Chiral Perturbation Theory

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Contents

Abstract

This is an abstract

Chapter 1

Introduction

(KLADD, INNTRODUKSJON)

The standard model describes, together with Einstein's general theory of relativity, all experiments we as humans are able to perform. It uses the language of quantum field theory to describe (X) particles, and their interactions. Quantum electrodynamics, or QED, is the description of electromagnetic interactions. It gives some of the most accurate predictions in science (EKSEMPEL). This is calculated using the techniques of Feynman diagrams, which describes how quantum fields interact as the sum of all possible ways patterns of interaction. When the interaction is weak, as is the case for QED, we get a sum that converges quickly, and can get highly accurate estimates by calculating a few orders of the sum. The weakness of QED is quantified in the fine structure constant $\alpha \approx 0.007297$.(CITE PDG) A Feynman diagram in QED is proportional to α^n, where n is the number of vertices in the diagram, which means the contributions from more complex diagrams with many vertices rapidly becomes insignificant

Quantum chromodynamics, or QCD, is the part of the standard model that describes quarks, the constituents of protons and neutrons, and how they interact via the strong nuclear force. When dealing with the strong force, the fact that the strength of interaction depend on the energy scale of the interaction becomes apparent. In high energy interactions, around $100\,\text{MeV}$ or more, the strong force equivalent to the fine structure constant is $\alpha_s \approx 0.1$. This makes it possible to do persuasive calculations using QCD. However, the strong force has its name for a reason. For scales around 1GeV and below, the perturbation method breaks down. In this case, the quark fields form *hadrons*. Hadrons are again subdivided in to *mesons* and *baryons*. Mesons are bosons, and none are stable. They do however play an important role in effective theories, which we will come back to. The most familiar of the baryons are the proton and neutrons. In this regime, we can not extract predictions of the theory using perturbation theory, at least not directly, and we are forced to come up with other techniques.

Effective field theories

A profound feature of physics is the possibility of describing a system by isolating the degrees of freedom of interest, and ignoring the rest. We can describe the motion of the plants in the solar system, massive and complex systems, by only their mass, velocity and position. In quantum field theory, this property manifests in the power of *effective field theory*. An effective field theory describes a system not by the fundamental, underlying particles, whatever that may mean, but by effective fields. A theory of two, interacting fields φ and ψ will be described by an action which depends on both fields, $S[\varphi, \psi]$. In the path integral formalism, predictions can be then be made by integrating over all possible states for these fields, such as for the vacuum transition amplitude,

$$Z = \int \mathcal{D}\varphi \mathcal{D}\psi \, \exp\{iS[\varphi, \psi]\}. \tag{1.1}$$

The effective description of only the φ-degrees of freedom by *integrating out* the ψ-degrees of freedom, which results in an effective action $S_{\text{eff}}[\varphi]$, defined by [1]

$$\int D\varphi \, \exp\{iS_{\text{eff}}[\varphi]\} = \int \mathcal{D}\varphi \mathcal{D}\psi \, \exp\{iS[\varphi, \psi]\}. \tag{1.2}$$

This gives us hope for describing low energy QCD as an effective theory of the particles we know from experiments appear. In this text, we will derive and explore an effective theory of the pionic degrees of freedom of low-energy QCD, called chiral perturbation theory of χPT.

Mesons, of which pions are the lightest, were first proposed by Hideki Yukawa as the mechanism to hold nucleons together to form the nucleus of atoms. Though first believed to appear in the showers of particles that comes from cosmic rays, they were decisively discovered in 1947, by Cecil F. Powell *et. al.* [2] Pions do not show up in the standard model, as quarks do, but rather as an effective degree of freedom at low temperature. The action of the standard model has a very particular, nice form. It is the integral over a local Lagrangian, that is

$$S[\varphi] = \int \mathrm{d}^4x \, \mathcal{L}[\varphi]. \tag{1.3}$$

Here, we denote all particles by φ. That the Lagrangian $\mathcal{L}$ is local means that it is made up of terms like $\varphi(x)\varphi(x)$, all interactions happens at one point in space-time, as opposed to a term such as $f(x,y)\varphi(x)\varphi(y)$. We can not a priori expect an effective action to take this form [1]. However, we have general principles we expect particles to obey, such Lorentz invariance and cluster decomposition. Cluster decomposition concerns a system of N sets of particles, α_i, that evolve into the sets β_i. That is,

$$|\alpha_1, \alpha_2, ...\alpha_N\rangle_{\text{in}} \longrightarrow |\beta_1, \beta_2, ...\beta_N\rangle_{\text{out}}. \tag{1.4}$$

It says that if the sets of particles α_i, β_i are located far enough apart, then the S-matrix factors as

$$\langle\beta_1, \beta_2, ...\beta_N|\alpha_1, \alpha_2, ...\alpha_N\rangle = \langle\beta_1|\alpha_1\rangle \langle\beta_2|\alpha_2\rangle ... \langle\beta_N|\alpha_N\rangle. \tag{1.5}$$

This is a familiar property, as it essentially says that wildly separated experiments do not interfere, and one that we expect all good effective description to have [3, 4].

The method we use to construct the effective action of χPTis one formulated by Weinberg. This method relies on, as Weinberg himself called it, a "theorem":

> [I]f one writes down the most general possible Lagrangian, including all terms consistent with assumed symmetry principles, and then calculates matrix elements with this Lagrangian to any given order of perturbation theory, the result will simply be the most general possible S-matrix consistent with analyticity, perturbative unitary, cluster decomposition and the assumed symmetry principles. [5]

In other words, if we write down the most general Lagrange density consistent with symmetries of the underlying theory, then we have not made any restrictions on the theory, other than some foundational assumptions. This Lagrangian will be on the form

$$\mathcal{L}_{\text{eff}}[\varphi] = \sum_i \lambda_i \mathcal{O}_i, \tag{1.6}$$

where $\mathcal{O}_i$ are local functions of the fields and its derivatives, and λ_i are coupling constants. The coupling constants are free parameters, which parametrizes the most general S-matrix consistent with fundational assumptions and the underlying theory. A Lagrangian with an infinite amount of free parameters might seem useless, however if we are able to find a consistent series expansion, then only a finite number of terms are needed to calculate to any given order in perturbation theory. In the case of χPT, the expansion is in the energy of the pions. We will detail this later in the text.

Stars

Although it might seem counterintuitive, stars are one of the objects we might hope to describe using QCD at low energies. Neutron stars, one of the most extreme objects in the universe, quickly cools down to

temperatures below 10^{10} K. This might be hot by almost all standards, however it corresponds to an energy of 0.862 MeV. This is well into the strong interaction regime of QCD, and the stars must therefore be described by an effective theory of interacting nuclear matter [6, 7]. Stars are modeled using the Tolman-Oppenheimer-Volkoff, or TOV, equation. The TOV equation is based on Einstein's general theory of relativity, and its solution gives the pressure of the star as a function of its radius. The only input needed is the *equation of state*, or EOS, of the material that makes up the star [8]. The equation of state of a material is the relationship between its energy density, u, and pressure P, i.e. a relationship of the form

$$f(P, u) = 0. \tag{1.7}$$

This is where QCD comes in. One way to compute the equation of state of QCD systems is using the lattice QCD, a numerical method. Here, space-time is approximated as finite and discrete, and Monte-Carlo importance sampling is used to perform the path integral. This method, however, fails for non-zero baryon densities in what is known as the fermion sign problem. Non-zero baryon density, or equivalently non-zero baryon chemical potential μ_B, corresponds to systems with a matter-antimatter asymmetry, such as neutron stars, or more generally all observed stars. Recently, it has been proposed that pions may condense and form a new type of compact, gravitationally bound object, i.e. a star. Pions have baryon chemical potential of zero, and pion condensates are thus amenable to lattice QCD simulations. This offer a way to model stellar objects from first principles, as well as by analytical methods using χPT [9, 10].

Pion condensate and the QCD phase diagram

(ILLUSTRASJON) A pioncondensate is a state with a non-zero expectation value of pions, $\langle \pi \rangle \neq 0$. The condensation happens when the pion chemical potential equals the pion mass, $\mu_I = m_\pi$. This is just one part of the rich phase structure of QCD. The full phase diagram is not yet known, due to the many difficulties, only some of which is mentioned in this text. Some, however, we do know. At low temperatures and chemical potentials, in the normal or vacumm phase, we get a hadronic gas. For high temperature or chemical potentials, we get a quark gluon plasma. In this phase, quarks are no longer tightly bound in hadrons, but together with gluons form as soup conjectured to be at the center of neutron stars.(SIKKER? KILDE) For high (HOW) baryon chemical potential, we expect that there forms a color super conductor (KILDE). Better understanding of the phase diagram of QCD is a large part of understanding the standard model and its consequences better. To be able to validate the techniques used, it is important to employ all possible, sound approaches. This makes it possible to crosscheck techniques, such as when comparing χPTwith lattice QCD.

Outline of thesis

The goal of this thesis is to calculate the next-to-leading order equation of state of a system at finite isospin chemical potential, using two-flavor chiral perturbation theory. We will also investigate the phase transition from the vacuum phase into a pion condensate phase. In Chapter 2, we take a survey of some general theory needed for χPT. We start by introducing the generating functional in the path integral formalism, and use this to define the one-pratcile-irreducible effective action, and the effective potential. This allows us to prove Goldstone theorem, an important result which provides the connection between the symmetries of a theory, and its low energy dynamics. The theorem states that a system which undergoes spontaneous symmetry breaking gives rise to massless particles. We then present the CCWZ construction, which provides a procedure to constructs an effective Lagrangian of Goldstone bosons. We also present some mathematical prerequisite, such as Lie groups and Lie algebras, and discuss the role and mathematical implementation of symmetry in physics in general and quantum field theory in particular. In chapter 3, we take the general theory of the last chapter, and apply it to QCD to get χPT. We start the chapter by discussing QCD, its constituent parts and their symmetries and the corresponding conserved currents. We then use the theory from last chapter to find the terms with makes up the Lagrangian of χPT, as well as how to incorporate explicit breaking of symmetry and external source currents, as well as a finite isospin chemical potential. This section also contains a discussion about how to order these terms in a well-defined series expansion, and avoid the need to include infinity many terms. With this, we construct the leading order and next-to-leading order Lagrangian, which is expanded in powers of the pion fields. We discuss how to identify possible redundant terms in the Lagrangian, and use our result to find properties of the pion such as their tree-level

mass and propagator. In chapter 4, we use our result to calculate the free energy density, to one loop using the leading order Lagrangian, and then use the tree level result at next-to-leading order to renormalize the result. We discuss the low energy parameters we use, and how to evaluate observable to the same order in perturbation theory consistently. With the free energy density, we explore the thermodynamics of pions at finite isospin chemical potential, and derive the equation of state. We also discuss the phase transition to the pion condensate phase using the Landau theory of phase transition. In chapter 5, we (SKRIV KAPITTEL!!!!)

The appendix is referenced throughout the text, to keep the thesis as self-contained as possible. In Appendix A, we review thermal field theory and the imaginary time formalism. This chapter contains calculations needed in the main part of the thesis, where they are referenced, and lays out the path integral approach as well as its connection to thermodynamics in more detail. We also discuss dimensional regularization, derive the Feynman rules for and interacting scalar and generalize thermal field theory to fermions. Appendix B summarize the convention and notation used in this thesis, as well as discussing some of the used in the thesis mathematics in more detail.

Chapter 2

Theory

In this section we will survey some general properties of quantum field theory that is necessary for chiral perturbation theory. First, we will introduce the path integral and the 1-particle irreducible effective action, as well as the effective action. We will derive Goldstone's theorem and present the CCWZ construction, which are the basis for χPT.

2.1 QFT via path integrals

The theory in this section is based on [11, 3, 4, 1]. Feynman diagrams are drawn using JaxoDraw [12].

In the path integral formalism, one evaluates quantum observable by summing or integrating over all possible configuration in space and time. If the system has specified initial and final configurations, this amounts to all possible paths the system might evolve between these, hence the name. We assume the reader has some familiarity with this formalism, however if a refresher is needed, section A.2 contains a derivation of the closely related imaginary-time formalism, and compares it with the path integral approach. In the path integral formalism, the vacuum-to-vacuum transition amplitude, i.e. the probability that that vacuum at $t = -\infty$ evolves to the vacuum at time $t = \infty$, is given by

$$Z = \lim_{T \to \infty} \langle \Omega, T/2 | -T/2, \Omega \rangle = \lim_{T \to \infty} \langle \Omega | e^{-iHT} | \Omega \rangle = \int \mathcal{D}\pi \mathcal{D}\varphi \, \exp\left\{ i \int \mathrm{d}^4 x \, (\pi \dot{\varphi} - \mathcal{H}[\pi, \varphi]) \right\}, \qquad (2.1)$$

where $|\Omega\rangle$ is the vacuum of the theory. By introducing a source term into the Hamiltonian density, $\mathcal{H} \to \mathcal{H} - J(x)\varphi(x)$, we get the generating functional

$$Z[J] = \int \mathcal{D}\pi \mathcal{D}\varphi \, \exp\left\{ i \int \mathrm{d}^4 x \, (\pi \dot{\varphi} - \mathcal{H}[\pi, \varphi] + J\varphi) \right\}. \qquad (2.2)$$

If $\mathcal{H}$ is quadratic in π, we can complete the square and integrate out π to obtain

$$Z[J] = C \int \mathcal{D}\varphi \, \exp\left\{ i \int \mathrm{d}^4 x \, (\mathcal{L}[\varphi] + J\varphi) \right\}. \qquad (2.3)$$

C is infinite, but constant, and will drop out of physical quantities. In scattering theory, the main objects of study are correlation functions $\langle \varphi(x_1)\varphi(x_2)... \rangle = \langle \Omega | T \{ \varphi(x_1)\varphi(x_2) \ldots \} | \Omega \rangle$, where T is the time ordering operator. These are given by functional derivatives of $Z[J]$,

$$\langle \varphi(x_1)\varphi(x_2)... \rangle = \frac{\int \mathcal{D}\varphi(x) \, (\varphi(x_1)\varphi(x_2)...) e^{iS[\varphi]}}{\int \mathcal{D}\varphi(x) \, e^{iS[\varphi]}} = \frac{1}{Z[0]} \prod_i \left(-i \frac{\delta}{\delta J(x_i)} \right) Z[J] \Big|_{J=0}, \qquad (2.4)$$

where

$$S[\varphi] = \int \mathrm{d}^4 x \, \mathcal{L}[\varphi] \qquad (2.5)$$

is the action of the theory. The functional derivative is described in section B.2. In a free theory, we are able to write

$$Z_0[J] = Z_0[0] \exp(iW_0[J]), \quad W_0[J] = \frac{1}{2} \int \mathrm{d}^4x \mathrm{d}^4y \, J(x) D_0(x-y) J(y), \tag{2.6}$$

where D_0 is the propagator of the free theory. Using this form of the generating functional, Eq. (2.4) becomes

$$\frac{1}{Z[0]}(-i)^n \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} Z_0[J]\Big|_{J=0} = (-i)^n \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} e^{iW_0[J]}\Big|_{J=0}$$

$$= (-i)^n \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_{n-1})} \left(i\frac{\delta W_0[J]}{\delta J(x_n)} \right) e^{iW_0[J]}\Big|_{J=0}$$

$$= (-i)^n \frac{\delta}{\delta J(x_2)} \cdots \frac{\delta}{\delta J(x_{n-1})} \left(i\frac{\delta^2 W_0[J]}{\delta J(x_{n-1})\delta J(x_n)} + i^2 \frac{\delta W_0[J]}{\delta J(x_{n-1})}\frac{\delta W_0[J]}{\delta J(x_n)} \right) e^{iW_0[J]}\Big|_{J=0} = \dots$$

$$= (-i)^{n/2} \sum_{(a,b)} \prod_{i=1}^{n/2} \frac{\delta^2 W_0[J]}{\delta J(x_{a(i)})\delta J(x_{b(i)})}\Big|_{J=0}.$$

In the last line we have introduced the functions a, b which define a way to pair up n elements. The domain of the functions are the integers between 1 and $n/2$, the image a subset of the integers between 1 and n of size $n/2$. A valid pairing is a set $\{(a(1), b(1)), \dots (a(n/2), b(n/2))\}$, where all elements $a(i)$ and $b(j)$ are different, such all integers up to and including n are featured. A pair is not directed, so $(a(i), b(i))$ is the same pair as $(b(i), a(i))$. The sum is over the set $\{(a, b)\}$ of all possible, unique pairings. If n is odd, the expression is equal to 0. This is Wick's theorem, and it can more simply be stated as *a correlation function is the sum of all possible pairings of 2-point functions*,

$$\left\langle \prod_{i=1}^{n} \varphi(x_i) \right\rangle_0 = \sum_{\{(a,b)\}} \prod_{i=1}^{n/2} \left\langle \varphi(x_{a(i)})\varphi(x_{b(i)}) \right\rangle_0. \tag{2.7}$$

The subscript on the expectation value indicates that it is evaluated in the free theory.

If we have an interacting theory, that is a theory with an action $S = S_0 + S_I$, where S_0 is a free theory, the generating functional can be written

$$Z[J] = Z_0[0] \left\langle \exp\left(iS_I + i \int \mathrm{d}^4x \, J(x)\varphi(x) \right) \right\rangle_0. \tag{2.8}$$

We can expand the exponential in power series, which means the expectation in Eq. (2.8) becomes

$$\sum_{n,m} \frac{1}{n!m!} \left\langle (iS_I)^n \left(i\int \mathrm{d}^4x \, J(x)\varphi(x) \right)^m \right\rangle_0. \tag{2.9}$$

The terms in this series are represented by Feynman-diagrams, which are constructed from the Feynman-rules, and can be read from the action. We will not go into further details on how the Feynman-rules are derived, which can be found in any of the main sources for this section [11, 3, 4, 1]. The source terms gives rise to an additional vertex

$$\longrightarrow \!\!\bullet \quad J(x). \tag{2.10}$$

The generating functional $Z[J]$ equals $Z_0[0]$ times *the sum of all diagrams with external sources $J(x)$*.

Consider a general diagram without external legs, built up of N different connected subdiagrams, where subdiagram i appears n_i times. As an illustration, a generic vacuum diagram in ϕ^4-theory has the form

$$V = \qquad \times \qquad \times \qquad \times \qquad \times \dots. \tag{2.11}$$

If sub-diagram i as a stand-alone diagram equals V_i, then each copy of that subdiagram contribute a factor V_i to the total diagram. However, due to the symmetry of permuting identical subdiagrams, one must divide

by the extra symmetry factor $s = n_i!$, which is the total number of permutation of all the copies of diagram i. The full diagram therefore equals

$$V = \prod_{i=1}^{N} \frac{1}{n_i!} V_i^{n_i}. \tag{2.12}$$

V is uniquely defined by a finite sequence of integers, $(n_1, n_2, \ldots n_N, 0, 0, \ldots)$, so the sum of all diagrams is the sum over the set S of all finite sequences of integers. This allows us to write the sum of all diagrams as

$$\sum_{(n_1,\ldots)\in S} \prod_i \frac{1}{n_i!} V_i^{n_i} = \prod_{i=1}^{\infty} \sum_{n_i=1}^{\infty} \frac{1}{n_i!} V_i^{n_i} = \exp\left(\sum_i V_i\right). \tag{2.13}$$

We showed that the generating functional $Z[J]$ were the $Z_0[0]$ times the sum of all diagrams due to external sources. Using Eq. (2.13), we see that the sum of all *connected* diagrams $W[J]$ is given by

$$Z[J] = Z_0[0] \exp(iW[J]). \tag{2.14}$$

We can see that this is trivially true for the free theory, the only connected diagram is

$$W_0[J] = {}_{J(x)}\bullet\!\!\longrightarrow\!\!\bullet{}_{J(y)} \ . \tag{2.15}$$

The 1PI effective action

The generating functional for connected diagrams, $W[J]$, is dependent on the external source current J. Analogously to what is done in thermodynamics and in Lagrangian and Hamiltonian mechanics, we can define a new quantity, with a different independent variable, using the Legendre transformation. The new independent variable is

$$\varphi_J(x) := \frac{\delta W[J]}{\delta J(x)} = \langle \varphi(x) \rangle_J. \tag{2.16}$$

The subscript J on the expectation value indicate that it is evaluated in the presence of a source. The Legendre transformation of W is then

$$\Gamma[\varphi_J] = W[J] - \int \mathrm{d}^4x \, J(x)\varphi_J(x). \tag{2.17}$$

Using the definition of φ_J, we have that

$$\frac{\delta}{\delta\varphi_J(x)}\Gamma[\varphi_J] = \int \mathrm{d}^4y \, \frac{\delta J(y)}{\delta\varphi_J(x)} \frac{\delta}{\delta J(y)} W[J] - \int \mathrm{d}^4y \, \frac{\delta J(y)}{\delta\varphi_J(x)}\varphi_J(y) - J(x) = -J(x). \tag{2.18}$$

If we compare this to the classical equations of motion of a field φ with the action S,

$$\frac{\delta S[\varphi]}{\delta\varphi(x)} = -J(x), \tag{2.19}$$

we see that Γ is an action that gives the equation of motion for the expectation value of the field, given a source current $J(x)$.

To interpret Γ further we observe what happens if we treat $\Gamma[\varphi]$ as a classical action with a coupling g. The generating functional in this new theory is

$$Z[J, g] = \int \mathcal{D}\varphi \exp\left\{ ig^{-1}\left(\Gamma[\varphi] + \int \mathrm{d}^4x\varphi(x)J(x)\right)\right\} \tag{2.20}$$

The free propagator in this theory will be proportional to g, as it is given by the inverse of the equation of motion for the free theory. All vertices in this theory, on the other hand, will be proportional to g^{-1}, as they are given by the higher order terms in the action $g^{-1}\Gamma$. This means that a diagram with V vertices and I internal lines is proportional to g^{I-V}. Regardless of what the Feynman-diagrams in this theory are, the

number of loops of a connected diagram is $L = I - V + 1$. [1] To see this, we first observe that one single loop must have equally many internal lines as vertices, so the formula holds for $L = 1$. If we add a new loop to a diagram with n loops by joining two vertices, the formula still holds. If we attach a new vertex with one line, the formula still holds, and as the diagram is connected, any more lines connecting the new vertex to the diagram will create additional loops. This ensures that the formula holds, by induction. As a consequence of this, any diagram is proportional to g^{L-1}. This means that in the limit $g \to 0$, the theory is fully described at the tree-level, i.e. by only considering diagrams without loops. In this limit, we may use the stationary phase approximation, as described in section B.3, which gives

$$Z[J, g \to 0] \approx C \det\left(-\frac{\delta^2 \Gamma[\varphi_J]}{\delta \varphi^2}\right) \exp\left\{ig^{-1}\left(\Gamma[\varphi_J] + \int \mathrm{d}^4 x J \varphi_J\right)\right\}. \tag{2.21}$$

This means that

$$-ig \ln(Z[J, g]) = g W[J, g] = \Gamma[\varphi_J] + \int \mathrm{d}^4 x\, J(x)\varphi_J(x) + \mathcal{O}(g), \tag{2.22}$$

which is exactly the Legendre transformation we started out with, modulo the factor g. Γ is therefore the action which describes the full theory at the tree level. For a free theory, the classical action S equals the effective action, as there are no loop diagrams.

The propagator $D(x, y)$, which is the connected 2 point function $\langle \varphi(x)\varphi(y)\rangle_J$, is given by the second functional derivative of $W[J]$, times $-i$. Using the chain rule, together with Eq. (2.18), we get

$$(-i) \int \mathrm{d}^4 z \frac{\delta^2 W[J]}{\delta J(x) \delta J(z)} \frac{\delta^2 \Gamma[\varphi_J]}{\delta \varphi_J(z) \varphi_J(y)} = (-i) \int \mathrm{d}^4 z \frac{\delta \varphi_J[z]}{\delta J(x)} \frac{\delta^2 \Gamma[\varphi_J]}{\delta \varphi_J(z) \varphi_J(y)} = \frac{\delta}{\delta J(x)} \frac{\delta \Gamma[\varphi_J]}{\delta \varphi_J(y)} = \delta(x - y). \tag{2.23}$$

This shows that the second functional derivative of the effective action is iD^{-1}, where D^{-1} is the inverse propagator. The inverse propagator is the sum of all one-particle-irreducible (1PI) diagrams, with two external vertices. More generally, Γ is the generating functional for 1PI diagrams, which is why it is called the 1PI effective action.

Γ may be viewed as an effective action in the sense it was used in the introduction. We define η as the fluctuations around the expectation value of the field, $\varphi(x) = \varphi_J(x) + \eta(x)$, and use this to change variables of integration in the path integral. In this case, φ_J is considered a constant, so

$$\int \mathcal{D}\varphi \, \exp\{iS[\varphi]\} = \int \mathcal{D}\eta \, \exp\{iS[\varphi_J + \eta]\} \tag{2.24}$$

By assumption, $\langle\eta\rangle_J = 0$, which means this is path integral is described by only 1PI diagrams. Comparing this to Eq. (1.2), we see that the 1PI effective potential corresponds to integrating out *all* degrees of freedom, an let the expectation value appear as a static background field,

$$\exp\{i\Gamma[\varphi_J]\} = \int \mathcal{D}\eta \, \exp\{iS[\varphi_J + \eta]\}. \tag{2.25}$$

Effective potential

For a constant field configuration $\varphi(x) = \varphi_0$, the effective action, which is a functional, becomes a regular function. We define the effective potential $\mathcal{V}_{\mathrm{eff}}$ by

$$\Gamma[\varphi_0] = -VT\,\mathcal{V}_{\mathrm{eff}}(\varphi_0), \tag{2.26}$$

VT is the volume of space-time. For a constant ground state, the effective potential will equal the energy of this state. To calculate the effective potential, we can expand the action around this state to calculate the effective action, by changing variables to $\varphi(x) = \varphi_0 + \eta(x)$. $\eta(x)$ now parametrizes fluctuations around the ground state, and has by assumption a vanishing expectation value. The generating functional becomes

$$Z[J] = \int \mathcal{D}(\varphi_0 + \eta) \, \exp\left\{iS[\varphi_0 + \eta] + i \int \mathrm{d}^4 x J(\varphi_0 + \eta)\right\} \tag{2.27}$$

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[1]This is a consequence of the Euler characteristic $\chi = V - E + F$.

The functional version of a Taylor expansion, as described in section B.2, is

$$S[\varphi_0 + \eta] = S[\varphi_0] + \int \mathrm{d}x\, \eta(x) \frac{\delta S[\varphi_0]}{\delta \varphi(x)} + \frac{1}{2} \int \mathrm{d}x \mathrm{d}y\, \eta(x)\eta(y) \frac{\delta^2 S[\varphi_0]}{\delta \varphi(x)\delta\varphi(y)} + \dots \tag{2.28}$$

The notation

$$\frac{\delta S[\varphi_0]}{\delta \varphi(x)} \tag{2.29}$$

indicates that the functional $S[\varphi]$ is differentiated with respect to $\varphi(x)$, then evaluated at $\varphi(x) = \varphi_0$. We use the notation

$$S_0[\eta] := \int \mathrm{d}^4x \mathrm{d}^4y\, \eta(x)\eta(y) \frac{\delta^2 S[\varphi_0]}{\delta\eta(x)\,\delta\eta(y)}, \tag{2.30}$$

$$S_i[\eta] := \int \mathrm{d}^4x \mathrm{d}^4y \mathrm{d}^4z\, \eta(x)\eta(y)\eta(z) \frac{\delta^3 S[\varphi_0]}{\delta\varphi(x)\,\delta\varphi(y)\,\delta\varphi(z)} + \dots, \tag{2.31}$$

where the dots indicate higher derivatives. When we insert this expansion into the generating functional $Z[J]$ we get

$$Z[J] = \int \mathcal{D}\eta \exp\left\{ i \int \mathrm{d}^4x\, (\mathcal{L}[\varphi_0] + J\varphi_0) + i \int \mathrm{d}^4x\, \eta(x) \left(\frac{\delta S[\varphi_0]}{\delta\varphi(x)} + J(x) \right) + iS_0[\eta] + iS_i[\eta] \right\}$$

The first term is constant with respect to η, and may therefore be taken outside the path integral. The second term gives rise to tadpole diagrams, which alter the expectation value of $\eta(x)$. For $J = 0$, this expectation value should vanish, so this term can be ignored. Furthermore, this means that the ground state must minimize the classical potential,

$$\frac{\partial \mathcal{V}(\varphi_0)}{\partial \varphi} = 0. \tag{2.32}$$

This leaves us with

$$-i \ln Z[J] = W[J] = \int \mathrm{d}^4x\, (\mathcal{L}[\varphi_0] + J\varphi_0) - i \ln\left(\int \mathcal{D}\eta \exp\{iS_0[\eta] + iS_i[\eta]\} \right) \tag{2.33}$$

We can now use the definition of the 1PI effective action to obtain a formula for the effective potential,

$$\mathcal{V}_{\text{eff}}(\varphi_0) = -\frac{1}{VT}\left(W[J] - \int \mathrm{d}^4x J\varphi_0 \right) = \mathcal{V}(\varphi_0) - i \ln\left(\int \mathcal{D}\eta \exp\{iS_0[\eta] + iS_i[\eta]\} \right). \tag{2.34}$$

In Eq. (2.17), we showed that the 1PI effective action had the property that it describes the whole quantum theory of the original action at the tree-level. This was done by inspecting a theory with the action proportional to g^{-1}, where Feynman diagrams with L loops are proportional to g^{L-1}. We can use the same argument to expand the effective potential in loops. This is done by modifying the action $S[\varphi] \to g^{-1} S[\varphi]$, and then expand in power of g. The first term in the effective potential is simply modified by $\mathcal{V} \to g^{-1}\mathcal{V}$, which means that it is made up of tree-level, i.e. zero loop terms. This is as expected, since the tree-level result corresponds to the classical result without any quantum corrections. The second term becomes

$$\ln\left(\int \mathcal{D}\eta e^{iS_0 + iS_i} \right) \longrightarrow \ln\left(\int \mathcal{D}\eta e^{ig^{-1}S_0 + ig^{-1}S_i} \right) = \ln\left(\int \mathcal{D}\eta\, e^{ig^{-1}S_0} \right) + \ln\left(\frac{\int \mathcal{D}\eta\, e^{ig^{-1}S_i}\, e^{ig^{-1}S_0}}{\int \mathcal{D}\eta\, e^{ig^{-1}S_0}} \right)$$

The first term is quadratic in η, and can therefore be evaluated as a generalized Gaussian integral, as described in section B.3,

$$\ln\left\{ \int \mathcal{D}\eta \exp\left(ig^{-1}\frac{1}{2} \int \mathrm{d}^4x \mathrm{d}^4y\, \eta(x)\eta(y) \frac{\delta^2 S[\varphi_0]}{\varphi(x)\varphi(y)} \right) \right\}$$

$$= \ln\left\{ \det\left(-g^{-1}\frac{\delta^2 S[\varphi_0]}{\delta\varphi(x)\,\delta\varphi(y)} \right)^{-1/2} \right\} = -\frac{1}{2} \text{Tr}\left\{ \ln\left(-\frac{\delta^2 S[\varphi_0]}{\delta\varphi(x)\,\delta\varphi(y)} \right) \right\} + \text{const.}$$

We then use the identity $\ln \det M = \operatorname{Tr} \ln M$. After we remove the constant term, this term proportional to g^0, i.e. it is made up of one-loop terms.

The last term can be evaluated by first expanding the exponential containing the S_i term, then using $\ln(1 + x) = \sum_n \frac{1}{n} x^n$. Notice that

$$\langle A \rangle = \frac{\int \mathcal{D}\varphi \, A \, e^{ig^{-1}S_0}}{\int \mathcal{D}\varphi \, e^{ig^{-1}S_0}} \tag{2.35}$$

with this, we can write

$$\ln \left[\frac{\int \mathcal{D}\eta \, e^{ig^{-1}S_i} e^{ig^{-1}S_0}}{\int \mathcal{D}\varphi \, e^{ig^{-1}S_0}} \right] = \ln \left(\sum_n \frac{1}{n!} \left\langle (ig^{-1}S_i)^n \right\rangle \right) \tag{2.36}$$

Expectation values of functionals of the field can be written as

$$\langle f[\varphi] \rangle = f \left[-i \frac{\delta}{\delta J} \right] Z[J] \Big|_{J=0}, \tag{2.37}$$

where the functional of a functional derivative is evaluated by its series expansion. From Wick's theorem, each term in this series is given by a product of second functional derivatives W_0, i.e. propagators. As argued earlier, the propagator of the interacting theory is proportional to g. We know that S_i is made up of terms that is third power or higher in the fields. Furthermore, as the expectation value of an *odd* number of fields are proportional to tadpole-diagrams, those can be ignored. This means that $\langle S_i^n \rangle \propto g^k$, where k is more or equal to $3n$, if n is even, $3n + 1$, if n is odd. This shows that $Z_i = \mathcal{O}\left(g^2\right)$, and thus that Z_1 contains *all* one loop contributions to $\ln Z$. The effective potential to 1 loop order is thus

$$\mathcal{V}_{\text{eff}}(\varphi_0) = \mathcal{V}(\varphi_0) - \frac{i}{VT} \frac{1}{2} \operatorname{Tr} \left\{ \ln \left(-\frac{\delta^2 S[\varphi_0]}{\delta \varphi(x) \, \delta \varphi(y)} \right) \right\}. \tag{2.38}$$

2.2   Symmetry and Goldstone's theorem

This section is based on [11, 3, 4, 13].

The symmetries of a theory are transformations of the physical state that leave the governing equations unchanged. A lot of physics is contained in the symmetries of a theory, such as the presence of conserved quantities and the systems low energy behavior. We distinguish between internal and external symmetries. An external symmetry is an active coordinate transformation, such as rotations or translations. They relate degrees of freedom at different space-time points, while internal symmetry transforms degrees of freedom at each space-time point independently of what happens at other points. A further distinction is between local and global symmetry transformations. Local transformations have one rule for how to transform degrees of freedom at each point, which is applied everywhere, while local transformations might themselves be functions of space-time.

In classical field theory, symmetries are encoded in how the Lagrangian changes due to a transformation of the fields. We will consider continuous transformations, which are can in general be written as

$$\varphi(x) \longrightarrow \varphi'(x) = f_t[\varphi](x), \quad t \in [0, 1]. \tag{2.39}$$

Here, $f_t[\varphi]$ is a functional in φ, and a smooth function of t, with the constraint that $f_0[\varphi] = \varphi$. This allows us to look at "infinitesimal" transformations,

$$\varphi'(x) = f_\epsilon[\varphi] \sim \varphi(x) + \epsilon \frac{\mathrm{d}f[\varphi]}{\mathrm{d}t} \Big|_{t=0}, \quad \epsilon \to 0. \tag{2.40}$$

We will consider internal, global transformations which act as linearly on φ. For N fields, φ_i, this can be written

$$\varphi'_i(x) = \varphi_i(x) + \epsilon \, i t_{ij} \varphi_j(x), \quad \epsilon \to 0. \tag{2.41}$$

t_{ij} is called the generator of the transformation. A symmetry of the system is then one in which the Lagrangian is unchanged by the transformation, or at most is different by a divergence-term. That is, a transformation $\varphi \rightarrow \varphi'$ is a symmetry if

$$\mathcal{L}[\varphi'] = \mathcal{L}[\varphi] + \partial_\mu K^\mu[\varphi], \tag{2.42}$$

where $K^\mu[\varphi]$ is a functional of φ.[2] This is a requirement for a symmetry in quantum field theory too. However, as physical quantities in quantum field theory are given not just by the action of a single state but the path integral, the integration measure $\mathcal{D}\varphi_i$ has to be invariant as well. If a classical symmetry fails due to the integration measure, it is called an anomaly.

To investigate the symmetry properties of a quantum theory, we explore what constraints a symmetry lies on the effective action. To that end, assume

$$\mathcal{D}\varphi'(x) = \mathcal{D}\varphi(x), \quad S[\varphi'] = S[\varphi]. \tag{2.43}$$

In the generating functional, such a transformation corresponds to a change of integration variable. Using the infinitesimal version of the transformation, we may write

$$Z[J] = \int \mathcal{D}\varphi \, \exp\left\{ iS[\varphi] + i\int \mathrm{d}^4 x J_i(x)\varphi_i(x) \right\} = \int \mathcal{D}\varphi' \, \exp\left\{ iS[\varphi'] + i\int \mathrm{d}^4 x J_i(x)\varphi'_i(x) \right\} \tag{2.44}$$

$$= Z[J] - \epsilon \int \mathrm{d}^4 x J_i(x) \int \mathcal{D}\varphi \, e^{iS[\varphi]}[t_{ij}\varphi_j(x)], \tag{2.45}$$

Using Eq. (2.18), we can write this as

$$\int \mathrm{d}^4 x \, \frac{\delta \Gamma[\varphi_J]}{\delta \varphi_i(x)} \, t_{ij} \, \langle \varphi_j(x) \rangle_J = 0. \tag{2.46}$$

This constraint will allow us to deduce the properties of a theory close to the ground state, only using information about the symmetries of the theory.

The arch typical example of an internal, global and continuous symmetry is the linear sigma model, which we will use as an example throughout this section. The linear sigma model is made up of N real scalar fields φ_i, which are governed by the Lagrangian

$$\mathcal{L}[\varphi] = \frac{1}{2}\partial_\mu \varphi_i(x)\partial^\mu \varphi_i(x) - \mathcal{V}(\varphi), \quad \mathcal{V}(\varphi) = -\frac{1}{2}\mu^2 \varphi_i(x)\varphi_i(x) + \frac{1}{4}\lambda[\varphi_i(x)\varphi_i(x)]^2. \tag{2.47}$$

This system is invariant under the rotation of the N fields into each other,

$$\varphi_i \longrightarrow \varphi'_i = O_{ij}\varphi_j, \quad O^{-1} = O^T. \tag{2.48}$$

The set of all such transformations form the Lie group $O(N)$. Lie groups will be discussed in the next section. For $N = 2$, $\mu^2, \lambda > 0$ we get the famous Mexican hat potential, as illustrated in Figure 2.1.

Lie groups

Lie groups are a natural structure to capture the symmetries of a theory. A Lie group is a smooth manifold, i.e. a space that is locally diffeomorphic to $\mathbb{R}^N$. This means that we can locally parametrize the space by N real numbers η_α, using smooth invertible functions. A Lie group is also equipped with group structure. A group is a set, G, together with a map

$$(\cdot, \cdot) : G \times G \longmapsto G, \tag{2.49}$$

$$(g_1, g_2) \longmapsto g_3, \tag{2.50}$$

called group multiplication. This map obeys the group axioms, which are the existence of an identity element $\mathbb{1}$, associativity and the existence of an inverse element g^{-1} for all $g \in G$. These can be written as

$$(g, \mathbb{1}) = g,$$
$$(g_1, (g_2, g_3)) = ((g_1, g_2), g_3),$$
$$(g, g^{-1}) = \mathbb{1}.$$

---

[2]Terms of the form $\partial_\mu K^\mu$ does not affect the physics, as variational principle $\delta S = 0$ do not vary the fields at infinity.
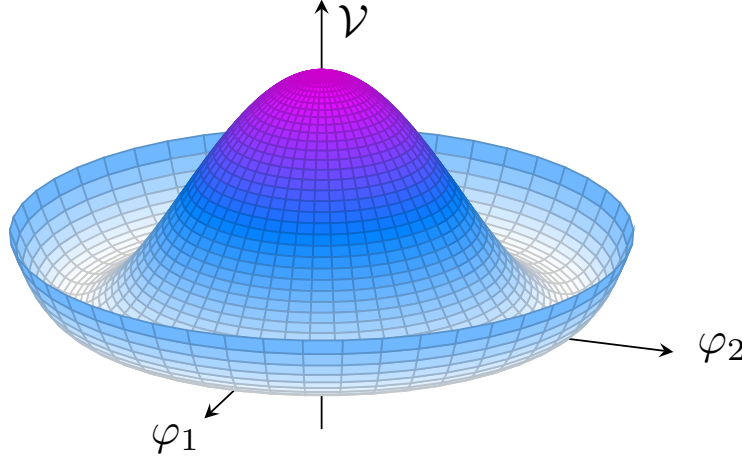
Figure 2.1: The Mexican hat potential is the classical potential $\mathcal{V}$ for the $N = 3$ linear sigma model.

In addition, we require that both the multiplication map and the inverse map, $g \mapsto g^{-1}$ are smooth. We describe the set of continuous symmetry transformations,

$$G = \left\{ g \mid g\varphi = \varphi', \, S[\varphi'] = S[\varphi], \mathcal{D}\varphi' = \mathcal{D}\varphi \right\}, \tag{2.51}$$

as a Lie group, where the group multiplication is composition, i.e. performing transformations in succession. This map is closed, as two symmetry transformations are another transformation. The identity map is a symmetry transformation, and composition is associative. This means that invertible symmetry transformations form a group.

We will focus on connected Lie groups, in which we all elements $g \in G$ is in the same connected piece as the identity map $\mathbb{1}\varphi = \varphi$. This means that for each $g \in G$, one can find a continuous path $\gamma(t)$ in the manifold, such that $\gamma(0) = \mathbb{1}$ and $\gamma(1) = g$. Given such a path, we can study transformations close to the identity. As the Lie group is a smooth manifold, we can write[3]

$$\gamma(\epsilon) = \mathbb{1} + i\epsilon V + \mathcal{O}\left(\epsilon^2\right) \tag{2.52}$$

V is a generator, and can be defined as

$$iV = \frac{\mathrm{d}\gamma}{\mathrm{d}t}\bigg|_{t=0}. \tag{2.53}$$

We can define a path γ by its path through parameter space, $\gamma(t) = g(\eta(t))$, which means that we can write the generator as

$$V = \frac{\mathrm{d}\gamma}{\mathrm{d}t}\bigg|_{t=0} = \frac{\mathrm{d}\eta_a}{\mathrm{d}t}\bigg|_{t=0} \frac{\partial g}{\partial \eta_a}\bigg|_{\eta=0} = \frac{\partial g}{\partial \eta_a}\bigg|_{\eta=0} T_\alpha, \quad T_\alpha = \frac{\partial g}{\partial \eta_\alpha}. \tag{2.54}$$

We see that the generators form a vector space, with the basis T_α, induced by the coordinates η_a. This vector space is called the tangent space of the identity element, $T_{\mathbb{1}}G$. Infinitesimal transformations can therefore be written as

$$\gamma(\epsilon) = \mathbb{1} + i\epsilon v_\alpha T_\alpha, \quad \epsilon \to 0. \tag{2.55}$$

The tangent space, together with the additional operation

$$[T_\alpha, T_\beta] = iC^\gamma_{\alpha\beta} T_\gamma, \tag{2.56}$$

called the Lie bracket, forms a Lie algebra denoted $\mathfrak{g}$. For matrix groups, which are what we deal with in this text, the Lie bracket is the commutator. $C^\gamma_{\alpha\beta}$ are called structure constants. They obey the Jacobi identity,

$$C^\gamma_{\alpha\beta} + C^\alpha_{\beta\gamma} + C^\beta_{\gamma\alpha} = 0, \tag{2.57}$$

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[3]The factor i is a physics convention, and differs from how mathematicians define generators of a lie group.

which mean that they are totally antisymmetric. A subset of the original Lie group, $H \subset G$, which is closed under the group action is called a subgroup. H then has its own Lie algebra $\mathfrak{h}$, with a set of $m = \dim H$ generators, x_i, which is a subset of the original generators T_α We denote the remaining set of generators t_a, such that x_i and t_a together span $\mathfrak{g}$.

Of special importance are one parameter subgroups. If a curve $\gamma(t)$ through G obey

$$\gamma(t)\gamma(s) = \gamma(t+s), \quad \gamma(0) = \mathbb{1}, \tag{2.58}$$

then all the points on this curve from a one parameter subgroup of G. This path is associated with a generator,

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t}\bigg|_{t=0} = i\eta_\alpha T_\alpha. \tag{2.59}$$

This allows us to define the exponential map,

$$\exp\{i\eta_\alpha T_\alpha\} := \gamma(1). \tag{2.60}$$

For connected and compact Lie groups, all elements of the Lie group $g \in G$ can be written as an exponential of elements in the corresponding Lie algebra $\eta_\alpha T_\alpha \in \mathfrak{g}$. For matrix groups, the exponential equals the familiar series expansion

$$\exp\{X\} = \sum_n \frac{1}{n!} X^n. \tag{2.61}$$

A subgroup $H \in G$ has its own Lie algebra $\mathfrak{h}$, with a set of $m = \dim H$ generators, x_i, which is a subset of the original generators T_α. We denote the remaining set of generators t_a, such that x_i and t_a together span $\mathfrak{g}$. The commutators of x_i must be closed, which means that we can write

$$[x_i, x_j] = iC_{ij}^k x_k, \tag{2.62}$$

$$[x_i, t_a] = iC_{ia}^b t_b, \tag{2.63}$$

$$[t_a, t_b] = iC_{ab}^k x_k + iC_{ab}^c t_c, \tag{2.64}$$

where ijk runs over the generators of $\mathfrak{h}$, and abc runs over the rest. The second formula can be derived using the Jacobi identity Eq. (2.57), which implies that $C_{ia}^k = -C_{ik}^a = 0$. This is called a Cartan decomposition.

Goldstone's theorem

The fact that a theory is left invariant under some symmetry transformation does not imply that the ground state is invariant under this transformation. The $N = 2$ linear sigma model illustrates this. If we assume the ground state φ_0 is translationally invariant, then it is given by minimizing the effective potential, of which the classical potential, $\mathcal{V}$, is the leading order approximation. This potential is illustrated in Figure 2.1. The ground state is therefore given by any of the values along the brim of the potential. If we, without loss of generality, choose $\varphi_0 = (0, v)$ as the ground state, then any symmetry transformation will change this state. We say that the symmetry has been *spontaneously broken*.

To explore this in a general context, assume a theory of φ_i real scalar fields are invariant under the actions of some Lie group, G. A symmetry $g \in G$ is broken if the vacuum expectation value of the original fields and the transformed fields differ. That is, if

$$\langle\varphi\rangle_0 \neq \langle\varphi'\rangle_0 = \langle g\varphi\rangle_0 \tag{2.65}$$

We can now exploit what we learned about Lie groups to write

$$\langle\varphi'\rangle_0 = \langle\varphi\rangle_0 + i\epsilon\eta_\alpha T_\alpha \langle\varphi\rangle_0. \tag{2.66}$$

Let t_a be the set of generators corresponding to broken symmetries, i.e.

$$t_\alpha \langle\varphi\rangle_0 \neq 0. \tag{2.67}$$

These are called the *broken generators*. The remaining set of generators x_i, which obey

$$x_i \langle \varphi \rangle_0 = 0, \tag{2.68}$$

are called unbroken, and generate a subgroup $H \subset G$ as the set of symmetry transformations of the vacuum is a group.

In Eq. (2.18) we found that the effective action obey

$$\int \mathrm{d}^4 x \frac{\delta \Gamma[\varphi_J]}{\delta \varphi_i} t_{ij} \langle \varphi_j \rangle_0 = 0. \tag{2.69}$$

We now differentiate this expression with respect to $\varphi_k(y)$ and evaluate it in the vacuum, which gives

$$\int \mathrm{d}^4 x \frac{\delta^2 \Gamma[\varphi_0]}{\delta \varphi_k(y) \, \delta \varphi_i(x)} t_{ij} \langle \varphi_j \rangle_0 = 0. \tag{2.70}$$

With the assumption that the ground state is constant, we get

$$\frac{\partial^2 \mathcal{V}_{\mathrm{eff}}}{\partial \varphi_k \, \partial \varphi_i} t_{ij} \langle \varphi_j \rangle_0 = 0. \tag{2.71}$$

This is trival for unbroken symmetries, as $t_{ij} \langle \varphi_j \rangle_0 = 0$ by definition. However, in the case of a broken symmetry, the second derivative of the effective potentail has an eigenvector $t_{ij}^\alpha \langle \varphi_j \rangle$ with a zero eigenvalue for each broken generator. In Eq. (2.23), we found that the second derivative of the effective action is the inverse propagator,

$$D_{ij}^{-1}(x,y) = -i \frac{\delta^2 \Gamma[\varphi_0]}{\delta \varphi_i(y) \, \delta \varphi_j(x)} = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} e^{-ip(x-y)} \tilde{D}_{ij}^{-1}(p). \tag{2.72}$$

Using this, we can write

$$\tilde{D}_{ij}^{-1}(p=0) \, t_{jk}^\alpha \langle \varphi_k \rangle_0 = 0. \tag{2.73}$$

Zeros of the inverse propagator corresponds to the physical mass of particles. In Lorentz-invariant systems, each zero-mass vector corresponds to a masses particles, called a Goldstone boson.[4] This means there are $n_G = \dim G - \dim H$ zero mass modes. In general, counting of massless modes is more complicated, and depends on the dispersion relation of the particles at low momenta. Systems with Goldstone bosons with quadratic dispersion relation, that is $E = |\vec{p}|^2$ when $\vec{p} \to 0$, often exhibit a lower number of massless modes. An example is ferromagnets, where the SU(2) rotational symmetry is broken down to U(1) when they align along one axis. This corresponds to two broken generators, yet the system only exhibits one massless mode [14].
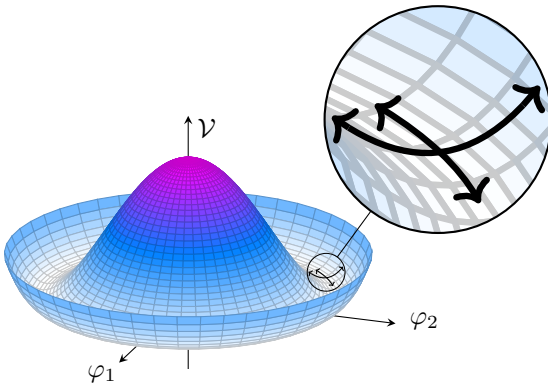
(a) Excitations along the brim does not cost any energy, as the potential is flat, unlike excitations in the radial direction.
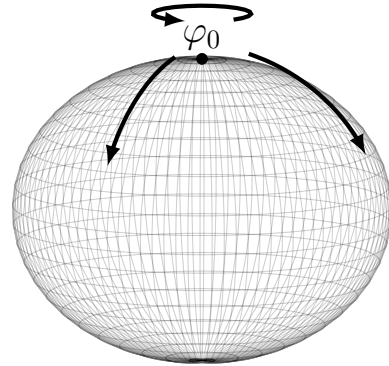
(b) Excitations for the $N = 3$ sigma model. Two of the symmetries are broken, while rotations around the groundstate leaves the system unchanged.

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[4]The particles are bosons due to the bosonic nature of the transformations, t. If the generators are Grassmann numbers, the resulting particle, called a goldstinos, are fermions.

The linear sigma model gives an intuition for the Goldstone mode. In the case of $N = 2$, the symmetry of the Lagrangian are rotations in the plane. As the ground state is a point along the "brim" of the hat, this rotational symmetry is broken. Any excitations in the rotational direction, however, does not cost any energy, which is indicative of a massless mode. This is illustrated in Figure 2.2a. In this example, the original symmetry group is one dimensional, so there are no unbroken symmetries. If we instead consider the $N = 3$ linear sigma model, which has a three-dimensional symmetry group, rotations of the sphere, we see that the ground state is left invariant under subgroup of the original symmetry transformations. The ground state manifold of this system, the set of all states that minimizes the effective potential, is then a sphere. When the system chooses one single ground state, this symmetry is broken, but only for two of the generators. The generator for rotations around the ground state leaves the that point unchanged, and is thus an unbroken symmetry. Any excitations in the direction of the broken symmetries does not cost energy, as it is in the ground state manifold. The unbroken symmetry, on the other hand, does not correspond to an excitation. This is illustrated in Figure 2.2b.

2.3 CCWZ construction

As Goldstone bosons are massless, they play a crucial role in the low energy dynamics. To best describe this limit, we seek a parametrization of the fields in which they are the degrees of freedom. This can be done using the CCWZ construction, named after Callan, Coleman, Wess and Zumino. As well as the original papers [15, 16], this section is based on [4, 17, 18] and[5]. (REFERER ORDENTLIG)

We saw that the Goldstone bosons corresponds to excitations within the vacuum manifold. The vacuum manifold corresponds to points in field space φ that can be reach from the vacuum φ_0 with a transformation $g \in G$. This means that we can write such excitations as

$$\varphi = \tilde{\Sigma}\varphi_0, \quad \tilde{\Sigma} = \tilde{\Sigma}(\eta) = \exp\{i\eta_\alpha T_\alpha\} \tag{2.74}$$

$\tilde{\Sigma}$ is thus a function from the parameter space, $\eta_\alpha \in \mathbb{R}^n$ to G,

$$\tilde{\Sigma} : \mathbb{R}^n \longmapsto G. \tag{2.75}$$

We then get space-time dependent field configurations by making the parameters dependent on spacetime. We will for now assume η_α is constant. This parametrization is highly redundant. There are $n = \dim G$ parameters η, but Goldstone's theorem says that there is one massless mode per broken generator. Two elements $\tilde{\Sigma}$ and $\tilde{\Sigma}'$, related by

$$\tilde{\Sigma}' = \tilde{\Sigma}e^{i\theta_a t_a} \tag{2.76}$$

results in the same φ. This is because $e^{i\theta_a t_a} = h \in H$, and $h\varphi_0 = \varphi_0$, by assumption. The set of all equivalent $\tilde{\Sigma}$'s is exactly the left coset, $gH = \{ gh \mid h \in H \}$. The set of cosets forms a new manifold, G/H, called the Goldstone manifold. This is a manifold of dimension $\dim(G/H) = \dim(G) - \dim(H)$, which is exactly the number of broken generators, and thus also the number of Goldstone modes. Membership of a certain coset from an equivalence relation, $g \sim g'$ if $g' = gh$. This means that the cosets gH form a partition of G, and that each element $g \in G$ belongs to one, and only one, coset. To remove the redundancy in the parametrization, we need to choose one representative element from each coset.

By the inverse function theorem, any mapping between manifolds $f : \mathcal{M} \mapsto \mathcal{N}$ that has a non-degenerate differential, that is an invertible Jacobian, at a point $p \in \mathcal{M}$, is invertible in a neighborhood of p. The map

$$\tilde{\Sigma}(\xi, \theta) = \exp\{i\xi_i x_i\} \exp\{i\theta_a t_a\} \tag{2.77}$$

is invertible at $p = (\xi_i = 0, \theta_a = 0)$, which is mapped to the identity, as the Jacobian is the identity matrix. This means that, in a neighborhood $U \subset G$ of the identity, each element g has a unique representation $g = \Sigma$ [13]. Furthermore, two elements $\tilde{\Sigma}'$ and $\tilde{\Sigma}$ related by $\tilde{\Sigma}' = \tilde{\Sigma}h$, $h \in H$ have the same ξ-arguments. We see that ξ_i parametrize G/H, in the neighborhood of the identity. We therefore demand that $\tilde{\Sigma}$ always appear in the standard form

$$\Sigma(\xi) = \tilde{\Sigma}(\xi, 0) = \exp\{i\xi_i x_i\}. \tag{2.78}$$

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[5]http://scipp.ucsc.edu/~haber/archives/physics251_17/PHYS251_Presentation_L_Morrison

The field $\varphi(x)$ can therefore be written as

$$\varphi(x) = \Sigma(x)\varphi_0 = \exp\{i\xi_i(x)x_i\}\varphi_0, \tag{2.79}$$

and $\xi_i(x)$ can be associated with the Goldstone bosons.

In the linear sigma model, the original $O(N)$ symmetry is broken down to $O(N-1)$, which transforms the remaining $N-1$ fields with vanishing expectation value into each other. However, $O(N)$ consists of two disconnected subsets, those matrices with determinant 1 and those with determinant -1. There is no continuous path that takes an element of $O(N)$ with determinant of -1 to an element with determinant 1.[6] The set of symmetries that are connected to the identity is

$$G = SO(N) = \{ M \in O(N) \mid \det M = 1 \}. \tag{2.80}$$

If we choose $\varphi_0 = (0, 0, ..., v)$, then it is apparent that the ground state is invariant under the rotations of the $N-1$ first fields, so the unbroken symmetry is $H = SO(N-1)$. The Goldstone manifold is $G/H = SO(N)/SO(N-1)$.

Consider the case of $N = 3$, which is illustrated in Figure 2.2b. G is the rotations of the sphere, while H is rotations around φ_0, $SO(2)$. The Goldstone manifold consists of the rotations of φ_0 to other points of the sphere, i.e. $G/H = SO(3)/S(2) = S^2$, the 2-sphere. This is not a Lie group, as translating φ in a closed path around the sphere may result in a rotation around the z-axis.

To check that ξ_i in fact are the Goldstone modes, we study the way they appear in the Lagrangian. As they are massless, no mass term of the form $M_{ij}\xi_i\xi_j$ should appear in the Lagrangian. The original Lagrangian $\mathcal{L}[\varphi]$ was invariant under global transformations $\varphi(x) \mapsto g\varphi(x)$. However, any terms that only depend on $\varphi(x)$, and not its derivatives, will also be invariant under a *local* transformation, $\varphi(x) \mapsto g(x)\varphi(x)$. Our parametrization of the fields, $\varphi(x) = \Sigma(x)\varphi_0$ is exactly such a transformation, which means that any such terms are independent of the Goldstone bosons. We can therefore write

$$\mathcal{L}[\varphi] = \mathcal{L}_{\text{kin}}[\varphi] + V(\varphi_0), \tag{2.81}$$

where all terms in $\mathcal{L}_{\text{kin}}$ are proportional to at least one derivative term, $\partial_\mu\varphi(x)$. Inserting the parametrization into this derivative term, we get

$$\partial_\mu\varphi(x) = \partial_\mu[\Sigma(x)\varphi_0] = \Sigma(x)[\Sigma(x)^{-1}\partial_\mu\Sigma(x)]\varphi_0. \tag{2.82}$$

The Lagrangian will therefore depend on ξ_i via terms of the term $\Sigma(x)^{-1}\partial_\mu\Sigma(x)$, which is called the Mauer-Cartan form. This is a $\mathfrak{g}$-valued function, which means that it can be written on the for

$$i\Sigma(x)^{-1}\partial_\mu\Sigma(x) = d_\mu(x) + e_\mu(x),$$
$$d_\mu = ix_i d_{ij}(\xi)\partial_\mu\xi_j,$$
$$e_\mu = it_a e_{ai}(\xi)\partial_\mu\xi_i.$$

where d_{ij} and d_{ai} are as-of-yet unknown real valued functions of ξ [4, 19].

Transformation properties of Goldstone bosons

We can deduce how the Goldstone bosons transforms under G from how φ transforms. In general,

$$\varphi' = g\varphi = (g\Sigma(\xi))\varphi_0 = \Sigma(\xi')\varphi_0 \quad g \in G. \tag{2.83}$$

While $\Sigma(\xi')$ has the standard form by assumption,

$$\Sigma(\xi') = \exp\{i\xi_i'x_i\}, \tag{2.84}$$

$g\Sigma(\xi)$ does not, in general.

---

[6]A simple proof of this is the fact that the determinant is a continuous function, while any path $\det M(t)$ such that $\det M(1) = -1$, $\det m(0) = 1$ must make a discontinuous jump.
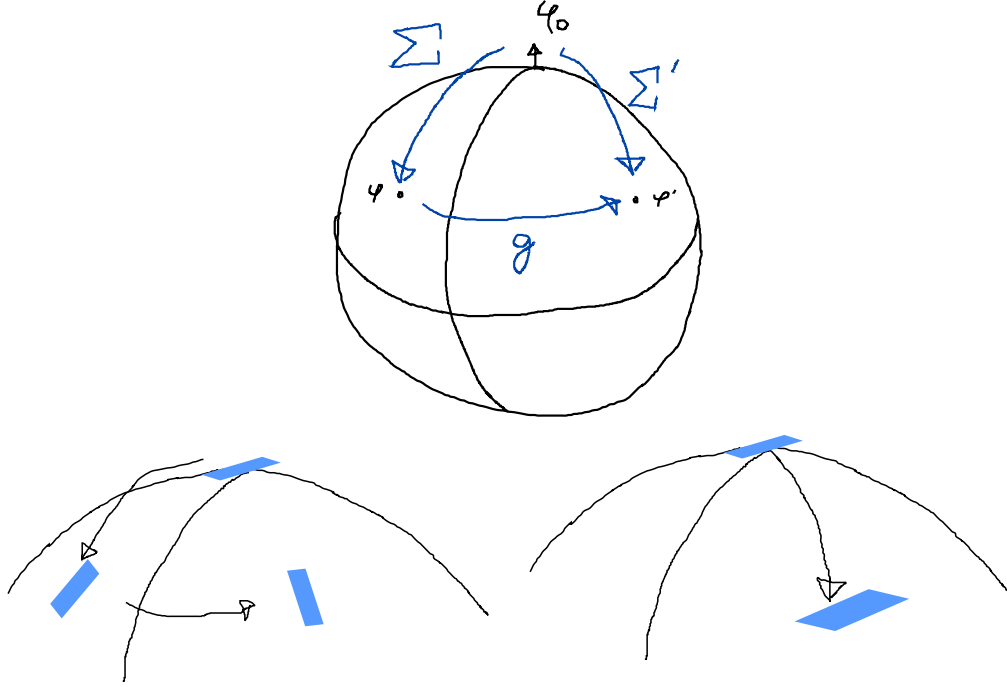
Figure 2.3: (KLADD) The top figure illustrates the transformation of φ_0 to φ and then φ, and the alternative transformation $\varphi_0 \to \varphi'$. The bottom figure illustrates how this can rotate a neighborhood of φ_0 differently.

Figure 2.3 illustrates this in the case of $G = SO(3)$. $\Sigma(\xi)$ transforms φ_0 to φ, then g transforms φ to $\varphi' = \Sigma(\xi')\varphi_0$. Assuming φ and φ' are close enough to φ_0, we can write $\Sigma(\xi)$ and $\Sigma(\xi')$ on the standard form. However, if we follow a small neighborhood around φ_0 as it is acted on by $\Sigma(\xi)$, then g, it will be rotated by the time it arrives at φ', when compared to the same neighborhood if it was acted on by $\Sigma(\xi')$.

$g\Sigma(\xi)$ and $\Sigma(\xi')$ are in the same coset, as they by assumption corresponds to the same physical state. This means that we can write $g\Sigma(\xi) = \Sigma(\xi')h(g,\xi)$, where $h(g,\xi) \in H$. The transformation rule of ξ under G is therefore implicitly defined by

$$\Sigma(\xi') = g\Sigma(\xi)[h(g,\xi)]^{-1}. \tag{2.85}$$

This is in general not a linear representation, which is why this construction also is called a *non-linear realization*. Using the transformation rule, we can obtain the transformation rule of the Maurer-Cartan form. We use the shorthand $\Sigma = \Sigma(\xi)$, $\Sigma' = \Sigma(\xi')$, and $h = h(g,\xi)$. This gives

$$\begin{aligned}
\Sigma^{-1}\partial_\mu\Sigma &\to \Sigma'^{-1}\partial_\mu\Sigma' \\
&= (g\Sigma h^{-1})^{-1}\partial_\mu(g\Sigma h^{-1}) \\
&= (h\Sigma^{-1}g^{-1})g[(\partial_\mu\Sigma)h^{-1} + \Sigma\partial_\mu h^{-1}] \\
&= h\Sigma^{-1}(\partial_\mu\Sigma)h^{-1} + h\partial_\mu h^{-1} \\
&= h(\Sigma^{-1}\partial_\mu\Sigma + \partial_\mu)h^{-1}.
\end{aligned}$$

In terms of d_μ and e_μ,

$$d_\mu \to hd_\mu h^{-1} \tag{2.86}$$

$$e_\mu \to h(e_\mu + i\partial_\mu)h^{-1}. \tag{2.87}$$

These are our building blocks for constructing a general, G-invariant effective Lagrangian. The trace of a product of d_μ's are invariant under G,

$$\text{Tr}\{d_\mu d_\nu \ldots d_\rho\} \to \text{Tr}\{hd_\mu h^{-1}hd_\nu h^{-1}h \ldots d_\rho h^{-1}h\} = \text{Tr}\{d_\mu d_\nu \ldots d_\rho\}, \tag{2.88}$$

where we have used the cyclic property of trace. However, the terms must also obey the other symmetries of the Lagrangian, such as C or P-parity and Lorentz invariance. The last criterion excludes any terms with free spacetime indices. The lowest order terms are therefore (REFFERER TIL POWERCOUNTING)

$$\mathrm{Tr}\{d_\mu\}\,\mathrm{Tr}\{d^\mu\}, \quad \mathrm{Tr}\{d_\mu d^\mu\}. \tag{2.89}$$

We see that e_μ transforms like a gauge field, with the gauge group H. If we include massive degrees of freedom, and not only the boson modes, e_μ is used to create a covariant derivative of the massive modes. We are only interested in the Goldstone, and will therefore be satisfied with d_μ. With these tools, we can create an effective theory of quantum chromodynamics at low energies.

Chapter 3

The effective theory of pions

3.1 QCD

(Inkluder alle kilder) This section is based on [11, 1]

Quantum chromodynamics, or QCD, is the theory of how quarks interact via the strong force. There are six flavors of quarks, labeled by the index f, called up, down, charm, strange, top and bottom. The up, charm and top quark, collectively called up-type quarks, have electrical charge $+\frac{2}{3}$, while the down-type quarks—down, strange and bottom—have an electrical charge of $-\frac{1}{3}$. Quarks have one additional degree of freedom called color and indexed by c, which takes on the value of red, green and blue for quarks, and anti-red, -green and -blue for anti-quarks. The analogy to colors is meant to capture the empirical fact that quarks are only ever observed in *colorless* configurations, that is either an equal amount of each color and its anti-color, or equal amounts of red, green and blue. Quarks are spin-$\frac{1}{2}$ particles, so each quark q_{fc} is a spinor. Free quarks obey the Dirac-equation,

$$\mathcal{L}[q, \bar{q}] = \sum_{f=1}^{N_f} \sum_{c=1}^{N_c} \bar{q}_{fc}(i\gamma^\mu \partial_\mu - m_f)q_{fc} = \bar{q}(i\slashed{\partial} - m)q, \tag{3.1}$$

where N_f is the number of flavors, and N_c the number of colors. In the last equality, we have hidden the indices to reduce clutter. The gamma matrices, γ^μ are described in section B.1, and we have employed the Feynman slash notation $\gamma^\mu A_\mu = \slashed{A}$. Furthermore, $\bar{q} = q^\dagger \gamma^0$. This Lagrangian is invariant under rations of the quarks in the color indices, i.e. the transformation

$$q_c \to q'_c = U_{cc'}q_{c'}, \quad \bar{q}'_c = \bar{U}^\dagger_{cc'}q_c \tag{3.2}$$

where $U_{cc'}$ is an $N_c \times N_c$ unitary matrix. The set of all $N_c \times N_c$ unitary matrices form the Lie group $U(N_c) = U(1) \times SU(N_c)_c$. $SU(N_c)_c$ is the group of all complex $N_c \times N_c$ matrices with determinant 1, and is the gauge group of the strong force. (HVORFOR IKKE U(N)?) A theory with a $SU(N)$ gauge group is called a Yang-Mills theory.

Yang-Mills theory

Given an element $U \in SU(N_c)$, we can write

$$U = \exp\{i\chi_\alpha \lambda_\alpha\}, \quad \chi_\alpha \lambda_\alpha \in \mathfrak{su}(N_c), \tag{3.3}$$

where $\mathfrak{su}(N_c)$ is the Lie algebra of $SU(N_c)$. We derive the Yang-Mills Lagrangian by including all normalizable and Gauge-invariant terms. A term is gauge invariant if it remains unchanged after a *local* $SU(N_c)$ transformation, i.e.

$$q \to \exp\{i\chi_\alpha(x)\lambda_\alpha\}q, \quad \bar{q} \to \exp\{-i\chi_\alpha(x)\lambda_\alpha\}q'. \tag{3.4}$$

The mass term $m_f \bar{q}_f q_f$ is gauge invariant, while the kinetic derivative term needs to be modified. For this term to be gauge-invariant, we have to introduce a covariant derivative, which transforms as

$$D_\mu q \to (D_\mu q)' = U(D_\mu q) = D'_\mu(Uq). \tag{3.5}$$

If we posit $D_\mu = \mathbb{1}\partial_\mu - igA_\mu^\alpha \lambda_\alpha$ where A_μ^α is a set of vector fields, and g a constant, then

$$(\mathbb{1}\partial_\mu - igA'_\mu)U = U(\mathbb{1}\partial_\mu - igA_\mu^\alpha \lambda_\alpha) \tag{3.6}$$

where $A_\mu = A_\mu^\alpha \lambda_\alpha$. This means that if we demand that A_μ transforms as

$$A_\mu \to U\left(A_\mu + \frac{i}{g}\partial_\mu\right)U^\dagger, \tag{3.7}$$

The second derivative operator,

$$D_\mu D_\nu = \partial_\mu \partial_\nu - ig(\partial_\mu A_\nu + A_\mu \partial_\nu + A_\nu \partial_\mu) - g^2 A_\mu A_\nu, \tag{3.8}$$

transforms in the same way as the first derivative. We see that the "operator-part" of this derivative is symmetric in the space-time indices, which means that the commutator will just a tensor, and not an operator. We define

$$F_{\mu\nu} := \frac{i}{g}[D_\mu, D_\nu] = (\partial_\mu A_\nu - \partial_\mu A_\nu) - ig[A_\mu, A_\nu] = (\partial_\mu A_\nu^\alpha - \partial_\nu A_\mu^\alpha + gC_{\beta\gamma}^\alpha A_\mu^\beta A_\nu^\gamma)\lambda_\alpha. \tag{3.9}$$

This transforms as

$$F_{\mu\nu} \to UF_{\mu\nu}U^\dagger. \tag{3.10}$$

We now need to include terms governing the gauge field A_μ in the Lagrangian. The tensor $F_{\mu\nu}$ allows construct all gauge invariant terms to dimension 4, which are

$$F_{\mu\nu}^a F_a^{\mu\nu}, \quad \varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}^a F_{\rho\sigma}^a. \tag{3.11}$$

Here, ε is the Levi-Civita symbol. This allows us to write down the most general gauge-invariant and renormalizable Lagrangian for a $SU(N_c)$ gauge theory, the Yang-Mills Lagrangian

$$\mathcal{L} = i\bar{q}(\not{D} - m)q + \frac{1}{4}F_{\mu\nu}^a F_a^{\mu\nu} + \theta\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}^a F_{\rho\sigma}^a, \tag{3.12}$$

where θ is a coupling constant. In the case of QCD, $N_c = 3$, and the generators λ_α, $\alpha \in \{1,...8\}$ are the 3×3 Gell-Mann matrices, and $\theta = 0$, or at least close enough that it does not matter.

Chiral symmetry

In addition to the color and flavor indices s and f, the quarks also have spinor indices, i, which are what the γ-matrices act on. We can define the projection operators,

$$P_\pm = \frac{1}{2}(\mathbb{1} \pm \gamma^5), \tag{3.13}$$

which obey $P_\pm^2 = P_\pm$, $P_+P_- = P_-P_+ = 0$ and $P_+ + P_- = 1$, as good projection operators should. Furthermore, $P_\pm^\dagger = P_\pm$. These project spinors down to their chiral components, called left- and right-handed spinors,

$$P_+q = q_R, \quad P_-q = q_L. \tag{3.14}$$

From section B.1, we have

$$\left\{\gamma^\mu, \gamma^5\right\} = 0, \tag{3.15}$$

which means that

$$\bar{q}P_\pm = (P_\mp q)^\dagger \gamma^0. \tag{3.16}$$

Using the chiral projection operators, we can write the QCD Lagrangian as

$$\bar{q}(i\slashed{D} - m)q = \bar{q}(P_+ + P_-)(P_+ + P_-)(i\slashed{D} - m)q = (qP_-)\gamma^0 P_+(i\slashed{D} - m)q + (qP_+)\gamma^0 P_-(i\slashed{D} - m)q$$
$$= \bar{q}_L(i\slashed{D})q_L + \bar{q}_R(i\slashed{D})q_R - \bar{q}_L m q_R - \bar{q}_R m q_L.$$

The mass-term mixes the two chiral components. In the case that $m_f = 0$, called the *chiral limit*, the QCD Lagrangian is invariant under the transformations

$$q_R \to e^{i\theta_R} U_R q_R, \quad q_L \to e^{i\theta_L} U_L q_L, \tag{3.17}$$

where U_L and U_R are complex $N_f \times N_f$ matrices with determinant 1 that act on the flavor indices, and θ_L and θ_R real numbers. The total set of such transformations form the Lie group $U(N_f)_L \times U(N_f)_R = U(1)_L \times U(1)_R \times SU(N_f)_L \times SU(N_f)_R$. We can write

$$U_R = P_- + \exp\{i\eta_\alpha^R T_\alpha^R\}P_+, \quad U_L = P_+ + \exp\{i\eta_\alpha^L T_\alpha^L\}P_-, \tag{3.18}$$

where T_α^L and T_α^R are the generator for $SU(N_f)_L$ and $SU(N_f)_R$. The conserved currents corresponding to these symmetries are

$$L_\alpha^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu q)}(-iT_\alpha^L q_L) = \bar{q}\gamma^\mu T_\alpha^L q_L = \bar{q}_L \gamma^\mu T_\alpha^L q_L, \tag{3.19}$$

$$R_\alpha^\mu = \bar{q}_R \gamma^\mu T_\alpha^R q_R, \tag{3.20}$$

$$J_R^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu q)}(-iq_L) = \bar{q}_R \gamma^\mu q_R \tag{3.21}$$

$$J_L^\mu = \bar{q}_L \gamma^\mu q_L \tag{3.22}$$

We have defined the currents with a positive sign to conform to common convention. We expect when we introduce a non-zero mass, the symmetry of the Lagrangian is explicitly broken. However, we see that when right and left-handed quarks are transformed in the same way, i.e. $\eta_R = \eta_L$, then the Lagrangian remain invariant. The diagonal subgroup of $SU(N_f)_L \times SU(N_f)_R$, $SU(N_f)_V$, remains unbroken. Elements of this group have the form

$$U_V = \exp\{i\eta_\alpha^V T_\alpha^V \mathbb{1}\}. \tag{3.23}$$

As one can check easily, this is a normal subgroup of $SU(N_f)_L \times SU(N_f)_R$, and the quotient $SU(N_f)_L \times SU(N_f)_R/SU(N_f)_V = SU(N_f)_A$ therefore forms a Lie group, with elements fo the form

$$U_A = \exp\{i\eta_\alpha^A T_\alpha^A \gamma^5\}, \tag{3.24}$$

Similarly, the diagonal subgroup of $U(1)_L \times U(1)_R$ have a unbroken, normal subgroup $U(1)_V$, with elements of the form $e^{i\theta_A \mathbb{1}}$, and a quotient group, $U(1)_A$, with elements of the form $e^{i\theta_V \gamma^5}$. In the chiral limit, these groups are all symmetries of the Lagrangian, and the corresponding conserved currents are

$$V_\alpha^\mu = R_\alpha^\mu + L_\alpha^\mu = \bar{q}\gamma^\mu T_\alpha^V q, \tag{3.25}$$

$$A_\alpha^\mu = R_\alpha^\mu - L_\alpha^\mu = \bar{q}\gamma^\mu \gamma^5 T_\alpha^A q, \tag{3.26}$$

$$J_V^\mu = J_R^\mu + L^\mu = \bar{q}\gamma^\mu q, \tag{3.27}$$

$$J_A^\mu = J_L^\mu - L^\mu = \bar{q}\gamma^\mu \gamma^5 q. \tag{3.28}$$

The currents corresponding to $U(1)_V$ and $U(1)_A$ are called the vector and axial currents, as they transform as vectors and axial-vectors under space-time transformations. The conserved charge corresponding to the vector current,

$$Q = \int \mathrm{d}^3 x\, \bar{q}\gamma^0 q, \tag{3.29}$$

is 3 times the *baryon number*. (HVORFOR?) Up until now, we have only made classical considerations. In the full quantum theory, the baryon number remains conserved. The axial current J_A^μ however is the subject to an anomaly. As the integration measure $\mathcal{D}\bar{q}\mathcal{D}q$ is not invariant under transformations in $U(1)_A$, as we saw earlier is required, the quantum correction to $\partial_\mu J_A^\mu$ are non-zero. The remaining symmetries, $G = SU(N_f)_L \times SU(N_f)_R = SU(N_f)_V \times SU(N_f)_A$ and how they are broken is the foundation for chiral perturbation theory, which we will consider int the next section.

3.2   Chiral perturbation theory

In this paper, we will consider the interaction of two quarks, the up and down quarks u and d, which means that $N_f = 2$. In this case, the approximate symmetry of the Lagrangian is $G = SU(2)_L \times SU(2)_R$, with the generators $T_\alpha = \tau_\alpha/2$, where τ_α are the Dirac matrices, as described in section B.1. The quarks are not massless, but have a non-zero mass matrix

$$m = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} = \frac{1}{2}(m_u + m_d)\mathbb{1} + \frac{1}{2}(m_u - m_d)\tau_3, \tag{3.30}$$

where $m_u \approx 2.16 \, \text{MeV}$, and $m_d \approx 4.67 \, \text{MeV}$ [20]. This means that the G symmetry is *explicitly* broken. In what is called the isospin limit, $m_u = m_d$, the subgroup $SU(2)_V$ remains intact. The difference between these masses is small, which is why isospin is a good quantum number. However, even though the underlying symmetry is only approximate, we can still apply the formalism from Goldstone's theorem and the CCWZ construction by including a small mass term for the Goldstone bosons, which in this case are called *Pseudo Goldstone bosons.*

The approximate $G = SU(2)_V \times SU(2)_A$ symmetry of the two-flavor QCD-Lagrangian is spontaneously broken by the ground state. As quarks q are spinors, a non-zero expectation value of the quark field would break Lorentz-invariance. Instead, the spontaneous symmetry breaking is characterized by the *scalar quark condensate*, $\langle \bar{q}q \rangle$. The scalar quantity $\bar{q}q$ is invariant under isospin transformations $H = SU(2)_V$, but not under $SU(2)_A$. The Goldstone manifold is therefore $G/H = SU(2)_L \times \mathrm{SU}(2)_R/SU(2)_V = SU(2)_A$. To model the low energy dynamics of QCD, we start with the massless QCD Lagrangian,

$$\mathcal{L}^0_{\text{QCD}}[q, \bar{q}, A_\mu] = i\bar{q}\slashed{D}q - \frac{1}{4}G^\alpha_{\mu\nu}G^{\mu\nu}_\alpha$$

Following [21, 22], we couple quarks to external currents. These can be used to model external fields, or to capture the symmetry breaking mass term. As found in section 3.1, the conserved currents are

$$V^\mu_a = \frac{1}{2}\bar{q}\gamma^\mu \tau_a q, \quad A^\mu_a = \frac{1}{2}\bar{q}\gamma^\mu\gamma^5 \tau_a q, \quad J^\mu = \bar{q}\gamma^\mu q. \tag{3.31}$$

In addition, external currents can couple to the scalar and pseudo-scalar quark bilinears

$$\bar{q}q, \quad \bar{q}\gamma^5 q, \quad \bar{q}\tau_a q, \quad \bar{q}\gamma^5\tau_a q. \tag{3.32}$$

The external Lagrangian is thus

$$\mathcal{L}_{\text{ext}} = -(\bar{q}q)\,s_0 + (i\bar{q}\gamma^5 q)\,p_0 - (\bar{q}\tau_a q)\,s_a + (i\bar{q}\gamma^5\tau_a q)\,p_a + \frac{1}{3}J_\mu v^\mu_{(s)} + V^a_\mu v^\mu_a + A^a_\mu a^\mu_a \tag{3.33}$$

$$= -\bar{q}\left(s + i\gamma^5 p\right)q + \bar{q}\left(\frac{1}{3}v^\mu_{(s)} + v^\mu + a^\mu\gamma^5\right)\gamma_\mu q. \tag{3.34}$$

Here, $v, v_{(s)}, a, s$ and p are the external currents, where we denote

$$s = s_0\mathbb{1} + s_a\tau_a, \quad p = p_0\mathbb{1} + p_a\tau_a, \quad v^\mu = \frac{1}{2}v^\mu_a\tau_a, \quad a^\mu = \frac{1}{2}a^\mu_a\tau_a. \tag{3.35}$$

Setting $v = v_{(s)} = a = s = p = 0$ recover the chiral limit. To include the effect of the quark masses, we set $s = m$. We denote all the external currents as $j = (v, v_{(s)}, a, s, p)$. The generating functional is then

$$Z[j] = \int \mathcal{D}\bar{q}\mathcal{D}q\mathcal{D}A_\mu \, \exp\left\{i\int \mathrm{d}^4x \left(\mathcal{L}^0_{\text{QCD}}[q, \bar{q}, A_\mu] + \mathcal{L}_{\text{ext}}[q, \bar{q}, j]\right)\right\} \tag{3.36}$$

As with the conserved currents, we define

$$v_\mu = \frac{1}{2}(r_\mu + l_\mu), \quad a_\mu = \frac{1}{2}(r_\mu - l_\mu). \tag{3.37}$$

With this, as well as

$$\bar{q}(s - i\gamma^5 p)q = \bar{q}_R(s - ip)q_L + \bar{q}_L(s + ip)q_R, \tag{3.38}$$

we can write the external Lagrangian as

$$\mathcal{L}_{\text{ext}} = -\bar{q}_R(s + ip)q_L - \bar{q}_L(s - ip)q_R + \frac{1}{3}(J_R^\mu + J_V^\mu)(v_{(s)})_\mu + R_\mu^a r_a^\mu + L_\mu^a l_a^\mu \tag{3.39}$$

We will now use the CCWZ construction and effective field theory to construct the effective Lagrangian, which obeys

$$Z[j] = \int \mathcal{D}\pi \, \exp\left\{ i \int \mathrm{d}^4x \, \mathcal{L}_{\text{eff}}[\pi] \right\}, \tag{3.40}$$

where π are the Goldstone bosons.

Weinberg's power counting scheme

(TODO: FIX;)

In last section, we found a parametrization that guarantees for a simple way the fields must appear in the Lagrangian. Thus, any combination of these building blocks $\mathcal{O}_i(\xi)$ that must appear in the Lagrangian. The effective Lagrangian is therefore

$$\mathcal{L}[\xi] = \sum_i c_i \mathcal{O}_i(\xi), \tag{3.41}$$

where c_i are free parameters.

This leaves a Lagrange density with infinitely many terms, and infinitely many free parameters. To be able to use this theory for anything one must have a method for ordering the terms in order of importance. As described in [22], by rescaling the external momenta $p_\mu \to t p_\mu$ and quark masses $m_i \to t^2 m_i$, each term in the Lagrangian obtains a factor t^D. The Lagrangian is then expanded as $\mathcal{L} = \sum_D \mathcal{L}_D$, where $\mathcal{L}_D$ contains all terms with a factor t^D.

Non-linear realization

The Goldstone manifold is $G/H = SU(2)_A$, which means that we parametrize the Goldstone modes as

$$\Sigma(x) = \exp\left\{ i \frac{\pi_a(x)\tau_a}{f} \right\}. \tag{3.42}$$

Here, f is the bare pion decay constant.

We start by fining a representative element of the Goldstone manifold $G/H = SU(2)_A$. Let $g \in G$. We write $g = (L, R)$, where $R \in SU(2)_R$, $L \in SU(2)_L$. Elements $h \in H$ are then of the form $h = (V, V)$. A general element g can be written as

$$g = (L, R) = (1, RL^\dagger)(L, L). \tag{3.43}$$

Since $(L, L) \in H$, this means that we can write the coset gH as $(1, RL^\dagger)H$, which gives a way to choose a representative element for each coset. We identify

$$\Sigma = RL^\dagger. \tag{3.44}$$

This is our standard form, and therefore makes it possible to obtain the transformation properties, which is given by the function $h(g, \xi)$. We have

$$\tilde{g}(1, \Sigma) = (\tilde{L}, \tilde{R})(1, RL^\dagger) = (1, \tilde{R}(RL^\dagger)\tilde{L}^\dagger)(\tilde{L}, \tilde{L}) = (1, \tilde{R}\Sigma\tilde{L})h. \tag{3.45}$$

This gives the transformation rule

$$\Sigma \to \Sigma' = R\Sigma L^\dagger. \tag{3.46}$$

Under transformations by $h = (V, V^\dagger) \in H$, we have

$$\Sigma \to \Sigma' = V\Sigma V^\dagger. \tag{3.47}$$

As $\partial_\mu(1, \Sigma) = (0, \partial_\mu \Sigma)$, the constituents of the Mauer-Cartan form are

$$d_\mu = i\Sigma(x)^\dagger \partial_\mu \Sigma(x), \quad e_\mu = 0. \tag{3.48}$$

Using $\partial_\mu[\Sigma(x)^\dagger \Sigma(x)] = 0$, we can write

$$d_\mu d^\mu = -\Sigma(x)^\dagger[\partial_\mu \Sigma(x)]\Sigma(x)^\dagger[\partial^\mu \Sigma(x)] = \Sigma(x)^\dagger[\partial_\mu \Sigma(x)][\partial^\mu \Sigma(x)^\dagger]\Sigma(x). \tag{3.49}$$

Inserting this into (REF. TIL LO TERM I CCWZ), we get

$$\mathrm{Tr}\{d_\mu d^\mu\} = \mathrm{Tr}\{\partial_\mu \Sigma(\partial^\mu \Sigma)\}, \tag{3.50}$$

where we have used the cyclic property of the trace.

External currents and explicit symmetry breaking

Using d_μ we are able to construct any terms in the effective Lagrangian corresponding to $j = 0$. To construct the effective Lagrangian when $j \neq 0$, which is needed to capture the effects of nonzero quark masses and external currents, we treat $SU(2)_L \times SU(2)_R \times U(1)$ as a gauge group, and the source currents as the corresponding gauge fields. The effective Lagrangian is then constructed as the most general Lagrangian that is invariant under *local* $SU(2)_L \times SU(2)_R \times U(1)$ transformations. The Ward-identities corresponding to the local symmetries of the Lagrangian are equivalent with the statement that $Z[j]$ is invariant under a gauge transformation of the external fields, in the absence of anomalies [23].

Following [22], we write the gauge transformation as

$$q_L \to e^{i\theta(x)/3} U_L(x) q_L, \quad q_R \to e^{i\theta(x)/3} U_R(x) q_R. \tag{3.51}$$

First, we consider the $U(1)_V$ transformation. The massless QCD Lagrangian then transforms as

$$\mathcal{L}_{\mathrm{QCD}}^0 = i\bar{q}\slashed{D}q \to i\bar{q}e^{-i\theta(x)/3}\slashed{D}e^{i\theta/3}q = i\bar{q}\slashed{D}q - \frac{1}{3}\bar{q}\gamma^\mu q \partial_\mu \theta(x). \tag{3.52}$$

This gives us the transformation rule

$$v_{(s)}^\mu \to v_{(s)}^\mu - \partial^\mu \theta(x). \tag{3.53}$$

Then, applying the $SU(2)_R$ transformation, we get

$$\bar{q}_R \slashed{D} q_R + (i\bar{q}_R \gamma^\mu \tau_a q_R)\, r_\mu^a \to i\bar{q}_R \slashed{D} q_R + (i\bar{q}_R \gamma^\mu q_R)\, U_R^\dagger \partial_\mu U_R + [\bar{q}_R \gamma^\mu (U_R^\dagger \tau_a U_r) q_R]\, r_\mu^a \tag{3.54}$$

This, and the similar expression for U_L, gives the gauge transformation rules

$$r_\mu^a \tau_a \to U_R^\dagger(r_\mu^a \tau_a + i\mathbb{1}\partial_\mu)U_R, \tag{3.55}$$

$$l_\mu^a \tau_a \to U_L^\dagger(l_\mu^a \tau_a + i\mathbb{1}\partial_\mu)U_L, \tag{3.56}$$

$$s + ip \to U_R^\dagger(s + ip)U_L, \tag{3.57}$$

$$s - ip \to U_L^\dagger(s - ip)U_R. \tag{3.58}$$

Goldstone fields now transform as

$$\Sigma(x) \to U_L(x)\Sigma(x)U_R(x)^\dagger, \tag{3.59}$$

and the derivative as

$$\begin{aligned}
\partial_\mu \Sigma &\to \partial_\mu(U_L \Sigma U_R) \\
&= U_L(\partial_\mu \Sigma)U_R^\dagger + (\partial_\mu U_L)\Sigma U_R^\dagger + U_L \Sigma(\partial_\mu U_R^\dagger) \\
&= U_L \left[\partial_\mu \Sigma + U_L^\dagger(\partial_\mu U_L)\Sigma + \Sigma(x)(\partial_\mu U_R(x)^\dagger)U_R\right] U_R^\dagger.
\end{aligned} \tag{3.60}$$

The covariant derivative is defined to transform in the same way as the object it is acting upon. This means that the definition of the covariant derivative depends on what it acts on. Assume A, B and Σ transforms as $A \to U_R A U_R^\dagger$, $B \to U_L A U_L^\dagger$, and $\Sigma \to U_L \Sigma U_R^\dagger$. In each of these cases, we define the covariant derivative as

$$\nabla_\mu A = \partial_\mu A - i[r_\mu, A], \tag{3.61}$$

$$\nabla_\mu B = \partial_\mu B - i[r_\mu, B], \tag{3.62}$$

$$\nabla_\mu \Sigma = \partial_\mu \Sigma - i r_\mu \Sigma + i \Sigma l_\mu. \tag{3.63}$$

It follows from Eq. (3.60) that $\nabla_\mu \Sigma$ transforms as Σ, and the proof for the other two are special cases. For quantities that do not transform under $SU(2)_L \times SU(2)_R$, the covariant derivative is the ordinary partial derivative. If A, B and $C = AB$ are all quantities with a well-defined covariant derivative, then

$$\nabla_\mu(AB) = (\nabla_\mu A)B + A(\nabla_\mu B). \tag{3.64}$$

We prove the special case of $a_\mu = 0$, which is what we use in this text, in which case

$$\nabla_\mu \Sigma = \partial_\mu \Sigma - i[v_\mu, \Sigma]. \tag{3.65}$$

Assume A, B transforms as Σ. Then,

$$\nabla_\mu(AB) = (\partial_\mu A)B + A(\partial_\mu B) - i\,[v_\mu, AB] = (\partial_\mu A - i\,[v_\mu, A])B + A(\partial_\mu B - i\,[v_\mu, B])$$
$$= (\nabla_\mu A)B + A(\nabla_\mu B).$$

The more general theorem follows by applying the definition of the various covariant derivatives [24]. Decomposing a 2-by-2 matrix M, as described in section B.1, shows that the trace of the commutator of τ_b and M is zero:

$$\mathrm{Tr}\{[\tau_a, M]]\} = M_b \,\mathrm{Tr}\{[\tau_a, \tau_b]\} = 0.$$

Together with the fact that $\mathrm{Tr}\{\partial_\mu A\} = \partial_\mu \mathrm{Tr}\{A\}$, this gives the product rule for invariant traces:

$$\mathrm{Tr}\{A\nabla_\mu B\} = \partial_\mu \mathrm{Tr}\{AB\} - \mathrm{Tr}\{(\nabla_\mu A)B\}.$$

This allows for the use of the divergence theorem when doing partial integration. Assume A, K^μ, and AK^μ have a well-defined covariant derivative, and that $\mathrm{Tr}\{AK^\mu\}$ is invariant under transformations by the gauge group. Let $\mathrm{Tr}\{K^\mu\}$ be a space-time vector, and $\mathrm{Tr}\{A\}$ scalar. Let Ω be the domain of integration, with coordinates x and $\partial\Omega$ its boundary, with coordinates y. Then,

$$\int_\Omega \mathrm{d}x\, \mathrm{Tr}\{A\nabla_\mu K^\mu\} = -\int_\Omega \mathrm{d}x\, \mathrm{Tr}\{(\nabla_\mu A)K^\mu\} + \int_{\partial\Omega} \mathrm{d}y\, n_\mu \,\mathrm{Tr}\{AK^\mu\},$$

where n_μ is the normal vector of $\partial\Omega$ [8]. By the assumption of no variation on the boundary, the last term is constant when varying the action, and may therefore be discarded.

We define the scalar current

$$\chi = 2B_0(s + ip), \tag{3.66}$$

where B_0 is defined by the up quark condensate in the chiral limit by

$$\langle \bar{u}u \rangle_{j=0} = -f^2 B_0. \tag{3.67}$$

Here, f is the bear pion decay constant. The pseudoscalar current is thus $\chi^\dagger = 2B_0(s - ip)$. These can be combined to give more gauge-invariant terms, such as

$$\mathrm{Tr}\{\chi^\dagger \Sigma\}, \quad \mathrm{Tr}\{\Sigma^\dagger \chi\}. \tag{3.68}$$

In the grand canonical ensemble, we modify the Lagrangian as

$$\mathcal{L} \to \mathcal{L} + \mu Q, \tag{3.69}$$

where Q is a conserved charge, and μ is the corresponding chemical component. We are interested systems with non-zero isospin,

$$Q_I = \int \mathrm{d}^3 x\, V_3^0 = \int \mathrm{d}^3 x\, \frac{1}{2}\bar{q}\gamma^0 \tau_3 q, \tag{3.70}$$

and the corresponding isospin chemical potential μ_I. We do this by considering the system with a external current

$$v_\mu = \frac{1}{2}\mu_I \delta^0_\mu \tau_3. \tag{3.71}$$

The most straight forward parametrization of the Goldstone manifold is

$$U(x) = \exp\left\{i\frac{\pi_a \tau_a}{2f}\right\}, \tag{3.72}$$

where f is the bare pion decay constant, π_a are the three Goldstone bosons. They are real functions, and are related to the pions π_0, π_+ and π_- by

$$\pi_a \tau_a = \begin{pmatrix} \pi_3 & \pi_1 - i\pi_2 \\ \pi_1 + i\pi_2 & -\pi_3 \end{pmatrix} = \begin{pmatrix} \pi_0 & \sqrt{2}\pi_+ \\ \sqrt{2}\pi_- & -\pi_0 \end{pmatrix}. \tag{3.73}$$

For $\mu_I = 0$, the ground state of the system will be the vacuum, i.e. at $\pi_a = 0$. For non-zero isospin chemical potential, however, we expect that the ground state may be rotated away from the vacuum. Following [25], we parametrize the Goldstone manifold as

$$\Sigma(x) = A_\alpha[U(x)\Sigma_0 U(x)]A_\alpha, \tag{3.74}$$

where

$$\Sigma_0 = \mathbb{1}, \ A_\alpha = \exp\left\{\frac{i\alpha}{2}\tau_1\right\}, \ U(x) = \exp\left\{i\frac{\tau_a \pi_a(x)}{2f}\right\}. \tag{3.75}$$

Here, α is a real number between 0 and 2π, that parametrizes the ground state of the system.

3.3   Leading order Lagrangian

When taking into account the discreete symmetries of the QCD Lagrangian, the most general first order Lagrangian is [22, 26]

$$\mathcal{L}_2 = \frac{1}{4}f^2 \operatorname{Tr}\{\nabla_\mu \Sigma (\nabla^\mu \Sigma)^\dagger\} + \frac{1}{4}f^2 \operatorname{Tr}\{\chi^\dagger \Sigma + \Sigma^\dagger \chi\}. \tag{3.76}$$

In this section, we will expand this Lagrangian in π/f, which we will use to calculate the free energy density. To get the series expansion of Σ in powers of π/f, we start by using the fact that $\tau_a^2 = \mathbb{1}$ to write

$$A_\alpha = \sum_n^\infty \frac{1}{n!}\left(\frac{i\alpha}{2}\tau_1\right)^n = \sum_n^\infty \left[\frac{\mathbb{1}}{(2n)!}\left(\frac{i\alpha}{2}\right)^{(2n)} + \frac{\tau_1}{(2n+1)!}\left(\frac{i\alpha}{2}\right)^{(2n+1)}\right] = \mathbb{1}\cos\frac{\alpha}{2} + i\tau_1 \sin\frac{\alpha}{2}. \tag{3.77}$$

The series expansion of U is

$$U = \exp\left(\frac{i\pi_a \tau_a}{2f}\right) = 1 + \frac{i\pi_a \tau_a}{2f} + \frac{1}{2}\left(\frac{i\pi_a \tau_a}{2f}\right)^2 + \frac{1}{6}\left(\frac{i\pi_a \tau_a}{2f}\right)^3 + \frac{1}{24}\left(\frac{i\pi_a \tau_a}{2f}\right)^4 + \mathcal{O}\left((\pi/f)^5\right),$$

which we use to calculate the expansion of the inner part of Σ, as given in Eq. (3.74),

$$\begin{aligned} U\Sigma_0 U &= \left(1 + \frac{i\pi_a \tau_a}{2f} + \frac{1}{2}\left(\frac{i\pi_a \tau_a}{2f}\right)^2 + \frac{1}{6}\left(\frac{i\pi_a \tau_a}{2f}\right)^3 + \frac{1}{24}\left(\frac{i\pi_a \tau_a}{2f}\right)^4\right) \\ &\quad \times \left(1 + \frac{i\pi_a \tau_a}{2f} + \frac{1}{2}\left(\frac{i\pi_a \tau_a}{2f}\right)^2 + \frac{1}{6}\left(\frac{i\pi_a \tau_a}{2f}\right)^3 + \frac{1}{24}\left(\frac{i\pi_a \tau_a}{2f}\right)^4\right) + \mathcal{O}\left((\pi/f)^5\right) \\ &= 1 + \frac{i\pi_a \tau_a}{f} + 2\left(\frac{i\pi_a \tau_a}{2f}\right)^2 + \frac{4}{3}\left(\frac{i\pi_a \tau_a}{2f}\right)^3 + \frac{2}{3}\left(\frac{i\pi_a \tau_a}{2f}\right)^4 + \mathcal{O}\left((\pi/f)^5\right). \end{aligned}$$

The symmetry of $\pi_a \pi_b$ means that

$$(\pi_a \tau_a)^2 = \pi_a \pi_b \frac{1}{2} \{\tau_a, \tau_b\} = \pi_a \pi_a, \quad (\pi_a \tau_a)^3 = \pi_a \pi_a \pi_b \tau_b, \quad (\pi_a \tau_a)^4 = \pi_a \pi_a \pi_b \pi_b.$$

This gives us the expression

$$U\Sigma_0 U = 1 + i\frac{\pi_a \tau_a}{f} - \frac{\pi_a^2}{2f^2} - i\frac{\pi_a^2 \pi_b \tau_b}{6f^3} + \frac{\pi_a^2 \pi_b^2}{24f^4} + \mathcal{O}\left((\pi/f)^5\right).$$

We combine this result with Eq. (3.77) to get an expression for Σ up to $\mathcal{O}\left((\pi/f)^5\right)$

$$\Sigma = \left(\cos\frac{\alpha}{2} + i\tau_1 \sin\frac{\alpha}{2}\right)\left(1 + i\frac{\pi_a \tau_a}{f} - \frac{\pi_a^2}{2f^2} - i\frac{\pi_a^2 \pi_b \tau_b}{6f^3} + \frac{\pi_a^2 \pi_b^2}{24f^4}\right)\left(\cos\frac{\alpha}{2} + i\tau_1 \sin\frac{\alpha}{2}\right)$$

$$= \left(1 + i\frac{\pi_a \tau_a}{f} - \frac{\pi_a^2}{2f^2} - i\frac{\pi_a^2 \pi_b \tau_b}{6f^3} + \frac{\pi_a^2 \pi_b^2}{24f^4}\right)\cos^2\frac{\alpha}{2}$$

$$- \left(1 + i\frac{\pi_a}{f}\tau_1 \tau_a \tau_1 - \frac{\pi_a^2}{2f^2} - i\frac{\pi_a^2 \pi_b}{6f^3}\tau_1 \tau_b \tau_1 + \frac{\pi_a^2 \pi_b^2}{24f^4}\right)\sin^2\frac{\alpha}{2}$$

$$+ i\left(2\tau_1 + i\frac{\pi_a}{f}\{\tau_1, \tau_a\} - 2\tau_1\frac{\pi_a^2}{2f^2} - i\frac{\pi_a^2 \pi_b}{6f^3}\{\tau_1, \tau_b\} + 2\tau_1\frac{\pi_a^2 \pi_b^2}{24f^4}\right)\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}.$$

Using trigonometric identities and the commutator,

$$\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2} = \cos\alpha, \quad 2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2} = \sin\frac{\alpha}{2}, \quad \tau_1 \tau_a \tau_1 = -\tau_a + 2\delta_{1a}\tau_1,$$

the final expression of Σ to $\mathcal{O}\left((\pi/f)^5\right)$ is

$$\Sigma = \left(1 - \frac{\pi_a^2}{2f^2} + \frac{\pi_a^2 \pi_b^2}{24f^4}\right)(\cos\alpha + i\tau_1 \sin\alpha) + \left(\frac{\pi_a}{f} - \frac{\pi_b^2 \pi_a}{6f^3}\right)\left(i\tau_a - 2i\delta_{a1}\tau_1 \sin^2\frac{\alpha}{2} - \delta_{a1}\sin\alpha\right). \quad (3.78)$$

The kinetic term in the χPTLagrangian is

$$\nabla_\mu \Sigma (\nabla^\mu \Sigma)^\dagger = \partial_\mu \Sigma \partial^\mu \Sigma^\dagger - i\left(\partial_\mu \Sigma \left[v^\mu, \Sigma^\dagger\right] - \text{h.c.}\right) - [v_\mu, \Sigma]\left[v_\mu, \Sigma^\dagger\right]. \quad (3.79)$$

Using Eq. (3.78) we find the expansion of the constitutive parts of the kinetic term to be

$$\partial_\mu \Sigma = \left(\frac{-1}{f^2} + \frac{\pi_b^2}{6f^4}\right)(\cos\alpha + i\tau_1 \sin\alpha)(\pi_a \partial_\mu \pi_a)$$

$$+ \left(\frac{\partial_\mu \pi_a}{f} - \frac{\pi_b^2 \partial_\mu \pi_a + 2\pi_a \pi_b \partial_\mu \pi_b}{6f^3}\right)\left(i\tau_a - 2i\delta_{a1}\tau_1 \sin^2\frac{\alpha}{2} - \delta_{1a}\sin\alpha\right)$$

$$= \left[\left(\frac{-1}{f^2} + \frac{\pi_b^2}{6f^4}\right)(\pi_a \partial_\mu \pi_a)\cos\alpha - \left(\frac{\partial_\mu \pi_1}{f} - \frac{\pi_b^2 \partial_\mu \pi_1 + 2\pi_1 \pi_b \partial_\mu \pi_b}{6f^3}\right)\sin\alpha\right]$$

$$- \left[\left(\frac{-1}{f^2} + \frac{\pi_b^2}{6f^4}\right)(\pi_a \partial_\mu \pi_a)\sin\alpha - \left(\frac{\partial_\mu \pi_1}{f} - \frac{\pi_b^2 \partial_\mu \pi_1 + 2\pi_1 \pi_b \partial_\mu \pi_b}{6f^3}\right)2\sin^2\frac{\alpha}{2}\right]i\tau_1$$

$$+ \left(\frac{\partial_\mu \pi_a}{f} - \frac{\pi_b^2 \partial_\mu \pi_a + 2\pi_a \pi_b \partial_\mu \pi_b}{6f^3}\right)i\tau_a, \quad (3.80)$$

and

$$[v_\mu, \Sigma] = \frac{1}{2}\mu_I \delta_\mu^0 \left[\left(1 - \frac{\pi_a^2}{2f^2} + \frac{\pi_a^2 \pi_b^2}{24f^4}\right)i\sin\alpha [\tau_3, \tau_1] + \left(\frac{\pi_a}{f} - \frac{\pi_b^2 \pi_a}{6f^3}\right)\left(i[\tau_a, \tau_3] - 2i\delta_{a1}\sin^2\frac{\alpha}{2}[\tau_3, \tau_1]\right)\right]$$

$$= -\mu_I \delta_\mu^0 \left\{\left(1 - \frac{\pi_a^2}{2f^2} + \frac{\pi_a^2 \pi_b^2}{24f^4}\right)\tau_2 \sin\alpha + \left(\frac{\pi_a}{f} - \frac{\pi_b^2 \pi_a}{6f^3}\right)\left[(\delta_{a1}\tau_2 - \delta_{a2}\tau_1) - 2\delta_{a1}\tau_2 \sin^2\frac{\alpha}{2}\right]\right\}$$

$$= -\mu_I \delta_\mu^0 \left\{\left[\left(1 - \frac{\pi_a^2}{2f^2} + \frac{\pi_a^2 \pi_b^2}{24f^4}\right)\sin\alpha + \left(\frac{\pi_1}{f} - \frac{\pi_b^2 \pi_1}{6f^3}\right)\cos\alpha\right]\tau_2 - \left(\frac{\pi_2}{f} - \frac{\pi_b^2 \pi_2}{6f^3}\right)\tau_1\right\}. \quad (3.81)$$

Combining Eq. (3.80) and Eq. (3.81) gives the following terms [1]

$$\mathrm{Tr}\{\partial_\mu \Sigma \partial^\mu \Sigma^\dagger\} = \frac{2}{f^2}\partial_\mu \pi_a \partial^\mu \pi_a + \frac{2}{3f^4}\left[(\pi_a\partial_\mu\pi_a)(\pi_b\partial^\mu\pi_b) - (\pi_a\partial_\mu\pi_b)(\pi_b\partial^\mu\pi_a)\right],$$

$$-i\,\mathrm{Tr}\{\partial^\mu\Sigma\left[v_\mu, \Sigma^\dagger\right] - \mathrm{h.c.}\} = 4\mu_I\frac{\partial_0\pi_2}{f} + 8\mu_I\frac{\pi_3}{3f^3}\sin\alpha(\pi_2\partial_0\pi_3 - \pi_3\partial_0\pi_2)\sin\alpha$$

$$+ \left(\frac{4\mu_I}{f^2}\cos\alpha - \frac{8\mu_I\pi_1}{3f^3}\sin\alpha - \frac{4\mu_I\pi_a\pi_a}{3f^4}\cos\alpha\right)(\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1),$$

$$-\mathrm{Tr}\{[v_\mu, \Sigma]\,[v^\mu, \Sigma^\dagger]\} = {\mu_I}^2\left[2\sin^2\alpha + \left(\frac{2}{f} - \frac{4\pi_a\pi_a}{3f^3}\right)\pi_1\sin 2\alpha + \left(\frac{2}{f^2} - \frac{2\pi_a\pi_a}{3f^4}\right)\pi_a\pi_b k_{ab}\right],$$

$$\mathrm{Tr}\{\Sigma + \Sigma^\dagger\} = 4\cos\alpha - \frac{4\pi_1}{f}\sin\alpha - \frac{2\pi_a\pi_a}{f^2}\cos\alpha + \frac{2\pi_1\pi_a\pi_a}{3f^3}\sin\alpha + \frac{(\pi_a\pi_a)^2}{6f^4}\cos\alpha,$$

where $k_{ab} = \delta_{a1}\delta_{b1}\cos 2\alpha + \delta_{a2}\delta_{b2}\cos^2\alpha - \delta_{a3}\delta_{b3}\sin^2\alpha$. If we write the Lagrangian as show in Eq. (3.76) as $\mathcal{L}_2 = \mathcal{L}_2^{(0)} + \mathcal{L}_2^{(1)} + \mathcal{L}_2^{(2)} + ...$, where $\mathcal{L}_2^{(n)}$ contains all terms of order $\mathcal{O}\left((\pi/f)^n\right)$, then the result of the series expansion is

$$\mathcal{L}_2^{(0)} = f^2\left(\bar{m}^2\cos\alpha + \frac{1}{2}\mu^2\sin^2\alpha\right), \tag{3.82}$$

$$\mathcal{L}_2^{(1)} = f(\mu_I^2\cos\alpha - \bar{m}^2)\pi_1\sin\alpha + f\mu_I\partial_0\pi_2\sin\alpha, \tag{3.83}$$

$$\mathcal{L}_2^{(2)} = \frac{1}{2}\partial_\mu\pi_a\partial^\mu\pi_a + \mu_I\cos\alpha\,(\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1) - \frac{1}{2}\bar{m}^2\pi_a\pi_a\cos\alpha + \frac{1}{2}\mu_I^2\pi_a\pi_b k_{ab}, \tag{3.84}$$

$$\mathcal{L}_2^{(3)} = \frac{\pi_a\pi_a\pi_1}{6f}(\bar{m}^2\sin\alpha - 2{\mu_I}^2\sin 2\alpha)$$

$$- \frac{2\mu_I}{3f}\left[\pi_1(\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1) + \pi_3(\pi_3\partial_0\pi_2 - \pi_2\partial_0\pi_3)\right]\sin\alpha, \tag{3.85}$$

$$\mathcal{L}_2^{(4)} = \frac{1}{6f^2}\left\{\frac{1}{4}\bar{m}^2(\pi_a\pi_a)^2\cos\alpha - [(\pi_a\pi_a)(\partial_\mu\pi_b\partial^\mu\pi_b) - (\pi_a\partial_\mu\pi_a)(\pi_b\partial^\mu\pi_b)]\right\}$$

$$- \frac{\mu_I\pi_a\pi_a}{3f^2}\left[(\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1)\cos\alpha + \frac{1}{2}\mu_I\pi_a\pi_b k_{ab}\right]. \tag{3.86}$$

We have introduced $\bar{m}^2 = B_0(m_u + m_d)$, the bare pion mass, and assumed $2\delta m^2 = m_u - m_d = 0$.

3.4   Equation of motion and redundant terms

Changing the field parametrization that appear in the Lagrangian does not affect any of the physics, as it corresponds to a change of variables in the path integral [22, 27, 28]. However, a change of variables can result in new terms in the Lagrangian. As a result of this, terms that on the face of it appear independent may be redundant. These terms can be eliminated by using the classical equation of motion. In this section we show first the derivation of the equation of motion, then use this result to identify redundant terms which need not be included in the most general Lagrangian.

We derive the equation of motion for the leading order Lagrangian using the principle of least action. Choosing the parametrization $\Sigma = \exp(i\pi_a\tau_a)$, a variation $\pi_a \to \pi_a + \delta\pi_a$ results in a variation in Σ, $\delta\Sigma = i\tau_a\delta\pi_a\Sigma$. The variation of the leading order action,

$$S_2 = \int_\Omega \mathrm{d}^4 x\,\mathcal{L}_2, \tag{3.87}$$

when varying π_a is

$$\delta S = \int_\Omega \mathrm{d}x\,\frac{f^2}{4}\,\mathrm{Tr}\{(\nabla_\mu\delta\Sigma)(\nabla^\mu\Sigma)^\dagger + (\nabla_\mu\Sigma)(\nabla^\mu\delta\Sigma)^\dagger + \chi\delta\Sigma^\dagger + \delta\Sigma\chi^\dagger\}.$$

---

[1]The scripts used to aid the calculation of the Lagrangian is available at `https://github.com/martkjoh/prosjektoppgave`

Using the properties of the covariant derivative to do partial integration, as show in section 3.2, as well as $\delta(\Sigma\Sigma^\dagger) = (\delta\Sigma)\Sigma^\dagger + \Sigma(\delta\Sigma)^\dagger = 0$, the variation of the action can be written

$$
\begin{aligned}
\delta S &= \frac{f^2}{4} \int_\Omega \mathrm{d}x \ \mathrm{Tr}\{-\delta\Sigma\nabla^2\Sigma^\dagger + (\nabla^2\Sigma)(\Sigma^\dagger\delta\Sigma\Sigma^\dagger) - \chi(\Sigma^\dagger\delta\Sigma\Sigma^\dagger) + \delta\Sigma\chi^\dagger\} \\
&= \frac{f^2}{4} \int_\Omega \mathrm{d}x \ \mathrm{Tr}\{\delta\Sigma\Sigma^\dagger \left[(\nabla^2\Sigma)\Sigma^\dagger - \Sigma\nabla^2\Sigma^\dagger - \chi\Sigma^\dagger + \Sigma\chi^\dagger\right]\} \\
&= i\frac{f^2}{4} \int_\Omega \mathrm{d}x \ \mathrm{Tr}\{\tau_a \left[(\nabla^2\Sigma)\Sigma^\dagger - \Sigma\nabla^2\Sigma^\dagger - \chi\Sigma^\dagger + \Sigma\chi^\dagger\right]\}\delta\pi_a = 0.
\end{aligned}
$$

As the variation is arbitrary, the equation of motion to leading order is

$$
\mathrm{Tr}\{\tau_a \left[(\nabla^2\Sigma)\Sigma^\dagger - \Sigma\nabla^2\Sigma^\dagger - \chi\Sigma^\dagger + \Sigma\chi^\dagger\right]\} = 0. \tag{3.88}
$$

This may be rewritten as a matrix equation. Using that

$$
\mathrm{Tr}\{(\nabla_\mu\Sigma)\Sigma^\dagger\} = \mathrm{Tr}\{i\tau_a(\partial_\mu\pi_a)\Sigma\Sigma^\dagger\} - i\,\mathrm{Tr}\{[v_\mu,\Sigma]\Sigma^\dagger\} = 0,
$$

we can see that $\mathrm{Tr}\{(\nabla^2\Sigma)\Sigma^\dagger - \Sigma\nabla^2\Sigma^\dagger\} = 0$, and the equation of motion may be written as

$$
\mathcal{O}_{\mathrm{EOM}}^{(2)}(\Sigma) = (\nabla^2\Sigma)\Sigma^\dagger - \Sigma\nabla^2\Sigma^\dagger - \chi\Sigma^\dagger + \Sigma\chi^\dagger - \frac{1}{2}\,\mathrm{Tr}\{\chi\Sigma^\dagger - \Sigma\chi^\dagger\} = 0. \tag{3.89}
$$

The next step in eliminating redundant terms is to change the parametrization of Σ by $\Sigma(x) \to \Sigma'(x)$. Here, $\Sigma(x) = e^{iS(x)}\Sigma'(x)$, $S(x) \in \mathfrak{su}(2)$. This change leads to a new Lagrange density, $\mathcal{L}[\Sigma] = \mathcal{L}[\Sigma'] + \Delta\mathcal{L}[\Sigma']$. We are free to choose $S(x)$, as long Σ' still obeys the required transformation properties. Any terms in the Lagrangian $\Delta\mathcal{L}$ due to a reparametrization can be neglected, as argued earlier. When demanding that Σ' obey the same symmetries as Σ, the most general transformation to second order in Weinberg's power counting scheme is [22]

$$
S_2 = i\alpha_2 \left[(\nabla^2\Sigma')\Sigma^\dagger - \Sigma'(\nabla^2\Sigma')^\dagger\right] + i\alpha_2 \left[\chi\Sigma'^\dagger - \Sigma'\chi^\dagger - \frac{1}{2}\,\mathrm{Tr}\{\chi\Sigma'^\dagger - \Sigma'\chi^\dagger\}\right]. \tag{3.90}
$$

α_1 and α_2 are arbitrary real numbers. As Eq. (3.90) is to second order, $\Delta\mathcal{L}$ is fourth order in Weinberg's power counting scheme. To leading order is given by

$$
\begin{aligned}
\mathcal{L}_2 \left[e^{iS_2}\Sigma'\right] &= \frac{f^2}{4}\,\mathrm{Tr}\{[\nabla_\mu(1+iS_2)\Sigma'][\nabla^\mu\Sigma'^\dagger(1-iS_2)]\} + \frac{f^2}{4}\,\mathrm{Tr}\{\chi\Sigma'^\dagger(1-iS_2) + (1+iS_2)\Sigma'\chi^\dagger\} \\
&= \mathcal{L}[\Sigma'] + i\frac{f^2}{4}\,\mathrm{Tr}\{[\nabla_\mu(S_2\Sigma')][\nabla^\mu\Sigma'^\dagger]^\dagger - [\nabla_\mu\Sigma'][\nabla^\mu(\Sigma'^\dagger S_2)]\} - i\frac{f^2}{4}\,\mathrm{Tr}\{\chi\Sigma'^\dagger S_2 - S_2\Sigma'\chi^\dagger\}
\end{aligned}
$$

Using the properties of the covariant derivative, as described in section 3.2, we may use the product rule and partial integration to write the difference between the two Lagrangians to fourth order as

$$
\begin{aligned}
\Delta\mathcal{L}[\Sigma'] &= i\frac{f^2}{4}\,\mathrm{Tr}\{(\nabla_\mu S_2)(\Sigma'\nabla^\mu\Sigma'^\dagger - (\nabla^\mu\Sigma')\Sigma'^\dagger)\} - i\frac{f^2}{4}\,\mathrm{Tr}\{\chi\Sigma'^\dagger S_2 - S_2\Sigma'\chi^\dagger\} \\
&= i\frac{f^2}{4}\,\mathrm{Tr}\{S_2 \left[\Sigma'^\dagger\nabla^2\Sigma' - (\nabla^2\Sigma')\Sigma'^\dagger - \chi\Sigma'^\dagger + \Sigma'\chi^\dagger\right]\}.
\end{aligned}
$$

Using the equation of motion Eq. (3.89), and the fact that $\mathrm{Tr}\{S_2\} = 0$, this difference can be written as

$$
\Delta\mathcal{L}[\Sigma'] = \frac{f^2}{4}\,\mathrm{Tr}\{iS_2\mathcal{O}_{\mathrm{EOM}}^{(2)}(\Sigma')\}. \tag{3.91}
$$

Any term that can be written in the form of Eq. (3.91) for arbitrary $\alpha_1, \alpha_2 \in \mathbb{R}$ is redundant, as we argued earlier, and may therefore be discarded. $\Delta\mathcal{L}_2$ is of fourth order, and it can thus be used to remove terms from $\mathcal{L}_4$ or higher order.

3.5   Next to leading order Lagrangian

The next to leading order Lagrangian density is, assuming no external fields From [22], we have the next-to-leading order Lagrangian

$$\mathcal{L}_4 = \frac{l_1}{4} \operatorname{Tr}\{\nabla_\mu \Sigma (\nabla^\mu \Sigma)^\dagger\}^2 + \frac{l_2}{4} \operatorname{Tr}\{\nabla_\mu \Sigma (\nabla_\nu \Sigma)^\dagger\} \operatorname{Tr}\{\nabla^\mu \Sigma (\nabla^\nu \Sigma)^\dagger\} + \frac{l_3 + h_1 - h_3}{16} \operatorname{Tr}\{\chi \Sigma^\dagger + \Sigma \chi^\dagger\}^2$$
$$+ \frac{l_4}{8} \operatorname{Tr}\{\nabla_\mu \Sigma (\nabla^\mu \Sigma)^\dagger\} \operatorname{Tr}\{\chi \Sigma^\dagger + \Sigma \chi^\dagger\} + \frac{h_1 - h_3 - l_4 - l_7}{16} \operatorname{Tr}\{\chi \Sigma^\dagger - \Sigma \chi^\dagger\}^2 + \frac{h_1 + h_3 - l_4}{4} \operatorname{Tr}\{\chi \chi^\dagger\}$$
$$- \frac{h_1 - h_3 - l_4}{8} \operatorname{Tr}\left\{(\chi \Sigma^\dagger)^2 + (\Sigma \chi^\dagger)^2\right\} \tag{3.92}$$

The parameters l_i and h_i are called low energy constants, or LEO. We can rewirte

$$\frac{h_1 - h_3 - l_4}{16} \left(\operatorname{Tr}\{\chi \Sigma^\dagger - \Sigma \chi^\dagger\}^2 - 2 \operatorname{Tr}\{(\chi \Sigma^\dagger)^2 + (\Sigma \chi^\dagger)^2\} \right) \tag{3.93}$$
$$= \frac{h_1 - h_3 - l_4}{16} \left(\operatorname{Tr}\{\chi \Sigma^\dagger\}^2 - 2 \operatorname{Tr}\{\chi \Sigma^\dagger\} \operatorname{Tr}\{\Sigma \chi^\dagger\} + \operatorname{Tr}\{\Sigma \chi^\dagger\}^2 - 2 \operatorname{Tr}\{(\chi \Sigma^\dagger)^2\} - 2 \operatorname{Tr}\{(\Sigma \chi^\dagger)^2\} \right) \tag{3.94}$$

Using $\operatorname{Tr}\{A^2\} = \operatorname{Tr}\{A\}^2 - \det(A) \operatorname{Tr}\{\mathbb{1}\}$, we get

$$= -\frac{h_1 - h_3 - l_4}{16} \left(\operatorname{Tr}\{\chi \Sigma^\dagger\}^2 + 2 \operatorname{Tr}\{\chi \Sigma^\dagger\} \operatorname{Tr}\{\Sigma \chi^\dagger\} + \operatorname{Tr}\{\Sigma \chi^\dagger\}^2 - 4 \det(\chi \Sigma^\dagger) - 4 \det(\Sigma \chi^\dagger) \right) \tag{3.95}$$
$$= -\frac{h_1 - h_3 - l_4}{16} \left(\operatorname{Tr}\{\chi \Sigma^\dagger + \Sigma \chi^\dagger\}^2 - 4 \det(\chi \Sigma^\dagger) - 4 \det(\Sigma \chi^\dagger) \right) \tag{3.96}$$

Furthermore, as $\det(\Sigma) = 1$,

$$\mathcal{L}_4 = \frac{l_1}{4} \operatorname{Tr}\{\nabla_\mu \Sigma (\nabla^\mu \Sigma)^\dagger\}^2 + \frac{l_2}{4} \operatorname{Tr}\{\nabla_\mu \Sigma (\nabla_\nu \Sigma)^\dagger\} \operatorname{Tr}\{\nabla^\mu \Sigma (\nabla^\nu \Sigma)^\dagger\} + \frac{l_3 + l_4}{16} \operatorname{Tr}\{\chi \Sigma^\dagger + \Sigma \chi^\dagger\}^2$$
$$+ \frac{l_4}{8} \operatorname{Tr}\{\nabla_\mu \Sigma (\nabla^\mu \Sigma)^\dagger\} \operatorname{Tr}\{\chi \Sigma^\dagger + \Sigma \chi^\dagger\} - \frac{l_7}{16} \operatorname{Tr}\{\chi \Sigma^\dagger - \Sigma \chi^\dagger\}^2 + \frac{h_1 + h_3 - l_4}{4} \operatorname{Tr}\{\chi \chi^\dagger\}$$
$$+ \frac{h_1 - h_3 - l_4}{4} (\det\{\chi\} + \det\{\chi^\dagger\}) \tag{3.97}$$

For real χ, we have $\operatorname{Tr}\{\chi \chi^\dagger\} = \det(\chi) + \det(\chi^\dagger)$, so

$$\mathcal{L}_4 = \frac{l_1}{4} \operatorname{Tr}\{\nabla_\mu \Sigma (\nabla^\mu \Sigma)^\dagger\}^2 + \frac{l_2}{4} \operatorname{Tr}\{\nabla_\mu \Sigma (\nabla_\nu \Sigma)^\dagger\} \operatorname{Tr}\{\nabla^\mu \Sigma (\nabla^\nu \Sigma)^\dagger\} + \frac{l_3 + l_4}{16} \operatorname{Tr}\{\chi \Sigma^\dagger + \Sigma \chi^\dagger\}^2$$
$$+ \frac{l_4}{8} \operatorname{Tr}\{\nabla_\mu \Sigma (\nabla^\mu \Sigma)^\dagger\} \operatorname{Tr}\{\chi \Sigma^\dagger + \Sigma \chi^\dagger\} - \frac{l_7}{16} \operatorname{Tr}\{\chi \Sigma^\dagger - \Sigma \chi^\dagger\}^2 + \frac{h_1 - l_4}{2} \operatorname{Tr}\{\chi \chi^\dagger\}$$
$$\tag{3.98}$$

To $\mathcal{L}_4$ to $\mathcal{O}\left((\pi/f)^3\right)$, we use the result from Eq. (3.80) and Eq. (3.81), up to and including $\mathcal{O}\left((\pi/f)^2\right)$, which gives

$$\operatorname{Tr}\{\partial_\mu \Sigma \partial_\nu \Sigma^\dagger\} = 2 \frac{\partial_\mu \pi_a \partial_\nu \pi_a}{f^2}$$

$$-i \operatorname{Tr}\{\partial_\mu \Sigma \left[v_\nu, \Sigma^\dagger\right] - \text{h.c.}\} = \frac{2\mu_I \pi_2}{f} (\delta_\mu^0 \partial_\nu + \delta_\nu^0 \partial_\mu) \sin\alpha + \frac{2\mu_I}{f^2} [\pi_1 (\delta_\mu^0 \partial_\nu + \delta_\nu^0 \partial_\mu)\pi_2 - \pi_2 (\delta_\mu^0 \partial_\nu + \delta_\nu^0 \partial_\mu)\pi_1] \cos\alpha$$

$$- \operatorname{Tr}\{[v_\nu, \Sigma] \left[v_\nu, \Sigma^\dagger\right]\} = 2\mu_I{}^2 \delta_\mu^0 \delta_\nu^0 \left[\sin^2\alpha + \frac{\pi_1}{f} \sin 2\alpha + \frac{\pi_a \pi_b}{f^2} k_{ab}\right].$$

Using the form of the Pauli matrices, we can write χ as

$$\chi = 2B_0 M = \bar{m}^2 \mathbb{1} + \Delta m^2 \tau_3,$$

where $\bar{m} = B_0(m_u + m_d)$, $\Delta m = B_0(m_u - m_d)/2$, which gives

$$\chi\Sigma^\dagger + \Sigma\chi^\dagger = 2(\bar{m}^2 + \Delta m^2 \tau_3)\left[\left(1 - \frac{\pi_a^2}{2f^2}\right)\cos\alpha - \frac{\pi_1}{f}\sin\alpha\right]$$

$$+ 2\Delta m^2\left[\left(1 - \frac{\pi_a^2}{2f^2}\right)\tau_2\sin\alpha + \frac{\pi_a}{f}\left(\delta_{a1}\tau_2\cos\alpha - \delta_{a2}\tau_1\right)\right],$$

$$\chi\Sigma^\dagger - \Sigma\chi^\dagger = -2i\bar{m}^2\left[\left(1 - \frac{\pi_a^2}{2f^2}\right)\tau_1\sin\alpha + \frac{\pi_a}{f}\left(\tau_a - 2\delta_{1a}\tau_1\sin^2\frac{\alpha}{2}\right)\right] - 2i\Delta m^2\frac{\pi_3}{f}.$$

Combining these results gives all the terms in $\mathcal{L}_4$, to $\mathcal{O}\left((\pi/f)^3\right)$:

$$\text{Tr}\left\{\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\right\}^2 = \text{Tr}\left\{\partial_\mu\Sigma\partial^\mu\Sigma^\dagger - i\left(\partial_\mu\Sigma\left[v^\mu,\Sigma^\dagger\right] - \text{h.c.}\right) - [v_\mu,\Sigma]\left[v^\mu,\Sigma^\dagger\right]\right\}^2$$

$$= \frac{8\mu_I^2}{f^2}(\partial_\mu\pi_a\partial^\mu\pi_a + 2\partial_\mu\pi_2\partial^\mu\pi_2)\sin^2\alpha$$

$$+ 16\mu_I^3\left[\frac{\partial_0\pi_2}{f}\sin^3\alpha + \frac{1}{f^2}\left(3\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1\right)\cos\alpha\sin^2\alpha\right]$$

$$+ 4\mu_I^4\left\{\sin^4\alpha + 2\sin^2\alpha\left[\frac{\pi_1}{f}\sin 2\alpha + \frac{\pi_a\pi_b}{f^2}\left(k_{ab} + 2\delta_{a1}\delta_{a2}\cos^2\alpha\right)\right]\right\}, \tag{3.99}$$

$$\text{Tr}\left\{\nabla_\mu\Sigma(\nabla_\nu\Sigma)^\dagger\right\}\text{Tr}\left\{\nabla^\mu\Sigma(\nabla^\nu\Sigma)^\dagger\right\}$$

$$= \frac{4\mu_I^2}{f^2}(\partial_0\pi_a\partial_0\pi_a + \partial_0\pi_2\partial_0\pi_2 + \partial_\mu\pi_2\partial^\mu\pi_2)\sin^2\alpha$$

$$+ 16\mu_I^3\left[\frac{\partial_0\pi_2}{f}\sin^3\alpha + \frac{1}{f^2}\left(3\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1\right)\cos\alpha\sin^2\alpha\right]$$

$$+ 4\mu_I^4\left\{\sin^4\alpha + 2\sin^2\alpha\left[\frac{\pi_1}{f}\sin 2\alpha + \frac{\pi_a\pi_b}{f^2}\left(k_{ab} + 2\delta_{a1}\delta_{a2}\cos^2\alpha\right)\right]\right\}, \tag{3.100}$$

$$\text{Tr}\left\{\nabla_\mu\Sigma(\nabla^\mu\Sigma)^\dagger\right\}\text{Tr}\left\{\chi\Sigma^\dagger + \Sigma\chi^\dagger\right\}$$

$$= 4\bar{m}^2\left\{2\frac{\partial_\mu\pi_a\partial^\mu\pi_a}{f^2}\cos\alpha + 4\mu_I\left[\frac{\partial_0\pi_2}{2f}\sin 2\alpha + \frac{1}{f^2}\left(\pi_1\partial_0\pi_2\cos 2\alpha - \pi_2\partial_0\pi_1\cos^2\alpha\right)\right]\right. \tag{3.101}$$

$$\left. + \mu_I^2\left[2\cos\alpha\sin^2\alpha - 2\frac{\pi_1}{f}\sin\alpha\left(2 - 3\sin^2\alpha\right) + \frac{1}{f^2}\left(\pi_1^2[2 - 9\sin^2\alpha] + \pi_2^2[2 - 3\sin^2\alpha] - 3\pi_3^2\sin^2\alpha\right)\cos\alpha\right]\right\},$$

$$\text{Tr}\left\{\chi\Sigma^\dagger + \Sigma\chi^\dagger\right\}^2 = 16\bar{m}^4\left[\cos^2\alpha - \frac{\pi_1}{f}\sin 2\alpha + \frac{1}{f^2}\left(\pi_1^2\sin^2\alpha - \pi_a\pi_a\cos^2\alpha\right)\right], \tag{3.102}$$

$$\text{Tr}\left\{\chi\Sigma^\dagger - \Sigma\chi^\dagger\right\}^2 = -16\left(\frac{\Delta m^2\pi_3}{f}\right)^2, \tag{3.103}$$

$$\text{Tr}\left\{\left(\chi\Sigma^\dagger\right)^2 + \left(\Sigma\chi^\dagger\right)^2\right\}$$

$$= 4\bar{m}^4\left(\cos 2\alpha - 2\frac{\pi_1}{f}\sin 2\alpha - 2\frac{\pi_a\pi_a}{f^2}\cos^2\alpha + 2\frac{\pi_1^2}{f^2}\sin^2\alpha\right) + 4\Delta m^4\left(1 - 2\frac{\pi_3^2}{f^2}\right), \tag{3.104}$$

$$\text{Tr}\left\{\chi^\dagger\chi\right\} = 2\bar{m}^4 + 2\Delta m^4. \tag{3.105}$$

The different terms of the NLO Lagrangian is

$$\mathcal{L}_4^{(0)} = (l_1 + l_2)\mu_I^4 \sin^4 \alpha + (l_3 + l_4)\bar{m}^2 \cos^2 \alpha + l_4 \bar{m}\mu_I^2 \cos\alpha \sin^2\alpha + (h_1 - l_4)\bar{m}^4 + h_3 \Delta m^4 \tag{3.106}$$

$$\mathcal{L}_4^{(1)} = 4\mu_I^3 \frac{l_1 + l_2}{f} (\partial_0\pi_2 + \mu_I \cos\alpha\pi_1)\sin^3\alpha - \frac{l_3 + l_4}{f}\bar{m}^4\pi_1 \sin 2\alpha$$

$$+ \bar{m}^2 \frac{l_4}{f} \left[\mu_I \partial_0\pi_2 \sin 2\alpha - \mu_I^2\pi_1 \sin\alpha\left(3\sin^2\alpha - 2\right)\right] \tag{3.107}$$

$$\mathcal{L}_4^{(2)} = 2\mu_I^2 \frac{l_1}{f^2}(\partial_\mu\pi_a\partial^\mu\pi_a + 2\partial_0\pi_2\partial_0\pi_2)\sin^2\alpha + \mu_I^2 \frac{l_2}{f^2}(\partial_\mu\pi_2\partial^\mu\pi_2 + 2\partial_0\pi_a\partial_0\pi_a + 2\partial_0\pi_2\partial_0\pi_2)\sin^2\alpha$$

$$+ 2\frac{l_1 + l_2}{f^2}\left[2\mu_I^3\left(3\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1\right)\cos\alpha + \mu_I^4\pi_a\pi_b\left(k_{ab} + 2\delta_{a1}\delta_{a2}\cos^2\alpha\right)\right]\sin^2\alpha$$

$$+ \frac{l_3 + l_4}{f^2}\bar{m}^2\left(\pi_1^2 \sin^2\alpha - \pi_a\pi_a \cos^2\alpha\right) + \frac{l_4}{f^2}\bar{m}^2\left[\partial_\mu\pi_a\partial^\mu\pi_a \cos\alpha + 4\mu_I\left(\pi_1\partial_0\pi_2 \cos 2\alpha - \pi_2\partial_0\pi_1 \cos^2\alpha\right)\right.$$

$$+ \frac{1}{2}\mu_I^2\left(\pi_1^2[2 - 9\sin^2\alpha] + \pi_2^2[2 - 3\sin^2\alpha] - 3\pi_3^2 \sin^2\alpha\right)\cos\alpha\right] + \frac{l_7}{f^2}\Delta m^2\pi_3^2 \tag{3.108}$$

3.6   Propagator

We may write the quadratic part of the Lagrangian Eq. (3.84) as [2]

$$\mathcal{L}_2^{(2)} = \frac{1}{2}\sum_a \partial_\mu\pi_a\partial^\mu\pi_a + \frac{1}{2}m_{12}(\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1) - \frac{1}{2}\sum_a m_a^2\pi_a^2, \tag{3.109}$$

where

$$m_1^2 = \bar{m}^2 \cos\alpha - \mu_I^2 \cos 2\alpha, \tag{3.110}$$

$$m_2^2 = \bar{m}^2 \cos\alpha - \mu_I^2 \cos^2\alpha, \tag{3.111}$$

$$m_3^2 = \bar{m}^2 \cos\alpha + \mu_I^2 \sin^2\alpha, \tag{3.112}$$

$$m_{12} = 2\mu_I \cos\alpha. \tag{3.113}$$

The components of the Euler-Lagrange equations of this field are

$$\frac{\partial\mathcal{L}}{\partial\pi_a} = \frac{1}{2}m_{12}(\delta_{a1}\partial_0\pi_2 - \delta_{a2}\partial_0\pi_1) - m_a^2\pi_a, \quad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\pi_a)} = \partial^\mu\pi_a - \frac{1}{2}m_{12}\delta_0^\mu(\delta_{a1}\pi_2 - \delta_{a2}\pi_1).$$

This gives the equation of motion for the field

$$\partial^\mu\partial_\mu\pi_a + m_a^2\pi_a = m_{12}(\delta_{a1}\partial_0\pi_2 - \delta_{a2}\partial_0\pi_1). \tag{3.114}$$

The propagator of the pion field is defined by

$$\left[\delta_{ab}(\partial^\mu\partial_\mu + m_a^2) - m_{12}(\delta_{a1}\delta_{b2} - \delta_{a2}\delta_{b1})\partial_0\right] D_{bc}(x, x') = -i\delta(x - x')\delta_{ac}. \tag{3.115}$$

The momentum space propagator, as defined in the section B.1, fulfills (MOTSATT FORTEGN FRA FRI ENERGI; FIKS!!!)

$$-\left[\delta_{ab}(p^2 - m_a^2) + ip_0 m_{12}(\delta_{a1}\delta_{b2} - \delta_{a2}\delta_{b1})\right] D_{bc}(p) := D_{ab}^{-1}D_{bc}(p) = -i\delta_{ac},$$

where

$$D^{-1} = -\begin{pmatrix} p^2 - m_1^2 & ip_0 m_{12} & 0 \\ -ip_0 m_{12} & p^2 - m_2^2 & 0 \\ 0 & 0 & p^2 - m_3^2 \end{pmatrix}.$$

The spectrum of the particles is given by solving $\det(D^{-1}) = 0$ for p^0. With $p = (p_0, \vec{p})$ as the four momentum, this gives

$$\det(D^{-1}) = D_{33}^{-1}\left(D_{11}^{-1}D_{22}^{-1} + (D_{12}^{-1})^2\right) = -\left(p^2 - m_3^2\right)\left[\left(p^2 - m_1^2\right)\left(p^2 - m_2^2\right) - p_0^2 m_{12}^2\right] = 0,$$

---

[2] Summation over isospin index (a, b, c) will be explicit in this section.

This equation has the solutions

$$E_0^2 = |\vec{p}|^2 + m_3^2, \tag{3.116}$$

$$E_\pm^2 = |\vec{p}|^2 + \frac{1}{2}\left(m_1^2 + m_2^2 + m_{12}^2\right) \pm \frac{1}{2}\sqrt{4|\vec{p}|^2 m_{12}^2 + \left(m_1^2 + m_2^2 + m_{12}^2\right)^2 - 4m_1^2 m_2^2}. \tag{3.117}$$

These are the energies of three particles π_0, π_+ and π_-. π_0 is π_3, while $\pi_\pm$ are linear combinations of π_1 and π_2. The (tree-level) masses of these particles are found by setting $\vec{p} = 0$, i.e. the rest-frame energy, and are

$$m_0^2 = m_3^2, \tag{3.118}$$

$$m_\pm^2 = \frac{1}{2}\left[m_1^2 + m_2^2 + m_{12}^2\right] \pm \frac{1}{2}\sqrt{\left(m_1^2 + m_2^2 + m_{12}^2\right)^2 - 4m_1^2 m_2^2}. \tag{3.119}$$

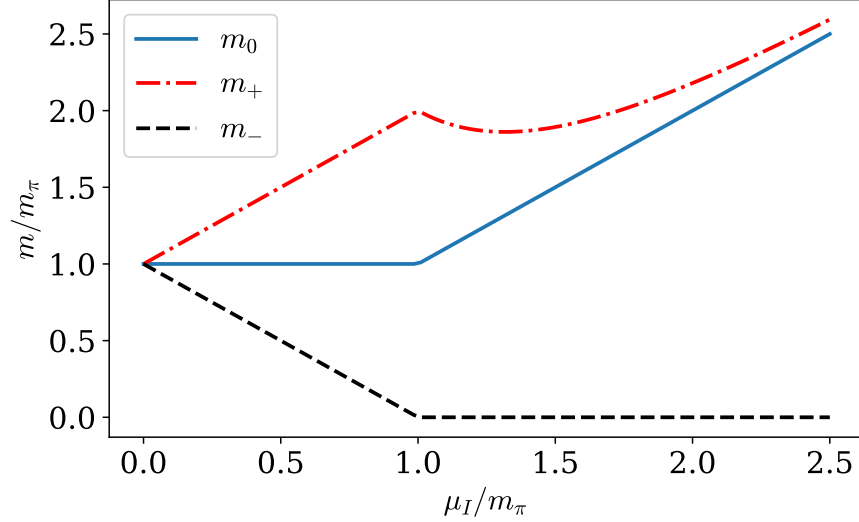Figure 3.1 shows the masses as functions of μ_I.

Figure 3.1: The masses of the three particles as functions of isosopin chemical potential.

With these energies we can write the determinant of the inverse propagator as

$$\det\left(D^{-1}\right) = -(p_0^2 - E_0^2)(p_0^2 - E_+^2)(p_0^2 - E_-^2). \tag{3.120}$$

The propagator may then be obtained as described in section B.1,

$$D = i(D^{-1})^{-1} = \frac{i}{\det(D^{-1})}\begin{pmatrix} D_{22}^{-1}D_{33}^{-1} & D_{12}^{-1}D_{33}^{-1} & 0 \\ -D_{12}^{-1}D_{33}^{-1} & D_{11}^{-1}D_{33}^{-1} & 0 \\ 0 & 0 & D_{11}^{-1}D_{22}^{-1} + (D_{12}^{-1})^2 \end{pmatrix}$$

$$= i\begin{pmatrix} \frac{p^2 - m_2^2}{(p_0^2 - E_+^2)(p_0^2 - E_-^2)} & \frac{-ip_0 m_{12}}{(p_0^2 - E_+^2)(p_0^2 - E_-^2)} & 0 \\ \frac{ip_0 m_{12}}{(p_0^2 - E_+^2)(p_0^2 - E_-^2)} & \frac{p^2 - m_1^2}{(p_0^2 - E_+^2)(p_0^2 - E_-^2)} & 0 \\ 0 & 0 & \frac{1}{p_0^2 - E_0^2} \end{pmatrix}. \tag{3.121}$$

Chapter 4

Thermodynamics

4.1 Free energy at lowest order

The equation of state (EOS) relates the thermodynamic variables of as system. In this section, we will obtain the equation of state of the pions by calculating their free energy. We use the effective Lagrangian found in **??** to find the leading-order contribution to one loop, and the next-to-leading order contribution at the tree-level, following the procedure used in [26, 25].[1] The free energy density of a homogenous system is

$$\mathcal{F} = -\frac{1}{V\beta} \ln Z. \tag{4.1}$$

Here, Z is the partition function, and V the volume of space. Using imaginary time formalism for thermal field theory, which is described in Appendix A, we find that the partition function is given by the path integral of the *Euclidean* Lagrange density, as shown in equation Eq. (A.14). In the zero temperature limit $\beta \to \infty$, the partition function is related to vacuum transition amplitude $Z_0 = Z[J = 0]$, as described in section 2.1, by a Wick rotation. The free energy density at zero temperature is therefore

$$\mathcal{F} = \frac{i}{VT} \ln Z_0, \tag{4.2}$$

where VT is the volume of space-time. This equals the effective potential in the ground state, which we found an explicit formula for in section 2.1, Eq. (2.38). We write $\mathcal{F} = \mathcal{F}^{(0)} + \mathcal{F}^{(1)} + \dots$, where $\mathcal{F}^{(n)}$ refers to the n-loop contributions to the free energy density.

Tree-level contribution

The tree-level contribution $\mathcal{F}^{(0)}$ is the classical potential, which is given by the static (π-independent) part of the Lagrangian. From Eq. (3.82) we have the leading order contribution,

$$\mathcal{F}_2^{(0)} = -\mathcal{L}_2^{(0)} = -f^2 \left(\bar{m}^2 \cos\alpha + \frac{1}{2}\mu^2 \sin^2\alpha \right), \tag{4.3}$$

where α parameterizes the ground state, which means that its value must minimize the free energy.

$$\frac{\partial}{\partial\alpha}\mathcal{F}_2^{(0)} = f^2 \left(\bar{m}^2 - \mu_I^2 \cos\alpha \right) \sin\alpha = 0.$$

This equation defines the relationship between the chemical potential μ_I, and the ground state parameter α, as illustrated in Figure 4.1. This gives the criterion

$$\alpha \in \{0, \pi\} \quad \text{or} \quad \cos\alpha = \frac{\bar{m}^2}{\mu_I{}^2}. \tag{4.4}$$

---

[1]Leading order and next-to-leading order, in this context, refers to Weinberg's power counting scheme.

As we see in the figure, $\alpha = \pi$ is a maximum, and thus unstable. This means that for all values $\mu_I^2 \leq \bar{m}^2$, we will have $\alpha = 0$, and the system will remain in its ground state.

In our discussion of the effective potential we also found that the ground state should minimize the classical potential, as shown by Eq. (2.32). This means that the linear part of the classical potential should vanish. The linear part of the classical potential is given by Eq. (3.83) to leading order, and reads $\mathcal{V}^{(1)} = f(\mu_I^2 \cos \alpha - \bar{m}^2) \sin \alpha \, \pi_1$, which vanishes given Eq. (4.4).
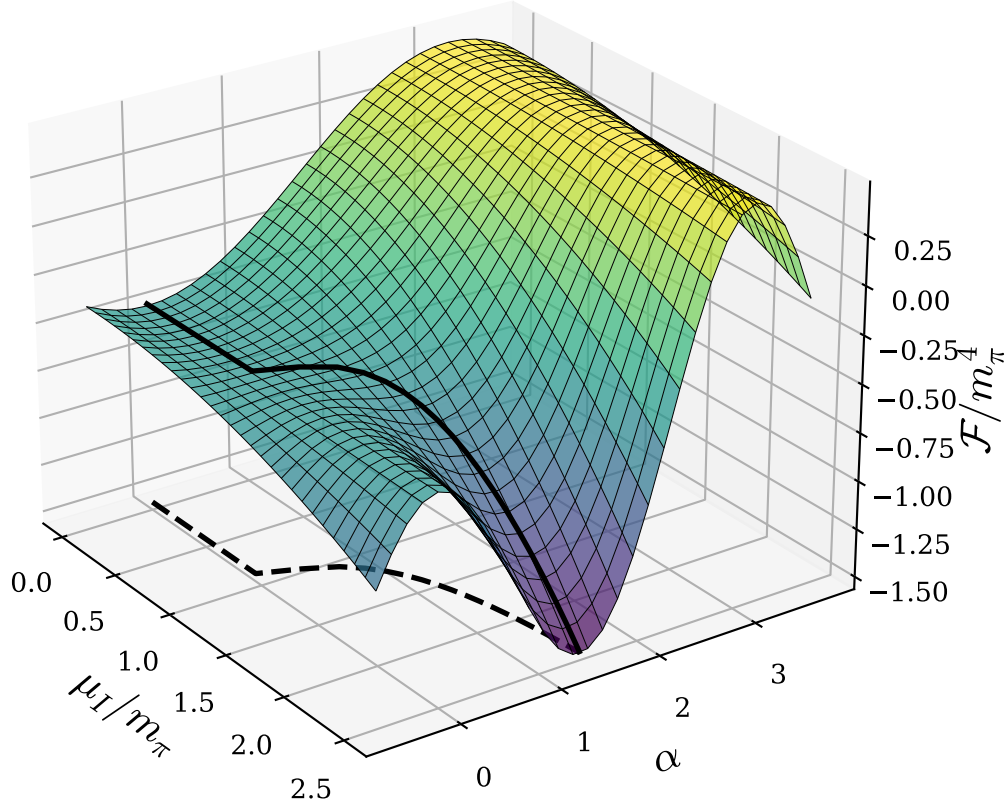

Figure 4.1: The surface gives free energy as a function of μ_I and α. α is the found by minimizing $\mathcal{F}$ for a given μ_I. This leads to a curve across the free energy surface, as show in the plot.

One-loop contribution

The one loop contribution to the free energy density is

$$\mathcal{F}^{(1)} = -\frac{i}{VT} \frac{1}{2} \operatorname{Tr} \left\{ \ln \left(-\frac{\delta^2 S[\pi = 0]}{\delta \pi_a(x) \, \delta \pi_b(y)} \right) \right\}. \tag{4.5}$$

This can be evaluated using the rules for functional differentiation given in section B.2. To leading order,

$$\frac{\delta^2 S[\pi = 0]}{\delta \pi_a(x) \, \delta \pi_b(y)} = \frac{\delta^2}{\delta \pi_a(x) \, \delta \pi_b(y)} \int \mathrm{d}^4 x \, \mathcal{L}_2^{(2)} = D_x^{-1} \delta(x - y). \tag{4.6}$$

Here, $\mathcal{L}_2^{(2)}$ is the quadratic part of the Lagrangian, as given in Eq. (3.109), and D_x^{-1} is the corresponding inverse propagator of the pion fields,

$$D_x^{-1} = -\left[\delta_{ab}(\partial_x^\mu \partial_{x,\mu} + m_a^2) - m_{12}(\delta_{a1}\delta_{b2} - \delta_{a2}\delta_{b1})\partial_{x,0} \right] \tag{4.7}$$

The inverse propagator is a matrix, which means that the determinant in Eq. (4.5) is both a matrix determinant, over the three pion indices, and a functional determinant. In section 3.6 we found the matrix part of the determinant in momentum space, which we can write using the dispersion relations of the pion fields

$$\det(-D^{-1}) = \det(-p_0^2 + E_0^2)\det(-p_0^2 + E_+^2)\det(-p_0^2 + E_-^2). \tag{4.8}$$

These dispersion relations are functions of the three-momentum $\vec{p}$, and are given in Eqs. (3.116) and (3.117). The functional determinant can therefore be evaluated as

$$
\begin{aligned}
\mathrm{Tr}\left\{\ln\left(-\frac{\delta^2 S[\pi = 0]}{\delta\pi_a(x)\,\delta\pi_b(y)}\right)\right\} &= \ln\det(-p_0^2 + E_0^2) + \ln\det(-p_0^2 + E_+^2) + \ln\det(-p_0^2 + E_-^2) \\
&= \mathrm{Tr}\left\{\ln(-p_0^2 + E_0^2) + \ln(-p_0^2 + E_+^2) + \ln(-p_0^2 + E_-^2)\right\} \\
&= (VT)\int\frac{\mathrm{d}^4 p}{(2\pi)^4}\left[\ln(-p_0^2 + E_0^2) + \ln(-p_0^2 + E_+^2) + \ln(-p_0^2 + E_-^2)\right],
\end{aligned} \tag{4.9}
$$

where we have used the identity $\ln\det M = \mathrm{Tr}\ln M$. These terms all have the form

$$I = \int\frac{\mathrm{d}^4 p}{(2\pi)^2}\ln(-p_0^2 + E^2), \tag{4.10}$$

where E is some function of the 3-momentum $\vec{p}$, but not p_0. We use the trick

$$\frac{\partial}{\partial\alpha}\left(-p_0^2 + E^2\right)^{-\alpha}\Big|_{\alpha=0} = \frac{\partial}{\partial\alpha}\exp\left[-\alpha\ln\left(-p_0^2 + E^2\right)\right]\Big|_{\alpha=0} = \ln\left(-p_0^2 + E^2\right), \tag{4.11}$$

and then perform a Wick-rotation of the p_0-integral to write the integral on the form

$$I = i\frac{\partial}{\partial\alpha}\int\frac{\mathrm{d}^4 p}{(2\pi)^4}\left(p_0^2 + E^2\right)^{-\alpha}\Big|_{\alpha=0}, \tag{4.12}$$

where p now is a Euclidean four-vector. The p_0 integral equals $\Phi_1(E, 1, \alpha)$, as defined in Eq. (A.33). The result is therefore given by Eq. (A.40),

$$\int\frac{\mathrm{d}p_0}{2\pi}\left(p_0^2 + E\right)^{-\alpha} = \frac{E^{1-2\alpha}}{\sqrt{4\pi}}\frac{\Gamma(\alpha - \frac{1}{2})}{\Gamma(\alpha)}. \tag{4.13}$$

The derivative of the Gamma function is $\Gamma'(\alpha) = \psi(\alpha)\Gamma(\alpha)$, where $\psi(\alpha)$ is the digamma function. Using

$$\frac{\partial}{\partial\alpha}\frac{\Gamma(\alpha - \frac{1}{2})}{\Gamma(\alpha)}\Big|_{\alpha=0} = \Gamma\left(\alpha - \frac{1}{2}\right)\frac{\psi(\alpha - \frac{1}{2}) - \psi(\alpha)}{\Gamma(\alpha)}\Big|_{\alpha=0} = \sqrt{4\pi}, \tag{4.14}$$

$$\frac{\Gamma(\alpha - \frac{1}{2})}{\Gamma(\alpha)}\Big|_{\alpha=0} = 0, \tag{4.15}$$

we get

$$I = i\int\frac{\mathrm{d}^3 p}{(2\pi)^3}E. \tag{4.16}$$

We see that the result is what we would expect physically, the total energy is the integral of the energy of each mode. This also agrees with the result from Appendix A in low temperature limit $\beta \to \infty$. This results in

$$\mathcal{F}^{(1)} = \frac{1}{2}\left[\int\frac{\mathrm{d}^3 p}{(2\pi)^3}E_0 + \int\frac{\mathrm{d}^3 p}{(2\pi)^3}(E_+ + E_-)\right] = \mathcal{F}_{\pi_0}^{(1)} + \mathcal{F}_{\pi_\pm}^{(1)}. \tag{4.17}$$

The first integral is identical to what we find for a free field in section A.3, in the zero temperature limit $\beta \to \infty$. These terms are all divergent, and must be regularized. We will use dimensional regularization, in which the integral is generalized to d dimensions, and the $\overline{\mathrm{MS}}$-scheme, as described in section A.3. Using the result for a free field Eq. (A.47), we get

$$\mathcal{F}_{\pi_0}^{(1)} = -\mu^{-2\epsilon}\frac{1}{4}\frac{m_3^4}{(4\pi)^2}\left(\frac{1}{\epsilon} + \frac{3}{2} + \ln\frac{\tilde{\mu}^2}{m_3^2}\right), \tag{4.18}$$

where μ is the renormalization scale, a parameter with mass-dimension 1 which is introduced to ensure the action integral remains dimensionless during dimensional regularization. $\tilde{\mu}$ is a related to μ as described in Eq. (A.46).

The contribution to the free energy from the π_+ and π_- particles is more complicated, as the dispersion relation is given by

$$E_\pm = \sqrt{|\vec{p}|^2 + \frac{1}{2}(m_1^2 + m_2^2 + m_{12}^2) \pm \frac{1}{2}\sqrt{4|\vec{p}|^2 m_{12}^2 + (m_1^2 + m_2^2 + m_{12}^2)^2 - 4m_1^2 m_2^2}}. \tag{4.19}$$

This is not an integral we can easily do in dimensional regularization. Instead, we will seek a function $f(|\vec{p}|)$ with the same UV-behavior, that is behavior for large $\vec{p}$, as $E_+ + E_-$. If we then add $0 = f(|\vec{p}|) - f(|\vec{p}|)$ to the integrand, we can isolate the divergent behavior

$$\mathcal{F}_{\pi_\pm}^{(1)} = \frac{1}{2}\int \frac{\mathrm{d}^3 p}{(2\pi)^3}[E_+ + E_- + f(|\vec{p}|) - f(|\vec{p}|)] = \mathcal{F}_{\mathrm{fin},\pi_\pm}^{(1)} + \mathcal{F}_{\mathrm{div},\pi_\pm}^{(1)}. \tag{4.20}$$

This results in a finite integral,

$$\mathcal{F}_{\mathrm{fin},\pi_\pm}^{(1)} = \frac{1}{2}\int \frac{\mathrm{d}^3 p}{(2\pi)^3}[E_+ + E_- - f(|\vec{p}|)], \tag{4.21}$$

which we can evaluate numerically, and isolate the divergence to

$$\mathcal{F}_{\mathrm{div},\pi_\pm}^{(1)} = \frac{1}{2}\int \frac{\mathrm{d}^3 p}{(2\pi)^3}f(|\vec{p}|), \tag{4.22}$$

which we hopefully will be able to do in dimensional regularization. We can explore the UV-behavior of $E_+ + E_-$ by expanding it in powers of $1/|\vec{p}|$,

$$E_+ + E_- = 2|\vec{p}| + \frac{m_{12} + 2(m_1^2 + m_2^2)}{4}|\vec{p}|^{-1} - \frac{m_{12}^4 + 4m_{12}^2(m_1^2 + m_2^2) + 8(m_1^4 + m_2^4)}{64}|\vec{p}|^{-3} + \mathcal{O}\left(|\vec{p}|^{-5}\right)$$

$$= a_1|\vec{p}| + a_2|\vec{p}|^{-1} + a_3|\vec{p}|^{-3} + \mathcal{O}\left(|\vec{p}|^{-5}\right). \tag{4.23}$$

We have defined the new constants a_i for brevity of notation. As

$$\int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3}|\vec{p}|^n = C\int_0^\infty \mathrm{d}p\, p^{2+n} \tag{4.24}$$

is UV divergent for $n \geq -3$, f need to match the expansion of $E_+ + E_-$ up to and including $\mathcal{O}\left(|\vec{p}|^{-3}\right)$ for $\mathcal{F}_{\mathrm{fin},\pi_\pm}^{(1)}$ to be finite. The most obvious choice for f is

$$f(|\vec{p}|) = a_1|\vec{p}| + a_2|\vec{p}|^{-1} + a_3|\vec{p}|^{-3}. \tag{4.25}$$

However, this introduces a new problem. f has the same UV-behavior as $E_+ + E_-$, but the last term diverges in the IR, that is for low $|\vec{p}|$. This can be amended by introducing a mass term. Let

$$|\vec{p}|^{-3} = \left(\frac{1}{\sqrt{|\vec{p}|^2}}\right)^3 \longrightarrow \left(\frac{1}{\sqrt{|\vec{p}|^2 + m^2}}\right)^3. \tag{4.26}$$

For $|\vec{p}|^2 \to \infty$, this term is asymptotic to $|\vec{p}|^{-3}$, so it retains its UV behavior. However, for $|\vec{p}| \to 0$, it now approaches m^{-3}, so the IR-divergence is gone. The cost of this technique is that we have introduced an arbitrary mass parameter. Any final result must thus be independent of the value of m to be acceptable.

We will instead regularize the integral by defining $E_i = \sqrt{|\vec{p}|^2 + \tilde{m}_i^2}$, and $\tilde{m}_i^2 = m_i^2 + \frac{1}{4}m_{12}^2$. Using the definition of the mass parameters, Eqs. (3.110) to (3.113), we get

$$m_3^2 = \bar{m}^2\cos\alpha + \mu_I^2\sin^2\alpha, \tag{4.27}$$

$$\tilde{m}_1^2 = m_1^2 + \mu^2\cos\alpha^2 = \bar{m}^2\cos\alpha + \mu_I^2\sin^2\alpha = m_3^2 \tag{4.28}$$

$$\tilde{m}_2^2 = m_2^2 + \mu^2\cos\alpha^2 = \bar{m}^2\cos\alpha. \tag{4.29}$$

Finally, we define $f(|\vec{p}|) = E_1 + E_2$ which differ from $E_+ + E_-$ by $\mathcal{O}\left(|\vec{p}|^{-5}\right)$, and is well-behaved in the IR. This leads to a divergent integral the same form as in the case of a free scalar. Thus, in the $\overline{\text{MS}}$-scheme,

$$\mathcal{F}^{(1)}_{\text{div},\pi_\pm} = -\mu^{-2\epsilon}\frac{1}{4}\frac{\tilde{m}_1^4}{(4\pi)^2}\left(\frac{1}{\epsilon}+\frac{3}{2}+\ln\frac{\tilde{\mu}^2}{\tilde{m}_1^2}\right) - \mu^{-2\epsilon}\frac{1}{4}\frac{\tilde{m}_2^4}{(4\pi)^2}\left(\frac{1}{\epsilon}+\frac{3}{2}+\ln\frac{\tilde{\mu}^2}{\tilde{m}_2^2}\right) + \mathcal{O}(\epsilon). \tag{4.30}$$

We define

$$\mathcal{F}^{(1)}_{\text{fin},\pi_\pm} = \frac{1}{2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}(E_+ + E_- - E_1 - E_2), \tag{4.31}$$

which is a finite integral. The total one-loop contribution is then, using Eqs. (4.28) and (4.29),

$$\mathcal{F}^{(2)} = \mathcal{F}^{(1)}_{\text{fin},\pi_\pm} - \mu^{-2\epsilon}\frac{1}{2}\frac{1}{(4\pi)^2}\left[\left(\frac{1}{\epsilon}+\frac{3}{2}+\ln\frac{\tilde{\mu}^2}{m_3^2}\right)m_3^4 + \frac{1}{2}\left(\frac{1}{\epsilon}+\frac{3}{2}+\ln\frac{\tilde{\mu}^2}{\tilde{m}_2^2}\right)\tilde{m}_2^4\right] \tag{4.32}$$

4.2 Next-to-leading order and renormalization

We have now regularized the divergences, so they can be handled in a well-defined way. However, they are still there. To get rid of them, we need to use renormalization. The power counting scheme used when constructing the Effective Lagrangians ensures that all terms in $\mathcal{L}_{2n}$ is of order p^{2n} in the pion momenta.[2] The tree level free energy from $\mathcal{L}_{2n}$ is thus of order p^{2n}. The m-loop correction to the tree level result is then suppressed by p^{2m} [21, 5]. Our one-loop calculation using $\mathcal{L}_2$ therefore contains divergences of order p^4. Since $\mathcal{L}_4$ is, by construction, the most general possible Lagrangian at order p^4, it contains coupling constants which can be renormalized to absorb all these divergences.

The renormalized coupling constants in $\mathcal{L}_4$ have been calculated for $\mu_I = 0$ [21]. They are independent of μ_I, and we can therefore use them in this calculation. The renormalized coupling constants in the $\overline{\text{MS}}$-scheme are related to the bare couplings through

$$l_i = l_i^r - \mu^{-2\epsilon}\frac{1}{2}\frac{\gamma_i}{(4\pi)^2}\left(\frac{1}{\epsilon}+1\right), \qquad i \in \{1,...7\} \tag{4.33}$$

$$h_i = h_i^r - \mu^{-2\epsilon}\frac{1}{2}\frac{\delta_i}{(4\pi)^2}\left(\frac{1}{\epsilon}+1\right), \qquad i \in \{1,...3\}. \tag{4.34}$$

Here, γ_i and δ_i are numerical constants that are used to match the divergences. The relevant terms are[3]

$$\gamma_1 = \frac{1}{3}, \quad \gamma_2 = \frac{2}{3}, \quad \gamma_3 = -\frac{1}{2}, \quad \gamma_4 = 2, \tag{4.35}$$

$$\delta_1 = 2, \quad \delta_3 = 0. \tag{4.36}$$

The bare coupling constants, though infinite, are independent of our renormalization scale μ. From this we obtain the renormalization group equations for the running coupling constants,

$$\mu\frac{\mathrm{d}l_i^r}{\mathrm{d}\mu} = -\mu^{-2\epsilon}\frac{\gamma_i}{(4\pi)^2}, \quad \mu\frac{\mathrm{d}h_i^r}{\mathrm{d}\mu} = -\mu^{-2\epsilon}\frac{\delta_i}{(4\pi)^2}. \tag{4.37}$$

These have the general solutions

$$l_i^r = \frac{1}{2}\mu^{-2\epsilon}\frac{\gamma_i}{(4\pi)^2}\left(\bar{l}_i - \ln\frac{\tilde{\mu}^2}{M^2}\right), \quad h_i^r = \frac{1}{2}\mu^{-2\epsilon}\frac{\gamma_i}{(4\pi)^2}\left(\bar{h}_i - \ln\frac{\tilde{\mu}^2}{M^2}\right), \tag{4.38}$$

where $\bar{l}_i$ and $\bar{h}_i$ are the values of the coupling constants (times a constant) measured at the energy M. This only applies if the numerical constants γ_i/δ_i are non-zero. If they are zero, then the renormalized constant is not running, and instead equal to its measured value at all energies. The bare couplings are thus given by

$$l_i = \mu^{-2\epsilon}\frac{1}{2}\frac{\gamma_i}{(4\pi)^2}\left(\bar{l}_i - 1 - \frac{1}{\epsilon} - \ln\frac{\tilde{\mu}^2}{M^2}\right), \quad h_i = \mu^{-2\epsilon}\frac{1}{2}\frac{\delta_i}{(4\pi)^2}\left(\bar{h}_i - 1 - \frac{1}{\epsilon} - \ln\frac{\tilde{\mu}^2}{M^2}\right) \tag{4.39}$$

---

[2]Remember that the bare pion mass $\bar{m} = B_0(m_u + m_d)$ is considered to be of order p^2.
[3]Some authors [25, 29] instead use $h_1' = h_1 - l_4$, with a corresponding $\delta_1' = \delta_1 - \gamma_1 = 0$.

The next-to-leading contribution to the free energy at tree-level is $\mathcal{F}_4^0 = -\mathcal{L}_4^{(0)}$, which is given by Eq. (3.106). When substituting Eq. (4.39) into the bare couplings, we get

$$
\begin{aligned}
\mathcal{F}_4^{(0)} = {}& -(l_1 + l_2)\mu_I^4 \sin^4 \alpha - (l_3 + l_4)\bar{m}^4 \cos^2 \alpha - l_4 \bar{m}^2 {\mu_I}^2 \cos \alpha \sin^2 \alpha - (h_1 - l_4)\bar{m}^4 - h_3 \Delta m^4 \\
= {}& -\mu^{-2\epsilon} \frac{1}{2} \frac{1}{(4\pi)^2} \left[\frac{1}{3} \left(\bar{l}_1 + 2\bar{l}_2 - 3 \right) \mu_I^4 \sin^4 \alpha + \frac{1}{2} \left(-\bar{l}_3 + 4\bar{l}_4 - 3 \right) \bar{m}^4 \cos^2 \alpha \right. \\
& + 2 \left(\bar{l}_4 - 1 \right) \bar{m}^2 \mu_I^2 \cos \alpha \sin^2 \alpha + 2(\bar{l}_4 - \bar{h}_1)\bar{m}^4 + \bar{h}_3 \Delta m^4 \\
& \left. - \left(\frac{1}{\epsilon} + \ln \frac{\tilde{\mu}^2}{M^2} \right) \left(\mu_I^4 \sin^4 \alpha + \frac{3}{2} \bar{m}^4 \cos^2 \alpha + 2\bar{m}^2 \mu_I^2 \cos \alpha \sin^2 \alpha \right) \right]
\end{aligned}
$$

Notice that the term proportional to ϵ^{-1} cancel exactly with the divergent term from $\mathcal{F}^{(2)}$, as we expected. Adding all the contribution to the free energy density, and taking the limit $\epsilon \to 0$, we get the next-to-leading order free energy density,

$$
\begin{aligned}
\mathcal{F}_{\mathrm{NLO}} = {}& -f^2 \left(\bar{m}^2 \cos \alpha + \frac{1}{2}\mu_I^2 \sin^2 \alpha \right) + \mathcal{F}_{\mathrm{fin},\pi_\pm}^{(1)} - \frac{1}{2}\frac{1}{(4\pi)^2} \left[\frac{1}{3} \left(\bar{l}_1 + 2\bar{l}_2 + \frac{3}{2} + 3\ln \frac{M^2}{m_3^2} \right) \mu_I^4 \sin^4 \alpha \right. \\
& \left. + \frac{1}{2}\left(-\bar{l}_3 + 4\bar{l}_4 + \frac{3}{2} + 2\ln\frac{M^2}{m_3^2} + \ln\frac{M^2}{\tilde{m}_2^2} \right) \bar{m}^4 \cos^2 \alpha + 2\left(\bar{l}_4 - \frac{1}{2} + \ln\frac{M^2}{m_3^2} \right) \bar{m}^2 \mu_I^2 \cos \alpha \sin^2 \alpha \right]. \quad (4.40)
\end{aligned}
$$

We have dropped the terms proportional to $\bar{l}_4 - \bar{h}_1$ and $\bar{h}_3$, as they only add an unobservable constant value to the free energy.

Parameters and consistent expansion

The coupling constants are free parameters, and can therefore not be calculated from first principles, but must be measured. The values for the pion mass and pion decay constants are (HVORFOR?)

$$
m_\pi = 131 \, \mathrm{MeV}, \quad f_\pi = \frac{1}{\sqrt{2}} 128 \, \mathrm{MeV} = 90.5 \, \mathrm{MeV}. \quad (4.41)
$$

This is the physical mass, m_π, which is defined as the pole of the propagator and thus the zero of the inverse propagator,

$$
D^{-1}(p^2 = m_\pi^2) = 0, \quad (4.42)
$$

This relates it to the bare mass $\bar{m}$. We found, in Eq. (3.110), that $m_\pi^2 = m_3^2(\mu_I = 0) = \bar{m}^2$ to leading order. Similarly, $f_\pi = f$ to leading order. However, in any NLO results we need $\bar{m}^2$ and f to NLO. This is given by [21]

$$
m_\pi^2 = \bar{m}^2 + \frac{\bar{l}_3}{2(4\pi)^2} \frac{\bar{m}^4}{f^2}, \quad (4.43)
$$

$$
f_\pi^2 = f^2 + \frac{2\bar{l}_4}{(4\pi)^2} \frac{\bar{m}^2}{f^2} \quad (4.44)
$$

All results in this text are given in units of m_π.

Table 4.1: The measured values and corresponding uncertainties of the relevant coupling constants, measured at $M = m_\pi$. (ELLER $M = \bar{m}$??)

	value	uncertainty	source
$\bar{l}_1$	-0.4	±0.6	[30]
$\bar{l}_2$	4.3	±0.1	[30]
$\bar{l}_3$	2.9	±2.4	[21]
$\bar{l}_4$	4.4	±0.2	[30]

The values for the coupling constants used in this text are given in Table 4.1. The constants $\bar{l}_1$, $\bar{l}_2$ and $\bar{l}_4$ are estimated using data from $\pi\pi$-scattering [30], while $\bar{l}_3$ is estimated using three flavor chiral perturbation

theory [21]. These are the same values as those used in [25]. Together with Eqs. (4.43) and (4.44), the NLO results for the bare mass and decay constant are

$$\bar{m} = 1.01136 \, m_\pi = 132.5 \, \text{MeV}, \tag{4.45}$$

$$f = 0.64835 \, m_\pi = 85.9 \, \text{MeV}. \tag{4.46}$$

In section 4.1, we found a relationship between α and μ_I, using the lowest order estimate of $\mathcal{F}$, given in Eq. (4.4). To calculate any thermodynamic quantities to leading order, at the tree-level, we must use this result. When using higher order corrections to the free energy, we must make sure to calculate this and other relationship to the same order, consistently. As we have seen earlier, when replacing the action by $S[\varphi] \to g^{-1}S[\varphi]$, the L-loop contribution is proportional to g^{L-1}. In Weinberg's power counting scheme, when replace $p \to tp$, and $m \to t^2 m$, the nth term in the expansion is proportional to t^{2n}. We can therefore expand the free energy as

$$\mathcal{F} = t^2 g^{-1} \mathcal{F}_2^{(0)} + t^2 \mathcal{F}_2^{(1)} + t^4 g^{-1} \mathcal{F}_4^{(0)} + \dots \tag{4.47}$$

We consider terms where $k = L + n$ to be of the same order. This expansion can be written as

$$\mathcal{F} = \sum_{k=0}^{\infty} \sum_{n+L=k} t^{2n} g^{L-1} \mathcal{F}_{2n}^{(L)}. \tag{4.48}$$

If we now define

$$\tilde{\mathcal{F}}_k = \sum_{n+L=k} t^{2n} g^{L-1} \mathcal{F}_{2n}^{(L)}, \tag{4.49}$$

then scale $t \to \sqrt{s}\, t$ and $g \to sg$, where s is some real number, then $t^{2n} g^{L-1}$ scales as $s^{n+L-1} = s^{m-1}$. All expansions are now done in this new parameter s. The free energy expansion is

$$\mathcal{F} = s^{-1} \sum_{k=0}^{\infty} \tilde{\mathcal{F}}_k \, s^k. \tag{4.50}$$

The same expansion holds for all quantities we calculate from this theory. As argued earlier, α must minimize the free energy, and therefore satisfy

$$\frac{\partial \mathcal{F}}{\partial \alpha} = 0 \tag{4.51}$$

to all orders. We expand this solution in s,

$$\alpha = \alpha_0 + \alpha_1 s + \dots. \tag{4.52}$$

Combining this, we get

$$
\begin{aligned}
0 &= \frac{\partial}{\partial \alpha} \left[s^{-1} \tilde{\mathcal{F}}_0 + \tilde{\mathcal{F}}_1 + \mathcal{O}\left(s^1\right) \right] \Bigg|_{\alpha = \alpha_0 + \alpha_1 s + \mathcal{O}(s^2)} \\
&= s^{-1} \left[\tilde{\mathcal{F}}_0'(\alpha_0) + (\alpha' - \alpha_0) \tilde{\mathcal{F}}_0''(\alpha_0) + \mathcal{O}\left(s^2\right) \right] + \tilde{\mathcal{F}}_1'(\alpha_0) + \mathcal{O}\left(s^1\right) \\
&= s^{-1} \tilde{\mathcal{F}}_1'(\alpha_0) + s^0 \left[\alpha_1 \tilde{\mathcal{F}}_0''(\alpha_0) + \tilde{\mathcal{F}}_1'(\alpha_0) \right] + \mathcal{O}\left(s^1\right)
\end{aligned}
$$

Here, the prime indicates partial derivatives with respect to α. This equality has to hold term for term. After setting $s = g = t = 1$, we get

$$\frac{\partial \tilde{\mathcal{F}}_0}{\partial \alpha} \Bigg|_{\alpha = \alpha_0} = 0, \quad \tilde{\mathcal{F}}_0 = \mathcal{F}_2^{(0)},$$

which is what we have used as the leading-order result. The first correction is on this result is

$$\alpha_1 = -\frac{\tilde{\mathcal{F}}_1'(\alpha_0)}{\tilde{\mathcal{F}}_0''(\alpha_0)}, \quad \tilde{\mathcal{F}}_1 = \mathcal{F}_4^{(0)} + \mathcal{F}_2^{(1)}. \tag{4.53}$$
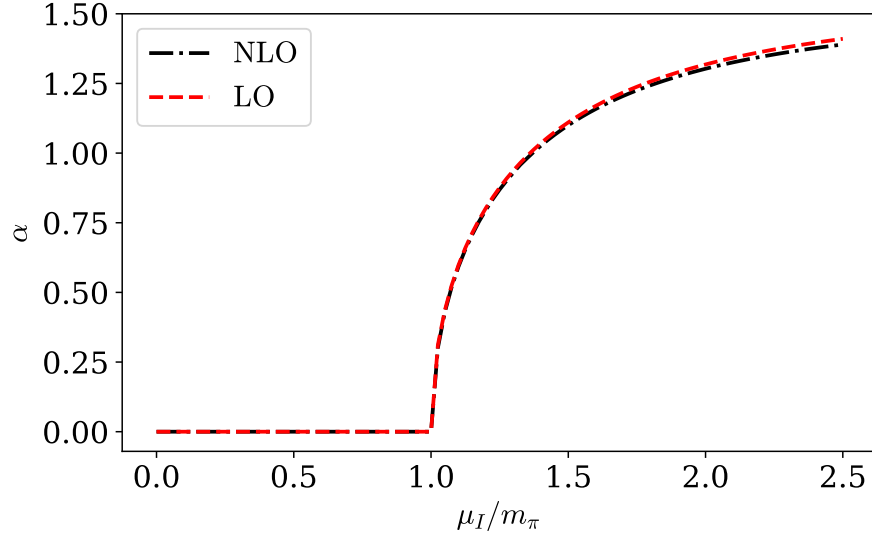
Figure 4.2: The leading order and next-to-leading order results for α as a function of μ_I, which is given in units of m_π.

The next to leading order results for the free energy and α are

$$\mathcal{F}_{\mathrm{NLO}} = \tilde{\mathcal{F}}_0 + \tilde{\mathcal{F}}_1, \quad \alpha_{\mathrm{NLO}} = \alpha_0 + \alpha_1. \tag{4.54}$$

If we use this to result to obtain an NLO-version of the equation for α, we get

$$\mathcal{F}'_{\mathrm{NLO}}(\alpha_{\mathrm{NLO}}) = \tilde{\mathcal{F}}'_0(\alpha_0) + \alpha_1 \tilde{\mathcal{F}}''_0(\alpha_0) + \cdots + \tilde{\mathcal{F}}'_1(\alpha_0) + \dots, \tag{4.55}$$

where the excluded terms are beyond next-to-leading order. Thus,

$$\left. \frac{\partial \mathcal{F}_{\mathrm{NLO}}}{\partial \alpha} \right|_{\alpha = \alpha_{\mathrm{NLO}}} = 0 \tag{4.56}$$

holds up to and including next-to-leading order. Figure 4.2, compares the leading order and next-to-leading order results for α. The NLO result was calculated numerically using Eq. (4.56).

4.3   Equation of state

The methods and analysis of this section is based on [11, 25, 31].

The free energy[4] is defined as

$$F = U - TS - \mu_I Q_I, \quad \mathrm{d}F = -S\mathrm{d}T - P\mathrm{d}V - Q_I \mathrm{d}\mu_I, \tag{4.57}$$

where Q_I is the isospin charge, and U is the energy. As we have seen earlier, our system is homogenous. This means that the free energy density is independent of volume, and thus $F = V\mathcal{F}$. From Eq. (4.57) we see that the pressure is given by

$$P = -\left(\frac{\partial F}{\partial V} \right)_{T,\mu_I} = -\mathcal{F}. \tag{4.58}$$

We are interested in the pressure relative to the state in which $\mu_I = 0$. We therefore normalize $P(\mu_I = 0) = 0$, which gives

$$P(\mu_I) = -[\mathcal{F}(\mu_I) - \mathcal{F}(\mu_I = 0)] \tag{4.59}$$

This is illustrated in Figure 4.3.

---

[4]As we are in the grand canonical ensemble, this is the grand canonical, or Landau, free energy, and not Helmholtz' free energy.
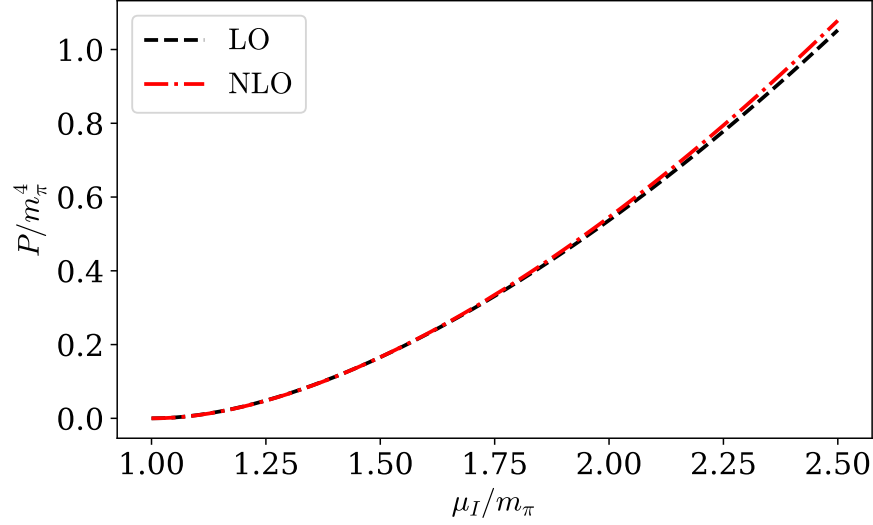
Figure 4.3: The NLO and LO result for the pressure, as a function of μ_I.

Likewise, the total isospin is proportional to volume, which means that the isospin density is

$$n_I = \frac{Q_I}{V} = -\frac{1}{V}\left(\frac{\partial F}{\partial \mu_I}\right)_{T,V} = -\frac{\partial \mathcal{F}}{\partial \mu_I}. \tag{4.60}$$

Using Eq. (4.40), this equals

$$n_I = f^2 \mu_I \sin^2 \alpha - \frac{\partial \mathcal{F}_{\text{fin}}}{\partial \mu_I}$$
$$+ \frac{1}{(4\pi)^2}\left[\left(2\bar{l}_4 + \ln\frac{M^2}{m_3^2}\right)\bar{m}^2 \mu_I \cos\alpha \sin^2\alpha + \frac{1}{3}\left(2\bar{l}_1 + 4\bar{l}_2 + 3\ln\frac{M^2}{m_3^2}\right)\mu_I^3 \sin^4\alpha\right] \tag{4.61}$$

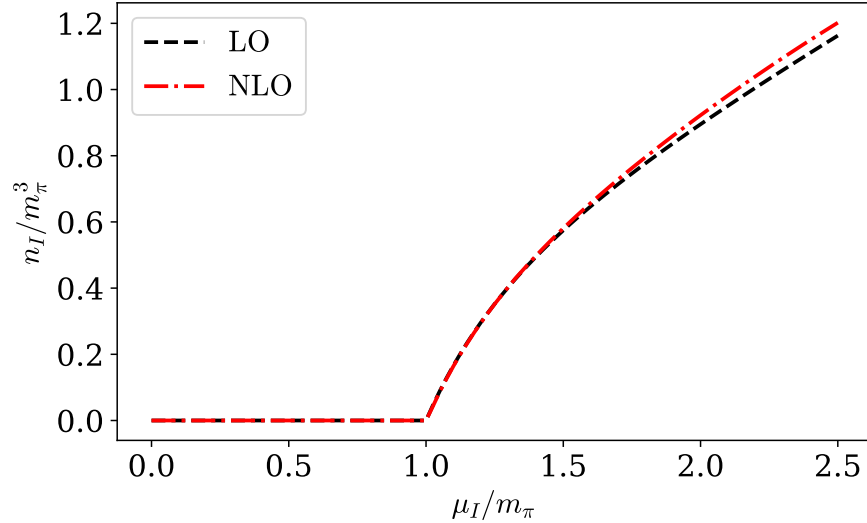The isospin density, as a function of μ_I, is shown in Figure 4.4.

Figure 4.4: The NLO and LO result for the isospin density, as a function of μ_I.

From Eq. (4.57) we get the energy density, $u = U/V$, at $T = 0$ is given by

$$u(\mu_I) = -P(\mu_I) + \mu_I n_I(\mu_I), \tag{4.62}$$

where we again have normalized so that $u(\mu_I = 0) = 0$. Now that we have both the dependence of pressure and energy density on the isospin chemical potential, we can trace out the line in the pressure-energy density plane, parametrized by μ_I. This is the equation of state of the system, and is shown in Figure 4.5.
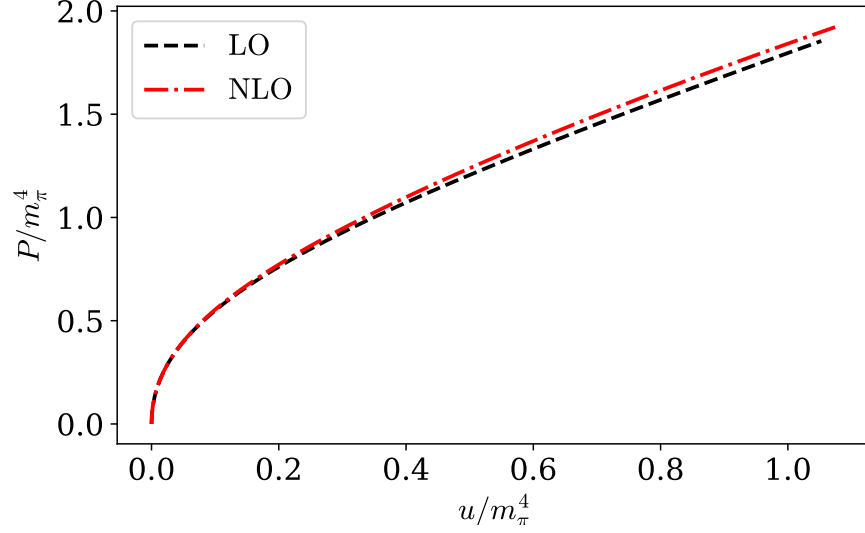

Figure 4.5: The leading and next-to-leading order equation of state. Both the pressure and energy density are given in units of m_π^4.

4.4 Phase transition

In our leading-order analysis, we saw that α was zero for $\mu_I \leq \bar{m}$, before it starts to increase for $\mu_I > \bar{m}$, and that $\bar{m} = m_\pi$. The behavior of α is illustrated in Figure 4.2. This is the hallmark of a phase transition, where α is the order parameter. The behavior systems near points of phase transition is described by Landau theory. Using Eq. (4.3), we can expand the leading-order free energy in α,

$$\mathcal{F} = -f^2\bar{m}^2 + f^2\frac{1}{2}(\bar{m}^2 - \mu_I^2)\alpha^2 - \frac{1}{24}f^2(\bar{m}^2 - 4\mu_I^2)\alpha^4 + \mathcal{O}\left(\alpha^5\right) \tag{4.63}$$

$$= \mathcal{F}(\alpha = 0) + a(\mu_I)\alpha^2 + \frac{1}{2}b(\mu_I)\alpha^4 + \mathcal{O}\left(\alpha^5\right), \tag{4.64}$$

Notice that near $\mu_I = \bar{m}$, $b > 0$. As earlier, the equation that governs α is

$$\frac{\partial \mathcal{F}}{\partial \alpha} = 2[a(\mu_I) + b(\mu_I)\alpha^2]\alpha = 0. \tag{4.65}$$

If $a > 0$, then $\alpha = 0$ will be the only solution, which gives us the criterion for a phase transition

$$a(\mu_I) = 0. \tag{4.66}$$

As expected, this criterion is fulfilled at $\mu_I = \bar{m}$. Near $\mu_I = \bar{m}$, we can write

$$a = -a_0(\mu_I - \bar{m}), \quad b = b_0, \tag{4.67}$$

where a_0 and b_0 are positive constants, so the solution to Eq. (4.65) for $\mu_I > \bar{m}$ is

$$\alpha(\mu_I) = \sqrt{\frac{a_0}{b_0}}(\mu_I - \bar{m})^{1/2}. \tag{4.68}$$

The free energy around the phase transition is illustrated in Figure 4.6.

The order parameter α changes continuously as the system undergoes phase-transition. This means we have a *second order* phase transition. The power law behavior, $\alpha \propto (\mu - \mu_C)^\beta$, is typical of systems near a phase
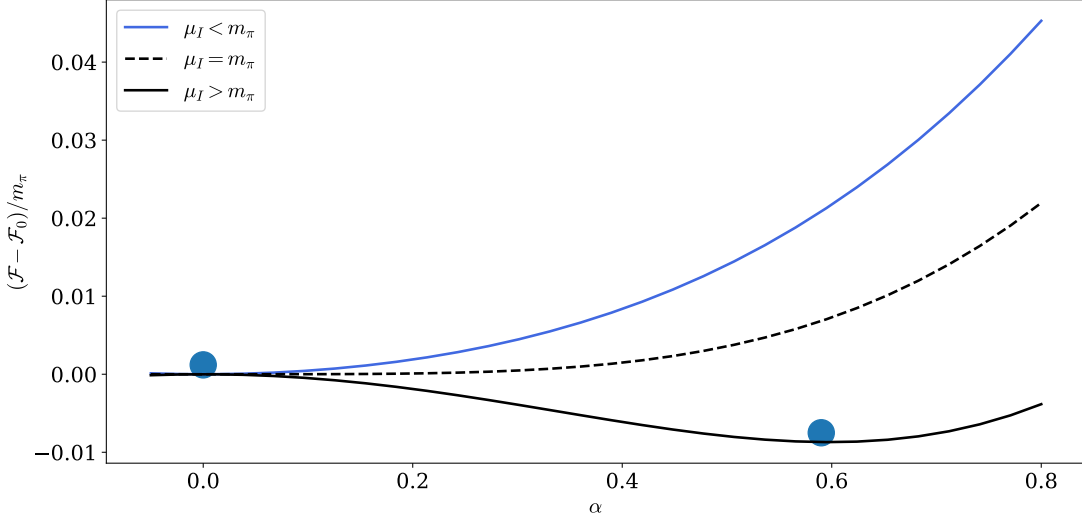
Figure 4.6: The plot shows normalized free energy density as a function of α, in the two different phases. Each line is a constant μ_I slice of the surface in Figure 4.1.

transition. The exponent β, which in this case equal $1/2$, is called a *critical exponent*. If we have a system where $b < 1$ near $\mu_I = \bar{m}$, then we must expand $\mathcal{F}$ further to show if the phase transition is continuous or not. **??** shows the free energy surface, but modified so that $b_0 < 0$, together with the corresponding value of α, which now changes discontinuously at the point of phase transition.

In the vacuum phase, $\alpha = 0$, the ground state is given by

$$\Sigma(\pi = 0) = \Sigma_0 = \mathbb{1}, \tag{4.69}$$

where we have use Eq. (3.74). Under $H = SU(2)_V$, Σ transforms as

$$\Sigma(x) \to \Sigma'(x) = V\Sigma(x)V^\dagger, \quad V = \exp\{i\theta_a\tau_a\}, \tag{4.70}$$

We see that the vacuum phase ground state is invariant under H. However, for $\alpha = 0$, the ground state is shifted to

$$\Sigma(\pi = 0) = A_\alpha \Sigma_0 A_\alpha = \exp\{i\alpha\tau_1\}. \tag{4.71}$$

This is not, in general, invariant under transformations in H, and generators τ_2 and τ_3 are broken. The new ground state corresponds to a condensate of the π_1-particle, as it is defined in the vacuum state. In Figure 3.1, we saw that the mass of the m_- particle vanishes, so we identify this particle with the corresponding Goldstone mode. There is only one Goldstone mode, however this is not a Lorentz invariant system, in which case we cannot guarantee one massless mode per broken generator. (ER DETTE RIKTIG?)

To find the value of μ_I to next-to-leading order, we must expand the NLO free energy in powers of α When we expand the static, second order Lagrangian to α^2, we get

$$\mathcal{F}_4^{(0)} = -(l_3 + l_4)\bar{m}^4 + [(l_3 + l_4)\bar{m}^4 - l_4\bar{m}^2\mu_I^2]\alpha^2 \tag{4.72}$$

$$= \text{const.} + \frac{\mu^{-2\epsilon}}{(4\pi)^2}\left[\left(\bar{l}_4 - \frac{1}{4}\bar{l}_3\right)\bar{m}^4 - \bar{l}_4\bar{m}^2\mu_I^2 - \left(1 + \frac{1}{\epsilon} + \ln\frac{\tilde{\mu}^2}{M^2}\right)\left(\frac{3}{4}\bar{m}^2 - \mu_I^2\right)\bar{m}^2\right]\alpha^2, \tag{4.73}$$

where const. is independent of α, and thus not of interest. From the one-loop correction, we have the contributions

$$\mathcal{F}_2^{(1)} = i\frac{1}{2}\int\frac{\mathrm{d}^4p}{(2\pi)^2}\ln(-p^2 + m_3^2) + i\frac{1}{2}\int\frac{\mathrm{d}^4p}{(2\pi)^2}\ln[(-p^2 + m_1^2)(-p^2 + m_2^2) - p_0^2m_{12}^2] \tag{4.74}$$

The first integral is the same free energy contribution from the π_0-particle as we have calculated earlier in Eq. (4.18), and it reads

$$\mathcal{F}_{\pi_0}^{(1)} = i\frac{1}{2}\int\frac{\mathrm{d}^4p}{(2\pi)^2}\ln(-p^2 + m_3^2) = -\mu^{-2\epsilon}\frac{1}{4}\frac{m_3^4}{(4\pi)^2}\left(\frac{1}{\epsilon} + \frac{3}{2} + \ln\frac{\tilde{\mu}^2}{m_3^2}\right). \tag{4.75}$$
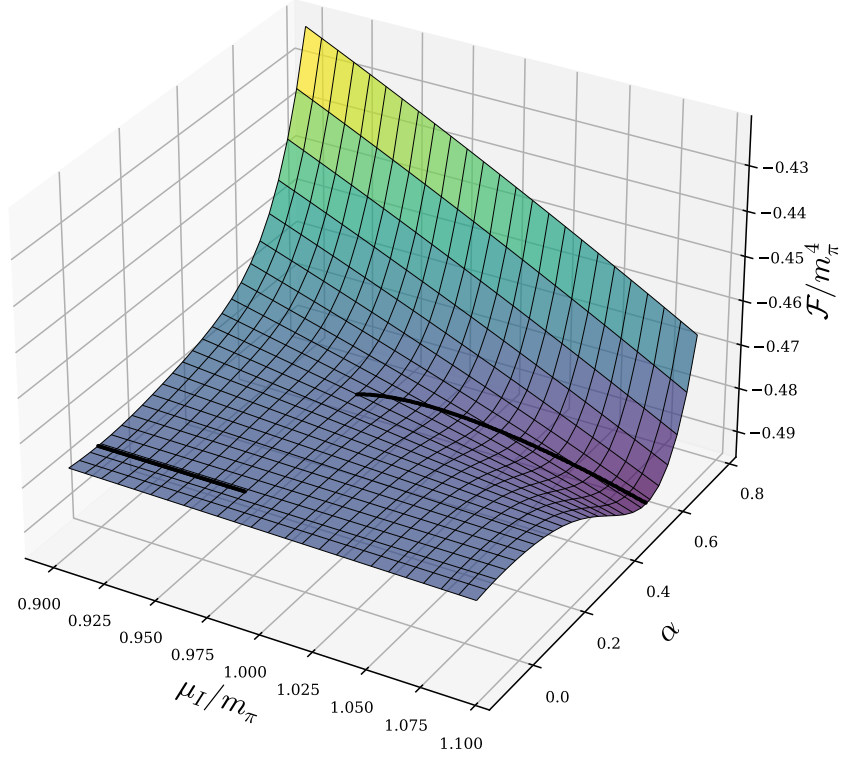
Figure 4.7: The surface illustrates the free energy, only modified so that $b_0 < 0$. The black line traces out the minimum for each value of μ_I, which jumps discontinuously at the point of phase transition.

The mass m_3 is dependent on α, and has the expansion

$$m_3^4 = \bar{m}^4 + \bar{m}^2(2\mu_I^2 - \bar{m}^2)\alpha^2 + \mathcal{O}\left(\alpha^3\right),$$

$$\ln \frac{\mu^2}{m_3^2} = \ln \frac{\mu^2}{\bar{m}_3^2} - \frac{1}{2}\frac{(2\mu_I^2 - \bar{m}^2)}{\bar{m}^2} + \mathcal{O}\left(\alpha^3\right).$$

In the second integral, we rewrite the argument of the logarithm as

$$(-p^2 + m_1^2)(-p^2 + m_2^2) - p_0^2 m_{12}^2 = \left[-p^2 + \frac{1}{2}(m_1^2 + m_2^2)\right]^2 - p_0^2 m_{12}^2 - \frac{1}{4}(m_1^2 - m_2^2)^2. \tag{4.76}$$

When we calculate the α dependence of the last term, we get $(m_1^2 - m_2^2)^2 = \mu^4 \sin^4 \alpha = \mathcal{O}\left(\alpha^4\right)$, which means that for our purposes, we can ignore this term. We further rewrite the remaining expression by factoring it,

$$\left[-p^2 + \frac{1}{2}(m_1^2 + m_2^2)\right]^2 - p_0^2 m_{12}^2 = \left[-p^2 + \frac{1}{2}(m_1^2 + m_2^2) - p_0 m_{12}\right]\left[-p^2 + \frac{1}{2}(m_1^2 + m_2^2) + p_0 m_{12}\right] \tag{4.77}$$

We then complete the square in each of the factors,

$$-p^2 + \frac{1}{2}(m_1^2 + m_2^2) \pm p_0 m_{12} = -\left(p_0 \mp \frac{1}{2}m_{12}\right)^2 + |\vec{p}|^2 + m_4^2, \tag{4.78}$$

where

$$m_4^2 = \frac{1}{2}\left(m_1^2 + m_2^2 + \frac{1}{2}m_{12}\right) = \bar{m}^2 \cos\alpha + \frac{1}{2}\mu_I^2 \sin^2\alpha, \tag{4.79}$$

$$m_4^4 = \bar{m}^4 - \bar{m}^2(m^2 + \mu_I^2)\alpha^2 + \mathcal{O}\left(\alpha^3\right), \tag{4.80}$$

$$\ln \frac{\mu^2}{m_4^2} = \ln \frac{\mu_I^2}{\bar{m}^2} + \frac{1}{2}\frac{\bar{m}^2 + \mu_I^2}{\bar{m}^2}\alpha^2 + \mathcal{O}\left(\alpha^3\right). \tag{4.81}$$

After a shift of variables, the integral has the same form as the logarithmic integrals we have calculated earlier, which gives us the result

$$\mathcal{F}_{\pi_\pm} = i \int \frac{d^4}{(2\pi)^4} \ln\left(-p^2 + m_4^2\right) = -\mu^{-2\epsilon} \frac{1}{2} \frac{m_4^4}{(4\pi)^2} \left(\frac{1}{\epsilon} + \frac{3}{2} + \ln\frac{\tilde{\mu}^2}{m_4^2}\right) \tag{4.82}$$

Combining these two contributions to the one-loop correction of the free energy gives

$$\mathcal{F}_2^{(1)} = \text{const.} + \frac{\mu^{-2\epsilon}}{(4\pi)^2}\left(1 + \frac{1}{\epsilon} + \ln\frac{\tilde{\mu}^2}{m^2}\right)\left(\frac{3}{4}m^2 - \mu_I^2\right)\bar{m}^2\alpha^2$$

We see that again, the ϵ^{-1} will cancel when we combine the NLO-terms. The NLO correction to the free energy becomes

$$\tilde{\mathcal{F}}_1 = \text{const.} + \frac{1}{(4\pi)^2}\left[\left(\bar{l}_4 - \frac{1}{4}\bar{l}_3\right)\bar{m}^4 - \bar{l}_4\bar{m}^2\mu_I^2 + \ln\frac{M^2}{\bar{m}^2}\left(\frac{3}{4}\bar{m}^2 - \mu_I^2\right)\bar{m}^2\right]\alpha^2 + \mathcal{O}\left(\alpha^3\right). \tag{4.83}$$

All coupling constants are measured at $M = m_\pi$. Using this in the logarithm gives,

$$\ln\frac{M^2}{\bar{m}^2} = \ln\frac{m_\pi^2}{\bar{m}^2} = \ln\frac{\bar{m}^2 + \mathcal{O}\left(\bar{m}^4\right)}{\bar{m}^2} = \mathcal{O}\left(\bar{m}^2\right). \tag{4.84}$$

The term proportional to the logarithm thus vanish to next-to-leading order. Combining these expressions give total NLO free energy, up to second order in α, is

$$\mathcal{F}_{\text{NLO}} = \mathcal{F}_{\text{NLO}}(\alpha = 0) + \frac{1}{2}f^2\bar{m}^2\left(1 - \frac{1}{2}\frac{\bar{l}_3 - 4\bar{l}_4}{(4\pi)^2}\frac{\bar{m}^2}{f^2}\right)\alpha^2 - \frac{1}{2}f^2\mu_I^2\left(1 + \frac{2\bar{l}_4}{(4\pi)^2}\frac{\bar{m}^2}{f^2}\right)\alpha^2 + \mathcal{O}\left(\alpha^3\right). \tag{4.85}$$

We now insert the physical constants f_π and m_π, by using the next-to-leading order expressions Eq. (4.43) and Eq. (4.44). First, notice that

$$f_\pi^2 m_\pi^2 = f^2\bar{m}^2\left(1 - \frac{1}{2}\frac{\bar{l}_3 - 4\bar{l}_4}{(4\pi)^2}\frac{\bar{m}^2}{f^2} + \mathcal{O}\left(\frac{\bar{m}^4}{f^4}\right)\right), \tag{4.86}$$

which means that the NLO free energy has the same structure as the leading order expression,

$$\mathcal{F}_{\text{NLO}} = \mathcal{F}_{\text{NLO}}(\alpha = 0) + \frac{1}{2}f_\pi^2(m_\pi^2 - \mu_I^2)\alpha^2 + \mathcal{O}\left(\alpha^4\right). \tag{4.87}$$

This shows that the phase transition happens at $\mu_I = m_\pi$, also at next-to-leading order.

Chapter 5

Conclusion and outlook

We conclude this conclusion by an outlook.

Appendix A

Thermal Field Theory

This section is based on [32, 33].

A.1 Statistical Mechanics

In classical mechanics, a thermal system at temperature $T = 1/\beta$ is described as an ensemble state, which have a probability P_n of being in state n, with energy E_n. In the canonical ensemble, the probability is proportional to $e^{-\beta E_n}$. The expectation value of some quantity A, with value A_n in state n is

$$\langle A \rangle = \sum_n A_n P_n = \frac{1}{Z} \sum_n A_n e^{-\beta E_n}, \quad Z = \sum_n e^{-\beta A_n}.$$

Z is the partition function. In quantum mechanics, an ensemble configuration is described by a non-pure density operator,

$$\hat{\rho} = C \sum_n P_n |n\rangle\langle n|,$$

where $|n\rangle$ is some basis for the relevant Hilbert space and C is a constant. Assuming $|n\rangle$ are energy eigenvectors, i.e. $\hat{H} |n\rangle = E_n |n\rangle$, the density operator for the canonical ensemble is

$$\hat{\rho} = C \sum_n e^{-\beta E_n} |n\rangle\langle n| = C e^{-\beta \hat{H}} \sum_n |n\rangle\langle n| = C e^{-\beta \hat{H}}.$$

The expectation value in the ensemble state of a quantity corresponding to the operator $\hat{A}$ is given by

$$\langle A \rangle = \frac{\text{Tr}\{\hat{\rho}\hat{A}\}}{\text{Tr}\{\hat{\rho}\}} = \frac{1}{Z} \text{Tr}\{\hat{A} e^{-\beta \hat{H}}\} \tag{A.1}$$

The partition function Z ensures that the probabilities adds up to 1, and is defined as

$$Z = \text{Tr}\{e^{-\beta \hat{H}}\}. \tag{A.2}$$

(REFFERER TIL THEROY-DELEN OM SYMMETRI; DETTE ER DOBBELT OPP) The grand canonical ensemble takes into account the conserved charges of the system. Conserved charges are a result of Nöther's theorem. Assume we have a set of fields φ_α. Nöther's theorem tells us that if the Lagrangian $\mathcal{L}[\varphi_\alpha]$ has a *continuous symmetry*, then there is a corresponding conserved current [11, 8]. To define a continuous symmetry of the Lagrangian, we need a one-parameter family of transformations,

$$\varphi_\alpha(x) \longrightarrow \varphi'_\alpha(x;\epsilon) \sim \varphi_\alpha(x) + \varepsilon\eta_\alpha(x), \; \varepsilon \to 0.$$

Here, $\eta_\alpha(x)$ is some arbitrary function which define the transformation as $\varepsilon \to 0$. Applying this transformation to the Lagrangian will in general change its form,

$$\mathcal{L}[\varphi_\alpha] \to \mathcal{L}[\varphi'_\alpha] \sim \mathcal{L}[\varphi_\alpha] + \delta\mathcal{L}, \ \varepsilon \to 0.$$

If the change in the Lagrangian can be written as a total derivative, i.e.

$$\delta\mathcal{L} = \varepsilon\partial_\mu K^\mu(x),$$

we say that the Lagrangian has a continuous symmetry. This is because a term of this form will result in a boundary term in the action integral, which does not contribute to the variation of the action. Nöther's theorem states more precisely that the current

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_\alpha)}\eta_\alpha - K^\mu \tag{A.3}$$

obeys the conservation law

$$\partial_\mu j^\mu = 0. \tag{A.4}$$

The flux of current through some space-like surface V, i.e. a surface with a time-like normal vector, defines a conserved charge. This surface is most commonly a surface of constant time in some reference frame. The conserved charge is then

$$Q = \int_V \mathrm{d}^3 x\, j^0.$$

Using the divergence theorem again, and assuming the current falls of quickly enough towards infinity, we can show that the total charge is conserved,

$$\frac{\partial}{\partial t}Q = -\int_V \mathrm{d}^3 x\, \nabla \cdot \vec{j} = -\int_{\partial V} \mathrm{d}^2 x\, n_\mu j^\mu = 0.$$

Here, n^μ is the time-like normal vector of the surface of V, ∂V.

In the grand canonical ensemble, a system with n conserved charges Q_i has probability proportional to $e^{-\beta(H-\mu_i Q_i)}$. μ_i are the chemical potentials corresponding to conserved charge Q_i. This leads to the partition function

$$Z = \mathrm{Tr}\left\{e^{-\beta(\hat{H}-\mu_i\hat{Q}_i)}\right\}. \tag{A.5}$$

A.2   Imaginary-time formalism

The partition function may be calculated in a similar way to the path integral approach, in what is called the imaginary-time formalism. This formalism is restricted to time independent problems, and is used to study fields in a volume V. This volume is taken to infinity in the thermodynamic limit. As an example, take a scalar quantum field theory with the Hamiltonian

$$\hat{H} = \int_V \mathrm{d}^3 x\, \hat{\mathcal{H}}\left[\hat{\varphi}(\vec{x}), \hat{\pi}(\vec{x})\right], \tag{A.6}$$

where $\hat{\varphi}(\vec{x})$ is the field operator, and $\hat{\pi}(\vec{x})$ is the corresponding canonical momentum operator. These field operators have time independent eigenvectors, $|\varphi\rangle$ and $|\pi\rangle$, defined by

$$\hat{\varphi}(\vec{x})\left|\varphi\right\rangle = \varphi(\vec{x})\left|\varphi\right\rangle, \quad \hat{\pi}(\vec{x})\left|\pi\right\rangle = \pi(\vec{x})\left|\pi\right\rangle. \tag{A.7}$$

In analogy with regular quantum mechanics, they obey the relations [1]

$$\mathbb{1} = \int \mathcal{D}\varphi(\vec{x})\left|\varphi\right\rangle\!\left\langle\varphi\right| = \int \mathcal{D}\pi(\vec{x})\left|\pi\right\rangle\!\left\langle\pi\right|, \tag{A.8}$$

$$\langle\varphi|\pi\rangle = \exp\left(i\int_V \mathrm{d}^3 x\, \varphi(\vec{x})\pi(\vec{x})\right), \tag{A.9}$$

$$\langle\pi_a|\pi_b\rangle = \delta(\phi_a - \phi_b), \quad \langle\varphi_a|\varphi_b\rangle = \delta(\varphi_a - \varphi_b). \tag{A.10}$$

---

[1]Some authors write $\mathcal{D}\pi/2\pi$. This extra factor 2π is a convention which in this text is left out for notational clarity.

The functional integral is defined by starting with M degrees of freedom, $\{\varphi_m\}_{m=1}^M$ located at a finite grid $\{\vec{x}_m\}_{m=1}^M \subset V$. The integral is then the limit of the integral over all degrees of freedom, as $M \to \infty$:

$$\int \mathcal{D}\varphi(\vec{x}) = \lim_{M\to\infty} \int \left(\prod_{m=1}^M \mathrm{d}\varphi_m \right).$$

The functional Dirac-delta $\delta(f) = \prod_x \delta(f(x))$ is generalization of the familiar Dirac delta function. Given a functional $\mathcal{F}[f]$, it is defined by the relation

$$\int \mathcal{D}f(x)\, \mathcal{F}[f]\delta(f-g) = \mathcal{F}[g]. \tag{A.11}$$

The Hamiltonian is the limit of a sum of Hamiltonians $\hat{H}_m$ for each point $\vec{x}_m$

$$\hat{H} = \lim_{M\to\infty} \sum_{m=1}^M \frac{V}{M} \hat{H}_m(\{\hat{\varphi}_m\}, \{\hat{\pi}_m\}).$$

H_m may depend on the local degrees of freedom $\hat{\varphi}_m$, $\hat{\pi}_m$ as well as those at neighboring points. By inserting the completeness relations Eq. (A.8) N times into the definition of the partition function, it may be written as

$$Z = \int \mathcal{D}\varphi(\vec{x}) \left\langle \varphi \middle| e^{-\beta\hat{H}} \middle| \varphi \right\rangle = \prod_{n=1}^N \left(\int \mathcal{D}\varphi_n(\vec{x}) \int \mathcal{D}\pi_n(\vec{x}) \right) \prod_{n=1}^N \langle\varphi_n|\pi_n\rangle \left\langle \pi_n \middle| e^{-\epsilon\hat{H}} \middle| \varphi_{n+1} \right\rangle \langle\varphi_1|\varphi_{N+1}\rangle,$$

where $\epsilon = \beta/N$. The last term ensures that $\varphi_1 = \varphi_{N+1}$. Strictly speaking, we only need to require $\varphi_1 = e^{i\theta}\varphi_{N+1}$, as the partition function is only defined up to a constant. As will be shown later, bosons such as the scalar field φ, follow the periodic boundary condition $\varphi(0, \vec{x}) = \varphi(\beta, \vec{x})$, i.e. $e^{i\theta} = 1$, while fermions follow the anti-periodic boundary condition $\psi(0, \vec{x}) = -\psi(\beta, \vec{x})$, i.e. $e^{i\theta} = -1$. We now want to exploit the fact that $|\pi\rangle$ and $|\varphi\rangle$ are the eigenvectors of the operators that define the Hamiltonian. In our case, as the Hamiltonian density $\mathcal{H}$ can be written as a sum of functions of φ and π separately, $\mathcal{H}[\varphi(\vec{x}), \pi(\vec{x})] = \mathcal{F}_1[\varphi(\vec{x})] + \mathcal{F}_2[\pi(\vec{x})]$ we may evaluate it as $\langle\pi_n|\mathcal{H}[\hat{\varphi}(\vec{x}), \hat{\pi}(\vec{x})]|\varphi_{n+1}\rangle = \mathcal{H}[\varphi_{n+1}(\vec{x}), \pi_n(\vec{x})] \langle\pi_n|\varphi_{n+1}\rangle$. This relationship does not, however, hold for more general functions of the field operators. In that case, one has to be more careful about the ordering of the operators, for example by using *Weyl ordering* [11]. By series expanding $e^{-\epsilon\hat{H}}$ and exploiting this relationship, the partition function can be written as, to second order in ϵ,

$$Z = \prod_{n=1}^N \left(\int \mathcal{D}\varphi_n(\vec{x}) \int \mathcal{D}\pi_n(\vec{x}) \right) \exp\left[-\epsilon \sum_{n=1}^N \int_V \mathrm{d}^3x \left(\mathcal{H}[\varphi_n(\vec{x}), \pi_n(\vec{x})] - i\pi_n(\vec{x}) \frac{\varphi_n(\vec{x}) - \varphi_{n+1}(\vec{x})}{\epsilon} \right) \right].$$

We denote $\varphi_n(\vec{x}) = \varphi(\tau_n, \vec{x})$, $\tau \in [0, \beta]$ and likewise with $\pi_n(\vec{x})$. In the limit $N \to \infty$, the expression for the partition function becomes

$$Z = \int_S \mathcal{D}\varphi(\tau, \vec{x}) \int \mathcal{D}\pi(\tau, \vec{x}) \exp\left\{ -\int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}\vec{x} \left\{ \mathcal{H}[\varphi(\tau, \vec{x}), \pi(\tau, \vec{x})] - i\pi(\tau, \vec{x})\dot{\varphi}(\tau, \vec{x}) \right\} \right\}, \tag{A.12}$$

where S is the set of field configurations φ such that $\varphi(\beta, \vec{x}) = \varphi(0, \vec{x})$. With a Hamiltonian density of the form $\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\varphi)^2 + \mathcal{V}(\varphi)$, we can evaluate the integral over the canonical momentum π by discretizing $\pi(\tau_n, \vec{x}_m) = \pi_{n,m}$,

$$\int \mathcal{D}\pi \exp\left\{ -\int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}^3x \left(\frac{1}{2}\pi^2 - i\pi\dot{\varphi} \right) \right\}$$

$$= \lim_{M,N\to\infty} \int \left(\prod_{m,n=1}^{M,N} \frac{\mathrm{d}\pi_{m,n}}{2\pi} \right) \exp\left\{ -\sum_{m,n} \frac{V\beta}{MN} \left[\frac{1}{2}(\pi_{m,n} - i\dot{\varphi}_{m,n})^2 + \frac{1}{2}\dot{\varphi}_{m,n}^2 \right] \right\}$$

$$= \lim_{M,N\to\infty} \left(\frac{MN}{2\pi V\beta} \right)^{MN/2} \exp\left\{ -\int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}^3x \frac{1}{2}\dot{\varphi}^2 \right\},$$

where $\dot{\varphi}_{m,n} = (\varphi_{m,n+1} - \varphi_{m,n})/\epsilon$. The partition function is then,

$$Z = C \int \mathcal{D}\varphi \exp\left\{ -\int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}^3x \left[\frac{1}{2}\left(\dot{\varphi}^2 + \nabla\varphi^2\right) + \mathcal{V}(\varphi) \right] \right\}. \tag{A.13}$$

Here, C is the divergent constant that results from the π-integral. In the last line, we exploited the fact that the variable of integration $\pi_{n,m}$ may be shifted by a constant without changing the integral, and used the Gaussian integral

$$\int_{-\infty}^\infty \mathrm{d}x\, e^{-ax^2/2} = \sqrt{\frac{2\pi}{a}}.$$

The partition function resulting from this procedure may also be obtained by starting with the ground state path integral

$$Z_g = \int \mathcal{D}\varphi \mathcal{D}\pi \exp\left\{ i \int_{\Omega'} \mathrm{d}^4x\, (\pi\dot{\varphi} - \mathcal{H}[\varphi,\pi]) \right\} = C' \int \mathcal{D}\varphi(x) \exp\left\{ i \int_{\Omega'} \mathrm{d}^4x\, \mathcal{L}[\varphi, \partial_\mu\varphi] \right\},$$

and follow a formal procedure. First, the action integral is modified by performing a Wick-rotation of the time coordinate t. This involves changing the domain of t from the real line to the imaginary line by closing the contour at infinity, and making the change of variable $it \to \tau$. The new variable is then restricted to the interval $\tau \in [0,\beta]$, and the domain of the functional integral $\int \mathcal{D}\varphi$ is restricted from *all* (smooth enough) field configurations $\varphi(t,\vec{x})$, to only those that obey $\varphi(\beta,\vec{x}) = e^{i\theta}\varphi(0,\vec{x})$, which is denoted S. This procedure motivates the introduction of the Euclidean Lagrange density, $\mathcal{L}_E(\tau,\vec{x}) = -\mathcal{L}(-i\tau,\vec{x})$, as well as the name "imaginary-time formalism". The result is the same partition function as before,

$$Z = C \int_S \mathcal{D}\varphi \int \mathcal{D}\pi \exp\left\{ -\int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}^3x\, [-i\dot{\varphi}\pi + \mathcal{H}(\varphi,\pi)] \right\}$$

$$= C' \int_S \mathcal{D}\varphi \exp\left\{ -\int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}^3x\, \mathcal{L}_E(\varphi,\pi) \right\}. \tag{A.14}$$

A.3   Free scalar field

This section uses notation as described in section B.1. The procedure for obtaining the thermal properties of an interacting scalar field is similar to that used in scattering theory. One starts with a free theory, which can be solved exactly. Then an interaction term is added, which is accounted for perturbatively by using Feynman diagrams. The Euclidean Lagrangian for a free scalar gas is, after integrating by parts,

$$\mathcal{L}_E = \frac{1}{2}\varphi(X)\left(-\partial_E^2 + m^2\right)\varphi(X) \tag{A.15}$$

Here, $X = (\tau,\vec{x})$ is the Euclidean coordinate which is a result of the Wick-rotation. We have also introduced the Euclidean Laplace operator, $\partial_E^2 = \partial_\tau^2 + \nabla^2$. Following the procedure as described in section A.2 to obtain the thermal partition function yields

$$Z = C \int_S \mathcal{D}\varphi(X) \exp\left\{ -\int_\Omega \mathrm{d}X \frac{1}{2}\varphi(X)\left(-\partial_E^2 + m^2\right)\varphi(X) \right\}. \tag{A.16}$$

Here, Ω is the domain $[0,\beta] \times V$. We then insert the Fourier expansion of φ, and change the functional integration variable to the Fourier components. The integration measures are related by

$$\mathcal{D}\varphi(X) = \det\left(\frac{\delta\varphi(X)}{\delta\tilde{\varphi}(K)}\right)\mathcal{D}\tilde{\varphi}(K),$$

where $K = (\omega_n, \vec{k})$ is the Euclidean Fourier-space coordinate. The determinant factor which appears may be absorbed into the constant C, as the integration variables are related by a linear transform. The action

becomes

$$S = -\int_\Omega \mathrm{d}X\, \mathcal{L}_e = -\frac{1}{2}V\beta \int_\Omega \mathrm{d}X \int_{\tilde\Omega} \mathrm{d}K \int_{\tilde\Omega} \mathrm{d}K'\, \tilde\varphi(K') \left(\omega_n^2 + \vec{k}^2 + m^2\right) \tilde\varphi(K) e^{iX\cdot(K-K')}$$

$$= -\frac{1}{2}V\beta^2 \int_{\tilde\Omega} \mathrm{d}K\, \tilde\varphi(K)^* \left(\omega_n^2 + \omega_k^2\right) \tilde\varphi(K),$$

where $\omega_k^2 = \vec{k}^2 + m^2$. $\tilde\Omega$ is the reciprocal space corresponding to Ω, as described in section B.1. We used the fact that φ is real, which implies that $\tilde\varphi(-K) = \tilde\varphi(K)^*$, as well as the identity Eq. (B.10). This gives the partition function

$$Z = C \int_{\tilde S} \mathcal{D}\tilde\varphi(K) \exp\left\{-\frac{1}{2}V \int_{\tilde\Omega} \mathrm{d}K\, \tilde\varphi(K)^* \left[\beta^2(\omega_n^2 + \omega_k^2)\right] \tilde\varphi(K)\right\}, \tag{A.17}$$

Going back to before the continuum limit, this integral can be written as a product of Gaussian integrals, and may therefore be evaluated

$$Z = C \prod_{n=-\infty}^{\infty} \prod_{k\in\tilde V} \left(\int \mathrm{d}\tilde\varphi_{n,\vec k}\, \exp\left\{-\frac{1}{2}\tilde\varphi_{n,\vec k}^* \left[\beta^2(\omega_n^2 + \omega_k^2)\right] \tilde\varphi_{n,\vec k}\right\}\right) = C \prod_{n=-\infty}^{\infty} \prod_{k\in\tilde V} \sqrt{\frac{2\pi}{\beta^2(\omega_n^2 + \omega_k^2)}}.$$

The partition function is related to Helmholtz free energy F through

$$\frac{F}{TV} = -\frac{\ln(Z)}{V} = \frac{1}{2}\int_{\tilde\Omega} \mathrm{d}K\frac{1}{2}\ln[\beta^2(\omega_n^2 + \omega_k^2)] + \frac{F_0}{TV}, \tag{A.18}$$

where F_0 is a constant.

A faster and more formal way to get to this result is to compare the partition function to the multidimensional version of the Gaussian integral [32, 11]. The partition function has the form

$$I_n = \int_{\mathbb{R}^n} \mathrm{d}^n x\, \exp\left\{-\frac{1}{2}\langle x, D_0^{-1}x\rangle\right\},$$

where D_0^{-1} is a linear operator, and $\langle\cdot,\cdot\rangle$ an inner product on the corresponding vector space. By diagonalizing D_0^{-1}, we get the result

$$I_n = \sqrt{\frac{(2\pi)^n}{\det(D_0^{-1})}}.$$

We may then use the identity

$$\det(D_0^{-1}) = \prod_i \lambda_i = \exp\left\{\mathrm{Tr}\left[\ln(D_0^{-1})\right]\right\}, \tag{A.19}$$

where λ_i are the eigenvalues of D_0^{-1}. The trace in this context is defined by the vector space D_0^{-1} acts on. For given an orthonormal basis x_n, such that $\langle x_n, x_n'\rangle = \delta_{nn'}$ the trace can be evaluated as $\mathrm{Tr}\{D_0^{-1}\} = \sum_n \langle x_n, D_0^{-1}x_n\rangle$. Identifying

$$\langle x, D_0^{-1}x\rangle = \int_\Omega \mathrm{d}X\varphi(X)\left(-\partial_E^2 + m^2\right)\varphi(X),$$

we get the formal result

$$Z = \det\left(-\partial_E^2 + m^2\right)^{-1/2},$$

and

$$\beta F = \frac{1}{2}\mathrm{Tr}\left\{\ln\left(-\partial_E^2 + m^2\right)\right\}.$$

The logarithm may then be evaluated by using the eigenvalues of the linear operator. This is found by diagonalizing the operator,

$$\langle x, D_0^{-1}x\rangle = \int_\Omega \mathrm{d}X\varphi(X)\left(-\partial_E^2 + m^2\right)\varphi(X) = V\int_{\tilde\Omega} \mathrm{d}K\tilde\varphi(K)^*[\beta^2(\omega_k^2 + \omega_n^2)]\tilde\varphi(K),$$

leaving us with the same result as we obtained in Eq. (A.18),

$$\beta F = \frac{1}{2}\mathrm{Tr}\left\{\ln\left(-\partial_E^2 + m^2\right)\right\} = \frac{1}{2}V\int_{\tilde\Omega} \mathrm{d}K\ln[\beta^2(\omega_n^2 + \omega_k^2)].$$

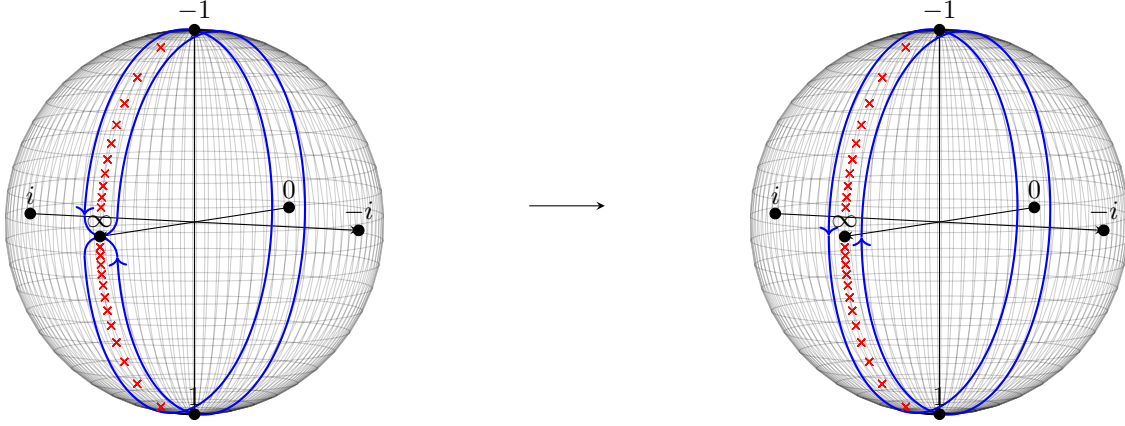Sums similar to this show up a lot, and it is show how to evaluate them in the next section.

Figure A.1: The integral contour γ, and the result of deforming it into to contours close to the real line. The red crosses illustrate the poles of n_B.

Thermal sum

When evaluating thermal integral, we will often encounter sums of the form

$$j(\omega, \mu) = \frac{1}{2\beta} \sum_{\omega_n} \ln\{\beta^2[(\omega_n + i\mu) + \omega^2]\} + g(\beta), \tag{A.20}$$

where the sum is over either the bosonic Matsubara frequencies $\omega_n = 2n\pi/\beta$, $n \in \mathbb{Z}$, or the fermionic ones, $\omega_n = (2n+1)\pi/\beta$, $n \in \mathbb{Z}$. $\mu \in \mathbb{R}$ is the chemical potential. g may be a function of β, but we assume it is independent of ω. Thus, the factor β^2 could strictly be dropped, but it is kept to make the argument within the logarithm dimensionless. We define the function

$$i(\omega, \mu) = \frac{1}{\omega} \frac{\mathrm{d}}{\mathrm{d}\omega} j(\omega, \mu) = \frac{1}{\beta} \sum_{\omega_n} \frac{1}{(\omega_n + i\mu)^2 + \omega^2}. \tag{A.21}$$

We will first work with the sum over bosonic Matsubara frequencies. Assume $f(z)$ is an analytic function, except perhaps on a set of isolated poles $\{z_i\}$ located outside the real line. By exploiting the properties of the Bose-distribution $n_B(z)$, as described in section B.1, we can rewrite the sum over Matsubara frequencies as a contour integral

$$\frac{1}{\beta} \sum_{\omega_n} f(\omega_n) = \oint_\gamma \frac{\mathrm{d}z}{2\pi i} f(z) i n_B(iz),$$

where γ is a contour that goes from $-\infty - i\epsilon$ to $+\infty - i\epsilon$, crosses the real line at ∞, goes from $+\infty - i\epsilon$ to $-\infty + i\epsilon$ before closing the curve. The contour γ, and the change of integral contours is illustrated in Figure A.1 This result exploits Cauchy's integral formula, by letting the poles of $i n_B(iz)$ at the Matsubara frequencies "pick out" the necessary residues. The integral over γ is equivalent to two integrals along $\mathbb{R} \pm i\epsilon$,

$$\frac{1}{\beta} \sum_{\omega_n} f(\omega_n) = \left(\int_{\infty+i\epsilon}^{-\infty+i\epsilon} \frac{\mathrm{d}z}{2\pi} + \int_{-\infty-i\epsilon}^{\infty-i\epsilon} \frac{\mathrm{d}z}{2\pi} \right) f(z) n_B(iz),$$

$$= \int_{-\infty-i\epsilon}^{\infty-i\epsilon} \frac{\mathrm{d}z}{2\pi} \{f(-z) + [f(z) + f(-z)] n_B(iz)\}$$

$$= \int_{-\infty}^{\infty} \frac{\mathrm{d}z}{2\pi} f(z) + \int_{-\infty-i\epsilon}^{\infty-i\epsilon} \frac{\mathrm{d}z}{2\pi} [f(z) - f(-z)] n_B(iz). \tag{A.22}$$

In the second line, we have changed variables $z \to -z$ in the first integral, and exploited the property $n_B(-iz) = -1 - n_B(iz)$. In the last line, we use the assumption that $f(z)$ is analytic on the real line, and therefore also in a neigbourhood of it. This allows us to shift the first integral back to the real line. As $n_B(iz)$

is analytic outside the real line, the result of the second integral is the sum of residues of $f(z) + f(-z)$ in the lower half plane. The function

$$f(z) = \frac{1}{(z+i\mu)^2 + \omega^2} = \frac{i}{2\omega}\left(\frac{1}{z+i(\mu+\omega)} - \frac{1}{z+i(\mu-\omega)}\right) \tag{A.23}$$

obeys the assumed properties, as it has poles at $z = -i(\mu \pm \omega)$, with residue $1/2\omega$, so the function defined in Eq. (A.21) may be written [2]

$$i(\omega, \mu) = \frac{1}{2\omega}[1 + n_B(\omega - \mu) + n_B(\omega + \mu)]. \tag{A.24}$$

Using the anti-derivative of the Bose distribution, we get the final form of Eq. (A.20)

$$j(\omega, \mu) = \int d\omega' \, \omega' i(\omega', \mu) = \frac{1}{2}\omega + \frac{1}{2\beta}\left[\ln\left(1 - e^{-\beta(\omega-\mu)}\right) + \ln\left(1 - e^{-\beta(\omega+\mu)}\right)\right] + g'(\beta). \tag{A.25}$$

The extra ω-independent term $g'(\beta)$ is an integration constant. We see there are temperature dependent terms, one due to the particle and one due to the anti-particle, and one due to the antiparticle, as they have opposite chemical potentials.

We now consider the sum over fermionic frequencies, which we for clarity denote $\tilde{\omega}_n$ in this chapter. The procedure in this case is the same, except that we have to use a function to with poles at the fermionic Matsubara frequencies. This is done by the Fermi distribution, $n_F(z)$, as described in section B.1. The result is

$$\frac{1}{\beta}\sum_{\tilde{\omega}_n} f(\tilde{\omega}_n) = -\int_{-\infty}^{\infty} \frac{dz}{2\pi} f(z) + \int_{-\infty-i\epsilon}^{\infty-i\epsilon} \frac{dz}{2\pi}\left[f(z) - f(-z)\right] n_F(iz), \tag{A.26}$$

and

$$i(\omega, \mu) = \frac{1}{2\omega}[-1 + n_F(\omega - \mu) + n_F(\omega + \mu)]. \tag{A.27}$$

Using the antiderivative of the Fermi-distribution, we get

$$j(\omega, \mu) = -\frac{1}{2}\omega - \frac{1}{2\beta}\left[\ln\left(1 + e^{-\beta(\omega-\mu)}\right) + \ln\left(1 + e^{-\beta(\omega+\mu)}\right)\right]. \tag{A.28}$$

Low temperature limit

Using the result from section A.3 on the result for the free energy density of the free scalar field, Eq. (A.14), we get

$$\mathcal{F} = \frac{\ln(Z)}{\beta V} = \frac{1}{2}\int_{\tilde{V}} \frac{d^3k}{(2\pi)^3}\left[\omega_k + \frac{2}{\beta}\ln\left(1 - e^{-\beta\omega_k}\right)\right]. \tag{A.29}$$

The free energy thus has two parts, the first part is dependent on temperature, the other is a temperature independent vacuum contribution. Noticing that the integral is spherically symmetric, we may write the two contributions as

$$\mathcal{F}_0 = \frac{1}{2}\frac{1}{2\pi^2}\int_{\mathbb{R}} dk \, k^2 \sqrt{k^2 + m^2}, \quad \mathcal{F}_T = \frac{T^4}{2\pi^2}\int_{\mathbb{R}} dx \, x^2 \ln\left(1 - e^{-\sqrt{x^2 + (m/T)^2}}\right), \tag{A.30}$$

The temperature-independent part, $\mathcal{F}_0$, is clearly divergent, and we must therefore impose a regulator, and then add counter-terms. $\mathcal{F}_T$, however, is convergent. To see this, we use the series expansion $\ln(1+\epsilon) \sim \epsilon + \mathcal{O}\left(\epsilon^2\right)$ to find the leading part of the integrand for large k's,

$$x^2 \ln\left(1 - e^{-\sqrt{x^2 + (\beta m)^2}}\right) \sim -x^2 e^{-x}, \tag{A.31}$$

which is exponentially suppressed, making the integral convergent. In the limit of $T \to \infty$, we get

$$\mathcal{F}_\infty \sim \frac{T^4}{2\pi^2}\int_{\mathbb{R}} dx \, x^2 \ln\left(1 - e^{-x}\right) = -\frac{T^4}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n}\frac{\partial^2}{\partial n^2}\int dx e^{-nx} = -\frac{T^4}{2\pi^2}\sum_{n=1}^{\infty}\frac{2}{n^4} = -\frac{T^4}{\pi^2}\zeta(4) = -T^4\frac{\pi^2}{90}. \tag{A.32}$$

Where ζ is the Riemann-zeta function.

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[2]Assuming $\omega > \mu$.

Regularization

Returning to the temperature-independent part, we use dimensional regularization to see its singular behavior. To that end, we define

$$\Phi_n(m, d, \alpha) = \int_{\tilde{\Omega}} \frac{\mathrm{d}^d k}{(2\pi)^d} (k^2 + m^2)^{-\alpha}, \tag{A.33}$$

so that $\mathcal{F}_0 = \Phi_3(m, 3, -1/2)/2$. Dimensional regularization takes uses the formula for integration of spherically symmetric function in d-dimensions,

$$\int_{\mathbb{R}^d} \mathrm{d}^d x \, f(r) = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_{\mathbb{R}} \mathrm{d}r \, r^{d-1} f(r), \tag{A.34}$$

where $r = \sqrt{x_i x_i}$ is the radial distance, and Γ is the Gamma function. The factor in the front of the integral comes from the solid angle. By extending this formula from integer-valued d to real numbers, the function we defined becomes

$$\Phi_n = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_{\mathbb{R}} \mathrm{d}k \, \frac{k^{d-1}}{(k^2 + m^2)^{\alpha}} = \frac{m^{n-2\alpha} m^{d-n}}{(4\pi)^{d/2} \Gamma(d/2)} 2 \int_{\mathbb{R}} \mathrm{d}z \, \frac{z^{d-1}}{(1+z)^{\alpha}}, \tag{A.35}$$

where we have made the change of variables $mz = k$. We make one more change of variable to the integral,

$$I = 2 \int_{\mathbb{R}} \mathrm{d}z \, \frac{z^{d-1}}{(1+z)^{\alpha}} \tag{A.36}$$

Let

$$z^2 = \frac{1}{s} - 1 \implies 2z\mathrm{d}z = -\frac{\mathrm{d}s}{s^2} \tag{A.37}$$

Thus,

$$I = \int_0^a \mathrm{d}s \, s^{\alpha - d/2 - 1}(1 - z)^{d/2 - 1}. \tag{A.38}$$

This is the beta function, which can be written in terms of Gamma functions [11]

$$I = B\left(\alpha - \frac{d}{2}, \frac{d}{2}\right) = \frac{\Gamma\left(\alpha - \frac{d}{2}\right) \Gamma\left(\frac{d}{2}\right)}{\Gamma(\alpha)}. \tag{A.39}$$

Combining this gives

$$\Phi_n(m, d, \alpha) = \mu^{n-d} \frac{m^{n-2\alpha}}{(4\pi)^{d/2}} \frac{\Gamma\left(\alpha - \frac{d}{2}\right)}{\Gamma(\alpha)} \left(\frac{m^2}{\mu^2}\right)^{(d-n)/2}. \tag{A.40}$$

In the last step, we have introduced a parameter μ with mass dimension 1, that is $[\mu] = [m]$. This is done to be able to series expand around $d - n$ in a dimensionless variable. This parameter is arbitrary, and should therefore not be present in any of the final results. We will shortly justify this parameter further.

Inserting $n = 3$, $d = 3 - 2\epsilon$ and $\alpha = -1/2$, we get

$$\Phi_3(m, 3 - 2\epsilon, -1/2) = \frac{m^4 \mu^{-2\epsilon}}{(4\pi)^{d/2} \Gamma(-1/2)} \Gamma(-2 + \epsilon) \left(\frac{m^2}{\mu^2}\right)^{-\epsilon} = -\mu^{-2\epsilon} \frac{m^4}{(4\pi)^2} \left(\frac{m^2}{4\pi\mu^2}\right)^{-\epsilon} \frac{\Gamma(\epsilon)}{(\epsilon - 2)(\epsilon - 1)}, \tag{A.41}$$

where we have used the defining property $\Gamma(z+1) = z\Gamma(z)$ and $\Gamma(1/2) = \sqrt{\pi}$. Expanding around $\epsilon = 0$ gives

$$\left(\frac{m^2}{4\pi\mu^2}\right)^{-\epsilon} \sim 1 + \epsilon \ln\left(4\pi \frac{\mu^2}{m^2}\right), \tag{A.42}$$

$$\Gamma(\epsilon) \sim \frac{1}{\epsilon} - \gamma, \tag{A.43}$$

$$\frac{1}{(\epsilon - 2)(\epsilon - 1)} \sim \frac{1}{2}\left(1 + \frac{3}{2}\epsilon\right). \tag{A.44}$$

The singular behavior of the time-independent term is therefore

$$\mathcal{F}_0 \sim -\mu^{-2\epsilon} \frac{1}{4} \frac{m^4}{(4\pi)^2} \left[\frac{1}{\epsilon} - \gamma + \frac{3}{2} + \ln\left(4\pi \frac{\mu^2}{m^2} \right) \right]. \tag{A.45}$$

With this regulator, one can then add counter-terms to cancel the ϵ^{-1}-divergence. The exact form of the counter-term is convention. One may also cancel the finite contribution due to the regulator. The minimal subtraction, or MS, scheme is to only subtract the divergent term, as the name suggest. We will use the modified minimal subtraction, or $\overline{\text{MS}}$, scheme. In this scheme, one also removes the $-\gamma$ and $\ln(4\pi)$ term, by defining a new mass parameter $\tilde{\mu}$ by

$$-\gamma + \ln\left(4\pi \frac{\mu^2}{m^2} \right) = \ln\left(4\pi e^{-\gamma} \frac{\mu^2}{m^2} \right) = \ln\left(\frac{\tilde{\mu}^2}{m^2} \right), \tag{A.46}$$

which leads to the expression

$$\mathcal{F}_0 \sim -\mu^{-2\epsilon} \frac{1}{4} \frac{m^4}{(4\pi)^2} \left(\frac{1}{\epsilon} + \frac{3}{2} + \ln\frac{\tilde{\mu}^2}{m^2} \right), \quad \epsilon \to 0. \tag{A.47}$$

Renormalization

Now that we have applied a regulator, we are able to handle the divergence in a well-defined way. When $\epsilon \neq 0$, we can subtract to terms which are proportional to ϵ^{-1}, and be left with a term that is finite in the limit $\epsilon \to 0$. Consider an arbitrary Lagrangian,

$$\mathcal{L}[\varphi] = \sum_n \lambda_n \mathcal{O}_n[\varphi]. \tag{A.48}$$

Here, $\mathcal{O}_n[\varphi]$ are operators consisting of φ and $\partial_\mu \varphi$, and λ_n are coupling constants. In d dimensions, the action integral is

$$S[\varphi] = \sum_n \int \mathrm{d}^d x \, \lambda_n \mathcal{O}_n[\varphi]. \tag{A.49}$$

The action must have mass dimension 0. This means that all terms $\lambda_n \mathcal{O}_n$ must have mass dimension d, as $[\mathrm{d}^d x] = -d$. We are free to choose the coupling constant corresponding $\mathcal{O}_0 = \partial_\mu \varphi \partial^\mu \varphi$ as $\lambda_0 = 1/2$ to get a canonically normalized field. This allows us to deduce the dimensionality of φ. As $[\partial_\mu] = 1$, we have that $[\varphi] = (d-2)/2$. Assume $\mathcal{O}_n$ consists of k_n factors of φ, and l_n factors of $\partial_\mu \varphi$. We must then have

$$[\lambda_n] + [\mathcal{O}_n] - d = [\lambda_n] + (k_n + l_n)(d-2)/2 + l_n - d = 0, \tag{A.50}$$

$$\implies D_n := [\lambda_n] = d - k_n \frac{d-2}{2} - l_n \frac{d}{2}. \tag{A.51}$$

From this formula, we recover that $[\lambda_0] = 0$, and if $\lambda_1 \varphi^2$, then $[\lambda_1] = 2$, which we recognize as the mass squared term. The mass dimensions of these coupling constants are independent of d. However, the coupling constant for the interaction term

$$-\frac{1}{4!} \lambda_3 \varphi^4 \tag{A.52}$$

has mass dimensions $[\lambda_3] = d - 4(d-2)/2 = 4 - 2d$. Our goal now is to exchange the bare coupling constants λ_n with renormalized ones, λ_n^r, and remove the divergent terms proportional to $(d-4)^{-m}$. We can always define the renormalized coupling constants as dimensionless, i.e. $[\lambda_n^r] = 0$, by measuring them in units of any mass scale. We therefore write

$$\lambda_n = \mu^{4-D_n} \left[\lambda_n^r + \sum_{m=1} \frac{a_m(\lambda_n^r)}{(d-4)^m} \cdot \right],$$

where we have introduced the dimensional parameter μ to ensure that λ_n has the correct mass dimension, so that the action integral stays dimensionless. The functions a_m are then determined to each order in perturbation theory by calculating Feynman diagrams. As μ again is arbitrary, λ_4' should not depend on this

parameter. In this case, we chose the same renormalization scale as we did when regulating the one-loop integral. This is only for our own convenience. This means that if we change $\mu \to \mu'$, then λ_i^r and a_m must adjust to compensate and keep λ_n constant [34].

The vacuum energy term absorbs the divergence in the one loop contribution to the free energy density. It is

$$\lambda_4 \mathcal{O}_4 = \lambda_4 = m^4 \lambda_4'. \tag{A.53}$$

Using the expansion in terms of renormalized coupling, we have, using $d = 4 - 2\epsilon$

$$\lambda_4' = \mu^{-2\epsilon} \left[\lambda_4^r + \frac{1}{2\epsilon} a_1 (\lambda_r^4) + ... \right] \tag{A.54}$$

After adding Eq. (A.53) to the Lagrangian of the free scalar, the temperature independent free energy density becomes

$$\mathcal{F}_0 \sim -\mu^{-2\epsilon} \frac{1}{4} \frac{m^4}{(4\pi)^2} \left[\frac{1}{\epsilon} + \frac{3}{2} + \ln \frac{\tilde{\mu}^2}{m^2} + (4(4\pi)^2) \left(\lambda_4^r + \frac{1}{2\epsilon} a_1(\lambda_4^r) \right) \right], \quad \epsilon \to 0. \tag{A.55}$$

Thus, if we choose $a_1 = -8(4\pi)^2 + \mathcal{O}((\lambda_4^r))$, and define $\lambda_4'^r = 4(4\pi)^2 \lambda_4^r$, we are able to cancel the divergence, and may take the limit $\epsilon \to 0$ safely. The free energy thus becomes

$$\mathcal{F} = -\frac{1}{4} \frac{m^4}{(4\pi)^2} \left(\frac{3}{2} + \lambda_4'^r + \ln \frac{\tilde{\mu}^2}{m^2} \right) + \frac{T^4}{2\pi^2} \int \mathrm{d}x \, x^2 \ln\left(1 - \exp\left\{ -\sqrt{x^2 + \beta^2 m^2} \right\} \right). \tag{A.56}$$

Notice that the choices we have made up until now, for example by defining $\lambda_4 = m^4 \lambda_4'$, and using the same renormalization scale μ has no impact on this result, as any other definition would just force us to define λ_4^r and a_4 differently.

A.4   Interacting scalar

We now study a scalar field with a $\lambda \varphi^4$ interaction term. We write the Lagrangian in the form

$$\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(I)}, \quad \mathcal{L}^{(0)} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - m^2 \varphi^2, \quad \mathcal{L}^{(I)} = -\frac{\lambda}{4!} \varphi^4$$

$\mathcal{L}^{(I)}$ is called the interaction term, and makes it impossible to exactly solve for the partition function. Instead, we turn to perturbation theory. The canonical partition function in this theory is

$$Z = \mathrm{Tr}\left\{ e^{-\beta \hat{H}} \right\} = \int_S \mathcal{D}\varphi \, \exp\left\{ -\int_\Omega \mathrm{d}X \left(\mathcal{L}_E^{(0)} + \mathcal{L}_E^{(I)} \right) \right\} = \int_S \mathcal{D}\varphi \, e^{-S_0} e^{-S_I}. \tag{A.57}$$

Here, S_0 and S_I denote the Euclidean action due to the free and interacting Lagrangian, respectively. The domain of integration S is again periodic field configurations $\varphi(\beta, \vec{x}) = \varphi(0, \vec{x})$. We may write the free energy as

$$-\beta F = \ln \left[\int_S \mathcal{D}\varphi \, e^{-S_0} \sum_n \frac{1}{n!} (-S_I)^n \right] = \ln[Z_0] + \ln[Z_I],$$

where Z_0 is the partition function of the free theory. The correction to the partition function is thus given by

$$Z_I = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \langle S_I{}^n \rangle_0, \tag{A.58}$$

where

$$\langle A \rangle_0 = \frac{\int_S \mathcal{D}\varphi \, A \, e^{-S_0}}{\int_S \mathcal{D}\varphi \, e^{-S_0}}. \tag{A.59}$$

To evaluate expectation values of the form $\langle \varphi(X_1)... \rangle_0$, we introduce the partition function with a source term

$$Z[J] = \int_S \mathcal{D}\varphi \, \exp\left\{ -\frac{1}{2} \int_\Omega \mathrm{d}X \, \varphi(-\partial_E^2 + m^2)\varphi + \int_\Omega \mathrm{d}X \, J\varphi \right\}. \tag{A.60}$$

Using the thermal Greens function $D_0(X, Y)$, as defined in section B.1, we may complete the square to write

$$Z[J] = Z[0] \exp\left\{ \frac{1}{2} \int_\Omega dX dY \, J(X) D_0(X, Y) J(Y) \right\} = Z[0] \exp(W[J]) \tag{A.61}$$

We can now write

$$\langle \varphi(X)\varphi(Y) \rangle_0 = \frac{1}{Z[0]} \frac{\delta}{\delta J(X)} \frac{\delta}{\delta J(Y)} Z[J]\Big|_{J=0} = D_0(X, Y), \tag{A.62}$$

This generalizes to higher order expectation values,

$$\langle \varphi(X_i) \dots \varphi(X_n) \rangle_0 = \left(\prod_{i=1}^{n} \frac{\delta}{\delta J(X_i)} \right) Z[J]\Big|_{J=0}, \tag{A.63}$$

Using Wick's theorem, as described in section 2.1, the expectation values we are evaluating can be written

$$\langle S_I{}^m \rangle = \left(-\frac{\lambda}{4!} \right)^m \int_\Omega dX_1 \dots dX_m \left\langle \varphi^4(X_1) \dots \varphi^4(X_m) \right\rangle$$

$$= \left(-\frac{\lambda}{4!} \right)^m \int_\Omega dX_1 \dots dX_m \sum_{\{a,b\}} \left\langle \varphi(X_{a(1)})\varphi(X_{b(1)}) \right\rangle \dots \left\langle \varphi(X_{a(2m)})\varphi(X_{b(2m)}) \right\rangle$$
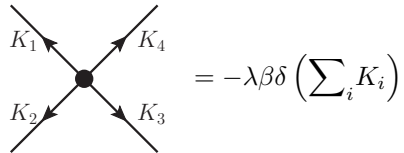
where X_i for $i > m$ is defined as X_j, where $j = i \mod m$. This means that $X_{m+i} = X_i$. The functions a, b represents a possible pairing, as described in section 2.1. Inserting the Fourier expansions of the field gives

$$\langle S_I{}^m \rangle$$

$$= \left(-\frac{\lambda}{4!} \right)^m \int_\Omega dX_1 \dots dX_m (V\beta)^2 \int_{\tilde{\Omega}} dK_1 \dots dK_{2m} \sum_{\{a,b\}}$$

$$\left\langle \varphi(K_{a(1)})\varphi(K_{b(1)}) \right\rangle \dots \left\langle \varphi(K_{a(2m)})\varphi(K_{b(2m)}) \right\rangle \exp\left(i \sum_{i=1}^{m} X_i \cdot K_i \right)$$

$$= \left(-\frac{\lambda}{4!} \right)^m \frac{(V\beta)^{2m}\beta^m}{(V\beta^2)^{2m}} \int_{\tilde{\Omega}} dK_1 \dots dK_{2m} \sum_{\{a,b\}}$$

$$\tilde{D}(K_{a(1)})\delta(K_{a(1)} + K_{b(1)}) \dots \tilde{D}(K_{a(2m)})\delta(K_{a(2m)} + K_{b(2m)}) \prod_{i=1}^{m} \delta\left(X_i \cdot \sum_{j=0}^{3} K_{i+jm} \right)$$

$$= \left(-\frac{\lambda\beta}{4!} \right)^m \prod_{i=1}^{2m} \int_{\tilde{\Omega}} \left(dK_i \frac{1}{\beta} \tilde{D}(K_i) \right) \prod_{i=1}^{m} \delta\left(X_i \cdot \sum_{j=0}^{3} K_{i+jm} \right) \sum_{\{a,b\}} \prod_{n=1}^{2m} \delta(K_{a(k)} + K_{b(k)})$$

Here we have used that $V\beta^2 \tilde{D}_0(K, P) = \tilde{D}_0(K)\delta(P + K)$, where $\tilde{D}_0(K)$ is the thermal propagator for the free field, as defined in section B.1. In this case, it is

$$\tilde{D}_0(K) = \tilde{D}_0(\omega_n, \vec{k}) = \frac{1}{\omega_k^2 + \omega_n^2}. \tag{A.64}$$

This expectation value can be represented graphically using Feynman diagrams. The thermal $\lambda\varphi^2$-theory gets the prescription

$$= -\lambda\beta\delta\left(\sum_i K_i \right), \tag{A.65}$$

$$= \frac{1}{\beta} D_0(K). \tag{A.66}$$

Lastly, one has to integrate over all internal momenta, and divide by a symmetry factor of the diagram s, which is described in detail in [11].

Calculating $\langle {S_I}^n \rangle_0$ boils down to the sum of all possible Feynman diagrams with n vertices. The first example is

$$\langle S_I \rangle = \frac{1}{8} \quad \text{} \tag{A.67}$$

In section 2.1, we saw that the sum of all vacuum diagrams is the exponential of the sum of all *connected* diagrams, so the free energy of the interacting theory is given by

$$-\beta F = \ln(Z_0) + \Sigma(\text{all connected diagrams}) \tag{A.68}$$

A.5 Fermions

The phase factor $e^{i\theta}$ that was introduced in section A.2 can be determined by studying the properties of the thermal Greens function. The thermal Greens function may be written

$$D(X_1, X_2) = D(\vec{x}, \vec{y}, \tau_1, \tau_2) = \left\langle e^{-\beta \hat{H}} \mathrm{T}\left\{ \hat{\varphi}(X_1)\hat{\varphi}(X_2) \right\} \right\rangle.$$

$\mathrm{T}\{...\}$ is time-ordering operator, and is defined as

$$\mathrm{T}\left\{ \varphi(\tau_1)\varphi(\tau_2) \right\} = \theta(\tau_1 - \tau_2)\varphi(\tau_1)\varphi(\tau_2) + \nu\theta(\tau_2 - \tau_1)\varphi(\tau_2)\varphi(\tau_1),$$

where $\nu = \pm 1$ for bosons and fermions respectively, and $\theta(\tau)$ is the Heaviside step function. In the same way that $i\hat{H}$ generates the time translation of a quantum field operator through $\hat{\varphi}(x) = \hat{\varphi}(t, \vec{x}) = e^{it\hat{H}}\hat{\varphi}(0, \vec{x})e^{-it\hat{H}}$, the imaginary-time formalism implies the relation

$$\hat{\varphi}(X) = \hat{\varphi}(\tau, \vec{x}) = e^{\tau\hat{H}}\hat{\varphi}(0, \vec{x})e^{-\tau\hat{H}}. \tag{A.69}$$

Using $\mathbb{1} = e^{\tau\hat{H}}e^{-\tau\hat{H}}$ and the cyclic property of the trace, we show that, assuming $\beta > \tau > 0$,

$$\begin{aligned} G(\vec{x}, \vec{y}, \tau, 0) &= \left\langle e^{-\beta\hat{H}} \mathrm{T}\left\{ \varphi(\tau, \vec{x})\varphi(0, \vec{y}) \right\} \right\rangle \\ &= \frac{1}{Z}\mathrm{Tr}\left\{ e^{-\beta\hat{H}}\varphi(\tau, \vec{x})\varphi(0, \vec{y}) \right\} \\ &= \frac{1}{Z}\mathrm{Tr}\left\{ \varphi(0, \vec{y})e^{-\beta\hat{H}}\varphi(\tau, \vec{x}) \right\} \\ &= \frac{1}{Z}\mathrm{Tr}\left\{ e^{-\beta\hat{H}}e^{\beta\hat{H}}\varphi(0, \vec{y})e^{-\beta\hat{H}}\varphi(\tau, \vec{x}) \right\} \\ &= \frac{1}{Z}\mathrm{Tr}\left\{ e^{-\beta\hat{H}}\varphi(\vec{y}, \beta)\varphi(\tau, \vec{x}) \right\} \\ &= \nu\left\langle e^{-\beta\hat{H}}\mathrm{T}\left\{ \varphi(\tau, \vec{x})\varphi(\beta, \vec{y}) \right\} \right\rangle. \end{aligned}$$

This implies that $\varphi(0, x) = \nu\varphi(\beta, \varphi)$, which shows that bosons are periodic in time, as stated earlier, while fermions are anti-periodic.

The Lagrangian density of a free fermion is

$$\mathcal{L} = \bar{\psi}\left(i\slashed{\partial} - m \right)\psi. \tag{A.70}$$

This Lagrangian is invariant under the transformation $\psi \to e^{-i\alpha}\psi$, which by Nöther's theorem results in a conserved current

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\delta\psi = \bar{\psi}\gamma^\mu\psi. \tag{A.71}$$

The corresponding conserved charge is

$$Q = \int_V \mathrm{d}^3x \, j^0 = \int_V \mathrm{d}^3x \, \bar{\psi}\gamma^0\psi. \tag{A.72}$$

We can now use our earlier result for the thermal partition function, Eq. (A.12), only with the substitution $\mathcal{H} \to \mathcal{H} - \mu\bar{\psi}\gamma^0\psi$, and integrate over anti-periodic ψ's:

$$Z = \mathrm{Tr}\left\{e^{-\beta(\hat{H}-\mu\hat{Q})}\right\} = \prod_{ab} \int \mathcal{D}\psi_a \mathcal{D}\pi_b \exp\left\{\int_\Omega \mathrm{d}X \left(i\dot{\psi}\pi - \mathcal{H}(\psi,\pi) + \mu\bar{\psi}\gamma^0\psi\right)\right\},$$

where a, b are the spinor indices. The canonical momentum corresponding to ψ is

$$\pi = \frac{\partial\mathcal{L}}{\partial(\partial_0\psi)} = i\bar{\psi}\gamma^0, \tag{A.73}$$

and the Hamiltonian density is

$$\mathcal{H} = \pi\dot{\psi} - \mathcal{L} = \bar{\psi}(-i\gamma^i\partial_i + m)\psi \tag{A.74}$$

which gives

$$\mathcal{L}_E = -i\pi\dot{\psi} + \mathcal{H}(\psi,\pi) - \mu\bar{\psi}\gamma^0\psi = \bar{\psi}[\gamma^0(\partial_\tau - \mu) - i\gamma^i\partial_i + m]\psi, \tag{A.75}$$

By using the Grassman-version of the Gaussian integral formula, the partition function can be written

$$Z = \prod_{ab} \int \mathcal{D}\psi_a \mathcal{D}\bar{\psi}_b \exp\left\{-\int_\Omega \mathrm{d}X \, \bar{\psi} \left[\tilde{\gamma}_0(\partial_\tau - \mu) - i\gamma^i\partial_i + m\right]\psi\right\}$$

$$= C \prod_{ab} \int \mathcal{D}\tilde{\psi}_a \mathcal{D}\tilde{\bar{\psi}}_b \exp\left\{-\int_{\tilde{\Omega}} \mathrm{d}K \, \tilde{\bar{\psi}} \left[i\tilde{\gamma}_0(\omega_n + i\mu) + i\gamma_i p_i + m\right]\tilde{\psi}\right\}$$

$$= C \prod_{ab} \int \mathcal{D}\tilde{\psi}_a \mathcal{D}\tilde{\bar{\psi}}_b e^{-\langle\tilde{\bar{\psi}}, D_0^{-1}\tilde{\psi}\rangle} = \det\left(D_0^{-1}\right).$$

In the second line, we have inserted the Fourier expansion of the field, as defined in section B.1, and changed variable of integration, as we did for the scalar field. The linear operator in this case is

$$D_0^{-1} = i\gamma^0(-i\partial_\tau + i\mu) - (-i\gamma^i)\partial_i + m = \beta[i\tilde{\gamma}_a p_a + m]. \tag{A.76}$$

This equality must be understood as an equality between linear operators, which are represented in different bases. We introduced the notation $p_{n;a} = (\omega_n + i\mu, p_i)$ and use the Euclidean gamma matrices, as defined in section B.1. We use the fact that

$$\det(i\tilde{\gamma}_a p_a + m) = \det\left(\gamma^5\gamma^5\right)\det(i\tilde{\gamma}_a p_a + m) = \det\left[\gamma^5(i\tilde{\gamma}_a p_a + m)\gamma^5\right] = \det(-i\tilde{\gamma}_a p_a + m),$$

Let $\tilde{D} = -i\tilde{\gamma}_a p_a + m$, which means we can write

$$Z = \sqrt{\det(D)\det\left(\tilde{D}\right)} = \sqrt{\det\left(D\tilde{D}\right)} = \det\left[\mathbb{1}(p_a p_a + m^2)\right]^{1/2}, \tag{A.77}$$

where we have used the anti-commutation rule for the Euclidean gamma-matrices, $\{\gamma_a, \gamma_b\} = 2\delta_{ab}$. It is important to keep in mind that the determinant here refers to linear operators on the space of spinor functions.

$$\ln(Z) = \ln\left\{\det\left[\mathbb{1}(p_a p_a + m^2)\right]^{1/2}\right\} = \frac{1}{2}\mathrm{Tr}\left\{\ln[\mathbb{1}(p_a p_a + m^2)]\right\}$$

$$= \frac{1}{2}\int_{\tilde{\Omega}} \mathrm{d}K \ln[\mathbb{1}\beta^2(p_a p_a + m^2)]_{aa} \tag{A.78}$$

As the matrix within the logarithm is diagonal, the matrix-part of the trace is trivial, and the free energy may be written

$$\beta\mathcal{F} = -2\int_{\tilde{\Omega}} \mathrm{d}X \, \ln\{\beta^2[(\omega_n + i\mu)^2 + \omega_k^2]\}. \tag{A.79}$$

Using the fermionic version of the thermal sum from section A.3 gives the answer

$$\mathcal{F} = 2\int \frac{\mathrm{d}^3p}{(2\pi)^3} \left[\beta\omega_k + \frac{1}{\beta}\ln\left(1 + e^{-\beta(\omega_k-\mu)}\right) + \frac{1}{\beta}\ln\left(1 + e^{-\beta(\omega_k+\mu)}\right)\right]. \tag{A.80}$$

We see again that the temperature-independent part of the integral diverges, and must be regulated. There are two temperature-dependent terms, one from the particle and one from the anti-particle.

Appendix B

Appendices

B.1   Conventions and notation

Throughout this text, natural units are employed, in which

$$\hbar = c = k_B = 1, \tag{B.1}$$

where $\hbar$ is the Planck reduced constant, k_B is the Boltzmann constant and c is the speed of light. The Minkowski metric convention used is the "mostly minus",

$$g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1). \tag{B.2}$$

The $\mathfrak{su}\,(2)$ basis used is the Pauli matrices,

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

They obey

$$[\tau_a, \tau_b] = 2i\varepsilon_{abc}\tau_c, \quad \{\tau_a, \tau_b\} = 2\delta_{ab}\mathbb{1}, \quad \mathrm{Tr}[\tau_a] = 0, \quad \mathrm{Tr}[\tau_a\tau_b] = 2\delta_{ab}\mathbb{1}.$$

Together with the identity matrix $\mathbb{1}$, the Pauli matrices form a basis for the vector space of all 2-by-2 matrices. An arbitrary 2-by-2 matrix M may be written

$$M = M_0\mathbb{1} + M_a\tau_a, \quad M_0 = \frac{1}{2}\,\mathrm{Tr}\{M\}, \; M_a = \frac{1}{2}\,\mathrm{Tr}\{\tau_a M\}. \tag{B.3}$$

The gamma matrices γ^μ, $\mu \in \{0, 1, 2, 3\}$, obey

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}. \tag{B.4}$$

The "fifth γ-matrix" is defined by

$$\gamma^5 = \frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = i\gamma^0\gamma^1\gamma^2\gamma^3. \tag{B.5}$$

The γ^5-matrix obey

$$\left\{\gamma^5, \gamma^\mu\right\} = 0, \quad (\gamma^5)^2 = \mathbb{1}, \quad \gamma^{0\dagger} = \gamma^0, \, \gamma^{i\dagger} = -\gamma^i \tag{B.6}$$

Their Euclidean counterpart obey

$$\{\tilde{\gamma}_a, \tilde{\gamma}_b\} = 2\delta_{ab}\mathbb{1}, \quad \tilde{\gamma}_a^\dagger = \tilde{\gamma}_a, \tag{B.7}$$

and they are related by $\tilde{\gamma}_0 = \gamma^0$, and $\tilde{\gamma}_j = -i\gamma^j$. The Euclidean $\tilde{\gamma}_5$ is defined as

$$\tilde{\gamma}_5 = \gamma_0\gamma_1\gamma_2\gamma_3 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5. \tag{B.8}$$

It thus also anti-commutes with the Euclidean γ-matrices.

The Fourier transform used in this text is defined by

$$\mathcal{F}\left\{f(x)\right\}(p) = \tilde{f}(p) = \int \mathrm{d}x\, e^{ipx} f(x), \quad \mathcal{F}^{-1}\left\{\tilde{f}(p)\right\}(x) = f(x) = \int \frac{\mathrm{d}p}{2\pi}\, e^{-ipx} \tilde{f}(p).$$

Fourier series

Imaginary-time formalism is set in a Euclidean space $\Omega = [0, \beta] \times V$, where $V = L_x L_y L_z$ is a space-like volume. The possible momenta in this space are

$$\tilde{V} = \left\{ \vec{k} \in \mathbb{R}^3 \,\middle|\, \vec{k} = \left(\frac{2\pi n_x}{L_x}, \frac{2\pi n_y}{L_y}, \frac{2\pi n_z}{L_z} \right) \right\}$$

ω_n are the Matsubara-frequencies, with $\omega_n = 2n\pi/\beta$ for bosons and $\omega_n = (2n+1)\pi/\beta$ for fermions. They together form the reciprocal space $\tilde{\Omega} = \{\omega_n\} \times \tilde{V}$. The Euclidean coordinates are denoted $X = (\tau, \vec{x})$ and $K = (\omega_n, \vec{K})$, and have the dot product $X \cdot P = \omega_n \tau + \vec{k} \cdot \vec{x}$. In the limit $V \to \infty$, we follow the prescription

$$\frac{1}{V} \sum_{\vec{p} \in \tilde{V}} \to \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3}.$$

The sum over all degrees of freedom, and the corresponding integrals for the thermodynamic limit are

$$\frac{\beta V}{NM} \sum_{n=1}^{N} \sum_{\vec{x}_m \in V} \xrightarrow{N,M \to \infty} \int_0^\beta \mathrm{d}\tau \int_{\mathbb{R}^3} \mathrm{d}^3 x = \int_\Omega \mathrm{d}X,$$

$$\frac{1}{V} \sum_{n=-\infty}^{\infty} \sum_{\vec{k} \in \tilde{V}} \xrightarrow{V \to \infty} \sum_{n=-\infty}^{\infty} \int_{R^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} = \int_{\tilde{\Omega}} \mathrm{d}K.$$

The convention used for the Fourier expansion of thermal fields is in accordance with [32]. The prefactor is chosen to make the Fourier components of the field dimensionless, which makes it easier to evaluate the trace correctly. For bosons, the Fourier expansion is

$$\varphi(X) = \sqrt{V\beta} \int_{\tilde{\Omega}} \mathrm{d}K\, \tilde{\varphi}(K) e^{iX \cdot K} = \sqrt{\frac{\beta}{V}} \sum_{n=-\infty}^{\infty} \sum_{\vec{k} \in \tilde{V}} \tilde{\varphi}_n(\vec{p}) \exp\left\{ i(\omega_n \tau + \vec{x} \cdot \vec{k}) \right\},$$

$$\tilde{\varphi}(K) = \sqrt{\frac{1}{V\beta^3}} \int_{\tilde{\Omega}} \mathrm{d}X\, \tilde{\varphi}(X) e^{-iX \cdot K}$$

while for Fermions it is

$$\psi(X) = \sqrt{V} \int_{\tilde{\Omega}} \mathrm{d}K\, \tilde{\psi}(K) e^{iX \cdot K} = \frac{1}{\sqrt{V}} \sum_{n=-\infty}^{\infty} \sum_{\vec{k} \in \tilde{V}} \psi(\omega_n, \vec{k}) \exp\left\{ i(\omega_n \tau + \vec{x} \cdot \vec{k}) \right\} \tag{B.9}$$

A often used identity is

$$\int_\Omega \mathrm{d}X e^{iX \cdot (K - K')} = \beta \delta_{nn'} (2\pi)^3 \delta^3(\vec{k} - \vec{k}') := \beta \delta(K - K'), \tag{B.10}$$

$$\int_{\tilde{\Omega}} \mathrm{d}K\, e^{iK(X - X')} = \beta \delta(\tau - \tau') \delta^3(\vec{x} - \vec{x}') := \beta \delta(X - X'). \tag{B.11}$$

Particle distributions

The Bose distribution is defined as

$$n_B(\omega) = \frac{1}{e^{\beta \omega} - 1}. \tag{B.12}$$

This function obeys

$$n_B(-i\omega) = -1 - n_B(i\omega). \tag{B.13}$$

We can expand it around the Bose Matsubara frequencies on the imaginary line:

$$in_B[i(\omega_n + \epsilon)] = \frac{i}{e^{i\beta\epsilon + 2\pi in} - 1} = i[i\beta\epsilon + \mathcal{O}(\epsilon^2)]^{-1} \sim \frac{1}{\epsilon\beta}. \tag{B.14}$$

This means that $in_B(i\omega)$ has a pole on all Matsubara-frequencies, with residue $1/\beta$. Furthermore, we have

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\ln\left(1 - e^{-\beta\omega}\right) = \beta n_B(\omega). \tag{B.15}$$

The Fermi distribution is

$$n_F(\omega) = \frac{1}{e^{\beta\omega} + 1}. \tag{B.16}$$

It obeys

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\ln\left(1 - e^{-\beta\omega}\right) = -\beta n_F(\omega), \tag{B.17}$$

$$n_F(-i\omega) = 1 - n_F(i\omega). \tag{B.18}$$

The two distributions are related by

$$2n_B(i\omega; 2\beta) - n_B(i\omega; \beta) = -n_F(i\omega; \beta). \tag{B.19}$$

Propagators

If $D^{-1}[f(x)] = 0$ is the equation of motion for some field f, where D^{-1} in general is a differential operator, then the propagator $D(x, x')$ for this field is defined by

$$D^{-1}[D(x, x')] = -i\delta(x - x')\mathbb{1}.$$

Assuming A is linear and independent of space, we may redefine $D(x - x', 0) \to D(x - x')$, and the Fourier transform with respect to both x and x' to obtain

$$\mathcal{F}\left\{D^{-1}[D(x, x')]\right\}(p, p') = \tilde{D}^{-1}(p)\,\tilde{D}(p)\,\delta(p + p') = -i\delta(p + p'),$$

meaning the momentum space propagator $\tilde{D}(p) = \mathcal{F}\{D(x)\}(p)$ is given by $\tilde{D} = -i(\tilde{D}^{-1})^{-1}$.

For some differential operator D^{-1}, the thermal propagator is defined as

$$D^{-1}D(X, Y) = \beta\delta(X - Y). \tag{B.20}$$

The Fourier transformed propagator is, assuming $D(X, Y) = D(X - Y, 0)$,

$$\tilde{D}(K, K') = \frac{1}{V\beta^3}\int_\Omega \mathrm{d}X\mathrm{d}Y\, D(X, Y)\exp(-i[X \cdot K + Y \cdot K']) \tag{B.21}$$

$$= \frac{1}{V\beta^3}\int_\Omega \mathrm{d}X'\mathrm{d}Y'\, D(X', 0)\exp\left(-i[X' \cdot \frac{1}{2}(K - K') + Y \cdot (K + K')]\right) \tag{B.22}$$

$$= \frac{1}{V\beta^2}\tilde{D}(K)\delta(K + K'), \tag{B.23}$$

where

$$\tilde{D}(K) = \int \mathrm{d}X e^{iK \cdot X} D(X, 0). \tag{B.24}$$

B.2 Functional Derivatives

Functional derivatives generalize the notion of a gradient and the directional derivative. A function $f(p)$, where p is point with coordinates $x_i = x_i(p)$, has a gradient

$$\mathrm{d}f_p = \frac{\partial f(p)}{\partial x_i}\mathrm{d}x_i. \tag{B.25}$$

The derivative in a particular direction $v = v^i\partial_i$ is

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}f(x_i + \epsilon v_i) = f(x) + \mathrm{d}f_x(v) = f(x) + \frac{\partial f}{\partial x^i}v_i. \tag{B.26}$$

This is generalized to functionals through the definition of functional derivative, and the variation of a functional. Let $F[f]$ be a functional, i.e. a machine that takes in a function, and returns a number. The obvious example in our case is the action, which takes in one or more field-configurations, and returns a single real number. We will assume here that the functions have the domain Ω, with coordinates x. The functional derivative is defined as

$$\delta F[f] = \frac{\mathrm{d}}{\mathrm{d}\epsilon}F[f + \epsilon\eta]\Big|_{\epsilon=0} = \int_\Omega \mathrm{d}x\, \frac{\delta F[f]}{\delta f(x)}\eta(x). \tag{B.27}$$

$\eta(x)$ is here an arbitrary function, but we will make the important assumption that it as well as all its derivatives are zero at the boundary of its domain Ω. This allows us to discard surface terms stemming from partial integration, which we will use frequently. We may use the definition to derive one of the fundamental relations of functional derivation. Take the functional $F[f] = f(x)$. Then,

$$\delta F[f] = \frac{\mathrm{d}}{\mathrm{d}\epsilon}[f(x) + \epsilon\eta(x)] = \eta(x) = \int \mathrm{d}y\, \delta(x - y)\eta(y) \tag{B.28}$$

This leads to the identity

$$\frac{\delta f(x)}{\delta f(y)} = \delta(x - y), \tag{B.29}$$

for any function f. Higher functional derivatives are defined similarly, by applying functional variation repeatedly

$$\delta^n F[f] = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\delta^{n-1}F[f + \epsilon\eta_n]\Big|_{\epsilon=0} = \int\left(\prod_{i=1}^n \mathrm{d}x_i\right)\frac{\delta^n F[f]}{\delta f(x_n)\dots\delta f(x_n)}\left(\prod_{i=1}^n \eta_i(x_i)\right). \tag{B.30}$$

A functional may be expanded in a generalization of the Fourier series, which has the form

$$F[f_0 + f] = F[f_0] + \int_\Omega \mathrm{d}x\, f(x)\frac{\delta F[f_0]}{\delta f(x)}\Big|_{f=f_0} + \frac{1}{2!}\int_\Omega \mathrm{d}x\mathrm{d}y\, f(x)f(y)\frac{\delta^2 F[f_0]}{\delta f(x)\delta f(y)} + \dots \tag{B.31}$$

As an example, the Klein-Gorodn action

$$S[\varphi] = -\frac{1}{2}\int_\Omega \mathrm{d}x\, \varphi(\partial^2 + m^2)\varphi(x) \tag{B.32}$$

can be evaluated quickly by using Eq. (B.28) and partial integration

$$\frac{\delta}{\delta\varphi(x)}S[\varphi] = -\frac{1}{2}\int_\Omega \mathrm{d}y\, [\delta(x - y)(\partial_y^2 + m^2)\varphi(y) + \varphi(y)(\partial_y^2 + m^2)\delta(x - y)]$$

$$= -\int_\Omega \mathrm{d}y\, \delta(x - y)(\partial_y^2 + m^2)\varphi(y) = (\partial_x^2 + m^2)\varphi(x) \tag{B.33}$$

The second derivative is

$$\frac{\delta^2 S[\varphi]}{\delta\varphi(x)\delta\varphi(y)} = \frac{\delta}{\delta\varphi(x)}(\partial_y^2 + m^2)\varphi(y) = (\partial_y^2 + m^2)\delta(x - y). \tag{B.34}$$
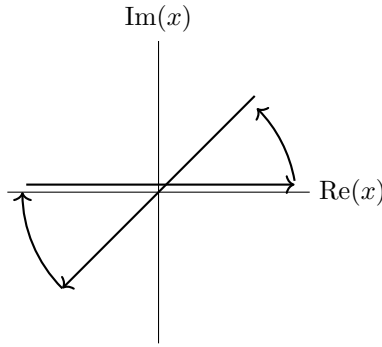
Figure B.1: Wick rotation

B.3 Integrals

A useful integral is the Gaussian integral,

$$\int_{\mathbb{R}} dz \, \exp\left(-\frac{1}{2}az^2\right) = \sqrt{\frac{2\pi}{a}}, \tag{B.35}$$

for $a \in \mathbb{R}$. The imaginary version,

$$\int_R dz \, \exp\left(i\frac{1}{2}az^2\right) \tag{B.36}$$

does not converge. However, if we change $a \to a + i\epsilon$, then the integrand is exponentially suppressed.

$$f(x) = \exp\left(i\frac{1}{2}ax^2\right) \to \exp\left(i\frac{1}{2}ax^2 - \frac{1}{2}\epsilon x^2\right), \tag{B.37}$$

As the integrand falls of exponentially for $x \to \infty$, and contains no poles in the upper right nor lower left quarter of the complex plane, we may perform a wick rotation by closing the contour as shown in Figure B.1. This gives the result

$$\int_{\mathbb{R}} dx \, \exp\left(i\frac{1}{2}(a + i\epsilon)x^2\right) = \int_{\sqrt{i}\mathbb{R}} dx \, \exp\left(i\frac{1}{2}ax^2\right) = \sqrt{i}\int_{\mathbb{R}} dy \, \exp\left(-\frac{1}{2}(-a)y^2\right) = \sqrt{\frac{2\pi i}{(-a)}} \tag{B.38}$$

where we have made the change of variable $y = (1 + i)/\sqrt{2}x = \sqrt{i}x$. In n dimensions, the Gaussian integral formula generalizes to

$$\int_{\mathbb{R}^n} d^nx \, \exp\left\{-\frac{1}{2}x_n A_{nm}x_m\right\} = \sqrt{\frac{(2\pi)^n}{\det(A)}}, \tag{B.39}$$

where A is a matrix with n real, positive eigenvalues. We may also generalize Eq. (B.38),

$$\int_{\mathbb{R}^n} d^nx \, \exp\left\{i\frac{1}{2}x_n(A_{nm} + i\epsilon\delta_{nm})x_m\right\} = \sqrt{\frac{(2\pi i)^n}{\det(-A)}}. \tag{B.40}$$

The final generalization is to functional integrals,

$$\int \mathcal{D}\varphi \, \exp\left(-\frac{1}{2}\int dx \, \varphi(x)A\varphi(x)\right) = C(\det(A))^{-1/2}, \int \mathcal{D}\varphi \, \exp\left(i\frac{1}{2}\int dx \, \varphi(x)A\varphi(x)\right) = C(\det(-A))^{-1/2}. \tag{B.41}$$

C is here a divergent constant, but will either fall away as we are only looking at the logarithm of I_∞ and are able to throw away additive constants, or ratios between quantities which are both multiplied by C.

The Gaussian integral can be used for the stationary phase approximation. In one dimension, it is

$$\int dx \, \exp(i\alpha f(x)) \approx \sqrt{\frac{2\pi}{f''(x_0)}} \exp(f(x_0)), \, f'(x) = 0, \, \alpha \to \infty \tag{B.42}$$

The functional generalization of this is

$$\int \mathcal{D}\varphi \exp\{iS[\varphi]\} \approx C \det\left(-\frac{\delta^2 S[\varphi_0]}{\delta\varphi^2}\right) \exp\{i\alpha S[\varphi_0]\}, \quad \frac{\delta S[\varphi_0]}{\delta\varphi} = 0 \tag{B.43}$$

Here, $S[\varphi]$ is a general functional of φ, and we have used the Taylor expansion, as described in section B.2, and φ_0 fulfills

$$\frac{\delta}{\delta\varphi(x)} S[\varphi_0] = 0, \tag{B.44}$$

Bibliography

[1] Matthew D. Schwartz. *Quantum Field Theory and the Standard Model.* Cambridge University Press, 3 2014.

[2] D. Griffiths. *Introduction to Elementary Particles.* Physics textbook. Wiley, 2008.

[3] Steven Weinberg. *The Quantum Theory of Fields*, volume 1. Cambridge University Press, 1995.

[4] Steven Weinberg. *The Quantum Theory of Fields*, volume 2. Cambridge University Press, 1996.

[5] Steven Weinberg. Phenomenological lagrangians. *Physica A: Statistical Mechanics and its Applications*, 96(1):327–340, 1979.

[6] N.K. Glendenning. *Compact Stars: Nuclear Physics, Particle Physics and General Relativity.* Astronomy and Astrophysics Library. Springer New York, 2000.

[7] Gordon Baym, Tetsuo Hatsuda, Toru Kojo, Philip D Powell, Yifan Song, and Tatsuyuki Takatsuka. From hadrons to quarks in neutron stars: a review. *Reports on Progress in Physics*, 81(5):056902, Mar 2018.

[8] Sean M. Carroll. *Spacetime and Geometry.* Cambridge University Press, 7 2019.

[9] B. B. Brandt, G. Endrődi, E. S. Fraga, M. Hippert, J. Schaffner-Bielich, and S. Schmalzbauer. New class of compact stars: Pion stars. *Physical Review D*, 98(9), Nov 2018.

[10] Jens O. Andersen and Patrick Kneschke. Bose-einstein condensation and pion stars, 2018.

[11] Michael E. Peskin and Daniel V. Schroeder. *An Introduction to quantum field theory.* Addison-Wesley, Reading, USA, 1995.

[12] D. Binosi and L. Theußl. Jaxodraw: A graphical user interface for drawing feynman diagrams. *Computer Physics Communications*, 161(1):76–86, 2004.

[13] John M. Lee. *Introduction to Smooth Manifolds.* Springer, 2002.

[14] Tomáš Brauner. Spontaneous symmetry breaking and nambu–goldstone bosons in quantum many-body systems. *Symmetry*, 2(2):609–657, Apr 2010.

[15] S. Coleman, J. Wess, and Bruno Zumino. Structure of phenomenological lagrangians. i. *Phys. Rev.*, 177:2239–2247, Jan 1969.

[16] Curtis G. Callan, Sidney Coleman, J. Wess, and Bruno Zumino. Structure of phenomenological lagrangians. ii. *Phys. Rev.*, 177:2247–2250, Jan 1969.

[17] Giuliano Panico and Andrea Wulzer. The composite nambu-goldstone higgs. *Lecture Notes in Physics*, 2016.

[18] Antonio Pich. Effective field theory with nambu-goldstone modes, 2020.

[19] Haruki Watanabe and Hitoshi Murayama. Effective lagrangian for nonrelativistic systems. *Physical Review X*, 4(3), Sep 2014.

[20] P.A. Zyla et al. Review of Particle Physics. *PTEP*, 2020(8):083C01, 2020.

[21] J Gasser and H Leutwyler. Chiral perturbation theory to one loop. *Annals of Physics*, 158(1):142–210, 1984.

[22] S. Scherer. Introduction to chiral perturbation theory. *Advances in Nuclear Physics*, 27:277–538, 2002.

[23] H. Leutwyler. On the foundations of chiral perturbation theory. *Annals of Physics*, 235(1):165–203, Oct 1994.

[24] H. W. Fearing and S. Scherer. Extension of the chiral perturbation theory meson lagrangian to order p^6. *Phys. Rev. D*, 53:315–348, Jan 1996.

[25] Prabal Adhikari, Jens O. Andersen, and Patrick Kneschke. Two-flavor chiral perturbation theory at nonzero isospin: Pion condensation at zero temperature. *Eur. Phys. J. C*, 79(10):874, 2019.

[26] Martin Aria Mojahed. Two-flavor chiral perturbation theory at finite isospin beyond leading order. Master's thesis, NTNU, 2020.

[27] J.S.R. Chisholm. Change of variables in quantum field theories. *Nuclear Physics*, 26(3):469–479, 1961.

[28] S. Kamefuchi, L. O'Raifeartaigh, and Abdus Salam. Change of variables and equivalence theorems in quantum field theories. *Nuclear Physics*, 28(4):529–549, 1961.

[29] P. Gerber and H. Leutwyler. Hadrons below the chiral phase transition. *Nuclear Physics B*, 321(2):387–429, 1989.

[30] G. Colangelo, J. Gasser, and H. Leutwyler. scattering. *Nuclear Physics B*, 603(1-2):125–179, Jun 2001.

[31] Jens O. Andersen. *Introduction to Statistical Mechanics*. Akademika forlag, 2012.

[32] J. I. Kapusta and Charles Gale. *Finite-temperature field theory: Principles and applications*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2011.

[33] Mikko Laine and Aleksi Vuorinen. Basics of thermal field theory. *Lecture Notes in Physics*, 2016.

[34] G. 't Hooft. Dimensional regularization and the renormalization group. *Nuclear Physics B*, 61:455–468, 1973.