# The effective pion Lagrangian.

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## 1 Effective Lagrangians

The technique used in $\chi$PT to obtain the effective Lagrangian of the pion relies on a "theorem", formulated by **WeinbergPhenom**:

> [I]f one writes down the most general possible Lagrangian, including all terms consistent with assumed symmetry principles, and then calculates matrix elements with this Lagrangian to any given order of perturbation theory, the result will simply be the most general possible S-matrix consistent with analyticity, perturbative unitarity, cluster decomposition and the assumed symmetry principles. [**WeinbergPhenom**]

In other words, if we write down the most general Lagrange density consistent with symmetries of the underlying theory, it will result in the most general S-matrix consistent with important physical assumptions and our underlying theory. All that is left to do is to tune the free parameters that this leaves. As this Lagrangian contains infinitely many terms, and thus infinitely many free parameters, a method for ordering them in terms of importance is needed. As described in [**Scherer2002IntroductionTC**], by rescaling the external momenta $p_\mu \to tp_\mu$ and quark masses $m_i \to t^2 m_i$, each term in the Lagrangian obtains a factor $t^D$. The Lagrangian is then expanded as $\mathcal{L} = \sum_D \mathcal{L}_D$, where $\mathcal{L}_D$ contains all terms with a factor $t^D$.

In our case, the underlying theory is QCD with two quarks, up and down, with masses

$$M = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}.$$

In the isospin limit, $m_u = m_d = m(= 0?)$, the theory is invariant under global transformations in the group $G' = \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_V$. All terms involving only pions is trivially invariant under $\mathrm{U}(1)_V$, (HVORFOR?) so we focus on the $G = \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ subgroup. This symmetry is spontaneously broken if the quark field has a non-zero ground state expectation value $\langle \bar{q}q \rangle$, leaving only a subgroup $H = \mathrm{SU}(2)_V \subseteq G$ of symmetry transformations of the vacuum state. The Goldstein manifold $G/H = \mathrm{SU}(2)_A$ is a three-dimensional Lie group, and therefore results in three (pseudo) Goldstein bosons, the pions. There exists an isomorphism from a subset $S \subseteq M_1$ of the set of all Goldstein-fields

$$M_1 = \left\{ \pi_a : \mathcal{M}_4 \longrightarrow \mathbb{R}^3 | \pi_a \,\mathrm{smooth} \right\}$$

close to the ground state, into fields taking values in the Goldstein manifold $G/H$. (BEVISE?)(HVA ER ISOMORFISME HER?). The $\chi$PT effective Lagrangian will be constructed using this map, through the parametrization

$$\begin{aligned} \Sigma : \mathcal{M}_4 &\longrightarrow \mathrm{SU}(2), \\ x &\longrightarrow \Sigma(x) = A_\alpha (U(x) \Sigma_0 U(x)) A_\alpha, \end{aligned} \tag{1}$$

where

$$\Sigma_0 = \mathbb{1},\ A_\alpha = \exp\left(\frac{i\alpha}{2}\tau_1\right),\ U(x) = \exp\left(i\frac{\tau_a \pi_a(x)}{2f}\right).$$

$\tau_a$ are the SU(2) generators, i.e. Pauli matrices, as described in Appendix A $\pi_a$, where $a \in \{1, 2, 3\}$, are the pion fields. These are real fields, meaning $\pi_a^\dagger = \pi_a$.

## 2 Leading order Lagrangian

The leading order Lagrangian in $\chi$PT is [**mojahed**, **Scherer2002IntroductionTC**]

$$\mathcal{L}_2 = \frac{f^2}{4}\,\mathrm{Tr}\left[\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\right] + \frac{f^2}{4}\,\mathrm{Tr}\left[\chi^\dagger\Sigma + \Sigma^\dagger\chi\right]. \tag{2}$$

$\chi$ and $f$ are the free parameters of the theory. $f$ is the pion decay constant, while $\chi = 2B_0M$. Here, $M$ is the mass matrix of the quarks, which in the isospin limit are taken to be equal. That is, $m = m_u = m_d$. $B_0$ is related to the quark condensate through $3f^2B_0 = -\langle\bar{q}q\rangle$. (ER DET SANT NÅR $\mu_I \neq 0$?) The covariant derivative is defined by

$$\nabla_\mu\Sigma = \partial_\mu\Sigma - i[v_\mu,\Sigma], \quad (\nabla_\mu\Sigma)^\dagger = \partial_\mu\Sigma^\dagger - i[v_\mu,\Sigma^\dagger], \quad v_\mu = \frac{1}{2}\mu_I\delta^0_\mu\tau_3,$$

where $\mu_I$ is the isospin chemical potential. This Lagrangian results in a pion mass of $M_\pi^2 = 2B_0m$. To get the series expansion of $\Sigma$ in powers of $\pi/f$, we start by using the fact that $\tau_a^2 = \mathbb{1}$ to write

$$A_\alpha = \sum_n^\infty \frac{1}{n!}\left(\frac{i\alpha}{2}\tau_1\right)^n = \sum_n^\infty\left[\frac{\mathbb{1}}{(2n)!}\left(\frac{i\alpha}{2}\right)^{(2n)} + \frac{\tau_1}{(2n+1)!}\left(\frac{i\alpha}{2}\right)^{(2n+1)}\right] = \mathbb{1}\cos\left(\frac{\alpha}{2}\right) + i\tau_1\sin\left(\frac{\alpha}{2}\right).$$

Using the expansion of the exponential,

$$U = \exp\left(\frac{i\pi_a\tau_a}{2f}\right) = 1 + \frac{i\pi_a\tau_a}{2f} + \frac{1}{2}\left(\frac{i\pi_a\tau_a}{2f}\right)^2 + \frac{1}{6}\left(\frac{i\pi_a\tau_a}{2f}\right)^3 + \frac{1}{24}\left(\frac{i\pi_a\tau_a}{2f}\right)^4 + \mathcal{O}\left((\pi/f)^5\right)$$

the inner part of the definition of $\Sigma$, as given in Eq. 1, has the expansion

$$\begin{aligned}
U\Sigma_0U &= \left(1 + \frac{i\pi_a\tau_a}{2f} + \frac{1}{2}\left(\frac{i\pi_a\tau_a}{2f}\right)^2 + \frac{1}{6}\left(\frac{i\pi_a\tau_a}{2f}\right)^3 + \frac{1}{24}\left(\frac{i\pi_a\tau_a}{2f}\right)^4\right)\\
&\quad\times\left(1 + \frac{i\pi_a\tau_a}{2f} + \frac{1}{2}\left(\frac{i\pi_a\tau_a}{2f}\right)^2 + \frac{1}{6}\left(\frac{i\pi_a\tau_a}{2f}\right)^3 + \frac{1}{24}\left(\frac{i\pi_a\tau_a}{2f}\right)^4\right) + \mathcal{O}\left((\pi/f)^5\right)\\
&= 1 + \frac{i\pi_a\tau_a}{f} + 2\left(\frac{i\pi_a\tau_a}{2f}\right)^2 + \frac{4}{3}\left(\frac{i\pi_a\tau_a}{2f}\right)^3 + \frac{2}{3}\left(\frac{i\pi_a\tau_a}{2f}\right)^4 + \mathcal{O}\left((\pi/f)^5\right)
\end{aligned}$$

The symmetry of $\pi_a\pi_b$ means that

$$(\pi_a\tau_a)^2 = \pi_a\pi_b\frac{1}{2}\{\tau_a,\tau_b\} = \pi_a\pi_a, \quad (\pi_a\tau_a)^3 = \pi_a\pi_a\pi_b\tau_b, \quad (\pi_a\tau_a)^4 = \pi_a\pi_a\pi_b\pi_b.$$

This gives us the expression

$$U\Sigma_0U = 1 + i\frac{\pi_a\tau_a}{f} - \frac{\pi_a^2}{2f^2} - i\frac{\pi_a^2\pi_b\tau_b}{6f^3} + \frac{\pi_a^2\pi_b^2}{24f^4} + \mathcal{O}\left((\pi/f)^5\right),$$

which we used to get the series expansion of $\Sigma$ up to $\mathcal{O}\left((\pi/f)^5\right)$

$$\begin{aligned}
\Sigma &= \left(\cos\left(\frac{\alpha}{2}\right) + i\tau_1\sin\left(\frac{\alpha}{2}\right)\right)\left(1 + i\frac{\pi_a\tau_a}{f} - \frac{\pi_a^2}{2f^2} - i\frac{\pi_a^2\pi_b\tau_b}{6f^3} + \frac{\pi_a^2\pi_b^2}{24f^4}\right)\left(\cos\left(\frac{\alpha}{2}\right) + i\tau_1\sin\left(\frac{\alpha}{2}\right)\right)\\
&= \left(1 + i\frac{\pi_a\tau_a}{f} - \frac{\pi_a^2}{2f^2} - i\frac{\pi_a^2\pi_b\tau_b}{6f^3} + \frac{\pi_a^2\pi_b^2}{24f^4}\right)\cos\left(\frac{\alpha}{2}\right)^2\\
&- \left(1 + i\frac{\pi_a}{f}\tau_1\tau_a\tau_1 - \frac{\pi_a^2}{2f^2} - i\frac{\pi_a^2\pi_b}{6f^3}\tau_1\tau_b\tau_1 + \frac{\pi_a^2\pi_b^2}{24f^4}\right)\sin\left(\frac{\alpha}{2}\right)^2\\
&+ i\left(2\tau_1 + i\frac{\pi_a}{f}\{\tau_1,\tau_a\} - 2\tau_1\frac{\pi_a^2}{2f^2} - i\frac{\pi_a^2\pi_b}{6f^3}\{\tau_1,\tau_b\} + 2\tau_1\frac{\pi_a^2\pi_b^2}{24f^4}\right)\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right).
\end{aligned}$$

Using trigonometric identities and the commutator,

$$\cos\Big(\frac{\alpha}{2}\Big)^2-\sin\Big(\frac{\alpha}{2}\Big)^2=\cos(\alpha),\quad 2\cos\Big(\frac{\alpha}{2}\Big)\sin\Big(\frac{\alpha}{2}\Big)=\sin\Big(\frac{\alpha}{2}\Big),\quad \tau_1\tau_a\tau_1=-\tau_a+2\delta_{1a}\tau_1,$$

the final expression of $\Sigma$ to $\mathcal{O}\left((\pi/f)^5\right)$ is

$$\Sigma=\left(1-\frac{\pi_a^2}{2f^2}+\frac{\pi_a^2\pi_b^2}{24f^4}\right)(\cos(\alpha)+i\tau_1\sin(\alpha))+\left(\frac{\pi_a}{f}-\frac{\pi_b^2\pi_a}{6f^3}\right)\left(i\tau_a-2i\delta_{a1}\tau_1\sin\Big(\frac{\alpha}{2}\Big)^2-\delta_{1a}\sin(\alpha)\right). \quad (3)$$

The kinetic term in the $\chi$PT Lagrangian is

$$\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger=\partial_\mu\Sigma\partial^\mu\Sigma^\dagger-i([v_\mu,\Sigma]\partial^\mu\Sigma^\dagger-\text{h.c.})+[v_\mu,\Sigma]\left([v^\mu,\Sigma]\right)^\dagger. \quad (4)$$

Using Eq. 3 we find the expansion of the constitutive parts of the kinetic term to be

$$\begin{aligned}
\partial_\mu\Sigma&=\left(\frac{-1}{f^2}+\frac{\pi_b^2}{6f^4}\right)(\cos(\alpha)+i\tau_1\sin(\alpha))(\pi_a\partial_\mu\pi_a)\\
&+\left(\frac{\partial_\mu\pi_a}{f}-\frac{\pi_b^2\partial_\mu\pi_a+2\pi_a\pi_b\partial_\mu\pi_b}{6f^3}\right)\left(i\tau_a-2i\delta_{a1}\tau_1\sin\Big(\frac{\alpha}{2}\Big)^2-\delta_{1a}\sin(\alpha)\right)\\
&=\left[\left(\frac{-1}{f^2}+\frac{\pi_b^2}{6f^4}\right)(\pi_a\partial_\mu\pi_a)\cos(\alpha)-\left(\frac{\partial_\mu\pi_1}{f}-\frac{\pi_b^2\partial_\mu\pi_1+2\pi_1\pi_b\partial_\mu\pi_b}{6f^3}\right)\sin(\alpha)\right]\\
&-\left[\left(\frac{-1}{f^2}+\frac{\pi_b^2}{6f^4}\right)(\pi_a\partial_\mu\pi_a)\sin(\alpha)-\left(\frac{\partial_\mu\pi_1}{f}-\frac{\pi_b^2\partial_\mu\pi_1+2\pi_1\pi_b\partial_\mu\pi_b}{6f^3}\right)2\sin\Big(\frac{\alpha}{2}\Big)^2\right]i\tau_1\\
&+\left(\frac{\partial_\mu\pi_a}{f}-\frac{\pi_b^2\partial_\mu\pi_a+2\pi_a\pi_b\partial_\mu\pi_b}{6f^3}\right)i\tau_a,
\end{aligned} \quad (5)$$

and

$$\begin{aligned}
[v_\mu,\Sigma^\dagger]&=-\frac{1}{2}\mu_I\delta_\mu^0\left[\left(1-\frac{\pi_a^2}{2f^2}+\frac{\pi_a^2\pi_b^2}{24f^4}\right)i\sin(\alpha)\,[\tau_3,\tau_1]+\left(\frac{\pi_a}{f}-\frac{\pi_b^2\pi_a}{6f^3}\right)\left(i\,[\tau_a,\tau_3]-2i\delta_{a1}\sin\Big(\frac{\alpha}{2}\Big)^2[\tau_3,\tau_1]\right)\right]\\
&=\mu_I\delta_\mu^0\left[\left(1-\frac{\pi_a^2}{2f^2}+\frac{\pi_a^2\pi_b^2}{24f^4}\right)\sin(\alpha)\tau_2+\left(\frac{\pi_a}{f}-\frac{\pi_b^2\pi_a}{6f^3}\right)\left((\delta_{a1}\tau_2-\delta_{a2}\tau_1)-2\delta_{a1}\sin\Big(\frac{\alpha}{2}\Big)^2\tau_2\right)\right]\\
&=\mu_I\delta_\mu^0\left\{\left[\left(1-\frac{\pi_a^2}{2f^2}+\frac{\pi_a^2\pi_b^2}{24f^4}\right)\sin(\alpha)+\left(\frac{\pi_1}{f}-\frac{\pi_b^2\pi_1}{6f^3}\right)\cos(\alpha)\right]\tau_2-\left(\frac{\pi_2}{f}-\frac{\pi_b^2\pi_2}{6f^3}\right)\tau_1\right\}.
\end{aligned} \quad (6)$$

Combining Eq. 5 and Eq. 6 gives the following terms [1]

$$\begin{aligned}
\mathrm{Tr}\big\{\partial_\mu\Sigma\partial^\mu\Sigma^\dagger\big\}&=\frac{2}{f^2}\partial_\mu\pi_a\partial^\mu\pi_a+\frac{2}{3f^4}\left[(\pi_a\partial_\mu\pi_a)(\pi_b\partial^\mu\pi_b)-(\pi_a\partial_\mu\pi_b)(\pi_b\partial^\mu\pi_a)\right],\\
-i\,\mathrm{Tr}\big\{[v_\mu,\Sigma]\,\partial^\mu\Sigma^\dagger-\text{h.c.}\big\}&=\frac{4\mu_I}{f}\partial_0\pi_2\sin(\alpha)+\frac{8\mu_I}{3f^3}\sin(\alpha)\pi_3(\pi_2\partial_0\pi_3-\pi_3\partial_0\pi_2)\\
&\quad+\left(\frac{4\mu_I}{f^2}\cos(\alpha)-\frac{8\mu_I\pi_1}{3f^3}\sin(\alpha)-\frac{4\mu_I\pi_a\pi_a}{3f^4}\cos(\alpha)\right)(\pi_1\partial_0\pi_2-\pi_2\partial_0\pi_1),\\
\mathrm{Tr}\Big\{[v_\mu,\Sigma]\,[v^\mu,\Sigma]^\dagger\Big\}&=\mu^2\left[2\sin(\alpha)^2+\left(\frac{2}{f}-\frac{4\pi_a\pi_a}{3f^3}\right)\pi_1\sin(2\alpha)+\left(\frac{2}{f^2}-\frac{2\pi_a\pi_a}{3f^4}\right)\pi_a\pi_b k_{ab}\right],\\
\mathrm{Tr}\big\{\Sigma+\Sigma^\dagger\big\}&=4\cos(\alpha)-\frac{4\pi_1}{f}\sin(\alpha)-\frac{2\pi_a\pi_a}{f^2}\cos(\alpha)+\frac{2\pi_1\pi_a\pi_a}{3f^3}+\frac{(\pi_a\pi_a)^2}{6f^4}\cos(\alpha),
\end{aligned}$$

where $k_{ab}=\delta_{1a}\delta_{1b}\cos(\alpha)^2+\delta_{2a}\delta_{2b}-\delta_{ab}\sin(\alpha)^2$. If we write the Lagrangian as show in Eq. 2 as $\mathcal{L}_2=\mathcal{L}_2^{(0)}+\mathcal{L}_2^{(1)}+\mathcal{L}_2^{(2)}+...$, where $\mathcal{L}_2^{(n)}$ contains all terms of order $\mathcal{O}\left((\pi/f)^n\right)$, then the result of the series

[1] The Mathematica script used to aid the calculation of the Lagrangian is available here: `https://github.com/martkjoh/prosjektopggave`

expansion is

$$\mathcal{L}_2^{(0)} = f^2\left(2B_0 m\cos(\alpha) + \frac{1}{2}\mu^2\sin^2(\alpha)\right), \tag{7}$$

$$\mathcal{L}_2^{(1)} = f(\mu_I^2\cos(\alpha) - 2B_0 m)\sin(\alpha)\pi_1 + f\mu_I\sin(\alpha)\partial_0\pi_2, \tag{8}$$

$$\mathcal{L}_2^{(2)} = \frac{1}{2}\partial_\mu\pi_a\partial^\mu\pi_a + \mu_I\cos(\alpha)\left(\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1\right) - B_0 m\cos(\alpha)\pi_a\pi_a + \frac{1}{2}\mu_I^2\pi_a\pi_b k_{ab}, \tag{9}$$

$$\mathcal{L}_2^{(3)} = \frac{\pi_a\pi_a\pi_1}{6f}(2B_0 m\sin(\alpha) - 2{\mu_I}^2\sin(2\alpha)) + \frac{2\mu_I}{3f}\left(\pi_2\pi_1\partial_0\pi_1 - \pi_1^2\partial_0\pi_2 - \pi_3^2\partial_0\pi_2 + \pi_2\pi_3\partial_0\pi_3\right)\sin(\alpha), \tag{10}$$

$$\begin{aligned}\mathcal{L}_2^{(4)} &= \frac{1}{6f^2}\left\{\frac{1}{2}B_0 m(\pi_a\pi_a)^2\cos(\alpha) - [(\pi_a\pi_a)(\partial_\mu\pi_b\partial^\mu\pi_b) - (\pi_a\partial_\mu\pi_a)(\pi_b\partial^\mu\pi_b)]\right\}\\ &\quad - \frac{\mu_I\pi_a\pi_a}{3f^2}\left[\left(\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1\right)\cos(\alpha) + \frac{1}{2}\mu_I^2\pi_a\pi_b k_{ab}\right].\end{aligned} \tag{11}$$

The equation of motion for the leading order Lagrangian is found by using the principle of least action. Choosing the parametrization $\Sigma = \exp(i\pi_a\tau_a)$, a variation $\pi_a \to \pi_a + \varepsilon\eta_a = \pi_a + \delta\pi_a$ gives a variation in $\Sigma$, $\delta\Sigma = i\tau_a\delta\pi_a\Sigma$. The variation of the action is then

$$\delta S = \int_\Omega \mathrm{d}x\,\frac{f^2}{4}\,\mathrm{Tr}\left\{(\nabla_\mu\delta\Sigma)(\nabla^\mu\Sigma)^\dagger + (\nabla_\mu\Sigma)(\nabla^\mu\delta\Sigma)^\dagger + \chi\delta\Sigma^\dagger + \delta\Sigma\chi^\dagger\right\}.$$

Using the properties of the covariant derivative to do partial integration, as show in Appendix B, as well as the $\delta(\Sigma\Sigma^\dagger) = (\delta\Sigma)\Sigma^\dagger + \Sigma(\delta\Sigma)^\dagger = 0$, the variation of the action can be written

$$\begin{aligned}\delta S &= \frac{f^2}{4}\int_\Omega \mathrm{d}x\ \mathrm{Tr}\left\{-\delta\Sigma\nabla^2\Sigma^\dagger + (\nabla^2\Sigma)(\Sigma^\dagger\delta\Sigma\Sigma^\dagger) - \chi(\Sigma^\dagger\delta\Sigma\Sigma^\dagger) + \delta\Sigma\chi^\dagger\right\}\\ &= \frac{f^2}{4}\int_\Omega \mathrm{d}x\ \mathrm{Tr}\left\{\delta\Sigma\Sigma^\dagger\left[(\nabla^2\Sigma)\Sigma^\dagger - \Sigma\nabla^2\Sigma^\dagger - \chi\Sigma^\dagger + \Sigma\chi^\dagger\right]\right\}\\ &= i\frac{f^2}{4}\int_\Omega \mathrm{d}x\ \mathrm{Tr}\left\{\tau_a\left[(\nabla^2\Sigma)\Sigma^\dagger - \Sigma\nabla^2\Sigma^\dagger - \chi\Sigma^\dagger + \Sigma\chi^\dagger\right]\right\}\delta\pi_a = 0\end{aligned}$$

As the variation is arbitrary, the equation of motion to leading order is

$$\mathrm{Tr}\left\{\tau_a\left[(\nabla^2\Sigma)\Sigma^\dagger - \Sigma\nabla^2\Sigma^\dagger - \chi\Sigma^\dagger + \Sigma\chi^\dagger\right]\right\} = 0. \tag{12}$$

This may be rewritten as a matrix equation. Using that

$$\mathrm{Tr}\left\{(\nabla_\mu\Sigma)\Sigma^\dagger\right\} = \mathrm{Tr}\left\{i\tau_a(\partial_\mu\pi_a)\Sigma\Sigma^\dagger\right\} - i\,\mathrm{Tr}\left\{[v_\mu,\Sigma]\Sigma^\dagger\right\} = 0,$$

we can se that $\mathrm{Tr}\left\{(\nabla^2\Sigma)\Sigma^\dagger - \Sigma\nabla^2\Sigma^\dagger\right\} = 0$, and the equation of motion may be written as

$$\mathcal{O}^{(2)}_{\mathrm{EOM}}(\Sigma) = (\nabla^2\Sigma)\Sigma^\dagger - \Sigma\nabla^2\Sigma^\dagger - \chi\Sigma^\dagger + \Sigma\chi^\dagger - \frac{1}{2}\,\mathrm{Tr}\left\{\chi\Sigma^\dagger - \Sigma\chi^\dagger\right\} = 0. \tag{13}$$

# 3 Next to leading order Lagrangian

The next to leading order Lagrangian density is, assuming no external fields (ER DET RIKTIG FORMULERING? GAUGE FIELDS?)

$$\begin{aligned}\mathcal{L}_4 &= \frac{l_1}{4}\,\mathrm{Tr}\left\{\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\right\}^2 + \frac{l_2}{4}\,\mathrm{Tr}\left\{\nabla_\mu\Sigma(\nabla_\nu\Sigma)^\dagger\right\}\mathrm{Tr}\left\{\nabla^\mu\Sigma(\nabla^\nu\Sigma)^\dagger\right\} + \frac{l_3 + h_1 - h_3}{16}\,\mathrm{Tr}\left\{\chi\Sigma^\dagger + \Sigma\chi^\dagger\right\}^2\\ &\quad + \frac{l_4}{4}\,\mathrm{Tr}\left\{\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\right\}\mathrm{Tr}\left\{\chi\Sigma^\dagger + \Sigma\chi^\dagger\right\} + \frac{h_1 - h_3 - l_4 - l_7}{16}\,\mathrm{Tr}\left\{\chi\Sigma^\dagger - \Sigma\chi^\dagger\right\}^2 + \frac{h_1 + h_2 - l_4}{4}\,\mathrm{Tr}\left\{\chi\chi^\dagger\right\}\\ &\quad - \frac{h_1 - h_3 - l_4}{8}\,\mathrm{Tr}\left\{\left(\chi\Sigma^\dagger\right)^2 + \left(\Sigma\chi^\dagger\right)^2\right\}\end{aligned} \tag{14}$$

The form of the Lagrangian is not unique. Changing the parametrization of $\Sigma$ by $\Sigma(x) \to \Sigma'(x), \quad \Sigma(x) = e^{iS(x)}\Sigma'(x)$, $S(x) \in \mathfrak{su}(2)$ leads to a new Lagrangian density, $\mathcal{L}[\Sigma] = \mathcal{L}[\Sigma'] + \Delta\mathcal{L}[\Sigma']$. By tuning $S(x)$ it is possible to eliminate terms that naively appeared to be independent. When demanding that $\Sigma'$ obey the same symmetries as $\Sigma$, the most general transformation to $\mathcal{O}\left(t^2\right)$ is

$$S_2 = i\alpha_2\left[(\nabla^2\Sigma')\Sigma^\dagger - \Sigma'(\nabla^2\Sigma')^\dagger\right] + i\alpha_2\left[\chi\Sigma'^\dagger - \Sigma'\chi^\dagger - \frac{1}{2}\operatorname{Tr}\left\{\chi\Sigma'^\dagger - \Sigma'\chi^\dagger\right\}\right].$$

$\alpha_1$ and $\alpha_2$ are arbitrary real numbers. To $\mathcal{O}\left(t^4\right)$, an arbitrary new parametrization gives the new Lagrangian

$$\begin{aligned}\mathcal{L}_2\left[e^{iS_2}\Sigma'\right] &= \frac{f^2}{f}\operatorname{Tr}\left\{[\nabla_\mu(1+iS_2)\Sigma'][\nabla^\mu\Sigma'^\dagger(1-iS_2)]\right\} + \frac{f^2}{4}\operatorname{Tr}\left\{\chi\Sigma'^\dagger(1-iS_2) + (1+iS_2)\Sigma'\chi^\dagger\right\}\\ &= \mathcal{L}[\Sigma'] + i\frac{f^2}{4}\operatorname{Tr}\left\{[\nabla_\mu(S_2\Sigma')][\nabla^\mu\Sigma']^\dagger - [\nabla_\mu\Sigma'][\nabla^\mu(\Sigma'^\dagger S_2)]\right\} - i\frac{f^2}{4}\operatorname{Tr}\left\{\chi\Sigma'^\dagger S_2 - S_2\Sigma'\chi^\dagger\right\}\end{aligned}$$

Using the properties of the covariant derivative, as described in Appendix B, we may use the product rule and partial integration to write the difference between the two Lagrangians as

$$\begin{aligned}\Delta\mathcal{L}[\Sigma'] &= i\frac{f^2}{4}\operatorname{Tr}\left\{(\nabla_\mu S_2)(\Sigma'\nabla^\mu\Sigma'^\dagger - (\nabla^\mu\Sigma')\Sigma'^\dagger)\right\} - i\frac{f^2}{4}\operatorname{Tr}\left\{\chi\Sigma'^\dagger S_2 - S_2\Sigma'\chi^\dagger\right\}\\ &= i\frac{f^2}{4}\operatorname{Tr}\left\{S_2\left[\Sigma'^\dagger\nabla^2\Sigma' - (\nabla^2\Sigma')\Sigma'^\dagger - \chi\Sigma'^\dagger + \Sigma'\chi^\dagger\right]\right\}.\end{aligned}$$

Using the equation of motion as given Eq. 13, and the fact that $\operatorname{Tr}\{S_2\} = 0$, we may write this difference as

$$\Delta\mathcal{L}[\Sigma'] = \frac{f^2}{4}\operatorname{Tr}\left\{iS_2\mathcal{O}^{(2)}_{\text{EOM}}(\Sigma')\right\}.$$

As the physics are unchanged under a reparametrization, any term that can be rewritten in the form $\Delta\mathcal{L}$, for any $\alpha_1, \alpha_2 \in \mathbb{R}$ can be removed from the Lagrangian. $\Delta\mathcal{L}_2$ is of order $\mathcal{O}\left(t^4\right)$, and it can thus be used to remove terms from $\mathcal{L}_4$.

To expand this $\mathcal{L}_4$ to $\mathcal{O}\left((\pi/f)^2\right)$, we us the result from Eq. 5 and Eq. 6,

$$\begin{aligned}\Sigma &= \left(1 - \frac{\pi_a^2}{2f^2}\right)(\cos(\alpha) + i\tau_1\sin(\alpha)) + \frac{\pi_a}{f}\left(i\tau_a - \delta_{1a}2i\sin\left(\frac{\alpha}{2}\right)^2\tau_1 - \delta_{1a}\sin(\alpha)\right),\\ \partial_\mu\Sigma &= -\frac{\pi_a\partial_\mu\pi_a}{f^2}(\cos(\alpha) + i\tau_1\sin(\alpha)) + \frac{\partial_\mu\pi_a}{f}\left(i\tau_a - 2i\delta_{a1}\sin\left(\frac{\alpha}{2}\right)^2\tau_1 - \delta_{a1}\sin(\alpha)\right),\\ [v_\mu, \Sigma^\dagger] &= \mu_I\delta^0_\mu\left[\left(1 - \frac{\pi_a^2}{2f^2}\right)\sin(\alpha)\tau_2 + \frac{\pi_a}{f}(\delta_{a1}\cos(\alpha)\tau_2 - \delta_{a2}\tau_1)\right].\end{aligned}$$

Up to and including $\mathcal{O}\left((\pi/f)^2\right)$, this gives

$$\begin{aligned}\operatorname{Tr}\left\{\partial_\mu\Sigma\partial_\nu\Sigma^\dagger\right\} &= 2\frac{\partial_\mu\pi_a\partial_\nu\pi_a}{f^2}\\ \operatorname{Tr}\left\{\partial_\mu\Sigma\left[v_\nu, \Sigma^\dagger\right] - \text{h.c.}\right\} &= 2i\mu_I\left[(\delta^0_\mu\partial_\nu + \delta^0_\nu\partial_\mu)\pi_2\sin(\alpha) + \cos(\alpha)\frac{\pi_1(\delta^0_\mu\partial_\nu + \delta^0_\nu\partial_\mu)\pi_2 - \pi_2(\delta^0_\mu\partial_\nu + \delta^0_\nu\partial_\mu)\pi_1}{f^2}\right]\\ \operatorname{Tr}\left\{[v_\nu, \Sigma]\left[v_\nu, \Sigma^\dagger\right]\right\} &= -2{\mu_I}^2\delta^0_\mu\delta^0_\nu\left[\sin(\alpha)^2 + \frac{\pi_1}{f}\sin(2\alpha) + \frac{\pi_a\pi_b}{f^2}k_{ab}\right].\end{aligned}$$

Using the form of the Pauli matrices, we may write $\chi$ as

$$\chi = 2B_0M = 2B_0(\bar{m}\mathbb{1} + \Delta m\tau_3),$$

where $\bar{m} = (m_u + m_d)/2$, $\Delta m = (m_u - m_d)/2$, which gives

$$\begin{aligned}
\chi\Sigma^\dagger + \Sigma\chi^\dagger = 4B_0\Bigg\{&(\bar{m} + \Delta m\tau_3)\left[\left(1 - \frac{\pi_a^2}{2f^2}\right)\cos(\alpha) - \frac{\pi_1}{f}\sin(\alpha)\right] \\
&+ \Delta m\left[\left(1 - \frac{\pi_a^2}{2f^2}\right)\sin(\alpha)\tau_2 + \frac{\pi_a}{f}\left(\delta_{a1}\cos(\alpha)\tau_2 - \delta_{a2}\tau_1\right)\right]\Bigg\}, \\
\chi\Sigma^\dagger - \Sigma\chi^\dagger = -4iB_0\Bigg\{&\bar{m}\left[\left(1 - \frac{\pi_a^2}{2f^2}\right)\sin(\alpha)\tau_1 + \frac{\pi_a}{f}\left(\tau_a - \delta_{1a}2\sin\Big(\frac{\alpha}{2}\Big)^2\tau_1\right)\right] + \Delta m\frac{\pi_3}{f}\Bigg\}.
\end{aligned}$$

This results in the terms, to $\mathcal{O}\left((\pi/f)^2\right)$

$$\begin{aligned}
&\mathrm{Tr}\big\{\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\big\}^2 = \mathrm{Tr}\big\{\partial_\mu\Sigma\partial^\mu\Sigma^\dagger - i(\partial_\mu\Sigma\left[v^\mu,\Sigma^\dagger\right] - \text{h.c.}) - \left[v_\mu,\Sigma\right]\left[v^\mu,\Sigma^\dagger\right]\big\}^2 \\
&\quad = \frac{8\mu_I^2}{f^2}(\partial_\mu\pi_a\partial^\mu\pi_a + 2\partial_\mu\pi_2\partial^\mu\pi_2)\sin(\alpha)^2 \\
&\quad + 16\mu_I^3\left[\frac{\partial_0\pi_2}{f}\ \sin(\alpha)^3 + \frac{3\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1}{f^2}\cos(\alpha)\sin(\alpha)^2\right] \\
&\quad + 4\mu_I^4\left\{\sin(\alpha)^4 + 2\sin(\alpha)^2\left[\frac{\pi_1}{f}\sin(2\alpha) + \frac{\pi_a\pi_b}{f^2}\left(k_{ab} + 2\cos(\alpha)^2\delta_{a1}\delta_{a2}\right)\right]\right\}, \\
&\mathrm{Tr}\big\{\nabla_\mu\Sigma(\nabla_\nu\Sigma)^\dagger\big\}\,\mathrm{Tr}\big\{\nabla^\mu\Sigma(\nabla^\nu\Sigma)^\dagger\big\} \\
&\quad = \frac{4{\mu_I}^2}{f^2}\left(\partial_0\pi_a\partial_0\pi_a + \partial_0\pi_2\partial_0\pi_2 + \partial_\mu\pi_2\partial^2\mu\pi_2\right)\sin(\alpha)^2 \\
&\quad + 16\mu_I^3\left[\frac{\partial_0\pi_2}{f}\ \sin(\alpha)^3 + \frac{3\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1}{f^2}\cos(\alpha)\sin(\alpha)^2\right] \\
&\quad + 4\mu_I^4\left\{\sin(\alpha)^4 + 2\sin(\alpha)^2\left[\frac{\pi_1}{f}\sin(2\alpha) + \frac{\pi_a\pi_b}{f^2}\left(k_{ab} + 2\cos(\alpha)^2\delta_{a1}\delta_{a2}\right)\right]\right\}, \\
&\mathrm{Tr}\big\{\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\big\}\,\mathrm{Tr}\big\{\chi\Sigma^\dagger + \Sigma\chi^\dagger\big\} \\
&\quad = 8B_0\bar{m}\Bigg\{2\cos(\alpha)\frac{\partial_\mu\pi_a\partial^\mu\pi_a}{f^2} + 4\mu_I\left[\frac{\partial_0\pi_2}{2f}\sin(2\alpha) + \frac{\pi_1\partial_0\pi_2\cos(2\alpha) - \pi_2\partial_0\pi_1\cos(\alpha)^2}{f^2}\right] \\
&\quad + {\mu_I}^2\left[2\cos(\alpha)\sin(\alpha)^2 - 2\sin(\alpha)\frac{\pi_1}{f}\left(3\sin(\alpha)^2 - 1\right) + \frac{\pi_1^2(2 - 9\sin(\alpha)^2) + \pi_2^2(2 - 3\sin(\alpha)^2) - 3\pi_3^2\sin(\alpha)^2}{f^2}\cos(\alpha)\right]\Bigg\}, \\
&\mathrm{Tr}\big\{\chi\Sigma^\dagger + \Sigma\chi^\dagger\big\}^2 = (8B_0\bar{m})^2\left[\cos(\alpha)^2 - \frac{\pi_1}{2f}\sin(2\alpha) + \frac{1}{f^2}\left(\pi_1^2\sin(\alpha)^2 - \pi_a\pi_a\cos(\alpha)^2\right)\right], \\
&\mathrm{Tr}\big\{\chi\Sigma^\dagger - \Sigma\chi^\dagger\big\}^2 = -16\left(\frac{2\Delta m B_0\pi_3}{f}\right)^2, \\
&\mathrm{Tr}\Big\{\left(\chi\Sigma^\dagger\right)^2 + \left(\Sigma\chi^\dagger\right)^2\Big\} = 16B_0\bar{m}\left(\cos(2\alpha) - \pi_1\sin(2\alpha) - \cos(\alpha)\pi_a\pi_a - \cos(2\alpha)\pi_1^2\right), \\
&\mathrm{Tr}\big\{\chi\chi^\dagger\big\} = \bar{m}^2 + \Delta m^2.
\end{aligned}$$

# 4 Minimizing energy

Minimizing the effective action $\Gamma[\pi]$ with respect to the expectation value $\pi_a(x) = \langle\hat{\pi}_a\rangle$ gives the ground state expectation value off the field. The first order approximation to this is give by the classical potential. (HVORFOR?) The static Hamiltonian density $\mathcal{H}^{(0)}$, which we get from Eq. 7 through

$$\mathcal{H}_2^{(0)} = -\mathcal{L}_2^{(0)} = -f^2\left(2B_0 m\cos(\alpha) + \frac{1}{2}\mu^2\sin^2(\alpha)\right),$$

is minimized with respect to $\alpha$. This is achieved at

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\mathcal{H}_2^{(0)} = f^2\left(2B_0m - \mu_I^2\cos(\alpha)\right)\sin(\alpha) = 0. \tag{15}$$

This gives the solution set and minimization criterion

$$\alpha = \pi n,\, n \in \mathbb{Z} \quad \text{or} \quad \cos(\alpha) = \frac{2B_0m}{{\mu_I}^2}. \tag{16}$$

We see that the linear part of the potential from Eq. 8, $\mathcal{V}^{(1)} = f({\mu_I}^2\cos(\alpha) - 2B_0m)\sin(\alpha)\pi_1 = 0$ if and only if the criterion for minimization is fulfilled, as we expect (HVORFOR??).

# 5 Propagator

We may write the quadratic part of the Lagrangian Eq. 9 as

$$\mathcal{L}^{(2)} = \frac{1}{2}\delta_{ab}\partial_\mu\pi_a\partial^\mu\pi_b + K_{ab}\pi_a\partial_0\pi_b - \frac{1}{2}M_{ab}^2\pi_a\pi_b, \tag{17}$$

where (Er det mer naturlig å ha $m_\pi^2 = 2B_0m$?)

$$M^2 = \mathrm{diag}\left(2B_0m\cos(\alpha) - \mu_I^2\cos(2\alpha),\quad 2B_0m\cos(\alpha) - \mu_I^2\cos(\alpha)^2,\quad 2B_0m\cos(\alpha) + \mu_I^2\sin(\alpha)^2\right),$$
$$K = \begin{pmatrix} 0 & \mu_I\cos(\alpha) & 0 \\ -\mu_I\cos(\alpha) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The components of the Euler-Lagrange equations of this field are

$$\frac{\partial\mathcal{L}}{\partial\pi_c} = K_{cb}\partial_0\pi_b - M_{cb}^2\pi_b, \quad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\pi_c)} = \partial^\mu\pi_c - K_{ca}\delta_0^\mu\pi_a.$$

This gives the equation of motion for the field

$$\partial^\mu\partial_\mu\pi_a + M_{ab}^2\pi_b = 2K_{ab}\partial_0\pi_b = 2\mu_I\cos(\alpha)(\delta_{a1}\pi_2 - \delta_{a2}\pi_1). \tag{18}$$

The propagator of the pion field is defined by

$$\left(\delta_{ab}\partial_\mu\partial^\mu - 2K_{ab}\partial_0 + M_{ab}^2\right)D_{bc}(x,x') = -i\delta(x-x')\delta_{ac}. \tag{19}$$

The momentum space propagator, as defined in the Appendix A, fulfills

$$-(\delta_{ab}p^2 - 2K_{ab}ip_0 - M_{ab}^2)\tilde{D}_{bc}(p) := A_{ab}\tilde{D}_{bc}(p) = -i\delta_{ac},$$

where

$$A = -\begin{pmatrix} p^2 - M_{11}^2 & -2ip_0K_{12} & 0 \\ 2ip_0K_{12} & p^2 - M_{22}^2 & 0 \\ 0 & 0 & p^2 - M_{33}^2 \end{pmatrix}.$$

The spectrum of the particles is given by solving $\det(A) = 0$ for $p^0$. With $p = (p_0, p_i) = (p_0, q)$ as the four momentum, this gives

$$\det(A) = A_{33}\left(A_{11}A_{22} + A_{12}^2\right) = \left(p^2 - M_{33}^2\right)\left[\left(p^2 - M_{11}^2\right)\left(p^2 - M_{22}^2\right) - \left(2p_0K_{12}\right)^2\right] = 0,$$

This equation has the solutions

$$E_0^2 = q^2 + M_{33}^2, \tag{20}$$
$$E_\pm^2 = q^2 + \frac{1}{2}\left[M_{11}^2 + M_{22}^2 + (2K_{12})^2\right] \pm \frac{1}{2}\sqrt{4q^2\left(2K_{12}\right)^2 + \left(M_{11}^2 + M_{22}^2 + 2K_{12}^2\right)^2 - 4M_{11}^2M_{22}^2}. \tag{21}$$

where $K_{12} = \mu_I \cos(\alpha)$. This gives the effective masses

$$m_0^2 = M_{33}^2, \tag{22}$$

$$m_\pm^2 = q\frac{1}{2}\left[M_{11}^2 + M_{22}^2 + (2K_{12})^2\right] \pm \frac{1}{2}\sqrt{\left(M_{11}^2 + M_{22}^2 + 2K_{12}^2\right)^2 - 4M_{11}^2 M_{22}^2}. \tag{23}$$

The propagator may then be obtained as described in Appendix A, (Sjekk fortegn off-diag.)

$$\begin{aligned} D = -iA^{-1} &= \frac{i}{\det(A)}\begin{pmatrix} A_{22}A_{33} & -A_{12}A_{33} & 0 \\ A_{12}A_{33} & A_{11}A_{33} & 0 \\ 0 & 0 & A_{11}A_{22} + A_{12}^2 \end{pmatrix} \\ &= i\begin{pmatrix} \frac{p^2 - M_{22}^2}{(p_0^2 - E_+^2)(p_-^2 - E_-^2)} & \frac{2ip_0K_{12}}{(p_0^2 - E_+^2)(p_-^2 - E_-^2)} & 0 \\ \frac{-2ip_0K_{12}}{(p_0^2 - E_+^2)(p_-^2 - E_-^2)} & \frac{p^2 - M_{11}^2}{(p_0^2 - E_+^2)(p_-^2 - E_-^2)} & 0 \\ 0 & 0 & \frac{1}{p_0^2 - E_0^2} \end{pmatrix}. \end{aligned} \tag{24}$$

# Appendix A Conventions and notation

The $\mathfrak{su}(2)$ basis used is the Pauli matrices,

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

They obey

$$[\tau_a, \tau_b] = 2i\varepsilon_{abc}\tau_c, \quad \{\tau_a, \tau_b\} = 2\delta_{ab}\mathbb{1}, \quad \mathrm{Tr}[\tau_a] = 0, \quad \mathrm{Tr}[\tau_a\tau_b] = 2\delta_{ab}\mathbb{1}.$$

Together with the identity matrix $\mathbb{1}$, the Pauli matrices form a basis for the vector space of all 2-by-2 matrices. An arbitrary 2-by-2 matrix $M$ may be written

$$M = M_0\mathbb{1} + M_a\tau_a, \quad M_0 = \frac{1}{2}\,\mathrm{Tr}\{M\}, \; M_a = \frac{1}{2}\,\mathrm{Tr}\{\tau_a M\}. \tag{25}$$

### Fourier transform

The Fourier transform used in this text is defined by

$$\mathcal{F}\left\{f(x)\right\}(p) = \tilde{f}(p) = \int \mathrm{d}x\, e^{ipx} f(x), \quad \mathcal{F}^{-1}\left\{\tilde{f}(p)\right\}(x) = f(x) = \int \frac{\mathrm{d}p}{2\pi}\, e^{-ipx}\tilde{f}(p).$$

### Propagators

If $A(x)[f(x)] = 0$ is the equation of motion for some field $f$, where $A(x)$ in general is a differential operator, then the propagator $D$ for this field is defined by

$$A(x)[D(x, x')] = -i\delta(x - x')\mathbb{1}.$$

Assuming $A$ is linear and independent of space, we may redefine $D(x - x', 0) \to D(x - x')$, and the Fourier transform with respect to both $x$ and $x'$ to obtain

$$\mathcal{F}\left\{A(x)[D(x - x')]\right\}_{ac}(p, p') = \tilde{A}_{ab}\tilde{A}_{bc}(p)\delta(p + p') = -i\delta(p + p')\delta_{ac},$$

meaning the momentum space propagator $\tilde{D}(p) = \mathcal{F}\left\{D(x)\right\}(p)$ is given by $\tilde{D} = -i\tilde{A}^{-1}$.

# Appendix B Covariant derivative

In $\chi$PT at finite isospin chemical potential $\mu_I$, the covariant derivative acts on functions $A(x) : \mathcal{M}_4 \to \mathrm{SU}(2)$, where $\mathcal{M}_4$ is the space-time manifold. It is defined as

$$\nabla_\mu A(x) = \partial_\mu A(x) - i\left[v_\mu, A(x)\right], \quad v_\mu = \frac{1}{2}\mu_I\delta^0_\mu\tau_3. \tag{26}$$

The covariant derivative obeys the product rule, as

$$\nabla_\mu(AB) = (\partial_\mu A)B + A(\partial_\mu B) - i\left[v_\mu, AB\right] = (\partial_\mu A - i\left[v_\mu, A\right])B + A(\partial_\mu B - i\left[v_\mu, B\right]) = (\nabla_\mu A)B + A(\nabla_\mu B).$$

Decomposing a 2-by-2 matrix $M$, as described in Appendix A, shows that the commutator of $\tau_b$ and $M$ is zero:

$$\mathrm{Tr}\{[\tau_a, M]]\} = M_b\,\mathrm{Tr}\{[\tau_a, \tau_b]\} = 0.$$

Together with the fact that $\mathrm{Tr}\{\partial_\mu A\} = \partial_\mu\,\mathrm{Tr}\{A\}$, this gives the product rule for invariant traces:

$$\mathrm{Tr}\{A\nabla_\mu B\} = \partial_\mu\,\mathrm{Tr}\{AB\} - \mathrm{Tr}\{(\nabla_\mu A)B\}.$$

This allows for the use of the divergence theorem when doing partial integration. Let $\mathrm{Tr}\{K^\mu\}$ be a space-time vector, and $\mathrm{Tr}\{A\}$ scalar. Let $\Omega$ be the domain of integration, with coordinates $x$ and $\partial\Omega$ its boundary, with coordinates $y$. Then,

$$\int_\Omega \mathrm{d}x\ \mathrm{Tr}\{A\nabla_\mu K^\mu\} = \int_{\partial\Omega} \mathrm{d}y\, n_\mu\, \mathrm{Tr}\{AK^\mu\} - \int_\Omega \mathrm{d}x\ \mathrm{Tr}\{(\nabla_\mu A)K^\mu\},$$

where $n_\mu$ is the normal vector of $\partial\Omega$. This makes it possible to do partial integration and discard surface terms in the $\chi$PT Lagrangian, given the assumption of no variation on the boundary.

# Glossary

$\chi$ Free parameter in the first order chiral Lagrangian. Related to the mass of the pion. 2