Bose-Einstein condensation and pion stars

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Pion stars consisting of Bose-Einstein condensed charged pions have recently been proposed as a new class of compact stars. We use the two-particle irreducible effective action to leading order in the $1/N$-expansion to describe charged and neutrals pions as well as the sigma particle. Tuning the parameters in the Lagrangian correctly, the onset of Bose-Einstein condesation of charged pions is exactly at $\mu_I = m_\pi$, where μ_I is the isospin chemical potential. We calculate the pressure, energy density, and equation of state, which are used as input to the Tolman-Oppenheimer-Volkoff equations. Solving these equations, we obtain the mass-radius relation for pion stars. Global electric charge neutrality is ensured by adding the contribution to the pressure and energy density from a gas of free relativistic leptons. We compare our results with those of recent lattice simulations and find good agreement. The masses of the pion stars are up to approximately 200 solar masses while the corresponding radii are of the order of 10^5 km.

I. INTRODUCTION

Apart from black holes, neutron stars are the most compact objects of the universe. Their masses are 1-2 solar masses and their radii are of the order of 10km. Ever since their existence was predicted by Landau in 1932 [1], have their properties been studied in detail. One of the properties of interest is the mass-radius relation of such compact objects. In order to obtain this relation, one must solve the Tolman-Oppenheimer-Volkoff (TOV) equations, which represent the generalization of hydrostatic equilibrium conditions in Newtonian gravity to general relativity [2]. The TOV equations require the equation of state (EoS) as input, i.e. one must know the EoS of nuclear matter at densities up to a few times saturation density. It is well known that lattice Monte Carlo techniques cannot be applied to systems with large baryon densities due to the infamous sign problem. Consequently, the EoS and the properties of neutron stars can only be derived from model calculations, see Ref. [3] for a recent review.

Bose-Einstein condensation (BEC) occurs in very different branches of physics ranging from condensation of atoms in harmonic traps and condensation of ^{4}He in superfluid Helium to condensation of pions and kaons in neutron stars [4]. It is basically the phenomenon that a macroscopic number of bosons occupy a specific single-particle state, which usually is a zero-momentum state. In the context of QCD, the onset of pion condensation at $T = 0$ is when the isospin chemical potential μ_I is

equal to the pion mass, $\mu_I^c = m_\pi$ [5, 6]. [1] In two-flavor QCD with equal quark masses, there is an $O(2)$ isospin symmetry, which gives rise to a conserved isospin charge Q_I. A pion condensate breaks this $O(2)$-symmetry and the phase transition from the vacuum state to a Bose-condensed state is of second order for all temperatures. [7–11]. In contrast to QCD at finite baryon chemical potential, there is no sign problem at finite isospin, and consequently one can carry out lattice simulations. Some early results can be found in Ref. [7, 8], while recent results on the phase diagram in the $\mu_I - T$ plane are reported in Ref. [9–11].

Various aspects of the QCD phase diagram at finite isospin chemical potential have been studied using chiral perturbation theory (CHPT) [5, 6, 12–15], the Nambu-Jona-Lasinio (NJL) model [16–28] and the quark-meson (QM) model [29–33] or their Polyakov-loop extended versions (PNJL and PQM). The applicability of Monte Carlo techniques offers the possibility of testing various models directly. In Ref. [33], it was shown that all the main features of the phase diagram mapped out in [9–11] could be reproduced using the PQM model. This includes the onset of charged pion condensation at $T = 0$ at a critical isospin chemical potential $\mu_I^c = m_\pi$, the second-order nature of this transition, and the merger of the transition lines for the chiral transition and the BEC transition line, as well as the BEC-BCS crossover at large values of μ_I.

Boson stars have a long history since they were proposed almost 50 years ago [34–36]. These stars are com-

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[1] Due to a another definition of the isospin chemical potential that differs by a factor of two, $\mu_I^c = \frac{1}{2}m_\pi$ is also frequently found in the literature.

posed of self-interacting bosons with e.g. a quartic interaction term [37] and coupled to gauge fields [38]. Axion stars are a special type of boson stars involving the hypothetical axion particle which originally was proposed to solve the strong CP problem of QCD [39]. Axions are pseudo-Goldstone bosons associated with the spontaneous breaking of a $U(1)$ symmetry and form a Bose-Einstein condensate in these stars [40].

Recently, it has been proposed that pions themselves may form a compact stellar object by condensing into a zero-momentum state [41]. Using the lattice results of Refs. [9–11], the authors of Ref. [41] calculated the EoS and the resulting mass-radius relation with and without electric charge neutrality imposed. Compared to a neutron star whose mass is of the order of one solar mass and whose radius is of the order of 10 km, these new objects are huge; their masses can be up to $M \approx 200$ solar masses and their radii as large as 10^5 km [41]. In this paper, we will study pion stars using the two-particle irreducible action formalism in the large-N limit, where N is the number of complex fields. Setting $N = 2$, this model reduces to the standard $O(4)$-symmetric linear sigma model for the sigma particle and the three pions.

The article is organized as follows. In Sec. II, we discuss a self-interacting Bose gas in the context of the two-particle irreducible (2PI) effective action formalism and the $1/N$ expansion. We derive the pressure, energy density, and the pion condensate as functions of the physical pion and sigma masses, the pion decay constant and isospin chemical potential. In Sec. III, we solve the Tolman-Oppenheimer-Volkoff equations to obtain the mass-radius relation of a pion star with and without electric charge neutrality. In the Appendix, we briefly review the renormalization of the thermodynamic functions and the matching of the parameters in the model in the $\overline{\text{MS}}$ scheme, and its relation to the parameters in the on-shell scheme.

II. THERMODYNAMICS OF AN INTERACTING BOSE GAS

We briefly discuss the application of the 2PI effective action formalism and the $1/N$-expansion of the pressure,

isospin density, and energy density. Some details of the renormalization procedure can be found in the Appendix as well as in Refs. [42–44].

The Euclidean Lagrangian for a Bose gas with N species of massive complex scalars is

$$\mathcal{L} = (\partial_\mu \Phi_i)^\dagger (\partial_\mu \Phi_i) - \frac{h}{\sqrt{2}} \left(\Phi_1^\dagger + \Phi_1 \right) + m^2 \Phi_i^\dagger \Phi_i + \frac{\lambda}{2N} \left(\Phi_i^\dagger \Phi_i \right)^2 , \tag{1}$$

where $i = 1, 2, \ldots, N$ and $\Phi_i = \frac{1}{\sqrt{2}}(\phi_{2i-1} + i\phi_{2i})$ are complex fields. If $h = 0$, the symmetry of the Lagrangian (1) is $O(2N)$, otherwise it is $O(2N - 1)$. For an $O(2N)$ symmetric theory, there are $(2N-1)N$ continuous symmetries and each continuous symmetry gives rise to a conserved charged Q_i. The maximum number of conserved charges that we can specify simultaneously is the maximum number of commuting generators, which is N, or $N - 1$ if $h \neq 0$. [45]. Eventually we are interested in $N = 2$ and the single chemical potential μ_I that in QCD corresponds to the conservation of isospin charge Q_I. The chemical potential is introduced by replacing the partial derivative with a covariant one, where μ_I is the zeroth component of the gauge field. Identifying the complex field Φ_2 with the charged pions, the recipe is $\partial_\mu \Phi_2 \to (\partial_\mu - \mu_I \delta_\mu^0)\Phi_2$. For $N = 2$, the Lagrangian (1) reduces to the linear sigma model describing the three pions and the sigma particle.

In order to allow for a pion condensate ρ_0 in addition to a chiral condensate ϕ_0, we write the two complex fields Φ_1 and Φ_2 as

$$\Phi_1 = \frac{1}{\sqrt{2}} \left(\phi_0 + \phi_1 + i\phi_2 \right) , \tag{2}$$

$$\Phi_2 = \frac{1}{\sqrt{2}} \left(\rho_0 + \phi_3 + i\phi_4 \right) . \tag{3}$$

After symmetry breaking and with a nonzero pion condensate, the thermodynamic potential Ω in the 2PI effective action formalism can be written as

$$\Omega = \frac{1}{2}m^2(\phi_0^2 + \rho_0^2) + \frac{\lambda}{8N}(\phi_0^2 + \rho_0^2)^2 - \frac{1}{2}\mu_I^2\rho_0^2 - h\phi_0 + \frac{1}{2}\text{Tr}\ln D^{-1} + \frac{1}{2}\text{Tr}D_0^{-1}D + \Phi[D] , \tag{4}$$

where D is the exact propagator, D_0 is the tree-level propagator, and $\Phi[D]$ is the sum of the two-particle irreducible diagrams. The traces are over field indices as well as space-time. The inverse tree-level propagator in Euclidean

space can be written as

$$D_0^{-1}(P) = \begin{pmatrix} P^2 + m_1^2 & 0 & \frac{\lambda}{N}\phi_0\rho_0 & 0 & 0 & \dots \\ 0 & P^2 + m_2^2 & 0 & 0 & 0 & \dots \\ \frac{\lambda}{N}\phi_0\rho_0 & 0 & P^2 + m_3^2 & -\mu_I P_0 & 0 & \dots \\ 0 & 0 & \mu_I P_0 & P^2 + m_4^2 & 0 & \dots \\ 0 & 0 & 0 & 0 & P^2 + m_2^2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} , \tag{5}$$

where $P^2 = \mathbf{P}^2 + P_0^2$.

The tree-level masses are

$$m_1^2 = m^2 + \frac{3\lambda}{2N}\phi_0^2 + \frac{\lambda}{2N}\rho_0^2 , \tag{6}$$

$$m_2^2 = m^2 + \frac{\lambda}{2N}\phi_0^2 + \frac{\lambda}{2N}\rho_0^2 , \tag{7}$$

$$m_3^2 = -\mu_I^2 + m^2 + \frac{\lambda}{2N}\phi_0^2 + \frac{3\lambda}{2N}\rho_0^2 , \tag{8}$$

$$m_4^2 = -\mu_I^2 + m^2 + \frac{\lambda}{2N}\phi_0^2 + \frac{\lambda}{2N}\rho_0^2 . \tag{9}$$

The terms in $\Phi[D]$ are $O(N)$ invariants and can be classified according to which order in the $1/N$-expansion they contribute. To leading order the only contribution comes from $\Phi_{\mathrm{LO}} = \frac{\lambda}{8N}(\mathrm{Tr}D)^2$, which diagrammatically corresponds to a double-bubble or figure-eight vacuum diagram. The coupling gives a factor of $1/N$, while each trace yields a factor of N. The expectation values ϕ_0 and ρ_0 satisfy the usual stationarity conditions, while

the exact propagator satisfies a variational equation:

$$\frac{\delta\Omega}{\delta\phi_0} = 0 , \qquad \frac{\delta\Omega}{\delta\rho_0} = 0 , \qquad \frac{\delta\Omega}{\delta D} = 0 . \tag{10}$$

Using that the self-energy $\Pi = D^{-1} - D_0^{-1}$, the variational gap equation can be written as $\Pi(D) = 2\frac{\delta\Phi}{\delta D}$. At leading order we can write the inverse propagator of each particle in the vacuum as $D_i^{-1}(P) = P^2 + m_i^2 + \Pi_{\mathrm{LO}}(D)$. The self-energy, which is obtained by cutting a propagator line, corresponds to a tadpole diagram. The leading self-energy contribution $\Pi_{\mathrm{LO}}(D)$ is therefore a momentum-independent constant

$$\Pi_{\mathrm{tniy\ LO}} = \lambda \int_Q \frac{1}{Q^2 + M^2} , \tag{11}$$

where M is a medium-dependent mass of the neutral pion and the integral in Euclidean space is

$$\int_Q = \left(\frac{e^{\gamma_E}\Lambda^2}{4\pi}\right)^\epsilon \int \frac{d^d q}{(2\pi)^d} . \tag{12}$$

Here $d = 4 - 2\epsilon$ and Λ is the renormalization scale associated with dimensional regularization.

To leading order in the $1/N$-expansion, the three gap equations (10) are

$$\frac{\delta\Omega}{\delta\phi_0} = m^2\phi_0 + \frac{\lambda}{2N}(\phi_0^2 + \rho_0^2)\phi_0 - h + \frac{\lambda}{2N}\phi_0\mathrm{Tr}D = m^2\phi_0 + \frac{\lambda}{2N}(\phi_0^2 + \rho_0^2)\phi_0 - h + \lambda\phi_0\int_Q \frac{1}{Q^2 + M^2} = 0 , \tag{13}$$

$$\frac{\delta\Omega}{\delta\rho_0} = (m^2 - \mu_I^2)\rho_0 + \frac{\lambda}{2N}(\phi_0^2 + \rho_0^2)\rho_0 + \lambda\rho_0\int_Q \frac{1}{Q^2 + M^2} = 0 , \tag{14}$$

$$\frac{\delta\Omega}{\delta D} = \frac{1}{2}\mathrm{Tr}\left(D^{-1} - D_0^{-1}\right) + \frac{\delta\Phi_{\mathrm{LO}}}{\delta D} = M^2 - m_2^2 - \lambda\int_Q \frac{1}{Q^2 + M^2} = 0 . \tag{15}$$

These equations are ultraviolet divergent and require nonperturbative renormalization. The details of this procedure can be found in Appendix A. At $T = 0$, the system can be in two different phases depending on the value of the isospin chemical potential μ_I. For $\mu_I \leq m_\pi$,

the system is in the vacuum phase, where $\phi_0 = f_\pi$ and $\rho = 0$. For $\mu_I > m_\pi$, the system is in the pion-condensed phase, where $\phi_0 = \frac{h}{\mu_I^2}$ and ρ_0 is nonzero. This is shown below. In the vacuum phase, one can identify M with the physical pion mass m_π, which follows from the fact

that the gap equation (15) is the same as the equation for the pole position of the pion mass (A25). In the pion-condensed phase, we find $M = \mu_I$. This follows directly from subtracting Eq. (14) from Eq. (15). In the appendix, we discuss in some detail the renormalization of the gap equation (15). The other gap equations as well as the thermodynamic potential Ω can be renormalized using the same techniques.

In the vacuum phase, the two nontrivial renormalized gap equations (13) and (15) are

$$m_{\overline{\text{MS}}}^2 \phi_0 + \frac{\lambda_{\overline{\text{MS}}}}{2N} \phi_0^3 - h_{\overline{\text{MS}}} - \frac{\lambda \phi_0 m_\pi^2}{(4\pi)^2} \left[\log \frac{\Lambda^2}{m_\pi^2} + 1 \right] = 0 \, , \tag{16}$$

$$m_\pi^2 - m_{\overline{\text{MS}}}^2 - \frac{\lambda_{\overline{\text{MS}}}}{2N} \phi_0^2 + \frac{\lambda m_\pi^2}{(4\pi)^2} \left[\log \frac{\Lambda^2}{m_\pi^2} + 1 \right] = 0 \, , \tag{17}$$

where the $m_{\overline{\text{MS}}}^2$, $\lambda_{\overline{\text{MS}}}$, and $h_{\overline{\text{MS}}}$ are running parameters in the $\overline{\text{MS}}$ renormalization scheme

$$m_{\overline{\text{MS}}}^2 = \frac{m_0^2}{1 - \frac{\lambda_0}{(4\pi)^2} \log \frac{\Lambda^2}{\Lambda_0^2}} \, , \tag{18}$$

$$\lambda_{\overline{\text{MS}}} = \frac{\lambda_0}{1 - \frac{\lambda_0}{(4\pi)^2} \log \frac{\Lambda^2}{\Lambda_0^2}} \, , \tag{19}$$

$$h_{\overline{\text{MS}}} = h \, , \tag{20}$$

and where m_0^2 and λ_0 are the values of the running parameters at the scale Λ_0. In the appendix it is shown that if we choose $\Lambda^2 = m_\pi^2/e$, the parameters m_0^2 and λ_0 coincide with the parameters m_{OS}^2 and λ_{OS} in the on-shell scheme. We note that h does not require renormalization and consequently $h_{\overline{\text{MS}}} = h = m_\pi^2 f_\pi$ (see Appendix).

Combining the two equations (16) and (17), we see that ϕ_0 satisfies $m_\pi^2 \phi_0 = h = m_\pi^2 f_\pi$ which implies that the minimum is at $\phi_0 = f_\pi$ as it should.

In the pion-condensed phase, the renormalized gap equations are

$$m_{\overline{\text{MS}}}^2 \phi_0 + \frac{\lambda_{\overline{\text{MS}}}}{2N} \phi_0(\phi_0^2 + \rho_0^2) - h_{\overline{\text{MS}}} - \frac{\lambda_{\overline{\text{MS}}} \phi_0 \mu_I^2}{(4\pi)^2} \left[\log \frac{\Lambda^2}{\mu_I^2} + 1 \right] = 0 \, , \tag{21}$$

$$(m_{\overline{\text{MS}}}^2 - \mu_I^2)\rho_0 + \frac{\lambda_{\overline{\text{MS}}}}{2N}(\phi_0^2 + \rho_0^2)\rho_0 - \frac{\lambda_{\overline{\text{MS}}} \rho_0 \mu_I^2}{(4\pi)^2} \left[\log \frac{\Lambda^2}{\mu_I^2} + 1 \right] = 0 \, , \tag{22}$$

$$\mu_I^2 - m_{\overline{\text{MS}}}^2 - \frac{\lambda_{\overline{\text{MS}}}}{2N}(\phi_0^2 + \rho_0^2) + \frac{\lambda_{\overline{\text{MS}}} \mu_I^2}{(4\pi)^2} \left[\log \frac{\Lambda^2}{\mu_I^2} + 1 \right] = 0 \, , \tag{23}$$

Combining Eqs. (21) and (23), we find $\phi_0 = \frac{h_{\overline{\text{MS}}}}{\mu_I^2}$. Using this result, Eq. (22) can be written as

$$\rho_0^2 = \frac{2N}{\lambda_{\overline{\text{MS}}}} \left(\mu_I^2 - m_{\overline{\text{MS}}}^2 \right) - \frac{h_{\overline{\text{MS}}}^2}{\mu_I^4} + \frac{2N\mu_I^2}{(4\pi)^2} \left[\log \frac{\Lambda^2}{\mu_I^2} + 1 \right] = \frac{(2\mu_I^2 + m_\sigma^2 - 3m_\pi^2)f_\pi^2}{m_\sigma^2 - m_\pi^2} - \frac{m_\pi^4 f_\pi^2}{\mu_I^4} + \frac{2N\mu_I^2}{(4\pi)^2} \left[\log \frac{m_\pi^2}{\mu_I^2} \right] \, . \tag{24}$$

We are interested in the pressure in the two phases. The pressure P is given by minus the thermodynamic potential Ω evaluated at the solutions to the gap equations in the two phases. Since we ultimately want zero pressure in the vacuum phase, we calculate the pressure difference of the two phases in the Appendix. The result is

$$P_{\text{eff}} = \frac{N}{2\lambda_{\overline{\text{MS}}}} \left(m_\pi^4 - \mu_I^4 \right) + \frac{1}{2}\mu_I^2\rho_0^2 + \frac{m_\pi^4 f_\pi^2}{\mu_I^2} - m_\pi^2 f_\pi^2 - \frac{N\mu_I^4}{2(4\pi)^2} \left[\log \frac{\Lambda^2}{\mu_I^2} + \frac{1}{2} \right] + \frac{Nm_\pi^4}{2(4\pi)^2} \left[\log \frac{\Lambda^2}{m_\pi^2} + \frac{1}{2} \right] \, . \tag{25}$$

Using the running coupling constant Eq. (19) and Eq. (24), we obtain

$$P_{\text{eff}} = \frac{1}{8} f_\pi^2 \frac{4m_\sigma^2(\mu_I^2 - 2m_\pi^2) + (2\mu_I^2 - 3m_\pi^2)^2 + 3m_\pi^4}{m_\sigma^2 - m_\pi^2} + \frac{1}{2} \frac{m_\pi^4 f_\pi^2}{\mu_I^2} + \frac{N\mu_I^4}{2(4\pi)^2} \left[\log \frac{m_\pi^2}{\mu_I^2} + \frac{1}{2} \right] - \frac{Nm_\pi^4}{4(4\pi)^2} \, . \tag{26}$$

The pressure difference vanishes at threshold, $\mu_I = m_\pi$ as it should.

We can now make contact with tree-level results in chiral perturbation theory as follows. First, we ignore renormalization effects, i.e. terms of order N and then we simply take the limit $m_\sigma \to \infty$. The results are

$$\rho_0^{\text{CHPT}} = f_\pi \sqrt{1 - \frac{m_\pi^4}{\mu_I^4}} \, , \tag{27}$$

$$P_{\text{eff}}^{\text{CHPT}} = \frac{1}{2} f_\pi^2 \mu_I^2 \left(1 - \frac{m_\pi^2}{\mu_I^2} \right)^2 \, . \tag{28}$$

In the pion-condensed phase, $\phi_0 = \frac{m_\pi^2 f_\pi}{\mu_I^2}$ implying that $\phi_0^2 + \rho_0^2 = f_\pi^2$ independent of μ_I. One can therefore think of pion condensation as a rotation of the chiral condensate into a pion condensate as the isospin chemical potential increases.

The pion-condensed phase is electrically charged, $n_Q \neq 0$. However, due to the Coulomb repulsion among the pions, there is an enormous energy cost of having bulk matter that is not electrically neutral [46]. We will therefore impose electric charge neutrality and do so by adding the free Lagrangian of a lepton field l of mass m_l

$$\mathcal{L}_{\text{lepton}} = \bar{l}\left[i\slashed{\partial} + \mu_l\gamma^0 + m_l\right]l , \qquad (29)$$

where μ_l is the lepton chemical potential. This yields an extra contribution to the pressure, which at $T = 0$ is

given by

$$
\begin{aligned}
P_l &= 2\int \frac{d^3p}{(2\pi)^3}\left(\mu_l - \sqrt{p^2+m_l^2}\right)\Theta\left(\mu_l - \sqrt{p^2+m_l^2}\right) \\
&= \frac{4}{3(4\pi)^2}\Big\{\mu_I\sqrt{\mu_l^2-m_l^2}(\mu_l^2 - \tfrac{5}{2}m_l^2) \\
&\quad + \frac{3}{2}m_l^4\log\frac{\mu_l + \sqrt{\mu_l^2-m_l^2}}{m_l}\Big\}\Theta(\mu_l - m_l) . \quad (30)
\end{aligned}
$$

The total pressure is then given by the sum of Eqs. (26) and (30), and it is denoted by $P(\mu_I, \mu_l)$. The isospin and lepton number densities are given by

$$n_I(\mu_I) = \frac{dP_{\text{eff}}}{d\mu_I} = \mu_I\rho_0^2 , \qquad (31)$$

$$n_l(\mu_l) = \frac{dP_l}{d\mu_l} = \frac{16}{3(4\pi)^2}\left(\mu_l^2 - m_l^2\right)^{\frac{3}{2}}\Theta(\mu_l - m_l) , \qquad (32)$$

where the pion condensate is given by Eq. (24). Finally, the energy density is given by

$$
\begin{aligned}
\epsilon(\mu_I, \mu_l) &= -P(\mu_I, \mu_l) + \mu_I n_I(\mu_I) + \mu_l n_l(\mu_l) \\
&= \frac{1}{2}f_\pi^2\frac{m_\sigma^2(\mu_I^2 + 2m_\pi^2) + 3\mu_I^2(\mu_I^2 - m_\pi^2) - 3m_\pi^4}{m_\sigma^2 - m_\pi^2} - \frac{3}{2}\frac{m_\pi^4 f_\pi^2}{\mu_I^2} + \frac{3N\mu_I^4}{2(4\pi)^2}\log\frac{m_\pi^2}{\mu_I^2} + \frac{N(m_\pi^4 - \mu_I^4)}{4(4\pi)^2} \\
&\quad + \frac{4}{(4\pi)^2}\left[\mu_l\sqrt{\mu_l^2 - m_l^2}(\mu_l^2 - \tfrac{1}{2}m_l^2) - \frac{1}{2}m_l^4\log\frac{\mu_l + \sqrt{\mu_l^2 - m_l^2}}{m_l}\right] . \quad (33)
\end{aligned}
$$

The isospin density and energy density in CHPT follows from Eqs. (28), (28) and (31). We can write the latter in terms of the pressure and we finally obtain

$$n_I = \mu_I f_\pi^2\left[1 - \frac{m_\pi^4}{\mu_I^4}\right] , \qquad \mu_I \geq m_\pi \qquad (34)$$

$$\frac{P}{\epsilon} = \frac{\mu_I^2 - m_\pi^2}{\mu_I^2 + 3m_\pi^3} , \qquad \mu_I \geq m_\pi . \qquad (35)$$

The onset of the isospin density is $\mu_I = m_\pi$ and it becomes linear for large values of μ_I. The ratio $\frac{P}{\epsilon}$ vanishes at threshold, $\mu_I = m_\pi$ and approaches unity rather quickly as μ_I increases.

III. NUMERICAL RESULTS AND DISCUSSION

In this section we will present and discuss our results. The lepton is either the electron, $l = e$, or the muon, $l = \mu$. The values for the meson and lepton masses, and

the pion-decay constant are

$$m_\sigma = 600 \text{ MeV} , \qquad m_\pi = 140 \text{ MeV} , \qquad (36)$$
$$m_e = 0.511 \text{ MeV} , \qquad m_\mu = 105 \text{ MeV} , \qquad (37)$$
$$f_\pi = 93 \text{ MeV} . \qquad (38)$$

The sigma particle is a broad resonance, whose mass is in the 400-800 MeV range. Unless otherwise stated, we will use the value $m_\sigma = 600$ MeV which is a fairly common choice. We will briefly discuss the m_σ dependence of our results.

In Fig. 1, we show the electric charge density n_Q normalized by m_π^3 as a function of the isospin chemical potential μ_I normalized by m_π. The red line is Eq. (31) and the blue line is from leading order in chiral perturbation theory, Eq. (34). The data points are from the lattice simulations of Brandt, Endrodi, and Schmalzbauer [9–11] (The data points have been scaled since their defintion of μ_I differs by a factor of two compared to ours). The charge density is zero all the way up to $\mu_I = \mu_I^c = m_\pi$. This reflects the socalled Silver Blaze

property, which is the independence of physical quantities, such as the isospin charge density, below some critical chemical potential. The vacuum state of the theory is therefore defined by $\mu_I \leq \mu_I^c$. The agreement between the results from the lattice simulations and those from CHPT and the linear sigma model is in general good, in particular for lower values of μ_I.
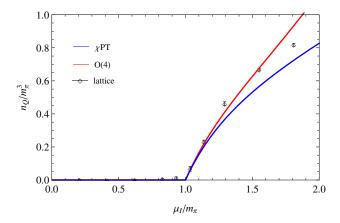

FIG. 1. Normalized electric charge density $\frac{n_Q}{m_\pi^3}$ as a function of $\frac{\mu_I}{m_\pi}$. See main text for details.

In Fig. 2 we show the normalized equation of state, i.e. the energy density as a function of the pressure, both normalized to m_π^4. The blue line is for a purely pionic system, while the other lines are obtained when imposing charge neutrality by the addition of muons (green) or electrons (red) to the system. We notice that the imposition of electric charge neutrality has a large effect on the EoS, although the mass dependence seems moderate given the the two order of magnitude between the electron and muon masses.

Charge neutrality is given by the equation

$$n_Q = \frac{1}{2}n_I + n_l = 0 . \tag{39}$$

Inserting the isospin and lepton charge densities, given by Eqs. (31) and (32), into Eq. (39) one obtains μ_I as a function of μ_l.

We next determine the mass-radius relation of the pion star using the Tolman-Oppenheimer-Volkoff equation. The radial dependence of the stellar mass is given by the energy density

$$\frac{dm}{dr} = 4\pi^2 r^2 \epsilon(\mu_I, \mu_l) , \tag{40}$$

while the TOV equation, which describes the pressure inside the star is rewritten for the lepton chemical potential [11]
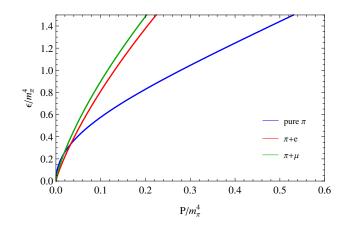
$$\frac{d\mu_l}{dr} = -G\mu_l \frac{m + 4\pi r^3 P}{r^2 - 2Grm}\left[1 + 2\frac{\mu_I}{\mu_l}\right]\left[1 + 4\frac{n_l'}{n_I'}\right]^{-1} \tag{41}$$

For a pure pion star, the system is characterized by the isospin chemical potential and Eq. (41) reduces to

$$\frac{d\mu_I}{dr} = -G\mu_I \frac{m + 4\pi r^3 P}{r^2 - 2Grm} . \tag{42}$$

In Fig. 3, we show the main result of the present paper, namely the mass-radius relation of pion stars. The blue lines are for the pure pion system, where dark blue indicates the stable solution of the TOV equation and light blue the unstable one. The dark and light green lines are obtained by imposing charge neutrality by adding a muon gas and the red and orange lines by adding electrons instead. For comparison we show the lattice results of [11] in black and grey, where we find overall good agreement with our model.

The central isospin chemical potentials for the heaviest stable stars are $\mu_I = 252.88$ MeV (pure pion star), $\mu_I = 142.0928$ MeV (pions+muons), and $\mu_I = 140.00008513$ MeV (pions+electrons). The onset for pion condensation is $\mu_I^c = 140$ MeV and therefore we find that the pion pressure is very small in the pion-lepton systems, which results in a larger star compared to the pure pion case.

In Fig. 4, we show the mass-radius relation for pure pion stars and different values of the sigma mass.

The blue line is the result from Fig. 3, i.e. for $m_\sigma = 600$ MeV. The solid red lines correspond to $m_\sigma = 500$ MeV (lower) and 700 MeV (upper). For comparison, the red dotted line shown the result from CHPT at tree level and the black dashed line the lattice results from Ref. [11]

FIG. 2. Energy density $\mathcal{E}$ normalized to m_π^4 as a function of the pressure normalized to m_π^4. See main text for details.
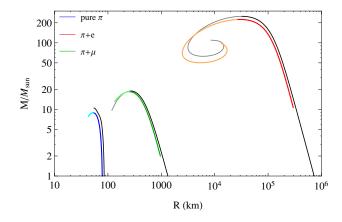
FIG. 3. Mass-radius relation of pion stars. See main text for details.

Similar calculations for the neutral systems, with either muons or electrons, show much smaller differences and are not shown here. Based on the Fig. 4 one might conclude that the result from CHPT, which is a model-independent one, is in best agreement with lattice data. This is true for the pure pion star, which is the least interesting case. It would be of interest to go to next-to-leading order in chiral perturbation theory to study the convergence of the results. Based on the values of the central isospin chemical potentials, one expects the largest corrections in the pure pion case.
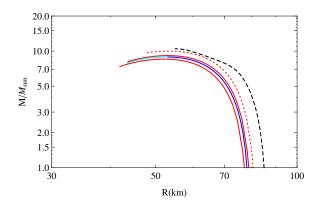

FIG. 4. Mass-radius relation of pion stars for different values of the sigma mass. See main text for details.

In Fig. 5, we show the pressure $P(r)$ (blue lines) and accumulated mass $m(r)$ (red lines) normalized to the central pressure P_c and the total mass M of the pion star, both as functions of the normalized distance from the center. The solid lines are for $\pi+e$, dashed lines for

$\pi+\mu$ and dotted lines for pions only. The central pressure and mass correspond to the maximum of the $M(R)$ curves in Fig. 3. The faster the pressure decreases, the faster the mass inside the star accumulates. The typical central pressure we find for the charged pion star is of order $P \sim 10^{33}$ Pa. For the $\pi + \mu$ and $\pi + e$ system it is much smaller, $P \sim 10^{31}$ Pa and $P \sim 10^{25}$ Pa respectively. For comparison, the central pressure of neutron stars is $P \sim 10^{34}$ Pa. Thus the higher the central pressure, the smaller the star.
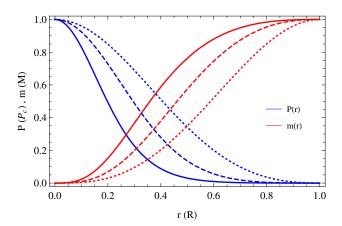

FIG. 5. Pressure and accumulated mass normalized to the central pressure P_c and the total mass M of the pion star. See main text for details.

ACKNOWLEDGEMENTS

Appendix A: Renormalization of gap equations and parameter fixing

In this Appendix, we carry out renormalization of the nonperturbative gap equations. We also briefly discuss how one can use the $\overline{\text{MS}}$ and the on-shell renormalization schemes to express the running parameters in terms of meson masses, the pion decay constant, and the renormalization scale.

In order to renormalize the gap equations and pressure, we need the following divergent integrals in dimen-

sional regularization for $d = 4 - 2\epsilon$ dimensions

$$
\begin{aligned}
C(m^2) &= \int_Q \log \left[Q^2 + m^2 \right] \\
&= -\frac{m^4}{2(4\pi)^2} \left(\frac{\Lambda^2}{m^2} \right)^{\epsilon} \left[\frac{1}{\epsilon} + \frac{3}{2} + \mathcal{O}(\epsilon) \right] \ , \quad \text{(A1)} \\
A(m^2) &= \int_Q \frac{1}{Q^2 + m^2} \\
&= -\frac{m^2}{(4\pi)^2} \left(\frac{\Lambda^2}{m^2} \right)^{\epsilon} \left[\frac{1}{\epsilon} + 1 + \mathcal{O}(\epsilon) \right] \ , \quad \text{(A2)}
\end{aligned}
$$

where Λ is the renormalization scale associated with dimensional regularization.

We first renormalize the gap equation (15), which we rewrite as

$$
M^2 - m_2^2 = \lambda \int_Q \frac{1}{Q^2 + m_2^2 + \Pi_{\mathrm{LO}}} \ , \quad \text{(A3)}
$$

where we have used that $M^2 = m_2^2 + \Pi_{\mathrm{LO}}$. Expanding the right-hand side in powers of Π_{LO}, we find

$$
\begin{aligned}
\frac{M^2}{\lambda} - \frac{m_2^2}{\lambda} &= \int_Q \frac{1}{Q^2 + m_2^2} - \Pi_{\mathrm{LO}} \int_Q \frac{1}{(Q^2 + m_2^2)^2} \\
&\quad + \Pi_{\mathrm{LO}}^2 \int_Q \frac{1}{(Q^2 + m_2^2)^3} + \dots \ . \quad \text{(A4)}
\end{aligned}
$$

The self-energy is written in a power series in λ, $\Pi_{\mathrm{LO}} = \Pi_{\mathrm{LO}}^{(1)} + \Pi_{\mathrm{LO}}^{(2)} + \dots$ where the superscript indicates the power of λ. Introducing renormalization constants δm_i^2 and $\delta \lambda_i$ for each order in the coupling, Eq. (A4) can be renormalized iteratively. The first iteration gives

$$
\Pi_{\mathrm{LO}}^{(1)} = \lambda \int_Q \frac{1}{Q^2 + m_2^2} \ , \quad \text{(A5)}
$$

and using Eq. (A2) we find

$$
\begin{aligned}
M^2 &= m^2 + \delta m_1^2 + \frac{\lambda + \delta \lambda_1}{2N} \left(\phi_0^2 + \rho_0^2 \right) \\
&\quad - \frac{\lambda m_2^2}{(4\pi)^2} \left[\frac{1}{\epsilon} + \log \frac{\Lambda^2}{m_2^2} + 1 \right] \ , \quad \text{(A6)}
\end{aligned}
$$

including the leading-order counterterms δm_1^2 and $\delta \lambda_1$. The divergences in (A6) are removed by choosing the counterterms in the $\overline{\mathrm{MS}}$ scheme

$$
\delta m_1^2 = \frac{\lambda m^2}{(4\pi)^2 \epsilon} \ , \qquad \delta \lambda_1 = \frac{\lambda^2}{(4\pi)^2 \epsilon} \ . \quad \text{(A7)}
$$

Carrying out renormalization order by order in λ, one finds that the n'th order mass and coupling constant counterterms are [42, 43]

$$
\delta m_n^2 = \frac{\lambda^n m^2}{(4\pi)^{2n} \epsilon^n} \ , \qquad \delta \lambda_n = \frac{\lambda^{n+1}}{(4\pi)^{2n} \epsilon^n} \ . \quad \text{(A8)}
$$

The coupling λ is renormalized by writing $\lambda \to \lambda_{\mathrm{b}} = \Lambda^{2\epsilon}(\lambda_{\overline{\mathrm{MS}}} + \delta \lambda)$, where λ_{b} is the bare coupling and

$$
\delta \lambda = \sum_{n=1}^{\infty} \delta \lambda_n = \frac{\lambda^2}{(4\pi)^2 \epsilon} \frac{1}{1 - \frac{\lambda}{(4\pi)^2 \epsilon}} \ . \quad \text{(A9)}
$$

Solving for $\frac{1}{\lambda_{\mathrm{b}}}$ yields

$$
\frac{1}{\lambda_{\mathrm{b}}} = \frac{\Lambda^{-2\epsilon}}{\lambda_{\overline{\mathrm{MS}}}} - \frac{\Lambda^{-2\epsilon}}{(4\pi)^2 \epsilon} \ . \quad \text{(A10)}
$$

It follows from Eq. (A8) that $\delta m^2 = \frac{m^2}{\lambda} \delta \lambda$ and therefore

$$
\frac{m_{\mathrm{b}}^2}{\lambda_{\mathrm{b}}} = \frac{m_{\overline{\mathrm{MS}}}^2}{\lambda_{\overline{\mathrm{MS}}}} . \quad \text{(A11)}
$$

Let us rewrite the gap equation (15) as

$$
\begin{aligned}
\frac{M^2}{\lambda} &= \frac{m^2}{\lambda} + \frac{\phi_0^2 + \rho_0^2}{2N} + \int_Q \frac{1}{Q^2 + M^2} \\
&= \frac{m^2}{\lambda} + \frac{\phi_0^2 + \rho_0^2}{2N} - \frac{M^2}{(4\pi)^2} \left[\frac{1}{\epsilon} + \log \frac{\Lambda^2}{M^2} + 1 \right] \ .
\end{aligned}
$$
$$\text{(A12)}$$

The gap equation (A12) is made finite by using Eq. (A11) and substituting Eq. (A10). The result is

$$
M^2 = m_{\overline{\mathrm{MS}}}^2 + \frac{\lambda_{\overline{\mathrm{MS}}}(\phi_0^2 + \rho_0^2)}{2N} - \frac{\lambda_{\overline{\mathrm{MS}}} M^2}{(4\pi)^2} \left[\log \frac{\Lambda^2}{M^2} + 1 \right] \ ,
$$
$$\text{(A13)}$$

which is Eq. (17) in the vacuum phase and Eq. (23) in the BEC phase. The gap equation (13) can be renormalized in the same manner and the result is given in Eq. (16) and (21), respectively. The renormalized version of Eq. (14) for the pion condensate is (22). Combining Eq. (16) and (17), we find $h = m_\pi^2 f_\pi$, i.e. the tree-level relation. Thus the parameter h is not renormalized.

Taking the derivative of Eq. (A10) with respect to the renormalization scale Λ and using that the bare coupling λ_b is independent of Λ, we find that the renormalized coupling $\lambda_{\overline{\mathrm{MS}}}$ satisfies a renormalization group equation. In the limit $\epsilon \to 0$, this equation reads

$$
\Lambda \frac{d\lambda_{\overline{\mathrm{MS}}}}{d\Lambda} = \frac{2\lambda_{\overline{\mathrm{MS}}}^2}{(4\pi)^2} \ , \quad \text{(A14)}
$$

whose solution is given by

$$
\lambda_{\overline{\mathrm{MS}}} = \frac{\lambda_0}{1 - \frac{\lambda_0}{(4\pi)^2} \log \frac{\Lambda^2}{\Lambda_0^2}} \ , \quad \text{(A15)}
$$

where the constant λ_0 is the value of the running parameter $\lambda_{\overline{\mathrm{MS}}}$ at the scale Λ_0. Using $\frac{m_{\mathrm{b}}^2}{\lambda_{\mathrm{b}}} = \frac{m_{\overline{\mathrm{MS}}}^2}{\lambda_{\overline{\mathrm{MS}}}}$ and that the bare mass and coupling are independent of the

renormalization scale, as well as Eq. (A14), one can show that the renormalized mass similarly satisfies the renormalization group equation,

$$\Lambda \frac{dm_{\overline{\rm MS}}^2}{d\Lambda} = \frac{2\lambda_{\overline{\rm MS}} m_{\overline{\rm MS}}^2}{(4\pi)^2} \; . \tag{A16}$$

The solution to Eq. (A16) is

$$m_{\overline{\rm MS}}^2 = \frac{m_0^2}{1 - \frac{\lambda_0}{(4\pi)^2} \log \frac{\Lambda^2}{\Lambda_0^2}} \; , \tag{A17}$$

where the constant m_0^2 is the value of the running mass parameter $m_{\overline{\rm MS}}^2$ at the scale Λ_0.

As mentioned in the main text we are ultimately interested in the pressure. We first write $\frac{1}{2}m^2(\phi_0^2 + \rho_0^2) + \frac{\lambda}{8N}(\phi_0^2 + \rho_0^2)^2$ as $\frac{N}{2\lambda}[m^2 + \frac{\lambda}{2N}(\phi_0^2 + \rho_0^2)]^2 - \frac{Nm^4}{2\lambda}$ in the potential Eq. (4) and then use the gap equation (15). The resulting unrenormalized pressure reads

$$P = \frac{N}{2\lambda}\left(m^4 - M^4\right) + \frac{1}{2}\mu_I^2 \rho_0^2 + h\phi_0$$
$$- N\int_Q \log\left[Q^2 + M^2\right] + NM^2 \int_Q \frac{1}{Q^2 + M^2} \; . \tag{A18}$$

The pressure difference of the two phases is then

$$P_{\rm eff} = \frac{N}{2\lambda}\left(m_\pi^4 - \mu_I^4\right) + \frac{1}{2}\mu_I^2 \rho_0^2 + \frac{m_\pi^4 f_\pi^2}{\mu_I^2} - m_\pi^2 f_\pi^2$$
$$- N\int_Q \log\left[Q^2 + \mu_I^2\right] + N\mu_I^2 \int_Q \frac{1}{Q^2 + \mu_I^2} \; .$$
$$+ N\int_Q \log\left[Q^2 + m_\pi^2\right] - Nm_\pi^2 \int_Q \frac{1}{Q^2 + m_\pi^2} \; . \tag{A19}$$

Using the integrals (A1)–(A2) and renormalizing the coupling according to Eq. (A10), the renormalized pressure difference is

$$P_{\rm eff} = \frac{N}{2\lambda_{\overline{\rm MS}}}\left(m_\pi^4 - \mu_I^4\right) + \frac{1}{2}\mu_I^2 \rho_0^2 + \frac{m_\pi^4 f_\pi^2}{\mu_I^2} - m_\pi^2 f_\pi^2$$
$$- \frac{N\mu_I^4}{2(4\pi)^2}\left[\log \frac{\Lambda^2}{\mu_I^2} + \frac{1}{2}\right]$$
$$+ \frac{Nm_\pi^4}{2(4\pi)^2}\left[\log \frac{\Lambda^2}{m_\pi^2} + \frac{1}{2}\right] \; . \tag{A20}$$

We finally discuss renormalization in the on-shell scheme [47–49]. The counterterms in this scheme are determined by demanding that the renormalized mass is equal to the physical mass, i.e. the pole mass, and that the residue of the propagator is unity,

$$\Pi_{\sigma,\pi}(P^2 = m_{\sigma,\pi}^2) + {\rm counterterms} = 0 \; , \tag{A21}$$

$$\frac{\partial}{\partial P^2}\Pi_{\sigma,\pi}(P^2)\big|_{P^2 = m_{\sigma,\pi}^2} + {\rm counterterms} = 0 \; , \tag{A22}$$

where $\Pi_{\sigma,\pi}(P^2)$ is the self-energy function. Eq. (A22) is trivially satisfied in the present case since the leading order self-energy is independent of the external momentum. The inverse propagator for the sigma and pion can be written as

$$P^2 + m_1^2 + \delta m_1^2 + \Pi_{\rm LO} \; , \tag{A23}$$
$$P^2 + m_2^2 + \delta m_2^2 + \Pi_{\rm LO} \; . \tag{A24}$$

Evaluating Eqs. (A23) and (A24) on-shell and suppressing the counterterms, the equations for the pion and sigma masses in the vacuum then become

$$m_\pi^2 = m^2 + \frac{\lambda}{2N}f_\pi^2 + \lambda\int_Q \frac{1}{Q^2 + m_\pi^2} \; , \tag{A25}$$

$$m_\sigma^2 = m^2 + \frac{3\lambda}{2N}f_\pi^2 + \lambda\int_Q \frac{1}{Q^2 + m_\pi^2} \; . \tag{A26}$$

We note in passing that Eq. (A25) is identical to Eq. (15) in the vacuum phase. At tree level, we can express the parameters m^2, λ, and h in terms of the physical sigma and pion masses, as well as the pion decay constant,

$$m^2 = -\frac{1}{2}(m_\sigma^2 - 3m_\pi^2) \; , \tag{A27}$$

$$\lambda = N\frac{(m_\sigma^2 - m_\pi^2)}{f_\pi^2} \; , \tag{A28}$$

$$h = m_\pi^2 f_\pi \; . \tag{A29}$$

We first consider Eq. (A25). Again we rewrite it as

$$\frac{m_\pi^2}{\lambda} = \frac{m^2}{\lambda} + \frac{f_\pi^2}{2N} + \int_Q \frac{1}{Q^2 + M^2}$$
$$= \frac{m^2}{\lambda} - \frac{f_\pi^2}{2N} - \frac{m_\pi^2}{(4\pi)^2}\left[\frac{1}{\epsilon} + \log \frac{\Lambda^2}{m_\pi^2} + 1\right] \tag{A30}$$

The self-energy correction $\Pi_{\rm LO}$ can be eliminated by the renormalization of the coupling, $\lambda_b = \Lambda^{2\epsilon}(\lambda_{\rm os} + \delta\lambda)$, with

$$\frac{1}{\lambda_{\rm b}} = \frac{\Lambda^{-2\epsilon}}{\lambda_{\rm os}} - \frac{\Lambda^{-2\epsilon}}{(4\pi)^2}\left[\frac{1}{\epsilon} + \log \frac{\Lambda^2}{m_\pi^2} + 1\right] \; . \tag{A31}$$

Furthermore, if we define $\delta m^2 = m^2 \delta\lambda$, i.e. $\frac{m_{\rm b}^2}{\lambda_{\rm b}} = \frac{m_{\rm os}^2}{\lambda_{\rm os}} = \frac{m^2}{\lambda}$, Eq. (A30) reduces to the tree-level expression, as it should in the OS-scheme. That this recipe is consistent can be seen by renormalizing Eq. (A30) iteratively as we did above. Then one finds

$$\delta m_n^2 = \frac{\lambda^n}{(4\pi)^{2n}}m^2\left[\frac{1}{\epsilon} + \log \frac{\Lambda^2}{m_\pi^2} + 1\right]^n \; , \tag{A32}$$

$$\delta\lambda_n = \frac{\lambda^{n+1}}{(4\pi)^{2n}}\left[\frac{1}{\epsilon} + \log \frac{\Lambda^2}{m_\pi^2} + 1\right]^n \; . \tag{A33}$$

It is straightforward to show that $\lambda_{\rm b}$ and $m_{\rm b}^2$ are independent of the renormalization scale Λ as they must be.

Moreover, using the fact that the bare parameters are independent of the renormalization scheme we can use Eqs. (A10) and (A31) to show that $\lambda_0 = \lambda$, given by Eq. (A28) if we choose $\Lambda_0^2 = \frac{m_\pi^2}{e}$. From Eqs. (A15) and (A17) it then follows that m_0^2 is given by the right-hand side of Eqs. (A27). After having renormalized the gap equation defining the pion mass nonperturbatively, we make some remarks regarding the equation for the sigma particle. While the divergence in Eq. (A26) is the same as in (A25), the tree-level term involving the coupling is three times as large. Thus it seems that one cannot renormalize the gap equation for the sigma mass using the counterterms given by Eqs. (A32)–(A33). The solution to this problem is to realize that one must in-

clude all counterterms that respect the symmetries. In the present case there is a counterterm proportional to $\text{Tr}(G^2)$ which is of order one, i.e. next-to-leading in the $1/N$-expansion [44]. One can then write the coupling constant counterterm in Eq. (A26) as [44]

$$\frac{\delta\lambda_A + 2\delta\lambda_B}{6N} f_\pi^2 , \qquad (A34)$$

where the two terms contribute at order N and one, respectively. The term $\delta\lambda_A$ is equal to the counterterm we have already found, while the term $\delta\lambda_B$ is used for renormalizing the gap equation at next-to-leading order. Finally, we remark that the parameter h does not require renormalization and therefore $h_{\overline{\text{MS}}} = h_{\text{os}} = m_\pi^2 f_\pi$.

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