Limiting Inequalities in Repeated House and Task Allocation*

Martin Aleksandrov

Freie Universität Berlin, Berlin, Germany
martin.aleksandrov@fu-berlin.de

Abstract. We consider house and task allocation markets over multiple rounds, where agents have endowments and valuations at each given round. The endowments encode allocations of agents at the previous rounds. The valuations encode the preferences of agents at the current round. For these problems, we define novel axiomatic norms, denoted as JFXRC, JFXRR, JF1RC, JF1RR, and JF1B, that limit inequalities in allocations gradually. When the endowments are equal, we prove that computing JFXRC and JFXRR allocations may take exponential time whereas computing JF1RC and JF1RR allocations takes polynomial time. However, when the endowments are unequal, JF1RC or JF1RR allocations may not exist whereas computing JF1B allocations takes polynomial time. Finally, our work offers a number of polynomial-time algorithms for limiting inequalities in repeated house and task allocation markets.

Keywords: Social Good · Individual Well-being · Algorithmic Decision Making.

1 Introduction

Limiting inequalities and ensuring that no one is left behind are integral to achieving the United Nations' Sustainable Development Goals. Despite some positive signs toward limiting inequality in some dimensions, such as limiting relative income inequality in some countries and preferential trade status benefiting lower-income countries, inequality still persists. Inequalities are also deepening for vulnerable populations in countries with weaker health systems and those facing existing humanitarian crises. Refugees and migrants, as well as indigenous peoples, older persons, and people with disabilities, are particularly at risk of facing increasing inequalities worldwide. Many of these people live in poor houses but have an increasing number of tasks in their daily routines. Hence, limiting inequalities in the associated house allocation [1] and task allocation [14] markets may promote social integration and life stability for those people.

Indeed, we expect that those with bigger houses would also have to do more tasks for maintaining those houses from one day to another. We study thus limiting inequalities in repeated settings, where agents have general valuations for houses and tasks. Unlike static settings [4], *repeated settings* enable us to balance inequalities not only within a fixed day but also over multiple days. For example, we might want to give

* The work was supported by the DFG Individual Research Grant on "Fairness and Efficiency in Emerging Vehicle Routing Problems" (497791398).

greater priority tomorrow to those who could not get houses today. Furthermore, unlike additive valuations where agents enjoy summing up their valuations for items in bundles [2], *general valuations* do not rely on assumptions about the function agents use to aggregate bundle valuations. For example, people might have negative time valuations for individual tasks, but they might prefer avoiding doing multiple tasks because this takes additional time as they get tired, in which case their valuations for bundles of tasks might be strictly lower than the sum of their valuations for the individual bundle tasks.

Social goods (e.g. items liked by everyone such as houses) and *social bads* (e.g. items not liked by everyone such as tasks) received some research attention [3,5]. If there were money as a divisible resource, we could use it to reduce potential inequalities over houses and tasks by paying more to agents who receive poorer houses but have more tasks. For this reason, we consider the more challenging case when there is no money and all items are *indivisible*. As a result, *jealousy*, caused by interpersonal comparisons of well-being among agents [11], is unavoidable. For example, if our neighbors have bigger houses than us, we might be jealous of them, but not if we know that they work much harder than us. Perceiving lower inequalities relates, therefore, to how well we balance our lives and how comfortable we feel living in a given area. As a consequence, limiting inequalities could improve our well-being!

2 Overview and Contributions

In Section 3, we present related works. In Section 4, we give the formal preliminaries. We allocate social goods and bads over multiple rounds. Pick a given allocation round. If we allocate tasks and their associated wages, we might be jealous of employees who get fewer tasks and higher wages than us but we get more tasks and lower wages than them. However, we might be able to eliminate this jealousy if they were getting lower wages or if we gave them some of our tasks. If we allocate houses and their associated taxes, we might be jealous of neighbors who get bigger houses than us because we also like bigger houses but get smaller houses. However, we might be able to eliminate this jealousy if they were getting smaller houses or if we did not have to pay taxes. In Section 5, we first define two novel axiomatic norms that reflect these observations:

1 *Jealousy-Freeness Limited To Every Non-zero Removed Good and Copied Bad (JFXRC)* requires that an agent's endowment plus valuation is not lower than any other agent's endowment plus valuation after any non-zero individual bad is *copied* from the former agent's bundle into the latter agent's bundle, and any non-zero individual good is *removed* from the latter agent's bundle. With zero endowments (i.e. agents are not allocated anything prior to the current round), we show that JFXRC allocations exist in problems with goods and bads (Theorem 1 in Sub-section 5.1).

2 *Jealousy-Freeness Limited To Every Non-zero Removed Good and Removed Bad (JFXRR)* requires that an agent's endowment plus valuation is at least as much as any other agent's endowment plus valuation after any given non-zero individual bad is *removed* from the former agent's bundle, and any given non-zero individual good is *removed* from the latter agent's bundle. When the endowments are zeros, we prove that JFXRR allocations exist in problems with either goods or bads (Theorem 2 in Sub-section 5.1).

Furthermore, we observe that computing JFXRC and JFXRR allocations might exhibit high computational complexity. For this reason, we weaken these norms and define two other novel axiomatic norms that also limit any agent-pairwise jealousy in allocations, but can be satisfied in polynomial time:

3 *Jealousy-Freeness Limited To One Removed Good or Copied Bad (JF1RC)* requires that the removal and copying operations in the formulation of JFXRC concern not every non-zero valued good and bad, but at least one, possibly zero-valued, item. With zero endowments, we prove that JF1RC allocations can be computed in polynomial time (Theorem 3 in Sub-section 5.2).
4 *Jealousy-Freeness Limited To One Removed Good or Removed Bad (JF1RR)* requires that the two removal operations in the formulation of JFXRR concern not every non-zero valued good and bad, but at least one, possibly zero-valued, item. With zero endowments, we also prove that JF1RR allocations can be computed in polynomial time (Theorem 4 in Sub-section 5.2).

For the case when the endowments are not necessarily zeros, we make the following two contributions in a given allocation round.

5 When the endowments are equal (i.e. agents are allocated items of equal, possibly non-zero, utility prior to the current round), we prove that JFXRC, JFXRR, JF1RC, and JF1RR allocations exist (Theorem 5 in Section 6).
6 When the endowments are unequal, we show that JF1RC or JF1RR allocations might not always exist (Theorem 6 in Section 6). As a response, we relax these properties and define an axiomatic norm for bounding agent pairwise jealousy in up to one item fashion by the corresponding absolute endowment difference (JF1B). We prove that JF1B allocations can be computed in polynomial time (Theorem 7 in Section 6).

Finally, we conclude and highlight several future directions in Section 7.

3 Related Works

Gourvès et al. [11] defined a notion called *near jealousy-freeness* in static instances without endowments and additive valuations for goods. Also, Freeman et al. [9,10] defined a notion called *equitability up to one item* in static instances without endowments and additive valuations for either goods or chores. Hence, neither near jealousy-freeness nor equitability up to one item is well-defined for repeated house and task allocations with endowments and general valuations. In fact, we are not aware of any prior work that studies limiting inequalities over multiple rounds. We are nevertheless aware of a very recent work [13] that studies limiting *envy*, another fairness metric of intrapersonal comparison of different consumption bundles [8], over multiple rounds. However, Herreiner and Puppe [12] found out through an experiment that people preferred allocations of fewer inequalities to allocations, say those limiting envy, that normally induce more inequalities. Our normative design reflects this experiment because JFXRC and JFXRR limit inequalities more than JF1RC and JF1RR, as well as JF1RC and JF1RR limit inequalities more than JF1B.

4 Formal Preliminaries

For given $p \in \mathbb{N}_{\geq 1}$, we let $[p] = \{1, \ldots, p\}$. We consider *repeated fair division instances* over rounds 1 to $t \in \mathbb{N}_{\geq 1}$. At each $\tau \in [t]$, we consider items from M_τ that must be allocated to agents from N_τ. We let $n_\tau = |N_\tau|$ and $m_\tau = |M_\tau|$ hold. We note that the sets of agents and items across different rounds might be different.

Pick τ. For each $a \in N_\tau$ and each bundle $M \subseteq M_\tau$, we let $v_a^\tau(M) \in \mathbb{R}$ denote their *general valuation* for M. We let $v_a^\tau(\emptyset) = 0$ hold. We write $v_a^\tau(o)$ for $v_a^\tau(\{o\})$. We say that $v_a^\tau(M)$ is *additive* if $v_a^\tau(M) = \sum_{o \in M} v_a^\tau(o)$ holds. Valuations can be elicited from agents in an experiment (see e.g. [12]), through a website (see e.g. [6]), or via an oracle (see e.g. [15]).

Pick $o \in M_\tau$. o is *good* for $a \in N_\tau$ with respect to (WRT) $M \subseteq M_\tau$ if $v_a^\tau(M \cup \{o\}) \geq v_a^\tau(M)$ holds, and *pure good* if $v_a^\tau(M \cup \{o\}) > v_a^\tau(M)$ holds. Also, o is *bad* for a WRT M if $v_a^\tau(M \cup \{o\}) \leq v_a^\tau(M)$ holds, and *pure bad* if $v_a^\tau(M \cup \{o\}) < v_a^\tau(M)$ holds. As we mentioned, we consider (social) goods and (social) bads. Hence, we can partition M_τ into $G_\tau = \{o \in M_\tau | \forall a \in N_\tau \forall M \subseteq M_\tau : v_a^\tau(M \cup \{o\}) \geq v_a^\tau(M)\}$ and $B_\tau = \{o \in M_\tau | \forall a \in N_\tau \forall M \subseteq M_\tau : v_a^\tau(M \cup \{o\}) \leq v_a^\tau(M)\}$.

A complete *allocation* at τ is $A^\tau = (A_1^\tau, \ldots, A_{n_\tau}^\tau)$, where (1) A_a^τ is the bundle of agent $a \in N_\tau$, (2) $\bigcup_{a \in N_\tau} A_a^\tau = M_\tau$ holds, and (3) $A_a^\tau \cap A_b^\tau = \emptyset$ holds for each $a, b \in N_\tau$ with $a \neq b$. We note that some agents might be allocated items across multiple rounds.

We trace therefore the valuations of agents over rounds. For this reason, we make use of *endowment* $e_a^\tau \in \mathbb{R}$ for each $a \in N_\tau$ that depends on the items allocated to a prior to τ. Thus, for $\tau = 1$, we let $e_a^1 = 0$ hold. But, for $\tau > 1$, the value of e_a^τ could be equal to the additive item number up to τ, i.e. $e_a^\tau = \sum_{\kappa \in 1:(\tau-1)} |A_a^\kappa|$, or the additive valuation sum up to τ, i.e. $e_a^\tau = \sum_{\kappa \in 1:(\tau-1)} v_a^\kappa(A_a^\kappa)$.

Thus, at τ, we let a receive *general utility* $u_a^\tau(A_a^\tau) = (e_a^\tau + v_a^\tau(A_a^\tau))$. With zero endowments at τ, we note that $u_a^\tau(A_a^\tau) = v_a^\tau(A_a^\tau)$ holds. We say that $a \in N_\tau$ is *jealousy-free* of $b \in N_\tau$ in A^τ if $u_a^\tau(A_a^\tau) \geq u_b^\tau(A_b^\tau)$ holds. Thus, A^τ is *jealousy-free* if, for each $a, b \in N_\tau$, a is jealousy-free of b in A^τ. With one round and zero endowments, giving one pure good to an agent makes any other agent feel jealous. Hence, it might be impossible to guarantee jealousy-freeness. For this reason, we propose to relax it.

5 One-round Instances

We begin with *one-round fair division instances* which are repeated fair division instances that repeat one round. That is, in such instances, we have that $t = 1$ holds. In this section, for simplicity, we **omit** the subscripts $_\tau$ and superscripts $^\tau$ from the model notation because $\tau = 1$ holds. Also, as there are no repetitions of the instance prior to round one, no agent $a \in N$ has been allocated anything prior to this round and, for this reason, we have that $e_a = 0$ hold. For example, when allocating tasks to employees today, we might want to make use of all employees equally regardless of whether they serviced unequal numbers of tasks in the previous days.

5.1 Jealousy Freeness Up To Every Non-zero Item

As jealousy-freeness might be too demanding, we next relax it. We say that *agent a is JFXRC of agent b* whenever a's utility is at least as much as b's utility, after hypothetically *coping* any given non-zero individual bad from a's bundle into b's bundle, and hypothetically *removing* any given non-zero individual good from b's bundle. Also, we say that *agent a is JFXRR of agent b* whenever a's utility is at least as much as b's utility, after hypothetically *removing* any given non-zero individual bad from a's bundle, and hypothetically *removing* any given non-zero individual good from b's bundle.

Definition 1 (JFXRC) *Allocation A is* JFXRC *if, $\forall a, b \in N$ such that a is not jealousy-free of b, (1) $\forall o \in A_a$ such that $u_a(A_a) < u_a(A_a \setminus \{o\})$: $u_a(A_a) \geq u_b(A_b \cup \{o\})$ and (2) $\forall o \in A_b$ such that $u_b(A_b) > u_b(A_b \setminus \{o\})$: $u_a(A_a) \geq u_b(A_b \setminus \{o\})$.*

Definition 2 (JFXRR) *Allocation A is* JFXRR *if, $\forall a, b \in N$ such that a is not jealousy-free of b, (1) $\forall o \in A_a$ such that $u_a(A_a) < u_a(A_a \setminus \{o\})$: $u_a(A_a \setminus \{o\}) \geq u_b(A_b)$ and (2) $\forall o \in A_b$ such that $u_b(A_b) > u_b(A_b \setminus \{o\})$: $u_a(A_a) \geq u_b(A_b \setminus \{o\})$.*

By definition, jealousy-freeness is *stronger* than JFXRC and JFXRR as such allocations satisfy JFXRC and JFXRR. However, unlike jealousy-free allocations that may not exist in some instances with pure goods, we prove that JFXRC and JFXRR allocations exist in every such instance. However, we first observe similarities between them.

When allocating n fixed-price houses among n agents, any allocation that gives one house to each agent is JFXRC and JFXRR. Furthermore, when allocating n fixed-time tasks among n agents, any allocation that gives one task to each agent is JFXRC and JFXRR. At the same time, there are fundamental differences between these two norms.

For example, achieving JFXRC might require giving unequal numbers of tasks to the agents, but thus optimize various welfares such as the Nash welfare and utilitarian welfare [6], whereas achieving JFXRR might require giving equal numbers of tasks to agents, but still may not optimize such welfares. We demonstrate this in Example 1.

Example 1 *Let us consider agents 1, 2, and 3 with zero endowments and valuations $(-\epsilon, -\epsilon, -\epsilon)$, $(-\epsilon, -\epsilon, -\epsilon)$, and $(-1, -1, -1)$, respectively, for three tasks. We let $\epsilon \in (0, 1/3)$ hold. Each JFXRR allocation gives one item to each agent and: minimizes the product of agents' valuations (i.e. Nash welfare) to $-\epsilon^2$; achieves a sum of agents' valuations (i.e. the utilitarian welfare) of $(-1 - 2\epsilon)$; returns a difference between the maximum and minimum agent's valuations (i.e. the inequality variance) of $(1 - \epsilon)$. By comparison, each JFXRC allocation shares all items only among agents 1 and 2 and: maximizes the Nash welfare to $0 > -\epsilon^2$; maximizes the utilitarian welfare to $-3\epsilon > (-1 - 2\epsilon)$; minimizes the inequality variance to $2\epsilon < (1 - \epsilon)$. Hence, we might prefer achieving JFXRC to achieving JFXRR, especially if we care about maximizing welfare and minimizing variance as secondary objectives.*

By Example 1, it follows that we might prefer achieving JFXRC to achieving JFXRR. For this reason, we start with JFXRC. We prove that JFXRC allocations are guaranteed to exist in every instance with goods and bads. Such allocations are returned by the *leximin++* solution [17].

To define this solution, we let $\overrightarrow{u}(A) \in \mathbb{R}^n$ denote the vector of agents' utilities in A, which are (re-)arranged in some non-decreasing order. We next write $A \succ_{++} B$ if there exists an index $i \leq n$ such that $\overrightarrow{u}(A)_j = \overrightarrow{u}(B)_j$ and $|A_j| = |B_j|$ for each $1 \leq j < i$, and either $\overrightarrow{u}(A)_i > \overrightarrow{u}(B)_i$ or, $\overrightarrow{u}(A)_i = \overrightarrow{u}(B)_i$ and $|A_i| > |B_i|$, hold. Thus, the leximin$++$ solution is defined as a maximal element under $\succ_{++}$. Informally, the leximin$++$ solution maximizes the least agent's utility, then maximizes the bundle's size of an agent with the least utility, before it maximizes the second least agent's utility and the bundle's size of an agent with the second least utility, and so on.

Theorem 1 *Pick a one-round instance. With zero endowments and general valuations for goods and bads, the leximin $++$ solution returns JFXRC allocations.*

We proceed with JFXRR. By definition, in instances with only goods, an allocation is JFXRR if and only if it is JFXRC. Hence, by Theorem 1, each leximin$++$ allocation satisfies JFXRR in instances with only goods. However, in instances with only bads, there is another solution for returning JFXRR allocations. This one is our Algorithm 1.

Algorithm 1 allocates the items one by one in rounds. At each round, Algorithm 1 picks the *max*imum utility *agent* given the current partial allocation. After allocating the current item to them, if the new partial allocation remains JFXRR then Algorithm 1 moves to the next round and, otherwise, Algorithm 1 picks an inclusion-wise *min*imal strict *subset* of the bundle of the selected agent, assigns this subset to them, and returns the remaining items from their bundle to the pool of unassigned items. Algorithm 1 runs in pseudo-polynomial time. This run-time complexity could be useful in practice whenever the utilities are specified in unary. This has been observed in settings where people divide service fares and house rent [6].

Algorithm 1 JFXRR in round instances with zero endowments and general valuations for bads.

procedure MAXAGENTMINSUBSET($N, M, (u_a)_n$)
  $O \leftarrow M, \forall a \in N : A_a \leftarrow \emptyset$
  while $O \neq \emptyset$ **do**
    $o \leftarrow$ an unallocated bad from O, $b \leftarrow \arg\max_{a \in N} u_a(A_a)$
    if $(A_1, \ldots, A_b \cup \{o\}, \ldots, A_n)$ is JFXRR **then**
      $O \leftarrow O \setminus \{o\}$, $A_b \leftarrow A_b \cup \{o\}$
    else
      $X \leftarrow$ inclusion-wise minimal strict subset of $A_b \cup \{o\}$ s.t. $u_b(X) < u_a(A_a)$ for some $a \in N$, $O \leftarrow O \cup [A_b \cup \{o\} \setminus X]$, $A_b \leftarrow X$
  return A

Theorem 2 *Pick a one-round instance. (1) With zero endowments and general valuations for goods, the leximin$++$ solution returns JFXRR allocations. (2) With zero endowments and general valuations for bads, Algorithm 1 returns JFXRR allocations.*

Proof. For instances with goods, the result for the leximin $++$ follows by the proof of Theorem 1. For instances with bads, the argument for Algorithm 1 is inductive. Let A denote the partial allocation. Suppose that A is JFXRR. We argue that the allocation after the current round remains JFXRR. If $(A_1, \ldots, A_b \cup \{o\}, \ldots, A_n)$ is JFXRR, then the proof is done. Otherwise, we claim that $(A_1, \ldots, X, \ldots, A_n)$ is JFXRR.

For this claim, we need to show that agent b and agent $c \in N \setminus \{b\}$ are JFXRR of each other. Every two agents where agent b is not involved are clearly JFXRR of each other because their allocations remain unchanged at the end of the current iteration.

Indeed, as $(A_1, \ldots, A_b \cup \{o\}, \ldots, A_n)$ violates JFXRR, $u_b(A_b \cup \{o\} \setminus \{g^-\}) < u_a(A_a)$ holds for some $g^- \in A_b, a \in N$. Thus, we can find an inclusion-wise minimal subset $X \subset A_b \cup \{o\}$ such that $u_b(X) < u_a(A_a)$ holds, e.g. $X = A_b \cup \{o\} \setminus \{g^-\}$. Even more, for a fixed $Y \subset X$, we have $u_b(Y) \geq u_c(A_c)$ for each $c \in N$.

Since X is a minimal subset of $A_b \cup \{o\}$ such that $u_b(X) < u_a(A_a)$ holds for some $a \in N$, we have $u_b(X \setminus \{g^-\}) \geq u_c(A_c)$ for each $g^- \in X$ and each $c \in N$. Therefore, agent b remains indeed JFXRR of agent $c \in N \setminus \{b\}$. We next also show that agent $c \in N \setminus \{b\}$ remains JFXRR of agent b.

As $u_b(X) < u_a(A_a)$ holds for some $a \in N$ and $u_c(A_c) \leq u_b(A_b)$ holds for each $c \in N$ by the choice of b, we conclude that $u_b(X) < u_b(A_b)$ holds. Thus, we derive $u_c(A_c \setminus \{g^-\}) \geq u_b(A_b) > u_b(X)$ for all $g^- \in A_c$, where the inequality $u_c(A_c \setminus \{g^-\}) \geq u_b(A_b)$ follows from A being JFXRR. The argument concludes.

At each iteration of the algorithm, either (1) one bad is allocated or (2) some bads are returned to the pool of currently unallocated bads but the overall sum of agents' utilities strictly decreases. As the number of bads is bounded from above by m and the minimum overall sum of agents' utilities is also bounded from below, the algorithm is guaranteed to terminate in a finite number of rounds.

Indeed, there can be at most m rounds where (1) holds. After that, either all bads are allocated or there is a round where (2) holds. We let U denote the maximum value of $|\sum_{a \in [n]} u_a(A_a)|$ and u denote the minimum value of $|u_a(X) - u_b(Y)| > 0$ for each $a, b \in N$ and each $X \subseteq M, Y \subseteq M$. After $O(m\frac{U}{u})$ rounds, either the sum of agents' utilities reaches $-U$ or all bads are allocated. $\square$

By Theorem 2, JFXRR allocations exist in instances with general utilities for either goods *or* bads. We hoped to prove that JFXRR allocations exist in instances with general utilities for goods *and* bads. Despite our efforts, we could not come up with a solution that satisfies JFXRR in this case. For this reason, we leave it as an *open* problem.

As a result, we might prefer JFXRC to JFXRR because JFXRC allocations exist in instances with general utilities for goods and bads, whereas JFXRR allocations exist in instances with general utilities for either goods or bads, and it remains unclear whether JFXRR allocations exist in instances with general utilities for goods and bads.

5.2 Jealousy Freeness Up To One Item

Each leximin++ allocation satisfies JFXRC. But, it can be computed in $O(n^m)$ time [17]. This run-time complexity might be fine for a constant value of m. However, m can be much larger than n in practice. For this reason, we propose to relax JFXRC.

Likewise, as we mentioned previously, we do not know whether JFXRR allocations exist in instances with general utilities for goods and bads. Nevertheless, we expect that a possible solution will inherit the complexity of computing leximin++ allocations. For this reason, we also propose to relax JFXRR.

Respectively, we say that *agent a is JF1RC or JF1RR of agent b* whenever a is not JFXRC or JFXRR, but the JFXRC or JFXRR conditions hold for at least one individual bad in a's bundle or at least one individual good in b's bundle.

Definition 3 (JF1RC) *A is* JF1RC *if, $\forall a, b \in N$ such that a is not JFXRC of b, (1) $\exists o \in A_a$ such that $u_a(A_a) \geq u_b(A_b \cup \{o\})$ or (2) $\exists o \in A_b$ such that $u_a(A_a) \geq u_b(A_b \setminus \{o\})$.*

Definition 4 (JF1RR) *A is* JF1RR *if, $\forall a, b \in N$ such that a is not JFXRR of b, (1) $\exists o \in A_a$ such that $u_a(A_a \setminus \{o\}) \geq u_b(A_b)$ or (2) $\exists o \in A_b$ such that $u_a(A_a) \geq u_b(A_b \setminus \{o\})$.*

By definition, JFXRC is stronger than JF1RC and JFXRR is stronger than JF1RR. Like JFXRC and JFXRR, JF1RC and JF1RR could exhibit substantial differences. To see this, let us recall Example 1. It is easy to observe that an allocation is JFXRC if and only if JF1RC; JFXRR if and only if JF1RR. Hence, in Example 1, each JF1RC allocation maximizes the Nash welfare and utilitarian welfare, whereas each JF1RR allocation does not do that.

However, unlike JFXRC and JFXRR whose satisfiability exhibits respectively exponential and pseudo-polynomial solutions, computing JF1RC and JF1RR allocations can be done in polynomial time by using our novel Algorithms 2 and 3, respectively.

Algorithm 2 allocates the items one by one. If the current item is good, then the algorithm gives it to the *min*imum utility *agent*. Otherwise, the algorithm runs a *pseudo* experiment, where a copy of the bad is allocated to every agent, and actually gives the bad to the *max*imum utility *agent* in this experiment.

By comparison, Algorithm 3 allocates the items one by one. If the current item is good, then the algorithm gives it to the *min*imum utility *agent*. Otherwise, the algorithm gives it to the *max*imum utility *agent*.

When allocating houses, both Algorithms 2 and 3 give priority to the more impacted people of the current unfair allocations. When allocating tasks, they give priority to the least impacted people of the current unfair allocations. These decision criteria could be used to explain to people how the algorithms limit inequalities.

Algorithm 2 JF1RC in round instances with zero endowments and general valuations.

1: **procedure** MINAGENTPSEUDOMAXAGENT($N, M, (u_a)_n$)
2: $O \leftarrow M, \forall a \in N : A_a \leftarrow \emptyset$
3: **while** $O \neq \emptyset$ **do**
4: $o \leftarrow$ an item from O
5: **if** o is social good **then**
6: $a \leftarrow \arg\min_{b \in N} u_b(A_b)$
7: **else** ▷ i.e. o is social bad
8: $a \leftarrow \arg\max_{b \in N} u_b(A_b \cup \{o\})$
9: $A_a \leftarrow A_a \cup \{o\}, O \leftarrow O \setminus \{o\}$
10: **return** A

Algorithm 3 JF1RR in round instances with zero endowments and general valuations.

1: **procedure** MINAGENTMAXAGENT($N, M, (u_a)_n$)
2: Copy lines 2-7 from Algorithm 2.
3: $a \leftarrow \arg\max_{b \in N} u_b(A_b)$
4: Copy lines 9-10 from Algorithm 2.

Theorem 3 *Pick a one-round instance. With zero endowments and general valuations for goods and bads, Algorithm 2 returns JF1RC allocations.*

Theorem 4 *Pick a one-round instance. With zero endowments and general valuations for goods and bads, Algorithm 3 returns JF1RR allocations.*

Proof. The proof is inductive. In the base case, no items are allocated. This allocation is JF1RR. In the hypothesis, we let A denote the partial allocation and assume that A is JF1RR. In the step case, we show that allocating o to agent a preserves JF1RR. By the hypothesis, agents $b \neq a$ and $c \neq a$ remain JF1RR after o is allocated to a because their allocations remain intact. We thus consider two cases.

Case 1: Let o be a social good. In this case, the proof is identical to the proof of Case 1 from Theorem 3. However, the conclusions are that: agent c remains JF1RR of agent a and agent a remains JF1RR of agent c.

Case 2: Let o be a social bad. In this case, $u_b(A_b \cup \{o\}) \leq u_b(A_b)$ and $u_b(A_b) \leq u_a(A_a)$ for each $b \in N$. Let us consider some agent $c \neq a$. If $u_a(A_a \cup \{o\}) < u_a(A_a)$, it follows that $u_c(A_c) \leq u_a(A_a \cup \{o\} \setminus \{o\})$ holds. If $u_a(A_a \cup \{o\}) = u_a(A_a)$, it follows that $u_c(A_c) \leq u_a(A_a \cup \{o\})$ holds. Agent a remains JF1RR of agent c.

In the opposite direction, as A is JF1RR, we conclude $u_c(A_c) \geq u_a(A_a \setminus \{g^+\})$ for some non-zero marginal good $g^+ \in A_a$ or $u_c(A_c \setminus \{g^-\}) \geq u_a(A_a)$ for some non-zero marginal bad $g^- \in A_c$. Thus, as item o is social bad, we derive $u_a(A_a \cup \{o\}) \leq u_a(A_a)$ and $u_a(A_a \cup \{o\} \setminus \{g^+\}) \leq u_a(A_a \setminus \{g^+\})$. Agent c remains JF1RR of agent a. □

The input of both Algorithms 2 and 3 is bounded by $O(mn)$. Their running times are dominated by computing a sorting of the n agents' bundle utilities for each of the m items, i.e. $O(m \cdot n)$.

To sum up, as both JF1RC and JF1RR allocations can be computed in polynomial time, we may be indifferent between such allocations or insist on a secondary objective such as envy-freeness [8] or Pareto optimality [16].

6 Repeated Instances

We end with repeated fair division instances that repeat at least one round. That is, in such instances, we have that $t > 1$ holds. Pick round $\tau \in [t]$. If $\tau = 1$, then $e_a^1 = 0$ holds for each $a \in N_1$. However, if $\tau > 1$, some agents might have been allocated items in previous rounds and, therefore, $e_a^\tau \neq 0$ might hold for some $a \in N_\tau$ at round τ. A special case is when the agent endowments are *equal* at round τ, i.e. $e_a^\tau = e^\tau$ holds for each $a \in N_\tau$. In this case, we can use the previously proposed solutions to return almost jealousy-free allocations.

Theorem 5 *Pick a repeated instance and round τ in it. (A) With equal endowments and general valuations for goods and bads at round τ, the leximin++ solution returns JFXRC allocations at τ. (B) With equal endowments and general valuations for goods at τ, the leximin++ solution returns JFXRR allocations at τ. With equal endowments and general valuations for bads at τ, Algorithm 1 returns JFXRR allocations at τ. (C) With equal endowments and general valuations for goods and bads at τ, Algorithm 2 returns JF1RC allocations at τ. (D) With equal endowments and general valuations for goods and bads at τ, Algorithm 3 returns JF1RR allocations at τ.*

Proof. The key argument is that, in a repeated instance and round τ in it, we can effectively reduce the round instance and allocations in it at τ where all endowments are the same to a one-round instance and allocations in it where all endowments are zeros. Then, by the definitions of the axiomatic norms, statement (A) follows by Theorem 1, statement (B) follows by Theorem 2, statement (C) follows by Theorem 3, and statement (D) follows by Theorem 4. □

As soon as two endowments are not equal at round $\tau > 1$ (i.e. $e_a^\tau \neq e_b^\tau$ holds for some $a, b \in N_\tau$ with $a \neq b$), we may no longer be able to achieve JF1RC or JF1RR.

Theorem 6 *Pick a repeated instance and round $\tau > 1$ in it. With two agents, unequal endowments, and additive valuations at round τ, it might be the case that* no *allocation at round τ satisfies JF1RC or JF1RR.*

Proof. Let us consider two goods g_1 and g_2, each giving a unit valuation to every agent. Further, let us define the endowments of agents a and b as $e_a^\tau = 4$ and $e_b^\tau = 1$, respectively. The result follows because there are no bads in the instance at round τ and, in each allocation $A^\tau = (A_a^\tau, A_b^\tau)$, we cannot eliminate the jealousy of b due to the fact that the following inequalities hold: $u_b^\tau(A_b^\tau) \leq u_b^\tau(\{g_1, g_2\}) = e_b^\tau + v_b^\tau(g_1) + v_b^\tau(g_2) = 3 < 4 = e_a^\tau + v_a^\tau(\emptyset) = e_a^\tau = u_a^\tau(\emptyset) \leq u_a^\tau(A_a^\tau \setminus \{g_1\}) = u_a^\tau(A_a^\tau \setminus \{g_2\})$. □

In response, we propose to place a bound on the agent-pairwise utility difference in a given allocation. Thus, for every pair of agents $a, b \in N_\tau$ where a is neither JF1RC nor JF1RR of b, we require that a *is JF1B of* b, i.e. the utility of b minus the utility of a is bounded from above by their absolute endowment difference plus the maximum marginal valuation for any meaningful item move, that either increases the utility of a or decreases the utility of b, within the bundles of a and b. We next formally extend this property to allocations.

Definition 5 (JF1B) *Allocation A^τ is* bounded jealousy-free *if, $\forall a, b \in N$ such that a is not JF1RC or JF1RR of b, $u_b^\tau(A_b^\tau) - u_a^\tau(A_a^\tau) \leq |e_b^\tau - e_a^\tau| + \max\{J_a^-, J_a^+, JF_b^-, JF_b^+\}$, where we have (a) $J_a^- = \max_{\forall o \in A_a^\tau : [v_a^\tau(A_a^\tau \setminus \{o\}) - v_a^\tau(A_a^\tau)] > 0}[v_a^\tau(A_a^\tau \setminus \{o\}) - v_a^\tau(A_a^\tau)]$ and (b) $J_a^+ = \max_{\forall o \in A_b^\tau : [v_a^\tau(A_a^\tau \cup \{o\}) - v_a^\tau(A_a^\tau)] > 0}[v_a^\tau(A_a^\tau \cup \{o\}) - v_a^\tau(A_a^\tau)]$, and (c) $JF_b^- = \max_{\forall o \in A_a^\tau : [v_b^\tau(A_b^\tau) - v_b^\tau(A_b^\tau \cup \{o\})] > 0}[v_b^\tau(A_b^\tau) - v_b^\tau(A_b^\tau \cup \{o\})]$ and (d) $JF_b^+ = \max_{\forall o \in A_b^\tau : [v_b^\tau(A_b^\tau) - v_b^\tau(A_b^\tau \setminus \{o\})] > 0}[v_b^\tau(A_b^\tau) - v_b^\tau(A_b^\tau \setminus \{o\})]$.*

By definition, if an allocation at round τ is JF1RC or JF1RR then it is also bounded jealousy-free. However, the reversed implication may not always be true. For example, in the allocation from Theorem 6, where the jealousy-free agent a gets no goods and the jealous agent b gets all goods, agent b is not JF1RC or JF1RR of agent a, but agent b is JF1B of agent a because $1 = u_a^\tau(\emptyset) - u_b^\tau(\{g_1, g_2\}) \leq |e_a^\tau - e_b^\tau| + \max\{0, 0, 0, 0\} = 3$ hold. Interestingly, we can compute JF1B allocations at round τ in polynomial time. For this purpose, we can use Algorithms 2 and 3.

Theorem 7 *Pick a repeated instance and round τ in it. With general endowments and general valuations for goods and bads, Algorithms 2 and 3 return JF1B allocations.*

Proof. The proof is inductive. In the base case, we let no items be allocated. This partial allocation is JF1B. In the hypothesis, we let some but not all items be allocated. We assume that the current allocation is JF1B. In the step case, we let the current item o be allocated to agent a. Thus, as only the bundle of agent a is changed, we only need to show that agents a and $b \neq a$ remain JF1B of each other because every other two agents remain JF1B of each other by the hypothesis.

Case 1 for Algorithms 2 and 3: Let o be a social good. Immediately after o is allocated to a, the utility of a does not decrease, and, hence, a remains JF1B of every b by the hypothesis. Also, immediately before o is allocated to a, every b is JF1RC and JF1RR of a because a is the minimum utility agent. Hence, immediately after o is allocated to a, every b that becomes neither JF1RC nor JF1RR of a is such that the utility of a minus the utility of b is at most $|e_a^\tau - e_b^\tau|$ plus the valuation of a with o minus the valuation of a without o, which is at most $|e_a^\tau - e_b^\tau|$ plus JF_a^+.

Case 2 for Algorithm 2: Let o be a social bad. As soon as o is allocated to a, the utility of a does not increase, and, hence, any other b remains JF1B of a by the hypothesis. Also, immediately before o is allocated to a, the utility of b with o is at most the utility of a with o because a is a pseudo maximum utility agent. Hence, immediately after o is allocated to a, every b for which a becomes neither JF1RC nor JF1RR of b is such that the utility of b minus the utility of a with o is at most $|e_b^\tau - e_a^\tau|$ plus the valuation of b without o minus the valuation of b with o, which is at most $|e_b^\tau - e_a^\tau|$ plus JF_b^-.

Case 2 for Algorithm 3: Let o be a social bad. As soon as o is allocated to a, the utility of a does not increase, and, hence, every b remains JF1B of a by the hypothesis. Also, immediately before o is allocated to a, a is JF1RC and JF1RR of every b because a is the maximum utility agent. Hence, immediately after o is allocated to a, every b for which a becomes neither JF1RC nor JF1RR of b is such that the utility of b minus the utility of a is at most $|e_b^\tau - e_a^\tau|$ plus the valuation of a without o minus the valuation of a with o, which is at most $|e_b^\tau - e_a^\tau|$ plus J_a^-. □

Finally, we ran Algorithms 2 and 3 in experiments. We generated instances where five agents compete for five items over five rounds[1]. The round valuations for houses were Borda [7], i.e. permutations of $(1, 2, 3, 4, 5)$, and for tasks were negative Borda, i.e. permutations of $(-1, -2, -3, -4, -5)$. The round valuations were additive valuation sums over items received in the current round. The round endowments were additive valuation sums over items received in the previous rounds. Thus, we considered three settings: in each round, (1) only Borda valuations were sampled (house allocations); (2) only negative Borda valuations were sampled (task allocations); (3) mixed Borda valuations (e.g. $(-1, 2, 3, -4, 5)$) were sampled (house and task allocations). Our performance measure was the value of the inequality variance – see Example 1 – at the end of each round. This value converged to zero with both algorithms but quicker with Algorithm 2 than with Algorithm 3. For this result, we refer the reader to Appendix D.

[1] Our instances extend the instances used in the experiment of Herreiner and Puppe [12], where there are two to three agents and three to four items.

7 Conclusions and Future Directions

We considered limiting inequalities in repeated house and task allocation problems. We looked at these problems because every individual in our society has the right to have better living and working conditions. For such problems, we proposed five axiomatic norms, denoted as JFXRC, JFXRR, JF1RC, JF1RR, and JF1B, that limit inequalities in allocations gradually, namely people prefer JFXRC and JFXRR to JF1RC and JF1RR, and JF1RC and JF1RR to JF1B. Our conclusions are two-fold: (1) computing JFXRC and JFXRR allocations might take time longer than the time people can wait to be allocated houses and tasks; (2) computing JF1RC, JF1RR, and JF1B allocations is faster but JF1RC and JF1RR allocations may not always exist, in which case people may accept JF1B allocations that limit inequalities over multiple rounds. In the future, we will run experiments and estimate the performance of the proposed solutions in practice in terms of their scalability. Finally, we will surely also extend our work to include items that could be good for some individuals and bad for other individuals.

References

1. Abdulkadiroğlu, A., Sönmez, T.: House allocation with existing tenants. Journal of Economic Theory **88**(2), 233–260 (1999). https://doi.org/https://doi.org/10.1006/jeth.1999.2553, https://www.sciencedirect.com/science/article/pii/S002205319992553X
2. Aziz, H., Moulin, H., Sandomirskiy, F.: A polynomial-time algorithm for computing a Pareto optimal and almost proportional allocation. Operations Research Letters **48**(5), 573–578 (2020). https://doi.org/https://doi.org/10.1016/j.orl.2020.07.005
3. Aziz, H., Rey, S.: Almost group envy-free allocation of indivisible goods and chores. In: Proceedings of the Twenty-Ninth International Joint Conference on Artificial Intelligence. pp. 39–45. IJCAI'20, IJCAI Press (2021), https://dl.acm.org/doi/abs/10.5555/3491440.3491446
4. Brams, S.J., Taylor, A.D.: Fair Division - from Cake-cutting to Dispute Resolution. Cambridge University Press (1996), https://doi.org/10.1017/CBO9780511598975
5. Caragiannis, I., Kaklamanis, C., Kanellopoulos, P., Kyropoulou, M.: The efficiency of fair division. Theory of Computing Systems **50**(4), 589–610 (May 2012). https://doi.org/10.1007/s00224-011-9359-y
6. Caragiannis, I., Kurokawa, D., Moulin, H., Procaccia, A.D., Shah, N., Wang, J.: The unreasonable fairness of maximum Nash welfare. ACM Trans. Econ. Comput. **7**(3) (September 2019). https://doi.org/10.1145/3355902
7. Darmann, A., Klamler, C.: Using the borda rule for ranking sets of objects. Social Choice and Welfare **53**(3), 399–414 (2019), http://www.jstor.org/stable/45212428
8. Foley, D.K.: Resource allocation and the public sector. Yale Economic Essays **7**(1), 45–98 (1967), https://www.proquest.com/docview/302230213?pq-origsite=gscholar&fromopenview=true
9. Freeman, R., Sikdar, S., Vaish, R., Xia, L.: Equitable allocations of indivisible goods. In: Proceedings of the 28th International Joint Conference on Artificial Intelligence. pp. 280–286. IJCAI'19, AAAI Press (2019), https://dl.acm.org/doi/10.5555/3367032.3367073
10. Freeman, R., Sikdar, S., Vaish, R., Xia, L.: Equitable allocations of indivisible chores. In: Proceedings of the 19th International Conference on Autonomous Agents and MultiAgent Systems. pp. 384–392. AAMAS '20, IFAAMAS, Richland, SC (2020), https://dl.acm.org/doi/10.5555/3398761.3398810

11. Gourvès, L., Monnot, J., Tlilane, L.: Near fairness in matroids. In: Proceedings of the Twenty-First European Conference on Artificial Intelligence. pp. 393–398. ECAI'14, IOS Press, NLD (2014), https://dl.acm.org/doi/10.5555/3006652.3006719
12. Herreiner, D., Puppe, C.: Envy freeness in experimental fair division problems. Theory and Decision **67** (January 2005). https://doi.org/10.1007/s11238-007-9069-8
13. Igarashi, A., Lackner, M., Nardi, O., Novaro, A.: Repeated fair allocation of indivisible items (2023), https://arxiv.org/abs/2304.01644
14. Krynke, M., Mielczarek, K., Vaško, A.: Analysis of the problem of staff allocation to work stations. Conference Quality Production Improvement – CQPI **1**(1), 545–550 (2019). https://doi.org/doi:10.2478/cqpi-2019-0073, https://doi.org/10.2478/cqpi-2019-0073
15. Lipton, R.J., Markakis, E., Mossel, E., Saberi, A.: On approximately fair allocations of indivisible goods. In: Proceedings of the 5th ACM Conference on Electronic Commerce, New York, USA, May 17-20, 2004. pp. 125–131 (2004), https://dl.acm.org/doi/10.1145/988772.988792
16. Pareto, V.: Cours d'Économie politique. Professeur à l'Université de Lausanne. Vol. I. Pp. 430. 1896. Vol. II. Pp. 426. 1897. Lausanne: F. Rouge (1897), https://www.cairn.info/cours-d-economie-politique-tomes-1-et-2--9782600040143.htm
17. Plaut, B., Roughgarden, T.: Almost envy-freeness with general valuations. SIAM Journal on Discrete Mathematics **34**(2), 1039–1068 (2020). https://doi.org/10.1137/19M124397X

A Proof of Theorem 1

Let A denote a leximin++ allocation. Suppose that A is not JFXRC for a given pair of agents $a \in N$ and $b \in N$ with $a \neq b$. That is, $u_a(A_a) < u_b(A_b)$. Also, (1) $u_a(A_a) < u_b(A_b \cup \{o\})$ holds for $o \in A_a$ with $u_a(A_a) < u_a(A_a \setminus \{o\})$ or (2) $u_a(A_a) < u_b(A_b \setminus \{o\})$ holds for $o \in A_b$ with $u_b(A_b) > u_b(A_b \setminus \{o\})$. Without loss of generality, we let $u_1(A_1) \leq \ldots \leq u_n(A_n)$ hold and define $k = \arg\max\{i \in N | u_i(A_i) \leq u_a(A_a)\}$. We note $a \leq k$ and $k < b$. We next consider two cases.

Case 1: Let (1) hold for bad $o \in A_a$. Let us move o from A_a to A_b. We let C denote this new allocation. That is, $C_a = A_a \setminus \{o\}$, $C_b = A_b \cup \{o\}$ and $C_c = A_c$ for each $c \in N \setminus \{a, b\}$. We argue that $C \succ_{++} A$ holds.

We note that $C_c = A_c$ holds for each $c \in [k] \setminus \{a\}$. We show $u_{a_k}(C_{a_k}) > u_k(A_k)$ where a_k is the kth agent in the utility order induced by C. If this agent is a, then $u_a(C_a) = u_a(A_a \setminus \{o\}) > u_a(A_a) = u_k(A_k)$ by (1). If this agent is b, then $u_b(C_b) = u_b(A_b \cup \{o\}) > u_a(A_a) = u_k(A_k)$ by (1). Otherwise, $u_{a_k}(C_{a_k}) = u_{k+1}(A_{k+1}) > u_k(A_k)$ by the choice of k. It follows in each case that $u_d(C_d) \geq u_{a_k}(C_{a_k}) > u_k(A_k)$ holds for each agent $d \in N \setminus [([k] \setminus \{a\}) \cup \{a_k\}]$. Therefore, A cannot be a leximin++ allocation. This observation leads to a contradiction with the choice of A.

Case 2: Let (2) hold for good $o \in A_b$. Let us move only item o from A_b to A_a. We let B denote this allocation: $B_a = A_a \cup \{o\}$, $B_b = A_b \setminus \{o\}$ and $B_c = A_c$ for each $c \in N \setminus \{a, b\}$. We next argue that $B \succ_{++} A$ holds.

As item o is good, it follows $u_a(B_a) \geq u_a(A_a)$. If $u_a(B_a) = u_a(A_a)$, then $B_c = A_c$ holds for each $c < a$, $|B_a| = |A_a| + 1$ and $B_d = A_d$ for each $d \in (a, k]$. Moreover, it follows that $u_e(B_e) > u_k(A_k) = u_a(A_a)$ holds for each agent $e \in N \setminus [k]$, including for agent b by (2). As the leximin++ solution maximizes the bundle size as a secondary objective, it follows that $B \succ_{++} A$ holds. Hence, A cannot be a leximin++ allocation. If $u_a(B_a) > u_a(A_a)$, then we can derive a contradiction as in Case 1 but we use in the proof condition (2) instead of condition (1).

B Proof of Theorem 3

The proof is inductive. In the base case, no items are allocated. This allocation is JF1RC. In the hypothesis, we let A denote the partial allocation and assume that A is JF1RC. In the step case, we show that allocating o to agent a preserves JF1RC. By the hypothesis, agents $b \neq a$ and $c \neq a$ remain JF1RC after o is allocated to a.

Case 1: Let o be a social good. In this case, $u_b(A_b \cup \{o\}) \geq u_b(A_b)$ and $u_b(A_b) \geq u_a(A_a)$ for each $b \in N$. Let us consider some agent $c \neq a$. If $u_a(A_a \cup \{o\}) > u_a(A_a)$, it follows that $u_c(A_c) \geq u_a(A_a \cup \{o\} \setminus \{o\})$ holds. If $u_a(A_a \cup \{o\}) = u_a(A_a)$, it follows that $u_c(A_c) \geq u_a(A_a \cup \{o\})$ holds. c remains JF1RC of a.

In the opposite direction, as A is JF1RC, we conclude $u_a(A_a) \geq u_c(A_c \setminus \{g^+\})$ for some non-zero marginal good $g^+ \in A_c$ or $u_a(A_a \setminus \{g^-\}) \geq u_c(A_c)$ for some non-zero marginal bad $g^- \in A_a$. Thus, as item o is social good, we derive $u_a(A_a \cup \{o\}) \geq u_a(A_a)$ and $u_a(A_a \cup \{o\} \setminus \{g^-\}) \geq u_a(A_a \setminus \{g^-\})$. a remains JF1RC of c.

Case 2: Let o be a social bad. In this case, $u_b(A_b \cup \{o\}) \leq u_b(A_b)$ and $u_b(A_b \cup \{o\}) \leq u_a(A_a \cup \{o\})$ for each $b \in N$. Let us consider some agent $c \neq a$. For the sake of contradiction, let us suppose that a is not JF1RC of c when $u_a(A_a \cup \{o\}) < u_c(A_c \cup \{o\})$ must hold. But, this fact contradicts the choice of a. a remains JF1RC of c.

In the opposite direction, as A is JF1RC, we conclude $u_c(A_c) \geq u_a(A_a \setminus \{g^+\})$ for some non-zero marginal good $g^+ \in A_a$ or $u_c(A_c) \geq u_a(A_a \cup \{g^-\})$ for some non-zero marginal bad $g^- \in A_c$. Thus, as item o is social bad, we derive $u_a(A_a \cup \{o\} \cup \{g^-\}) \leq u_a(A_a \cup \{g^-\})$ and $u_a(A_a \cup \{o\} \setminus \{g^+\}) \leq u_a(A_a \setminus \{g^+\})$. c remains JF1RC of a.

C Full Proof of Theorem 5

The proof of Theorem 5 in the main text contains our short proof of this result. We next give the complete proof of it. Pick two bundles $M_1 \subseteq M_\tau$ and $M_2 \subseteq M_\tau$ and two agents $a, b \in N_\tau$ with $a \neq b$. We have the following equivalences due to the fact that the endowments are equal: (1) $u^\tau_a(M_1) < u^\tau_a(M_2)$ if and only if $v^\tau_a(M_1) < v^\tau_a(M_2)$; (2) $u^\tau_a(M_1) = u^\tau_a(M_2)$ if and only if $v^\tau_a(M_1) = v^\tau_a(M_2)$; (3) $u^\tau_a(M_1) \leq u^\tau_a(M_2)$ if and only if $v^\tau_a(M_1) \leq v^\tau_a(M_2)$; (4) $u^\tau_a(M_1) < u^\tau_b(M_2)$ if and only if $v^\tau_a(M_1) < v^\tau_b(M_2)$; (5) $u^\tau_a(M_1) = u^\tau_b(M_2)$ if and only if $v^\tau_a(M_1) = v^\tau_b(M_2)$; (6) $u^\tau_a(M_1) \leq u^\tau_b(M_2)$ if and only if $v^\tau_a(M_1) \leq v^\tau_b(M_2)$. Thus, for a given A^τ, A^τ is JFXRC, JFXRR, JF1RC, or JF1RR with equal endowments and general valuations at round τ if and only if A^τ is JFXRC, JFXRR, JF1RC, or JF1RR with zero endowments and general valuations at round τ if and only if, in a newly constructed one-round instance with agents from $N_1 = N_\tau$ and items from $M_1 = M_\tau$, $A^1 = A^\tau$ is JFXRC, JFXRR, JF1RC, or JF1RR with zero endowments and general valuations. Thus, (A) follows by Theorem 1, (B) follows by Theorem 2, (C) follows by Theorem 3, and (D) follows by Theorem 4.

D Experiments

Figure 1 shows the performance of Algorithms 2 and 3.

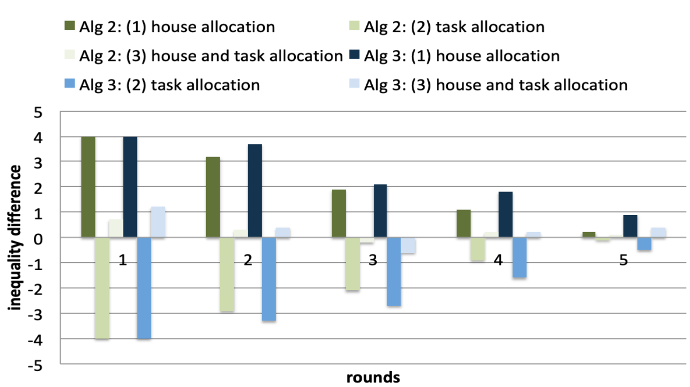

Fig. 1. The performance of Algorithms 2 and 3.