Pion Lagrangian

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September 1, 2021

1 Second order χPT Lagrangian

The leading order Lagrangian in χPT is [1, 2]

$$\mathcal{L} = \frac{f^2}{4}\,\mathrm{Tr}\left[\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\right] + \frac{f^2m^2}{4}\,\mathrm{Tr}\left[\Sigma+\Sigma^\dagger\right].$$

m and f are the pion mass and the pion decay constant, which are both free parameters of the theory. The covariant derivative is defined by

$$\nabla_\mu\Sigma = \partial_\mu\Sigma - i[v_\mu,\Sigma], \quad (\nabla_\mu\Sigma)^\dagger = \partial_\mu\Sigma^\dagger - i[v_\mu,\Sigma^\dagger], \quad v_\mu = \frac{1}{2}\mu_I\delta^0_\mu\tau_3,$$

while Σ is defined by

$$\begin{aligned}
&\Sigma = A_\alpha(U\Sigma_0U)A_\alpha,\\
&\Sigma_0 = \mathbb{1},\ A_\alpha = \exp\left(\frac{i\alpha}{2}\tau_1\right),\ U = \exp\left(i\frac{\tau_a\pi_a}{2f}\right).
\end{aligned}$$

τ_a are the SU(2) generators, i.e. Pauli matrices, and obey

$$[\tau_a,\tau_b] = 2i\varepsilon_{abc}\tau_c, \quad \{\tau_a,\tau_b\} = 2\delta_{ab}\mathbb{1}, \quad \mathrm{Tr}[\tau_a]=0, \quad \mathrm{Tr}[\tau_a\tau_b] = 2\delta_{ab}\mathbb{1}.$$

(FORKLAE HVA π_a, AT DET ER REELT, OG DERMED OGSÅ Σ) As $\tau_a^2 = \mathbb{1}$, we may write

$$A_\alpha = \sum_n^\infty \frac{1}{n!}\left(\frac{i\alpha}{2}\tau_1\right)^n = \sum_n^\infty\left[\frac{1}{(2n)!}\left(\frac{i\alpha}{2}\right)^{(2n)} + \frac{\tau_1}{(2n+1)!}\left(\frac{i\alpha}{2}\right)^{(2n+1)}\right] = \mathbb{1}\cos\left(\frac{\alpha}{2}\right) + i\tau_1\sin\left(\frac{\alpha}{2}\right).$$

Expanding the inner term of Σ to order $(\pi/f)^2$ gives

$$\begin{aligned}
U\Sigma_0U &= \exp\left(i\frac{\tau_a\pi_a}{2f}\right)\mathbb{1}\exp\left(i\frac{\tau_b\pi_b}{2f}\right) = \left(1 + i\frac{\tau_a\pi_a}{2f} + \frac{1}{2}\left(i\frac{\tau_a\pi_a}{2f}\right)^2\right)\mathbb{1}\left(1 + i\frac{\tau_a\pi_b}{2f} + \frac{1}{2}\left(i\frac{\tau_b\pi_b}{2f}\right)^2\right)\\
&= 1 + i\frac{\tau_a\pi_a}{f} - \frac{\pi_a\pi_b\tau_a\tau_b}{2f^2}.
\end{aligned}$$

The symmetry of $\pi_a\pi_b$ means that

$$\pi_a\pi_b\tau_a\tau_b = \pi_a\pi_b\frac{1}{2}\{\tau_a,\tau_b\} = \pi_a\pi_a\mathbb{1}.$$

Thus, to $\mathcal{O}\left((\pi/f)^3\right)$,

$$\begin{aligned}
\Sigma &= \left(\cos\left(\frac{\alpha}{2}\right) + i\tau_1\sin\left(\frac{\alpha}{2}\right)\right)\left(1 + \frac{\tau_a\pi_a}{f} - \frac{\pi_a\pi_a}{2f^2}\right)\left(\cos\left(\frac{\alpha}{2}\right) + i\tau_1\sin\left(\frac{\alpha}{2}\right)\right)\\
&= \left[\cos\left(\frac{\alpha}{2}\right)\left(1 + \frac{\tau_a\pi_a}{f} - \frac{\pi_a\pi_a}{2f^2}\right) + i\sin\left(\frac{\alpha}{2}\right)\left(\tau_1 + \frac{\pi_a\tau_a}{f}\tau_1 - \tau_1\frac{\pi_a\pi_a}{2f^2}\right)\right]\left(\cos\left(\frac{\alpha}{2}\right) + i\tau_1\sin\left(\frac{\alpha}{2}\right)\right)\\
&= \cos\left(\frac{\alpha}{2}\right)^2\left(1 + \frac{\pi_a\tau_a}{f} - \frac{\pi_a\pi_a}{2f^2}\right) - \sin\left(\frac{\alpha}{2}\right)^2\left(1 + i\frac{\pi_a\tau_a}{f}\tau_1 - \frac{\pi_a\pi_a}{2f^2}\right)\\
&+ \cos\left(\frac{\alpha}{2}\right)\sin\left(\frac{\alpha}{2}\right)\left(2i\tau_1 - \frac{\pi_a}{f}\{\tau_a,\tau_1\} - i\frac{\pi_a\pi_a\tau_1}{f^2}\right)
\end{aligned}$$

Using trigonometric identities and the commutator,

$$\cos\left(\frac{\alpha}{2}\right)^2 - \sin\left(\frac{\alpha}{2}\right)^2 = \cos(\alpha), \quad 2\cos\left(\frac{\alpha}{2}\right)\sin\left(\frac{\alpha}{2}\right) = \sin\left(\frac{\alpha}{2}\right), \quad \tau_1\tau_a\tau_1 = -\tau_a + 2\delta_{1a}\tau_1$$

we may write Σ as

$$\begin{aligned}
&\Sigma = \\
&(\cos(\alpha) + i\tau_1\sin(\alpha))\left(1 - \frac{\pi_a\pi_a}{2f^2}\right) + i\frac{\pi_a}{f}\left(\tau_a\cos\left(\frac{\alpha}{2}\right)^2 - (2\delta_{1a}\tau_1 - \tau_a)\sin\left(\frac{\alpha}{2}\right)^2\right) + \sin(\alpha)\left(-\frac{\pi_a}{f}\delta_{a1}\right) \\
&= (\cos(\alpha) + i\tau_1\sin(\alpha))\left(1 - \frac{\pi_a\pi_a}{2f^2}\right) + i\frac{\pi_a}{f}\tau_a - \frac{\pi_a}{f}\delta_{1a}\left(2i\tau_1\sin\left(\frac{\alpha}{2}\right)^2 + \sin(\alpha)\right).
\end{aligned}$$

The kinetic term in the Lagrangian is

$$\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger = (\partial_\mu\Sigma - i[v_\mu,\Sigma])(\partial^\mu\Sigma - i[v^\mu,\Sigma])^\dagger = \partial_\mu\Sigma\partial^\mu\Sigma^\dagger - i([v_\mu,\Sigma]\partial^\mu\Sigma^\dagger - \text{h.c.}) + [v_\mu,\Sigma]\left([v^\mu,\Sigma]\right)^\dagger.$$

The terms needed to calculate this is, to $\mathcal{O}\left((\pi/f)^3\right)$

$$\begin{aligned}
\partial_\mu\Sigma &= -\frac{\pi_a\partial_\mu\pi_a}{f^2}(\cos(\alpha) + i\tau_1\sin(\alpha)) + \frac{\partial_\mu\pi_a}{f}\left(i\tau_a - \delta_{1a}\left[2i\tau_1\sin\left(\frac{\alpha}{2}\right)^2 + \sin(\alpha)\right]\right) \\
&= -\frac{\pi_a\partial_\mu\pi_a}{f^2}\cos(\alpha) - \frac{\partial_\mu\pi_1}{f}\sin(\alpha) - \left(i\frac{\pi_a\partial_\mu\pi_a}{f^2}\sin(\alpha) + 2i\frac{\partial_\mu\pi_1}{f}\sin\left(\frac{\alpha}{2}\right)^2\right)\tau_1 + \frac{\partial_\mu\pi_a}{f}i\tau_a
\end{aligned}$$

and (using $v_\mu = \frac{1}{2}\mu_I\delta^0_\mu\tau_3$, and $[\tau_3,\tau_a] = 2i\varepsilon_{3ac}\tau_c = 2i(\delta_{a1}\tau_2 - \delta_{a2}\tau_1)$, $[\tau_3,\tau_1] = 2i\tau_2$)

$$\begin{aligned}
[v_\mu,\Sigma] &= \frac{1}{2}\mu_I\delta^0_\mu\left[i\sin(\alpha)\left(1 - \frac{\pi_a\pi_a}{2f^2}\right)[\tau_3,\tau_1] + i\frac{\pi_a}{f}[\tau_3,\tau_a] - 2i\frac{\pi_a}{f}\delta_{1a}\sin\left(\frac{\alpha}{2}\right)^2[\tau_3,\tau_1]\right] \\
&= -\delta^0_\mu\mu_I\left[\sin(\alpha)\left(1 - \frac{\pi_a\pi_a}{2f^2}\right)\tau_2 + \frac{\pi_a}{f}(\delta_{a1}\tau_2 - \delta_{a2}\tau_1) - 2\frac{\pi_a}{f}\delta_{1a}\sin\left(\frac{\alpha}{2}\right)^2\tau_2\right] \\
&= -\delta^0_\mu\mu_I\left[\left(\sin(\alpha)\left(1 - \frac{\pi_a\pi_a}{2f^2}\right) + \frac{\pi_1}{f}\cos(\alpha)\right)\tau_2 - \frac{\pi_2}{f}\tau_1\right]
\end{aligned}$$

The first term is, to second order

$$\begin{aligned}
\partial_\mu\Sigma\partial^\mu\Sigma^\dagger &= \frac{\partial_\mu\pi_a\partial^\mu\pi_b}{f^2}\left[i\left(\tau_a - \delta_{1a}2\tau_1\sin\left(\frac{\alpha}{2}\right)^2\right) - \delta_{1a}\sin(\alpha)\right]\left[i\left(\tau_b - \delta_{1b}2\tau_1\sin\left(\frac{\alpha}{2}\right)^2\right) - \delta_{1b}\sin(\alpha)\right] \\
&= \frac{\partial_\mu\pi_a\partial^\mu\pi_b}{f^2}\Bigg[\tau_a\tau_b - 2\sin\left(\frac{\alpha}{2}\right)^2(\delta_{1a}\tau_1\tau_b + \delta_1\tau_a\tau_1) + \delta_{1a}\delta_{1b}\left(4\sin\left(\frac{\alpha}{2}\right)^2 + \sin(\alpha)^2\right) \\
&\qquad - i\sin(\alpha)\left(\tau_a\delta_{1b} + \tau_b\delta_{1a} - 4\delta_{1a}\delta_{1b}\tau_1\sin\left(\frac{\alpha}{2}\right)^2\right)\Bigg] \\
\operatorname{Tr}\left\{\partial_\mu\Sigma\partial^\mu\Sigma^\dagger\right\} &= \frac{\partial_\mu\pi_a\partial^\mu\pi_b}{f^2}\left[2\delta_{ab} - 2\sin\left(\frac{\alpha}{2}\right)^2 4\delta_{a1}\delta_{b1} + 2\delta_{1a}\delta_{1b}\left(4\sin\left(\frac{\alpha}{2}\right)^4 + \sin(\alpha)^2\right)\right] \\
&= 2\frac{\partial_\mu\pi_a\partial^\mu\pi_a}{f^2}.
\end{aligned}$$

Expanding the second term in the kinetic part of the Lagrangian yields (FORTEGN PGA KONJUGAT!!!!)

$$\begin{aligned}
&\operatorname{Tr}\left\{[v^\mu,\Sigma]\partial_\mu\Sigma^\dagger\right\} \\
&= \mu_I\operatorname{Tr}\Bigg\{\left(\left[\sin(\alpha)\left(1 - \frac{\pi_a\pi_a}{2f^2}\right) + \frac{\pi_1}{f}\cos(\alpha)\right]\tau_2 - \frac{\pi_a}{f}\delta_{a2}\tau_1\right) \\
&\qquad\times\left(\frac{\pi_a\partial_0\pi_a}{f^2}\cos(\alpha) + \frac{\partial_0\pi_1}{f}\sin(\alpha) + \left[i\frac{\pi_a\partial_0\pi_a}{f^2}\sin(\alpha) + 2i\frac{\partial_0\pi_1}{f}\sin\left(\frac{\alpha}{2}\right)^2\right]\tau_1 - i\frac{\partial_0\pi_a}{f}\tau_a\right)\Bigg\} \\
&= \mu_I\left\{\left(2i\frac{\partial_0\pi_1}{f}\sin\left(\frac{\alpha}{2}\right)^2\right)\left(-\frac{\pi_2}{f}\right)\operatorname{Tr}\{\tau_1\tau_1\} - i\frac{\partial_0\pi_a}{f}\left[\left(\sin(\alpha) + \frac{\pi_1}{f}\cos(\alpha)\right)\operatorname{Tr}\{\tau_a\tau_2\} - \frac{\pi_2}{f}\operatorname{Tr}\{\tau_a\tau_1\}\right]\right\} \\
&= -2i\mu_I\left[\frac{\partial_0\pi_2}{f}\sin(\alpha) + \left(\frac{\pi_1\partial_0\pi_2}{f^2} - \frac{\pi_2\partial_0\pi_1}{f^2}\right)\cos(\alpha)\right] = -2i\frac{\mu_I}{f^2}\left[f\partial_0\pi_2\sin(\alpha) + (\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1)\cos(\alpha)\right]
\end{aligned}$$

Lastly, we have

$$
\begin{aligned}
&\mathrm{Tr}\big[[v_\mu,\Sigma][v^\mu,\Sigma^\dagger]\big] \\
&= \mu_I^2\,\mathrm{Tr}\left[\left(\left[\sin(\alpha)\left(1-\frac{\pi_a\pi_a}{2f^2}\right)+\frac{\pi_1}{f}\cos(\alpha)\right]\tau_2-\frac{\pi_2}{f}\tau_1\right)\left(\left[\sin(\alpha)\left(1-\frac{\pi_a\pi_a}{2f^2}\right)+\frac{\pi_1}{f}\cos(\alpha)\right]\tau_2-\frac{\pi_2}{f}\tau_1\right)^\dagger\right] \\
&= 2\mu_I^2\left[\sin(\alpha)^2\left(1-\frac{\pi_a\pi_a}{2f^2}\right)^2+2\sin(\alpha)\cos(\alpha)\frac{\pi_1}{f}\left(1-\frac{\pi_a\pi_a}{2f^2}\right)+\cos(\alpha)^2\frac{\pi_1^2}{f^2}+\frac{\pi_2^2}{f^2}\right] \\
&= 2\frac{\mu_I^2}{f^2}\left[f^2\sin(\alpha)^2+f\pi_1\sin(2\alpha)-\pi_a\pi_a\sin(\alpha)^2+\pi_1^2\cos(\alpha)^2+\pi_2^2\right] \\
&= 2\frac{\mu_I^2}{f^2}\left[f^2\sin(\alpha)^2+f\pi_1\sin(2\alpha)+\pi_1^2\cos(2\alpha)+\pi_2^2\cos(\alpha)^2-\pi_3^2\sin(\alpha)^2\right]
\end{aligned}
$$

The second term of the Lagrangian is

$$
\mathrm{Tr}\big[\Sigma+\Sigma^\dagger\big]=\cos(\alpha)\left(1-\frac{\pi_a\pi_a}{2f^2}\right)\mathrm{Tr}[\mathbb{1}]-\frac{\pi_a}{f}\delta_{1a}\sin(\alpha)\,\mathrm{Tr}[\mathbb{1}]+\text{h.c.}
$$

This means that the total Lagrangian is

$$
\begin{aligned}
\mathcal{L}=&\frac{1}{2}\partial_\mu\pi_a\partial^\mu\pi_a+\mu_I\left[f\partial_0\pi_2\sin(\alpha)+(\pi_1\partial_0\pi_2-\pi_2\partial_0\pi_1)\cos(\alpha)\right] \\
&+\frac{\mu_I^2}{2}\left[f^2\sin(\alpha)^2+f\pi_1\sin(2\alpha)+\pi_1^2\cos(2\alpha)+\pi_2^2\cos(\alpha)^2-\pi_3^2\sin(\alpha)^2\right] \\
&+m^2f^2\cos(\alpha)\left(1-\frac{\pi_a\pi_a}{2f^2}\right)-m^2f\pi_1\sin(\alpha)
\end{aligned}
$$

Splitting the Lagrangian up into $\mathcal{L}=\mathcal{L}_{\text{static}}+\mathcal{L}_{\text{linear}}+\mathcal{L}_{\text{quad}}$ gives

$$
\begin{aligned}
\mathcal{L}_{\text{static}}&=m^2f^2\cos(\alpha)+\frac{1}{2}\mu_I^2f^2\sin(\alpha)^2 \\
\mathcal{L}_{\text{linear}}&=\mu_If\sin(\alpha)\partial_0\pi_2+f\left(\frac{1}{2}\mu_I^2\sin(2\alpha)-m^2\sin(\alpha)\right)\pi_1 \\
\mathcal{L}_{\text{quad}}&=\frac{1}{2}\partial_\mu\pi_a\partial^\mu\pi_a+\mu_I\cos(\alpha)(\pi_1\partial_0\pi_2-\pi_2\partial_0\pi_1)-\frac{1}{2}m^2\cos(\alpha)\pi_a\pi_a+\mu_I^2(\cos(2\alpha)\pi_1^2+\cos(\alpha)^2\pi_2^2-\sin(\alpha)^2\pi_3^2) \\
&=\frac{1}{2}\partial_\mu\pi_a\partial^\mu\pi_a+\mu_I\cos(\alpha)(\pi_1\partial_0\pi_2-\pi_2\partial_0\pi_1)-\frac{1}{2}M_a\pi_a\pi_a,
\end{aligned}
$$

where $M_a=\left(m^2\cos(\alpha)-\mu_I^2\cos(2\alpha),\ m^2\cos(\alpha)-\mu_I^2\cos(\alpha)^2, m^2\cos(\alpha)+\mu_I^2\sin(\alpha)^2\right)$

2 3. and 4. order

Further expansion of Σ gives

$$
U=\exp\left(\frac{i\pi_a\tau_a}{2f}\right)=1+\frac{i\pi_a\tau_a}{2f}+\frac{1}{2}\left(\frac{i\pi_a\tau_a}{2f}\right)^2+\frac{1}{6}\left(\frac{i\pi_a\tau_a}{2f}\right)^3+\frac{1}{24}\left(\frac{i\pi_a\tau_a}{2f}\right)^4+\mathcal{O}\left((\pi/f)^5\right)
$$

$$
\begin{aligned}
U\Sigma_0U=&\left(1+\frac{i\pi_a\tau_a}{2f}+\frac{1}{2}\left(\frac{i\pi_a\tau_a}{2f}\right)^2+\frac{1}{6}\left(\frac{i\pi_a\tau_a}{2f}\right)^3+\frac{1}{24}\left(\frac{i\pi_a\tau_a}{2f}\right)^4\right) \\
&\times\left(1+\frac{i\pi_a\tau_a}{2f}+\frac{1}{2}\left(\frac{i\pi_a\tau_a}{2f}\right)^2+\frac{1}{6}\left(\frac{i\pi_a\tau_a}{2f}\right)^3+\frac{1}{24}\left(\frac{i\pi_a\tau_a}{2f}\right)^4\right)+\mathcal{O}\left((\pi/f)^5\right) \\
=&\,1+2\left(\frac{i\pi_a\tau_a}{2f}\right)^1+2\left(\frac{i\pi_a\tau_a}{2f}\right)^2+\frac{4}{3}\left(\frac{i\pi_a\tau_a}{2f}\right)^3+\frac{2}{3}\left(\frac{i\pi_a\tau_a}{2f}\right)^4+\mathcal{O}\left((\pi/f)^5\right)
\end{aligned}
$$

$(\pi_a\tau_a)^2=\pi_a\pi_a,\ (\pi_a\tau_a)^3=\pi_a\pi_a\pi_b\tau_b,\ (\pi_a\tau_a)^4=\pi_a\pi_a\pi_b\pi_b.$

$$
U\Sigma_0U=1+i\frac{\pi_a\tau_a}{f}-\frac{\pi_a^2}{2f^2}-i\frac{\pi_a^2\pi_b\tau_b}{6f^3}+\frac{\pi_a^2\pi_b^2}{24f^4}+\mathcal{O}\left((\pi/f)^5\right)
$$

This means that (SIGMA) to the fourth order is

$$
\begin{aligned}
\Sigma &= \left(\cos\!\left(\frac{\alpha}{2}\right) + i\tau_1 \sin\!\left(\frac{\alpha}{2}\right)\right)\left(1 + i\frac{\pi_a\tau_a}{f} - \frac{\pi_a^2}{2f^2} - i\frac{\pi_a^2\pi_b\tau_b}{6f^3} + \frac{\pi_a^2\pi_b^2}{24f^4}\right)\left(\cos\!\left(\frac{\alpha}{2}\right) + i\tau_1 \sin\!\left(\frac{\alpha}{2}\right)\right)\\
&= \left(1 + i\frac{\pi_a\tau_a}{f} - \frac{\pi_a^2}{2f^2} - i\frac{\pi_a^2\pi_b\tau_b}{6f^3} + \frac{\pi_a^2\pi_b^2}{24f^4}\right)\cos\!\left(\frac{\alpha}{2}\right)^2\\
&- \left(1 + i\frac{\pi_a}{f}\tau_1\tau_a\tau_1 - \frac{\pi_a^2}{2f^2} - i\frac{\pi_a^2\pi_b}{6f^3}\tau_1\tau_b\tau_1 + \frac{\pi_a^2\pi_b^2}{24f^4}\right)\sin\!\left(\frac{\alpha}{2}\right)^2\\
&+ i\left(2\tau_1 + i\frac{\pi_a}{f}\{\tau_1,\tau_a\} - 2\tau_1\frac{\pi_a^2}{2f^2} - i\frac{\pi_a^2\pi_b}{6f^3}\{\tau_1,\tau_b\} + 2\tau_1\frac{\pi_a^2\pi_b^2}{24f^4}\right)\sin\!\left(\frac{\alpha}{2}\right)\cos\!\left(\frac{\alpha}{2}\right)\\
&= \left(1 - \frac{\pi_a^2}{2f^2} + \frac{\pi_a^2\pi_b^2}{24f^4}\right)(\cos(\alpha) + i\tau_1\sin(\alpha)) + \left(\frac{\pi_a}{f} - \frac{\pi_b^2\pi_a}{6f^3}\right)\left(i\tau_a - 2i\delta_{a1}\tau_1\sin\!\left(\frac{\alpha}{2}\right)^2 - \delta_{1a}\sin(\alpha)\right)
\end{aligned}
$$

$$
\begin{aligned}
\partial_\mu\Sigma &= \left(\frac{-1}{f^2} + \frac{\pi_b^2}{6f^4}\right)(\cos(\alpha) + i\tau_1\sin(\alpha))(\pi_a\partial_\mu\pi_a)\\
&+ \left(\frac{\partial_\mu\pi_a}{f} - \frac{\pi_b^2\partial_\mu\pi_a + 2\pi_a\pi_b\partial_\mu\pi_b}{6f^3}\right)\left(i\tau_a - 2i\delta_{a1}\tau_1\sin\!\left(\frac{\alpha}{2}\right)^2 - \delta_{1a}\sin(\alpha)\right)
\end{aligned}
$$

$$
\begin{aligned}
[\Sigma, v_\mu] &= \frac{1}{2}\mu_I\delta_\mu^0\left[\left(1 - \frac{\pi_a^2}{2f^2} + \frac{\pi_a^2\pi_b^2}{24f^4}\right)i\sin(\alpha)[\tau_3,\tau_1] + \left(\frac{\pi_a}{f} - \frac{\pi_b^2\pi_a}{6f^3}\right)\left(i[\tau_a,\tau_3] - 2i\delta_{a1}\sin\!\left(\frac{\alpha}{2}\right)^2[\tau_3,\tau_1]\right)\right]\\
&= -\mu_I\delta_\mu^0\left[\left(1 - \frac{\pi_a^2}{2f^2} + \frac{\pi_a^2\pi_b^2}{24f^4}\right)\sin(\alpha)\tau_2 + \left(\frac{\pi_a}{f} - \frac{\pi_b^2\pi_a}{6f^3}\right)\left((\delta_{a1}\tau_2 - \delta_{a2}\tau_1) - 2\delta_{a1}\sin\!\left(\frac{\alpha}{2}\right)^2\tau_2\right)\right]\\
&= -\mu_I\delta_\mu^0\left[\left\{\left(1 - \frac{\pi_a^2}{2f^2} + \frac{\pi_a^2\pi_b^2}{24f^4}\right)\sin(\alpha) + \left(\frac{\pi_1}{f} - \frac{\pi_b^2\pi_1}{6f^3}\right)\cos(\alpha)\right\}\tau_2 - \left(\frac{\pi_2}{f} - \frac{\pi_b^2\pi_2}{6f^3}\right)\tau_1\right]
\end{aligned}
$$

This gives Lagrangian term

$$
\begin{aligned}
\mathcal{L}_2^{(0)} &= \frac{1}{2}f^2\mu^2\sin^2(\alpha) + f^2m^2\cos(\alpha)\\
\mathcal{L}_2^{(1)} &= f(\mu_I^2\sin(\alpha)\cos(\alpha) - m^2\sin(\alpha))\pi_1 + f\mu_I\sin(\alpha)\partial_0\pi_2\\
\mathcal{L}_2^{(2)} &= \frac{1}{2}\partial_\mu\pi_a\partial^\mu\pi_a + \mu_I\left(\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1\right)\cos(\alpha) - \frac{1}{2}m^2\pi_a\pi_a\cos(\alpha) + \frac{\mu_I^2}{2}\left(\pi_2^2\cos(\alpha)^2 + \pi_1^2\cos(2\alpha) - \pi_3^2\sin(\alpha)^2\right)\\
\mathcal{L}_2^{(3)} &= \frac{\pi_a\pi_a\pi_1}{6f}(m^2\sin(\alpha) - 2\mu^2\sin(2\alpha)) + \frac{2\mu}{3f}\left(\pi_2\pi_1\partial_0\pi_1 - \pi_1^2\partial_0\pi_2 - \pi_3^2\partial_0\pi_2 + \pi_2\pi_3\partial_0\pi_3\right)\sin(\alpha)\\
\mathcal{L}_2^{(4)} &= \frac{1}{6f^2}\left[\frac{1}{4}m^2(\pi_a\pi_a)^2\cos(\alpha) - (\pi_a\pi_a\partial_\mu\pi_b\partial^\mu\pi_b - \pi_a\partial_\mu\pi_a\pi_b\partial^\mu\pi_b)\right]\\
&+ \frac{\mu\pi_a\pi_a}{3f^2}\left\{\left(\pi_2\partial_0\pi_1 - \pi_1\partial_0\pi_2\right)\cos(\alpha) - \frac{\mu}{2}\left(\cos(2\alpha)\pi_1^2 + \cos(\alpha)^2\pi_2^2 - \sin(\alpha)^2\pi_3^2\right)\right\}a
\end{aligned}
$$

References

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