Gravgård

Gravgården er hvor tekster går for å dø. Ting som er fjernet fra main, men som jeg ikke helt klarer å slette enda

1 path integral

By Wick's theorem, an n-point correlated is given by the sum of all Feynman diagrams with n external vertices. The factor $Z[0]^{-1}$ divides out all *vacuum bubbles*, that is diagrams without external vertices. We can show this by considering

Here we defined the *generating functional for connected diagrams*, $W[J]$. The reason for the name will become apparent later. (HUSK Å REFFERE TILBAKE) The expectation value of some function of the field-configuration, $A = A[\varphi]$, in the precesense of the source J is

$$\langle A \rangle_J = \frac{1}{Z[J]} A\left(-i\frac{\delta}{\delta J}\right) Z[J]. \tag{1}$$

(DEFINE FUNCTIONAL DERIVATIVE) The expectation value of the field defines a functional,

$$\varphi[J](x) = \langle \varphi(x) \rangle_J = \frac{1}{Z[J]} \left(-i\frac{\delta}{\delta J}\right) Z[J] = \frac{\delta}{\delta J(x)} W[J], \tag{2}$$

and is sometimes called the *classical field*. The notation $\mathcal{F}[f](x)$ means that $\mathcal{F}$ is a functional which takes in a function f, and returns the new function $(\mathcal{F}[f])(x)$. One example is the Lagrangian density, which takes in a field, and returns a function which has a value for each point in space-time. We can reverse this relationship, by defining the functional $J[\varphi](x)$ as *the current which causes the classical field* φ. That is, if $\varphi[J_0](x) = \varphi_0(x)$ for some source J_0, then $J[\varphi_0] = J_0$

2 free scalar

Comparing with the definitions of the thermal propagator in **??**, we can write the free energy compactly as

$$\beta F = \frac{1}{2}\operatorname{Tr}\left\{\ln[D_0^{-1}(K, K')]\right\} = \frac{1}{2}\operatorname{Tr}\left\{\ln[\beta^2 D_0^{-1}(K)]\right\}. \tag{3}$$

3 interacting scalar

Notice that the constant factor from the Jacobian due to the change of variable $\varphi \to \tilde{\varphi}$ does not affect the expectation value, as the same factor is in both the numerator and denominator. If the quantity A is a function of the momentum-space fields, $A = A[\tilde{\varphi}(K)]$, then this expectation value takes the form

$$\langle A \rangle_0 = \frac{\int_{\tilde{S}} \mathcal{D}\tilde{\varphi}(K)\, f[\tilde{\varphi}(K)]\, \exp\left\{-\frac{1}{2}\langle \tilde{\varphi}^*, D\tilde{\varphi}\rangle\right\}}{\int_{\tilde{S}} \mathcal{D}\tilde{\varphi}(K)\, \exp\left\{-\frac{1}{2}\langle \tilde{\varphi}^*, D\tilde{\varphi}\rangle\right\}}. \tag{4}$$

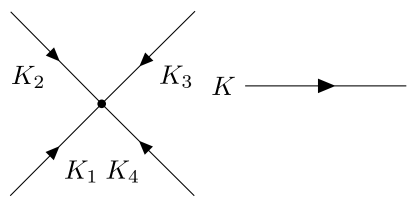
where, as before,

$$\langle \tilde{\varphi}^*, D\tilde{\varphi} \rangle = \int_\Omega \tilde{\mathrm{d}}K \, [\beta^2(\omega_n^2 + \omega_n^2)] |\tilde{\varphi}(K)|^2 \tag{5}$$

The exponential form of $Z[J]$ leads straight forwardly to Wick's theorem, which states that an expectation value of $2n$ fields is a sum of *all possible, distinct* combination of n propagators. To write this in a formal way, we define the functions a and b, which define a way to pair up $2m$ elements. The domain of the functions are the integers between 1 and m, the image a subset of the integers between 1 and $2m$ of size m. A valid pairing is a set $\{(a(1), b(1)), \ldots (a(m), b(m))\}$, where all elements $a(i)$ and $b(j)$ are different, such all integers up to and including $2m$ are featured. A pair is not directed, so $(a(i), b(i))$ is the same pair as $(b(i), a(i))$. Wick theorem states that,

$$\left\langle \prod_{i=1}^{2m} \varphi(X_i) \right\rangle_0 = \sum_{\{(a,b)\}} \left\langle \varphi(X_{a(i)}) \varphi(X_{b(i)}) \right\rangle, \tag{6}$$

where the sum is over all tuples (a, b) that define a valid and unique pairing.



(7)

The expression is the integrated over all *internal* momenta. The factor 1/4! is removed as a general Feynman diagram represent all diagrams with the same form, but different pairing of the momenta. Some diagrams are more symmetric, such that an exchange of momenta still gives *the same pairing*.

4 effective action

In free theory, we may write

$$W[J] = \frac{1}{2} \int \mathrm{d}^4x \mathrm{d}^4y J(x) D_0(x-y) J(y), \tag{8}$$

where D_0 is the free propagator. We may reverse the relation Eq. (2) to write the source in terms of the field,

$$J = D_0^{-1} \varphi(x) \tag{9}$$

This is the field equation for the free field with a source. For the scalar Klein-Gordon field, $D_0^{-1} = \partial^2 + m^2$ Inserting these two relation into the definition of the effective action, and assuming we can do partial integration with D_0^{-1}, we get

$$\Gamma[\varphi] = W[J] - \int \mathrm{d}^4x J(x)\varphi(x) = \int \mathrm{d}x (\frac{1}{2} \int \mathrm{d}y (D_0^{-1}\varphi) D_0 (D_0^{-1}\varphi) - (D_0^{-1}\varphi)\varphi) = -\frac{1}{2} \int \mathrm{d}^4x \varphi(x) D_0^{-1} \varphi(x) \tag{10}$$

This is the classical action. Thus, the effective action Γ and the classical action S are the same to first order in perturbation theory.

Let φ^* solve the quantum mechanical version of the equation of motion, i.e.

$$\frac{\delta \Gamma[\varphi^*]}{\delta \varphi} = 0. \tag{11}$$

We can Taylor-expand the classical action around this point, by setting $\varphi(x) = \varphi^*(x) + \eta(x)$ for some function η. The generating functional becomes

$$Z[J] = \int \mathcal{D}(\varphi^* + \eta) \, \exp\left\{ iS[\varphi^* + \eta] + i \int \mathrm{d}^4x J(\varphi^* + \eta) \right\} \tag{12}$$

The functional version of a Taylor expansion is

$$S[\varphi^* + \eta] = S[\varphi^*] + \int \mathrm{d}x \frac{\delta S[\varphi^*]}{\delta\varphi(x)}\eta(x) + \frac{1}{2}\int \mathrm{d}x\mathrm{d}y\, \frac{\delta^2 S[\varphi^*]}{\delta\varphi(x)\delta\varphi(y)}\eta(x)\eta(y) + \dots \tag{13}$$

Inserting this into $Z[J]$, with S_I to denote the derivatives of higher order than 2, we get

$$Z[J] =$$
$$\int \mathcal{D}\eta \exp\left\{ i\int \mathrm{d}^4x\, (\mathcal{L}[\varphi^*] + J\varphi^*) + i\int \mathrm{d}x \left(\frac{\delta S[\varphi^*]}{\delta\varphi(x)} + J(x)\right)\eta(x) + i\frac{1}{2}\int \mathrm{d}x\mathrm{d}y\, \frac{\delta^2 S[\varphi^*]}{\delta\varphi(x)\delta\varphi(y)}\eta(x)\eta(y) + iS_I[\eta] \right\}$$

In the first term we used the definition of the classical action. This term is constant with respect to η, and may therefore be taken outside the path integral. The next term is the classical equation of motion with a source,

$$\frac{\delta S[\varphi]}{\delta\varphi(x)} = -J(x), \tag{14}$$

evaluated at φ^*.

$$\frac{\delta S[\varphi^*]}{\delta\varphi(x)} + J(x) = \frac{\delta\Gamma[\varphi^*]}{\delta\varphi(x)} + J(x) + \left(\frac{\delta S[\varphi^*]}{\delta\varphi(x)} - \frac{\delta\Gamma[\varphi^*]}{\delta\varphi(x)}\right) = \left(\frac{\delta S[\varphi^*]}{\delta\varphi(x)} - \frac{\delta\Gamma[\varphi^*]}{\delta\varphi(x)}\right) := \delta J \tag{15}$$

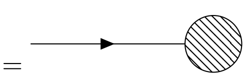
The second to last term is a Gaussian integral, and may be evaluated as described in **??**,

$$\int \mathcal{D}\eta\, \exp\left(i\frac{1}{2}\int \mathrm{d}x\mathrm{d}y\, \frac{\delta^2 S[\varphi^*]}{\varphi(x)\varphi(y)}\eta(x)\eta(y)\right) = C\det\left(\frac{\delta^2 S[\varphi^*]}{\delta\varphi^2}\right)^{-1/2} \tag{16}$$

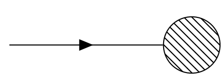
This leaves us with

$$W[J] = -i\ln(Z) \tag{17}$$
$$= \int \mathrm{d}^4x\, \left(\mathcal{L}[\varphi^*] + J\varphi^*\right) - \frac{1}{2}\mathrm{Tr}\left\{\ln\left(-\frac{\delta^2 S[\varphi^*]}{\delta\varphi^2}\right)\right\} + \int \mathcal{D}\eta\, \exp\left\{ i\int \mathrm{d}^4x \delta J(x)\eta \right\} + \int \mathcal{D}\eta\, e^{iS_I} \tag{18}$$

δJ is ultimately dependent on our choice of J to define φ. It contributes to the expectation value of η, through tadpole diagrams

$\langle\eta\rangle_{j=0} =$  (19)

This can be removed by using the renormalization condition

 $= 0.$ (20)

FULL LAGRANGIAN !!!!!

$$
\begin{aligned}
&\mathcal{L} = \\
&f^2\left(2B_0 m\cos\alpha + \frac{1}{2}\mu^2\sin^2\alpha\right), \\
&+ f(\mu_I^2\cos\alpha - 2B_0 m)\pi_1\sin\alpha + f\mu_I\partial_0\pi_2\sin\alpha, \\
&+ \frac{1}{2}\partial_\mu\pi_a\partial^\mu\pi_a + \mu_I\cos\alpha\,(\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1) - B_0 m\pi_a\pi_a\cos\alpha + \frac{1}{2}\mu_I^2\pi_a\pi_b k_{ab}, \\
&+ \frac{\pi_a\pi_a\pi_1}{6f}(2B_0 m\sin\alpha - 2{\mu_I}^2\sin 2\alpha) \\
&- \frac{2\mu_I}{3f}\left[\pi_1(\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1) + \pi_3(\pi_3\partial_0\pi_2 - \pi_2\partial_0\pi_3)\right]\sin\alpha, \\
&+ \frac{1}{6f^2}\left\{\frac{1}{2}B_0 m(\pi_a\pi_a)^2\cos\alpha - [(\pi_a\pi_a)(\partial_\mu\pi_b\partial^\mu\pi_b) - (\pi_a\partial_\mu\pi_a)(\pi_b\partial^\mu\pi_b)]\right\} \\
&- \frac{\mu_I\pi_a\pi_a}{3f^2}\left[(\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1)\cos\alpha + \frac{1}{2}\mu_I\pi_a\pi_b k_{ab}\right]. \\
&+ \frac{l_1}{4}\Bigg(\frac{8\mu_I^2}{f^2}(\partial_\mu\pi_a\partial^\mu\pi_a + 2\partial_\mu\pi_2\partial^\mu\pi_2)\sin^2\alpha + 16\mu_I^3\left[\frac{\partial_0\pi_2}{f}\sin^3\alpha + \frac{1}{f^2}\left(3\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1\right)\cos\alpha\sin^2\alpha\right] \\
&+ 4\mu_I^4\left\{\sin^4\alpha + 2\sin^2\alpha\left[\frac{\pi_1}{f}\sin 2\alpha + \frac{\pi_a\pi_b}{f^2}\left(k_{ab} + 2\delta_{a1}\delta_{a2}\cos^2\alpha\right)\right]\right\}\Bigg) \\
&+ \frac{l_2}{4}\Bigg(\frac{4{\mu_I}^2}{f^2}\left(\partial_0\pi_a\partial_0\pi_a + \partial_0\pi_2\partial_0\pi_2 + \partial_\mu\pi_2\partial^\mu\pi_2\right)\sin^2\alpha + 16\mu_I^3\left[\frac{\partial_0\pi_2}{f}\sin^3\alpha + \frac{1}{f^2}\left(3\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1\right)\cos\alpha\sin^2\alpha\right] \\
&+ 4\mu_I^4\left\{\sin^4\alpha + 2\sin^2\alpha\left[\frac{\pi_1}{f}\sin 2\alpha + \frac{\pi_a\pi_b}{f^2}\left(k_{ab} + 2\delta_{a1}\delta_{a2}\cos^2\alpha\right)\right]\right\}\Bigg) \\
&+ \frac{l_3 + h_1 - h_3}{16}\left((8B_0\bar{m})^2\left[\cos^2\alpha - \frac{\pi_1}{f}\sin 2\alpha + \frac{1}{f^2}\left(\pi_1^2\sin^2\alpha - \pi_a\pi_a\cos^2\alpha\right)\right]\right) \\
&+ \frac{l_4}{4}\Bigg(8B_0\bar{m}\Bigg\{2\frac{\partial_\mu\pi_a\partial^\mu\pi_a}{f^2}\cos\alpha + 4\mu_I\left[\frac{\partial_0\pi_2}{2f}\sin 2\alpha + \frac{1}{f^2}\left(\pi_1\partial_0\pi_2\cos 2\alpha - \pi_2\partial_0\pi_1\cos^2\alpha\right)\right] \\
&\quad + {\mu_I}^2\left[2\cos\alpha\sin^2\alpha - 2\frac{\pi_1}{f}\sin\alpha\left(3\sin^2\alpha - 1\right) + \frac{1}{f^2}\left(\pi_1^2[2 - 9\sin^2\alpha] + \pi_2^2[2 - 3\sin^2\alpha] - 3\pi_3^2\sin^2\alpha\right)\cos\alpha\right]\Bigg\}\Bigg) \\
&+ \frac{h_1 - h_3 - l_4 - l_7}{16}\left(-16\left(\frac{2\Delta m B_0\pi_3}{f}\right)^2\right) + \frac{h_1 + h_2 - l_4}{4}\left(8B_0^2\left(\bar{m}^2 + \Delta m^2\right)\right) \\
&- \frac{h_1 - h_3 - l_4}{8}\left(\left(\cos 2\alpha - 2\frac{\pi_1}{f}\sin 2\alpha - 2\frac{\pi_a\pi_a}{f^2}\cos^2\alpha + 2\frac{\pi_1^2}{f^2}\sin^2\alpha\right) + 16B_0^2\Delta m^2\left(1 - 2\frac{\pi_3^2}{f^2}\right)\right) \\
&+ \mathcal{O}\left[t^6\left(\frac{\pi}{f}\right)^5\right]
\end{aligned}
$$

The different terms of the NLO Lagrangian is

$$
\begin{aligned}
\mathcal{L}_4^{(0)} &= \frac{l_1}{4}4\mu_I^4\sin^4\alpha + \frac{l_2}{4}4\mu_I^4\sin^4\alpha + \frac{l_3+h_1-h_3}{16}(8B_0\bar{m})^2\cos^2\alpha + \frac{l_4}{8}8B_0\bar{m}{\mu_I}^2 2\cos\alpha\sin^2\alpha \\
&+ \frac{h_1+h_3-l_4}{4}\left(8B_0^2\left(\bar{m}^2+\Delta m^2\right)\right) - \frac{h_1-h_3-l_4}{8}\left(16B_0^2\bar{m}^2\cos 2\alpha + 16B_0^2\Delta m^2\right) \\
&= (l_1+l_2)\mu_I^4\sin^4\alpha + l_3(2B_0\bar{m})^2\cos^2\alpha + l_4[2B_0\bar{m}{\mu_I}^2\cos\alpha\sin^2\alpha - 2B_0^2(\bar{m}^2(1-\cos 2\alpha))] \\
&+ h_1[2B_0^2\bar{m}^2(1-\cos 2\alpha) + (2B_0\bar{m})^2\cos^2\alpha] + h_3[2B_0^2\bar{m}^2(1+\cos 2\alpha) - (2B_0\bar{m})^2\cos^2(\alpha) + 4B_0\Delta m^2] \\
&= (l_1+l_2)\mu_I^4\sin^4\alpha + l_3(2B_0\bar{m})^2\cos^2\alpha + l_4[2B_0\bar{m}{\mu_I}^2\cos\alpha - (2B_0\bar{m})^2]\sin^2\alpha + h_1(2B_0\bar{m})^2 + h_3(2B_0\Delta m)^2 \\
&= (l_1+l_2)\mu_I^4\sin^4\alpha + (l_3+l_4)(2B_0\bar{m})^2\cos^2\alpha + l_4(2B_0\bar{m}){\mu_I}^2\cos\alpha\sin^2\alpha - l_4(2B_0\bar{m})^2 + h_1(2B_0\bar{m})^2 + h_3(2B_0\Delta m)^2 \\
\mathcal{L}_4^{(1)} &= \frac{l_1}{4}\left(16\mu_I^3\left[\frac{\partial_0\pi_2}{f}\sin^3\alpha\right] + 4\mu_I^4\left\{2\sin^2\alpha\left[\frac{\pi_1}{f}\sin 2\alpha\right]\right\}\right) + \frac{l_2}{4}\left(16\mu_I^3\left[\frac{\partial_0\pi_2}{f}\sin^3\alpha\right] + 4\mu_I^4\left\{2\sin^2\alpha\left[\frac{\pi_1}{f}\sin 2\alpha\right]\right\}\right) \\
&+ \frac{l_3+h_1-h_3}{16}\left((8B_0\bar{m})^2\left[-\frac{\pi_1}{f}\sin 2\alpha\right]\right) + \frac{l_4}{4}\left(8B_0\bar{m}\left\{4\mu_I\left[\frac{\partial_0\pi_2}{2f}\sin 2\alpha\right] + {\mu_I}^2\left[-2\frac{\pi_1}{f}\sin\alpha\left(3\sin^2\alpha-1\right)\right]\right\}\right) \\
&- \frac{h_1-h_3-l_4}{8}\left(16B_0^2\bar{m}^2\left(-2\frac{\pi_1}{f}\sin 2\alpha\right)\right) \\
&= (l_1+l_2)\left(4\mu_I^3\frac{\partial_0\pi_2}{f}\sin^3\alpha + \mu_I^4 2\sin^2\alpha\frac{\pi_1}{f}\sin 2\alpha\right) \\
&- (l_3+h_1-h_3)(2B_0\bar{m})^2\frac{\pi_1}{f}\sin 2\alpha + l_4 2B_0\bar{m}\left\{4\mu_I\left[\frac{\partial_0\pi_2}{2f}\sin 2\alpha\right] - 2{\mu_I}^2\frac{\pi_1}{f}\sin\alpha\left(3\sin^2\alpha-1\right)\right\} \\
&+ (h_1-h_3-l_4)(2B_0\bar{m})^2\frac{\pi_1}{f}\sin 2\alpha \\
&= (l_1+l_2)\frac{1}{f}\left(4\mu_I^3\partial_0\pi_2\sin^3\alpha + \mu_I^4 2\sin^2\alpha\pi_1\sin 2\alpha\right) - (l_3+l_4)\frac{1}{f}(2B_0\bar{m})^2\pi_1\sin 2\alpha \\
&+ l_4 2B_0\bar{m}\frac{1}{f}\left[2\mu_I\partial_0\pi_2\sin 2\alpha - 2{\mu_I}^2\pi_1\sin\alpha\left(3\sin^2\alpha-1\right)\right] \\
\mathcal{L}_4^{(2)} &= \frac{l_1}{4}\left(\frac{8\mu_I^2}{f^2}(\partial_\mu\pi_a\partial^\mu\pi_a + 2\partial_\mu\pi_2\partial^\mu\pi_2)\sin^2\alpha + 16\mu_I^3\left[(3\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1)\cos\alpha\sin^2\alpha\right]\right. \\
&\left.+ 4\mu_I^4\left\{2\sin^2\alpha\left[\frac{\pi_a\pi_b}{f^2}\left(k_{ab} + 2\delta_{a1}\delta_{a2}\cos^2\alpha\right)\right]\right\}\right) \\
&+ \frac{l_2}{4}\left(\frac{4{\mu_I}^2}{f^2}\left(\partial_0\pi_a\partial_0\pi_a + \partial_0\pi_2\partial_0\pi_2 + \partial_\mu\pi_2\partial^\mu\pi_2\right)\sin^2\alpha + 16\mu_I^3\left[\frac{1}{f^2}\left(3\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1\right)\cos\alpha\sin^2\alpha\right]\right. \\
&\left.+ 4\mu_I^4\left\{2\sin^2\alpha\left[+\frac{\pi_a\pi_b}{f^2}\left(k_{ab} + 2\delta_{a1}\delta_{a2}\cos^2\alpha\right)\right]\right\}\right) \\
&+ \frac{l_3+h_1-h_3}{16}\left((8B_0\bar{m})^2\left[+\frac{1}{f^2}\left(\pi_1^2\sin^2\alpha - \pi_a\pi_a\cos^2\alpha\right)\right]\right) \\
&+ \frac{l_4}{4}\left(8B_0\bar{m}\left\{2\frac{\partial_\mu\pi_a\partial^\mu\pi_a}{f^2}\cos\alpha + 4\mu_I\left[+\frac{1}{f^2}\left(\pi_1\partial_0\pi_2\cos 2\alpha - \pi_2\partial_0\pi_1\cos^2\alpha\right)\right]\right.\right. \\
&\left.\left.+ {\mu_I}^2\left[+\frac{1}{f^2}\left(\pi_1^2[2-9\sin^2\alpha] + \pi_2^2[2-3\sin^2\alpha] - 3\pi_3^2\sin^2\alpha\right)\cos\alpha\right]\right\}\right) \\
&+ \frac{h_1-h_3-l_4-l_7}{16}\left(-16\left(\frac{2\Delta m B_0\pi_3}{f}\right)^2\right) \\
&- \frac{h_1-h_3-l_4}{8}\left(16B_0^2\bar{m}^2\left(-2\frac{\pi_a\pi_a}{f^2}\cos^2\alpha + 2\frac{\pi_1^2}{f^2}\sin^2\alpha\right) + 16B_0^2\Delta m^2\left(-2\frac{\pi_3^2}{f^2}\right)\right) \\
&= l_1\frac{2\mu_I^2}{f^2}(\partial_\mu\pi_a\partial^\mu\pi_a + 2\partial_\mu\pi_2\partial^\mu\pi_2)\sin^2\alpha + l_2\frac{4{\mu_I}^2}{f^2}\left(\partial_0\pi_a\partial_0\pi_a + \partial_0\pi_2\partial_0\pi_2 + \partial_\mu\pi_2\partial^\mu\pi_2\right)\sin^2\alpha \\
&+ \frac{l_1+l_2}{f^2}\left[> 4\mu_I^3\left(3\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1\right)\cos\alpha\sin^2\alpha + 2\mu_I^4\sin^2\alpha\,\pi_a\pi_b\left(k_{ab} + 2\delta_{a1}\delta_{a2}\cos^2\alpha\right)\right] \\
&+ \frac{l_3+l_4}{f^2}(2B_0\bar{m})^2\left(\pi_1^2\sin^2\alpha - \pi_a\pi_a\cos^2\alpha\right) + \frac{l_4}{f^2}2B_0\bar{m}\Big[2\partial_\mu\pi_a\partial^\mu\pi_a\cos\alpha + 4\mu_I\left(\pi_1\partial_0\pi_2\cos 2\alpha - \pi_2\partial_0\pi_1\cos^2\alpha\right) \\
&+ {\mu_I}^2\left(\pi_1^2[2-9\sin^2\alpha] + \pi_2^2[2-3\sin^2\alpha] - 3\pi_3^2\sin^2\alpha\right)\cos\alpha\Big] + \frac{l_7}{f^2}\left(2\Delta m B_0\right)^2\pi_3^2
\end{aligned}
$$

Calculating the free energy density:

$$\mathcal{F} = -f^2 \left(m_\pi^2 \cos\alpha + \frac{1}{2}\mu_I^2 \sin^2\alpha \right) + \mathcal{F}^{(1)}_{\text{fin},\pi_\pm} \tag{21}$$

$$- \frac{1}{2}\frac{1}{(4\pi)^2}\left[\frac{3}{2}\left(\frac{3}{2} m_\pi^4 \cos^4\alpha + m_\pi^2\mu_I^2 \cos\alpha \sin^2\alpha + \mu_I^4 \sin^4\alpha \right) + \frac{1}{3}\left(\bar{l}_1 + 2\bar{l}_2 - 3 \right) \mu_I^4 \sin^4\alpha + \right. \tag{22}$$

$$\left. \frac{1}{2}\left(-\bar{l}_3 + 4\bar{l}_4 - 3 \right) m_\pi^4 \cos^2\alpha + 2\left(\bar{l}_4 - 1 \right) m_\pi^2 \mu_I^2 \cos\alpha \sin^2\alpha \right] \tag{23}$$

$$- \frac{1}{2}\frac{1}{(4\pi)^2}\left[\frac{1}{\epsilon}\left(\frac{3}{2} m_\pi^4 \cos^4\alpha + m_\pi^2\mu_I^2 \cos\alpha \sin^2\alpha + \mu_I^4 \sin^4\alpha \right) + \left(\ln\frac{\mu^2}{m_3^2} + \frac{1}{2}\ln\frac{\mu^2}{\tilde{m}_2^2} \right) m_\pi^3 \cos^2\alpha \right. \tag{24}$$

$$\left. + \ln\frac{\mu}{m_3^2}(\mu_I^4 \sin^4\alpha + 2m_\pi^2\mu_I^2 \cos\alpha \sin^2\alpha) - \left(\frac{1}{\epsilon} + \ln\frac{\mu^2}{M^2} \right)\left(\mu_I^4 \sin^4\alpha + \frac{3}{2} m_\pi^4 \cos^2\alpha + 2m_\pi^2\mu_I^2 \cos\alpha \sin^2\alpha \right) \right] \tag{25}$$

$$= -f^2 \left(m_\pi^2 \cos\alpha + \frac{1}{2}\mu_I^2 \sin^2\alpha \right) + \mathcal{F}^{(1)}_{\text{fin},\pi_\pm} \tag{26}$$

$$- \frac{1}{2}\frac{1}{(4\pi)^2}\left[\frac{1}{3}\left(\bar{l}_1 + 2\bar{l}_2 - 3 + \frac{3^2}{2} \right) \mu_I^4 \sin^4\alpha + \frac{1}{2}\left(-\bar{l}_3 + 4\bar{l}_4 - 3 + \frac{3^2}{2} \right) m_\pi^4 \cos^2\alpha + 2\left(\bar{l}_4 - 1 + \frac{3}{4} \right) m_\pi^2\mu_I^2 \cos\alpha \sin^2\alpha \right. \tag{27}$$

$$\left. \left(\mu_I^4 \sin^4\alpha + m_\pi^4 \cos^2\alpha + 2m_\pi^2\mu_I^2 \cos\alpha \sin^2\alpha \right) \ln\frac{M}{m_3^2} + \frac{1}{2} m_\pi^2 \cos^2\alpha \, \ln\frac{M^2}{\tilde{m}_2^2} \right] \tag{28}$$

$$= -f^2 \left(m_\pi^2 \cos\alpha + \frac{1}{2}\mu_I^2 \sin^2\alpha \right) + \mathcal{F}^{(1)}_{\text{fin},\pi_\pm} - \frac{1}{2}\frac{1}{(4\pi)^2}\left[\frac{1}{3}\left(\bar{l}_1 + 2\bar{l}_2 + \frac{3}{2} + 3\ln\frac{M^2}{m_3^2} \right) \mu_I^4 \sin^4\alpha \right. \tag{29}$$

$$\left. + \frac{1}{2}\left(-\bar{l}_3 + 4\bar{l}_4 + \frac{3}{2} + 2\ln\frac{M^2}{m_3^2} + \ln\frac{M^2}{\tilde{m}_2^2} \right) m_\pi^4 \cos^2\alpha + 2\left(\bar{l}_4 - \frac{1}{4} + \frac{1}{2}\ln\frac{M^2}{m_3^2} \right) m_\pi^2\mu_I^2 \cos\alpha \sin^2\alpha \right]. \tag{30}$$