') = \exp\{i\xi_i'x_i\}, \tag{1.82}$$

$g\Sigma(\xi)$ does not, in general.

Figure 1.3 illustrates this in the case of $G = SO(3)$. $\Sigma(\xi)$ transforms φ_0 to φ, then g transforms φ to $\varphi' = \Sigma(\xi')\varphi_0$. Assuming φ and φ' are close enough to φ_0, we can write $\Sigma(\xi)$ and $\Sigma(\xi')$ on the standard form. However, if we follow a small neighborhood around φ_0 as it is acted on by $\Sigma(\xi)$, then g, it will be rotated by the time it arrives at φ', when compared to the same neighborhood if it was acted on by $\Sigma(\xi')$.

$g\Sigma(\xi)$ and $\Sigma(\xi')$ are in the same coset, as they by assumption corresponds to the same physical state. This means that we can write $g\Sigma(\xi) = \Sigma(\xi')h(g,\xi)$, where $h(g,\xi) \in H$. The transformation rule of ξ under G is therefore implicitly defined by

$$\Sigma(\xi') = g\Sigma(\xi)[h(g,\xi)]^{-1}. \tag{1.83}$$

This is in general not a linear representation, which is why this construction also is called a *non-linear realization*. Using the transformation rule, we can obtain the transformation rule of the Maurer-Cartan form. We use the shorthand $\Sigma = \Sigma(\xi)$, $\Sigma' = \Sigma(\xi')$, and $h = h(g,\xi)$. This gives

$$\begin{aligned} \Sigma^{-1}\partial_\mu\Sigma \to &\Sigma'^{-1}\partial_\mu\Sigma' \\ &= (g\Sigma h^{-1})^{-1}\partial_\mu(g\Sigma h^{-1}) \\ &= (h\Sigma^{-1}g^{-1})g[(\partial_\mu\Sigma)h^{-1} + \Sigma\partial_\mu h^{-1}] \\ &= h\Sigma^{-1}(\partial_\mu\Sigma)h^{-1} + h\partial_\mu h^{-1} \\ &= h(\Sigma^{-1}\partial_\mu\Sigma + \partial_\mu)h^{-1}. \end{aligned}$$

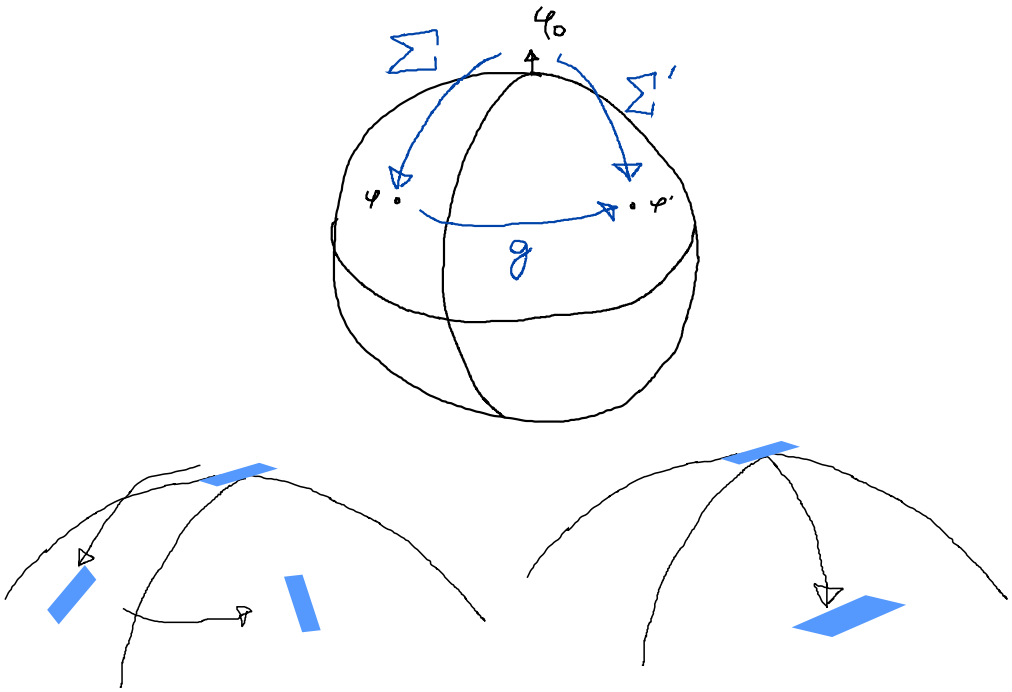


Figure 1.3: (KLADD) The top figure illustrates the transformation of φ_0 to φ and then φ, and the alternative transformation $\varphi_0 \to \varphi'$. The bottom figure illustrates how this can rotate a neighborhood of φ_0 differently.

In terms of D_μ and E_μ,

$$D_\mu \to h D_\mu h^{-1} \tag{1.84}$$

$$E_\mu \to h(E_\mu + i\partial_\mu)h^{-1}. \tag{1.85}$$

We see that E_μ transforms like a gauge field, with the gauge group H. These are the peicees we need to construct a Lagrangian in terms of the Goldstone bosons. (MANGLER: Inkluder massiver partikler, med $\varphi(x) = \Sigma(x)\tilde{\varphi}(x)$, og hvordan det transformerer. Skriver om representasjoner? (adj, r pi))

1.4 Effective field theories

In section 1.1, we studied the effective action, and found that it gave the equation of motion for the expectation value of the field in the full quantum theory. Let $\varphi^*(x) = \langle\varphi(x)\rangle$, and $\varphi(x) = \varphi^*(x) + \eta(x)$. We can write this as

$$\exp\{i\Gamma[\varphi^*]\} = \int \mathcal{D}\eta \, \exp\{iS[\varphi^* + \eta]\}. \tag{1.86}$$

As, by assumption, $\langle\eta\rangle = 0$, this only includes 1PI diagrams. We say that the degree of freedom η has been *integrated out*. More generally, we can integrate out some of the degrees of freedom of a system, to get an effective theory for what is left. If we have two sets of fields, φ and ψ, and a Lagrangian $\mathcal{L}[\varphi, \psi]$, then the effective theory of the φ fields are defined by

$$\int \mathcal{D}\varphi\mathcal{D}\psi \, \exp\left\{ i \int \mathrm{d}x \, \mathcal{L}[\varphi, \psi] \right\} = \int \mathcal{D}\varphi \exp\{iS_{\text{eff}}[\varphi]\}. \tag{1.87}$$

(EKSEMPLER? WILSON RENORMALISERING; FERMI TEORI)

The effective action can not in general be written as a single integral over a power series in the field, it migh for example be non-local [4]. To construct the effective a effective theory of Goldstone bosons, such as chiral pertubation theory, we rely on a "theorem", as formulated by Weinberg:

> [I]f one writes down the most general possible Lagrangian, including all terms consistent with assumed symmetry principles, and then calculates matrix elements with this Lagrangian to any given order of perturbation theory, the result will simply be the most general possible S-matrix consistent with analyticity, perturbative unitary, cluster decomposition and the assumed symmetry principles. [12]

In other words, if we write down the most general Lagrange density consistent with symmetries of the underlying theory, it will result in the most general S-matrix consistent with that theory, and important physical assumptions. In last section, we found a parametrization that guarantees for a simple way the fields must appear in the Lagrangian. Thus, any combination of these building blocks $\mathcal{O}_i(\xi)$ that must appear in the Lagrangian. The effective Lagrangian is therefore

$$\mathcal{L}[\xi] = \sum_i c_i \mathcal{O}_i(\xi), \tag{1.88}$$

where c_i are free parameters.

This leaves a Lagrange density with infinitely many terms, and infinitely many free parameters. To be able to use this theory for anything one must have a method for ordering the terms in order of importance. As described in [13], by rescaling the external momenta $p_\mu \to tp_\mu$ and quark masses $m_i \to t^2 m_i$, each term in the Lagrangian obtains a factor t^D. The Lagrangian is then expanded as $\mathcal{L} = \sum_D \mathcal{L}_D$, where $\mathcal{L}_D$ contains all terms with a factor t^D.

Chapter 2

The effective theory of pions

2.1 QCD

Quantum chromodynamics, or QCD, is the theory of how quarks interact using the strong force. It has its name due to the fact that the charges of QCD are called red, green and blue, which of course is only an analogy to colors. The two main quantities are the quark spinors, q_f if we have N_f free quarks, with N_c colors, then the Lagrangian is given by (HUSK i!)

$$\mathcal{L}[q] = \sum_{f=1}^{N_f} \sum_{c=1}^{N_c} i\bar{q}_{fc}(\gamma^\mu \partial_\mu - m_f)q_{fc} = i\bar{q}(\not{\partial} - m)q. \tag{2.1}$$

In the last equality, we have hidden the indices to reduce clutter. Each element q_{fc} is a spinor, and γ^μ are the gamma matrices, as described in section B.1. Furthermore, $\bar{q} = q^\dagger \gamma^0$. This Lagrangian is invariant under rations of the quarks in the color indices, i.e. the transformation

$$q_c \to q'_c = U_{cc'} q_{c'}, \quad \bar{q}_c = \bar{U}^\dagger_{cc'} q'_c \tag{2.2}$$

where $U_{cc'}$ is an $N_c \times N_c$ unitary matrix. The set of all $N_c \times N_c$ unitary matrices form the Lie group $U(N_c)$. All unitary matrices U can be written as $e^{i\theta}U'$, where θ is a real number, and U' is a matrix with determinant, which means we can decompose the Lie group into $U(1) \times SU(N_c)_c$. We recognize $U(1)$ as the gauge group of the electromagnetic field, which we will ignore for the time being. (ER DETTE SANT? HVORDAN FUNGERER DETTE?) $SU(N_c)$ is the group of all complex $N_c \times N_c$ matrices with determinant 1, and the subscript c specifies that this is the set of color transformation, and not just some abstract group.

The Yang-Mills Lagrangian

$SU(N_c)$ is the gauge group for the strong force. Given an element $U \in SU(N_c)$, we can write

$$U = \exp\{i\chi_\alpha \lambda_\alpha\}, \quad \chi_\alpha \lambda_\alpha \in \mathfrak{su}\left(N_c\right)_c, \tag{2.3}$$

where $\mathfrak{su}\left(N_c\right)_c$ is the Lie algebra of $SU(N_c)_c$. We derive the full, interacting Lagrangian of QCD by demanding that it remain invariant under a *local* $SU(N_c)_c$ transformation, i.e.

$$q \to \exp\{i\chi_\alpha(x)\lambda_\alpha\}q, \quad \bar{q} \to \exp\{-i\chi_\alpha(x)\lambda_\alpha\}q'. \tag{2.4}$$

This is no problem for the mass term, however, we need to modify the kinetic derivative term. For the Lagrangian to be gauge-invariant, this has to obey

$$D_\mu q \to (D_\mu q)' = U D_\mu q. \tag{2.5}$$

As $q' = Uq$, this implies that

$$D'_\mu = UD_\mu U^\dagger. \tag{2.6}$$

If we posit $D_\mu = \mathbb{1}\partial_\mu + A^\alpha_\mu \lambda_\alpha$, then

$$D'_\mu = UD_\mu U^\dagger = U(U^\dagger \partial_\mu + i\partial_\mu \chi_\alpha(x)\lambda_\alpha U^\dagger) + UA_\mu U^\dagger = \partial_\mu + U(A_\mu + i\partial_\mu \chi_\alpha(x))U^\dagger, \tag{2.7}$$

where $A_\mu = A^\alpha_\mu \lambda_\alpha$. This means that if we demand that A_μ transforms as

$$A_\mu \to U(A_\mu - i\partial_\mu \chi)U^\dagger, \tag{2.8}$$

then our new derivative operator, called the *covariant derivative*, follows the desired transformation rule. The second derivative operator,

$$D_\mu D_\nu = [\partial_\mu \partial_\nu - i(\partial_\mu A_\nu + A_\mu \partial_\nu + A_\nu \partial_\mu) - A_\mu A_\nu], \tag{2.9}$$

transforms in the same way as the first derivative. We see that the "operator-part" of this derivative is symmetric in the space-time indices, which means that the commutator will just a tensor, and not an operator. We define

$$F_{\mu\nu} := i[D_\mu, D_\nu] = (\partial_\mu A_\nu - \partial_\mu A_\nu) - i[A_\mu, A_\nu] = (\partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu + C^\alpha_{\beta\gamma} A^\beta_\mu A^\gamma_\nu)\lambda_\alpha. \tag{2.10}$$

This transforms as

$$F_{\mu\nu} \to UF_{\mu\nu}U^\dagger. \tag{2.11}$$

We now need to include terms governing the gauge field A_μ in the Lagrangian. The tensor $F_{\mu\nu}$ allows construct all gauge invariant terms to dimension 4, which are

$$F^a_{\mu\nu}F^{\mu\nu}_a, \quad \varepsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma}. \tag{2.12}$$

Here, ε is the Levi-Civita symbol. This allows us to write down the most general gauge-invariant Lagrangian for a $SU(N_c)$ gauge theory, the Yang-Mills Lagrangian

$$\mathcal{L} = i\bar{q}(\not{D} - m)q + \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a + c\varepsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma}. \tag{2.13}$$

Chiral symmetry

In addition to the color and flavor indices s and f, the quarks also have spinor indices, i, which are what the γ-matrices act on. We can define the projection operators,

$$P_\pm = \frac{1}{2}(1 \pm \gamma^5), \tag{2.14}$$

which obey $P^2_\pm = P_\pm$, $P_+P_- = P_-P_+ = 0$ and $P_+ + P_- = 1$, as good projection operators should. Furthermore, $P^\dagger_\pm = P_\pm$. These project spinors down to their chiral components, called left- and right-handed spinors,

$$P_+ q = q_R, \quad P_- q = q_L. \tag{2.15}$$

From section B.1, we have

$$\{\gamma^\mu, \gamma^5\} = 0, \tag{2.16}$$

which means that

$$\bar{q}P_\pm = (P_\mp q)^\dagger \gamma^0. \tag{2.17}$$

This means that we can write the quark part of the Lagrangian as

$$\begin{aligned}\bar{q}(i\not{D} - m)q &= \bar{q}(P_+ + P_-)(P_+ + P_-)(i\not{D} - m)q = (qP_-)\gamma^0 P_+(i\not{D} - m)q + (qP_+)\gamma^0 P_-(i\not{D} - m)q \\ &= \bar{q}_L(i\not{D})q_L + \bar{q}_R(i\not{D})q_R - \bar{q}_L m q_R - \bar{q}_R m q_L.\end{aligned}$$

We see that the mass-term mixes the two chiral components. In the limit $m \to 0$, called the *chiral limit*, we gain two new symmetries,

$$q_R \to U_R q_R, \quad q_L \to U_L q_L, \tag{2.18}$$

where U_L and U_R are unitary matrices which act on the flavor indices. The total set of such transformations form the Lie group $U(N_f)_L \times U(N_f)_R = U_V(1)$. We can write

$$U_R = \exp\{i\eta_\alpha^R T_\alpha^R\} e^{i\theta_R}, \quad U_L = \exp\{i\eta_\alpha^L T_\alpha^L\} e^{i\theta_L}, \tag{2.19}$$

by again decomposing $U(N_f) = U(1) \times SU(N_f)$. The conserved currents corresponding to these symmetries are

$$L_\alpha^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu q}(iT_\alpha^L q_L) = \bar{q}\gamma^\mu T_\alpha^L q_L = \bar{q}_L \gamma^\mu T_\alpha^L q_L, \tag{2.20}$$

$$R_\alpha^\mu = \bar{q}_R \gamma^\mu T_\alpha^R q_R, \tag{2.21}$$

$$J_R^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu q}(iq_L) = -\bar{q}_R \gamma^\mu q_R \tag{2.22}$$

$$J_L^\mu = -\bar{q}_L \gamma^\mu q_L \tag{2.23}$$

We expect when we introduce a non-zero mass, the symmetry is broken. However, we see that when right and left-handed quarks are transformed in the same way, i.e. $\eta_R = \eta_L$, then the Lagrangian remain invariant. We can write a general transformation in $U_R(N_f) \times U_L(N_f)$ as a matrix

$$U = \begin{pmatrix} U_L & 0 \\ 0 & U_R \end{pmatrix}, \tag{2.24}$$

which act on spinors $q = (q_L, q_R)^T$. The subspace of transformations that remains unbroken for non-zero masses are

$$U_V = \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} = \exp\{i\eta_\alpha^V T_\alpha^V \mathbb{1}\} e^{i\theta_V \mathbb{1}} \tag{2.25}$$

while the remaining set of transformation are of the form

$$U_A = \begin{pmatrix} U^\dagger & 0 \\ 0 & U \end{pmatrix} = \exp\{i\eta_\alpha^V T_\alpha^V \gamma^5\} e^{i\theta_A \gamma^5}. \tag{2.26}$$

These sets of transformations form the Lie groups $U(1)_V \times SU(N_f)_V$ and $U(1)_A \times SU(N_f)_A$. The corresponding conserved currents are

$$V_\alpha^\mu = R_\alpha^\mu + L_\alpha^\mu = \bar{q}\gamma^\mu T_\alpha^V q, \tag{2.27}$$

$$A_\alpha^\mu = R_\alpha^\mu - L_\alpha^\mu = \bar{q}\gamma^\mu \gamma^5 T_\alpha^A q, \tag{2.28}$$

$$J_V^\mu = \bar{R}^\mu + L^\mu = \bar{q}\gamma^\mu q, \tag{2.29}$$

$$J_A^\mu = \bar{R}^\mu - L^\mu = \bar{q}\gamma^\mu \gamma^5 q. \tag{2.30}$$

The currents corresponding to $U(1)_V$ and $U(1)_A$ are called the vector and axial currents, as they transform as vectors and axial-vectors under space-time transformations. The conserved charge corresponding to the vector current,

$$Q = \int \mathrm{d}^3x \, \bar{q}\gamma^0 q, \tag{2.31}$$

is 1/3 times the *baryon number*. (HVORFOR?) We have up to now only made classical considerations. In the full quantum theory, the baryon number remains conserved. The axial current J_A^μ however is the subject to an anomaly. As the integration measure $\mathcal{D}\bar{q}\mathcal{D}q$ is not invariant under transformations in $U(1)_A$, as we saw earlier is required, the quantum correction to $\partial_\mu J_A^\mu$ are non-zero. The remaining symmetry, $G = SU(N_f)_L \times SU(N_f)_R = SU(N_f)_V \times SU(N_f)_A$ and how it is broken is the foundation for chiral perturbation theory, which we will consider int the next section.

2.2 Chiral perturbation theory

In this paper, we will consider the interaction of the two lightest quarks, the up and down quarks u and d. These are not massless, but have a non-zero mass matrix

$$m = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}. \tag{2.32}$$

These masses are estimated to be (FINN TALL). This means that the $SU(2)_V \times SU(2)_A$ symmetry is *explicitly* broken. In what is called the isospin limit, $m_u = m_d$, the subgroup $SU(2)_V$ remains intact. The difference between these masses is small, which is why isospin is a good quantum number. However, even the underlying symmetry is only approximate, we can still apply the formalism from Gold stones theorem CCWZ construction, with the only change that wee need to include small mass terms in for the Goldstone bosons, no called *Pseudo Goldstone bosons*.

The (approximate) $G = SU(2)_V \times SU(2)_A$ symmetry of the two-flavor QCD-Lagrangian is spontaneously broken by the ground state. As quarks q are spinors, a non-zero expectation value of the quark field would break Lorentz-invariance. Instead, the spontaneous symmetry breaking is characterized by the *scalar quark condensate*, $\langle \bar{q}q \rangle$. In the isospin-limit, one can show that $\langle \bar{u}u \rangle = \langle \bar{d}d \rangle = \langle \bar{q}q \rangle /2$. The scalar quantity $\bar{q}q$ is invariant under isospin transformation $SU(2)_V$, but not under $SU(2)_A$. The Goldstone boson is therefore $G/H = SU(2)_L \times \mathrm{SU}(2)_R / SU(2)_V = SU(2)_A$.

The χPT effective Lagrangian will be constructed using the parametrization

$$\Sigma(x) = A_\alpha (U(x) \Sigma_0 U(x)) A_\alpha, \tag{2.33}$$

where

$$\Sigma_0 = \mathbb{1},\ A_\alpha = \exp\left(\frac{i\alpha}{2}\tau_1\right),\ U(x) = \exp\left(i\frac{\tau_a \pi_a(x)}{2f}\right).$$

τ_a are the SU(2) generators, i.e. Pauli matrices, as described in section B.1. π_a, where $a \in \{1,2,3\}$, are the pion fields. These are real fields, meaning $\pi_a^\dagger = \pi_a$.

2.3 Leading order Lagrangian

The leading order Lagrangian in χPT is [13, 14]

$$\mathcal{L}_2 = \frac{f^2}{4} \operatorname{Tr}\left[\nabla_\mu \Sigma (\nabla^\mu \Sigma)^\dagger\right] + \frac{f^2}{4} \operatorname{Tr}\left[\chi^\dagger \Sigma + \Sigma^\dagger \chi\right]. \tag{2.34}$$

χ and f are the free parameters of the theory. f is the pion decay constant, while $\chi = 2B_0 M$. Here, M is the mass matrix Eq. (2.32), and B_0 is related to the quark condensate through $f^2 B_0 = -\langle \bar{u}u \rangle$. The covariant derivative is defined by

$$\nabla_\mu \Sigma = \partial_\mu \Sigma - i\left[v_\mu, \Sigma\right], \quad (\nabla_\mu \Sigma)^\dagger = \partial_\mu \Sigma^\dagger - i\left[v_\mu, \Sigma^\dagger\right], \quad v_\mu = \frac{1}{2}\mu_I \delta^0_\mu \tau_3,$$

where μ_I is the isospin chemical potential. To get the series expansion of Σ in powers of π/f, we start by using the fact that $\tau_a^2 = \mathbb{1}$ to write

$$A_\alpha = \sum_n^\infty \frac{1}{n!}\left(\frac{i\alpha}{2}\tau_1\right)^n = \sum_n^\infty \left[\frac{\mathbb{1}}{(2n)!}\left(\frac{i\alpha}{2}\right)^{(2n)} + \frac{\tau_1}{(2n+1)!}\left(\frac{i\alpha}{2}\right)^{(2n+1)}\right] = \mathbb{1}\cos\frac{\alpha}{2} + i\tau_1 \sin\frac{\alpha}{2}. \tag{2.35}$$

The series expansion of U is

$$U = \exp\left(\frac{i\pi_a\tau_a}{2f}\right) = 1 + \frac{i\pi_a\tau_a}{2f} + \frac{1}{2}\left(\frac{i\pi_a\tau_a}{2f}\right)^2 + \frac{1}{6}\left(\frac{i\pi_a\tau_a}{2f}\right)^3 + \frac{1}{24}\left(\frac{i\pi_a\tau_a}{2f}\right)^4 + \mathcal{O}\left((\pi/f)^5\right),$$

which we use to calculate the expansion of the inner part of Σ, as given in Eq. (2.33),

$$\begin{aligned}U\Sigma_0 U &= \left(1+\frac{i\pi_a\tau_a}{2f}+\frac{1}{2}\left(\frac{i\pi_a\tau_a}{2f}\right)^2+\frac{1}{6}\left(\frac{i\pi_a\tau_a}{2f}\right)^3+\frac{1}{24}\left(\frac{i\pi_a\tau_a}{2f}\right)^4\right)\\ &\times\left(1+\frac{i\pi_a\tau_a}{2f}+\frac{1}{2}\left(\frac{i\pi_a\tau_a}{2f}\right)^2+\frac{1}{6}\left(\frac{i\pi_a\tau_a}{2f}\right)^3+\frac{1}{24}\left(\frac{i\pi_a\tau_a}{2f}\right)^4\right)+\mathcal{O}\left((\pi/f)^5\right)\\ &= 1+\frac{i\pi_a\tau_a}{f}+2\left(\frac{i\pi_a\tau_a}{2f}\right)^2+\frac{4}{3}\left(\frac{i\pi_a\tau_a}{2f}\right)^3+\frac{2}{3}\left(\frac{i\pi_a\tau_a}{2f}\right)^4+\mathcal{O}\left((\pi/f)^5\right).\end{aligned}$$

The symmetry of $\pi_a\pi_b$ means that

$$(\pi_a\tau_a)^2=\pi_a\pi_b\frac{1}{2}\{\tau_a,\tau_b\}=\pi_a\pi_a,\quad (\pi_a\tau_a)^3=\pi_a\pi_a\pi_b\tau_b,\quad (\pi_a\tau_a)^4=\pi_a\pi_a\pi_b\pi_b.$$

This gives us the expression

$$U\Sigma_0U=1+i\frac{\pi_a\tau_a}{f}-\frac{\pi_a^2}{2f^2}-i\frac{\pi_a^2\pi_b\tau_b}{6f^3}+\frac{\pi_a^2\pi_b^2}{24f^4}+\mathcal{O}\left((\pi/f)^5\right).$$

We combine this result with Eq. (2.35) to get an expression for Σ up to $\mathcal{O}\left((\pi/f)^5\right)$

$$\begin{aligned}\Sigma &= \left(\cos\frac{\alpha}{2}+i\tau_1\sin\frac{\alpha}{2}\right)\left(1+i\frac{\pi_a\tau_a}{f}-\frac{\pi_a^2}{2f^2}-i\frac{\pi_a^2\pi_b\tau_b}{6f^3}+\frac{\pi_a^2\pi_b^2}{24f^4}\right)\left(\cos\frac{\alpha}{2}+i\tau_1\sin\frac{\alpha}{2}\right)\\ &= \left(1+i\frac{\pi_a\tau_a}{f}-\frac{\pi_a^2}{2f^2}-i\frac{\pi_a^2\pi_b\tau_b}{6f^3}+\frac{\pi_a^2\pi_b^2}{24f^4}\right)\cos^2\frac{\alpha}{2}\\ &-\left(1+i\frac{\pi_a}{f}\tau_1\tau_a\tau_1-\frac{\pi_a^2}{2f^2}-i\frac{\pi_a^2\pi_b}{6f^3}\tau_1\tau_b\tau_1+\frac{\pi_a^2\pi_b^2}{24f^4}\right)\sin^2\frac{\alpha}{2}\\ &+i\left(2\tau_1+i\frac{\pi_a}{f}\{\tau_1,\tau_a\}-2\tau_1\frac{\pi_a^2}{2f^2}-i\frac{\pi_a^2\pi_b}{6f^3}\{\tau_1,\tau_b\}+2\tau_1\frac{\pi_a^2\pi_b^2}{24f^4}\right)\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}.\end{aligned}$$

Using trigonometric identities and the commutator,

$$\cos^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{2}=\cos\alpha,\quad 2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}=\sin\frac{\alpha}{2},\quad \tau_1\tau_a\tau_1=-\tau_a+2\delta_{1a}\tau_1,$$

the final expression of Σ to $\mathcal{O}\left((\pi/f)^5\right)$ is

$$\Sigma=\left(1-\frac{\pi_a^2}{2f^2}+\frac{\pi_a^2\pi_b^2}{24f^4}\right)(\cos\alpha+i\tau_1\sin\alpha)+\left(\frac{\pi_a}{f}-\frac{\pi_b^2\pi_a}{6f^3}\right)\left(i\tau_a-2i\delta_{a1}\tau_1\sin^2\frac{\alpha}{2}-\delta_{a1}\sin\alpha\right).\tag{2.36}$$

The kinetic term in the χPT Lagrangian is

$$\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger=\partial_\mu\Sigma\partial^\mu\Sigma^\dagger-i\left(\partial_\mu\Sigma\left[v^\mu,\Sigma^\dagger\right]-\text{h.c.}\right)-\left[v_\mu,\Sigma\right]\left[v_\mu,\Sigma^\dagger\right].\tag{2.37}$$

Using Eq. (2.36) we find the expansion of the constitutive parts of the kinetic term to be

$$\begin{aligned}\partial_\mu\Sigma &= \left(\frac{-1}{f^2}+\frac{\pi_b^2}{6f^4}\right)(\cos\alpha+i\tau_1\sin\alpha)(\pi_a\partial_\mu\pi_a)\\ &+\left(\frac{\partial_\mu\pi_a}{f}-\frac{\pi_b^2\partial_\mu\pi_a+2\pi_a\pi_b\partial_\mu\pi_b}{6f^3}\right)\left(i\tau_a-2i\delta_{a1}\tau_1\sin^2\frac{\alpha}{2}-\delta_{1a}\sin\alpha\right)\\ &=\left[\left(\frac{-1}{f^2}+\frac{\pi_b^2}{6f^4}\right)(\pi_a\partial_\mu\pi_a)\cos\alpha-\left(\frac{\partial_\mu\pi_1}{f}-\frac{\pi_b^2\partial_\mu\pi_1+2\pi_1\pi_b\partial_\mu\pi_b}{6f^3}\right)\sin\alpha\right]\\ &-\left[\left(\frac{-1}{f^2}+\frac{\pi_b^2}{6f^4}\right)(\pi_a\partial_\mu\pi_a)\sin\alpha-\left(\frac{\partial_\mu\pi_1}{f}-\frac{\pi_b^2\partial_\mu\pi_1+2\pi_1\pi_b\partial_\mu\pi_b}{6f^3}\right)2\sin^2\frac{\alpha}{2}\right]i\tau_1\\ &+\left(\frac{\partial_\mu\pi_a}{f}-\frac{\pi_b^2\partial_\mu\pi_a+2\pi_a\pi_b\partial_\mu\pi_b}{6f^3}\right)i\tau_a,\end{aligned}\tag{2.38}$$

and

$$\begin{aligned}
[v_\mu, \Sigma] &= \frac{1}{2}\mu_I \delta^0_\mu \left[\left(1 - \frac{\pi_a^2}{2f^2} + \frac{\pi_a^2\pi_b^2}{24f^4}\right) i \sin\alpha \, [\tau_3, \tau_1] + \left(\frac{\pi_a}{f} - \frac{\pi_b^2 \pi_a}{6f^3}\right) \left(i\,[\tau_a, \tau_3] - 2i\delta_{a1} \sin^2\frac{\alpha}{2}\,[\tau_3, \tau_1]\right)\right] \\
&= -\mu_I \delta^0_\mu \left\{ \left(1 - \frac{\pi_a^2}{2f^2} + \frac{\pi_a^2\pi_b^2}{24f^4}\right) \tau_2 \sin\alpha + \left(\frac{\pi_a}{f} - \frac{\pi_b^2\pi_a}{6f^3}\right) \left[(\delta_{a1}\tau_2 - \delta_{a2}\tau_1) - 2\delta_{a1}\tau_2 \sin^2\frac{\alpha}{2}\right]\right\} \\
&= -\mu_I \delta^0_\mu \left\{ \left[\left(1 - \frac{\pi_a^2}{2f^2} + \frac{\pi_a^2\pi_b^2}{24f^4}\right) \sin\alpha + \left(\frac{\pi_1}{f} - \frac{\pi_b^2\pi_1}{6f^3}\right)\cos\alpha \right] \tau_2 - \left(\frac{\pi_2}{f} - \frac{\pi_b^2\pi_2}{6f^3}\right)\tau_1 \right\}. \qquad (2.39)
\end{aligned}$$

Combining Eq. (2.38) and Eq. (2.39) gives the following terms [1]

$$\begin{aligned}
\mathrm{Tr}\{\partial_\mu \Sigma \partial^\mu \Sigma^\dagger\} &= \frac{2}{f^2}\partial_\mu\pi_a\partial^\mu\pi_a + \frac{2}{3f^4}\left[(\pi_a\partial_\mu\pi_a)(\pi_b\partial^\mu\pi_b) - (\pi_a\partial_\mu\pi_b)(\pi_b\partial^\mu\pi_a)\right], \\
-i\,\mathrm{Tr}\{\partial^\mu\Sigma\,[v_\mu, \Sigma^\dagger] - \text{h.c.}\} &= 4\mu_I\frac{\partial_0\pi_2}{f} + 8\mu_I\frac{\pi_3}{3f^3}\sin\alpha(\pi_2\partial_0\pi_3 - \pi_3\partial_0\pi_2)\sin\alpha \\
&\quad + \left(\frac{4\mu_I}{f^2}\cos\alpha - \frac{8\mu_I\pi_1}{3f^3}\sin\alpha - \frac{4\mu_I\pi_a\pi_a}{3f^4}\cos\alpha\right)(\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1), \\
-\mathrm{Tr}\{[v_\mu, \Sigma]\,[v^\mu, \Sigma^\dagger]\} &= {\mu_I}^2\left[2\sin^2\alpha + \left(\frac{2}{f} - \frac{4\pi_a\pi_a}{3f^3}\right)\pi_1\sin 2\alpha + \left(\frac{2}{f^2} - \frac{2\pi_a\pi_a}{3f^4}\right)\pi_a\pi_b k_{ab}\right], \\
\mathrm{Tr}\{\Sigma + \Sigma^\dagger\} &= 4\cos\alpha - \frac{4\pi_1}{f}\sin\alpha - \frac{2\pi_a\pi_a}{f^2}\cos\alpha + \frac{2\pi_1\pi_a\pi_a}{3f^3}\sin\alpha + \frac{(\pi_a\pi_a)^2}{6f^4}\cos\alpha,
\end{aligned}$$

where $k_{ab} = \delta_{a1}\delta_{b1}\cos 2\alpha + \delta_{a2}\delta_{b2}\cos^2\alpha - \delta_{a3}\delta_{b3}\sin^2\alpha$. If we write the Lagrangian as show in Eq. (2.34) as $\mathcal{L}_2 = \mathcal{L}_2^{(0)} + \mathcal{L}_2^{(1)} + \mathcal{L}_2^{(2)} + ...$, where $\mathcal{L}_2^{(n)}$ contains all terms of order $\mathcal{O}\left((\pi/f)^n\right)$, then the result of the series expansion is

$$\mathcal{L}_2^{(0)} = f^2\left(\bar{m}^2\cos\alpha + \frac{1}{2}\mu^2\sin^2\alpha\right), \qquad (2.40)$$

$$\mathcal{L}_2^{(1)} = f(\mu_I^2\cos\alpha - \bar{m}^2)\pi_1\sin\alpha + f\mu_I\partial_0\pi_2\sin\alpha, \qquad (2.41)$$

$$\mathcal{L}_2^{(2)} = \frac{1}{2}\partial_\mu\pi_a\partial^\mu\pi_a + \mu_I\cos\alpha\,(\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1) - \frac{1}{2}\bar{m}^2\pi_a\pi_a\cos\alpha + \frac{1}{2}\mu_I^2\pi_a\pi_b k_{ab}, \qquad (2.42)$$

$$\begin{aligned}
\mathcal{L}_2^{(3)} &= \frac{\pi_a\pi_a\pi_1}{6f}(\bar{m}^2\sin\alpha - 2{\mu_I}^2\sin 2\alpha) \\
&\quad - \frac{2\mu_I}{3f}\left[\pi_1(\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1) + \pi_3(\pi_3\partial_0\pi_2 - \pi_2\partial_0\pi_3)\right]\sin\alpha, \qquad (2.43)
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_2^{(4)} &= \frac{1}{6f^2}\left\{\frac{1}{4}\bar{m}^2(\pi_a\pi_a)^2\cos\alpha - \left[(\pi_a\pi_a)(\partial_\mu\pi_b\partial^\mu\pi_b) - (\pi_a\partial_\mu\pi_a)(\pi_b\partial^\mu\pi_b)\right]\right\} \\
&\quad - \frac{\mu_I\pi_a\pi_a}{3f^2}\left[(\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1)\cos\alpha + \frac{1}{2}\mu_I\pi_a\pi_b k_{ab}\right]. \qquad (2.44)
\end{aligned}$$

We have introduced $\bar{m}^2 = B_0(m_u + m_d)$, the bare pion mass, and assumed $2\delta m^2 = m_u - m_d = 0$.

2.4 Equation of motion and redundant terms

Changing the field parametrization that appear in the Lagrangian does not affect any of the physics, as it corresponds to a change of variables in the path integral [13, 15, 16]. However, a change of variables can result in new terms in the Lagrangian. As a result of this, terms that on the face of it appear independent may be redundant. These terms can be eliminated by using the classical equation of motion. In this section we show first the derivation of the equation of motion, then use this result to identify redundant terms which need not be included in the most general Lagrangian.

[1] The scripts used to aid the calculation of the Lagrangian is available at `https://github.com/martkjoh/prosjektopggave`

We derive the equation of motion for the leading order Lagrangian using the principle of least action. Choosing the parametrization $\Sigma = \exp(i\pi_a\tau_a)$, a variation $\pi_a \to \pi_a + \delta\pi_a$ results in a variation in Σ, $\delta\Sigma = i\tau_a\delta\pi_a\Sigma$. The variation of the leading order action,

$$S_2 = \int_\Omega \mathrm{d}^4x\, \mathcal{L}_2, \tag{2.45}$$

when varying π_a is

$$\delta S = \int_\Omega \mathrm{d}x\, \frac{f^2}{4} \operatorname{Tr}\big\{(\nabla_\mu\delta\Sigma)(\nabla^\mu\Sigma)^\dagger + (\nabla_\mu\Sigma)(\nabla^\mu\delta\Sigma)^\dagger + \chi\delta\Sigma^\dagger + \delta\Sigma\chi^\dagger\big\}.$$

Using the properties of the covariant derivative to do partial integration, as show in section B.2, as well as $\delta(\Sigma\Sigma^\dagger) = (\delta\Sigma)\Sigma^\dagger + \Sigma(\delta\Sigma)^\dagger = 0$, the variation of the action can be written

$$\begin{aligned}
\delta S &= \frac{f^2}{4}\int_\Omega \mathrm{d}x\, \operatorname{Tr}\big\{-\delta\Sigma\nabla^2\Sigma^\dagger + (\nabla^2\Sigma)(\Sigma^\dagger\delta\Sigma\Sigma^\dagger) - \chi(\Sigma^\dagger\delta\Sigma\Sigma^\dagger) + \delta\Sigma\chi^\dagger\big\} \\
&= \frac{f^2}{4}\int_\Omega \mathrm{d}x\, \operatorname{Tr}\big\{\delta\Sigma\Sigma^\dagger\left[(\nabla^2\Sigma)\Sigma^\dagger - \Sigma\nabla^2\Sigma^\dagger - \chi\Sigma^\dagger + \Sigma\chi^\dagger\right]\big\} \\
&= i\frac{f^2}{4}\int_\Omega \mathrm{d}x\, \operatorname{Tr}\big\{\tau_a\left[(\nabla^2\Sigma)\Sigma^\dagger - \Sigma\nabla^2\Sigma^\dagger - \chi\Sigma^\dagger + \Sigma\chi^\dagger\right]\big\}\delta\pi_a = 0.
\end{aligned}$$

As the variation is arbitrary, the equation of motion to leading order is

$$\operatorname{Tr}\big\{\tau_a\left[(\nabla^2\Sigma)\Sigma^\dagger - \Sigma\nabla^2\Sigma^\dagger - \chi\Sigma^\dagger + \Sigma\chi^\dagger\right]\big\} = 0. \tag{2.46}$$

This may be rewritten as a matrix equation. Using that

$$\operatorname{Tr}\big\{(\nabla_\mu\Sigma)\Sigma^\dagger\big\} = \operatorname{Tr}\big\{i\tau_a(\partial_\mu\pi_a)\Sigma\Sigma^\dagger\big\} - i\operatorname{Tr}\big\{[v_\mu,\Sigma]\Sigma^\dagger\big\} = 0,$$

we can see that $\operatorname{Tr}\big\{(\nabla^2\Sigma)\Sigma^\dagger - \Sigma\nabla^2\Sigma^\dagger\big\} = 0$, and the equation of motion may be written as

$$\mathcal{O}^{(2)}_{\mathrm{EOM}}(\Sigma) = (\nabla^2\Sigma)\Sigma^\dagger - \Sigma\nabla^2\Sigma^\dagger - \chi\Sigma^\dagger + \Sigma\chi^\dagger - \frac{1}{2}\operatorname{Tr}\big\{\chi\Sigma^\dagger - \Sigma\chi^\dagger\big\} = 0. \tag{2.47}$$

The next step in eliminating redundant terms is to change the parametrization of Σ by $\Sigma(x) \to \Sigma'(x)$. Here, $\Sigma(x) = e^{iS(x)}\Sigma'(x)$, $S(x) \in \mathfrak{su}(2)$. This change leads to a new Lagrange density, $\mathcal{L}[\Sigma] = \mathcal{L}[\Sigma'] + \Delta\mathcal{L}[\Sigma']$. We are free to choose $S(x)$, as long Σ' still obeys the required transformation properties. Any terms in the Lagrangian $\Delta\mathcal{L}$ due to a reparametrization can be neglected, as argued earlier. When demanding that Σ' obey the same symmetries as Σ, the most general transformation to second order in Weinberg's power counting scheme is [13]

$$S_2 = i\alpha_2\left[(\nabla^2\Sigma')\Sigma^\dagger - \Sigma'(\nabla^2\Sigma')^\dagger\right] + i\alpha_2\left[\chi\Sigma'^\dagger - \Sigma'\chi^\dagger - \frac{1}{2}\operatorname{Tr}\big\{\chi\Sigma'^\dagger - \Sigma'\chi^\dagger\big\}\right]. \tag{2.48}$$

α_1 and α_2 are arbitrary real numbers. As Eq. (2.48) is to second order, $\Delta\mathcal{L}$ is fourth order in Weinberg's power counting scheme. To leading order is given by

$$\begin{aligned}
\mathcal{L}_2\left[e^{iS_2}\Sigma'\right] &= \frac{f^2}{4}\operatorname{Tr}\big\{[\nabla_\mu(1+iS_2)\Sigma'][\nabla^\mu\Sigma'^\dagger(1-iS_2)]\big\} + \frac{f^2}{4}\operatorname{Tr}\big\{\chi\Sigma'^\dagger(1-iS_2) + (1+iS_2)\Sigma'\chi^\dagger\big\} \\
&= \mathcal{L}[\Sigma'] + i\frac{f^2}{4}\operatorname{Tr}\big\{[\nabla_\mu(S_2\Sigma')][\nabla^\mu\Sigma']^\dagger - [\nabla_\mu\Sigma'][\nabla^\mu(\Sigma'^\dagger S_2)]\big\} - i\frac{f^2}{4}\operatorname{Tr}\big\{\chi\Sigma'^\dagger S_2 - S_2\Sigma'\chi^\dagger\big\}
\end{aligned}$$

Using the properties of the covariant derivative, as described in section B.2, we may use the product rule and partial integration to write the difference between the two Lagrangians to fourth order as

$$\begin{aligned}
\Delta\mathcal{L}[\Sigma'] &= i\frac{f^2}{4}\operatorname{Tr}\big\{(\nabla_\mu S_2)(\Sigma'\nabla^\mu\Sigma'^\dagger - (\nabla^\mu\Sigma')\Sigma'^\dagger)\big\} - i\frac{f^2}{4}\operatorname{Tr}\big\{\chi\Sigma'^\dagger S_2 - S_2\Sigma'\chi^\dagger\big\} \\
&= i\frac{f^2}{4}\operatorname{Tr}\big\{S_2\left[\Sigma'^\dagger\nabla^2\Sigma' - (\nabla^2\Sigma')\Sigma'^\dagger - \chi\Sigma'^\dagger + \Sigma'\chi^\dagger\right]\big\}.
\end{aligned}$$

Using the equation of motion Eq. (2.47), and the fact that $\mathrm{Tr}\{S_2\} = 0$, this difference can be written as

$$\Delta\mathcal{L}[\Sigma'] = \frac{f^2}{4}\,\mathrm{Tr}\Big\{iS_2\mathcal{O}^{(2)}_{\mathrm{EOM}}(\Sigma')\Big\}. \tag{2.49}$$

Any term that can be written in the form of Eq. (2.49) for arbitrary $\alpha_1, \alpha_2 \in \mathbb{R}$ is redundant, as we argued earlier, and may therefore be discarded. $\Delta\mathcal{L}_2$ is of fourth order, and it can thus be used to remove terms from $\mathcal{L}_4$ or higher order.

2.5 Next to leading order Lagrangian

The next to leading order Lagrangian density is, assuming no external fields

$$\begin{aligned}\mathcal{L}_4 = {}& \frac{l_1}{4}\,\mathrm{Tr}\big\{\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\big\}^2 + \frac{l_2}{4}\,\mathrm{Tr}\big\{\nabla_\mu\Sigma(\nabla_\nu\Sigma)^\dagger\big\}\,\mathrm{Tr}\big\{\nabla^\mu\Sigma(\nabla^\nu\Sigma)^\dagger\big\} + \frac{l_3+h_1-h_3}{16}\,\mathrm{Tr}\big\{\chi\Sigma^\dagger+\Sigma\chi^\dagger\big\}^2 \\ & + \frac{l_4}{8}\,\mathrm{Tr}\big\{\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\big\}\,\mathrm{Tr}\big\{\chi\Sigma^\dagger+\Sigma\chi^\dagger\big\} + \frac{h_1-h_3-l_4-l_7}{16}\,\mathrm{Tr}\big\{\chi\Sigma^\dagger-\Sigma\chi^\dagger\big\}^2 + \frac{h_1+h_3-l_4}{4}\,\mathrm{Tr}\big\{\chi\chi^\dagger\big\} \\ & - \frac{h_1-h_3-l_4}{8}\,\mathrm{Tr}\Big\{\big(\chi\Sigma^\dagger\big)^2+\big(\Sigma\chi^\dagger\big)^2\Big\}\end{aligned} \tag{2.50}$$

To $\mathcal{L}_4$ to $\mathcal{O}\left((\pi/f)^3\right)$, we use the result from Eq. (2.38) and Eq. (2.39), up to and including $\mathcal{O}\left((\pi/f)^2\right)$, which gives

$$\begin{aligned}\mathrm{Tr}\big\{\partial_\mu\Sigma\partial_\nu\Sigma^\dagger\big\} &= 2\frac{\partial_\mu\pi_a\partial_\nu\pi_a}{f^2} \\ -i\,\mathrm{Tr}\big\{\partial_\mu\Sigma\left[v_\nu,\Sigma^\dagger\right] - \mathrm{h.c.}\big\} &= \frac{2\mu_I\pi_2}{f}(\delta^0_\mu\partial_\nu+\delta^0_\nu\partial_\mu)\sin\alpha + \frac{2\mu_I}{f^2}[\pi_1(\delta^0_\mu\partial_\nu+\delta^0_\nu\partial_\mu)\pi_2 - \pi_2(\delta^0_\mu\partial_\nu+\delta^0_\nu\partial_\mu)\pi_1]\cos\alpha \\ -\mathrm{Tr}\big\{[v_\nu,\Sigma]\left[v_\nu,\Sigma^\dagger\right]\big\} &= 2{\mu_I}^2\delta^0_\mu\delta^0_\nu\left[\sin^2\alpha + \frac{\pi_1}{f}\sin 2\alpha + \frac{\pi_a\pi_b}{f^2}k_{ab}\right].\end{aligned}$$

Using the form of the Pauli matrices, we can write χ as

$$\chi = 2B_0M = \bar{m}^2\mathbb{1} + \Delta m^2\tau_3,$$

where $\bar{m} = B_0(m_u+m_d)$, $\Delta m = B_0(m_u-m_d)/2$, which gives

$$\begin{aligned}\chi\Sigma^\dagger+\Sigma\chi^\dagger &= 2(\bar{m}^2+\Delta m^2\tau_3)\left[\left(1-\frac{\pi_a^2}{2f^2}\right)\cos\alpha - \frac{\pi_1}{f}\sin\alpha\right] \\ &\quad + 2\Delta m^2\left[\left(1-\frac{\pi_a^2}{2f^2}\right)\tau_2\sin\alpha + \frac{\pi_a}{f}\left(\delta_{a1}\tau_2\cos\alpha - \delta_{a2}\tau_1\right)\right], \\ \chi\Sigma^\dagger-\Sigma\chi^\dagger &= -2i\bar{m}^2\left[\left(1-\frac{\pi_a^2}{2f^2}\right)\tau_1\sin\alpha + \frac{\pi_a}{f}\left(\tau_a - 2\delta_{1a}\tau_1\sin^2\frac{\alpha}{2}\right)\right] - 2i\Delta m^2\frac{\pi_3}{f}.\end{aligned}$$

Combining these results gives all the terms in $\mathcal{L}_4$, to $\mathcal{O}\left((\pi/f)^3\right)$:

$$\begin{aligned}&\mathrm{Tr}\big\{\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\big\}^2 = \mathrm{Tr}\big\{\partial_\mu\Sigma\partial^\mu\Sigma^\dagger - i\left(\partial_\mu\Sigma\left[v^\mu,\Sigma^\dagger\right] - \mathrm{h.c.}\right) - [v_\mu,\Sigma]\left[v^\mu,\Sigma^\dagger\right]\big\}^2 \\ &= \frac{8\mu_I^2}{f^2}(\partial_\mu\pi_a\partial^\mu\pi_a + 2\partial_\mu\pi_2\partial^\mu\pi_2)\sin^2\alpha \\ &+ 16\mu_I^3\left[\frac{\partial_0\pi_2}{f}\sin^3\alpha + \frac{1}{f^2}\left(3\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1\right)\cos\alpha\sin^2\alpha\right] \\ &+ 4\mu_I^4\left\{\sin^4\alpha + 2\sin^2\alpha\left[\frac{\pi_1}{f}\sin 2\alpha + \frac{\pi_a\pi_b}{f^2}\left(k_{ab}+2\delta_{a1}\delta_{a2}\cos^2\alpha\right)\right]\right\},\end{aligned} \tag{2.51}$$

$$\begin{aligned}&\mathrm{Tr}\big\{\nabla_\mu\Sigma(\nabla_\nu\Sigma)^\dagger\big\}\,\mathrm{Tr}\big\{\nabla^\mu\Sigma(\nabla^\nu\Sigma)^\dagger\big\} \\ &= \frac{4{\mu_I}^2}{f^2}\left(\partial_0\pi_a\partial_0\pi_a + \partial_0\pi_2\partial_0\pi_2 + \partial_\mu\pi_2\partial^\mu\pi_2\right)\sin^2\alpha\end{aligned}$$

$$+ 16\mu_I^3 \left[\frac{\partial_0 \pi_2}{f} \sin^3\alpha + \frac{1}{f^2}\left(3\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1\right)\cos\alpha \sin^2\alpha \right]$$
$$+ 4\mu_I^4 \left\{ \sin^4\alpha + 2\sin^2\alpha \left[\frac{\pi_1}{f}\sin 2\alpha + \frac{\pi_a\pi_b}{f^2}\left(k_{ab} + 2\delta_{a1}\delta_{a2}\cos^2\alpha\right)\right]\right\}, \tag{2.52}$$

$$\mathrm{Tr}\{\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\}\,\mathrm{Tr}\{\chi\Sigma^\dagger + \Sigma\chi^\dagger\}$$
$$= 4\bar{m}^2\left\{ 2\frac{\partial_\mu\pi_a\partial^\mu\pi_a}{f^2}\cos\alpha + 4\mu_I\left[\frac{\partial_0\pi_2}{2f}\sin 2\alpha + \frac{1}{f^2}\left(\pi_1\partial_0\pi_2\cos 2\alpha - \pi_2\partial_0\pi_1\cos^2\alpha\right)\right]\right. \tag{2.53}$$
$$\left. + {\mu_I}^2\left[2\cos\alpha\sin^2\alpha - 2\frac{\pi_1}{f}\sin\alpha\left(2-3\sin^2\alpha\right) + \frac{1}{f^2}\left(\pi_1^2[2-9\sin^2\alpha] + \pi_2^2[2-3\sin^2\alpha] - 3\pi_3^2\sin^2\alpha\right)\cos\alpha\right]\right\},$$

$$\mathrm{Tr}\left\{\chi\Sigma^\dagger + \Sigma\chi^\dagger\right\}^2 = 16\bar{m}^4\left[\cos^2\alpha - \frac{\pi_1}{f}\sin 2\alpha + \frac{1}{f^2}\left(\pi_1^2\sin^2\alpha - \pi_a\pi_a\cos^2\alpha\right)\right], \tag{2.54}$$

$$\mathrm{Tr}\left\{\chi\Sigma^\dagger - \Sigma\chi^\dagger\right\}^2 = -16\left(\frac{\Delta m^2\pi_3}{f}\right)^2, \tag{2.55}$$

$$\mathrm{Tr}\left\{\left(\chi\Sigma^\dagger\right)^2 + \left(\Sigma\chi^\dagger\right)^2\right\}$$
$$= 4\bar{m}^4\left(\cos 2\alpha - 2\frac{\pi_1}{f}\sin 2\alpha - 2\frac{\pi_a\pi_a}{f^2}\cos^2\alpha + 2\frac{\pi_1^2}{f^2}\sin^2\alpha\right) + 4\Delta m^4\left(1 - 2\frac{\pi_3^2}{f^2}\right), \tag{2.56}$$

$$\mathrm{Tr}\{\chi^\dagger\chi\} = 2\bar{m}^4 + 2\Delta m^4. \tag{2.57}$$

The different terms of the NLO Lagrangian is

$$\mathcal{L}_4^{(0)} = (l_1+l_2)\mu_I^4\sin^4\alpha + (l_3+l_4)\bar{m}^2\cos^2\alpha + l_4\bar{m}{\mu_I}^2\cos\alpha\sin^2\alpha + (h_1-l_4)\bar{m}^4 + h_3\Delta m^4 \tag{2.58}$$

$$\mathcal{L}_4^{(1)} = 4\mu_I^3\frac{l_1+l_2}{f}\left(\partial_0\pi_2 + \mu_I\cos\alpha\pi_1\right)\sin^3\alpha - \frac{l_3+l_4}{f}\bar{m}^4\pi_1\sin 2\alpha$$
$$+ \bar{m}^2\frac{l_4}{f}\left[\mu_I\partial_0\pi_2\sin 2\alpha - {\mu_I}^2\pi_1\sin\alpha\left(3\sin^2\alpha - 2\right)\right] \tag{2.59}$$

$$\mathcal{L}_4^{(2)} = 2\mu_I^2\frac{l_1}{f^2}(\partial_\mu\pi_a\partial^\mu\pi_a + 2\partial_0\pi_2\partial_0\pi_2)\sin^2\alpha + {\mu_I}^2\frac{l_2}{f^2}\left(\partial_\mu\pi_2\partial^\mu\pi_2 + 2\partial_0\pi_a\partial_0\pi_a + 2\partial_0\pi_2\partial_0\pi_2\right)\sin^2\alpha$$
$$+ 2\frac{l_1+l_2}{f^2}\left[2\mu_I^3\left(3\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1\right)\cos\alpha + \mu_I^4\pi_a\pi_b\left(k_{ab} + 2\delta_{a1}\delta_{a2}\cos^2\alpha\right)\right]\sin^2\alpha$$
$$+ \frac{l_3+l_4}{f^2}\bar{m}^2\left(\pi_1^2\sin^2\alpha - \pi_a\pi_a\cos^2\alpha\right) + \frac{l_4}{f^2}\bar{m}^2\left[\partial_\mu\pi_a\partial^\mu\pi_a\cos\alpha + 4\mu_I\left(\pi_1\partial_0\pi_2\cos 2\alpha - \pi_2\partial_0\pi_1\cos^2\alpha\right)\right.$$
$$\left. + \frac{1}{2}{\mu_I}^2\left(\pi_1^2[2-9\sin^2\alpha] + \pi_2^2[2-3\sin^2\alpha] - 3\pi_3^2\sin^2\alpha\right)\cos\alpha\right] + \frac{l_7}{f^2}\Delta m^2\pi_3^2 \tag{2.60}$$

2.6 Propagator

We may write the quadratic part of the Lagrangian Eq. (2.42) as [2]

$$\mathcal{L}_2^{(2)} = \frac{1}{2}\sum_a \partial_\mu\pi_a\partial^\mu\pi_a + \frac{1}{2}m_{12}(\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1) - \frac{1}{2}\sum_a m_a^2\pi_a^2, \tag{2.61}$$

where

$$m_1^2 = \bar{m}^2\cos\alpha - \mu_I^2\cos 2\alpha, \tag{2.62}$$
$$m_2^2 = \bar{m}^2\cos\alpha - \mu_I^2\cos^2\alpha, \tag{2.63}$$
$$m_3^2 = \bar{m}^2\cos\alpha + \mu_I^2\sin^2\alpha, \tag{2.64}$$
$$m_{12} = 2\mu_I\cos\alpha. \tag{2.65}$$

[2] Summation over isospin index (a, b, c) will be explicit in this section.

The components of the Euler-Lagrange equations of this field are

$$\frac{\partial \mathcal{L}}{\partial \pi_a} = \frac{1}{2} m_{12}(\delta_{a1}\partial_0\pi_2 - \delta_{a2}\partial_0\pi_1) - m_a^2\pi_a, \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \pi_a)} = \partial^\mu \pi_a - \frac{1}{2} m_{12}\delta_0^\mu(\delta_{a1}\pi_2 - \delta_{a2}\pi_1).$$

This gives the equation of motion for the field

$$\partial^\mu\partial_\mu\pi_a + m_a^2\pi_a = m_{12}(\delta_{a1}\partial_0\pi_2 - \delta_{a2}\partial_0\pi_1). \tag{2.66}$$

The propagator of the pion field is defined by

$$\left[\delta_{ab}(\partial^\mu\partial_\mu + m_a^2) - m_{12}(\delta_{a1}\delta_{b2} - \delta_{a2}\delta_{b1})\partial_0\right] D_{bc}(x,x') = -i\delta(x-x')\delta_{ac}. \tag{2.67}$$

The momentum space propagator, as defined in the section B.1, fulfills (MOTSATT FORTEGN FRA FRI ENERGI; FIKS!!!)

$$-\left[\delta_{ab}(p^2 - m_a^2) + ip_0 m_{12}(\delta_{a1}\delta_{b2} - \delta_{a2}\delta_{b1})\right] D_{bc}(p) := D_{ab}^{-1} D_{bc}(p) = -i\delta_{ac},$$

where

$$D^{-1} = -\begin{pmatrix} p^2 - m_1^2 & ip_0 m_{12} & 0 \\ -ip_0 m_{12} & p^2 - m_2^2 & 0 \\ 0 & 0 & p^2 - m_3^2 \end{pmatrix}.$$

The spectrum of the particles is given by solving $\det\left(D^{-1}\right) = 0$ for p^0. With $p = (p_0, \vec{p})$ as the four momentum, this gives

$$\det\left(D^{-1}\right) = D_{33}^{-1}\left(D_{11}^{-1}D_{22}^{-1} + (D_{12}^{-1})^2\right) = -\left(p^2 - m_3^2\right)\left[\left(p^2 - m_1^2\right)\left(p^2 - m_2^2\right) - p_0^2 m_{12}^2\right] = 0,$$

This equation has the solutions

$$E_0^2 = |\vec{p}|^2 + m_3^2, \tag{2.68}$$

$$E_\pm^2 = |\vec{p}|^2 + \frac{1}{2}\left(m_1^2 + m_2^2 + m_{12}^2\right) \pm \frac{1}{2}\sqrt{4|\vec{p}|^2 m_{12}^2 + \left(m_1^2 + m_2^2 + m_{12}^2\right)^2 - 4m_1^2 m_2^2}. \tag{2.69}$$

These are the energies of three particles π_0, π_+ and π_-. π_0 is π_3, while $\pi_\pm$ are linear combinations of π_1 and π_2. The (tree-level) masses of these particles are found by setting $\vec{p} = 0$, i.e. the rest-frame energy, and are

$$m_0^2 = m_3^2, \tag{2.70}$$

$$m_\pm^2 = \frac{1}{2}\left[m_1^2 + m_2^2 + m_{12}^2\right] \pm \frac{1}{2}\sqrt{\left(m_1^2 + m_2^2 + m_{12}^2\right)^2 - 4m_1^2 m_2^2}. \tag{2.71}$$

Figure 2.1 shows the masses as functions of μ_I.

With these energies we can write the determinant of the inverse propagator as

$$\det\left(D^{-1}\right) = -(p_0^2 - E_0^2)(p_0^2 - E_+^2)(p_0^2 - E_-^2). \tag{2.72}$$

The propagator may then be obtained as described in section B.1,

$$\begin{aligned} D = i(D^{-1})^{-1} &= \frac{i}{\det(D^{-1})}\begin{pmatrix} D_{22}^{-1}D_{33}^{-1} & D_{12}^{-1}D_{33}^{-1} & 0 \\ -D_{12}^{-1}D_{33}^{-1} & D_{11}^{-1}D_{33}^{-1} & 0 \\ 0 & 0 & D_{11}^{-1}D_{22}^{-1} + (D_{12}^{-1})^2 \end{pmatrix} \\ &= i\begin{pmatrix} \frac{p^2 - m_2^2}{(p_0^2 - E_+^2)(p_0^2 - E_-^2)} & \frac{-ip_0 m_{12}}{(p_0^2 - E_+^2)(p_0^2 - E_-^2)} & 0 \\ \frac{ip_0 m_{12}}{(p_0^2 - E_+^2)(p_0^2 - E_-^2)} & \frac{p^2 - m_1^2}{(p_0^2 - E_+^2)(p_0^2 - E_-^2)} & 0 \\ 0 & 0 & \frac{1}{p_0^2 - E_0^2} \end{pmatrix}. \end{aligned} \tag{2.73}$$

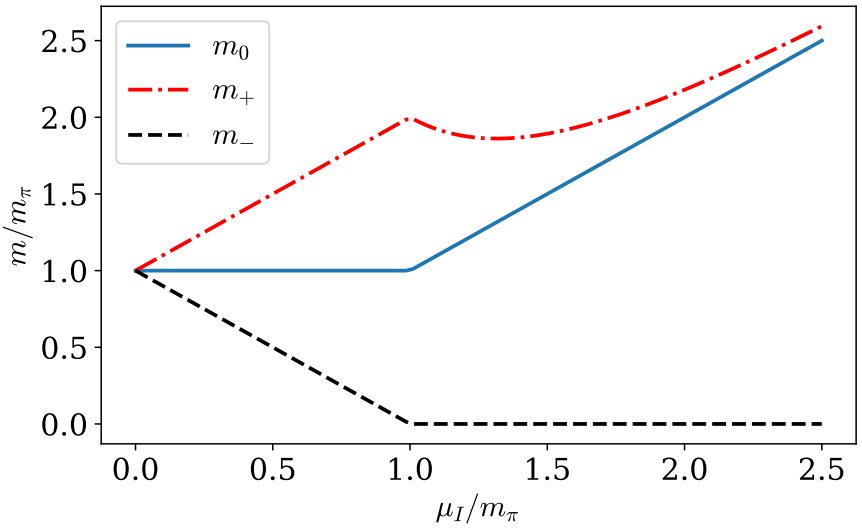


Figure 2.1: The masses of the three particles as functions of isosopin chemical potential.

Chapter 3

Equation of state for pions

3.1 Free energy at lowest order

The equation of state (EOS) relates the thermodynamic variables of as system. In this section, we will obtain the equation of state of the pions by calculating their free energy. We use the effective Lagrangian found in chapter 2 to find the leading-order contribution to one loop, and the next-to-leading order contribution at the tree-level, following the procedure used in [14, 17].[1] The free energy density of a homogenous system is

$$\mathcal{F} = -\frac{1}{V\beta} \ln Z. \tag{3.1}$$

Here, Z is the partition function, and V the volume of space. Using imaginary time formalism for thermal field theory, which is described in Appendix A, we find that the partition function is given by the path integral of the *Euclidean* Lagrange density, as shown in equation Eq. (A.14). In the zero temperature limit $\beta \to \infty$, the partition function is related to vacuum transition amplitude $Z_0 = Z[J = 0]$, as described in section 1.1, by a Wick rotation. The free energy density at zero temperature is therefore

$$\mathcal{F} = \frac{i}{VT} \ln Z_0, \tag{3.2}$$

where VT is the volume of space-time. This equals the effective potential in the ground state, which we found an explicit formula for in section 1.1, Eq. (1.36). We write $\mathcal{F} = \mathcal{F}^{(0)} + \mathcal{F}^{(1)} + \dots$, where $\mathcal{F}^{(n)}$ refers to the n-loop contributions to the free energy density.

Tree-level contribution

The tree-level contribution $\mathcal{F}^{(0)}$ is the classical potential, which is given by the static (π-independent) part of the Lagrangian. From Eq. (2.40) we have the leading order contribution,

$$\mathcal{F}_2^{(0)} = -\mathcal{L}_2^{(0)} = -f^2 \left(\bar{m}^2 \cos\alpha + \frac{1}{2}\mu^2 \sin^2\alpha \right), \tag{3.3}$$

where α parameterizes the ground state, which means that its value must minimize the free energy.

$$\frac{\partial}{\partial\alpha}\mathcal{F}_2^{(0)} = f^2 \left(\bar{m}^2 - \mu_I^2 \cos\alpha \right) \sin\alpha = 0.$$

This equation defines the relationship between the chemical potential μ_I, and the ground state parameter α, as illustrated in Figure 3.1. This gives the criterion

$$\alpha \in \{0, \pi\} \quad \text{or} \quad \cos\alpha = \frac{\bar{m}^2}{{\mu_I}^2}. \tag{3.4}$$

[1] Leading order and next-to-leading order, in this context, refers to Weinberg's power counting scheme.

As we see in the figure, $\alpha = \pi$ is a maximum, and thus unstable. This means that for all values $\mu_I^2 \leq \bar{m}^2$, we will have $\alpha = 0$, and the system will remain in its ground state.

In our discussion of the effective potential we also found that the ground state should minimize the classical potential, as shown by Eq. (1.30). This means that the linear part of the classical potential should vanish. The linear part of the classical potential is given by Eq. (2.41) to leading order, and reads $\mathcal{V}^{(1)} = f(\mu_I{}^2 \cos\alpha - \bar{m}^2) \sin\alpha\, \pi_1$, which vanishes given Eq. (3.4).

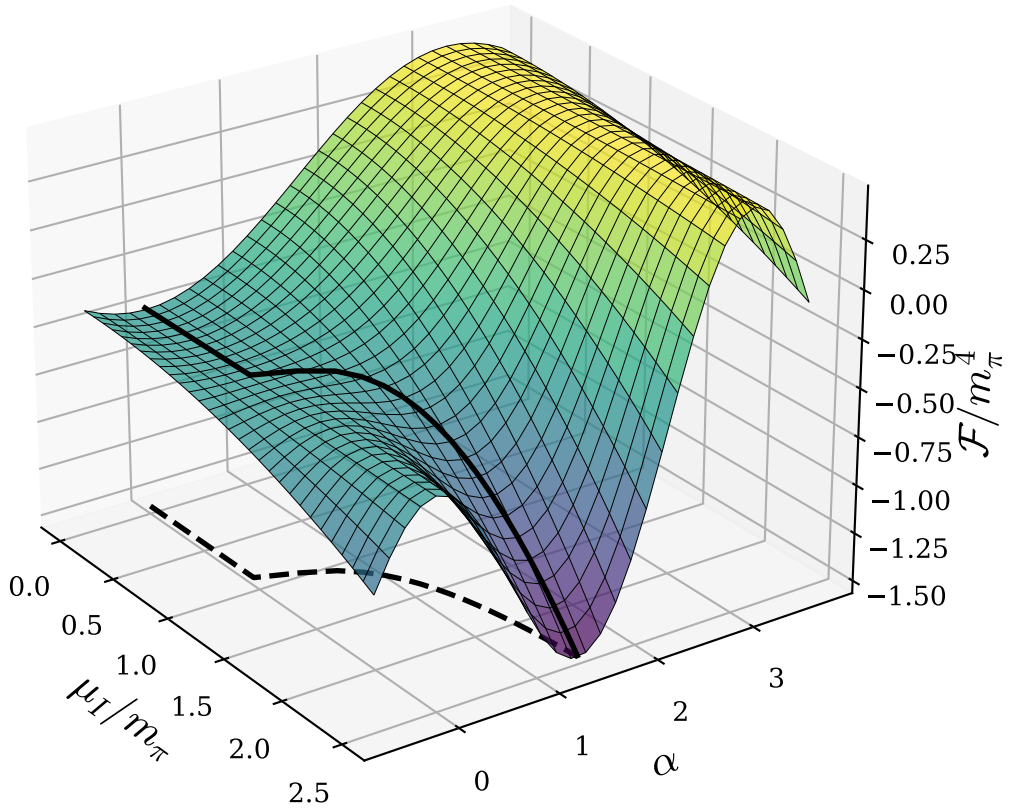


Figure 3.1: The surface gives free energy as a function of μ_I and α. α is the found by minimizing $\mathcal{F}$ for a given μ_I. This leads to a curve across the free energy surface, as show in the plot.

One-loop contribution

The one loop contribution to the free energy density is

$$\mathcal{F}^{(1)} = -\frac{i}{VT}\frac{1}{2}\,\mathrm{Tr}\left\{\ln\left(-\frac{\delta^2 S[\pi=0]}{\delta\pi_a(x)\,\delta\pi_b(y)}\right)\right\}. \tag{3.5}$$

This can be evaluated using the rules for functional differentiation given in section B.4. To leading order,

$$\frac{\delta^2 S[\pi=0]}{\delta\pi_a(x)\,\delta\pi_b(y)} = \frac{\delta^2}{\delta\pi_a(x)\,\delta\pi_b(y)}\int \mathrm{d}^4x\, \mathcal{L}_2^{(2)} = D_x^{-1}\delta(x-y). \tag{3.6}$$

Here, $\mathcal{L}_2^{(2)}$ is the quadratic part of the Lagrangian, as given in Eq. (2.61), and D_x^{-1} is the corresponding inverse propagator of the pion fields,

$$D_x^{-1} = -\left[\delta_{ab}(\partial_x^\mu \partial_{x,\mu} + m_a^2) - m_{12}(\delta_{a1}\delta_{b2} - \delta_{a2}\delta_{b1})\partial_{x,0}\right] \tag{3.7}$$

The inverse propagator is a matrix, which means that the determinant in Eq. (3.5) i both a matrix determinant, over the three pion indices, as well as a functional determinant. In section 2.6 we found the matrix part of the determinant in momentum space, which we can write using the dispersion relations of the pion fields

$$\det\left(-D^{-1}\right) = \det\left(-p_0^2 + E_0^2\right)\det\left(-p_0^2 + E_+^2\right)\det\left(-p_0^2 + E_-^2\right). \tag{3.8}$$

These dispersion relations are functions of the three-momentum $\vec{p}$, and are given in Eqs. (2.68) and (2.69). The functional determinant can therefore be evaluated as

$$\begin{aligned}\mathrm{Tr}\left\{\ln\left(-\frac{\delta^2 S[\pi=0]}{\delta\pi_a(x)\,\delta\pi_b(y)}\right)\right\} &= \ln\det\left(-p_0^2 + E_0^2\right) + \ln\det\left(-p_0^2 + E_+^2\right) + \ln\det\left(-p_0^2 + E_-^2\right)\\ &= \mathrm{Tr}\left\{\ln\left(-p_0^2 + E_0^2\right) + \ln\left(-p_0^2 + E_+^2\right) + \ln\left(-p_0^2 + E_-^2\right)\right\}\\ &= (VT)\int\frac{\mathrm{d}^4p}{(2\pi)^4}\left[\ln\left(-p_0^2 + E_0^2\right) + \ln\left(-p_0^2 + E_+^2\right) + \ln\left(-p_0^2 + E_-^2\right)\right],\end{aligned} \tag{3.9}$$

where we have used the identity $\ln\det M = \mathrm{Tr}\ln M$. These terms all have the form

$$I = \int\frac{\mathrm{d}^4p}{(2\pi)^2}\ln\left(-p_0^2 + E^2\right), \tag{3.10}$$

where E is some function of the 3-momentum $\vec{p}$, but not p_0. We use the trick

$$\frac{\partial}{\partial\alpha}\left(-p_0^2 + E^2\right)^{-\alpha}\Big|_{\alpha=0} = \frac{\partial}{\partial\alpha}\exp\left[-\alpha\ln\left(-p_0^2 + E^2\right)\right]\Big|_{\alpha=0} = \ln\left(-p_0^2 + E^2\right), \tag{3.11}$$

and then perform a Wick-rotation of the p_0-integral to write the integral on the form

$$I = i\frac{\partial}{\partial\alpha}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\left(p_0^2 + E^2\right)^{-\alpha}\Big|_{\alpha=0}, \tag{3.12}$$

where p now is a Euclidean four-vector. The p_0 integral equals $\Phi_1(E, 1, \alpha)$, as defined in Eq. (A.33). The result is therefore given by Eq. (A.40),

$$\int\frac{\mathrm{d}p_0}{2\pi}(p_0^2 + E)^{-\alpha} = \frac{E^{1-2\alpha}}{\sqrt{4\pi}}\frac{\Gamma(\alpha - \frac{1}{2})}{\Gamma(\alpha)}. \tag{3.13}$$

The derivative of the Gamma function is $\Gamma'(\alpha) = \psi(\alpha)\Gamma(\alpha)$, where $\psi(\alpha)$ is the digamma function. Using

$$\frac{\partial}{\partial\alpha}\frac{\Gamma(\alpha - \frac{1}{2})}{\Gamma(\alpha)}\Big|_{\alpha=0} = \Gamma\left(\alpha - \frac{1}{2}\right)\frac{\psi(\alpha - \frac{1}{2}) - \psi(\alpha)}{\Gamma(\alpha)}\Big|_{\alpha=0} = \sqrt{4\pi}, \tag{3.14}$$

$$\frac{\Gamma(\alpha - \frac{1}{2})}{\Gamma(\alpha)}\Big|_{\alpha=0} = 0, \tag{3.15}$$

we get

$$I = i\int\frac{\mathrm{d}^3p}{(2\pi)^3}E. \tag{3.16}$$

We see that the result is what we would expect physically, the total energy is the integral of the energy of each mode. This also agrees with the result from Appendix A in low temperature limit $\beta \to \infty$. This results in

$$\mathcal{F}^{(1)} = \frac{1}{2}\left[\int\frac{\mathrm{d}^3p}{(2\pi)^3}E_0 + \int\frac{\mathrm{d}^3p}{(2\pi)^3}(E_+ + E_-)\right] = \mathcal{F}^{(1)}_{\pi_0} + \mathcal{F}^{(1)}_{\pi_\pm}. \tag{3.17}$$

The first integral is identical to what we find for a free field in section A.3, in the zero temperature limit $\beta \to \infty$. These terms are all divergent, and must be regularized. We will use dimensional regularization, in which the integral is generalized to d dimensions, and the $\overline{\mathrm{MS}}$-scheme, as described in section A.3. Using the result for a free field Eq. (A.47), we get

$$\mathcal{F}^{(1)}_{\pi_0} = -\mu^{-2\epsilon}\frac{1}{4}\frac{m_3^4}{(4\pi)^2}\left(\frac{1}{\epsilon} + \frac{3}{2} + \ln\frac{\tilde{\mu}^2}{m_3^2}\right), \tag{3.18}$$

where μ is the renormalization scale, a parameter with mass-dimension 1 which is introduced to ensure the action integral remains dimensionless during dimensional regularization. $\tilde{\mu}$ is a related to μ as described in Eq. (A.46).

The contribution to the free energy from the π_+ and π_- particles is more complicated, as the dispersion relation is given by

$$E_\pm = \sqrt{|\vec{p}|^2 + \frac{1}{2}\left(m_1^2 + m_2^2 + m_{12}^2\right) \pm \frac{1}{2}\sqrt{4|\vec{p}|^2 m_{12}^2 + \left(m_1^2 + m_2^2 + m_{12}^2\right)^2 - 4m_1^2 m_2^2}}. \tag{3.19}$$

This is not an integral we can easily do in dimensional regularization. Instead, we will seek a function $f(|\vec{p}|)$ with the same UV-behavior, that is behavior for large $\vec{p}$, as $E_+ + E_-$. If we then add $0 = f(|\vec{p}|) - f(|\vec{p}|)$ to the integrand, we can isolate the divergent behavior

$$\mathcal{F}^{(1)}_{\pi_\pm} = \frac{1}{2}\int \frac{\mathrm{d}^3 p}{(2\pi)^3}[E_+ + E_- + f(|\vec{p}|) - f(|\vec{p}|)] = \mathcal{F}^{(1)}_{\mathrm{fin},\pi_\pm} + \mathcal{F}^{(1)}_{\mathrm{div},\pi_\pm}. \tag{3.20}$$

This results in a finite integral,

$$\mathcal{F}^{(1)}_{\mathrm{fin},\pi_\pm} = \frac{1}{2}\int \frac{\mathrm{d}^3 p}{(2\pi)^3}[E_+ + E_- - f(|\vec{p}|)], \tag{3.21}$$

which we can evaluate numerically, and isolate the divergence to

$$\mathcal{F}^{(1)}_{\mathrm{div},\pi_\pm} = \frac{1}{2}\int \frac{\mathrm{d}^3 p}{(2\pi)^3} f(|\vec{p}|), \tag{3.22}$$

which we hopefully will be able to do in dimensional regularization. We can explore the UV-behavior of $E_+ + E_-$ by expanding it in powers of $1/|\vec{p}|$,

$$\begin{aligned} E_+ + E_- &= 2|\vec{p}| + \frac{m_{12} + 2(m_1^2 + m_2^2)}{4}|\vec{p}|^{-1} - \frac{m_{12}^4 + 4m_{12}^2(m_1^2 + m_2^2) + 8(m_1^4 + m_2^4)}{64}|\vec{p}|^{-3} + \mathcal{O}\left(|\vec{p}|^{-5}\right) \\ &= a_1|\vec{p}| + a_2|\vec{p}|^{-1} + a_3|\vec{p}|^{-3} + \mathcal{O}\left(|\vec{p}|^{-5}\right). \end{aligned} \tag{3.23}$$

We have defined the new constants a_i for brevity of notation. As

$$\int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3}|\vec{p}|^n = C\int_0^\infty \mathrm{d}p\, p^{2+n} \tag{3.24}$$

is UV divergent for $n \geq -3$, f need to match the expansion of $E_+ + E_-$ up to and including $\mathcal{O}\left(|\vec{p}|^{-3}\right)$ for $\mathcal{F}^{(1)}_{\mathrm{fin},\pi_\pm}$ to be finite. The most obvious choice for f is

$$f(|\vec{p}|) = a_1|\vec{p}| + a_2|\vec{p}|^{-1} + a_3|\vec{p}|^{-3}. \tag{3.25}$$

However, this introduces a new problem. f has the same UV-behavior as $E_+ + E_-$, but the last term diverges in the IR, that is for low $|\vec{p}|$. This can be amended by introducing a mass term. Let

$$|\vec{p}|^{-3} = \left(\frac{1}{\sqrt{|\vec{p}|^2}}\right)^3 \longrightarrow \left(\frac{1}{\sqrt{|\vec{p}|^2 + m^2}}\right)^3. \tag{3.26}$$

For $|\vec{p}|^2 \to \infty$, this term is asymptotic to $|\vec{p}|^{-3}$, so it retains its UV behavior. However, for $|\vec{p}| \to 0$, it now approaches m^{-3}, so the IR-divergence is gone. The cost of this technique is that we have introduced an arbitrary mass parameter. Any final result must thus be independent of the value of m to be acceptable.

We will instead regularize the integral by defining $E_i = \sqrt{|\vec{p}|^2 + \tilde{m}_i^2}$, and $\tilde{m}_i^2 = m_i^2 + \frac{1}{4}m_{12}^2$. Using the definition of the mass parameters, Eqs. (2.62) to (2.65), we get

$$m_3^2 = \bar{m}^2\cos\alpha + \mu_I^2\sin^2\alpha, \tag{3.27}$$

$$\tilde{m}_1^2 = m_1^2 + \mu^2\cos\alpha^2 = \bar{m}^2\cos\alpha + \mu_I^2\sin^2\alpha = m_3^2 \tag{3.28}$$

$$\tilde{m}_2^2 = m_2^2 + \mu^2\cos\alpha^2 = \bar{m}^2\cos\alpha. \tag{3.29}$$

Finally, we define $f(|\vec{p}|) = E_1 + E_2$ which differ from $E_+ + E_-$ by $\mathcal{O}\left(|\vec{p}|^{-5}\right)$, and is well-behaved in the IR. This leads to a divergent integral the same form as in the case of a free scalar. Thus, in the $\overline{\text{MS}}$-scheme,

$$\mathcal{F}^{(1)}_{\text{div},\pi_\pm} = -\mu^{-2\epsilon}\frac{1}{4}\frac{\tilde{m}_1^4}{(4\pi)^2}\left(\frac{1}{\epsilon}+\frac{3}{2}+\ln\frac{\tilde{\mu}^2}{\tilde{m}_1^2}\right) - \mu^{-2\epsilon}\frac{1}{4}\frac{\tilde{m}_2^4}{(4\pi)^2}\left(\frac{1}{\epsilon}+\frac{3}{2}+\ln\frac{\tilde{\mu}^2}{\tilde{m}_2^2}\right)+\mathcal{O}(\epsilon). \tag{3.30}$$

We define

$$\mathcal{F}^{(1)}_{\text{fin},\pi_\pm} = \frac{1}{2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}(E_+ + E_- - E_1 - E_2), \tag{3.31}$$

which is a finite integral. The total one-loop contribution is then, using Eqs. (3.28) and (3.29),

$$\mathcal{F}^{(2)} = \mathcal{F}^{(1)}_{\text{fin},\pi_\pm} - \mu^{-2\epsilon}\frac{1}{2}\frac{1}{(4\pi)^2}\left[\left(\frac{1}{\epsilon}+\frac{3}{2}+\ln\frac{\tilde{\mu}^2}{m_3^2}\right)m_3^4 + \frac{1}{2}\left(\frac{1}{\epsilon}+\frac{3}{2}+\ln\frac{\tilde{\mu}^2}{\tilde{m}_2^2}\right)\tilde{m}_2^4\right] \tag{3.32}$$

3.2 Next-to-leading order and renormalization

We have now regularized the divergences, so they can be handled in a well-defined way. However, they are still there. To get rid of them, we need to use renormalization. The power counting scheme used when constructing the Effective Lagrangians ensures that all terms in $\mathcal{L}_{2n}$ is of order p^{2n} in the pion momenta.[2] The tree level free energy from $\mathcal{L}_{2n}$ is thus of order p^{2n}. The m-loop correction to the tree level result is then suppressed by p^{2m} [18, 12]. Our one-loop calculation using $\mathcal{L}_2$ therefore contains divergences of order p^4. Since $\mathcal{L}_4$ is, by construction, the most general possible Lagrangian at order p^4, it contains coupling constants which can be renormalized to absorb all these divergences.

The renormalized coupling constants in $\mathcal{L}_4$ have been calculated for $\mu_I = 0$ [18]. They are independent of μ_I, and we can therefore use them in this calculation. The renormalized coupling constants in the $\overline{\text{MS}}$-scheme are related to the bare couplings through

$$l_i = l_i^r - \mu^{-2\epsilon}\frac{1}{2}\frac{\gamma_i}{(4\pi)^2}\left(\frac{1}{\epsilon}+1\right), \qquad i\in\{1,...7\} \tag{3.33}$$

$$h_i = h_i^r - \mu^{-2\epsilon}\frac{1}{2}\frac{\delta_i}{(4\pi)^2}\left(\frac{1}{\epsilon}+1\right), \qquad i\in\{1,...3\}. \tag{3.34}$$

Here, γ_i and δ_i are numerical constants that are used to match the divergences. The relevant terms are[3]

$$\gamma_1 = \frac{1}{3}, \quad \gamma_2 = \frac{2}{3}, \quad \gamma_3 = -\frac{1}{2}, \quad \gamma_4 = 2, \tag{3.35}$$

$$\delta_1 = 2, \quad \delta_3 = 0. \tag{3.36}$$

The bare coupling constants, though infinite, are independent of our renormalization scale μ. From this we obtain the renormalization group equations for the running coupling constants,

$$\mu\frac{\mathrm{d}l_i^r}{\mathrm{d}\mu} = -\mu^{-2\epsilon}\frac{\gamma_i}{(4\pi)^2}, \quad \mu\frac{\mathrm{d}h_i^r}{\mathrm{d}\mu} = -\mu^{-2\epsilon}\frac{\delta_i}{(4\pi)^2}. \tag{3.37}$$

These have the general solutions

$$l_i^r = \frac{1}{2}\mu^{-2\epsilon}\frac{\gamma_i}{(4\pi)^2}\left(\bar{l}_i - \ln\frac{\tilde{\mu}^2}{M^2}\right), \quad h_i^r = \frac{1}{2}\mu^{-2\epsilon}\frac{\gamma_i}{(4\pi)^2}\left(\bar{h}_i - \ln\frac{\tilde{\mu}^2}{M^2}\right), \tag{3.38}$$

where $\bar{l}_i$ and $\bar{h}_i$ are the values of the coupling constants (times a constant) measured at the energy M. This only applies if the numerical constants γ_i/δ_i are non-zero. If they are zero, then the renormalized constant is not running, and instead equal to its measured value at all energies. The bare couplings are thus given by

$$l_i = \mu^{-2\epsilon}\frac{1}{2}\frac{\gamma_i}{(4\pi)^2}\left(\bar{l}_i - 1 - \frac{1}{\epsilon} - \ln\frac{\tilde{\mu}^2}{M^2}\right), \quad h_i = \mu^{-2\epsilon}\frac{1}{2}\frac{\delta_i}{(4\pi)^2}\left(\bar{h}_i - 1 - \frac{1}{\epsilon} - \ln\frac{\tilde{\mu}^2}{M^2}\right) \tag{3.39}$$

[2] Remember that the bare pion mass $\bar{m} = B_0(m_u + m_d)$ is considered to be of order p^2.

[3] Some authors [17, 19] instead use $h_1' = h_1 - l_4$, with a corresponding $\delta_1' = \delta_1 - \gamma_1 = 0$.

The next-to-leading contribution to the free energy at tree-level is $\mathcal{F}_4^0 = -\mathcal{L}_4^{(0)}$, which is given by Eq. (2.58). When substituting Eq. (3.39) into the bare couplings, we get

$$\begin{aligned}\mathcal{F}_4^{(0)} &= -(l_1+l_2)\mu_I^4\sin^4\alpha - (l_3+l_4)\bar{m}^4\cos^2\alpha - l_4\bar{m}^2{\mu_I}^2\cos\alpha\sin^2\alpha - (h_1-l_4)\bar{m}^4 - h_3\Delta m^4 \\ &= -\mu^{-2\epsilon}\frac{1}{2}\frac{1}{(4\pi)^2}\left[\frac{1}{3}\left(\bar{l}_1+2\bar{l}_2-3\right)\mu_I^4\sin^4\alpha + \frac{1}{2}\left(-\bar{l}_3+4\bar{l}_4-3\right)\bar{m}^4\cos^2\alpha\right. \\ &\qquad + 2\left(\bar{l}_4-1\right)\bar{m}^2\mu_I^2\cos\alpha\sin^2\alpha + 2(\bar{l}_4-\bar{h}_1)\bar{m}^4 + \bar{h}_3\Delta m^4 \\ &\qquad \left. - \left(\frac{1}{\epsilon}+\ln\frac{\tilde{\mu}^2}{M^2}\right)\left(\mu_I^4\sin^4\alpha + \frac{3}{2}\bar{m}^4\cos^2\alpha + 2\bar{m}^2\mu_I^2\cos\alpha\sin^2\alpha\right)\right]\end{aligned}$$

Notice that the term proportional to ϵ^{-1} cancel exactly with the divergent term from $\mathcal{F}^{(2)}$, as we expected. Adding all the contribution to the free energy density, and taking the limit $\epsilon \to 0$, we get the next-to-leading order free energy density,

$$\begin{aligned}\mathcal{F}_{\text{NLO}} = -f^2\left(\bar{m}^2\cos\alpha + \frac{1}{2}\mu_I^2\sin^2\alpha\right) + \mathcal{F}^{(1)}_{\text{fin},\pi_\pm} - \frac{1}{2}\frac{1}{(4\pi)^2}\left[\frac{1}{3}\left(\bar{l}_1+2\bar{l}_2+\frac{3}{2}+3\ln\frac{M^2}{m_3^2}\right)\mu_I^4\sin^4\alpha\right. \\ \left. + \frac{1}{2}\left(-\bar{l}_3+4\bar{l}_4+\frac{3}{2}+2\ln\frac{M^2}{m_3^2}+\ln\frac{M^2}{\tilde{m}_2^2}\right)\bar{m}^4\cos^2\alpha + 2\left(\bar{l}_4-\frac{1}{2}+\ln\frac{M^2}{m_3^2}\right)\bar{m}^2\mu_I^2\cos\alpha\sin^2\alpha\right]. \quad (3.40)\end{aligned}$$

We have dropped the terms proportional to $\bar{l}_4 - \bar{h}_1$ and $\bar{h}_3$, as they only add an unobservable constant value to the free energy.

Parameters, and consistent expansion

The coupling constants are free parameters, and can therefore not be calculated from first principles, but must be measured. The values for the pion mass and pion decay constants are (HVORFOR?)

$$m_\pi = 131\,\text{MeV}, \quad f_\pi = \frac{1}{\sqrt{2}}128\,\text{MeV}. \quad (3.41)$$

This is the physical mass, m_π, which is defined as the pole of the propagator and thus the zero of the inverse propagator,

$$D^{-1}(p^2 = m_\pi^2) = 0, \quad (3.42)$$

This relates it to the bare mass $\bar{m}$. We found, in Eq. (2.62), that $m_\pi^2 = m_3^2(\mu_I = 0) = \bar{m}^2$ to leading order. Similarly, $f_\pi = f$ to leading order. However, in any NLO results we need $\bar{m}^2$ and f to NLO. This is given by [18]

$$m_\pi^2 = \bar{m}^2 + \frac{\bar{l}_3}{2(4\pi)^2}\frac{\bar{m}^4}{f^2}, \quad (3.43)$$

$$f_\pi^2 = f^2 + \frac{2\bar{l}_4}{(4\pi)^2}\frac{\bar{m}^2}{f^2} \quad (3.44)$$

All results in this text are given in units of m_π.

Table 3.1: The measured values and corresponding uncertainties of the relevant coupling constants, measured at $M = m_\pi$. (ELLER $M = \bar{m}$??)

	value	uncertainty	source
$\bar{l}_1$	-0.4	± 0.6	[20]
$\bar{l}_2$	4.3	± 0.1	[20]
$\bar{l}_3$	2.9	± 2.4	[18]
$\bar{l}_4$	4.4	± 0.2	[20]

The values for the coupling constants used in this text are given in Table 3.1. The constants $\bar{l}_1$, $\bar{l}_2$ and $\bar{l}_4$ are estimated using data from $\pi\pi$-scattering [20], while $\bar{l}_3$ is estimated using three flavor chiral perturbation

theory [18]. These are the same values as those used in [17]. Together with Eqs. (3.43) and (3.44), the NLO results for the bare mass and decay constant are

$$\bar{m}/m_\pi = 1.01136, \tag{3.45}$$
$$f/m_\pi = 0.64835. \tag{3.46}$$

In section 3.1, we found a relationship between α and μ_i, using the lowest order estimate of $\mathcal{F}$, given in Eq. (3.4). To calculate any thermodynamic quantities to leading order, at the tree-level, we must use this result. When calculating higher order corrections to the free energy, we must make sure to calculate this and other relationship to the same order, consistently. As we have seen earlier, when replacing the action by $S[\varphi] \to g^{-1}S[\varphi]$, the L-loop contribution is proportional to g^{L-1}. In Weinberg's power counting scheme, when replace $p \to tp$, and $m \to t^2 m$, the nth term in the expansion is proportional to t^{2n}. We can therefore expand the free energy as

$$\mathcal{F} = t^2 g^{-1}\mathcal{F}_2^{(0)} + t^2\mathcal{F}_2^{(1)} + t^4 g^{-1}\mathcal{F}_4^{(0)} + ... \tag{3.47}$$

We consider terms where $m = L + n$ to be of the same order. This expansion can be written as

$$\mathcal{F} = \sum_{m=0}^{\infty} \sum_{n+L=m} t^{2n} g^{L-1} \mathcal{F}_{2n}^{(L)}. \tag{3.48}$$

If we now define

$$\tilde{\mathcal{F}}_m = \sum_{n+L=m} t^{2n} g^{L-1} \mathcal{F}_{2n}^{(L)}, \tag{3.49}$$

then scale $t \to \sqrt{s}\, t$ and $g \to sg$, where s is some real number, then $t^{2n}g^{L-1}$ scales as $s^{n+L-1} = s^{m-1}$. We can therefore expand the free energy as

$$s\mathcal{F} = \sum_{m=0}^{\infty} \tilde{\mathcal{F}}_m\, s^m. \tag{3.50}$$

The same expansion holds for all quantities we calculate from this theory. Let α' be the solution to

$$\left.\frac{\partial \mathcal{F}}{\partial \alpha}\right|_{\alpha=\alpha'} = 0 \tag{3.51}$$

to all orders. We expand this solution in s,

$$\alpha' = \alpha_0 + \alpha_1 s + \dots. \tag{3.52}$$

Combining this, we get

$$\begin{aligned} 0 &= \frac{\partial}{\partial \alpha}\left[s^{-1}\tilde{\mathcal{F}}_0 + \tilde{\mathcal{F}}_1 + \mathcal{O}\left(s^1\right)\right]\Big|_{\alpha=\alpha_0+\alpha_1 s+\mathcal{O}(s^2)} \\ &= s^{-1}\left[\tilde{\mathcal{F}}_0'(\alpha_0) + (\alpha'-\alpha_0)\tilde{\mathcal{F}}_0''(\alpha_0) + \mathcal{O}\left(s^2\right)\right] + \tilde{\mathcal{F}}_1'(\alpha_0) + \mathcal{O}\left(s^1\right) \\ &= s^{-1}\tilde{\mathcal{F}}_1'(\alpha_0) + s^0\left[\alpha_1\tilde{\mathcal{F}}_0''(\alpha_0) + \tilde{\mathcal{F}}_1'(\alpha_0)\right] + \mathcal{O}\left(s^1\right) \end{aligned}$$

Here, the prime indicates partial derivatives with respect to α. This equality has to hold term for term. After setting $s = g = t = 1$, we get

$$\left.\frac{\partial \tilde{\mathcal{F}}_0}{\partial \alpha}\right|_{\alpha=\alpha_0} = 0, \quad \tilde{\mathcal{F}}_0 = \mathcal{F}_2^{(0)},$$

which is what we have used as the leading-order result. The next to leading order result is

$$\alpha_1 = -\frac{\tilde{\mathcal{F}}_1'(\alpha_0)}{\tilde{\mathcal{F}}_0''(\alpha_0)}, \quad \tilde{\mathcal{F}}_1 = \mathcal{F}_4^{(0)} + \mathcal{F}_2^{(1)}. \tag{3.53}$$

The next to leading order results for the free energy and α' are

$$\mathcal{F}_{\text{NLO}} = \tilde{\mathcal{F}}_0 + \tilde{\mathcal{F}}_1, \quad \alpha_{\text{NLO}} = \alpha_0 + \alpha_1. \tag{3.54}$$

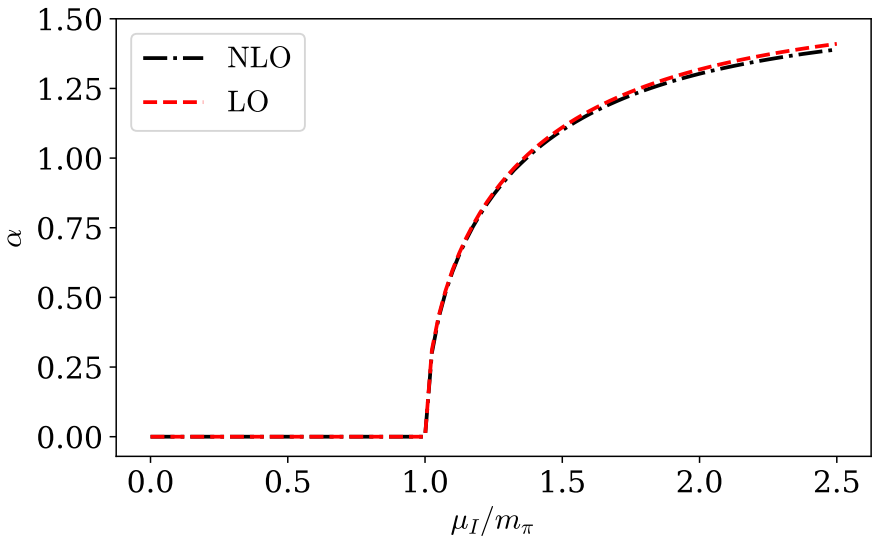


Figure 3.2: The leading order and next-to-leading order results for α as a function of μ_I.

If we use this to result to obtain an NLO-version of the equation for α, we get

$$\mathcal{F}'_{\text{NLO}}(\alpha_{\text{NLO}}) = \tilde{\mathcal{F}}'_0(\alpha_0) + \alpha_1 \tilde{\mathcal{F}}''_0(\alpha_0) + \cdots + \tilde{\mathcal{F}}'_1(\alpha_0) + \ldots, \tag{3.55}$$

where the terms that are not included are beyond next-to-leading order. Thus,

$$\left.\frac{\partial \mathcal{F}_{\text{NLO}}}{\partial \alpha}\right|_{\alpha=\alpha'} = 0 \tag{3.56}$$

will give the correct answer to α' to next-to-leading order.

In Figure 3.2, the leading order and next-to-leading order results for α' is shown. The NLO result was calculated numerically using Eq. (3.56).

3.3 Thermodynamics

(KIILDER)

The free energy[4] is defined as

$$F = U - TS - \mu_I Q_I, \quad \mathrm{d}F = -S\mathrm{d}T - P\mathrm{d}V - Q_I \mathrm{d}\mu_I, \tag{3.57}$$

where Q_I is the isospin charge, and U is the energy. As we have seen earlier, our system is homogenous. This means that the free energy density is independent of volume, and thus $F = V\mathcal{F}$. From Eq. (3.57) we see that the pressure is given by

$$P = -\left(\frac{\partial F}{\partial V}\right)_{T,\mu_I} = -\mathcal{F}. \tag{3.58}$$

We are interested in the pressure relative to the state in which $\mu_I = 0$. We therefore normalize $P(\mu_I = 0) = 0$, which gives

$$P(\mu_I) = -(\mathcal{F}(\mu_I) - \mathcal{F}(\mu_I = 0)) \tag{3.59}$$

This is illustrated in Figure 3.3.

Likewise, the total isospin is proportional to volume, which means that the isospin density is

$$n_I = \frac{Q_I}{V} = -\frac{1}{V}\left(\frac{\partial F}{\partial \mu_I}\right)_{T,V} = -\frac{\partial \mathcal{F}}{\partial \mu_I}. \tag{3.60}$$

[4]As we are in the grand canonical ensemble, this is the *grand canonical* free energy, and not Helmholtz' free energy.

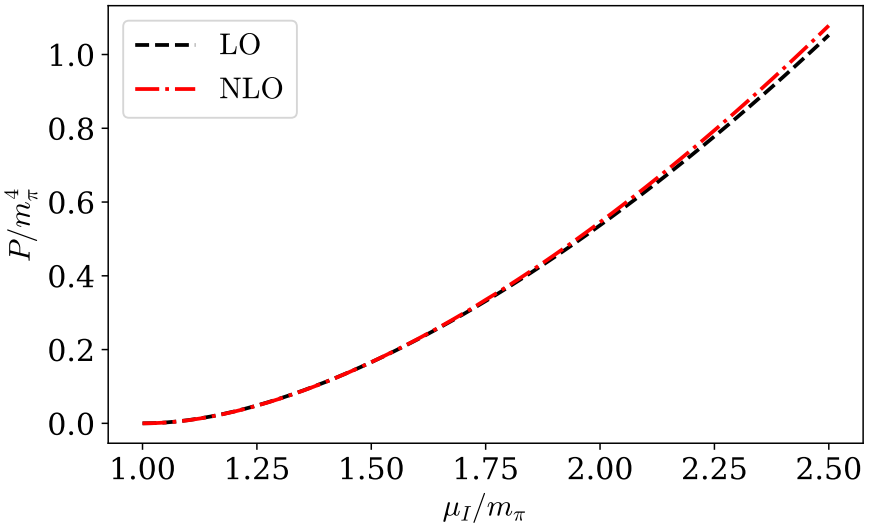


Figure 3.3: The NLO and LO result for the pressure of the pions, as a function of μ_I.

This gives

$$n_I = f^2\mu_I \sin^2\alpha - \frac{\partial \mathcal{F}_{\text{fin}}}{\partial \mu_I} + \frac{1}{(4\pi)^2}\left[\left(2\bar{l}_4 + \ln\frac{M^2}{m_3^2}\right)\bar{m}^2\mu_I\cos\alpha\sin^2\alpha + \frac{1}{3}\left(2\bar{l}_1 + 4\bar{l}_2 + 3\ln\frac{M^2}{m_3^2}\right)\mu_I^3\sin^4\alpha\right] \tag{3.61}$$

The isospin density, as a function of μ_I, is shown in Figure 3.4.

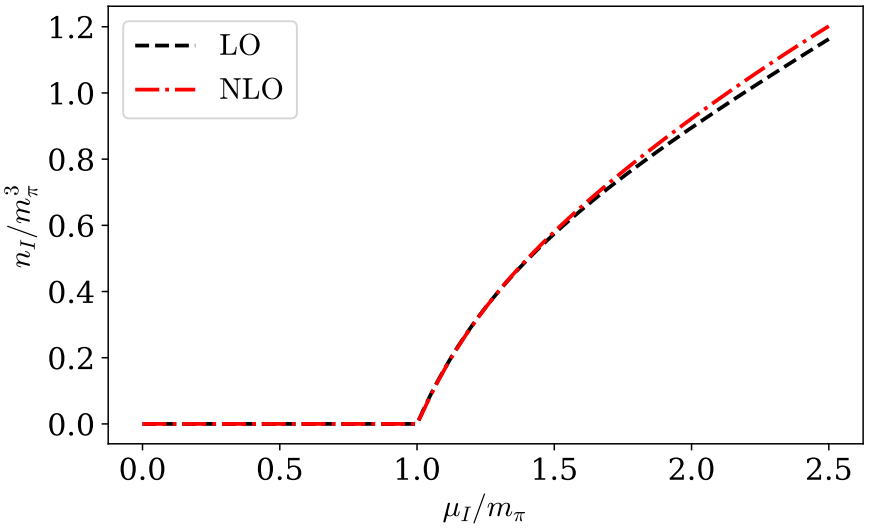


Figure 3.4: The NLO and LO result for the isosopin densisty, as a function of μ_I.

From Eq. (3.57) we get the energy density, $u = U/V$, at $T = 0$ is given by

$$u(\mu_I) = -P(\mu_I) + \mu_I n_I(\mu_I), \tag{3.62}$$

where we again have normalized so that $u(\mu_I = 0) = 0$. Now that we have both the dependence of pressure and energy density on the isospin chemical potential, we can trace out the line in the pressure-energy density plane, parametrized by μ_I. This is shown in Figure 3.5. This is the equation of state of the system.

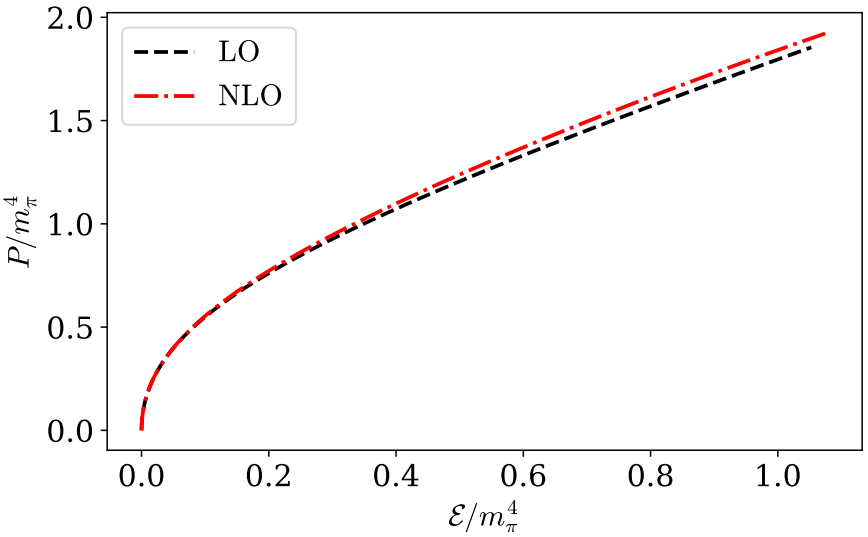


Figure 3.5: The equation of state of the system.

Phase transition

(KILDER)

In our leading-order analysis, we saw that α was zero for $\mu_I \leq \bar{m}$, before it starts to increase for $\mu_I > \bar{m}$, and that $\doteq m_\pi$. The behavior of α is illustrated in Figure 3.2. This is the hallmark of a phase transition, where α is the order parameter. The behavior systems near points of phase transition is described by Landau theory. Using Eq. (3.3), we can expand the leading-order free energy in α,

$$\mathcal{F} = -f^2\bar{m}^2 + f^2\frac{1}{2}(\bar{m}^2 - \mu_I^2)\alpha^2 - \frac{1}{24}f^2(\bar{m}^2 - 4\mu_I^2)\alpha^4 + \mathcal{O}\left(\alpha^5\right) \tag{3.63}$$

$$= \mathcal{F}(\alpha = 0) + a(\mu_I)\alpha^2 + \frac{1}{2}b(\mu_I)\alpha^4 + \mathcal{O}\left(\alpha^5\right), \tag{3.64}$$

to leading order. Notice that near $\mu_I = \bar{m}$, $b > 0$. As earlier, the equation that governs α' is

$$\left.\frac{\partial\mathcal{F}}{\partial\alpha}\right|_{\alpha=\alpha'} = 2[a(\mu_I) + b(\mu_I)\alpha^2]\alpha = 0. \tag{3.65}$$

If $a > 0$, then $\alpha' = 0$ will be the only solution, which gives us the criterion for a phase transition

$$a(\mu_I) = 0, \tag{3.66}$$

which again gives $\mu_I = \bar{m}$. Near $\mu_I = \bar{m}$, we can write

$$a = -a_0(\mu_I - \bar{m}), \quad b = b_0, \tag{3.67}$$

where a_0 and b_0 are positive constants, so the solution to Eq. (3.65) for $\mu_I > \bar{m}$

$$\alpha(\mu_I) = \sqrt{\frac{a_0}{b_0}}(\mu_I - \bar{m})^{1/2}. \tag{3.68}$$

The order parameter α changes continuously as the system undergoes phase-transition. This means we have a *second order* phase transition. However, if $b < 1$ near $\mu_I = \bar{m}$, this is not a valid solution, so we must expand $\mathcal{F}$ further to show if the phase transition is continuous or not. The difference is illustrated in Figure 3.6.

In the vacuum phase, $\alpha' = 0$, the ground state is given by

$$\Sigma(\pi = 0) = \Sigma_0 = \mathbb{1}, \tag{3.69}$$

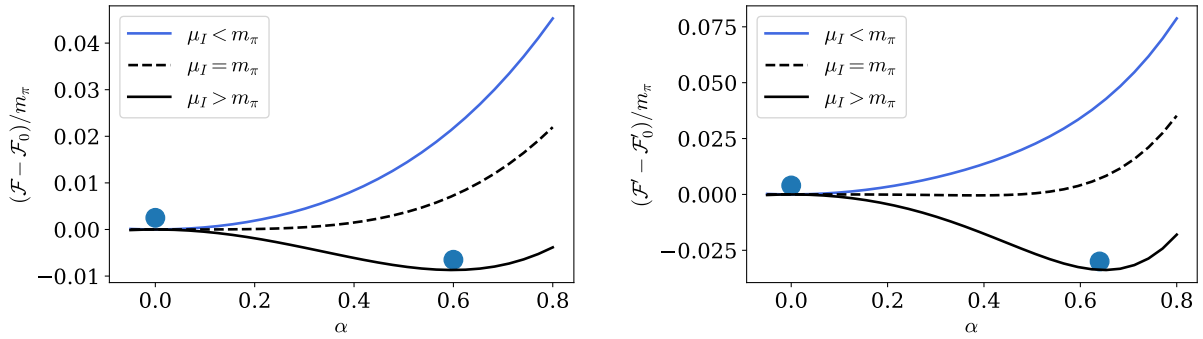


Figure 3.6: The plot on the left shows normalized free energy density as a function of α, in the two different phases. The plot on the right shows the same, but with a free energy $\mathcal{F}'$ that have been modified so that $b(\mu_I = \bar{m}) < 0$. In the first case, the minima transitions continuous, while in the second it jumps.

where we have use Eq. (2.33). Under $H = SU(2)_V$, Σ transforms as

$$\Sigma(x) \to \Sigma'(x) = V\Sigma(x)V^\dagger, \quad V = \exp\{i\theta_a\tau_a\}, \tag{3.70}$$

which means that the vacuum phase ground state is invariant under H. However, for $\alpha = 0$, the ground state is shifted to

$$\Sigma(\pi = 0) = A_\alpha \Sigma_0 A_\alpha = \exp\{i\alpha\tau_1\}. \tag{3.71}$$

The generators τ_2 and τ_3 are broken. The new ground state corresponds to a condensate of the π_1-particle, as it is defined in the vacuum state. In Figure 2.1, we saw that the mass of the m_- particle vanishes, so we identify this particle with the corresponding Goldstone mode. There is only one Goldstone mode, however this is not a Lorentz invariant system, in which case we cannot guarantee one massless mode per broken generator. (ER DETTE RIKTIG?)

To find the value of μ_I to next-to-leading order, we must expand the NLO free energy in powers of α When we expand the static, second order Lagrangian to α^2, we get

$$\mathcal{F}_4^{(0)} = -(l_3 + l_4)\bar{m}^4 + [(l_3 + l_4)\bar{m}^4 - l_4\bar{m}^2\mu_I^2]\alpha^2 \tag{3.72}$$

$$= \text{const.} + \frac{\mu^{-2\epsilon}}{(4\pi)^2}\left[\left(\bar{l}_4 - \frac{1}{4}\bar{l}_3\right)\bar{m}^4 - \bar{l}_4\bar{m}^2\mu_I^2 - \left(1 + \frac{1}{\epsilon} + \ln\frac{\tilde{\mu}^2}{M^2}\right)\left(\frac{3}{4}\bar{m}^2 - \mu_I^2\right)\bar{m}^2\right]\alpha^2, \tag{3.73}$$

where const. is independent of α, and thus not of interest. From the one-loop correction, we have the contributions

$$\mathcal{F}_2^{(1)} = i\frac{1}{2}\int\frac{\mathrm{d}^4p}{(2\pi)^2}\ln(-p^2 + m_3^2) + i\frac{1}{2}\int\frac{\mathrm{d}^4p}{(2\pi)^2}\ln[(-p^2 + m_1^2)(-p^2 + m_2^2) - p_0^2 m_{12}^2] \tag{3.74}$$

The first integral is the same free energy contribution from the π_0-particle as we have calculated earlier, and it reads

$$\mathcal{F}_{\pi_0} = i\frac{1}{2}\int\frac{\mathrm{d}^4p}{(2\pi)^2}\ln(-p^2 + m_3^2) = -\mu^{-2\epsilon}\frac{1}{4}\frac{m_3^4}{(4\pi)^2}\left(\frac{1}{\epsilon} + \frac{3}{2} + \ln\frac{\tilde{\mu}^2}{m_3^2}\right). \tag{3.75}$$

The mass m_3 is dependent on α, and has the expansion

$$m_3^4 = \bar{m}^4 + \bar{m}^2(2\mu_I^2 - \bar{m}^2)\alpha^2 + \mathcal{O}\left(\alpha^3\right),$$
$$\ln\frac{\mu^2}{m_3^2} = \ln\frac{\mu^2}{\bar{m}_3^2} - \frac{1}{2}\frac{(2\mu_I^2 - \bar{m}^2)}{\bar{m}^2} + \mathcal{O}\left(\alpha^3\right).$$

In the second integral, we rewrite the argument of the logarithm as

$$(-p^2 + m_1^2)(-p^2 + m_2^2) - p_0^2 m_{12}^2 = \left[-p^2 + \frac{1}{2}(m_1^2 + m_2^2)\right]^2 - p_0^2 m_{12}^2 - \frac{1}{4}(m_1^2 - m_2^2)^2. \tag{3.76}$$

When we calculate the α dependence of the last term, we get $(m_1^2 - m_2^2)^2 = \mu^4 \sin^4\alpha = \mathcal{O}\left(\alpha^4\right)$, which means that for our purposes, we can ignore this term. We further rewrite the remaining expression by factoring it,

$$\left[-p^2 + \frac{1}{2}(m_1^2 + m_2^2)\right]^2 - p_0^2 m_{12}^2 = \left[-p^2 + \frac{1}{2}(m_1^2 + m_2^2) - p_0 m_{12}\right]\left[-p^2 + \frac{1}{2}(m_1^2 + m_2^2) + p_0 m_{12}\right] \tag{3.77}$$

We then complete the square in each of the factors,

$$-p^2 + \frac{1}{2}(m_1^2 + m_2^2) \pm p_0 m_{12} = -\left(p_0 \mp \frac{1}{2}m_{12}\right)^2 + |\vec{p}|^2 + m_4^2, \tag{3.78}$$

where

$$m_4^2 = \frac{1}{2}\left(m_1^2 + m_2^2 + \frac{1}{2}m_{12}\right) = \bar{m}^2\cos\alpha + \frac{1}{2}\mu_I^2\sin^2\alpha, \tag{3.79}$$

$$m_4^4 = \bar{m}^4 - \bar{m}^2(m^2 + \mu_I^2)\alpha^2 + \mathcal{O}\left(\alpha^3\right), \tag{3.80}$$

$$\ln\frac{\mu^2}{m_4^2} = \ln\frac{\mu_I^2}{\bar{m}^2} + \frac{1}{2}\frac{\bar{m}^2 + \mu_I^2}{\bar{m}^2}\alpha^2 + \mathcal{O}\left(\alpha^3\right). \tag{3.81}$$

After a shift of variables, the integral has the same form as the logarithmic integrals we have calculated earlier, which gives us the result

$$\mathcal{F}_{\pi_\pm} = i\int\frac{\mathrm{d}^4}{(2\pi)^4}\ln\left(-p^2 + M^2\right) = -\mu^{-2\epsilon}\frac{1}{2}\frac{M^4}{(4\pi)^2}\left(\frac{1}{\epsilon} + \frac{3}{2} + \ln\frac{\tilde{\mu}^2}{M^2}\right) \tag{3.82}$$

Combining these two contributions to the one-loop correction of the free energy gives

$$\mathcal{F}_2^{(1)} = \text{const.} + \frac{\mu^{-2\epsilon}}{(4\pi)^2}\left(1 + \frac{1}{\epsilon} + \ln\frac{\tilde{\mu}^2}{m^2}\right)\left(\frac{3}{4}m^2 - \mu_I^2\right)\bar{m}^2\alpha^2$$

We see that again, the ϵ^{-1} will cancel when we combine the NLO-terms, and we get

$$\tilde{\mathcal{F}}_1 = \text{const.} + \frac{1}{(4\pi)^2}\left[\left(\bar{l}_4 - \frac{1}{4}\bar{l}_3\right)\bar{m}^4 - \bar{l}_4\bar{m}^2\mu_I^2 + \ln\frac{M^2}{\bar{m}^2}\left(\frac{3}{4}\bar{m}^2 - \mu_I^2\right)\bar{m}^2\right]\alpha^2 + \mathcal{O}\left(\alpha^3\right). \tag{3.83}$$

The total NLO free energy, up to second order in α, is (HVOR BLE DET AV LOG? TROR DET ER 1 + HØYERE ORDEN. SJEKK!)

$$\mathcal{F}_{\text{NLO}} = \mathcal{F}_{\text{NLO}}(\alpha = 0) + \frac{1}{2}f^2\bar{m}^2\left(1 - \frac{1}{2}\frac{\bar{l}_3 - 4\bar{l}_4}{(4\pi)^2}\frac{\bar{m}^2}{f^2}\right)\alpha^2 - \frac{1}{2}f^2\mu_I^2\left(1 + \frac{2\bar{l}_4}{(4\pi)^2}\frac{\bar{m}^2}{f^2}\right)\alpha^2 + \mathcal{O}\left(\alpha^3\right). \tag{3.84}$$

We now insert the physical constants f_π and m_π, by using the next-to-leading order expressions Eq. (3.43) and Eq. (3.44). First, notice that

$$f_\pi^2 m_\pi^2 = f^2\bar{m}^2\left(1 - \frac{1}{2}\frac{\bar{l}_3 - 4\bar{l}_4}{(4\pi)^2}\frac{\bar{m}^2}{f^2} + \mathcal{O}\left(\frac{\bar{m}^4}{f^4}\right)\right) \tag{3.85}$$

This means that, to leading order,

$$\mathcal{F}_{\text{NLO}} = \mathcal{F}_{\text{NLO}}(\alpha = 0) + \frac{1}{2}f_\pi^2(m_\pi^2 - \mu_I^2)\alpha^2 + \mathcal{O}\left(\alpha^3\right). \tag{3.86}$$

This shows that the phase transition happens at $\mu_I = m_\pi$, also at next-to-leading order.

Appendix A

Thermal Field Theory

This section is based on [21, 22].

A.1 Statistical Mechanics

In classical mechanics, a thermal system at temperature $T = 1/\beta$ is described as an ensemble state, which have a probability P_n of being in state n, with energy E_n. In the canonical ensemble, the probability is proportional to $e^{-\beta E_n}$. The expectation value of some quantity A, with value A_n in state n is

$$\langle A \rangle = \sum_n A_n P_n = \frac{1}{Z} \sum_n A_n e^{-\beta E_n}, \quad Z = \sum_n e^{-\beta A_n}.$$

Z is the partition function. In quantum mechanics, an ensemble configuration is described by a non-pure density operator,

$$\hat{\rho} = C \sum_n P_n \left| n \right\rangle \!\left\langle n \right|,$$

where $|n\rangle$ is some basis for the relevant Hilbert space and C is a constant. Assuming $|n\rangle$ are energy eigenvectors, i.e. $\hat{H} |n\rangle = E_n |n\rangle$, the density operator for the canonical ensemble is

$$\hat{\rho} = C \sum_n e^{-\beta E_n} \left| n \right\rangle \!\left\langle n \right| = C e^{-\beta \hat{H}} \sum_n \left| n \right\rangle \!\left\langle n \right| = C e^{-\beta \hat{H}}.$$

The expectation value in the ensemble state of a quantity corresponding to the operator $\hat{A}$ is given by

$$\langle A \rangle = \frac{\mathrm{Tr}\left\{ \hat{\rho} \hat{A} \right\}}{\mathrm{Tr}\{\hat{\rho}\}} = \frac{1}{Z} \mathrm{Tr}\left\{ \hat{A} e^{-\beta \hat{H}} \right\} \tag{A.1}$$

The partition function Z ensures that the probabilities adds up to 1, and is defined as

$$Z = \mathrm{Tr}\left\{ e^{-\beta \hat{H}} \right\}. \tag{A.2}$$

(REFFERER TIL THEROY-DELEN OM SYMMETRI; DETTE ER DOBBELT OPP) The grand canonical ensemble takes into account the conserved charges of the system. Conserved charges are a result of Nöther's theorem. Assume we have a set of fields φ_α. Nöther's theorem tells us that if the Lagrangian $\mathcal{L}[\varphi_\alpha]$ has a *continuous symmetry*, then there is a corresponding conserved current [1, 23]. To define a continuous symmetry of the Lagrangian, we need a one-parameter family of transformations,

$$\varphi_\alpha(x) \longrightarrow \varphi'_\alpha(x; \epsilon) \sim \varphi_\alpha(x) + \varepsilon \eta_\alpha(x), \; \varepsilon \to 0.$$

Here, $\eta_\alpha(x)$ is some arbitrary function which define the transformation as $\varepsilon \to 0$. Applying this transformation to the Lagrangian will in general change its form,

$$\mathcal{L}[\varphi_\alpha] \to \mathcal{L}[\varphi'_\alpha] \sim \mathcal{L}[\varphi_\alpha] + \delta\mathcal{L},\ \varepsilon \to 0.$$

If the change in the Lagrangian can be written as a total derivative, i.e.

$$\delta\mathcal{L} = \varepsilon\partial_\mu K^\mu(x),$$

we say that the Lagrangian has a continuous symmetry. This is because a term of this form will result in a boundary term in the action integral, which does not contribute to the variation of the action. Nöther's theorem states more precisely that the current

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_\alpha)}\eta_\alpha - K^\mu \tag{A.3}$$

obeys the conservation law

$$\partial_\mu j^\mu = 0. \tag{A.4}$$

The flux of current through some space-like surface V, i.e. a surface with a time-like normal vector, defines a conserved charge. This surface is most commonly a surface of constant time in some reference frame. The conserved charge is then

$$Q = \int_V \mathrm{d}^3x\, j^0.$$

Using the divergence theorem again, and assuming the current falls of quickly enough towards infinity, we can show that the total charge is conserved,

$$\frac{\partial}{\partial t}Q = -\int_V \mathrm{d}^3x\, \nabla\cdot\vec{j} = -\int_{\partial V} \mathrm{d}^2x\, n_\mu j^\mu = 0.$$

Here, n^μ is the time-like normal vector of the surface of V, ∂V.

In the grand canonical ensemble, a system with n conserved charges Q_i has probability proportional to $e^{-\beta(H-\mu_i Q_i)}$. μ_i are the chemical potentials corresponding to conserved charge Q_i. This leads to the partition function

$$Z = \mathrm{Tr}\Big\{e^{-\beta(\hat{H}-\mu_i\hat{Q}_i)}\Big\}. \tag{A.5}$$

A.2 Imaginary-time formalism

The partition function may be calculated in a similar way to the path integral approach, in what is called the imaginary-time formalism. This formalism is restricted to time independent problems, and is used to study fields in a volume V. This volume is taken to infinity in the thermodynamic limit. As an example, take a scalar quantum field theory with the Hamiltonian

$$\hat{H} = \int_V \mathrm{d}^3x\, \hat{\mathcal{H}}\left[\hat{\varphi}(\vec{x}), \hat{\pi}(\vec{x})\right], \tag{A.6}$$

where $\hat{\varphi}(\vec{x})$ is the field operator, and $\hat{\pi}(\vec{x})$ is the corresponding canonical momentum operator. These field operators have time independent eigenvectors, $|\varphi\rangle$ and $|\pi\rangle$, defined by

$$\hat{\varphi}(\vec{x})\,|\varphi\rangle = \varphi(\vec{x})\,|\varphi\rangle\,, \quad \hat{\pi}(\vec{x})\,|\pi\rangle = \pi(\vec{x})\,|\pi\rangle\,. \tag{A.7}$$

In analogy with regular quantum mechanics, they obey the relations [1]

$$\mathbb{1} = \int \mathcal{D}\varphi(\vec{x})\,|\varphi\rangle\langle\varphi| = \int \mathcal{D}\pi(\vec{x})\,|\pi\rangle\langle\pi|\,, \tag{A.8}$$

$$\langle\varphi|\pi\rangle = \exp\Big(i\int_V \mathrm{d}^3x\,\varphi(\vec{x})\pi(\vec{x})\Big), \tag{A.9}$$

$$\langle\pi_a|\pi_b\rangle = \delta(\phi_a - \phi_b), \quad \langle\varphi_a|\varphi_b\rangle = \delta(\varphi_a - \varphi_b). \tag{A.10}$$

[1] Some authors write $\mathcal{D}\pi/2\pi$. This extra factor 2π is a convention which in this text is left out for notational clarity.

The functional integral is defined by starting with M degrees of freedom, $\{\varphi_m\}_{m=1}^M$ located at a finite grid $\{\vec{x}_m\}_{m=1}^M \subset V$. The integral is then the limit of the integral over all degrees of freedom, as $M \to \infty$:

$$\int \mathcal{D}\varphi(\vec{x}) = \lim_{M\to\infty} \int \left(\prod_{m=1}^{M} \mathrm{d}\varphi_m \right).$$

The functional Dirac-delta $\delta(f) = \prod_x \delta(f(x))$ is generalization of the familiar Dirac delta function. Given a functional $\mathcal{F}[f]$, it is defined by the relation

$$\int \mathcal{D}f(x)\, \mathcal{F}[f]\delta(f-g) = \mathcal{F}[g]. \tag{A.11}$$

The Hamiltonian is the limit of a sum of Hamiltonians $\hat{H}_m$ for each point $\vec{x}_m$

$$\hat{H} = \lim_{M\to\infty} \sum_{m=1}^{M} \frac{V}{M} \hat{H}_m(\{\hat{\varphi}_m\}, \{\hat{\pi}_m\}).$$

H_m may depend on the local degrees of freedom $\hat{\varphi}_m$, $\hat{\pi}_m$ as well as those at neighboring points. By inserting the completeness relations Eq. (A.8) N times into the definition of the partition function, it may be written as

$$Z = \int \mathcal{D}\varphi(\vec{x}) \left\langle \varphi \middle| e^{-\beta\hat{H}} \middle| \varphi \right\rangle = \prod_{n=1}^{N} \left(\int \mathcal{D}\varphi_n(\vec{x}) \int \mathcal{D}\pi_n(\vec{x}) \right) \prod_{n=1}^{N} \langle \varphi_n|\pi_n\rangle \left\langle \pi_n \middle| e^{-\epsilon\hat{H}} \middle| \varphi_{n+1} \right\rangle \langle \varphi_1|\varphi_{N+1}\rangle,$$

where $\epsilon = \beta/N$. The last term ensures that $\varphi_1 = \varphi_{N+1}$. Strictly speaking, we only need to require $\varphi_1 = e^{i\theta}\varphi_{N+1}$, as the partition function is only defined up to a constant. As will be shown later, bosons such as the scalar field φ, follow the periodic boundary condition $\varphi(0,\vec{x}) = \varphi(\beta,\vec{x})$, i.e. $e^{i\theta} = 1$, while fermions follow the anti-periodic boundary condition $\psi(0,\vec{x}) = -\psi(\beta,\vec{x})$, i.e. $e^{i\theta} = -1$. We now want to exploit the fact that $|\pi\rangle$ and $|\varphi\rangle$ are the eigenvectors of the operators that define the Hamiltonian. In our case, as the Hamiltonian density $\mathcal{H}$ can be written as a sum of functions of φ and π separately, $\mathcal{H}[\varphi(\vec{x}),\pi(\vec{x})] = \mathcal{F}_1[\varphi(\vec{x})] + \mathcal{F}_2[\pi(\vec{x})]$ we may evaluate it as $\langle\pi_n|\mathcal{H}[\hat{\varphi}(\vec{x}),\hat{\pi}(\vec{x})]|\varphi_{n+1}\rangle = \mathcal{H}[\varphi_{n+1}(\vec{x}),\pi_n(\vec{x})]\langle\pi_n|\varphi_{n+1}\rangle$. This relationship does not, however, hold for more general functions of the field operators. In that case, one has to be more careful about the ordering of the operators, for example by using *Weyl ordering* [1]. By series expanding $e^{-\epsilon\hat{H}}$ and exploiting this relationship, the partition function can be written as, to second order in ϵ,

$$Z = \prod_{n=1}^{N} \left(\int \mathcal{D}\varphi_n(\vec{x}) \int \mathcal{D}\pi_n(\vec{x}) \right) \exp\left[-\epsilon \sum_{n=1}^{N} \int_V \mathrm{d}^3x \left(\mathcal{H}[\varphi_n(\vec{x}),\pi_n(\vec{x})] - i\pi_n(\vec{x}) \frac{\varphi_n(\vec{x}) - \varphi_{n+1}(\vec{x})}{\epsilon} \right) \right].$$

We denote $\varphi_n(\vec{x}) = \varphi(\tau_n,\vec{x})$, $\tau \in [0,\beta]$ and likewise with $\pi_n(\vec{x})$. In the limit $N \to \infty$, the expression for the partition function becomes

$$Z = \int_S \mathcal{D}\varphi(\tau,\vec{x}) \int \mathcal{D}\pi(\tau,\vec{x}) \exp\left\{ -\int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}\vec{x}\, \{\mathcal{H}[\varphi(\tau,\vec{x}),\pi(\tau,\vec{x})] - i\pi(\tau,\vec{x})\dot{\varphi}(\tau,\vec{x})\} \right\}, \tag{A.12}$$

where S is the set of field configurations φ such that $\varphi(\beta,\vec{x}) = \varphi(0,\vec{x})$. With a Hamiltonian density of the form $\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\varphi)^2 + \mathcal{V}(\varphi)$, we can evaluate the integral over the canonical momentum π by discretizing $\pi(\tau_n,\vec{x}_m) = \pi_{n,m}$,

$$\begin{aligned}
&\int \mathcal{D}\pi \exp\left\{ -\int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}^3x \left(\frac{1}{2}\pi^2 - i\pi\dot{\varphi} \right) \right\} \\
&= \lim_{M,N\to\infty} \int \left(\prod_{m,n=1}^{M,N} \frac{\mathrm{d}\pi_{m,n}}{2\pi} \right) \exp\left\{ -\sum_{m,n} \frac{V\beta}{MN} \left[\frac{1}{2}(\pi_{m,n} - i\dot{\varphi}_{m,n})^2 + \frac{1}{2}\dot{\varphi}_{m,n}^2 \right] \right\} \\
&= \lim_{M,N\to\infty} \left(\frac{MN}{2\pi V\beta} \right)^{MN/2} \exp\left\{ -\int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}^3x\, \frac{1}{2}\dot{\varphi}^2 \right\},
\end{aligned}$$

where $\dot{\varphi}_{m,n} = (\varphi_{m,n+1} - \varphi_{m,n})/\epsilon$. The partition function is then,

$$Z = C \int \mathcal{D}\varphi \exp\left\{ - \int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}^3x \left[\frac{1}{2} \left(\dot{\varphi}^2 + \nabla\varphi^2 \right) + \mathcal{V}(\varphi) \right] \right\}. \tag{A.13}$$

Here, C is the divergent constant that results from the π-integral. In the last line, we exploited the fact that the variable of integration $\pi_{n,m}$ may be shifted by a constant without changing the integral, and used the Gaussian integral

$$\int_{-\infty}^{\infty} \mathrm{d}x\, e^{-ax^2/2} = \sqrt{\frac{2\pi}{a}}.$$

The partition function resulting from this procedure may also be obtained by starting with the ground state path integral

$$Z_g = \int \mathcal{D}\varphi \mathcal{D}\pi \exp\left\{ i \int_{\Omega'} \mathrm{d}^4x \ (\pi\dot{\varphi} - \mathcal{H}[\varphi, \pi]) \right\} = C' \int \mathcal{D}\varphi(x) \exp\left\{ i \int_{\Omega'} \mathrm{d}^4x\, \mathcal{L}[\varphi, \partial_\mu \varphi] \right\},$$

and follow a formal procedure. First, the action integral is modified by performing a Wick-rotation of the time coordinate t. This involves changing the domain of t from the real line to the imaginary line by closing the contour at infinity, and making the change of variable $it \to \tau$. The new variable is then restricted to the interval $\tau \in [0, \beta]$, and the domain of the functional integral $\int \mathcal{D}\varphi$ is restricted from *all* (smooth enough) field configurations $\varphi(t, \vec{x})$, to only those that obey $\varphi(\beta, \vec{x}) = e^{i\theta}\varphi(0, \vec{x})$, which is denoted S. This procedure motivates the introduction of the Euclidean Lagrange density, $\mathcal{L}_E(\tau, \vec{x}) = -\mathcal{L}(-i\tau, \vec{x})$, as well as the name "imaginary-time formalism". The result is the same partition function as before,

$$\begin{aligned} Z &= C \int_S \mathcal{D}\varphi \int \mathcal{D}\pi \exp\left\{ - \int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}^3x \ [-i\dot{\varphi}\pi + \mathcal{H}(\varphi, \pi)] \right\} \\ &= C' \int_S \mathcal{D}\varphi \exp\left\{ - \int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}^3x\, \mathcal{L}_E(\varphi, \pi) \right\}. \end{aligned} \tag{A.14}$$

A.3 Free scalar field

This section uses notation as described in section B.1. The procedure for obtaining the thermal properties of an interacting scalar field is similar to that used in scattering theory. One starts with a free theory, which can be solved exactly. Then an interaction term is added, which is accounted for peturbatively by using Feynman diagrams. The Euclidean Lagrangian for a free scalar gas is, after integrating by parts,

$$\mathcal{L}_E = \frac{1}{2} \varphi(X) \left(-\partial_E^2 + m^2 \right) \varphi(X) \tag{A.15}$$

Here, $X = (\tau, \vec{x})$ is the Euclidean coordinate which is a result of the Wick-rotation. We have also introduced the Euclidean Laplace operator, $\partial_E^2 = \partial_\tau^2 + \nabla^2$. Following the procedure as described in section A.2 to obtain the thermal partition function yields

$$Z = C \int_S \mathcal{D}\varphi(X) \exp\left\{ - \int_\Omega \mathrm{d}X \frac{1}{2} \varphi(X) \left(-\partial_E^2 + m^2 \right) \varphi(X) \right\}. \tag{A.16}$$

Here, Ω is the domain $[0, \beta] \times V$. We then insert the Fourier expansion of φ, and change the functional integration variable to the Fourier components. The integration measures are related by

$$\mathcal{D}\varphi(X) = \det\left(\frac{\delta\varphi(X)}{\delta\tilde{\varphi}(K)} \right) \mathcal{D}\tilde{\varphi}(K),$$

where $K = (\omega_n, \vec{k})$ is the Euclidean Fourier-space coordinate. The determinant factor which appears may be absorbed into the constant C, as the integration variables are related by a linear transform. The action

becomes

$$\begin{aligned} S &= -\int_\Omega \mathrm{d}X\, \mathcal{L}_e = -\frac{1}{2}V\beta \int_\Omega \mathrm{d}X \int_{\tilde{\Omega}} \mathrm{d}K \int_{\tilde{\Omega}} \mathrm{d}K'\, \tilde{\varphi}(K') \left(\omega_n^2 + \vec{k}^2 + m^2\right) \tilde{\varphi}(K) e^{iX\cdot(K-K')} \\ &= -\frac{1}{2}V\beta^2 \int_{\tilde{\Omega}} \mathrm{d}K\, \tilde{\varphi}(K)^* \left(\omega_n^2 + \omega_k^2\right) \tilde{\varphi}(K), \end{aligned}$$

where $\omega_k^2 = \vec{k}^2 + m^2$. $\tilde{\Omega}$ is the reciprocal space corresponding to Ω, as described in section B.1. We used the fact that φ is real, which implies that $\tilde{\varphi}(-K) = \tilde{\varphi}(K)^*$, as well as the identity Eq. (B.9). This gives the partition function

$$Z = C \int_{\tilde{S}} \mathcal{D}\tilde{\varphi}(K) \exp\left\{-\frac{1}{2}V \int_{\tilde{\Omega}} \mathrm{d}K\, \tilde{\varphi}(K)^* \left[\beta^2(\omega_n^2 + \omega_k^2)\right] \tilde{\varphi}(K)\right\}, \tag{A.17}$$

Going back to before the continuum limit, this integral can be written as a product of Gaussian integrals, and may therefore be evaluated

$$Z = C \prod_{n=-\infty}^{\infty} \prod_{k\in\tilde{V}} \left(\int \mathrm{d}\tilde{\varphi}_{n,\vec{k}}\, \exp\left\{-\frac{1}{2}\tilde{\varphi}^*_{n,\vec{k}} \left[\beta^2(\omega_n^2 + \omega_k^2)\right] \tilde{\varphi}_{n,\vec{k}}\right\}\right) = C \prod_{n=-\infty}^{\infty} \prod_{k\in\tilde{V}} \sqrt{\frac{2\pi}{\beta^2(\omega_n^2 + \omega_k^2)}}.$$

The partition function is related to Helmholtz free energy F through

$$\frac{F}{TV} = -\frac{\ln(Z)}{V} = \frac{1}{2}\int_{\tilde{\Omega}} \mathrm{d}K \frac{1}{2} \ln[\beta^2(\omega_n^2 + \omega_k^2)] + \frac{F_0}{TV}, \tag{A.18}$$

where F_0 is a constant.

A faster and more formal way to get to this result is to compare the partition function to the multidimensional version of the Gaussian integral [21, 1]. The partition function has the form

$$I_n = \int_{\mathbb{R}^n} \mathrm{d}^n x\, \exp\left\{-\frac{1}{2}\langle x, D_0^{-1} x\rangle\right\},$$

where D_0^{-1} is a linear operator, and $\langle\cdot,\cdot\rangle$ an inner product on the corresponding vector space. By diagonalizing D_0^{-1}, we get the result

$$I_n = \sqrt{\frac{(2\pi)^n}{\det\left(D_0^{-1}\right)}}.$$

We may then use the identity

$$\det\left(D_0^{-1}\right) = \prod_i \lambda_i = \exp\left\{\mathrm{Tr}\left[\ln\left(D_0^{-1}\right)\right]\right\}, \tag{A.19}$$

where λ_i are the eigenvalues of D_0^{-1}. The trace in this context is defined by the vector space D_0^{-1} acts on. For given an orthonormal basis x_n, such that $\langle x_n, x'_n\rangle = \delta_{nn'}$ the trace can be evaluated as $\mathrm{Tr}\left\{D_0^{-1}\right\} = \sum_n \langle x_n, D_0^{-1} x_n\rangle$. Identifying

$$\langle x, D_0^{-1} x\rangle = \int_\Omega \mathrm{d}X \varphi(X) \left(-\partial_E^2 + m^2\right) \varphi(X),$$

we get the formal result

$$Z = \det\left(-\partial_E^2 + m^2\right)^{-1/2},$$

and

$$\beta F = \frac{1}{2}\,\mathrm{Tr}\left\{\ln\left(-\partial_E^2 + m^2\right)\right\}.$$

The logarithm may then be evaluated by using the eigenvalues of the linear operator. This is found by diagonalizing the operator,

$$\langle x, D_0^{-1} x\rangle = \int_\Omega \mathrm{d}X \varphi(X) \left(-\partial_E^2 + m^2\right) \varphi(X) = V \int_{\tilde{\Omega}} \mathrm{d}K \tilde{\varphi}(K)^* [\beta^2(\omega_k^2 + \omega_n^2)] \tilde{\varphi}(K),$$

leaving us with the same result as we obtained in Eq. (A.18),

$$\beta F = \frac{1}{2}\,\mathrm{Tr}\left\{\ln\left(-\partial_E^2 + m^2\right)\right\} = \frac{1}{2} V \int_{\tilde{\Omega}} \mathrm{d}K \ln[\beta^2(\omega_n^2 + \omega_k^2)].$$

Sums similar to this show up a lot, and it is show how to evaluate them in the next section.

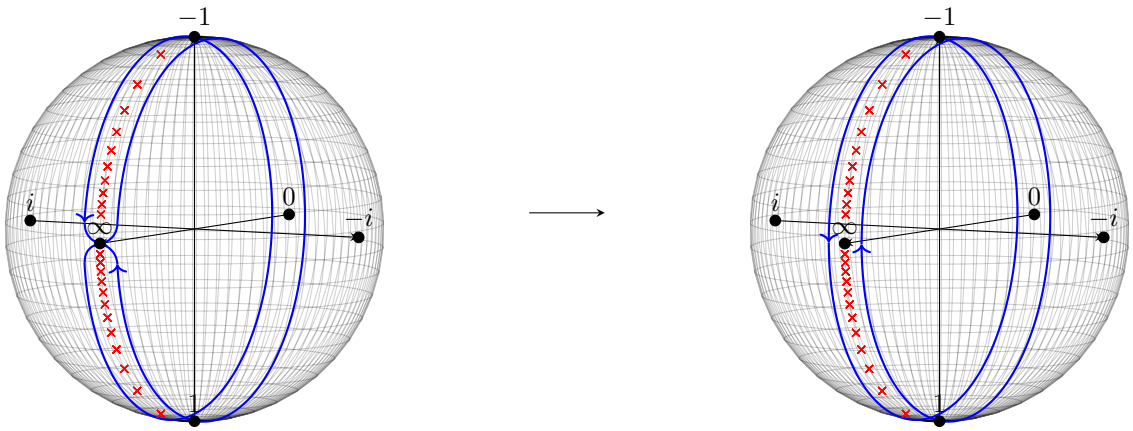


Figure A.1: The integral contour γ, and the result of deforming it into to contours close to the real line. The red crosses illustrate the poles of n_B.

Thermal sum

When evaluating thermal integral, we will often encounter sums of the form

$$j(\omega,\mu) = \frac{1}{2\beta}\sum_{\omega_n} \ln\{\beta^2[(\omega_n + i\mu) + \omega^2]\} + g(\beta), \tag{A.20}$$

where the sum is over either the bosonic Matsubara frequencies $\omega_n = 2n\pi/\beta$, $n \in \mathbb{Z}$, or the fermionic ones, $\omega_n = (2n+1)\pi/\beta$, $n \in \mathbb{Z}$. $\mu \in \mathbb{R}$ is the chemical potential. g may be a function of β, but we assume it is independent of ω. Thus, the factor β^2 could strictly be dropped, but it is kept to make the argument within the logarithm dimensionless. We define the function

$$i(\omega,\mu) = \frac{1}{\omega}\frac{\mathrm{d}}{\mathrm{d}\omega} j(\omega,\mu) = \frac{1}{\beta}\sum_{\omega_n} \frac{1}{(\omega_n + i\mu)^2 + \omega^2}. \tag{A.21}$$

We will first work with the sum over bosonic Matsubara frequencies. Assume $f(z)$ is an analytic function, except perhaps on a set of isolated poles $\{z_i\}$ located outside the real line. By exploiting the properties of the Bose-distribution $n_B(z)$, as described in section B.1, we can rewrite the sum over Matsubara frequencies as a contour integral

$$\frac{1}{\beta}\sum_{\omega_n} f(\omega_n) = \oint_\gamma \frac{\mathrm{d}z}{2\pi i} f(z) i n_B(iz),$$

where γ is a contour that goes from $-\infty - i\epsilon$ to $+\infty - i\epsilon$, crosses the real line at ∞, goes from $+\infty - i\epsilon$ to $-\infty + i\epsilon$ before closing the curve. The contour γ, and the change of integral contours is illustrated in Figure A.1 This result exploits Cauchy's integral formula, by letting the poles of $in_B(iz)$ at the Matsubara frequencies "pick out" the necessary residues. The integral over γ is equivalent to two integrals along $\mathbb{R} \pm i\epsilon$,

$$\begin{aligned}
\frac{1}{\beta}\sum_{\omega_n} f(\omega_n) &= \left(\int_{\infty+i\epsilon}^{-\infty+i\epsilon} \frac{\mathrm{d}z}{2\pi} + \int_{-\infty-i\epsilon}^{\infty-i\epsilon} \frac{\mathrm{d}z}{2\pi}\right) f(z) n_B(iz), \\
&= \int_{-\infty-i\epsilon}^{\infty-i\epsilon} \frac{\mathrm{d}z}{2\pi} \left\{f(-z) + [f(z) + f(-z)]\, n_B(iz)\right\} \\
&= \int_{-\infty}^{\infty} \frac{\mathrm{d}z}{2\pi} f(z) + \int_{-\infty-i\epsilon}^{\infty-i\epsilon} \frac{\mathrm{d}z}{2\pi} \left[f(z) - f(-z)\right] n_B(iz).
\end{aligned} \tag{A.22}$$

In the second line, we have changed variables $z \to -z$ in the first integral, and exploited the property $n_B(-iz) = -1 - n_B(iz)$. In the last line, we use the assumption that $f(z)$ is analytic on the real line, and therefore also in a neigbourhood of it. This allows us to shift the first integral back to the real line. As $n_B(iz)$

is analytic outside the real line, the result of the second integral is the sum of residues of $f(z)+f(-z)$ in the lower half plane. The function

$$f(z)=\frac{1}{(z+i\mu)^2+\omega^2}=\frac{i}{2\omega}\left(\frac{1}{z+i(\mu+\omega)}-\frac{1}{z+i(\mu-\omega)}\right) \tag{A.23}$$

obeys the assumed properties, as it has poles at $z=-i(\mu\pm\omega)$, with residue $1/2\omega$, so the function defined in Eq. (A.21) may be written [2]

$$i(\omega,\mu)=\frac{1}{2\omega}[1+n_B(\omega-\mu)+n_B(\omega+\mu)]. \tag{A.24}$$

Using the anti-derivative of the Bose distribution, we get the final form of Eq. (A.20)

$$j(\omega,\mu)=\int \mathrm{d}\omega'\,\omega' i(\omega',\mu)=\frac{1}{2}\omega+\frac{1}{2\beta}\left[\ln\left(1-e^{-\beta(\omega-\mu)}\right)+\ln\left(1-e^{-\beta(\omega+\mu)}\right)\right]+g'(\beta). \tag{A.25}$$

The extra ω-independent term $g'(\beta)$ is an integration constant. We see there are temperature dependent terms, one due to the particle and one due to the anti-particle, and one due to the antiparticle, as they have opposite chemical potentials.

We now consider the sum over fermionic frequencies, which we for clarity denote $\tilde{\omega}_n$ in this chapter. The procedure in this case is the same, except that we have to use a function to with poles at the fermionic Matsubara frequencies. This is done by the Fermi distribution, $n_F(z)$, as described in section B.1. The result is

$$\frac{1}{\beta}\sum_{\tilde{\omega}_n} f(\tilde{\omega}_n)=-\int_{-\infty}^{\infty}\frac{\mathrm{d}z}{2\pi}f(z)+\int_{-\infty-i\epsilon}^{\infty-i\epsilon}\frac{\mathrm{d}z}{2\pi}\left[f(z)-f(-z)\right]n_F(iz), \tag{A.26}$$

and

$$i(\omega,\mu)=\frac{1}{2\omega}[-1+n_F(\omega-\mu)+n_F(\omega+\mu)]. \tag{A.27}$$

Using the antiderivative of the Fermi-distribution, we get

$$j(\omega,\mu)=-\frac{1}{2}\omega-\frac{1}{2\beta}\left[\ln\left(1+e^{-\beta(\omega-\mu)}\right)+\ln\left(1+e^{-\beta(\omega+\mu)}\right)\right]. \tag{A.28}$$

Low temperature limit

Using the result from section A.3 on the result for the free energy density of the free scalar field, Eq. (A.14), we get

$$\mathcal{F}=\frac{\ln(Z)}{\beta V}=\frac{1}{2}\int_{\tilde{V}}\frac{\mathrm{d}^3k}{(2\pi)^3}\left[\omega_k+\frac{2}{\beta}\ln\left(1-e^{-\beta\omega_k}\right)\right]. \tag{A.29}$$

The free energy thus has two parts, the first part is dependent on temperature, the other is a temperature independent vacuum contribution. Noticing that the integral is spherically symmetric, we may write the two contributions as

$$\mathcal{F}_0=\frac{1}{2}\frac{1}{2\pi^2}\int_{\mathbb{R}}\mathrm{d}k\,k^2\sqrt{k^2+m^2},\quad \mathcal{F}_T=\frac{T^4}{2\pi^2}\int_{\mathbb{R}}\mathrm{d}x\,x^2\ln\left(1-e^{-\sqrt{x^2+(m/T)^2}}\right), \tag{A.30}$$

The temperature-independent part, $\mathcal{F}_0$, is clearly divergent, and we must therefore impose a regulator, and then add counter-terms. $\mathcal{F}_T$, however, is convergent. To see this, we use the series expansion $\ln(1+\epsilon)\sim\epsilon+\mathcal{O}\left(\epsilon^2\right)$ to find the leading part of the integrand for large k's,

$$x^2\ln\left(1-e^{-\sqrt{x^2+(\beta m)^2}}\right)\sim -x^2e^{-x}, \tag{A.31}$$

which is exponentially suppressed, making the integral convergent. In the limit of $T\to\infty$, we get

$$\mathcal{F}_\infty\sim\frac{T^4}{2\pi^2}\int_{\mathbb{R}}\mathrm{d}x\,x^2\ln\left(1-e^{-x}\right)=-\frac{T^4}{2\pi^2}\sum_{n=1}\frac{1}{n}\frac{\partial^2}{\partial n^2}\int\mathrm{d}x e^{-nx}=-\frac{T^4}{2\pi^2}\sum_{n=1}\frac{2}{n^4}=-\frac{T^4}{\pi^2}\zeta(4)=-T^4\frac{\pi^2}{90}. \tag{A.32}$$

Where ζ is the Riemann-zeta function.

[2] Assuming $\omega>\mu$.

Regularization

Returning to the temperature-independent part, we use dimensional regularization to see its singular behavior. To that end, we define

$$\Phi_n(m,d,\alpha) = \int_{\tilde{\Omega}} \frac{\mathrm{d}^d k}{(2\pi)^d}(k^2+m^2)^{-\alpha}, \tag{A.33}$$

so that $\mathcal{F}_0 = \Phi_3(m,3,1/2)/2$. Dimensional regularization takes uses the formula for integration of spherically symmetric function in d-dimensions,

$$\int_{\mathbb{R}^d} \mathrm{d}^d x\, f(r) = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_{\mathbb{R}} \mathrm{d}r\, r^{d-1} f(r), \tag{A.34}$$

where $r = \sqrt{x_i x_i}$ is the radial distance, and Γ is the Gamma function. The factor in the front of the integral comes from the solid angle. By extending this formula from integer-valued d to real numbers, the function we defined becomes

$$\Phi_n = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_{\mathbb{R}} \mathrm{d}k\, \frac{k^{d-1}}{(k^2+m^2)^\alpha} = \frac{m^{n-2\alpha} m^{d-n}}{(4\pi)^{d/2}\Gamma(d/2)} 2 \int_{\mathbb{R}} \mathrm{d}z\, \frac{z^{d-1}}{(1+z)^\alpha}, \tag{A.35}$$

where we have made the change of variables $mz = k$. We make one more change of variable to the integral,

$$I = 2\int_{\mathbb{R}} \mathrm{d}z\, \frac{z^{d-1}}{(1+z)^\alpha} \tag{A.36}$$

Let

$$z^2 = \frac{1}{s} - 1 \implies 2z\mathrm{d}z = -\frac{\mathrm{d}s}{s^2} \tag{A.37}$$

Thus,

$$I = \int_0^a \mathrm{d}s\, s^{\alpha - d/2 - 1}(1-z)^{d/2-1}. \tag{A.38}$$

This is the beta function, which can be written in terms of Gamma functions [1]

$$I = B\left(\alpha - \frac{d}{2}, \frac{d}{2}\right) = \frac{\Gamma\left(\alpha - \frac{d}{2}\right)\Gamma\left(\frac{d}{2}\right)}{\Gamma(\alpha)}. \tag{A.39}$$

Combining this gives

$$\Phi_n(m,d,\alpha) = \mu^{n-d} \frac{m^{n-2\alpha}}{(4\pi)^{d/2}} \frac{\Gamma\left(\alpha - \frac{d}{2}\right)}{\Gamma(\alpha)} \left(\frac{m^2}{\mu^2}\right)^{(d-n)/2}. \tag{A.40}$$

In the last step, we have introduced a parameter μ with mass dimension 1, that is $[\mu] = [m]$. This is done to be able to series expand around $d-n$ in a dimensionless variable. This parameter is arbitrary, and should therefore not be present in any of the final results. We will shortly justify this parameter further.

Inserting $n = 3$, $d = 3 - 2\epsilon$ and $\alpha = -1/2$, we get

$$\Phi_3(m, 3-2\epsilon, -1/2) = \frac{m^4 \mu^{-2\epsilon}}{(4\pi)^{d/2}\Gamma(-1/2)} \Gamma(-2+\epsilon) \left(\frac{m^2}{\mu^2}\right)^{-\epsilon} = -\mu^{-2\epsilon} \frac{m^4}{(4\pi)^2} \left(\frac{m^2}{4\pi\mu^2}\right)^{-\epsilon} \frac{\Gamma(\epsilon)}{(\epsilon-2)(\epsilon-1)}, \tag{A.41}$$

where we have used the defining property $\Gamma(z+1) = z\Gamma(z)$ and $\Gamma(1/2) = \sqrt{\pi}$. Expanding around $\epsilon = 0$ gives

$$\left(\frac{m^2}{4\pi\mu^2}\right)^{-\epsilon} \sim 1 + \epsilon \ln\left(4\pi\frac{\mu^2}{m^2}\right), \tag{A.42}$$

$$\Gamma(\epsilon) \sim \frac{1}{\epsilon} - \gamma, \tag{A.43}$$

$$\frac{1}{(\epsilon-2)(\epsilon-1)} \sim \frac{1}{2}\left(1 + \frac{3}{2}\epsilon\right). \tag{A.44}$$

The singular behavior of the time-independent term is therefore

$$\mathcal{F}_0 \sim -\mu^{-2\epsilon}\frac{1}{4}\frac{m^4}{(4\pi)^2}\left[\frac{1}{\epsilon}-\gamma+\frac{3}{2}+\ln\left(4\pi\frac{\mu^2}{m^2}\right)\right]. \tag{A.45}$$

With this regulator, one can then add counter-terms to cancel the $\frac{1}{\epsilon}$-divergence. The exact form of the counter-term is convention. One may also cancel the finite contribution due to the regulator. The minimal subtraction, or MS, scheme, is to only subtract the divergent term, as the name suggest. We will use the modified minimal subtraction, or $\overline{\text{MS}}$, scheme. In this scheme, one also removes the $-\gamma$ and $\ln(4\pi)$ term, by defining a new mass parameter $\tilde{\mu}$ by

$$-\gamma+\ln\left(4\pi\frac{\mu^2}{m^2}\right)=\ln\left(4\pi e^{-\gamma}\frac{\mu^2}{m^2}\right)=\ln\left(\frac{\tilde{\mu}^2}{m^2}\right), \tag{A.46}$$

which leads to the expression

$$\mathcal{F}_0 \sim -\mu^{-2\epsilon}\frac{1}{4}\frac{m^4}{(4\pi)^2}\left(\frac{1}{\epsilon}+\frac{3}{2}+\ln\frac{\tilde{\mu}^2}{m^2}\right), \quad \epsilon\to 0. \tag{A.47}$$

Renormalization

Now that we have applied a regulator, we are able to handle the divergence in a well-defined way. When $\epsilon \neq 0$, we can subtract to terms which are proportional to ϵ^{-1}, and be left with a term that is finite in the limit $\epsilon \to 0$. Consider an arbitrary Lagrangian,

$$\mathcal{L}[\varphi]=\sum_n \lambda_n \mathcal{O}_n[\varphi]. \tag{A.48}$$

Here, $\mathcal{O}_n$ are operators consisting of φ and $\partial_\mu\varphi$, and λ_n are coupling constants. In d dimensions, the action integral is

$$S[\varphi]=\sum_n \int \mathrm{d}^d x\, \lambda_n \mathcal{O}_n[\varphi]. \tag{A.49}$$

The action must have mass dimension 0. This means that all terms $\lambda_n\mathcal{O}_n$ must have mass dimension d, as $[\mathrm{d}^d x] = -d$. We are free to choose the coupling constant corresponding $\mathcal{O}_0 = \partial_\mu\varphi\partial^\mu\varphi$ as $\lambda_0 = 1/2$ to get a cannocially normalized field. This allows us to deduce the dimensionality of φ. As $[\partial_x] = 1$, we have that $[\varphi] = (d-2)/2$. Assume $\mathcal{O}_n$ consists of k_n factors of φ, and l_n factors of $\partial_\mu\varphi$. We must then have

$$[\lambda_n]+[\mathcal{O}_n]-d=[\lambda_n]+(k_n+l_n)(d-2)/2+l_n-d=0, \tag{A.50}$$

$$\implies D_n := [\lambda_n] = d-k_n\frac{d-2}{2}-l_n\frac{d}{2}. \tag{A.51}$$

From this formula, we recover that $[\lambda_0] = 0$, and if $\lambda_1\varphi^2$, then $[\lambda_1] = 2$, which we recognize as the mass squared term. The mass dimensions of these coupling constants are independent of d. However, the coupling constant for the interaction term

$$-\frac{1}{4!}\lambda_3\varphi^4 \tag{A.52}$$

has mass dimensions $[\lambda_3] = d-4(d-2)/2 = 4-2d$. Our goal now is to exchange the bare coupling constants λ_n with renormalized ones, λ_n^r, and remove the divergent terms proportional to $(d-4)^{-m}$. We can always define the renormalized coupling constants as dimensionless, i.e. $[\lambda_n^r] = 0$, by measuring them in units of any mass scale. We therefore write

$$\lambda_n = \mu^{4-D_n}\left[\lambda_n^r+\sum_{m=1}\frac{a_m(\lambda_n^r)}{(d-4)^m}.\right],$$

where we have introduced the dimensional parameter μ to ensure that λ_n has the correct mass dimension, so that the action integral stays dimensionless. The functions a_m are then determined to each order in perturbation theory by calculating Feynman diagrams. As μ again is arbitrary, λ_4' should not depend on this

parameter. In this case, we chose the same renormalization scale as we did when regulating the one-loop integral. This is only for our own convenience. This means that if we change $\mu \to \mu'$, then λ_i^r and a_m must adjust to compensate and keep λ_n constant [24].

The term that is used to absorb the divergence in the one loop contribution to the free energy density is the vacuum energy

$$\lambda_4 \mathcal{O}_4 = \lambda_4 = m^4 \lambda_4'. \tag{A.53}$$

Using the expansion in terms of renormalized coupling, we have, using $d = 4 - 2\epsilon$

$$\lambda_4' = \mu^{-2\epsilon} \left[\lambda_4^r + \frac{1}{2\epsilon} a_1(\lambda_r^4) + ... \right] \tag{A.54}$$

If we add Eq. (A.53) to the Lagrangian of the free term, which we are allowed to as to Lagrangian that differ by just a constant result in the same physics, then the temperature independent free energy density becomes

$$\mathcal{F}_0 \sim -\mu^{-2\epsilon} \frac{1}{4} \frac{m^4}{(4\pi)^2} \left[\frac{1}{\epsilon} + \frac{3}{2} + \ln \frac{\tilde{\mu}^2}{m^2} + (4(4\pi)^2) \left(\lambda_4^r + \frac{1}{2\epsilon} a_1(\lambda_4^r) \right) \right], \quad \epsilon \to 0. \tag{A.55}$$

Thus, if we choose $a_1 = -8(4\pi)^2 + \mathcal{O}((\lambda_4^r))$, and define $\lambda_4'^r = 4(4\pi)^2 \lambda_4^r$, we are able to cancel the divergence, and may take the limit $\epsilon \to 0$ safely. The free energy thus becomes

$$\mathcal{F} = -\frac{1}{4} \frac{m^4}{(4\pi)^2} \left(\frac{3}{2} + \lambda_4'^r + \ln \frac{\tilde{\mu}^2}{m^2} \right) + \frac{T^4}{2\pi^2} \int \mathrm{d}x \, x^2 \ln\Big(1 - \exp\Big\{ -\sqrt{x^2 + \beta^2 m^2} \Big\}\Big). \tag{A.56}$$

Notice that the choices we have made up until now, for example by defining $\lambda_4 = m^4 \lambda_4'$, and using the same renormalization scale μ has no impact on this result, as any other definition would just force us to define λ_4^r and a_4 differently.

A.4 Interacting scalar

We now study a scalar field with a $\lambda\varphi^4$ interaction term. We write the Lagrangian in the form

$$\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(I)}, \quad \mathcal{L}^{(0)} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - m^2 \varphi^2, \quad \mathcal{L}^{(I)} = -\frac{\lambda}{4!} \varphi^4$$

$\mathcal{L}^{(I)}$ is called the interaction term, and makes it impossible to exactly solve for the partition function. Instead, we turn to perturbation theory. The canonical partition function in this theory is

$$Z = \mathrm{Tr}\Big\{ e^{-\beta \hat{H}} \Big\} = \int_S \mathcal{D}\varphi \, \exp\Big\{ -\int_\Omega \mathrm{d}X \left(\mathcal{L}_E^{(0)} + \mathcal{L}_E^{(I)} \right) \Big\} = \int_S \mathcal{D}\varphi \, e^{-S_0} e^{-S_I}. \tag{A.57}$$

Here, S_0 and S_I denote the Euclidean action due to the free and interacting Lagrangian, respectively. The domain of integration S is again periodic field configurations $\varphi(\beta, \vec{x}) = \varphi(0, \vec{x})$. We may write the free energy as

$$-\beta F = \ln \left[\int_S \mathcal{D}\varphi \, e^{-S_0} \sum_n \frac{1}{n!} (-S_I)^n \right] = \ln[Z_0] + \ln\left[Z_I \right],$$

where Z_0 is the partition function of the free theory. The correction to the partition function is thus given by

$$Z_I = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left\langle {S_I}^n \right\rangle_0, \tag{A.58}$$

where

$$\langle A \rangle_0 = \frac{\int_S \mathcal{D}\varphi \, A \, e^{-S_0}}{\int_S \mathcal{D}\varphi \, e^{-S_0}}. \tag{A.59}$$

To evaluate expectation values of the form $\langle\varphi(X_1)...\rangle_0$, we introduce the partition function with a source term

$$Z[J] = \int_S \mathcal{D}\varphi \, \exp\left\{-\frac{1}{2}\int_\Omega \mathrm{d}X \, \varphi(-\partial_E^2 + m^2)\varphi + \int_\Omega \mathrm{d}X \, J\varphi\right\}. \tag{A.60}$$

Using the thermal Greens function $D_0(X,Y)$, as defined in section B.1, we may complete the square to write

$$Z[J] = Z[0]\exp\left\{\frac{1}{2}\int_\Omega \mathrm{d}X\mathrm{d}Y \, J(X)D_0(X,Y)J(Y)\right\} = Z[0]\exp(W[J]) \tag{A.61}$$

We can now write

$$\langle\varphi(X)\varphi(Y)\rangle_0 = \frac{1}{Z[0]}\frac{\delta}{\delta J(X)}\frac{\delta}{\delta J(Y)}Z[J]\Big|_{J=0} = D_0(X,Y), \tag{A.62}$$

This generalizes to higher order expectation values,

$$\langle\varphi(X_i)\dots\varphi(X_n)\rangle_0 = \left(\prod_{i=1}^n \frac{\delta}{\delta J(X_i)}\right) Z[J]\Big|_{J=0}, \tag{A.63}$$

Using Wick's theorem, as described in section 1.1, the expectation values we are evaluating can be written

$$\begin{aligned}\langle {S_I}^m\rangle &= \left(-\frac{\lambda}{4!}\right)^m \int_\Omega \mathrm{d}X_1\dots\mathrm{d}X_m \left\langle\varphi^4(X_1)\dots\varphi^4(X_m)\right\rangle \\ &= \left(-\frac{\lambda}{4!}\right)^m \int_\Omega \mathrm{d}X_1\dots\mathrm{d}X_m \sum_{\{a,b\}} \left\langle\varphi(X_{a(1)})\varphi(X_{b(1)})\right\rangle\dots\left\langle\varphi(X_{a(2m)})\varphi(X_{b(2m)})\right\rangle\end{aligned}$$

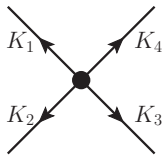
where X_i for $i > m$ is defined as X_j, where $j = i \mod m$. This means that $X_{m+i} = X_i$. The functions a, b represents a possible pairing, as described in section 1.1. Inserting the Fourier expansions of the field gives

$$\begin{aligned}&\langle {S_I}^m\rangle \\ &= \left(-\frac{\lambda}{4!}\right)^m \int_\Omega \mathrm{d}X_1\dots\mathrm{d}X_m (V\beta)^2 \int_{\tilde\Omega} \mathrm{d}K_1...\mathrm{d}K_{2m} \sum_{\{a,b\}} \\ &\qquad \left\langle\varphi(K_{a(1)})\varphi(K_{b(1)})\right\rangle\dots\left\langle\varphi(K_{a(2m)})\varphi(K_{b(2m)})\right\rangle \exp\left(i\textstyle\sum_{i=1}^m X_i\cdot K_i\right) \\ &= \left(-\frac{\lambda}{4!}\right)^m \frac{(V\beta)^{2m}\beta^m}{(V\beta^2)^{2m}} \int_{\tilde\Omega} \mathrm{d}K_1...\mathrm{d}K_{2m} \sum_{\{a,b\}} \\ &\qquad \tilde D(K_{a(1)})\delta(K_{a(1)} + K_{b(1)})\dots\tilde D(K_{a(2m)})\delta(K_{a(2m)} + K_{b(2m)}) \prod_{i=1}^m \delta\left(X_i\cdot\textstyle\sum_{j=0}^3 K_{i+jm}\right) \\ &= \left(-\frac{\lambda\beta}{4!}\right)^m \prod_{i=1}^{2m}\int_{\tilde\Omega}\left(\mathrm{d}K_i\frac{1}{\beta}\tilde D(K_i)\right)\prod_{i=1}^m \delta\left(X_i\cdot\textstyle\sum_{j=0}^3 K_{i+jm}\right)\sum_{\{a,b\}}\prod_{n=1}^{2m}\delta(K_{a(k)} + K_{b(k)})\end{aligned}$$

Here we have used that $V\beta^2\tilde D_0(K,P) = \tilde D_0(K)\delta(P+K)$, where $\tilde D_0(K)$ is the thermal propagator for the free field, as defined in section B.1. In this case, it is

$$\tilde D_0(K) = \tilde D_0(\omega_n,\vec k) = \frac{1}{\omega_k^2 + \omega_n^2}. \tag{A.64}$$

This expectation value can be represented graphically using Feynman diagrams. The thermal $\lambda\varphi^2$-theory gets the prescription



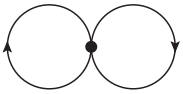
$$= -\lambda\beta\delta\left(\sum\nolimits_i K_i\right), \tag{A.65}$$

$$= \frac{1}{\beta} D_0(K). \tag{A.66}$$

Lastly, one has to integrate over all internal momenta, and divide by a symmetry factor of the diagram s, which is described in detail in [1].

Calculating $\langle {S_I}^n \rangle_0$ boils down to the sum of all possible Feynman diagrams with n vertices. The first example is

$$\langle S_I \rangle = \frac{1}{8}$$

 (A.67)

In section 1.1, we saw that the sum of all vacuum diagrams is the exponential of the sum of all *connected* diagrams, so the free energy of the interacting theory is given by

$$-\beta F = \ln(Z_0) + \Sigma(\text{all connected diagrams}) \tag{A.68}$$

A.5 Fermions

The phase factor $e^{i\theta}$ that was introduced in section A.2 can be determined by studying the properties of the thermal Greens function. The thermal Greens function may be written

$$D(X_1, X_2) = D(\vec{x}, \vec{y}, \tau_1, \tau_2) = \left\langle e^{-\beta\hat{H}} \mathrm{T}\left\{\hat{\varphi}(X_1)\hat{\varphi}(X_2)\right\}\right\rangle.$$

$\mathrm{T}\{...\}$ is time-ordering operator, and is defined as

$$\mathrm{T}\left\{\varphi(\tau_1)\varphi(\tau_2)\right\} = \theta(\tau_1 - \tau_2)\varphi(\tau_1)\varphi(\tau_2) + \nu\theta(\tau_2 - \tau_1)\varphi(\tau_2)\varphi(\tau_1),$$

where $\nu = \pm 1$ for bosons and fermions respectively, and $\theta(\tau)$ is the Heaviside step function. In the same way that $i\hat{H}$ generates the time translation of a quantum field operator through $\hat{\varphi}(x) = \hat{\varphi}(t, \vec{x}) = e^{it\hat{H}}\hat{\varphi}(0, \vec{x})e^{-it\hat{H}}$, the imaginary-time formalism implies the relation

$$\hat{\varphi}(X) = \hat{\varphi}(\tau, \vec{x}) = e^{\tau\hat{H}}\hat{\varphi}(0, \vec{x})e^{-\tau\hat{H}}. \tag{A.69}$$

Using $\mathbb{1} = e^{\tau\hat{H}}e^{-\tau\hat{H}}$ and the cyclic property of the trace, we show that, assuming $\beta > \tau > 0$,

$$\begin{aligned}
G(\vec{x}, \vec{y}, \tau, 0) &= \left\langle e^{-\beta\hat{H}} \mathrm{T}\left\{\varphi(\tau, \vec{x})\varphi(0, \vec{y})\right\}\right\rangle \\
&= \frac{1}{Z}\,\mathrm{Tr}\Big\{e^{-\beta\hat{H}}\varphi(\tau, \vec{x})\varphi(0, \vec{y})\Big\} \\
&= \frac{1}{Z}\,\mathrm{Tr}\Big\{\varphi(0, \vec{y})e^{-\beta\hat{H}}\varphi(\tau, \vec{x})\Big\} \\
&= \frac{1}{Z}\,\mathrm{Tr}\Big\{e^{-\beta\hat{H}}e^{\beta\hat{H}}\varphi(0, \vec{y})e^{-\beta\hat{H}}\varphi(\tau, \vec{x})\Big\} \\
&= \frac{1}{Z}\,\mathrm{Tr}\Big\{e^{-\beta\hat{H}}\varphi(\vec{y}, \beta)\varphi(\tau, \vec{x})\Big\} \\
&= \nu\left\langle e^{-\beta\hat{H}} \mathrm{T}\left\{\varphi(\tau, \vec{x})\varphi(\beta, \vec{y})\right\}\right\rangle.
\end{aligned}$$

This implies that $\varphi(0,x) = \nu\varphi(\beta,\varphi)$, which shows that bosons are periodic in time, as stated earlier, while fermions are anti-periodic.

The Lagrangian density of a free fermion is

$$\mathcal{L} = \bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi. \tag{A.70}$$

This Lagrangian is invariant under the transformation $\psi \to e^{-i\alpha}\psi$, which by Nöther's theorem results in a conserved current

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta\psi = \bar{\psi}\gamma^\mu\psi. \tag{A.71}$$

The corresponding conserved charge is

$$Q = \int_V \mathrm{d}^3x\, j^0 = \int_V \mathrm{d}^3x\, \bar{\psi}\gamma^0\psi. \tag{A.72}$$

We can now use our earlier result for the thermal partition function, Eq. (A.12), only with the substitution $\mathcal{H} \to \mathcal{H} - \mu\bar{\psi}\gamma^0\psi$, and integrate over anti-periodic ψ's:

$$Z = \mathrm{Tr}\Big\{e^{-\beta(\hat{H}-\mu\hat{Q})}\Big\} = \prod_{ab}\int \mathcal{D}\psi_a\mathcal{D}\pi_b \exp\Big\{\int_\Omega \mathrm{d}X\,\Big(i\dot{\psi}\pi - \mathcal{H}(\psi,\pi) + \mu\bar{\psi}\gamma^0\psi\Big)\Big\},$$

where a, b are the spinor indices. The canonical momentum corresponding to ψ is

$$\pi = \frac{\partial\mathcal{L}}{\partial(\partial_0\psi)} = i\bar{\psi}\gamma^0, \tag{A.73}$$

and the Hamiltonian density is

$$\mathcal{H} = \pi\dot{\psi} - \mathcal{L} = \bar{\psi}(-i\gamma^i\partial_i + m)\psi \tag{A.74}$$

which gives

$$\mathcal{L}_E = -i\pi\dot{\psi} + \mathcal{H}(\psi,\pi) - \mu\bar{\psi}\gamma^0\psi = \bar{\psi}[\gamma^0(\partial_\tau - \mu) - i\gamma^i\partial_i + m]\psi, \tag{A.75}$$

By using the Grassman-version of the Gaussian integral formula, the partition function can be written

$$\begin{aligned} Z &= \prod_{ab}\int \mathcal{D}\psi_a\mathcal{D}\bar{\psi}_b \exp\Big\{-\int_\Omega \mathrm{d}X\,\bar{\psi}\left[\tilde{\gamma}_0(\partial_\tau - \mu) - i\gamma^i\partial_i + m\right]\psi\Big\} \\ &= C\prod_{ab}\int \mathcal{D}\tilde{\psi}_a\mathcal{D}\tilde{\bar{\psi}}_b \exp\Big\{-\int_{\tilde{\Omega}} \mathrm{d}K\,\tilde{\bar{\psi}}\left[i\tilde{\gamma}_0(\omega_n + i\mu) + i\gamma_i p_i + m\right]\tilde{\psi}\Big\} \\ &= C\prod_{ab}\int \mathcal{D}\tilde{\psi}_a\mathcal{D}\tilde{\bar{\psi}}_b e^{-\langle\tilde{\bar{\psi}}, D_0^{-1}\psi\rangle} = \det\big(D_0^{-1}\big). \end{aligned}$$

In the second line, we have inserted the Fourier expansion of the field, as defined in section B.1, and changed variable of integration, as we did for the scalar field. The linear operator in this case is

$$D_0^{-1} = i\gamma^0(-i\partial_\tau + i\mu) - (-i\gamma^i)\partial_i + m = \beta[i\tilde{\gamma}_a p_a + m]. \tag{A.76}$$

This equality must be understood as an equality between linear operators, which are represented in different bases. We introduced the notation $p_{n;a} = (\omega_n + i\mu, p_i)$ and use the Euclidean gamma matrices, as defined in section B.1. We use the fact that

$$\det(i\tilde{\gamma}_a p_a + m) = \det\big(\gamma^5\gamma^5\big)\det(i\tilde{\gamma}_a p_a + m) = \det\big[\gamma^5(i\tilde{\gamma}_a p_a + m)\gamma^5\big] = \det(-i\tilde{\gamma}_a p_a + m),$$

Let $\tilde{D} = -i\tilde{\gamma}_a p_a + m$, which means we can write

$$Z = \sqrt{\det(D)\det\Big(\tilde{D}\Big)} = \sqrt{\det\Big(D\tilde{D}\Big)} = \det\big[\mathbb{1}(p_a p_a + m^2)\big]^{1/2}, \tag{A.77}$$

where we have used the anti-commutation rule for the Euclidean gamma-matrices, $\{\gamma_a, \gamma_b\} = 2\delta_{ab}$. It is important to keep in mind that the determinant here refers to linear operators on the space of spinor functions.

$$\begin{aligned}\ln(Z) &= \ln\left\{\det\left[\mathbb{1}(p_a p_a + m^2)\right]^{1/2}\right\} = \frac{1}{2}\,\mathrm{Tr}\left\{\ln[\mathbb{1}(p_a p_a + m^2)]\right\} \\ &= \frac{1}{2}\int_{\tilde{\Omega}} \mathrm{d}K\, \ln[\mathbb{1}\beta^2(p_a p_a + m^2)]_{aa} \end{aligned} \tag{A.78}$$

As the matrix within the logarithm is diagonal, the matrix-part of the trace is trivial, and the free energy may be written

$$\beta\mathcal{F} = -2\int_{\tilde{\Omega}} \mathrm{d}X\ \ln\{\beta^2[(\omega_n + i\mu)^2 + \omega_k^2]\}. \tag{A.79}$$

Using the fermionic version of the thermal sum from section A.3 gives the answer

$$\mathcal{F} = 2\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\left[\beta\omega_k + \frac{1}{\beta}\ln\left(1 + e^{-\beta(\omega_k - \mu)}\right) + \frac{1}{\beta}\ln\left(1 + e^{-\beta(\omega_k + \mu)}\right)\right]. \tag{A.80}$$

We see again that the temperature-independent part of the integral diverges, and must be regulated. There are two temperature-dependent terms, one from the particle and one from the anti-particle.

Appendix B

Appendices

B.1 Conventions and notation

Throughout this text, natural units are employed, in which

$$\hbar = c = k_B = 1, \tag{B.1}$$

where $\hbar$ is the Planck reduced constant, k_B is the Boltzmann constant and c is the speed of light. The Minkowski metric convention used is the "mostly minus", $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.

The $\mathfrak{su}(2)$ basis used is the Pauli matrices,

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

They obey

$$[\tau_a, \tau_b] = 2i\varepsilon_{abc}\tau_c, \quad \{\tau_a, \tau_b\} = 2\delta_{ab}\mathbb{1}, \quad \mathrm{Tr}[\tau_a] = 0, \quad \mathrm{Tr}[\tau_a\tau_b] = 2\delta_{ab}\mathbb{1}.$$

Together with the identity matrix $\mathbb{1}$, the Pauli matrices form a basis for the vector space of all 2-by-2 matrices. An arbitrary 2-by-2 matrix M may be written

$$M = M_0\mathbb{1} + M_a\tau_a, \quad M_0 = \frac{1}{2}\,\mathrm{Tr}\{M\},\ M_a = \frac{1}{2}\,\mathrm{Tr}\{\tau_a M\}. \tag{B.2}$$

The gamma matrices γ^μ, $\mu \in \{0, 1, 2, 3\}$, obey

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}. \tag{B.3}$$

The "fifth γ-matrix" is defined by

$$\gamma^5 = \frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = i\gamma^0\gamma^1\gamma^2\gamma^3. \tag{B.4}$$

The γ^5-matrix obey

$$\left\{\gamma^5, \gamma^\mu\right\} = 0, \quad (\gamma^5)^2 = \mathbb{1}, \quad {\gamma^0}^\dagger = \gamma^0,\ {\gamma^i}^\dagger = -\gamma^i \tag{B.5}$$

Their Euclidean counterpart obey

$$\{\tilde{\gamma}_a, \tilde{\gamma}_b\} = 2\delta_{ab}\mathbb{1}, \quad \tilde{\gamma}_a^\dagger = \tilde{\gamma}_a, \tag{B.6}$$

and they are related by $\tilde{\gamma}_0 = \gamma^0$, and $\tilde{\gamma}_j = -i\gamma^j$. The Euclidean $\tilde{\gamma}_5$ is defined as

$$\tilde{\gamma}_5 = \gamma_0\gamma_1\gamma_2\gamma_3 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5. \tag{B.7}$$

It thus also anti-commutes with the Euclidean γ-matrices.

B.1.1 Fourier transform

The Fourier transform used in this text is defined by

$$\mathcal{F}\left\{f(x)\right\}(p) = \tilde{f}(p) = \int \mathrm{d}x\, e^{ipx} f(x), \quad \mathcal{F}^{-1}\left\{\tilde{f}(p)\right\}(x) = f(x) = \int \frac{\mathrm{d}p}{2\pi}\, e^{-ipx}\tilde{f}(p).$$

B.1.2 Fourier series

Imaginary-time formalism is set in a Euclidean space $\Omega = [0,\beta] \times V$, where $V = L_x L_y L_z$ is a space-like volume. The possible momenta in this space are

$$\tilde{V} = \left\{ \vec{k} \in \mathbb{R}^3 \,\middle|\, \vec{k} = \left(\frac{2\pi n_x}{L_x}, \frac{2\pi n_y}{L_y}, \frac{2\pi n_z}{L_z} \right) \right\}$$

ω_n are the Matsubara-frequencies, with $\omega_n = 2n\pi/\beta$ for bosons and $\omega_n = (2n+1)\pi/\beta$ for fermions. They together form the reciprocal space $\tilde{\Omega} = \{\omega_n\} \times \tilde{V}$. The Euclidean coordinates are denoted $X = (\tau, \vec{x})$ and $K = (\omega_n, \vec{K})$, and have the dot product $X \cdot P = \omega_n \tau + \vec{k} \cdot \vec{x}$. In the limit $V \to \infty$, we follow the prescription

$$\frac{1}{V} \sum_{\vec{p} \in \tilde{V}} \to \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3}.$$

The sum over all degrees of freedom, and the corresponding integrals for the thermodynamic limit are

$$\frac{\beta V}{NM} \sum_{n=1}^{N} \sum_{\vec{x}_m \in V} \xrightarrow{N,M\to\infty} \int_0^\beta \mathrm{d}\tau \int_{\mathbb{R}^3} \mathrm{d}^3 x = \int_\Omega \mathrm{d}X,$$
$$\frac{1}{V} \sum_{n=-\infty}^{\infty} \sum_{\vec{k} \in \tilde{V}} \xrightarrow{V\to\infty} \sum_{n=-\infty}^{\infty} \int_{R^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} = \int_{\tilde{\Omega}} \mathrm{d}K.$$

The convention used for the Fourier expansion of thermal fields is in accordance with [21]. The prefactor is chosen to make the Fourier components of the field dimensionless, which makes it easier to evaluate the trace correctly. For bosons, the Fourier expansion is

$$\varphi(X) = \sqrt{V\beta} \int_{\tilde{\Omega}} \mathrm{d}K\, \tilde{\varphi}(K) e^{iX\cdot K} = \sqrt{\frac{\beta}{V}} \sum_{n=-\infty}^{\infty} \sum_{\vec{k} \in \tilde{V}} \tilde{\varphi}_n(\vec{p}) \exp\Big\{ i(\omega_n \tau + \vec{x} \cdot \vec{k}) \Big\},$$
$$\tilde{\varphi}(K) = \sqrt{\frac{1}{V\beta^3}} \int_{\tilde{\Omega}} \mathrm{d}X\, \tilde{\varphi}(X) e^{-iX\cdot K}$$

while for Fermions it is

$$\psi(X) = \sqrt{V} \int_{\tilde{\Omega}} \mathrm{d}K\, \tilde{\psi}(K) e^{iX\cdot K} = \frac{1}{\sqrt{V}} \sum_{n=-\infty}^{\infty} \sum_{\vec{k} \in \tilde{V}} \psi(\omega_n, \vec{k}) \exp\Big\{ i(\omega_n \tau + \vec{x} \cdot \vec{k}) \Big\} \tag{B.8}$$

A often used identity is

$$\int_\Omega \mathrm{d}X e^{iX\cdot(K-K')} = \beta \delta_{nn'} (2\pi)^3 \delta^3(\vec{k} - \vec{k}') := \beta\delta(K - K'), \tag{B.9}$$
$$\int_{\tilde{\Omega}} \mathrm{d}K\, e^{iK(X-X')} = \beta\delta(\tau - \tau')\delta^3(\vec{x} - \vec{x}') := \beta\delta(X - X'). \tag{B.10}$$

B.1.3 Particle distributions

The Bose distribution is defined as

$$n_B(\omega) = \frac{1}{e^{\beta\omega} - 1}. \tag{B.11}$$

This function obeys

$$n_B(-i\omega) = -1 - n_B(i\omega). \tag{B.12}$$

We can expand it around the Bose Matsubara frequencies on the imaginary line:

$$in_B[i(\omega_n + \epsilon)] = \frac{i}{e^{i\beta\epsilon + 2\pi i n} - 1} = i[i\beta\epsilon + \mathcal{O}\left(\epsilon^2\right)]^{-1} \sim \frac{1}{\epsilon\beta}. \tag{B.13}$$

This means that $in_B(i\omega)$ has a pole on all Matsubara-frequencies, with residue $1/\beta$. Furthermore, we have

$$\frac{\mathrm{d}}{\mathrm{d}\omega} \ln\left(1 - e^{-\beta\omega}\right) = \beta n_B(\omega). \tag{B.14}$$

The Fermi distribution is

$$n_F(\omega) = \frac{1}{e^{\beta\omega} + 1}. \tag{B.15}$$

It obeys

$$\frac{\mathrm{d}}{\mathrm{d}\omega} \ln\left(1 - e^{-\beta\omega}\right) = -\beta n_F(\omega), \tag{B.16}$$

$$n_F(-i\omega) = 1 - n_F(i\omega). \tag{B.17}$$

The two distributions are related by

$$2n_B(i\omega; 2\beta) - n_B(i\omega; \beta) = -n_F(i\omega; \beta). \tag{B.18}$$

B.1.4 Propagators

If $D^{-1}[f(x)] = 0$ is the equation of motion for some field f, where D^{-1} in general is a differential operator, then the propagator $D(x, x')$ for this field is defined by

$$D^{-1}[D(x, x')] = -i\delta(x - x')\mathbb{1}.$$

Assuming A is linear and independent of space, we may redefine $D(x - x', 0) \to D(x - x')$, and the Fourier transform with respect to both x and x' to obtain

$$\mathcal{F}\left\{D^{-1}[D(x, x')]\right\}(p, p') = \tilde{D}^{-1}(p)\,\tilde{D}(p)\,\delta(p + p') = -i\delta(p + p'),$$

meaning the momentum space propagator $\tilde{D}(p) = \mathcal{F}\left\{D(x)\right\}(p)$ is given by $\tilde{D} = -i(\tilde{D}^{-1})^{-1}$.

For some differential operator D^{-1}, the thermal propagator is defined as

$$D^{-1}D(X, Y) = \beta\delta(X - Y). \tag{B.19}$$

The Fourier transformed propagator is, assuming $D(X, Y) = D(X - Y, 0)$,

$$\tilde{D}(K, K') = \frac{1}{V\beta^3} \int_\Omega \mathrm{d}X \mathrm{d}Y\, D(X, Y) \exp(-i[X \cdot K + Y \cdot K']) \tag{B.20}$$

$$= \frac{1}{V\beta^3} \int_\Omega \mathrm{d}X' \mathrm{d}Y'\, D(X', 0) \exp\left(-i[X' \cdot \frac{1}{2}(K - K') + Y \cdot (K + K')]\right) \tag{B.21}$$

$$= \frac{1}{V\beta^2} \tilde{D}(K)\delta(K + K'), \tag{B.22}$$

where

$$\tilde{D}(K) = \int \mathrm{d}X e^{iK \cdot X} D(X, 0). \tag{B.23}$$

B.2 Covariant derivative

In χPT at finite isospin chemical potential μ_I, the covariant derivative acts on functions $A(x) : \mathcal{M}_4 \to \mathrm{SU}(2)$, where $\mathcal{M}_4$ is the space-time manifold. It is defined as

$$\nabla_\mu A(x) = \partial_\mu A(x) - i\left[v_\mu, A(x)\right], \quad v_\mu = \frac{1}{2}\mu_I \delta^0_\mu \tau_3. \tag{B.24}$$

The covariant derivative obeys the product rule, as

$$\nabla_\mu(AB) = (\partial_\mu A)B + A(\partial_\mu B) - i\left[v_\mu, AB\right] = (\partial_\mu A - i\left[v_\mu, A\right])B + A(\partial_\mu B - i\left[v_\mu, B\right]) = (\nabla_\mu A)B + A(\nabla_\mu B).$$

Decomposing a 2-by-2 matrix M, as described in section B.1, shows that the trace of the commutator of τ_b and M is zero:

$$\mathrm{Tr}\{[\tau_a, M]]\} = M_b \,\mathrm{Tr}\{[\tau_a, \tau_b]\} = 0.$$

Together with the fact that $\mathrm{Tr}\{\partial_\mu A\} = \partial_\mu \,\mathrm{Tr}\{A\}$, this gives the product rule for invariant traces:

$$\mathrm{Tr}\{A\nabla_\mu B\} = \partial_\mu \,\mathrm{Tr}\{AB\} - \mathrm{Tr}\{(\nabla_\mu A)B\}.$$

This allows for the use of the divergence theorem when doing partial integration. Let $\mathrm{Tr}\{K^\mu\}$ be a space-time vector, and $\mathrm{Tr}\{A\}$ scalar. Let Ω be the domain of integration, with coordinates x and $\partial\Omega$ its boundary, with coordinates y. Then,

$$\int_\Omega \mathrm{d}x \;\mathrm{Tr}\{A\nabla_\mu K^\mu\} = \int_{\partial\Omega} \mathrm{d}y \, n_\mu \,\mathrm{Tr}\{AK^\mu\} - \int_\Omega \mathrm{d}x \;\mathrm{Tr}\{(\nabla_\mu A)K^\mu\},$$

where n_μ is the normal vector of $\partial\Omega$.[23] This makes it possible to do partial integration and discard surface terms in the χPT Lagrangian, given the assumption of no variation on the boundary.

B.3 Integrals

A useful integral is the Gaussian integral,

$$\int_{\mathbb{R}} \mathrm{d}z \, \exp\left(-\frac{1}{2}az^2\right) = \sqrt{\frac{2\pi}{a}}, \tag{B.25}$$

for $a \in \mathbb{R}$. The imaginary version,

$$\int_R \mathrm{d}z \, \exp\left(i\frac{1}{2}az^2\right) \tag{B.26}$$

does not converge. However, if we change $a \to a + i\epsilon$, then the integrand is exponentially suppressed.

$$f(x) = \exp\left(i\frac{1}{2}ax^2\right) \to \exp\left(i\frac{1}{2}ax^2 - \frac{1}{2}\epsilon x^2\right), \tag{B.27}$$

As the integrand falls of exponentially for $x \to \infty$, and contains no poles in the upper right nor lower left quarter of the complex plane, we may perform a wick rotation by closing the contour as shown in Figure B.1. This gives the result

$$\int_{\mathbb{R}} \mathrm{d}x \, \exp\left(i\frac{1}{2}(a+i\epsilon)x^2\right) = \int_{\sqrt{i}\mathbb{R}} \mathrm{d}x \, \exp\left(i\frac{1}{2}ax^2\right) = \sqrt{i}\int_{\mathbb{R}} \mathrm{d}y \, \exp\left(-\frac{1}{2}(-a)y^2\right) = \sqrt{\frac{2\pi i}{(-a)}} \tag{B.28}$$

where we have made the change of variable $y = (1+i)/\sqrt{2}x = \sqrt{i}x$. In n dimensions, the Gaussian integral formula generalizes to

$$\int_{\mathbb{R}^n} \mathrm{d}^n x \, \exp\left\{-\frac{1}{2}x_n A_{nm} x_m\right\} = \sqrt{\frac{(2\pi)^n}{\det(A)}}, \tag{B.29}$$

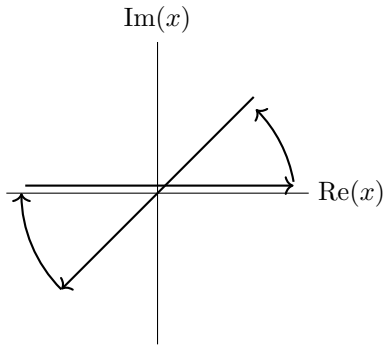


Figure B.1: Wick rotation

where A is a matrix with n real, positive eigenvalues. We may also generalize Eq. (B.28),

$$\int_{\mathbb{R}^n} \mathrm{d}^n x \, \exp\left\{ i\frac{1}{2} x_n (A_{nm} + i\epsilon\delta_{nm}) x_m \right\} = \sqrt{\frac{(2\pi i)^n}{\det(-A)}}. \tag{B.30}$$

The final generalization is to functional integrals,

$$\int \mathcal{D}\varphi \, \exp\left(-\frac{1}{2}\int \mathrm{d}x \, \varphi(x) A \varphi(x)\right) = C(\det(A))^{-1/2}, \int \mathcal{D}\varphi \, \exp\left(i\frac{1}{2}\int \mathrm{d}x \, \varphi(x) A \varphi(x)\right) = C(\det(-A))^{-1/2}. \tag{B.31}$$

C is here a divergent constant, but will either fall away as we are only looking at the logarithm of I_∞ and are able to throw away additive constants, or ratios between quantities which are both multiplied by C.

The Gaussian integral can be used for the stationary phase approximation. In one dimension, it is

$$\int \mathrm{d}x \, \exp(i\alpha f(x)) \approx \sqrt{\frac{2\pi}{f''(x_0)}} \exp(f(x_0)), \; f'(x) = 0, \; \alpha \to \infty \tag{B.32}$$

The functional generalization of this is

$$\int \mathcal{D}\varphi \exp\{iS[\varphi]\} \approx C \det\left(-\frac{\delta^2 S[\varphi_0]}{\delta\varphi^2}\right) \exp\{i\alpha S[\varphi_0]\}, \quad \frac{\delta S[\varphi_0]}{\delta\varphi} = 0 \tag{B.33}$$

Here, $S[\varphi]$ is a general functional of φ, and we have used the Taylor expansion, as described in section B.4, and φ_0 fulfills

$$\frac{\delta}{\delta\varphi(x)} S[\varphi_0] = 0, \tag{B.34}$$

B.4 Functional Derivatives

Functional derivatives generalize the notion of a gradient and the directional derivative. A function $f(p)$, where p is point with coordinates $x_i = x_i(p)$, has a gradient

$$\mathrm{d}f_p = \frac{\partial f(p)}{\partial x_i} \mathrm{d}x_i. \tag{B.35}$$

The derivative in a particular direction $v = v^i \partial_i$ is

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon} f(x_i + \epsilon v_i) = f(x) + \mathrm{d}f_x(v) = f(x) + \frac{\partial f}{\partial x^i} v_i. \tag{B.36}$$

This is generalized to functionals through the definition of functional derivative, and the variation of a functional. Let $F[f]$ be a functional, i.e. a machine that takes in a function, and returns a number. The

obvious example in our case is the action, which takes in one or more field-configurations, and returns a single real number. We will assume here that the functions have the domain Ω, with coordinates x. The functional derivative is defined as

$$\delta F[f] = \frac{\mathrm{d}}{\mathrm{d}\epsilon} F[f + \epsilon\eta]\Big|_{\epsilon=0} = \int_\Omega \mathrm{d}x \, \frac{\delta F[f]}{\delta f(x)} \eta(x). \tag{B.37}$$

$\eta(x)$ is here an arbitrary function, but we will make the important assumption that it as well as all its derivatives are zero at the boundary of its domain Ω. This allows us to discard surface terms stemming from partial integration, which we will use frequently. We may use the definition to derive one of the fundamental relations of functional derivation. Take the functional $F[f] = f(x)$. Then,

$$\delta F[f] = \frac{\mathrm{d}}{\mathrm{d}\epsilon}[f(x) + \epsilon\eta(x)] = \eta(x) = \int \mathrm{d}y \, \delta(x-y)\eta(y) \tag{B.38}$$

This leads to the identity

$$\frac{\delta f(x)}{\delta f(y)} = \delta(x-y), \tag{B.39}$$

for any function f. Higher functional derivatives are defined similarly, by applying functional variation repeatedly

$$\delta^n F[f] = \frac{\mathrm{d}}{\mathrm{d}\epsilon} \delta^{n-1} F[f + \epsilon\eta_n]\big|_{\epsilon=0} = \int \left(\prod_{i=1}^n \mathrm{d}x_i \right) \frac{\delta^n F[f]}{\delta f(x_n) \dots \delta f(x_n)} \left(\prod_{i=1}^n \eta_i(x_i) \right). \tag{B.40}$$

A functional may be expanded in a generalization of the Fourier series, which has the form

$$F[f_0 + f] = F[f_0] + \int_\Omega \mathrm{d}x \, f(x) \frac{\delta F[f_0]}{\delta f(x)}\bigg|_{f=f_0} + \frac{1}{2!} \int_\Omega \mathrm{d}x \mathrm{d}y \, f(x) f(y) \frac{\delta^2 F[f_0]}{\delta f(x) \delta f(y)} + \dots \tag{B.41}$$

As an example, the Klein-Gorodn action

$$S[\varphi] = -\frac{1}{2} \int_\Omega \mathrm{d}x \, \varphi(\partial^2 + m^2)\varphi(x) \tag{B.42}$$

can be evaluated quickly by using Eq. (B.38) and partial integration

$$\begin{aligned} \frac{\delta}{\delta\varphi(x)} S[\varphi] &= -\frac{1}{2} \int_\Omega \mathrm{d}y \, [\delta(x-y)(\partial_y^2 + m^2)\varphi(y) + \varphi(y)(\partial_y^2 + m^2)\delta(x-y)] \\ &= -\int_\Omega \mathrm{d}y \, \delta(x-y)(\partial_y^2 + m^2)\varphi(y) = (\partial_x^2 + m^2)\varphi(x) \end{aligned} \tag{B.43}$$

The second derivative is

$$\frac{\delta^2 S[\varphi]}{\delta\varphi(x)\delta\varphi(y)} = \frac{\delta}{\delta\varphi(x)}(\partial_y^2 + m^2)\varphi(y) = (\partial_y^2 + m^2)\delta(x-y). \tag{B.44}$$

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