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(Offline) Fair Division is a fundamental problem in which a number of resources are allocated to a number of agents supposing *complete* information about the preferences of the agents for the resources. By comparison, in Online Fair Division, the resources arrive over time and are allocated to agents only as they become available supposing *limited* information about resources that could possibly arrive in the future. In this thesis, I study the *problem* of online fair division with indivisible resources.

I consider a simple model for online fair division in which an indivisible item arrives at a given time step, the agents report their preferences for the item and then the item is allocated to an agent by using an allocation mechanism. The online nature plays a significant role in the design of mechanisms for online fair division compared to mechanisms for offline fair division. An allocation mechanism for online fair division must make decisions supposing limited problem input. The online nature also has an impact on characterizing important fair division axiomatic properties such as strategy-proofness, envy-freeness, Pareto efficiency, etc. For example, we cannot use offline mechanisms to characterize these properties. Also, there are less envy-free and Pareto efficient online mechanisms than offline mechanisms and more strategy-proof online mechanisms than offline mechanisms. The online nature further challenges the competitive worst-case performance of online mechanisms. An online allocation mechanism is less likely to perform as well as an offline allocation mechanism.

I conduct throughout the thesis an extensive comparison of my results with existing results in offline fair division. My thesis thus has a number of contributions to the field of Fair Division. I first describe the model of *online fair division* and propose six *online mechanisms* for it. I prove that most of these mechanisms are *characterizing* for online fair division. I next study the computational complexity of the *expected outcomes and manipulations* of these mechanisms. For example, agents may lie about their preferences for items and thus manipulate the expected outcomes of mechanisms in their favour. I therefore further consider computational questions around the *pure equilibrium* strategies of agents for items. Interestingly, we cannot have fairness and efficiency in expectation in online fair division but we can have both of these properties in (offline) fair division. For this reason, I discuss how often these mechanisms return *fair and efficient* actual outcomes, and how to compute these outcomes. Finally, I analyse the competitive performance of these mechanisms and show that they are *most competitive* for common fair division objectives such as the utilitarian welfare, egalitarian welfare and Nash welfare.

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DOCTORAL THESIS

Online Fair Division with Indivisible Items

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Abstract

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Online Fair Division with Indivisible Items

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Originality Statement

I hereby declare that this submission is my own work and to the best of my knowledge it contains no materials previously published or written by another person, or substantial proportions of material which have been accepted for the award of any other degree or diploma at UNSW or any other educational institution, except where due acknowledgement is made in the thesis. Any contribution made to the research by others, with whom I have worked at UNSW or elsewhere, is explicitly acknowledged in the thesis. I also declare that the intellectual content of this thesis is the product of my own work, except to the extent that assistance from others in the project's design and conception or in style, presentation and linguistic expression is acknowledged.

Signed:

Date:

To my parents Damyan Aleksandrov and Svetla Aleksandrova, my brother Iliyan Aleksandrov and my cute nephew Damyan Aleksandrov.

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Preface

This thesis is based on a number of conference publications. I next list these publications and relate them to the chapters of the thesis.

1 Martin Aleksandrov, Haris Aziz, Serge Gaspers, Toby Walsh: *Online Fair Division: Analysing a Food Bank Problem.* IJCAI 2015 **Outstanding Student Paper Award**: pages 2540-2546
2 Martin Aleksandrov, Toby Walsh: *Pure Nash Equilibria in Online Fair Division.* IJCAI 2017: pages 42-48
3 Martin Aleksandrov, Toby Walsh: *Expected Outcomes and Manipulations in Online Fair Division.* KI 2017 **Best Paper Award**: pages 29-43
4 Martin Aleksandrov, Toby Walsh: *Most Competitive Mechanisms in Online Fair Division.* KI 2017: pages 44-57

In Chapter 1, I motivate my work with a real-world application of online fair division: the food bank problem.

In Chapter 2, I describe the formal model for online fair division and define three axiomatic properties that are commonly studied in (offline) fair division: strategy-proofness, envy-freeness and Pareto efficiency. These properties are still well-defined in online fair division. However, due to the absence of information, I also propose three new online versions of these three (offline) axiomatic properties.

In Chapter 3, I analyse completely all axiomatic properties. I also propose two existing - LIKE and BALANCED LIKE - and four new - ONLINE SERIAL DICTATORSHIP, ONLINE RANDOM PRIORITY, MAXIMUM LIKE and PARETO LIKE - mechanisms for online fair division, and show that some of these mechanisms satisfy most of the axiomatic properties subject to a number of impossibility results. I thus extend some existing results from [1].

Chapter 4 is based on my work from [3]. I study in this paper how to compute outcomes returned by the LIKE and BALANCED LIKE mechanisms as well as issues related to manipulating these two mechanisms. In this chapter, I extend these results to the other four mechanisms.

With some of the proposed online mechanisms, the agents in our online fair division model can benefit by misreporting their preferences for items. Therefore, in Chapter 5, I turn attention to questions around computing and counting manipulation strategies of agents for items. This chapter is based on my work from [2].

In Chapter 6, I discuss my new results regarding computing envy-free and Pareto efficient outcomes of these online mechanisms. Interestingly, we cannot have both properties in expectation and, for this reason, I focus on actual allocations returned by online mechanisms.

In Chapter 7, I analyse the competitive performance of these online mechanisms. This chapter is based on my work from [4]. I proposed there the MAXIMUM LIKE mechanism for the first time and analysed its competitive performance together with the performance of the LIKE and BALANCED LIKE mechanisms. In this paper, I only analysed mechanisms from a utilitarian and an egalitarian perspectives. I extend these results with new results from a Nash perspective.

Finally, I discuss several extensions of my work in Chapter 8 and I close this thesis in Chapter 9.

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List of Symbols

A	set $\{a_1, \ldots, a_n\}$ of n agents
O	set $\{o_1, \ldots, o_m\}$ of m items
o	ordering $(o_1, \ldots, o_m)$ of m items
u_{ij} (v_{ij})	utility (bid) value of a_i for o_j
$U = (u_{ij})_{n \times m}$	utility matrix: row i to a_i, column j to o_j
$V = (v_{ij})_{n \times m}$	bid matrix: row i to a_i, column j to o_j
U_j (V_j)	utility (bid) matrix of the first j columns in U (V)
U_{ij} (V_{ij})	utility (bid) vector of the row i in U_j (V_j)
U_{-ij} (V_{-ij})	utility (bid) matrix U_j (V_j) except vector U_{ij} (V_{ij})
$\mathcal{I} = (A, O, o, V)$	instance of online fair division
$\mathcal{M}$	mechanism for online fair division
$\pi(j)$	allocation of o_1 to o_j
$\pi_i(j)$	allocation $\pi(j)$ in which a_i receives o_j
$p(\pi(j))$	probability of $\pi(j)$
$u(i, k, \pi(j))$	utility of a_i for the items of a_k in $\pi(j)$
$u(i, \pi(j))$	utility of a_i for their items in $\pi(j)$
$\Pi(j)$	set of allocations of o_1 to o_j
$\overline{p}(i, \Pi(j))$	probability of a_i for o_j in $\Pi(j)$
$\overline{u}(i, k, \Pi(j))$	expected utility of a_i for the items of a_k in $\Pi(j)$
$\overline{u}(i, \Pi(j))$	expected utility of a_i for their items in $\Pi(j)$
$\overline{u}(i, \mathcal{I})$	expected utility of a_i for all items in $\mathcal{I}$
$\overline{u}(\mathcal{I})$	utilitarian welfare $\sum_{i=1}^{n} \overline{u}(i, \mathcal{I})$
$\overline{e}(\mathcal{I})$	egalitarian welfare $\min_{i=1}^{n} \overline{u}(i, \mathcal{I})$
$\overline{n}(\mathcal{I})$	Nash welfare $\prod_{i=1}^{n} \overline{u}(i, \mathcal{I})$

Chapter 1

Introduction

A greatly investigated topic in resource allocation is (offline) fair division. Fair division normally stands for the problem of *fairly* dividing resources amongst multiple parties. The problem of fair division was presented for the first time around the fifties of the previous century (Steinhaus, 1948). Since then, many models and algorithms for fair division problems had been proposed. Most of these models had mainly been used to solve economical rather than technological problems (Campbell, 1987). As a result, *efficiency* had been preferred to *equity* over the years which had often been a reason why many classical algorithms for fair division often return allocations that are more efficient than fair. Although such models provide a technical language limited in its expressiveness, they are still of great importance because they lay down the basis for future research. However, due to many environmental, economic and technological challenges that we face nowadays, there are ever increasing pressures on the fair division of resources. Fair division problems become more and more demanding and advances in computer science can support the design of more complex fair division models and algorithms in order to tackle such pressures. To achieve this, we need however to be able to measure the allocations returned by such algorithms both quantitatively and qualitatively. As a result, nowadays equity has slowly become as preferable as efficiency (Walsh, 2014a). For example, suppose that each party receives a portion of the resource they most value. While this is efficient allocation it is unlikely to be equitable. In contrast, suppose the resource is allocated uniformly at random amongst the parties. This allocation is likely to be equitable but it is less likely to be efficient. Equity and efficiency are very important (Koopmans, 1951; 1951). For this reason, many normative measures are proposed in order to measure the quality and quantity of resource allocations, e.g. anonymity, neutrality, false-name-proofness, monotonicity, strategy-proofness, envy-freeness, Pareto optimality, etc. (Clippel, 2008; Kalinowski, Narodytska, Walsh, and Xia, 2013; Conitzer, 2007; Conitzer and Yokoo, 2010; Yokoo, Sakurai, and Matsubara, 2001; Tsuruta, Oka, Todo, Sakurai, and Yokoo, 2015; Satterthwaite, 1975; Bei, Chen, Hua, Tao, and Yang, 2012; Brams and Fishburn, 2000).

There is a large body of work done in fair division since the fifties. In this section, we consider how we might possibly characterize fair division problems with respect to several parameters. For example, the resource can be *divisible* or *indivisible*. Cake cutting is a classical fair division setting with a divisible resource which is to be distributed fairly amongst a finite number of users taking into account their preferences (Steinhaus, 1948; Weller, 1985; Malik Magdon-Ismail and Krishnamoorthy, 2003). A challenging task related to cake cutting is determining the minimum number of cuts that is required in order to provide a fair division of the cake. This is an especially important task when an exponential number of cuts is required to achieve a fair distribution of the cake (Aziz and Mackenzie, 2017). House allocation is another classical fair division setting in which the resource is represented by a collection of indivisible "houses" and they need to be allocated to tenants (Shapley and Scarf, 1974; Abdulkadiroglu and Sönmez, 1999; Sönmez and

Ünver, 2010). In general, the allocation of indivisible items offers additional challenges than the allocation of divisible items. For example, we can share a cake between two people but we cannot share a house between two people. The case of 2 agents often receives a special attention within the general case of multiple agents (Brams and Fishburn, 2000; Edelman and Fishburn, 2001).

Fair division problems can further be *centralized* or *decentralized*. In a centralized problem, there is a governing authority that monitors the allocation of resources. We could think of such an authority as a chair who often wants to maximize say the social welfare or exercise some form of control on the allocation. For example, the chair in a food bank regulates which and what food is donated to charities (Walsh, 2014b; Aleksandrov, Aziz, Gaspers, and Walsh, 2015). Or, they might need to take into account that different charities are often entitled to different resouces and thus they might prefer allocations that reflect these entitlements to other allocations (Aleksandrov, Gaspers, and Walsh, 2015). By comparison, in a decentralized problem, it is very often the case that the allocation of the resource is achieved through say negotiation and communication amongst the parties involved involved in the allocation process (Ronen and Talisman, 2005). For example, many resources happen to be naturally allocated to the parties. However, such allocation may not be efficient and, for this reason, agents need to exchange them (Laurent Gourvés, 2017).

An important aspect of fair division is related to how we value resources. We often have some valuation for each resource and possibly different valuations for different resources. In fact, the utility function that describes our valuations over the resources could be quite arbitrary. For this reason, the domain of this function is very often used to categorize fair division problems. For instance, there are problems in which the utility function is *cardinal* (i.e. assigns a number to each part of the resource) or *ordinal* (i.e. assigns a rank to each part of the resource), *additive* (i.e. we get more value if we receive more resouces) or *non-additive* (e.g. super-additive, sub-modular, etc.), *linear* or *non-linear*, etc.(Chevaleyre, Endriss, Estivie, and Maudet, 2008a; Assaad, Ben-Ameur, and Hamid, 2014; Bashar and Ding, 2009; Miralles, 2012; Iwai, 1972; ParandehGheibi, Eryilmaz, Ozdaglar, and Médard, 2008). The preference structure depends on the utility function and normally satisfies a number of properties that lead to the design of algorithms specifically tailored to the utility function. Fair division problems in which the parties have valuations over the resources are normally called *one-sided*. By comparison, there are also *two-sided* settings in which there are two sets of agents and they need to be matched to each other (Rysman, 2009). A popular example of a two-sided problem is stable marrige. A stable marriage involves men and women which are to be matched according to their preferences (Gale and Shapley, 1962).

Fair division problems can additionally be *static* or *dynamic*. In a static problem, we suppose that the entire resource is available and all preferences are known in advance. In a dynamic problem, we consider the possibility that the preferences may change over time. For instance, Freeman et al. recently considered a dynamic fair division setting in which agents and items are fixed but the preferences of agents for items change over time (Freeman, Zahedi, and Conitzer, 2017). Also, Kash et al. proposed a dynamic model of fair division in which agents arrive over time and the resource is viewed as a collection of homogeneous divisible goods (Kash, Procaccia, and Shah, 2013). Settings of this kind motivate the development of dynamic models that anticipate the preferences of agents for resources as well as their demand for these resources. Moreover, it might be costly for the chair to guarantee a good allocation. In response, there are many settings in which the resource is not free-of-charge and the its allocation takes place subject to say some sort of *monetary transfers* or *budgetary constraints* (Green, Ali, Naeem, and Ross, 2000). We normally use some kind of combinatorial auction mechanism to allocate resources in such settings (Vries and Vohra, 2003; Cramton, Shoham, and Steinberg, 2007; Mares and Swinkels, 2014).

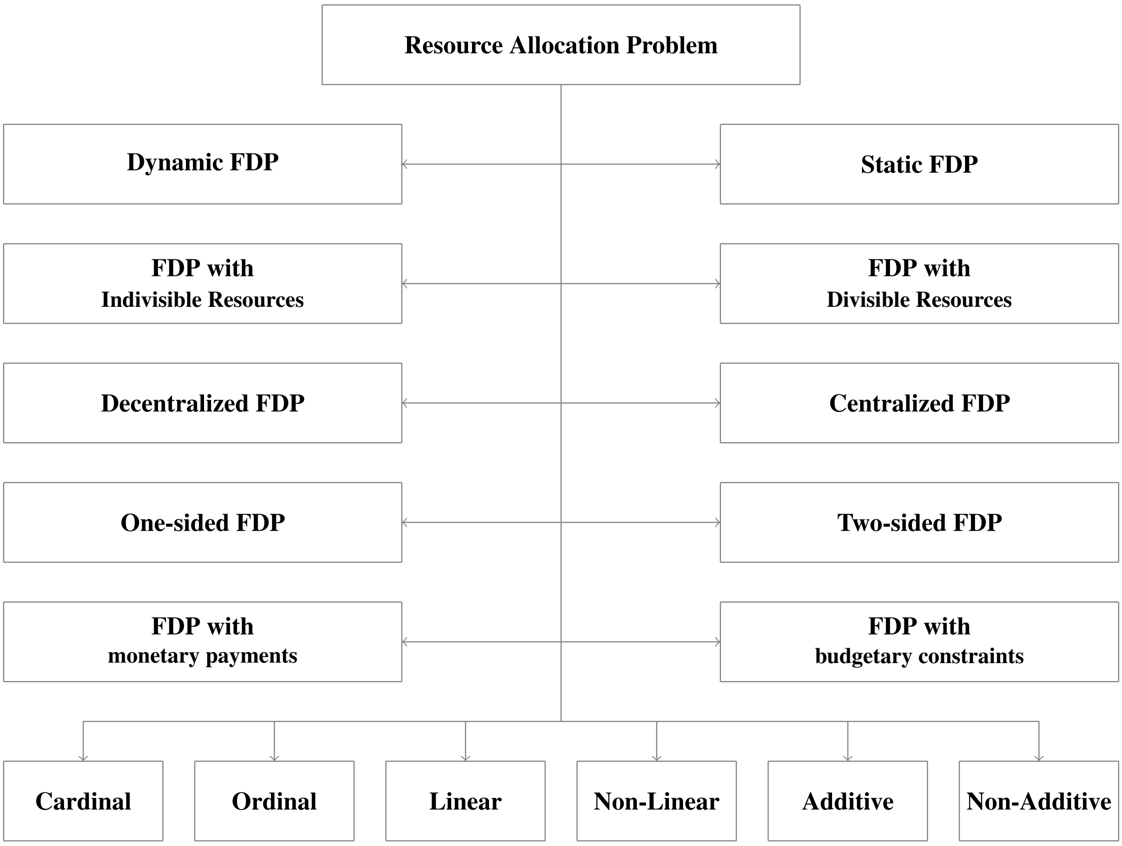

FIGURE 1.1: A classification of Fair Division Problems (FDP).

We briefly discussed some of the most common parameters of fair division problems. In Figure 1.1, we provide a classification of such problems that is based on these parameters which are usually considered in theory. What about allocations in practice? In practice, surprisingly, only a limited number of works introduce the challenges that resource allocation offers. For instance, a consistent approach towards allocating medical resources is used thus diminishing the differences between the micro- and macro-allocation paradigms in US (Liebert, 1983). Also, health care resources can be allocated subject to various ethical or religious considerations (Rangel, 2007; Benn and Hyder, 2002). This is the place at which justice comes into interaction with the allocation of those resources (Gillon, 1985). Yet, game-theoretic algorithms are used to describe the optimal allocation of security resources at the Los Angeles International Airport (LAX) (Kiekintveld, Jain, Tsai, Pita, Ordóñez, and Tambe, 2009). These algorithms outperform the existing algorithms and scale up to problems of several orders of magnitude larger than the fastest known algorithms for this setting. Further, there are many issues related to the secure allocation of the resource when this resources is food. For example, in Nasarawa, Northern Nigeria (and in Africa in general), the food production often requires farming of high quality which itself requires regular watering, access to the lands and supervision (Ibrahim, Bello, and Ibrahim, 2009). Additionally, modern applications such as asymmetric and radio-spectrum auctions have hugely benefited from combinatorial auctions (Decarolis, 2013). In this thesis, our work is also motivated by a practical problem in which donated food is allocated to charities. We next shortly describe the problem.

The food bank problem Consider the classical fair division problem. Fair division problems can be categorised along several orthogonal dimensions: divisible or indivisible goods, centralised or decentralised mechanisms, cardinal or ordinal preferences, etc. Such categories are, however, not able to capture the richness of a practical resource allocation problem which came to our attention recently. FoodBank Local is a social startup founded by students from the University of New South Wales that is working with food bank charities around the world, as well as with us, to improve the efficiency of their operations. FoodBank Local won the Microsoft Imagine Cup in Australia in 2013, and were finalists worldwide for their novel and innovative approach to using technology for social good. In cooperation with FoodBank Local, we have been helping Food Bank Australia develop technologies to operate more effectively. So far, this has involved building an app to help collect and deliver donated food. This app uses our vehicle routing solver Indigo to route their trucks (Aleksandrov, Barahona, Kilby, and Walsh, 2013). We are now turning our attention to how the donated food is allocated to different charities. After supermarkets, catering companies and the public have donated food, the food bank must allocate and distribute this food to charities in the local area. This requires solving a fair division problem. Fairness is important as charities often cater to different sectors of the population (geographical, social and ethnic), whilst efficiency is important to maximise the overall good. This is an interesting fair division problem. It has many traditional features. For example, we want to allocate food *fairly* between the different charities that feed different sectors of the community. Goods are mostly *indivisible*. The allocation does not use money as these are all charities. By comparison, this fair division problem has several new dimensions, rarely considered in the literature in combination.

Online: Offers of donated food arrive throughout the day. The food bank cannot wait till the end of the day before deciding how to allocate the donated food. They further cannot always store the food as this increases their operational costs. This means we have an online problem, where new food items must be allocated in the absence of any information about any future items.

Unequal entitlements: The different charities that work with the food bank have different abilities at feeding their clients. The allocation of food needs to reflect these abilities.

Combinatorial: The different charities have complex, combinatorial preferences over the donated food. A charity might want the donated apples or the bananas but not both. Models based on simple additive utilities, like those often considered in the literature, are simply not rich enough to describe their true preferences.

Constrained: There are various constraints over the allocations. For example, we might have to use the same truck to deliver to the charities the food products that require refrigeration. As a second example, certain combinations of food cannot be put together for health and safety or religious reasons.

Mixed: Each allocation problem induces a new pickup and delivery routing problem. This means that we have a mixed problem that combines resource allocation and logistics. We therefore need to satisfy multiple allocation and transportation objectives at the same time.

The food bank problem is only one of the many fair division problems in practice. An especially challenging form of such a fair division problem is when we are allocating resources in an online fashion with only partial knowledge of the future. Consider allocating students to courses, doctors to hospitals, deceased organs to patients, donated food to charities, viewing slots to astronomers, vehicles to charging stations, tenants to houses, water quotas to farmers, landing slots to airlines, etc (Budish and Cantillon, 2007; Krysta, Manlove, Rastegari, and Zhang, 2014). We often cannot wait till all resources are available, preferences known or agents present before starting to allocate the resources. For example, when a kidney is donated, it must be allocated to a patient within a few hours, and we cannot wait until some more kidneys are donated (Mattei, Saffidine, and Walsh, 2017; Mattei, Saffidine, and Walsh, 2018). As a second example, when managing a river, we might start allocating irrigation water to farmers today, not knowing how much it will rain next month (Ibrahim, Bello, and Ibrahim, 2009). As a third example, when allocating charging slots to electric cars, we may not know when or where cars will arrive for charging in the future (Gerding, Robu, Stein, Parkes, Rogers, and Jennings, 2011).

In this thesis, we focus on this new type of *online fair division* problems. In (offline) fair division, the entire information about the agents, resources and preferences is normally available, and an offline algorithm can allocate the resources using this information. In online fair division, the agents and resources (and therefore the preferences) arrive over time and thus information about them is revealed step by step. The online setting offers additional challenges to the classical offline algorithms because they now need to make decisions with only limited input about the problem of interest. Indeed, we argue that existing mechanisms for (offline) fair division cannot be applied to our online setting. As a consequence, existing axiomatic characterizations change in the online setting. Additionally, computing allocation outcomes of online mechanisms is hard in the online setting. Moreover, online mechanisms are likely to perform much less competitive than offline mechanisms simply because they make decisions with limited input. Finally, in this thesis, our goal is to address some of these challenges in a greater depth.

Chapter 2

Online Fair Division

We use the simple model for centralized online fair division proposed for first time by Walsh (Walsh, 2014b; Walsh, 2015). This model is motivated by the food bank problem. In this model, there are agents (i.e. charities), items (i.e. food products), ordering of items (i.e. the stochastic sequence in which the food products are donated) and preferences of agents for items (i.e. how much charities value the food donations). We next present our assumptions in regard to this model in Section 2.1. In Section 2.2, we define instances and mechanisms for online fair division. A given mechanism allocates items to agents in a given instance and thus returns a number of allocation outcomes. We define these outcomes in Section 2.3. We finally conclude in Section 2.4.

2.1 Assumptions

We make the following four assumptions about the model throughout the entire thesis. First, we suppose that the set of agents is *known* to the central authority. In practice, agents can arrive to and leave from the market over time. However, we can model this feature by setting their utilities to some default value for the time moments when they are not in the market. For example, in the food bank problem, some charities work only in the morning and therefore we can assume that they do not bid for items in the afternoon.

Assumption 1 *The agents are* known.

Second, we assume the items are *indivisible* for a number of reasons. For example, in the food bank problem, the chair prefers to dispatch items as soon as they are donated simply because storing them is expensive in terms of space and costs, and (un)packing them is expensive in terms of time and costs. As a second example, in the organ matching problem, an organ cannot be shared amongst multiple patients. For such reasons, we believe that the indivisible case can often be much harder than the divisible case.

Assumption 2 *The items are* indivisible.

Third, we further assume that the items in our model arrive in an *online* faishion from some ordering. This fair division setting is novel because only *some* unallocated items are available in one moment in time and must be given to agents in this particular moment. By comparison, in (offline) fair division, *all* unallocated items are available in one moment in time. We therefore focus to the special case when the ordering of items is *strict* and *deterministic*. A strict ordering means that items arrive one by one. Such an ordering is further deterministic if each next item arrives with probability one from some stochastic distribution of the items.

Assumption 3 *The ordering of items is* strict *and* deterministic.

Fourth, we assume *additive and monotonic* preferences. That is, the overall valuation of an agent for a set of items is equal to the sum of their valuations for the items in this set. And, whenever a new item is added to this set of items, their valuation for this new set of items cannot be less than their valuation for the old set of items. In the food bank problem, there are hundreds of people fed by the charities every day and most of these people would gladly accept each next meal. Charities do not charge people in need for the donated food. They can thus be ranked by various "moneyless" performance measures. For example, one such measure is the number of donated meals. Based on this measure, charities can apply for various sponsorships which can partially cover their operational costs. Charities are therefore incentivized to gladly accept each donated meal. In the organ matching problem, the average valuation of a patient in the waiting list increases supposing organs of better quality for them arrive over time. On the one hand, hospitals are often incentivized to perform more local matches than global matches on a national level simply because local matches is less costly for them than global matches in terms of logistics. On the other hand, hospitals are likely to receive organs of better quality for their local patients if they participate in the national organ exchange programme. In fair division, additivity and monotonicity of the preferences offers an elegant compromise between simplicity and expressivity in our model as well as in many other models (Beviá, 1998; Brams, Edelman, and Fishburn, 1998; Chevaleyre, Endriss, Estivie, and Maudet, 2008b; Keijzer, Bouveret, Klos, and Zhang, 2009). In economics, incomes and wealth are additive for the population (Gini, 1912; Atkinson, 1970; Hoover, 1936).

Assumption 4 *The preferences are* additive *and* monotone.

Fifth, in practice, the items normally arrive according to some probability distribution. We consider however just two such distributions depending on how much information about items is available. In the first case, we suppose *complete* information about the items, i.e. the ordering of items (and therefore the items) and the preferences are known only to the central authority. In the second case, we suppose *no* information about the items, i.e. the items (and therefore the ordering of items) and the preferences are unknown to anyone. The case of no information is more likely to occur in practice. However, the case of complete information also happens in practice. In the food bank problem, some items are donated on a regular basis and delivered to the food bank at specific times, and therefore the chair knows what items will arrive and when these items will arrive. Moreover, some charities request items on a regular basis and therefore the chair knows what are the utilities of these charities for such items. Further, when matching deceased organs to patients, doctors in Australia can calculate the utilities of patients for organs of a certain quality using a simple formula and they can even do this for organs that have not yet arrived. Moreover, there is a strong correlation between the car accidents in the country and the donated deceased donors which means that hospitals could estimate with high probability the arrival order of the next few organs.

Assumption 5 *The information about the items is* complete *or* none.

Finally, in the thesis, we consider *only* these two extreme cases of *complete* or *no* information about the future items. We however submit that there are *many* other cases in between these two extremes (Lumet, Bouveret, and Lemaître, 2012). We leave these cases for future work.

2.2 Instances and mechanisms

In this section, we define more formally instances and mechanisms for online fair division within our assumptions.

Definition 1 (**Instance for online fair division**) *An* instance $\mathcal{I}$ *of an online fair division problem has (1) a set A of* agents $a_1, \ldots, a_n$*, (2) a set O of indivisible* items $o_1, \ldots, o_m$*, (3) an ordering $o = (o_1, \ldots, o_m)$ of these items, and (4) a* bid *matrix $V = (v_{ij})_{n \times m}$ where $v_{ij} \in \mathbb{Q}^{\geq 0}$ is a bounded value that agent a_i reports (i.e. bids) for item o_j. We denote $\mathcal{I}$ as the 4-tuple (A, O, o, V).*□

An online instance $\mathcal{I}$.

o	o_1	o_2	$\ldots$	o_n
a_1	v_{11}	v_{12}	$\ldots$	v_{1n}
a_2	v_{21}	v_{22}	$\ldots$	v_{2n}
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
a_n	v_{n1}	v_{n2}	$\ldots$	v_{nn}

Each agent a_i has some (private) bounded utility value $u_{ij} \in \mathbb{Q}^{\geq 0}$ for each item o_j. We say that agent a_i *likes* item o_j if $u_{ij} > 0$. We suppose that at least one agent likes every item (otherwise we can simply eliminate the item) and that each agent likes at least one item (otherwise we can simply eliminate the agent). We might not know if agent a_i likes item o_j simply because they report the bid v_{ij} for it which might be different from their utility u_{ij} for this item. We consider *binary* and *general* cardinal valuations (i.e. utilities and bids). Binary valuations could be appropriate when we merely care of how much agents like items. Instead, we are only interested in whether they like or dislike items. General valuations are related to the fact that an agent might have indeed different valuations for different items they like, and also to how much these valuations differ from one another. We are well aware that eliciting preferences could be a challenging problem. However, in our model, the chair needs to elicit at most $m \cdot n$ preference values which might not always be too costly. Moreover, both types of valuations happen to be elicited in our practical applications. For example, in the food bank problem, charities often submit utilities for items only a couple of times during the day, e.g. morning, afternoon, evening. As another example, in the organ matching problem, the valuations of patients for organs are calculated explicitly via a formula.

We use online mechanims to allocate each indivisible item in a given instance to exactly one agent. In this thesis, we focus on *responsive* such mechanisms that allocate all items only to agents that like them. We further assume that the decisions of online mechanisms for past items are *irrevocable* for a number of reasons. From a theoretical perspective, this is the worst case when current decisions cannot be changed at some later moment in time. From a practical perspective, we cannot ask agents at some later moment in time to return items they have already been allocated. In the food bank problem, charities might have consumed products they have received earlier. In the organ matching problem, we cannot simply request post the transplantation an organ matched earlier to a patient.

Definition 2 (**Mechanism for online fair division**) *For $j \in [1, m]$, an* online *mechanism allocates item o_j supposing* complete information *about the past items o_1 to o_{j-1} and* no information *about any of the future items o_{j+1} to o_m, or even if any more items will arrive (i.e. m is uknown to anyone).*□

2.3 Allocation outcomes

In this section, we define the outcomes returned by a given mechanism when allocate items in a given instance. We consider both deterministic and randomized mechanisms. A deterministic mechanism returns a single actual (*ex post*) allocation. A randomized mechanism returns a single expected (*ex ante*) allocation, i.e. a probability distribution over ex post allocations. A deterministic mechanism can therefore be viewed as a special case of a randomized mechanism that returns a single ex post allocation with probability one. We next introduce notations for both ex post and ex ante outcomes returned by mechanisms when allocate items in instances.

Consider an instance $\mathcal{I} = (A, O, o, V)$. For $j \in [1, m]$, let V_j denote the sub-matrix of V that contains the first j rows in V, V_{ij} denote the ith row in V_j and V_{-ij} denote all rows in V_j except the ith one. Similarly, we define U_j, U_{ij} and U_{-ij} for the matrix of sincere utilities U. We sometimes denote V_{ij} as $(v_{i1}, \ldots, v_{ij})$ and U_{ij} as $(u_{i1}, \ldots, u_{ij})$. We also sometimes omit index j from these notations whenever it is equal to m, i.e. $V = V_m$, $V_i = V_{im}$, $V_{-i} = V_{-im}$.

Consider an online mechanism and suppose we run it on the instance $\mathcal{I}$. For $j \in [1, m]$, let $\pi(j)$ denote the ex post allocation of items o_1 to o_j to agents returned by the mechanism. We view $\pi(j)$ as a set of "agent-item" pairs, i.e. $\{(a_i, o_h) | h \in [1, j]\}$. Let $\pi_i(j)$ denote such an allocation $\pi(j)$ in which agent a_i receives item o_j. We write $u(i, k, \pi(j))$ for the additive *utility* of agent a_i for the items that agent a_k receives in $\pi(j)$, i.e. $u(i, k, \pi(j)) = \sum_{(a_k, o_h) \in \pi(j)} u_{ih}$. We write $u(i, \pi(j))$ for the additive utility of agent a_i for their own items in $\pi(j)$, i.e. $u(i, \pi(j)) = \sum_{(a_i, o_h) \in \pi(j)} u_{ih}$. We again omit index j from these notations whenever it is equal to m, i.e. $\pi = \pi(m)$.

Further, for $j \in [1, m]$, let $\Pi(j)$ denote the set of ex post allocations of items o_1 to o_j to agents returned by the mechanism with positive probabilities. A deterministic mechanism returns $\pi(j) \in \Pi(j)$ with probability 1 whilst a randomized mechanism returns $\pi(j)$ with *probability* $p(\pi(j))$. We have that $\sum_{\pi(j) \in \Pi(j)} p(\pi(j)) = 1$ holds because the mechanism is responsive. Let $\pi_i(j)$ denote such an allocation $\pi(j)$ in which agent a_i receives item o_j. We write $\overline{p}(i, \Pi(j))$ for the additive *probability* of agent a_i for item o_j, i.e. $\overline{p}(i, \Pi(j)) = \sum_{\pi_i(j) \in \Pi(j)} p(\pi_i(j))$. We also write $\overline{u}(i, k, \Pi(j))$ for the additive *expected utility* of agent a_i for the items that agent a_k receives in each allocation in $\Pi(j)$, i.e. $\overline{u}(i, k, \Pi(j)) = \sum_{\pi(j) \in \Pi(j)} p(\pi(j)) \cdot u(i, k, \pi(j))$. And, we write $\overline{u}(i, \Pi(j))$ for the additive expected utility of agent a_i for their own items in each allocation in $\Pi(j)$, i.e. $\overline{u}(i, \Pi(j)) = \sum_{j=1}^{m} \overline{p}(i, \Pi(j)) \cdot u_{ij}$. Again, we omit index j from these notations whenever it is equal to m, i.e. $\Pi = \Pi(m)$.

We use throughout the thesis several alternative notations for these expected outcomes. For example, we sometimes write $\overline{p}(i, V_{ij}, V_{-ij})$ whenever we want to stress that this probability depends on the bids of the agents for items o_1 to o_j, $\overline{p}(i, V_{ij})$ whenever we want to stress that this probability depends on the bids of agent a_i supposing the other bids are fixed or simply $\overline{p}(i, j)$ whenever all these bids are clear from the context. Analogously, we use $\overline{u}(i, k, V_{ij}, V_{-ij})$ and $\overline{u}(i, V_{ij}, V_{-ij})$, $\overline{u}(i, k, V_{ij})$ and $\overline{u}(i, V_{ij})$, and $\overline{u}(i, k)$ and $\overline{u}(i)$.

2.4 Conclusions

In this chapter, we described our assumptions in this thesis: (1) the agents are known, (2) the items are indivisible, (3) the ordering of items is strict and deterministic, (4) the preferences of agents for items are additive and monotone, and (5) the information about the items is complete or none. Further, we defined instances and mechanisms for online fair division. Finally, we defined allocation outcomes returned by mechanisms when allocate items in instances.

Chapter 3

Strategy-Proofness, Envy-Freeness and Pareto Efficiency

There could be plenty of mechanisms for online fair division. However, we would most likely be not satisfied with an arbitrary mechanism and, instead, with mechanism that comply with some normative standards. Such standards would be useful for a number of reasons such as evaluating a given mechanism or even compare multiple mechanisms. In this way, we could focus on mechanisms that comply with such standards and possibly even be able to suggest which mechanisms to use in practice. One way to standardize mechanisms in (offline) fair division is to axiomatize them. Three of the most studied axiomatic concepts in (offline) fair division are *strategy-proofness*, *envy-freeness* and *Pareto efficiency*. Strategy-proofness captures the ability of agents to manipulate the outcome of mechanisms in their favor. Envy-freeness is based on the idea of one agent being envious of another agent. Pareto efficiency is based on the idea that we cannot improve the current allocation returned by a mechanism through say re-allocating items in this allocation.

In online setting, we could use these three properties to axiomatize online mechanisms with respect to their outcomes over all items when the entire information about the these items is available. However, we do not know in advance when the allocation will terminate and, for this reason, we might further want to axiomatize online mechanisms with respect to the partial allocations they return at each point in time without the knowledge if any more items will arrive. This means that an online mechanism would have to be strategy-proof, envy-free or Pareto efficient at each moment in time. For instance, in deciding if agents have any incentive to misreport preferences to an online mechanism, we may need to account for the fact that their past decisions are fixed, and that the future is unknown. We do not know what items will arrive next, or indeed if any more items will even arrive. This leads to an *online* and weaker form of strategy-proofness. It is weaker because agents have committed to their past decisions and because uncertainty about the future reduces their options to be strategic. On the plus side, this means that it is easier to achieve strategy-proofness in the online setting than in the offline setting. On the minus side, this uncertainty means that it is might be harder in the online setting to achieve other properties such as envy-freeness and Pareto efficiency. Interestingly, we argue that no new *online* notions of envy-freeness and Pareto efficiency needed. If an online mechanism is envy-free (Pareto efficient), then it is envy-free (Pareto efficient) in each moment in time.

The *online* nature of fair division problems may have an impact not only on axiomatic properties of online mechanisms but it may also make it impossible to use offline mechanisms from (offline) fair division. Many such existing mechanisms are no longer well-defined and can no longer be applied in the online setting as they assume that all unallocated items are available at every point in time. For example, consider the well known *sequential* allocation mechanism used frequently in (offline) fair division (Brams and King, 2005). This mechanism has agents following a picking sequence and taking turns to pick their most preferred item from the pool of available items. In online fair division, their most preferred item may not be currently (or even ever) available. A similar observation holds for other popular (offline) mechanisms such as *serial dictatorship*, *random priority* or *probabilistic serial rule* (Bogomolnaia and Moulin, 2001). We hence conclude that the absence of information about future items can have a significant impact on characterizations of axiomatic properties in terms of the mechanisms used for such characterizations.

In this chapter, as a response, we develop new *online* mechanisms that take account of the fact that past or future items are not always available. More precisely, we propose two existing mechanisms - LIKE and BALANCED LIKE - and four new mechanisms - ONLINE SERIAL DICTATORSHIP, ONLINE RANDOM PRIORITY, MAXIMUM LIKE and PARETO LIKE - for online fair division. We study (online) strategy-proofness, envy-freeness and Pareto eficiency of these six mechanisms. Some results already exist. Aleksandrov et al. studied the axiomatic properties of the LIKE and BALANCED LIKE mechanisms (Aleksandrov, Aziz, Gaspers, and Walsh, 2015). For example, the LIKE mechanism is strategy-proof whereas the BALANCED LIKE mechanism is not. Also, the LIKE mechanism is fairer in expectation than the BALANCED LIKE mechanism. One disadvantage of the LIKE mechanism is that it could give all items to a single agent with positive probability and thus return a very unfair allocation. In contrast, the BALANCED LIKE mechanism bounds the unfairness of the ex post allocation because it tries to give the same number of items to agents. Aleksandrov et al. further expirically compared these two mechanisms and concluded that the BALANCED LIKE mechanism might be fairer than the LIKE mechanism in practice (Aleksandrov, Gaspers, and Walsh, 2015). We add to these results in this thesis. For example, we show that agents are less strategic with *no* information about future items compared to the case with *complete* information about future items. Indeed, we show that the BALANCED LIKE mechanism is online strategy-proof which contrasts the result that it is not strategy-proof. We give similar axiomatization results for the other mechanisms as well.

We also consider how we might possibly characterize each axiomatic property in isolation. We thus identify classes of online mechanisms which are guaranteed to be (online) strategy-proof, and to return envy-free or Pareto efficient allocations. For example, each mechanism that is envy-free ex ante and strategy-proof allocates items to agents with the same probabilities as the LIKE mechanism. In addition, we consider which mechanisms satisfy combinations of axioms. With general utilities, one of our new online mechanisms, the ONLINE RANDOM PRIORITY mechanism is strategy-proof, and is guaranteed to return allocations that are online envy-free ex ante and Pareto efficient ex post. Unfortunately, this mechanism may not be very fair as we cannot bound the envy ex post one agent may have for another. Indeed, it can even fail to return an envy-free ex post allocation even when an exponential number of such allocations exist. With 0/1 utilities, we can do even better. The BALANCED LIKE mechanism returns allocations which are online strategy-proof, envy-free ex ante, bounded envy-free ex post with 1 utility unit, and Pareto efficient both ex ante and ex post. Bounded envy-freeness ex post requires that no agent envies another agent's allocation by more than 1 unit of utility (in each moment in time). Finally, we prove several impossibility results. For example, *no* online mechanism is envy-free ex post even with 0/1 utilities and *no* online mechanism is bounded envy-free ex post with general utilities. Also, in offline fair division, there are offline mechanisms that can return allocations which are *always* Pareto efficient *and* envy-free ex ante Bogomolnaia and Moulin, 2001. By comparison, in online fair division, we prove that this is *impossible* for online mechanisms. Online fair division requires you to choose one or the other property. Moreover, even with 0/1 utilities, *no* online mechanism is strategy-proof and bounded envy-free ex post with 1 utility unit.

The chapter is organized as follows. We next discuss related work in Section 3.1. We then give preliminaries in Section 3.2. In Section 3.3, we define the mechanisms. We present the axiomatic characterizations of (online) strategy-proofness, envy-freeness and Pareto efficiency in Sections 3.4, 3.5 and 3.6, respectively. We also analyse the mechanisms with respect to these properties thus extending several existing results. In Section 3.7, we further report which of these online mechanisms satisfy combinations of axiomatic properties subject to our impossibility results. Finally, we conclude in Section 3.8.

3.1 Related work

The large literature on fair division mostly considers static fair division problems. There are, however, some exceptions. For instance, Walsh (Walsh, 2011) has proposed an online model of cake cutting in which goods are divisible (unlike here where the goods are indivisible) and the agents arrive over time (unlike here where the goods arrive over time). Kash, Procaccia and Shah (Kash, Procaccia, and Shah, 2013) have proposed a dynamic model of fair division in which agents again arrive over time (and not items as here), but there are now multiple, homogeneous *divisible* goods (and not multiple, heterogeneous indivisible goods as here). There has been also some considerations of the dynamic nature of fair division markets in settings like kidney exchange (Dickerson and Sandholm, 2015; Dickerson, Procaccia, and Sandholm, 2013; Dickerson, Procaccia, and Sandholm, 2012). However, unlike the markets we consider here where agents can receive multiple items, these markets involve agents receiving a single item (a kidney). Typically, these markets require us to find a maximum-weight matching consisting of cycles (and more recently chains) in an appropriate bipartite matching graph. In addition, our setting also has some similarities to some dynamic settings in voting (Chevaleyre, Lang, Maudet, Monnot, and Xia, 2012; Xia and Conitzer, 2010; Xia, Conitzer, and Lang, 2011). However, none of the results in these settings resembles our axiomatc results. Our analyses can further be applied to other online settings, e.g. online scheduling, online bin packing, etc. (György, Lugosi, and Ottucsák, 2010; Jaillet and Lu, 2014; Kalyanasundaram and Pruhs, 2000; Kapelner and Krieger, 2014; Karp, Vazirani, and Vazirani, 1990; Khuller, Mitchell, and Vazirani, 1994; Pruhs, Sgall, and Torng, 2004).

It is interesting to contrast our results with results in (offline) fair division. For example, in (offline) fair division, the sequential allocation mechanisms characterize Pareto efficiency ex post in isolation (Brams and King, 2005). This remains to hold in our online setting with complete information and supposing that all items will arrive. However, this characterization result does not give us any procedural way of computing such allocations. Moreover, we cannot apply sequential allocation mechanisms in our online setting. Instead, we propose an online mechanism - PARETO LIKE - that computes all Pareto efficient ex post allocations and thus characterizes Pareto efficiency ex post. Our mechanism further accounts for the fact that some items may not arrive. Another interesting result from (offline) fair division is that each expected allocation that is Pareto efficient ex ante can be represented as a convex convex combination of Pareto efficient ex post allocations (Bogomolnaia and Moulin, 2001). This again suppose that all items are available and will arrive. Further, in (offline) fair division, the *"... dictatorships are the only strategyproof mechanisms within the class of random priority mechanisms (i.e., mechanisms where agents take turns choosing objects in some random order)."* (Budish and Cantillon, 2012) (page 2204). In our online setting, the LIKE mechanism is strategy-proof but it cannot be generated by any dictatorship mechanism.

In addition, our results are in-line with several impossibility results in (offline) fair division. For example, we proved that the ONLINE RANDOM PRIORITY mechanism is strategy-proof and Pareto efficient ex post, but not Pareto efficient ex ante. Additionally, this mechanism provides equal treatment of equals (i.e. each two agents with the same utilities for items receive the same expected utilities). We recall the impossibility results of (Bogomolnaia and Moulin, 2001) and (Zhou, 1990) for (offline) fair division which state that no mechanism is strategy-proof, Pareto efficient ex ante, and provides equal treatment of equals. Finally, all our mechanisms do not use quotas which have attracted some attention in the research literature. We could add quotas or modify these online mechanisms in other ways to balance the number of items allocated to agents, or the utility allocated to agents. However, this would quickly destroy many desirable normative properties such as strategy-proofness, envy-freeness and Pareto efficiency. For example, supposing that agents can misreport their quotas, it is easy to see that any mechanism that is responsive to quotas is not strategy-proof. To achieve strategy-proofness with quotas, we indeed need to be irresponsive to the quotas of the agents such as in (Hosseini and Larson, 2015).

3.2 Preliminaries

We start with strategic behaviour. With complete information, agents report their bids for items simultaneuosly. Strategy-proofness captures the ability of agents to misreport their utilities for multiple items simultaneously. With no information, agents report their bids for the current item simultaneously. Online strategy-proofness captures the ability of agents to misreport their utilities for the current item simultaneously.

Definition 3 (Strategy-proofness vs Online strategy-proofness) *An online mechanism is* strategy-proof *if, for each instance, no agent a_i can misreport their utilities for items o_1 to o_m and thus strictly increase their expected utility for items o_1 to o_m supposing each other agent reports sincerely their utilities for items o_1 to o_m.*

$$\forall a_i, \forall (v_{i1}, \dots, v_{im}) : \overline{u}(i, (v_{i1}, \dots, v_{im})) < \overline{u}(i, (u_{i1}, \dots, u_{im})) \tag{3.1}$$

An online mechanism is online strategy-proof *if, for each instance and round $j \in [1, m]$, no agent a_i can misreport their utility for item o_j and thus strictly increase their expected utility for items o_1 to o_j supposing each other agent reports sincerely their utilities for items o_1 to o_j and agent a_i reports sincerely their utilities for items o_1 to o_{j-1}.*

$$\forall a_i, \forall v_{ij} : \overline{u}(i, (u_{i1}, \dots, u_{ij-1}, v_{ij})) < \overline{u}(i, (u_{i1}, \dots, u_{ij-1}, u_{ij})) \tag{3.2}$$

Strategy-proofness implies online strategy-proofness. To see this, let us consider a mechanism which is strategy-proof and an instance with m items. But, then the mechanism is strategy-proof on the sub-instance of this instance in which only the first j items arrive. This holds for each $j \in [1, m]$. Hence, no agent has an incentive to misreport their utility for o_j in this sub-instance. The mechanism is therefore online strategy-proof. Online strategy-proofness however does not imply strategy-proofness. To see this, observe that online strategy-proofness allows agents to misreport their utility for one item only. However, an agent might prefer to misreport their utilities for multiple items in order to increase their expected utility over all items.

Observation 1 *Strategy-proofness strictly implies online strategy-proofness.*

We next proceed with envy-freeness. An agent a_i *envies ex post* another agent a_k if agent a_i assigns greater utility to agent a_k for their allocation than to their own allocation. An agent a_i *envies ex ante* another agent a_k if agent a_i assigns greater expected utility to agent a_k for their expected allocation than to their own expected allocation.

Definition 4 (Envy-freeness) *An online mechanism is* envy-free ex post *if, for each instance, no agent a_i envies ex post another agent a_k in each ex post allocation of $\pi(m)$ returned by the mechanism.*

$$\forall a_i, \forall a_k, \forall \pi_m : u(i, \pi(m)) \geq u(i, k, \pi(m)) \tag{3.3}$$

An online mechanism is envy-free ex ante *if, for each instance, no agent a_i envies ex ante another agent a_k in the ex ante allocation of $\Pi(m)$ returned by the mechanism.*

$$\forall a_i, \forall a_k : \overline{u}(i, \Pi(m)) \geq \overline{u}(i, k, \Pi(m)) \tag{3.4}$$

The uncertainty about future items means that it is harder in the online setting to achieve envy-freeness. Similarly to strategy-proofness, we might therefore ask whether we need new *online* notions - *online envy-freeness ex post* and *online envy-free ex ante* - that require an online mechanism to be envy-free in each point in time, i.e. envy-free for each $j \in [1, m]$. Interestingly, we argue that no such notions are needed. To see this, consider an online mechanism that is envy-free. This mechanism is thus envy-free on a given instance and it is also envy-free on the sub-instance of the first j items for each $j \in [1, m]$. Therefore, envy-freeness of the mechanism already guarantees that, at every time step, the partial allocations constructed so far are envy-free.

Observation 2 *Envy-freeness is equivalent to online envy-freeness.*

We finally consider Pareto efficiency. An ex post allocation *Pareto dominates* another ex post allocation if each agent a_i receives utility in the former allocation that is at least as their utility in the latter allocation, and some agent a_k receives utility in the former allocation that is strictly greater than their utility in the latter allocation. Similarly, we can define Pareto dominance between two ex ante allocations with respect to the expected utilities of agents in these allocations.

Definition 5 (Pareto efficiency) *An online mechanism is* Pareto efficient ex post *if, for each instance, there is no ex post allocation $\pi'(m)$ that Pareto dominates a given ex post allocation $\pi(m)$ returned by the mechanism.*

$$(\forall a_i : u(i, \pi'(m)) \geq u(i, \pi(m))) \text{ and } (\exists a_k : u(k, \pi'(m)) > u(k, \pi(m))) \tag{3.5}$$

An online mechanism is Pareto efficient ex ante *if, for each instance, there is no other expected allocation $\Pi'(m)$ that Pareto dominates the expected allocation $\Pi(m)$ returned by the mechanism.*

$$(\forall a_i : \overline{u}(i, \Pi'(m)) \geq \overline{u}(i, \Pi(m))) \text{ and } (\exists a_k : \overline{u}(k, \Pi'(m)) > \overline{u}(k, \Pi(m))) \tag{3.6}$$

Similarly to envy-freeness, we argue again that no new *online* notions - *online Pareto efficiency ex post* and *online Pareto efficiency ex ante* - of Pareto efficiency are needed, i.e. Pareto efficiency for each $j \in [1, m]$. For ex post, we use an argument similar to the one for envy-freeness ex post. If a mechanism is Pareto efficient ex post on an instance with m items, then it is Pareto efficient ex post on sub-instance of the first j items for each $j \in [1, m]$. For ex ante, we use a different argument that relies on the fact that each Pareto efficient ex ante allocation can be represented as a convex combination over Pareto efficient ex post allocations (Bogomolnaia and Moulin, 2001). Let us pick such a combination for the expected allocation of m items returned by a given Pareto efficient ex ante mechanism. Further, in the combination, let us pick an allocation $\pi(m)$ and let its weight be $w(\pi(m))$. This allocation is Pareto efficient ex post of all m items. For $j \in [1, m]$, let $\pi(j)$ be the sub-allocation of $\pi(m)$ of the first j items. This sub-allocation is also Pareto efficient ex post of the first j items. For $j \in [1, m]$, let us consider the convex combination in which, for each $\pi(m)$ in the initial combination, we consider the sub-allocation $\pi(j)$ with weight equal to $w(\pi(j)) = w(\pi(m))$. This convex combination is over Pareto efficient ex post allocations of the first j items. Moreover, this combination represents the expected allocation of these items returned by the mechanism. Consequently, for $j \in [1, m]$, the mechanism is Pareto efficient ex ante on the instance of the first j items.

Observation 3 *Pareto efficiency is equivalent to online Pareto efficiency.*

3.3 Online mechanisms

In this section, we define six mechanisms for online fair division. We use two existing online mechanisms that were proposed previously: the LIKE and BALANCED LIKE mechanisms. And, we also propose four new online mechanisms: the ONLINE SERIAL DICTATORSHIP, ONLINE RANDOM PRIORITY, MAXIMUM LIKE and PARETO LIKE mechanisms. For each item o_j, each of these six mechanisms computes a set of agents feasible for item o_j given the allocation $\pi(j-1)$ of past items o_1 to o_{j-1} is fixed. Let f_j denote the number of agents feasible for item o_j given the allocation $\pi(j-1)$ of items o_1 to o_{j-1}. Each mechanism then allocates item o_j to a feasible agent with probability $\frac{1}{f_j}$ that is basically uniform with respect to the other feasible agents. We say that this probability is *conditional* because it may depend on the allocation of past items. In fact, this is the case with the BALANCED LIKE mechanism.

- The LIKE mechanism decides that agent a_i is feasible for item o_j if, given $p(j-1)$, $v_{ij} > 0$.
- The BALANCED LIKE mechanism decides that agent a_i is feasible for item o_j if, given $p(j-1)$, $v_{ij} > 0$ and agent a_i has fewest items in $\pi(j-1)$ amongst those agents bidding positively for item o_j.
- The ONLINE SERIAL DICTATORSHIP mechanism uses a strict priority ordering $\sigma = (a_1, \ldots, a_n)$ of the agents and decides that agent a_i is feasible for item o_j if, given $\pi(j-1)$, $v_{ij} > 0$ and agent a_i have highest priority in σ amongst those agents bidding positively for item o_j.
- The ONLINE RANDOM PRIORITY mechanism draws a strict priority ordering $\sigma = (a_1, \ldots, a_n)$ uniformly at random with probability $\frac{1}{n!}$ prior to round 1 and runs ONLINE SERIAL DICTATORSHIP with it.
- The MAXIMUM LIKE mechanism decides that agent a_i is feasible for item o_j if, given $\pi(j-1)$, $v_{ij} > 0$ and $v_{ij} = \max_{k \in [1,n]} v_{kj}$.
- The PARETO LIKE mechanism decides that agent a_i is feasible for item o_j if, given $\pi(j-1)$, $v_{ij} > 0$ and the ex post allocation $\pi(j-1) \cup \{(a_i, o_j)\}$ is Pareto efficient ex post.

The LIKE and BALANCED LIKE mechanisms are randomized and return a distribution of actual allocations. Note that the LIKE mechanism does not keep track on past allocations and thus "blindly" allocates each next item. By comparison, the BALANCED LIKE mechanism keeps track on past allocations and tries to give the same number of items to each agent. These two mechanisms are simple and we already know that they satisfy many nice axiomatic properties (Aleksandrov, Aziz, Gaspers, and Walsh, 2015). The ONLINE RANDOM PRIORITY and MAXIMUM LIKE mechanisms are also randomized and return a probability distribution of allocations whilst ONLINE SERIAL DICTATORSHIP is deterministic and returns a single allocation. The MAXIMUM LIKE mechanism greedily allocates each next item to arrive to one of the agents with the greatest bid for this item. This mechanism is reminiscent of a first-bid auction mechanism but with the difference that we do not consider payments in our setting. Note that ONLINE SERIAL DICTATORSHIP is similar to but nevertheless different from the serial dictatorship mechanism from offline fair division (Gibbard, 1973; Bade, 2015). In an online context, the serial dictatorship becomes much more interesting. To see this, consider 2 agents and 2 items. If agent 2 comes after agent 1, on the one hand perhaps agent 1 will have preferred item 2, but on the other hand, there may be new items arriving in between item 1 and item 2. That is to say that coming late is sometimes compensated by a larger choice set. By comparison, our ONLINE SERIAL DICTATORSHIP is a different mechanism as it is equivalent to repeteadly applying the serial dictatorship mechanism with the same priority ordering for each next item. If the first agent in the priority ordering likes all items, the ONLINE SERIAL DICTATORSHIP mechanism will give all items to this agent. Similarly, the ONLINE RANDOM PRIORITY mechanism resembles the popular random priority mechanism in offline fair division (Bogomolnaia and Moulin, 2001). The latter priority mechanism is a uniform distribution over dictatorship mechanisms. And, again, the ONLINE RANDOM PRIORITY mechanism is memoryless whereas the random priority mechanism is not. Further, interestingly, the PARETO LIKE mechanism resembles the sequential mechanism for (offline) fair division. The latter characterizes Pareto efficiency ex post with *some* picking sequence. However, the sequential allocation may return an ex post allocation that is not Pareto efficient ex post with a *fixed* picking sequence. Instead, as we show, the PARETO LIKE always returns only and all Pareto efficient ex post allocations. We proceed with our characterization results.

3.4 Strategy-proofness

We begin with strategy-proofness. It is harder for an agent to be strategic in an online setting with no information as the uncertain future may preclude a strategic bid that could be successful otherwise in the online setting with complete information. It is also harder to be strategic with no information as agents are committed to decisions made in the past. Recall that strategy-proofness implies online strategy-proofness but the reverse direction does not hold. Online strategy-proofness is thus the weaker notion. We highlight this difference in Example 1.

Example 1 (**strategy-proofness vs online strategy-proofness**) *Consider the following instance and the* BALANCED LIKE *mechanism.*

o	o_1	o_2
a_1	1	2
a_2	2	1

Suppose now that agents have complete *information about both items prior to the first round. By bidding sincerely, agent a_1 receives expected utility of $3/2$. By bidding strategically 0 for item o_1, agent a_1 receives strictly higher expected utility of 2.*

Suppose next that agents have no *information about whether item o_2 will arrive or not prior to the first round. From their perspective, the problem has just item o_1 and each agent has no allocated items. The dominant strategy of each agent is then sincerity. Item o_1 is allocated to either agent a_1 or agent a_2. Then, item o_2 suddenly arrives. We consider two cases. In the first case, let item o_1 be allocated to agent a_1. From their perspective, they do not get a chance to pick item o_2 and therefore may regret they have reported a positive bid for item o_1. However, their bid for item o_1 is in the past and cannot be changed. Hence, sincerity is a dominant strategy of agent a_1 for item o_2. Similarly, for agent a_2 because they receive item o_2 with probability of 1. In the second case, let item o_1 be allocated to agent a_2. We can use similar lines of reasoning as in the first case and conclude that the sincerity is a dominant strategy of each agent for item o_2.*□

We can generalize the argument we used in this simple example to any number of agents, any number of items and general utilities. Therefore, we derive the following simple proposition which contrasts with the fact that the BALANCED LIKE mechanism is not strategy-proof even with 0/1 utilities except when there are just 2 agents (Aleksandrov, Aziz, Gaspers, and Walsh, 2015).

Proposition 1 *With general utilities, the* BALANCED LIKE *mechanism is online strategy-proof.*

Proof. For a given ex post allocation $\pi(j-1)$ of items o_1 to o_{j-1}, we consider two cases for a given agent a_i. In the first case, agent a_i is not feasible for item o_j. Sincerity for this item is then their dominant strategy. In the second case, agent a_i is feasible for item o_j. If agent a_i strategically bid $v_{ij} \neq u_{ij}$ instead of $u_{ij} > 0$ for item o_j, their conditional expected utility over items o_1 to o_j given $\pi(j-1)$ cannot increase. Even worse, it can decrease if $v_{ij} = 0$. This simple observation holds for each $\pi(j-1)$. Hence, they cannot increase their expected utility over items o_1 to o_j (over all allocations of these items) by misreporting their sincere utility for o_j.□

Online strategy-proofness is characterized in terms of *step* mechanisms. An online mechanism is *step* if, for each instance, agent a_i and item o_j, the expected probability $\overline{p}(i,j)$ is 0 if $v_{ij} = 0$ and admits the same value in $[0,1]$ for each positive v_{ij} if $v_{ij} > 0$. Note that the probability $\overline{p}(i,j)$ could admit the same value for each positive value of v_{ij} but still different values for different collections of bids for past items o_1 to o_{j-1} or collections of bids for the current item o_j. We can observe this with the BALANCED LIKE mechanism. The probability $\overline{p}(i,j)$ with this mechanism depends on the bids of agents for past items and therefore admits different values for different values of these bids. However, supposing that these bids are fixed, the probability $\overline{p}(i,j)$ admits the same value for each $v_{ij} > 0$ and it zero for $v_{ij} = 0$.

Theorem 1 *With general utilities, a mechanism is online strategy-proof iff it is a step.*

Proof. Suppose the mechanism is a step. The proof uses induction on the number of rounds. In the step case, consider agent a_i and the last item o_j. If $u_{ij} > 0$, agent a_i has no incentive to report 0 for it as their expected utility can only decrease, and also has no incentive to report any positive value $v_{ij} \neq u_{ij}$ as their expected utility cannot change because their ex ante probability for item o_j is a constant of v_{ij}. If $u_{ij} = 0$, then agent a_i has no incentive to report $v_{ij} > 0$ as their expected utility cannot increase. For bids for earlier items, we can appeal to the induction hypothesis. We conclude that the mechanism is online strategy-proof.

Suppose the mechanism is not a step, and that the probability of some agent a_i and some item o_j is not a constant of v_{ij} for the sincere bids of agents for items o_1 to o_{j-1} and the sincere bids of other agents for item o_j. Without loss of generality, we can suppose o_j is the last item to arrive. We can also suppose $u_{ij} > 0$. Agent a_i has an incentive to report $v_{ij} > u_{ij}$ or $v_{ij} < u_{ij}$ and thus increase their probability for item o_j and therefore their total expected utility for items o_1 to o_j. We conclude therefore that the mechanism is not online strategy-proof. □

With 0/1 utilities, each mechanism is a step mechanism and therefore online strategy-proof. Also, beside the BALANCED LIKE mechanism, it follows immediately that the ONLINE SERIAL DICTATORSHIP, ONLINE RANDOM PRIORITY, LIKE mechanisms are all online strategy-proof with general utilities. This holds because they are ordinal in the sense that they do not take into account the positive magnitudes of the bids of agents for items. On the other hand, the MAXIMUM LIKE and PARETO LIKE mechanisms are not online strategy-proof. They are non-step mechanisms and agents have an incentive to overreport their utility for an item so as to hold the winning bid for this item or to simulate Pareto efficiency of the allocation of the new item. However, with 0/1 utilities, these two mechanisms are even strategy-proof.

Strategy-proof mechanisms are characterized in terms of step and memoryless mechanisms. Apparently, we need to have additional constraints beyond the mechanism being just constant. We observe that each mechanism that uses the past allocation in order to bias the future allocation is not strategy-proof. The key idea is that any such mechanism might prevent an agent from receiving their most preferred item at a given future moment in time. This agent would then act strategically for items at earlier rounds in order to maximize their chance for this greatly valued item. Let us formalize this concept. An online mechanism is *memoryless* iff, for each instance, agent a_i and item o_j, the expected probability $\overline{p}(i,j)$ admits the same value for each collection of bids v_{i1} to $v_{i(j-1)}$ of agent a_i. Note that the value of $\overline{p}(i,j)$ could still depend on the bids of other agents for past items. A step mechanism is the BALANCED LIKE mechanism which is not strategy-proof because it is not memoryless. In contrast, the MAXIMUM LIKE mechanism is memoryless but it is not a step and hence it is not strategy-proof. Therefore, none of these two propeties in isolation can characterize all strategy-proof mechanisms. They need to be combined.

Theorem 2 *With general utilities, a mechanism is strategy-proof iff it is a memoryless step.*

Proof. Let the mechanism be strategy-proof. Suppose now that it is not a step. Hence, it is not online strategy-proof and strategy-proof. Suppose next that the mechanism is not memoryless. Consider an online fair division instance $\mathcal{I}$ in which the positive bids of agent a_i for items are $u_{ij_1} = \frac{\epsilon}{m.(m-1)}, \ldots, u_{ij_k} = \frac{\epsilon}{m.(m-1)}$ and $u_{im} = 1$, and each other their bid is equal to 0. WLOG, let this be probability $\overline{p}(i,m)$ which depends on the bids of agent a_i for items o_1 to o_{m-1}. Let us denote $\overline{p}(i,m)$ by $\overline{p}(i,(v_{i1},\ldots,v_{im-1}))$. Hence, there is a vector v_{i1} to v_{im-1} of bids of agent a_i for items o_1 to o_{m-1} such that $\overline{p}(i,(v_{i1},\ldots,v_{im-1})) > \overline{p}(i,(u_{i1},\ldots,u_{im-1}))$. Supposing sincere play of all agents, the expected utility of agent a_i is equal to $\sum_{j=1}^{m-1}\overline{p}(i,j)\cdot u_{ij} + \overline{p}(i,(u_{i1},\ldots,u_{im-1}))$. Supposing only agent a_i plays strategically v_{i1} to v_{im-1}, the expected utility of agent a_i: $\sum_{j=1}^{m-1}\overline{q}(i,j)\cdot u_{ij} + \overline{p}(i,(v_{i1},\ldots,v_{im-1}))$. The probability $\overline{q}(i,j)$ might be different than the probability $\overline{p}(i,j)$ for a fixed $j \in [1,m-1]$. Consider now the difference between their strategic and sincere expected utility. It is equal to $\sum_{j=1}^{m-1}\overline{q}(i,j)\cdot v_{ij} - \sum_{j=1}^{m-1}\overline{p}(i,j)\cdot u_{ij} + (\overline{p}(i,(v_{i1},\ldots,v_{im-1})) - \overline{p}(i,(u_{i1},\ldots,u_{im-1})))$. The minimum value of this difference is equal to $-\frac{\epsilon}{m}+\epsilon$ and is achieved when $\overline{q}(i,j)=0$ and $\overline{p}(i,j)=1$ for each $j \in [1,m-1]$. This value is strictly greater than 0 which means that agent a_i receives strictly higher expected utility supposing they play strategically v_{i1} to v_{im-1} and not sincerely u_{i1} to u_{im-1} for items o_1 to o_{m-1}. Therefore, the mechanism is not strategy-proof.

Let the mechanism be step and memoryless. As it is memoryless, the probability $\overline{p}(i,j)$ of each agent a_i for each item o_j does not depend on their bids for items o_1 to o_{j-1}. Hence, it depends only on the bids of agents for item o_j or bids of other agents for items o_1 to o_{j-1}. The setting is competitive and the sincere bids of other agents are fixed. We conclude that agent a_i can increase $\overline{p}(i,j)$ only by misreporting v_{ij}. As the mechanism is a step, $\overline{p}(i,j)$ is 0 if v_{ij} is 0 and it admits the same value if $v_{ij} > 0$. Consider and instance $\mathcal{I}$ and agent a_i. Suppose sincere play of all agents. Now, agent a_i bids their utilities u_{i1} to u_{im}. Let their expected utility be $\sum_{j=1}^{m} \overline{p}(i,j) \cdot u_{ij}$. Suppose strategic play only of agent a_i who bids v_{i1} to v_{im}. Let their expected utility be $\sum_{j=1}^{m} \overline{q}(i,j) \cdot v_{ij}$. For each item o_j with $v_{ij} = u_{ij}$, we have that $\overline{q}(i,j) \cdot u_{ij} = \overline{p}(i,j) \cdot u_{ij}$ as the mechanism is a memoryless step. For each item o_j with $v_{ij} > 0$ and $u_{ij} = 0$, we have that $\overline{q}(i,j) \cdot u_{ij} = \overline{p}(i,j) \cdot u_{ij} = 0$. For each item o_j with $v_{ij} = 0$ and $u_{ij} > 0$, we have that $\overline{q}(i,j) \cdot u_{ij} = 0$ and $\overline{p}(i,j) \cdot u_{ij} \geq 0$. We conclude that the expected utility of agent a_i cannot increase supposing they play strategically v_{i1} to v_{im} and not sincerely u_{i1} to u_{im} for items o_1 to o_m. The mechanism is therefore strategy-proof. □

From our work in (Aleksandrov, Aziz, Gaspers, and Walsh, 2015), we already know that the LIKE mechanism is strategy-proof. On the other hand, the BALANCED LIKE mechanism is not strategy-proof unless there are just 2 agents and they have simple 0/1 utilities for items. In terms of our characterization result, the LIKE mechanism is a memoryless step. And whilst the BALANCED LIKE mechanism is a atep mechanism, it is not memoryless because its allocation outcomes depend on the ordering of items. Further, the ONLINE SERIAL DICTATORSHIP and ONLINE RANDOM PRIORITY mechanisms are also memoryless step mechanisms. Hence they are strategy-proof. On the other hand, the MAXIMUM LIKE mechanism is memoryless but it is not a step mechanism so it is not strategy-proof. The same holds for the PARETO LIKE mechanism. However, this mechanism is neither memoryless nor a step. The only case when these mechanisms are strategy-proof is the 0/1 case.

To conclude, if we want a mechanism that is strategy-proof for online fair division then the ONLINE SERIAL DICTATORSHIP, ONLINE RANDOM PRIORITY, or LIKE mechanisms are possible candidates. However, if we are willing to have just online strategy-proofness then we might as well use the BALANCED LIKE mechanism.

3.5 Envy-freeness

We next move to envy-freeness. Envy-freeness of the allocation outcomes returned by a given mechanism depend on whether agents are strategic or sincere. Suppose that agents bid sincerely. This might be because we use a mechanism that is strategy-proof, or because we suppose that the future is unknown to the agents and, being risk averse, they act sincerely. We previously showed that no new online notion of envy-freeness. Additionally, no mechanism is envy-free ex post even in a bounded sense. We illustrate this in Example 2.

Example 2 *Consider the fair division of item o_1 to agents a_1, a_2 with the following utilities.*

o	o_1
a_1	u
a_2	u

Consider a mechanism that allocates the item to one of the agents. As a result, the other agent envies ex post with u the agent who received the item. Even worse, their envy is unbounded as u could go to ∞.□

We further consider envy in terms of expected utilities of agents for items. For example, we know that the LIKE mechanism is envy-free ex ante (Aleksandrov, Aziz, Gaspers, and Walsh, 2015). We say that two online mechanisms are *probabilistically equivalent* if, for each instance, agent a_i and item o_j, the probabilities of a_i for item o_j are equal with both mechanisms. Two online mechanisms could be probabilistically equivalent and *not* envy-free ex ante at the same time. Interestingly, any mechanism that is probabilistically equivalent to the LIKE mechanism is envy-free ex ante but the opposite direction does not hold. We illustrate this in Example 3.

Example 3 *Consider the fair division of items o_1, o_2 to agents a_1, a_2 with the following utilities.*

o	o_1	o_2
a_1	1	1
a_2	0	1

Further, consider the mechanism that works as the LIKE *mechanism on each instance except on this one in which it gives item o_2 to agent a_2 with some probability in $(\frac{1}{2}, 1]$. This mechanism is envy-free ex ante but it is not ex ante equivalent to the* LIKE *mechanism.* □

Example 3 suggests that the class of envy-free ex ante mechanisms is not trivial to characterize in general. Moreover, there could be infinitely many mechanisms that are envy-free ex ante and no pair of them are probabilistically equivalent to each other, and also none of them is probabilistically equivalent to the LIKE mechanism. Interestingly, any mechanism that is probabilistic equivalent to the LIKE mechanism is further strategy-proof but the opposite direction does not hold. The ONLINE SERIAL DICTATORSHIP mechanism is strategy-proof but is not envy-free ex ante. Hence, it is not probabilistically equivalent to the LIKE mechanism. Consequently, this mechanism does not characterize envy-freeness or strategy-proofness in isolation. Surprisingly, it however characterizes these two axiomatic properties in combination.

Theorem 3 *With general utilities, a mechanism is strategy-proof and envy-free ex ante iff it is probabilistically equivalent to the* LIKE *mechanism.*

Proof. If a mechanism is probabilistically equivalent to the LIKE mechanism, then it is envy-free ex ante and a memoryless step. By Theorem 2, the mechanism is strategy-proof. If a mechanism is envy-free ex ante and strategy-proof, then it is a memoryless step. We show that it is problablistically equivalent to the LIKE mechanism by induction on the round number j. The most interesting case is the step case. We consider round j and suppose the mechanism is ex ante equivalent to the LIKE mechanism for items o_1 to o_{j-1} and is not for item o_j. That is, there are two agents a_i, a_k that like item o_j and suppose that $\overline{p}(i, \Pi(j)) < \overline{p}(k, \Pi(j))$. As the mechanism is envy-free ex ante up to round $(j-1)$, we have that $\overline{u}(i, \Pi(j-1)) \geq \overline{u}(i, k, \Pi(j-1))$. As the mechanism is memoryless step, we can suppose that $u_{ij} = 1-(\overline{u}(i, k, \Pi(j-1))-\overline{u}(i, \Pi(j-1)))/(\overline{p}(k, \Pi(j))-\overline{p}(i, \Pi(j))) > 0$. We hence obtain that $\overline{u}(i, k, \Pi(j-1)) - \overline{u}(i, \Pi(j-1)) + (\overline{p}(k, \Pi(j)) - \overline{p}(i, \Pi(j))) \cdot u_{ij} > 0$, or a_i envies ex ante a_k for o_1 to o_j. This contradicts that the mechanism is envy-free ex ante up to round j. Hence, $\overline{p}(i, \Pi(j)) = \overline{p}(k, \Pi(j))$. □

With general utilities, we conclude that the MAXIMUM LIKE and PARETO LIKE mechanism are not envy-free ex ante because they are not probabilistically equivalent to the LIKE mechanism. With 0/1 utilities, they are however probabilistically equivalent to the LIKE mechanism and so are envy-free ex ante. Also, the ONLINE RANDOM PRIORITY mechanism is envy-free ex ante because it is probabilistically equivalent to the LIKE mechanism.

We might hope that we could characterize all envy-free ex ante mechanisms on some restricted utility domains. However, by Example 3, we conclude that this is a difficult task even with 0/1 utilities. For example, the BALANCED LIKE mechanism is envy-free ex ante in this setting (Aleksandrov, Aziz, Gaspers, and Walsh, 2015). Moreover, there are 0/1 instances on which this mechanism is *not* probabilistically equivalent to the LIKE mechanism. Another simple mechanism that is envy-free ex ante is BALANCED2LIKE: it (1) computes the subset of agents that like the item and have fewest or fewest plus one items amongst those agents bidding positively for the current item, and (2) allocates the item uniformly at random to one of the agents from this subset.

Proposition 2 *With 0/1 utilities, the* BALANCED2LIKE *mechanism is envy-free ex ante.*

Proof. The proof uses induction on the number j of rounds. In the base case, consider j to be 1. Prior round 1, each agent is envy-free ex ante because they have 0 items. Therefore, each agent that likes o_1 receives with the same positive probability. Each other agent receives it with probability 0. It is each to see that no agent envies ex ante another for this item. In the hypothesis case, consider round $j-1$ and suppose that each agent is envy-free ex ante of each other agent prior to this round. In the step case, consider round j and a given allocation of items o_1 to o_{j-1}. There are two cases for each agent a_i. In the first case, agent a_i is not feasible for item o_j. Hence, they have at least 2 more items than each agent that is feasible for this item. By the hypothesis, agent a_i is envy-free ex ante prior to round $j-1$ of each other agent that not feasible for this item. This envy-freeness remains prior to round j when item o_j is allocated. Agent a_i assigns additional expected utility of $\frac{1}{f_j}$ to each feasible agent because each such agent might receive item o_j with probability $\frac{1}{f_j}$. The value of f_j is the number of feasible agents given this particular allocation of past items. But, agent a_i has at least 2 utility units more than each of these agents. Therefore, they do not envy ex ante prior to round j any feasible agent even when item o_j is allocated. In the second case, agent a_i is feasible for item o_j. By the hypothesis, they are envy-free ex ante prior to round $j-1$ of each agent that is not feasible for o_j. This envy-freeness remains prior to round j when item o_j is allocated. The increase in their utility is $\frac{1}{f_j}$ but so is to each other feasible agent. Note that some feasible agents might have 1 more item than others. However, as agent a_i was envy-free ex ante prior to round j of each other feasible agent, they do not envy ex ante such agents prior to round j. The result follows.□

Interestingly, we can obtain a complete characterization of envy-freeness ex ante supposing that each agent likes each item. For example, in the food bank problem, the chair often purchases items that charities like in case not enough such items are donated during the day. In the organ matching problem, a patient would gladly accept any compatible for them organ even if the organ is not currently available match for them.

Theorem 4 *With positive utilities, a mechanism is envy-free ex ante iff it is probabilistically equivalent to* LIKE.

Proof. If a mechanism is probabilistically equivalent to the LIKE mechanism, then it is envy-free ex ante. We next the opposite direction. The proof is by induction on round j. In the step case, we consider a_i, a_k that like o_j. Assume $\overline{p}(i, \Pi(j)) < \overline{p}(k, \Pi(j))$. We have that $\overline{u}(i, k, \Pi(j-1)) = \overline{u}(i, \Pi_(j-1))$ and $\overline{u}(k, i, \Pi(j-1)) = \overline{u}(k, \Pi(j-1))$ because the cardinal utilities are positive and the mechanism is ex ante equivalent to LIKE for o_1 to o_{j-1} by the hypothesis. Hence, $\overline{p}(i, \Pi(j)) = \overline{p}(k, \Pi(j))$ as the mechanism is envy-free ex ante up to round j. This contradicts our assumption.□

Even in (offline) fair division, there are only a few attractive envy-free ex ante mechanisms to the best of our knowledge. For example, the popular probabilistic serial rule is envy-free ex ante because it satisfies a second-order of stochastic dominance envy-freeness which is equivalent to necessary envy-freeness (i.e. envy-freeness with respect to any choice of cardinal values consistant with the ordinal preferences of agents for items) (Aziz, Gaspers, Mackenzie, and Walsh, 2014; Bogomolnaia and Moulin, 2001). For this reason, we may decide to apply an online mechanism to an offline problem. For example, we can pick (perhaps at random) an ordering of the items in the offline instance and run the LIKE or ONLINE RANDOM PRIORITY mechanism with general utilities or the BALANCED LIKE mechanism with 0/1 utilities in order to return an envy-free ex ante outcome.

Finally, suppose that agents bid strategically with general utilities. For the MAXIMUM LIKE and PARETO LIKE mechanisms, it is not difficult to see that the dominant strategy of each agent is to bid the maximum possible bid for each item. As a result, the outcomes of these mechanisms coincide with the outcome of the LIKE mechanism and therefore they are envy-free ex ante. For the BALANCED LIKE mechanism, there are instances on which the mechanism is not envy-free ex ante supposing agents act strategically. If we are interested in a mechanism that is envy-free ex ante for general utilities, we can use the LIKE or ONLINE RANDOM PRIORITY mechanism. In the 0/1 setting, the ONLINE SERIAL DICTATORSHIP mechanism is not envy-free ex ante whereas the BALANCED LIKE mechanism is envy-free ex ante and bounded envy-free ex post with 1.

3.6 Pareto efficiency

We finally consider Pareto efficiency. We might prefer to use a mechanism that returns ex post allocations that cannot be improved. For similar reasons as with envy-freeness, we again suppose that agents act sincerely. Recall that no new online notion of Pareto efficiency is needed. In this section, we consider how we might characterize Pareto efficiency in the online setting with no information. In the case of 0/1 utilities, it is easy to characterize the online mechanisms that are Pareto efficient. Any responsive mechanism is Pareto efficient.

With general utilities, a more limited set of mechanisms are Pareto efficient. We start with Pareto efficiency ex post. In offline fair division, as we pointed out earlier, each Pareto efficient ex post allocation can be generated by the (offline) sequential mechanism for *some* picking sequence supposing agents pick sincerely (Brams and King, 2005). However, note that sequential allocation mechanisms may however return allocations that are not Pareto efficient ex post for a *fixed* picking sequence. We next show this observation in Example 4.

Example 4 *Consider the following instance and the sequential mechanism with picking sequence* (a_1, a_2, a_2, a_1).

o	o_1	o_2	o_3	o_4
a_1	10	9	8	1
a_2	10	3	2	1

The mechanism returns the allocation $\pi_1 = \{(a_1, o_1), (a_2, o_2), (a_2, o_3), (a_1, o_4)\}$. *In* π_1*, the agents receive utilities 11 and 5, respectively. The allocation* π_1 *is Pareto dominated by the allocation* $\pi_2 = \{(a_2, o_2), (a_1, o_2), (a_1, o_3), (a_2, o_4)\}$ *because, in* π_2*, the agents receive utilities 17 and 11, respectively.*$\square$

In any case, sequential allocation mechanisms are not well-defined in our online setting as not all items are available in the same moment. We therefore conclude that we need a new online mechanism that accounts for the fact that the allocation process may terminate at any moment in time, and to ensure that the partially constructed allocation up to that moment is Pareto efficient ex post. One such mechanism is PARETO LIKE. Interestingly, this mechanism is responsive, and it returns all and only Pareto efficient ex post allocations.

Proposition 3 *With general utilities, the* PARETO LIKE *mechanism is responsive.*

Proof. We need to show that it allocates all items. Hence, given an allocation of the past items $\pi(j-1)$ for a given round j, we need to show that the set of agents feasible for item o_j that is computed by the mechanism is not empty. The proof is by contradiction. Suppose that the set of agents feasible for item o_j given $\pi(j-1)$ is empty. Hence, we have that $\pi(j-1) \cup \{(a_i, o_j)\}$ is not Pareto efficient ex post for each agent a_i that likes item o_j. Let us pick agent, say agent a_i. The allocation $\pi(j-1) \cup \{(a_i, o_j)\}$ is Pareto dominated by some other allocation of items o_1 to o_j. Let this be $\pi'(j-1) \cup \{(a_k, o_j)\}$. We now consider two cases. In the first one, agent a_k is in fact the same agent as agent a_i. We therefore obtain that $\pi'(j-1)$ Pareto dominates $\pi(j-1)$. This is a contradiction with the Pareto optimality of $\pi(j-1)$. In the second case, agent a_k is some agent that likes item o_j and is different from agent a_i. Therefore, the allocation $\pi'(j-1) \cup \{(a_k, o_j)\}$ also Pareto dominates $\pi(j-1)$ because agent a_i in $\pi'(j-1)$ receives at least as much ex post utility as in $\pi(j-1)$. This is again in contradiction with the Pareto optimality of $\pi(j-1)$. Therefore, it cannot be the case that $\pi(j-1) \cup \{(a_i, o_j)\}$ is not Pareto optimal for each agent a_i. □

Proposition 4 *With general utilities, the* PARETO LIKE *mechanism returns only and all Pareto efficient ex post allocations.*

Proof. By the definition of the mechanism, we cannot have an allocation that is returned by it and is not Pareto efficent ex post. Hence, it returns only Pareto efficient ex post allocations. We show next that it returns all of them. Consider a Pareto efficient ex post allocation π of all items and suppose it is not returned by the mechanism. Let us run the mechanism and follow the allocation π till the first round when some agent a_i is not feasible for some item o_j. There must be such a round with $j < m$ as the allocation is not returned by the mechanism, i.e. agent a_i receives item o_j with probability 0 given $\pi(j-1)$ that is the sub-allocation of π on items o_1 to o_{j-1}. Note that $\pi(j-1)$ is further Pareto efficient ex post for items o_1 to o_{j-1} as otherwise the mechanism would not propage till round j by following π. Moreover, the allocation $\pi(j-1) \cup \{(a_i, o_j)\}$ is Pareto efficient ex post. Otherwise, there is another allocation of items o_1 to o_j that Pareto dominates it. We can use this new allocation and add to it the allocations from π for items o_{j+1} to o_m. But, then this new allocation of all items Pareto dominates π. This contradicts the fact that π is Pareto efficient ex post. We end up with the two Pareto efficient ex post allocations $\pi(j-1) \cup \{(a_i, o_j)\}$ and $\pi(j-1)$ and we supposed that agent a_i is not feasible for item o_j given $\pi(j-1)$. This is a contradiction with the definition of the mechanism. We conclude that π is returned by the mechanism. Since it was chosen arbitrarily, the result follows.□

We conclude that the PARETO LIKE mechanism is Pareto efficient ex post. There are other mechanisms which are online Pareto efficient ex post. For example, the ONLINE SERIAL DICTATORSHIP and MAXIMUM LIKE are all Pareto efficient ex post. We immediately conclude that the ONLINE RANDOM PRIORITY mechanism is also Pareto efficient ex post. Unfortunately, it is easy to derive that LIKE and BALANCED LIKE are not Pareto efficient ex post with general utilities. To see this, consider the instance in Example 1.

Proposition 5 *With general utilities, the* ONLINE SERIAL DICTATORSHIP *and* MAXIMUM LIKE *mechanisms are Pareto efficient ex post.*

Proof. For the ONLINE SERIAL DICTATORSHIP mechanism, the first agent in the priority ordering cannot do better. They are allocated every item for which they have non-zero utility. Therefore we can focus on all other agents and items that the first agent does not like. By a similar argument, the second agent in the priority ordering cannot do better by reallocating the remaining items, etc. Hence the allocation returned by the ONLINE SERIAL DICTATORSHIP mechanism is Pareto efficient ex post.

For the MAXIMUM LIKE mechanism, consider the allocation returned by the mechanism of items o_1 to o_j. Also, consider agents a_i and a_k and items o_j and o_h, and suppose that agent a_i receive item o_j and agent a_k receive item o_h in this allocation. There are 2 cases. In the first one, agent a_k have the same utility as agent a_i for item o_j. This utility might be either lower or greater than their utility for item o_h. If it is lower, they do not benefit by exchanging item o_h with agent a_i for item o_j. If it is greater, there are two sub-cases. In the first sub-case, agent a_i have the same utility as agent a_k for item o_h. Now, at least one of the agents do not benefit by exchanging items o_h and o_j. In the second sub-case, agent a_i have lower utility for item o_h than agent a_k. Hence, agent a_i does not benefit by exchanging their item o_j with agent a_k for item o_h. This allocation of items o_1 to o_j is Pareto efficient ex post. Similarly, we conduct a case analysis in the second case when agent a_k have lower utility for o_j.□

There are instances on which each of these two online mechanisms can generate allocations that the other mechanism cannot. We observe this in Example 5.

Example 5 *Consider again the instance from Example 1. With the* MAXIMUM LIKE *mechanism, only the allocation* $\pi_1 = \{(a_1, o_2), (a_2, o_1)\}$. *This allocation is Pareto efficient ex post and cannot be returned by any instance of the* ONLINE SERIAL DICTATORSHIP *mechanism. For each priority order, the* ONLINE SERIAL DICTATORSHIP *mechanism allocates both items to a single agent. Such an allocation is also Pareto efficient ex post and cannot be returned by the* MAXIMUM LIKE *mechanism.* □

We might hope that the simple and greedy MAXIMUM LIKE and ONLINE SERIAL DICTATORSHIP mechanisms could be used to characterize all online mechanisms that are Pareto efficient ex post. This is not the case. There are Pareto efficient ex post allocations that cannot be returned by either mechanism. However, they are returned by the PARETO LIKE mechanism. We show this in Example 6.

Example 6 *Consider the following fair division instance.*

o	o_1	o_2
a_1	3	2
a_2	2	1

Each ex post allocation is Pareto efficient. However, let us consider the following allocation of items to agents: $\pi = \{(a_1, o_1), (a_2, o_2)\}$. *It is returned by the* PARETO LIKE *mechanism but by none of the* MAXIMUM LIKE *and* ONLINE SERIAL DICTATORSHIP *mechanisms.* □

We continue with our characterization result which quickly follows by Propositions 3 and 4.

Corollary 1 *With general utilities, a mechanism is Pareto efficient ex post iff it returns a probability distribution over a subset of the allocations returned by the* PARETO LIKE *mechanism.*

Another form of Pareto efficiency is to consider not the utility of each allocation returned by a mechanism but the utility in expectation across the distribution of allocations it returns. In offline fair division, an expected allocation is Pareto efficient ex ante if it can be represented as a convex combination of Pareto efficient ex post allocations (Bogomolnaia and Moulin, 2001). By this result, it follows directly that a mechanism for online fair division is Pareto efficient ex ante if its expected allocation can be represented as a convex combination of allocations returned by the PARETO LIKE mechanism. However, a Pareto efficient ex ante mechanism can return a convex combination in which the allocations that make up the distribution are not necessarily Pareto efficient ex post. This implies that a mechanism could be Pareto efficient ex ante but not Pareto efficient ex post. We show this in Example 7.

Example 7 *Consider again the instance from Example 1 and the following mechanism: item o_1 is allocated to agent a_1 with probability 1, and item o_2 is allocated to agent a_1 with probability $1 - \epsilon$ and to agent a_2 with probability ϵ where $\epsilon > 0$.*

Now agent a_1 gets expected utility $3 - 2\epsilon$ whilst agent a_2 gets expected utility ϵ. This distribution of allocations is Pareto efficient ex ante for $\epsilon < 1/2$. If $\epsilon = 1/2$, it is dominated by the one in which each agent gets their most valued item. But one ex post allocation in this distribution allocates o_1 to a_1 and o_2 to a_2. This allocation is not Pareto efficient ex post. Hence, we have a distribution of allocations that is Pareto efficient ex ante and contains an allocation that is not Pareto efficient ex post. $\square$

In Example 8, we give a convex combination of Pareto efficient ex post allocations that corresponds to the Pareto efficient ex ante mechanism in Example 7.

Example 8 *Consider again the instance and mechanism in Example 7. The expected allocation of this mechanism is Pareto efficient ex ante and can be represented as the following convex combination over its ex post allocations: $\{(a_1, o_1), (a_1, o_2)\}$ with $(1 - \epsilon)$ and $\{(a_1, o_1), (a_2, o_2)\}$ with ϵ. But this convex combination contains allocations which are not Pareto efficient ex post. The following combination represent the same expected allocation but it contains only Pareto efficient ex post allocations: $\{(a_1, o_1), (a_1, o_2)\}$ with $(1 - \frac{2\epsilon}{3})$ and $\{(a_2, o_1), (a_2, o_2)\}$ with $\frac{\epsilon}{3}$.* $\square$

In the other direction, not every convex combination of Pareto efficient ex post allocations is Pareto efficient ex ante. For example, the ONLINE RANDOM PRIORITY mechanism is not Pareto efficient ex ante on the instance from Example 1 but is Pareto efficient ex post in general. Another example of such a mechanism is the PARETO LIKE mechanism. It is Pareto efficient ex post but not Pareto efficient ex ante. To see this, consider again the instance in Example 1 and suppose that both agents like the second item with two. By comparison, deterministic mechanism such as the ONLINE SERIAL DICTATORSHIP mechanism returns a Pareto efficient ex post allocation and so is Pareto efficient ex ante. Interestingly, randomized mechanisms can also be Pareto efficient ex ante. The MAXIMUM LIKE mechanism is one such mechanism. We conclude that there is a Pareto efficient ex post mechanism for each given Pareto efficient ex ante distribution. This observation brings us to the our partial characterization result for Pareto efficiency ex ante.

Corollary 2 *With general utilities, a mechanism is Pareto efficient ex ante iff its returned ex ante allocation can be represented as a convex combination of the allocations returned by the* PARETO LIKE *mechanism.*

Proof. Since the expected allocation of the mechanism is Pareto efficient ex ante, it then can be represented as a convex combination of Pareto efficient ex post allocations. These allocations are returned by the PARETO LIKE mechanism. However, the allocations in such a convex combination form a subset of the allocations returned by the PARETO LIKE mechanism. Hence, we can extend any such Pareto efficient ex ante combination to all allocations returned by the PARETO LIKE mechanism by assigning zero weights to those allocations returned by the PARETO LIKE mechanism that are not in the initial combination. □

Finally, suppose that agents bid strategically with general utilities. The outcomes of the MAXIMUM LIKE and PARETO LIKE mechanism coincide with the outcome of the LIKE mechanism and therefore these outcomes are not Pareto efficient ex post or even ante. For the BALANCED LIKE mechanism, there are instances on which the strategic bids of all agents for a single item are all zeros which means that the item is "wasted" (i.e. not allocated to any agent). Therefore, we conclude that Pareto efficiency is compromised. If we want simple mechanisms for Pareto efficiency ex post with general utilities we might consider either the PARETO LIKE or ONLINE RANDOM PRIORITY mechanism. If we are interested in Pareto efficiency ex ante, we can continue to consider the MAXIMUM LIKE or ONLINE SERIAL DICTATORSHIP mechanism. The LIKE and BALANCED LIKE mechanisms are not Pareto efficient ex post or ex ante with general utilities but are Pareto efficient ex post and ex ante on 0/1 utilities.

3.7 Multiple axioms

We now consider combinations of axioms. We begin with a simple but important impossibility result. In offline fair division, Pareto efficiency and envy-freeness are always possible simultaneously. An offline mechanism such as the probabilistic serial rule mechanism returns allocations that are Pareto efficient and envy-free ex ante. In online fair division, uncertainty about future items makes it more challenging to return Pareto efficient or envy-free allocations. We may, for instance, commit to decisions now which cause envy in this uncertain future. Indeed, it is not just more challenging to return Pareto efficient and envy-free allocations in the online setting. The two goals conflict with each other.

Proposition 6 *With general utilities,* **no** *mechanism is envy-free and Pareto efficient ex ante.*

Proof. Consider an online mechanism that is envy-free ex ante. We can run this mechanism on the instance in Example 1. As the utilities of agents for items are positive, the mechanism allocates each item to an agent with probability $1/2$. This follows by Theorem 4. Hence, each agent receives expected utility of $3/2$. This outcome is not Pareto efficient ex ante because it is dominated by the one in which each agent receives the item they value with 2.□

When limited to 0/1 utilities, there are online mechanisms that achieve Pareto efficiency and envy-freeness simultaneously. For example, the LIKE and BALANCED LIKE mechanisms are both Pareto efficient and envy-free ex ante in this case. There are also mechanisms that satisfy many other combinations of axioms. For example, the ONLINE RANDOM PRIORITY mechanism satisfies strategy-proofness, envy-freeness ex ante and Pareto efficiency ex post. However, this mechanism may only return ex post allocations which are unfair even when exponentially many envy-free ex post allocations exist and say the BALANCED LIKE mechanism returns all of them. To see this, consider n agents, n items and suppose that each agent has the same utility for each item. There are $n!$ allocations which are envy-free ex post. The ONLINE RANDOM PRIORITY mechanism returns none of these envy-free ex post allocations whilst the BALANCED LIKE mechanism returns only these allocations. The LIKE mechanism also returns all of these envy-free ex post allocations but it also returns all $n^n - n!$ allocations which are not envy-free ex post. Surprisingly, the BALANCED LIKE mechanism satisfies many axioms with 0/1 utilities. In particular, it bounds the envy one agent has for another to 1 utility unit. This comes at a cost as this mechanism is not strategy-proof but we might be satisfied with its online strategy-proofness.

Proposition 7 *With 0/1 utilities, the* BALANCED LIKE *mechanism is online strategy-proof, envy-free ex ante, bounded envy-free ex post with 1, and Pareto efficient ex post and ex ante.*

We next ask if there is a mechanism that satisfies all axiomatic properties with 0/1 utilities, including strategy-proofness. We show that this is impossible in terms of the following result.

Proposition 8 *With 0/1 utilities,* **no** *mechanism is strategy-proof and bounded envy-free ex post with 1.*

Proof. Let us assume that is bounded envy-free ex post with 1. Given an allocation of items o_1 and o_{j-1}. We can show by induction that the mechanism must then allocate item o_j to an agent who like the item and has previously received fewest items amongst the agents that like item o_j. Consequently, the mechanism is not memoryless as the value of the probability of some agent for item o_j is different for different allocations of past items (i.e. bids of agents for past items). Therefore, the mechanism is not strategy-proof by Theorem 2. □

Finally, with general utilities, we might use the LIKE mechanism if we want envy-freeness ex ante, the ONLINE RANDOM PRIORITY and ONLINE SERIAL DICTATORSHIP mechanisms if we want Pareto efficiency ex post, or the MAXIMUM LIKE or PARETO LIKE mechanisms if we want Pareto efficiency ex ante. With 0/1 utilities, we might also consider the BALANCED LIKE mechanism if we want to bound the envy ex post between agents. Strategy-proofness is achieved by LIKE, ONLINE SERIAL DICTATORSHIP and ONLINE RANDOM PRIORITY. However, we can also use BALANCED LIKE whenever we have limited information because it is online strategy-proof. We summarize these results in Table 3.1.

mechanism	SP	online SP	EF ante	BEF post	PE ante	PE post
	general utilities					
ON. RAND. PRIO.	✓	✓	✓	×	×	✓
ON. SERIAL DICT.	✓	✓	×	×	✓	✓
MAXIMUM LIKE	×	×	×	×	✓	✓
PARETO LIKE	×	×	×	×	×	✓
LIKE	✓	✓	✓	×	×	×
BALANCED LIKE	×	✓	×	×	×	×
	binary utilities					
ON. RAND. PRIO.	✓	✓	✓	×	✓	✓
ON. SERIAL DICT.	✓	✓	×	×	✓	✓
MAXIMUM LIKE	✓	✓	✓	×	✓	✓
PARETO LIKE	✓	✓	✓	×	×	✓
LIKE	✓	✓	✓	×	✓	✓
BALANCED LIKE	×	✓	✓	✓	✓	✓

TABLE 3.1: Strategy-Proofness, Envy-Freeness and Pareto Efficiency.

3.8 Conclusions

To conclude, we have proposed several new mechanisms for online fair division. We identified mechanisms and classes of mechanisms which are (online) strategy-proof, envy-free and Pareto efficient. In the case of strategy-proofness, we proposed a new, weaker definition of strategy-proofness applicable to the online setting where past decisions are fixed and future items are uncertain. We further argued that no such online notions are needed for envy-freeness and Pareto efficiency. We report mechanisms that satisfy combinations of Pareto efficiency, envy-freeness and (online) strategy-proofness. We finally identified an important tradeoff between envy-freeness and Pareto efficiency in general. Unlike the offline setting, in online fair division we have to choose between envy-freeness and Pareto efficiency ex ante. For these reasons, when allocating items in an online setting, we might want to consider envy-free mechanisms such as LIKE and BALANCED LIKE, Pareto efficient ex post mechanisms such as ONLINE SERIAL DICTATORSHIP and ONLINE RANDOM PRIORITY, or Pareto efficient ex ante mechanisms such as MAXIMUM LIKE and PARETO LIKE. With 0/1 utilities, this tension disappears and it is possible to have mechanisms that are envy-free and Pareto efficient ex ante. In future, we are also interested in characterizing other axiomatic properties of online mechanisms such as group strategy-proofness, proportionality, anonymity, neutrality, equal treatment of equals, false-name proofness, etc. For example, we already showed that there is *no* group strategy-proof mechanism in general (Aleksandrov and Walsh, 2017c).

Chapter 4

Expected Outcomes and Manipulations

We studied in the previous chapter how we could possibly characterize (online) strategy-proofness, envy-freeness and Pareto efficiency. For example, we might want a mechanism that is robust to manipulations, and that returns an envy-free and Pareto efficient outcome supposing agents act sincerely. However, agents may manipulate the outcomes of online mechanisms. Such manipulations are directly related to the way we compute outcomes of online mechanisms. For this reason, we turn attention in this chapter to possible, necessary and exact outcomes and manipulations of online mechanisms with complete and no information about the items. For example, the chair might be concerned that agents receive enough utility or particular essential items. Alternatively, the chair might want to be sure that a favored agent gets a particular item. Also, they might even want to give similar utility to each agent or bias the future allocation in case some agents receive only a few items and are promised to receive more in expectation. In practice, it may be difficult to query the agents each time an item arrives. The chair will often collect the preferences of the agents in advance, and allocate items to agents as they arrive. There are several settings where it is reasonable to suppose that the chair does that. For instance, in the food bank problem, a good proxy for the utility of an item to a charity that likes it might simply be its retail price. This is public information. As a second example, in deceased organ matching, the utility of allocating an organ to a patient might be computed from a simple formula that takes account of the age of the organ, the age of the patient and a number of other medical factors. This is public information.

There are two sources of uncertainty in deciding outcomes of online mechanisms. First, such mechanisms could be randomized. Second, as the problem is online, the arrival order of items is typically unknown. We often have no information about future items or even if any more items will arrive but we can have information about past items. The agents might then be interested whether they could have *possibly* or *necessarily* picked their most favourite past items, and what are their *exact* probabilities or utilities for these items. For example, they might go elsewhere to request items on the next day if their expected utilities or probabilities for these items were very low today. In particular, we focus on computing whether an agent can possibly or necessarily receive a given expected utility. These results easily translate into whether an agent can possibly or necessarily receive a given item. We simply give most of the agent's utility to that item. Also, as all our results hold in the case of binary utilities, they can also be viewed as computing whether an agent can possibly or necessarily receive a given expected number of items. Whilst some of our results consider general utilities, such utilities are mainly used to compare outcomes and do not need to be elicited explicitly. General utilities could be used when bidding or allocating items. However, such “like" and “not like” reporting has advantages. It is simple, does not require costly eliciting of utilities of agents for items and it also leads to mechanisms with nice axioms. The possible, necessary and exact expected outcomes of online mechanisms naturally provide quarantees to agents for items and we study the computational complexity of calculating these quarantees.

We next discuss related literature in Section 4.1. We then give several preliminary notes in Section 4.2. For example, computing the outcomes of some of our mechanisms is tractable because they do not keep track on the past allocation. In Section 4.3, we report our general complexity results. We compare these results with our results for the case of 2 agents in Section 4.4. We then consider problems of computing possible, necessary and exact manipulations of our mechanisms in Section 4.5. Finally, we summarize our results in Section 4.6.

4.1 Related work

Our study is in-line with many similar results. For example, the possible and necessary allocations of the popular sequential allocation mechanism from (offline) fair division are intractable in general (Aziz, Brill, Fischer, Harrenstein, Lang, and Seedig, 2015). As a result, agents might decide to act sincerely. Sincere play is vital for this mechanism as it characterizes Pareto efficiency ex post in (offline) fair division (Brams and King, 2005). Partial tournaments is another domain where computing possible and necessary outcomes might be vital for determining winners of the tournament, especially when there is only limited information about the players (Aziz, Brill, Fischer, Harrenstein, Lang, and Seedig, 2015; Bachrach, Betzler, and Faliszewski, 2010). There are several considerations on how partial orders could impact the computational complexity of determining such outcomes (Xia and Conitzer, 2011).

Our hardness results also reveal an interesting connection between matching and fair division problems because our reductions are from popular computational matching problems (Demange and Ekim, 2008; Valiant, 1979a). We further study the complexity of computing exact outcomes of online mechanisms which is in-line with similar results for mechanisms in (offline) fair division. For example, Aziz et al. show that computing the exact outcomes of the popular random priority mechanism is intractable (Aziz, Brandt, and Brill, 2013). They showed their results in parallel and independent from (Sabán and Sethuraman, 2015). Online fair division is related to (online) house allocation (i.e. as many agents as items) with indifferences (Sabán and Sethuraman, 2013a). From this perspective, our hardness results put lower bounds on similar questions for house allocations.

Our results in regard to manipulations also provide a stepping stone towards better understanding the strategic behavior of agents. A number of works already considered such behavior for offline mechanisms. For example, Bouveret and Lang investigate how a single agent can manipulate the picking sequence of the sequential allocation (Bouveret and Lang, 2014). We also study unilateral manipulations. These manipulations are intractable for the BALANCED LIKE and PARETO LIKE mechanisms. Interestingly, some results are however tractable for the BALANCED LIKE mechanism in the case of 2 agents.

4.2 Preliminaries

Consider an instance and a mechanism. The mechanism returns a number of expected outcomes when run on the instance. For example, recall that $\overline{p}(i,j)$ denotes the probability of agent a_i for item o_j, and $\overline{u}(i)$ denotes their expected utility for all items. We say that the probability (expected utility) of agent a_i for item o_j (all items) is *exactly* k if its exact value is equal to some given $k \in \mathbb{Q}$. We further say that this probability (expected utility) is *necessarily* k if its exact value is at least some given $k \in \mathbb{Q}$. And, we say that this probability (expected utility) is *possibly* positive if its exact value is at least 0. We study in this chapter the complexity of computing the exact, necessary and possible outcomes of our online mechanisms.

We make a number of observations. For any mechanism that allocates all items to agents that like them, possible and necessary probabilities are related. All our mechanisms are such mechanisms. We next show this relation. Suppose we ask if $\overline{p}(i,j) > 0$ holds. This is true iff there is an allocation $\pi_i(j)$ of the first j items in which agent a_i receives item o_j. Such an allocation occurs however with some positive probability. Let the minimum value of this probability be $\epsilon > 0$. We have that $\epsilon \in (0, \frac{1}{n^j}]$ for all our mechanisms. We therefore conclude that $\overline{p}(i,j) > 0$ iff $\overline{p}(i,j) \geq \epsilon$. Unfortunately, there is no such a relation between possible and necessary expected utilities. For example, given agent a_i, we have that $\overline{u}(i) > 0$ holds iff agent a_i bids positively for at least one item and at least one item arrives. Consequently, we can easily decide if their expected utility is positive. However, as we show in our work, we cannot always decide easily if $\overline{u}(i) \geq k$ holds. Additionally, there is also a relation between necessary and exact outcomes. If we can compute outcomes easily, then we can compare outcomes easily. Consequently, tractability of exact outcomes would imply tractability of necessary outcomes. The reverse direction does not hold. This tractability relation transits to possible outcomes as well. We make the following simple observation based on these relations.

Observation 4 *With 2 or more agents and general utilities, the possible, necessary and exact outcomes of the* ONLINE SERIAL DICTATORSHIP*,* ONLINE RANDOM PRIORITY*,* MAXIMUM LIKE *mechanisms are tractable.*

The ONLINE SERIAL DICTATORSHIP and ONLINE RANDOM PRIORITY mechanisms are dictatorial mechanisms that use strict priority orderongs of agents and greedily allocate each next item to the agent that likes the item and has the highest priority. The MAXIMUM LIKE mechanism is also a greedy mechanism because it allocates each next item to one of the agents that bid most for it. Computing outcomes with these mechanisms is tractable for any number of agents because none of these mechanisms keeps track of past allocations. We therefore focus on the necessary and exact outcomes (i.e. NECESSARYUTILITY and EXACTUTILITY) of the LIKE, BALANCED LIKE and PARETO LIKE mechanisms. For example, computing the outcomes of the LIKE mechanism is tractable whereas the ones of the BALANCED LIKE mechanism is intractable even when agents have simple “like” and “not like” preferences. By comparison, with such 0/1 preferences, the PARETO LIKE mechanism returns the same distribution of expected outcomes as the LIKE mechanism and so these outcomes are tractable. However, with general utilities, we show that it is intractable to decide if an agent is feasible for an item with the PARETO LIKE mechanism.

Exact, necessary and possible outcomes are closely related to *exact*, *necessary* and *possible* manipulations, respectively. The manipulation problems that we consider are verification problems and, given an agent and a vector of bids of the agent, they ask if the agent can possibly or necessarily increase their sincere expected utility by reporting the bids in the vector supposing that all other agents bid sincerely for items, or how much is the exact gain of the manipulator by doing such misreports. We exploit a similar connection between possible and necessary manipulations as for possible and necessary outcomes and therefore focus only on necessary and exact expected utility manipulations (i.e. NECESSARYMANIPULATION and EXACTMANIPULATION). Again, tractable exact manipulations imply tractable necessary and possible manipulations. For example, recall that the ONLINE SERIAL DICTATORSHIP, ONLINE RANDOM PRIORITY and LIKE mechanisms are strategy-proof and hence agents cannot manipulate these mechanisms. However, the MAXIMUM LIKE and PARETO LIKE are not strategy-proof in general. We show that unilateral manipulations of the PARETO LIKE mechanism are not tractable as its outcomes are not tractable. By comparison, we observe that unilateral manipulations of the MAXIMUM LIKE mechanism are tractable as its outcomes are tractable.

Observation 5 *With 2 or more agents and general utilities, the possible, necessary and exact manipulations of the* MAXIMUM LIKE *mechanism are tractable.*

We are mainly interested in computing the possible, necessary and exact expected outcomes and manipulations of the PARETO LIKE and BALANCED LIKE mechanisms. However, we complement our results with those for the LIKE mechanism. For example, with 2 or more agents, computing each of these expected outcomes is tractable for the LIKE mechanism and intractable for any of the PARETO LIKE and BALANCED LIKE mechanisms even when the ordering of items is fixed. Interestingly, we also obtained some tractable results for the BALANCED LIKE mechanism in this case. For example, computing outcomes with this mechanism becomes tractable with 2 agents. The PARETO LIKE and BALANCED LIKE mechanisms are not strategy-proof in general. For this reason, at the end, we show that necessary and exact manipulations of these mechanisms are intractable. Our results are worst-case and indicate that agents might give up manipulating these two mechanisms. We summarize our results in Table 4.1.

mechanism	n **agents**	**utilities**	EXACTUTIL.	EXACTMAN.	NEC.UTIL.	NEC.MAN.
ON. SERIAL DICT.	$n \geq 2$	general	P	–	P	–
ON. RAND PRIO.	$n \geq 2$	general	P	–	P	–
PARETO LIKE	$n > 2$	general	coNP-hard	coNP-hard	coNP-hard	coNP-hard
PARETO LIKE	$n \geq 2$	binary	P	–	P	–
MAXIMUM LIKE	$n \geq 2$	general	P	P	P	P
LIKE	$n \geq 2$	general	P	–	P	–
BALANCED LIKE	$n = 2$	general	P	P	P	P
BALANCED LIKE	$n > 2$	binary	#P-hard	#P-hard	coNP-hard	coNP-hard

TABLE 4.1: Expected Outcomes and Manipulations.

Graph complexity We use problem reductions and classes from computational complexity. For example, we use *Karp*, *Turing*, *parsimonious* and *arithmetic* reductions in our proofs. (Bürgisser, 2000; Turing, 1936; Valiant, 1979a). We also use classes such as P, NP, coNP and #P. A decision problem belongs to NP if verifying a given candidate answer to the problem is in P. A decision problem is in NP-*hard* if there is a Karp reduction from another popular NP-hard problem to it. A decision problem is NP-*complete* if it belongs to NP and it is in NP-hard. The class coNP contains decision problems whose "negated" versions are in NP. The class #P contains counting problems that correspond to decision problems. The classes of decision problems are closed under polynomially-bounded Turing reductions. The classes of counting problems are closed under polynomially-bounded arithmetic reductions.

Our hardness results are closely related to popular problems for bipartite graphs. For this reason, we need some notation from graph theory. Let G be an undirected bipartite graph. A *matching* μ in G is a set of vertex-disjoint edges. We say that μ *matches* a vertex if there is an edge in it that is incident with the vertex. Matching μ is *maximal* if it is no longer a matching once some other edge is added to it. Matching μ is *perfect* if it matches all vertices in the graph. We can now define the graph problems. Given a graph G, the *counting perfect matchings problem* is to output the number of perfect matchings in G. This problem is proven to be in #P-hard on various bipartite graphs (Okamoto, Uehara, and Uno, 2009; Valiant, 1979b). Given a graph G and an integer r, the *minimum size maximal matching problem* is to decide if there is a maximal matching μ in G with $|\mu| \leq r$. This problem is shown to be NP-hard on various bipartite graphs (Demange and Ekim, 2008; Sabán and Sethuraman, 2015).

#PERFECT MATCHINGS	MINIMUM MAXIMAL MATCHING
Input: G.	Input: G, $r \in \mathbb{N}$.
Output: $\#$ of perfect matchings in G.	Question: is there μ in G with $\|\mu\| \leq r$?

4.3 Expected outcomes: the case of n agents

We suppose the agents act sincerely and next consider the case that the chair knows the utilities *and* the arrival ordering of items. We argued previously that these could be future or past items. We next define the computational problems for the necessary and exact outcomes of mechanisms. In general, there could be many agents bidding for items. We start with the exact outcomes of the LIKE and BALANCED LIKE mechanisms and proceed with their necessary outcomes.

EXACTUTILITY	NECESSARYUTILITY
Input: $\mathcal{I}$, a_i.	Input: $\mathcal{I}$, a_i, $k \in \mathbb{Q}$.
Output: $\overline{u}(i, U_i)$.	Question: $\overline{u}(i, U_i) \geq k$?

4.3.1 Exact outcomes

The mechanism that we now consider is the LIKE mechanism. This mechanism does not keep track on the allocation of past items. Moreover an agent is feasible for an item if they simply bid positively for this item. The probability of an agent for an item then solely depends on the number of agents bidding positively for the item. Consequently, all of its exact outcomes are tractable.

Proposition 9 *With general utilities and the* LIKE *mechanism, problem* EXACTUTILITY *is in* P.

Proof. The probability $\overline{p}(i, j)$ of agent a_i for item o_j is $1/n_j$ where n_j is the number of agents that like this item. Their additive expected utility $\overline{u}(i)$ can be given as $\sum_{j=1}^{m}(1/n_j) \cdot u_{ij}$. □

We next argue that even agent feasibility is intractable with the PARETO LIKE mechanism and general utilities. This is because the feasibility check of this mechanism requires us to check if a given partially constructed allocation is Pareto efficient ex post. The latter problem is in coNP-hard with additive utilities (Keijzer, Bouveret, Klos, and Zhang, 2009). We just give a Turing reduction for this problem.

Proposition 10 *With general utilities and the* PARETO LIKE *mechanism, problem* EXACTUTILITY *is in* coNP*-hard under Turing reductions.*

Proof. Let us consider ex post allocation π returned by the mechanism. Suppose that agent a_{i_j} is allocated item o_j in π. By the definition of the mechanism, we have that the sub-allocation $\pi(j-1) \cup \{(a_{i_j}, o_j)\}$ is verified for Pareto efficiency ex post over items o_1 to o_j. The jth verification corresponds to checking if agent a_{i_j} is feasible for item o_j, i.e. whether their conditional probability given $\pi(j-1)$ is positive. We thus conclude that $\pi(j-1) \cup \{(a_{i_j}, o_j)\}$ is Pareto efficient iff the value of this probability is positive iff it is equal to $\frac{1}{f_j}$ iff it is at least $\frac{1}{f_j}$ where f_j is the number of feasible agents given $\pi(j-1)$. If these verification checks can be performed in polynomial-time for the arbitrary allocation π, then we could verify Pareto efficiency ex post of an arbitrary allocation in polynomial-time. This is in contradiction the result that verifying Pareto efficiency ex post is coNP-hard. Consequently, computing the exact outcomes of this mechanism is at least as hard as verifying Pareto efficiency.□

We continue with the BALANCED LIKE mechanism. This mechanism is biased towards agents with fewest items which leads to intractability of its outcomes. To see this, note that an agent is feasible for an item with the BALANCED LIKE mechanism given *some* possible allocations of past items. By comparison, they are feasible for this item with the LIKE mechanism given *all* possible allocations of past items. We give for our intractability result a parsimonious reduction from the counting #PERFECT MATCHINGS problem to the EXACTUTILITY problem. The counting problem is shown to be #P-hard even on 3-regular undirected bipartite graphs (Dagum and Luby, 1992). We next present the reduction.

Reduction 1 Let G be a 3-regular bipartite graph, $u_1, \ldots, u_N$ be the vertices from one of its partitions and $v_1, \ldots, v_N$ be the vertices from the other one of its partitions. Let v_{i1}, v_{i2}, v_{i3} denote the vertices in the graph that are incident with vertex u_i with $i \in [1, N]$. Further, let $e_{3\cdot(i-1)+1} = (u_i, v_{i1})$, $e_{3\cdot(i-1)+2} = (u_i, v_{i2})$, $e_{3\cdot(i-1)+3} = (u_i, v_{i3})$ denote the edges in the graph that are incident with vertex u_i with $i \in [1, N]$. Each edge e_k for $k \in [1, 3 \cdot N]$ can be represented as (u_i, v_j) for some $u_i \in \{u_1, \ldots, u_N\}$ and $v_j \in \{v_{i1}, v_{i2}, v_{i3}\}$. We next construct the online fair division instance $\mathcal{E}_G$ from the graph G as follows:

- **Agents:** 1 agent per edge e_k and 1 special agent $e_{3\cdot N+1}$
- **Items:** 1 item per vertex v_j, 2 items u_{i1}, u_{i2} per vertex u_i and 2 items x, y
- **Non-zero utilities:**
  - **agent** $e_{3\cdot(i-1)+j}$ with $j \in \{1, 2, 3\}$ has utility 1 for items v_{ij}, u_{i1}, u_{i2}, y
  - **agent** $e_{3\cdot N+1}$ has utility 1 for items x, y
- **Ordering:** $o = (v_1 \ldots v_N u_{11} u_{12} \ldots u_{N1} u_{N2} x y)$

Reduction 1 is very insightful and provides a very tight bound on the complexity of the EXACTUTILITY problem for the BALANCED LIKE mechanism (e.g. 0/1 utilities, each agent likes at most 4 items, each item except one is liked by at most 3 agents, each pair of agents like at most 3 items in common, the ordering of items is fixed, etc.). Our result relies on the fact that counting *perfect* allocations in which each agent receives exactly one item is in #P-hard.

Lemma 1 *With the* BALANCED LIKE *mechanism, the number of allocations in $\mathcal{E}_G$ in which agent $e_{3\cdot N+1}$ is feasible for item y is equal to 2^N times the number of perfect matchings in G.*

Proof. By construction, each item v_j is liked by exactly three different agents and, hence, each allocation of items $v_1, \ldots, v_N$ gives these items to N different agents among $e_1, \ldots, e_{3\cdot N}$. Consider then an ex post allocation $\pi(N)$ of items $v_1, \ldots, v_N$ such that, for each vertex u_i, either agent $e_{3\cdot(i-1)+1}$ gets item v_{i1} or agent $e_{3\cdot(i-1)+2}$ gets item v_{i2} or agent $e_{3\cdot(i-1)+3}$ gets item v_{i3}. We say that such an allocation $\pi(N)$ has *perfect* matches for vertices $u_1, \ldots, u_N$ because exactly one agent per triplet $e_{3\cdot(i-1)+1}, e_{3\cdot(i-1)+2}, e_{3\cdot(i-1)+3}$ gets an item among $v_1, \ldots, v_N$. In fact, there is a perfect matching in the graph G between vertices $v_1, \ldots, v_N$ and vertices $u_1, \ldots, u_N$ iff there is an allocation in the instance $\mathcal{E}_G$ of items $v_1, \ldots, v_N$ that has perfect matches for vertices $u_1, \ldots, u_N$. Furthermore, this is a 1-to-1 parsimonious correspondence. We also show that there is a 1-to-2^N parsimonious correspondence between the ex post perfect allocations of the first $3 \cdot N + 1$ items to all $3 \cdot N + 1$ agents in $\mathcal{E}_G$ and the allocations of items $v_, \ldots, v_N$ that have perfect matches for vertices $u_1, \ldots, u_N$. We use these two correspondences and conclude that the number of perfect allocations of the first $3 \cdot N + 1$ items to all $3 \cdot N + 1$ agents in $\mathcal{E}_G$ is equal to 2^N times the number of perfect matchings in G. We next present the proof.

First, let us consider one discrete allocation $\pi'(N)$ in $\mathcal{E}_G$ of items $v_1, \ldots, v_N$ that has perfect matches for vertices $u_1, \ldots, u_N$. WLOG, suppose that $\pi'(N)$ is such that, for each vertex u_i, agent $e_{3\cdot(i-1)+1}$ receives their corresponding item v_{i1}. The allocation $\pi'(N)$ can be extended by the mechanism to two discrete allocations w.r.t. each vertex u_i: (1) agent $e_{3\cdot(i-1)+2}$ gets item u_{i1} and agent $e_{3\cdot(i-1)+3}$ gets item u_{i2} or (2) agent $e_{3\cdot(i-1)+2}$ gets item u_{i2} and agent $e_{3\cdot(i-1)+3}$ gets item u_{i1}. By the preference structure, the allocation $\pi'(N)$ can therefore be extended by the mechanism to 2^N perfect allocations of the first $3\cdot N+1$ items to all $3\cdot N+1$ agents in $\mathcal{E}_G$. Note that each of these perfect allocations necessarily gives item x to agent $e_{3\cdot N+1}$ because only they like it. Second, consider one perfect allocation $\pi(3\cdot N+1)$ of the first $3\cdot N+1$ items to all $3\cdot N+1$ agents in $\mathcal{E}_G$. It must be the case that $\pi(3\cdot N+1)$ extends some ex post allocation of items $v_1, \ldots, v_N$ that has perfect matches for vertices $u_1, \ldots, u_N$. To show this, consider a discrete allocation $\pi''(N)$ of items $v_1, \ldots, v_N$ that has not perfect matches for vertices $u_1, \ldots, u_N$. Hence, the allocation $\pi''(N)$ is such that at least two of the agents $e_{3\cdot(i-1)+1}, e_{3\cdot(i-1)+2}, e_{3\cdot(i-1)+3}$ for some vertex u_i receive their corresponding items v_{i1}, v_{i2}, v_{i3} among $v_1, \ldots, v_N$. Therefore, each allocation of the first $3 \cdot N + 1$ items to all $3 \cdot N + 1$ agents that extends $\pi''(N)$ by using the mechanism gives item u_{i1} or item u_{i2} to one of the agents $e_{3\cdot(i-1)+1}, e_{3\cdot(i-1)+2}, e_{3\cdot(i-1)+3}$ as their second item. As a consequence, in each such allocation, there is another agent with zero items after round $3 \cdot N + 1$. We conclude that each such extension of the allocation $\pi''(N)$ is not a perfect allocation in $\mathcal{E}_G$ of the first $3 \cdot N + 1$ items to all $3 \cdot N + 1$ agents.

Finally, note that agent $e_{3\cdot N+1}$ is feasible for item y given each perfect allocation of the first $3\cdot N+1$ items to all $3\cdot N+1$ agents, and they are not feasible for item y given any other allocation of the first $3 \cdot N + 1$ items to all $3 \cdot N + 1$ agents. These statements hold because all $3 \cdot N + 1$ agents like item y.$\square$

To understand better the combinatorial nature of the counting EXACTUTILITY problem, we next depict in Example 9 the instance from the proof of Lemma 1 on a small graph.

Example 9 *(***Instance** $\mathcal{E}_G$*) Let us consider the 3-regular bipartite graph G with vertices from $U = \{u_1, u_2, u_3, u_4\}$ of degree exactly 3 and $V = \{v_1, v_2, v_3, v_4\}$ of degree exactly 3. Consider the edges: $(u_1, v_1), (u_1, v_2), (u_1, v_4)$, $(u_2, v_2), (u_2, v_3), (u_2, v_4)$, $(u_3, v_1), (u_3, v_3), (u_3, v_4)$, $(u_4, v_1), (u_4, v_2), (u_4, v_3)$. The instance $\mathcal{E}_G$ that corresponds to the graph G has 13 agents e_1 to e_{13}, 14 items v_1 to v_4, u_{i1} and u_{i2} for $i \in [1, 4]$, x and y, and ordering $o = (v_1 v_2 v_3 v_4\ u_{11} u_{12}\ u_{21} u_{22}\ u_{31} u_{32}\ u_{41} u_{42}\ xy)$. The utilities are depicted next.*

o	v_1	v_2	v_3	v_4	u_{11}	u_{12}	u_{21}	u_{22}	u_{31}	u_{32}	u_{41}	u_{42}	x	y
e_1	**1**	0	0	0	1	1	0	0	0	0	0	0	0	1
e_2	0	1	0	0	**1**	1	0	0	0	0	0	0	0	1
e_3	0	0	0	1	1	**1**	0	0	0	0	0	0	0	1
e_4	0	**1**	0	0	0	0	1	1	0	0	0	0	0	1
e_5	0	0	1	0	0	0	**1**	1	0	0	0	0	0	1
e_6	0	0	0	1	0	0	1	**1**	0	0	0	0	0	1
e_7	1	0	0	0	0	0	0	0	1	**1**	0	0	0	1
e_8	0	0	1	0	0	0	0	0	**1**	1	0	0	0	1
e_9	0	0	0	**1**	0	0	0	0	1	1	0	0	0	1
e_{10}	1	0	0	0	0	0	0	0	0	0	1	**1**	0	1
e_{11}	0	1	0	0	0	0	0	0	0	0	**1**	1	0	1
e_{12}	0	0	**1**	0	0	0	0	0	0	0	1	1	0	1
e_{13}	0	0	0	0	0	0	0	0	0	0	0	0	**1**	**1**

The allocation $\pi(4) = \{(e_1, v_1), (e_4, v_2), (e_{12}, v_3), (e_9, v_4)\}$ has perfect matches for vertices u_1 to u_4 and corresponds to a perfect matching in the graph. The "red" allocation is a perfect allocation that extends $\pi(4)$ and gives exactly one item to each agent. The number of perfect allocations of the first 13 items to all 13 agents in $\mathcal{E}_G$ is 2^4 the number of perfect matchings in G. The latter number is 9 and therefore the number of perfect such allocations is $16 \cdot 9 = 144$. Note that agent e_{13} is feasible for the "orange" item y given the "red" allocation.$\square$

We are now ready to prove one of our main results for the BALANCED LIKE mechanism. Namely, computing exact outcomes is intractable for this mechanism even when agents have simple binary utilities for items.

Theorem 5 *Even with 0/1 utilities and the* BALANCED LIKE *mechanism, problem* EXACTUTILITY *is in* $\#\mathsf{P}$*-hard under arithmetic reductions.*

Proof. Let us again consider instance $\mathcal{E}_G$. Recall that a perfect allocation gives one item to each agent that likes the item. Let $\pi(3 \cdot N+1)$ denote such a perfect allocation of the first $3 \cdot N+1$ items to all $3 \cdot N+1$ agents. This allocation is returned by the mechanism with probability $p(\pi(3 \cdot N+1)) = (\frac{1}{3^N}) \cdot (\frac{1}{2^N})$. The conditional probability that agent $a_{3 \cdot N+1}$ is feasible for item y is $\frac{1}{3 \cdot N+1}$ given such a perfect allocation because all agents like item y. Therefore, each allocation in which agent $e_{3 \cdot N+1}$ receives item y occurs with probability $(\frac{1}{3^N}) \cdot (\frac{1}{2^N}) \cdot (\frac{1}{3 \cdot N+1})$. In contrast, the conditional probability of agent $e_{3 \cdot N+1}$ for item y is 0 given any other allocation of the first $3 \cdot N+1$ items to all $3 \cdot N+1$ agents. In such an allocation, note that agent $e_{3 \cdot N+1}$ receives item x and at least one other agent has zero items. Also, note that both agent $e_{3 \cdot N+1}$ and the agent with zero items like item y. Therefore, agent $e_{3 \cdot N+1}$ is not feasible for it in such an allocation of past items. We conclude that the expected probability $\overline{p}((3 \cdot N+1), y)$ is equal to $(\frac{1}{3^N}) \cdot (\frac{1}{2^N}) \cdot (\frac{1}{3 \cdot N+1})$ times the number of perfect allocations of the first $3 \cdot N+1$ items to all $3 \cdot N+1$ agents. The expected utility of agent $e_{3 \cdot N+1}$ is equal to $\overline{p}((3 \cdot N+1), x) + \overline{p}((3 \cdot N+1), y)$. We have that $\overline{p}((3 \cdot N+1), x) = 1$ because only agent $e_{3 \cdot N+1}$ likes item x and the mechanism allocates each item to an agent. The result follows by Lemma 1 and under arithmetic reductions. □

To conclude, with an unbounded number of agents, the exact outcomes are tractable for the LIKE mechanism and intractable for the BALANCED LIKE and PARETO LIKE mechanisms. We supposed sincere play. Interestingly, supposing agents bid strategically, all agents with the PARETO LIKE mechanism bid the same maximal bid for each item. Hence, its exact outcomes become the same as outcomes of the LIKE mechanism and so are tractable. The outcomes of BALANCED LIKE remain intractable in this case.

4.3.2 Necessary outcomes

Let us again start with the LIKE mechanism. We can compute outcomes efficiently with this mechanism and therefore we can compare these outcomes efficiently. Computing exact outcomes of the PARETO LIKE mechanism are intractable even in a given ex post allocation. But, in such an allocation, its exact outcomes coincide with its necessary outcomes. Consequently, we immediately conclude the next two results.

Corollary 3 *With general utilities and the* LIKE *mechanism, problem* NECESSARYUTILITY *is in* P.

Corollary 4 *With general utilities and the* PARETO LIKE *mechanism, problem* NECESSARYUTILITY *is in* coNP*-hard under Turing reductions.*

We therefore next again consider the BALANCED LIKE mechanism. The intractability of its exact outcomes does not give us any insight about the complexity of computing its necessary outcomes. One reason for this is because its exact outcomes are related to counting problems and its necessary outcomes are related to decision problems.

On the one hand, there are counting problems which are computationally intractable and whose corresponding decision versions are computationally tractable. For example, the counting perfect matching problem we introduced earlier is in #P-hard and the problem of deciding whether a maximum matching exist is in P (Hopcroft and Karp, 1973). For problems which are hard to cound and easy to decide, there are polynomial approximation schemes that allows us to approximate the counting (Ausiello, Protasi, Marchetti-Spaccamela, Gambosi, Crescenzi, and Kann, 1999). On the other hand, there are counting problems which are computationally intractable and whose corresponding decision versions are also computationally intractable. For example, computing the probability with the popular serial dictatorship is in #P-hard and the problem of deciding whether an agent possibly receives an item with this mechanism is in NP-hard. For problems which are hard both to count and to decide, it is believed that there is no polynomial approximation scheme meaning that it is also hard to approximate them (Jerrum and Sinclair, 1989). Hence, as we show that the necessary outcomes of the BALANCED LIKE mechanism are in NP-hard in general, we doubt that a polynomial approximation scheme for its counting version exist. However, we can use a fast *sampling scheme* in order to distinguish with high probabilty between its low and high expected outcomes of agents for items (Sabán and Sethuraman, 2015). To show our hardness result, we give a Karp reduction from the decision MINIMUM MAXIMAL MATCHING problem to the negation of the NECESSARYUTILITY problem. The minimum size maximal matching problem is shown to be NP-hard on 3-regular undirected bipartite graphs (Demange and Ekim, 2008). We next present the reduction.

Reduction 2 Let us consider the graph G from Reduction 1 and an integer r. Recall there are $3 \cdot N$ edges in the graph. Let e_k is one such edge and let it connect vertex u_i and v_j with $i, j \in [1, N]$. WLOG, let vertex u_i be connected to v_{i1}, v_{i2} and v_{i3} via edges $e_{3\cdot(i-1)+1}$, $e_{3\cdot(i-1)+2}$ and $e_{3\cdot(i-1)+3}$, respectively. We next construct the online fair division instance $\mathcal{P}_{G,r}$ from the graph G and the integer r as follows:

- **Agents:** 1 agent per edge e_k, agents $a_1, \ldots, a_{2\cdot(N-r)}$ and 2 special agents b, c
- **Items:** 1 item for each vertex v_j, items $x_1^k, \ldots, x_N^k, y_1^k, \ldots, y_N^k$ for $k \in \{1, 2\}$, $z_1, \ldots, z_{N-r}$ and 1 special item w
- **Non-zero utilities:**
  - **agent** $e_{3\cdot(i-1)+j}$ with $j \in \{1, 2, 3\}$ has utility 1 for items x_i^k, v_{ij}, y_i^k with $k \in \{1, 2\}$ and 1 for items $z_1, \ldots, z_{N-r}$
  - **agent** a_i has utility 1 for items $x_1^k, \ldots, x_N^k$ with $k \in \{1, 2\}$
  - **agent** b has each utility 1 for item w
  - **agent** c has utility 1 for items z_{N-r}, w
- **Ordering:** $o = (x_1^1 \ldots x_N^1 x_1^2 \ldots x_N^2 v_1 \ldots v_M y_1^1 \ldots y_N^1 y_1^2 \ldots y_N^2 z_1 \ldots z_{N-r} w)$

Let us focus on the special agents. Agent b likes only item w which means that their expected probability for it is equal to their expected utility for all items. Therefore, they receive this item with expected utility of at least 1 iff their probability for it is exactly equal to 1 iff the probability of agent c for item w is equal to 0. This means that it is not that case that agent c receives positive probability for item w. We show the probability of agent c for item w in $\mathcal{P}_{G,r}$ is positive iff there is a maximal matching in G of size at most r. We therefore conclude the following result.

Theorem 6 *Even with 0/1 utilities and the* BALANCED LIKE *mechanism, problem* NECESSARYUTILITY *is in* coNP*-hard under Turing reductions.*

Proof. We show that there is a maximal matching in G of cardinality at most r iff there is an allocation in $\mathcal{P}_{G,r}$ in which agent c receives item w iff $\overline{p}(c,w) > 0$. The second "iff" statement is trivial. We, therefore, show only the first "iff" statement. First, suppose that $\mu = \{(u_1, v_{13}), \dots, (u_l, v_{l3})\}$ is a maximal matching in G of cardinality l that is at most r. We construct one allocation π_μ in $\mathcal{P}_{G,r}$ in which agent c gets item w.

1. For each v_{i3} matched to u_i in μ, we allocate x_i^1 to $e_{3(i-1)+1}$, x_i^2 to $e_{3(i-1)+2}$ and v_{i3} to $e_{3(i-1)+3}$. By the maximality of μ, the unallocated items among $v_1, \dots, v_M$ are liked only by agents among $e_{3(i-1)+1}, e_{3(i-1)+2}, e_{3(i-1)+3}$ for each $i \in [1, l]$. For each such unallocated item, we just allocated one item among $x_1^1, \dots, x_l^1$ and one item among $x_l^2, \dots, x_l^2$ to each agent that likes this unallocated item. Items $x_1^1, \dots, x_l^1$ and $x_l^2, \dots, x_l^2$ are allocated.
2. After step 1, items $x_{l+1}^1, \dots, x_N^1$ and $x_{l+1}^2, \dots, x_N^2$ are unallocated. Their number is $2 \cdot (N - l)$ which is greater than $2 \cdot (N - r)$ because $l \le r$. WLOG, for $i \in [1, N-r]$, let us allocate item x_{l+i}^1 to agent a_i and x_{l+i}^2 to agent $a_{i+(N-r)}$. WLOG, for $i \in [N-r+1, N-l]$, let us allocate item x_{l+i}^1 to agent $e_{3(l+i-1)+1}$ and x_{l+i}^2 to agent $e_{3(l+i-1)+2}$. Items $x_{l+1}^1, \dots, x_N^1$ and $x_{l+1}^2, \dots, x_N^2$ are now allocated.
3. After step 2, $N-l$ items among $v_1, \dots, v_N$ are unallocated and only agents among $e_{3(i-1)+1}$, $e_{3(i-1)+2}, e_{3(i-1)+3}$ for each $i \in [1, l]$ are feasible for these items. By the preferences, each of these agents like at most one of these unallocated items. Hence, there are different feasible agents for each such unallocated item. Let us run the mechanism and allocate each such unallocated item to one of the agents feasible for this item. Items $v_1, \dots, v_N$ are allocated.
4. After step 3, there are three cases for each triple of agents $e_{3(i-1)+1}, e_{3(i-1)+2}$, $e_{3(i-1)+3}$ for each $i \in [1, l]$: (*i*) one of them is feasible for item y_i^1, (*ii*) two of them are feasible for item y_i^1 and (*iii*) three of them are feasible for item y_i^1. We allocate items y_i^1 and y_i^2 to these agents in each of these cases subject to the feasibility constraints. WLOG, for $i \in [1, N-r]$, let us allocate item y_{l+i}^1 to agent $e_{3(i-1)+1}$ and item y_{l+i}^2 to agent $e_{3(i-1)+2}$. Furthermore, agents $e_{3\cdot(i-1)+1}, e_{3\cdot(i-1)+2}, e_{3\cdot(i-1)+3}$ for $i \in [N-r+1, N-l]$ have no items as well. WLOG, for $i \in [N-r+1, N-l]$, let us allocate item y_{l+i}^1 to agent $e_{3\cdot(i-1)+2}$ and item y_{l+i}^2 to agent $e_{3\cdot(i-1)+3}$. Items $y_1^1, \dots, y_N^1$ and $y_1^2, \dots, y_N^2$ are allocated.
5. After step 4, agents $e_{3\cdot(i-1)+1}$ for each $i \in [N-r+1, N-l]$ have no items. WLOG, for each $i \in [1, N-r]$, let us allocate z_i to $e_{3\cdot(i-1)+1}$. Items $z_1, \dots, z_{N-r}$ are allocated.
6. After step 5, agent c have no items. Let us then allocate item w to them. Items are allocated.

We leave to the inquisitive reader to check that π_μ occurs with positive probability. Second, suppose next that π is a discrete allocation of all items in $\mathcal{P}_{G,r}$ in which agent c holds items w.

1. Item w is allocated in π to agent c as their first item. To see this, suppose they also get some item z_{N-r} in π. Now, they would not be feasible when item w arrives as agent b has zero items in π at this moment and the mechanism would have given w to b and not to c.
2. Prior to item w in π, agent c have received zero items. Hence, items $z_1, \dots, z_{N-r}$ are allocated in π to $N - r$ agents as their first items. By the preferences, these agents are from different triples among $e_{3\cdot(i-1)+1}, e_{3\cdot(i-1)+2}, e_{3\cdot(i-1)+3}$ for each $i \in [1, N]$ because, for each fixed triplet of such agents, two of the agents are forced to get items y_i^1 and y_i^2. WLOG, let us assume that $e_{3\cdot(i-1)+1}$ for each $i \in [1, N-r]$ get $z_1, \dots, z_{N-r}$ in π.
3. Prior to item z_1 in π, agents $e_{3\cdot(i-1)+1}$ for each $i \in [1, N-r]$ have zero items. Hence, $N - r$ items among $y_1^k, \dots, y_N^k$ for $k \in \{1, 2\}$ are allocated in π to $e_{3\cdot(i-1)+2}$ for each $i \in [1, N-r]$ and $e_{3\cdot(i-1)+3}$ for each $i \in [1, N-r]$ as their first items. WLOG, let these items be $y_1^1, \dots, y_{N-r}^1$ and $y_1^2, \dots, y_{N-r}^2$. For $i \in [N-r+1, N]$, we note that item y_i^k with $k \in \{1, 2\}$ is allocated in π to either $e_{3\cdot(i-1)+2}$ or $e_{3\cdot(i-1)+3}$ as their first or second item.

4. Prior to item y_1^1 in π, agents $e_{3\cdot(i-1)+2}$ and $e_{3\cdot(i-1)+3}$ for each $i \in [1, N-r]$ have zero items. By the preferences, agents $a_1, \ldots, a_{2\cdot(N-r)}$ must then receive $2 \cdot (N-r)$ items in π among the first $2 \cdot N$ items in the ordering. These are items $x_1^k, \ldots, x_{(N-r)}^k$ for $k \in \{1, 2\}$. For $i \in [N-r+1, N]$, items x_i^1, x_i^2 are allocated in π to $e_{3\cdot(i-1)+2}$ and $e_{3\cdot(i-1)+3}$ as their first items.

We conclude that agents $e_{3\cdot(i-1)+1}$ for each $i \in [N-r+1, N]$ have zero items prior to item v_1 in π. Moreover, only agents $e_{3\cdot(i-1)+1}, e_{3\cdot(i-1)+2}, e_{3\cdot(i-1)+3}$ for each $i \in [N-r+1, N]$ receive items $v_1, \ldots, v_M$ in π. Finally, only $l \le r$ agents among $e_{3\cdot(i-1)+1}$ for each $i \in [N-r+1, N]$ get items in π among $v_1, \ldots, v_M$ as first items as some of these agents might like the same items among $v_1, \ldots, v_M$. WLOG, let these agents be $e_{3\cdot(i-1)+1}$ for each $i \in [N-l+1, N]$ and they are allocated in π items $v_1, \ldots, v_l$ as first items.

The set $\mu_\pi = \{(e_{3\cdot(N-l+1)+1}, v_1), \ldots, (e_{3\cdot(N-1)+1}, v_l)\}$ contains only edges from the graph G which are vertex-disjoint. Therefore, this set is a matching in G. Moreover, the cardinality of this set is l at most r. We next show that μ_π is a maximal matching. For the sake of contradiction, suppose that μ_π remains a matching if we add a new edge to it, say (u, v). The edge (u, v) is vertex-disjoint with the edges in μ_π. This means that vertex u is not among $u_{N-l+1}, \ldots, u_N$ and vertex v is not among $v_1, \ldots, v_l$. Hence, vertex u is among $u_1, \ldots, u_{N-l}$. In the allocation π, agents $e_{3\cdot(i-1)+2}$ and $e_{3\cdot(i-1)+3}$ for each $i \in [1, N-r]$ do not receive any items among $v_1, \ldots, v_M$ because they have zero items prior to the round when item y_1^1 is revealed. This implies that all these agents are feasible for the items they like among $v_1, \ldots, v_M$ but they do not get them in π. As agents $e_{3\cdot(i-1)+1}$ for each $i \in [N-l+1, N]$ get items $v_1, \ldots, v_l$ as their first items, we conclude that some agents among $e_{3\cdot(i-1)+1}, e_{3\cdot(i-1)+2}, e_{3\cdot(i-1)+3}$ for each $i \in [N-l+1, N]$ receive items $v_{l+1}, \ldots, v_M$ as their second items. Therefore, it must be the case that all agents $e_{3\cdot(i-1)+1}, e_{3\cdot(i-1)+2}, e_{3\cdot(i-1)+3}$ for $i \in [1, N-r]$ do not like any item among $v_{l+1}, \ldots, v_M$. Otherwise, the mechanism would allocate some of these items to agents among $e_{3\cdot(i-1)+1}, e_{3\cdot(i-1)+2}, e_{3\cdot(i-1)+3}$ for $i \in [1, N-r]$. This follows by the definition of the mechanism as, for each item, it favors the agents like the item and have fewest items. And, we reached a contradiction with the existence of the allocation π. Finally, in the graph G, vertices $u_1, \ldots, u_{N-r}$ are connected only to vertices among $v_1, \ldots, v_l$. Hence, v is among $v_1, \ldots, v_l$. This fact contradicts that $\mu_\pi \cup \{(u, v)\}$ is a matching. □

Finally, in general, the outcomes are tractable for the LIKE mechanism and intractable for the BALANCED LIKE and PARETO LIKE mechanisms. We supposed sincere play. Interestingly, supposing strategic play, the outcomes of the PARETO LIKE mechanism are equal to the outcomes of the LIKE mechanism, and so are tractable, as each agent reports the maximal bid for each item they like. The outcomes of the BALANCED LIKE mechanism remains intractable in this case.

4.4 Expected outcomes: the case of 2 agents

The outcomes of the LIKE mechanism are tractable. Surprisingly, in contrast to Theorems 5 and 6, the outcomes of BALANCED LIKE become tractable with a constant number n of agents (Aleksandrov, Aziz, Gaspers, and Walsh, 2015). In fact, we can compute them in $O(m^n)$ time using dynamic programming. Interestingly, with 2 agents, we can do even better and compute outcomes in $O(m)$ time when the ordering of items is fixed. Tractability of exact outcomes in this case with 2 agents implies tractability of necessary outcomes.

Proposition 11 *With 2 agents, general utilities and the* BALANCED LIKE *mechanism, problems* EXACTUTILITY *and* NECESSARYUTILITY *are in* P.

Proof. We use a dynamic program. Each state $s = (p, q)$ in it encodes that agent a_1 has p items, agent a_2 has q items, and its expected probability $\overline{p}(s)$. By induction, we show that there are at most 2 different states after each allocation round. In the base case, consider round 1. There are at most 2 states after this round depending on whether both a_1 and a_2 or only one of them like the first item. In the hypothesis, consider round j and suppose there are at most two states after round j. In the step case, consider round $j + 1$. Now, there are two cases. In the first one, there is only one state after round j. The result follows by the base case. In the second case, there are two states after round j. Let these be (p, q) and $(p - 1, q + 1)$ where $p + q = j$. If only one agent likes item o_{j+1}, each state transits into a new state and the result follows. If both a_1 and a_2 like item o_{j+1}, we consider four sub-cases depending on the difference $p - q$: (1) (p, q) and $(p - 1, q + 1)$ for $p - q > 2$, (2) $(q + 2, q)$ and $(q + 1, q + 1)$ for $p - q = 2$, (3) $(q + 1, q)$ and $(q, q + 1)$ for $p - q = 1$ and (4) (q, q) and $(q - 1, q + 1)$ for $p - q = 0$. For sub-case (1), each state transits into one new state with the same probability. For sub-case (2), $(q + 2, q)$ transits into $(q + 2, q + 1)$, and $(q+1, q+1)$ into $(q+2, q+1)$ and $(q+1, q+2)$. For sub-case (3), both states transit into the same new state with probability 1. For sub-case (4), (q, q) transits into $(q, q + 1)$ and $(q + 1, q)$, and $(q - 1, q + 1)$ into $(q, q + 1)$. We conclude that there are at most two different states after round $j + 1$ in each sub-case.

Let s_j^1 and s_j^2 be the states after round j. Further, let $p(s_{j+1}^l | s_j^k)$ be the transition probability from s_j^k to s_{j+1}^l for $l, k \in \{1, 2\}$ and agent a_1. This is essentially the conditional probability that agent a_1 is feasible in s_j^k. Its value is equal to (1) 0 if agent a_1 is not feasible in s_j^k, (2) 1 if agent a_1 is only feasible in s_j^k and (3) $\frac{1}{2}$ if agent a_1 is feasible in s_j^k together with agent a_2. Hence, agent a_1 receives item o_{j+1} with expected probability $\overline{p}(1, (j+1)) = \sum_{l=1}^{2} \sum_{k=1}^{2} \overline{p}(s_{j+1}^l) \cdot p(s_{j+1}^l | s_j^k)$. Similarly, we can compute this probability if there is a single state after round j. $\square$

We are also aware of other tractable cases for the BALANCED LIKE mechanism and the case when the ordering is fixed. For example, the case of identical preferences is tractable. As another example, the case when each agent likes at most two items and each item is liked by at most two agents is also tractable. For the case when the ordering is uncertain, computing outcomes becomes hard with 2 agents even when we use the LIKE mechanism (Aleksandrov and Walsh, 2017a).

4.5 Expected manipulations

In this section, we consider how agents can act strategically and *possibly*, *necessarily* or *exactly* manipulate the outcomes of mechanisms. We suppose competing and self-interested agents. We also suppose rational agents. For this reason, we believe that a risk-loving agent a_i could try to report strategically a bid vector V_i and thus increase their truthful expected utility $\overline{u}(i, U_i)$ supposing other agents bid sincerely. We next present the necessary and exact manipulation problems. These problems are verification problems and ask whether a given agent increases their sincere outcome supposing they bid according to a given strategy while the strategies of all other agents are fixed.

EXACTMANIPULATION	NECESSARYMANIPULATION
Input: $\mathcal{I}$, a_i, U_i, V_i.	Input: $\mathcal{I}$, a_i, U_i, V_i, $k \in \mathbb{Q}$.
Output: $\overline{u}(i, V_i) - \overline{u}(i, U_i)$.	Question: $\overline{u}(i, V_i) - \overline{u}(i, U_i) \geq k$?

The ONLINE SERIAL DICTATORSHIP, ONLINE RANDOM PRIORITY and LIKE are strategy-proof and hence agents have an incentive to bid sincerely for items. The BALANCED LIKE mechanism is online strategy-proof and so agents cannot manipulate the outcome with no information about future items. Moreover, in this setting, the manipulations of the MAXIMUM LIKE and PARETO LIKE mechanisms coincide with the manipulations of these mechanisms with complete information. We therefore just look into necessary and exact manipulations in the setting with complete information. For example, the BALANCED LIKE mechanism is not strategy-proof and agents can have an incentive to bid strategically for items; see Theorem 2 from (Aleksandrov, Aziz, Gaspers, and Walsh, 2015). We also know that the MAXIMUM LIKE and PARETO LIKE mechanisms are not strategy-proof. We thus focus on strategic misreporting of bids with these three mechanisms. In particular, we study the worst case when the utilities and the ordering of the items are known to the misreporting agent. Our hardness results therefore provide lower bounds on the complexity of these manipulation problems in the case of partial or probabilistic information. These results also highlight the relations between the possible, necessary and exact outcomes and the possible, necessary and exact manipulations of these mechanisms. For the MAXIMUM LIKE mechanism, tractability of its outcomes implies tractability of its manipulations. For the PARETO LIKE mechanisms, intractability of its outcomes implies intractability of its manipulations. We just therefore report our results for the BALANCED LIKE mechanism.

Theorem 7 *Even with 0/1 utilities and the* BALANCED LIKE *mechanism, problem* EXACTMANIPULATION *is in* $\#$P*-hard under arithmetic reductions.*

Proof. Let us consider instance $\mathcal{E}_G$ from Reduction 1. Let us add one new item z between items x and y and let only agent $e_{3 \cdot N+1}$ like item z. Suppose now that all agents bid sincerely. In this setting, agent $e_{3 \cdot N+1}$ receives each of the items x and z with probability 1 because they are the only agent who likes them. As a result, they have 2 items in each possible allocation and some other agent has 1 item before item y is revealed. This means that agent $e_{3 \cdot N+1}$ is never feasible for item y and receives it with probability 0. Therefore, their expected utility is equal to 2. Suppose now that all agents bid sincerely except agent $e_{3 \cdot N+1}$ who only bids strategically for item z and they bid 0 for this item. Hence, by Theorem 5, we have that their expected utility is equal to $1 + \overline{p}((3 \cdot N+1), x)$. The problem instance of EXACTMANIPULATION uses as input the instance $\mathcal{E}_G$ with item z, agent $e_{3 \cdot N+1}$ and the sincere and strategic bidding vectors of agent $e_{3 \cdot N+1}$. The hardness of this problem therefore follows by Theorem 5. □

Observe that the truthful report of the manipulator in the proof of Theorem 7 leads to their utility being 2 whereas their insincere report leads to their utility being at most 2. Hence, their strategic move cannot lead to an increase in their utility but the computation of the exact difference in utility is still intractable in this setting. However, as we discuss next, computing an exact profitable insincere report that leads to such an increase is also intractable.

Necessary manipulations might be easy even when exact manipulations are hard. For example, in the proof of Theorem 7, suppose that the manipulator has cardinal utility for item y that is strictly greater than $(3^N).(3N+1)$. If they bid sincerely, their expected utility is 2. If they bid strategically zero for item z, their expected utility is strictly greater than 2. This *necessary* increase can be decided in polynomial time but computing the *exact* profitable increase is intractable. However, as we confirm next, necessary manipulations are also in general not always easy even if we ask merely for any increase in the expected utility of a given agent.

Theorem 8 *Even with 0/1 utilities and the* BALANCED LIKE *mechanism, problem* NECESSARYMANIPULATION *is in* coNP*-hard under Turing reductions.*

Proof. Let us consider instance $\mathcal{P}_{G,r}$ from Reduction 2. Suppose now that all agents bid sincerely. Hence, the expected utility of agent c is equal to $\overline{p}(c, z_{N-r}) + \overline{p}(c, w)$. Suppose next that all agents bid sincerely except agent c who only bids strategically 0 for item w. We obtain that their expected utility is now equal to $\overline{p}(c, z_{N-r})$. The problem instance of NECESSARYMANIPULATION uses as input the instance $\mathcal{P}_{G,r}$, agent c, their sincere and strategic bidding vectors and the rational number 0. Consequently, the difference in their expected utilities supposing they bid strategically and sincerely is non-negative iff $\overline{p}(c, w) = 0$. We close the result by Theorem 6. □

Another definition of the manipulation problem is whether a player can *possibly* increase their expected utility by insincere reporting, rather than computing the necessary or exact gain. Observe that in the proof of Theorem 8, we have that the difference in the expected utilities of the manipulator supposing they bid sincerely and strategically is positive iff their probability for item w is positive. We conclude that possible manipulations are also intractable in general by the argument we used right before Theorem 6 and the proof of Theorem 6.

Finally, by Proposition 11, we conclude that unilateral manipulations of the BALANCED LIKE mechanism are tractable with just two agents and items arriving from a fixed ordering. However, necessary and exact manipulations become computationally hard even with two agents when the arrival order of items is uncertain (Aleksandrov and Walsh, 2017a).

4.6 Conclusions

We studied the computational complexity of possible, necessary and exact outcomes returned by six mechanisms for online fair divison: ONLINE SERIAL DICTATORSHIP, ONLINE RANDOM PRIORITY, MAXIMUM LIKE, PARETO LIKE, LIKE and BALANCED LIKE. The outcomes of the ONLINE SERIAL DICTATORSHIP, ONLINE RANDOM PRIORITY, MAXIMUM LIKE and LIKE mechanisms are tractable and the ones of the PARETO LIKE and BALANCED LIKE mechanism are intractable. With the ONLINE SERIAL DICTATORSHIP, ONLINE RANDOM PRIORITY or LIKE mechanisms, there is no benefit for agents to act strategically. With the MAXIMUM LIKE, PARETO LIKE or BALANCED LIKE mechanism, the agents might be strategic but we proved that computing a manipulation is computationally tractable for the MAXIMUM LIKE mechanism and intractable for the PARETO LIKE and BALANCED LIKE mechanisms in general. We are also aware that our results are worst-case and do not necessary imply that agents could not manipulate the outcome of these mechanisms in any reasonable or even practical way. As we pointed out, there are some tractable cases. One such case is the setting with 2 agents and fixed ordering. Another case is the setting with identical preferences. Finally, an interesting future direction would be to look at some different parameters of these problems that allow us to speed up these computations (Downey and Fellows, 2013).

Chapter 5

Pure Nash Equilibria

In the previous chapter, we looked at questions about manipulations. In particular, we studied how a single agent could possible, necessary and exactly manipulate the outcome of online mechanisms. But, if a single agent can do that, then why not the other agents as well? This brings us to our next topic, namely the *pure Nash equilibria* of mechanisms. Fair division of indivisible items without monetary transfers is a particularly difficult case because items cannot be divided to satisfy multiple agents and because we cannot use money to compensate agents receiving less utility. Competing agents are then likely to behave strategically, especially if they have some information about what items remain to be allocated. Most of the fair division mechanisms developed in the literature are responsive to the preferences of the agents. This is the minimum requirement for Pareto efficiency. However, at the same time, this immediately implies that agents may try to take advantage of mechanisms and try to bias the allocation in their favor. It is therefore perhaps not surprising that many models and mechanisms in fair division have game-theoretic underpinnings. For this reason, many research works borrow concepts from game theory. One such concept is *Nash equilibrium* (Nash, 1950). Nash equilibria are often used to predict the behavior of competing agents even in sport tournaments (Chiappori, Levitt, and Groseclose, 2002). This is particularly difficult in randomized or repeated settings (as in online fair division) because agents may have multiple strategies to adopt (Neyman and Okada, 2000; Shapley, 1953).

Nash equilibrium is a game-theoretic concept commonly used to understand the behavior of competing agents. If there is a unique pure Nash equilibrium, the main focus is usually around computational questions. Computing equilibria is more relevant for agent control when agents behave strategically. For example, one may be interested in what is the equilibrium strategy of a given agent, or whether a given pure strategy is played in any equilibrium (Gilboa and Zemel, 1989). If there are multiple pure Nash equilibria, the focus is then normally shifted to questions about selecting equilibria with some desirable properties such as envy-freeness or Pareto efficiency. Selecting equilibria is more relevant for chair control when they want to enforce say a Pareto efficient or an envy-free equilibrium once the items start to arrive. For example, at Harvard Business School, students submit their sincere preferences for courses and a “proxy draft" mechanism allocates courses to them by using a Pareto efficient equilibrium strategy that is computed based on the submitted preferences (Budish and Cantillon, 2012). As a second example, one may argue that it is impossible to have a good overview of all the Nash equilibria of a game if one cannot even count equilibria efficiently. This, however, does not preclude us from making predictions about equilibria. Counting complexity is thus a stepping stone for us in order to better understand selection complexity. If counting equilibria is intractable in general, then enumerating equilibria is intractable. If enumerating equilibria is intractable in general, then selecting equilibria is intractable.

We study Nash equilibria in both online settings with complete and no information. With complete information, all agents bid for all items simultaneously. From the perspective of the chair, the agents play a one-shot game. It is not always practical to query agents repeatedly as every item arrives so we often collect their preferences in advance. For example, hundreds of items arrive in the food bank problem and, for this reason, charities request food only a few times per day when they reveal their utilities. In addition, we might ask agents to play the one-shot game by submitting their utilities in advance. In this way, the chair may reduce their strategic options as there are fewer equilibria in the "one-shot" game than in the repeated game in general. Nevertheless, the problem remains online because the items arrive over time, they need to be allocated promptly and as we do not know in advance what items actually will be donated. With no information, at a given round, the past is fixed and all agents bid simultaneously only for the current item as they do not know if any more items will arrive. From the perspective of the agents, they play a repeated game. On the one hand, ONLINE SERIAL DICTATORSHIP, ONLINE RANDOM PRIORITY, BALANCED LIKE and LIKE are online strategy-proof and agents have an incentive to report their sincere utilities for items. On the other hand, MAXIMUM LIKE and PARETO LIKE are not online strategy-proof and agents have an incentive to overreport these utilities. However, the sub-game perfect Nash equilibria of these two mechanisms are not so interesting when the agents do not have information about future items. Nevertheless, they are related to one-shot equilibria. Each pure Nash equilibrium is also a sub-game perfect Nash equilibrium but the opposite direction does not hold.

In particular, in both online settings with complete and no information, we study pure Nash equilibria of our manipulable online mechanisms: MAXIMUM LIKE, PARETO LIKE and BALANCED LIKE. For example, we show that both the MAXIMUM LIKE and PARETO LIKE mechanisms admit a unique pure Nash equilibrium whereas the BALANCED LIKE mechanism could admit exponentially many different pure Nash equilibria. As another example, we also show that the pure Nash equilibrium of the MAXIMUM LIKE and PARETO LIKE mechanisms can be computed in polynomial-time as well as the best responses of agents for items whereas even best response computations are intractable for the BALANCED LIKE mechanism. We also count the equilibria of the BALANCED LIKE mechanism. This is again intractable to achieve and indicates that the chair might have to select a pure Nash equilibrium with some desirable properties among exponentially many such equilibria. Our intractability results have many interesting consequences. For example, we show that to decide if there exist a pure Nash equilibrium of the BALANCED LIKE mechanism with egalitarian welfare (i.e. minimum expected utility of an agent) of at least k is intractable. As a second example, we show that counting envy-free pure Nash equilibria is also intractable.

We discuss some related work in Section 5.1 and give some preliminaries in Section 5.2. For example, the ONLINE SERIAL DICTATORSHIP, ONLINE RANDOM PRIORITY and LIKE mechanisms admit an unique equilibrium which happens to be when the agents are sincere. In Section 5.3, we show that the sincere profile might not be an equilibrium with the MAXIMUM LIKE and PARETO LIKE mechanisms. Surprisingly, these mechanisms also admit an unique pure Nash equilibrium. By comparison, there could be an unique equilibrium or multiple equilibria with the BALANCED LIKE mechanism. We also discuss Pareto efficient and envy-free pure Nash equilibria in Sections 5.4 and 5.5. We then study the complexity of computing equilibria in Section 5.6 and selecting equilibria in Section 5.7 of these mechanisms. It is tractable to compute the qeuilibria of the MAXIMUM LIKE and PARETO LIKE mechanisms whereas it is intractable to compute equilibria with the BALANCED LIKE mechanism. It is also computationally hard to count equilibria with this mechanism. We conclude in Section 5.8.

5.1 Related work

We studied previously how a single agent could possibly, necessarily or exactly manipulate the outcomes of online mechanisms (Aleksandrov and Walsh, 2017a). Manipulations by multiple agents are also investigsted in the (offline) fair division literature. For example, Aziz et al. study the complexity of manipulating the popular probabilistic serial rule by multiple agents, the so called *equilibria* manipulations (Aziz, Gaspers, Mackenzie, Mattei, Narodytska, and Walsh, 2015). Another example of manipulations in fair division can be found in (Aziz, Bouveret, Lang, and Mackenzie, 2017). The authors extend their past work on single-agent manipulations of the popular sequenatial allocation mechanism to multiple-agent manipulations.

It is very often the case that predictions about the behavior of agents is a challenging computational task. In response, Sandholm et al. consider using mixed-integer programs for such computations (Sandholm, Gilpin, and Conitzer, 2005). Their programs significantly outperform famous and previously known algorithms such as Lemke-Howson on real-world data and Porter-Nudelman-Shoham on games whose equilibria has medium-sized support. Some of their mixed-integer formulations yield provable performance for computing almost exact equilibria. Another example of fast equilibrium computations is provided by the work in (Porter, Nudelman, and Shoham, 2008). The authors present two simple search methods for computing a sample Nash equilibrium in a normal-form game: one for 2-player games and one for n-player games. These simple methods outperform the state-of-the-art Lemke-Howson algorithm for 2-player games, and Simplicial Subdivision and Govindan-Wilson for n-player games.

In our work, we focus on pure Nash equilibria of online mechanisms such as ONLINE SERIAL DICTATORSHIP, MAXIMUM LIKE and BALANCED LIKE under the assumption of complete information. There are several reasons why we look at pure equilibria. First, pure Nash equilibria always exist in our repeated setting. For example, Tomala considers a re-peated game with public observation which is characterized in terms of a one shot-game. Complete information in their setting allows them to study of the effect of undetectable deviations which allows them to define new types of punishments using approachability techniques (Tomala, 1998). Second, our results are just a stepping stone towards better understanding behavior of agents in more general settings with partial information. Even with complete information, we show that computing equilibria with the BALANCED LIKE mechanism is intractable. It is therefore harder to compute equilibria of this mechanism in settings with partial information. This might be good news because this mechanism bounds the envy ex post one agent has for another but it is not strategy-proof. Our results simply suggest that agents might decide to act sincerely.

We were also inspired by the weak and strict Nash equilibria of the vector-payoff games presented in (Shapley and Rigby, 1959). Indeed, as we show, there could be exponentially many weakly beneficial strategies which motivates why we look also at counting, enumerating and selecting problems about pure Nash equilibria. We show however that the MAXIMUM LIKE and PARETO LIKE mechanisms admit a unique equilibrium. For this reason, we study the complexity of counting equilibria only with the BALANCED LIKEṪhe presentation of our counting results was inspired by the hardness results for two agents presented in (Conitzer and Sandholm, 2008). Unfortunately, we could not inherit any of these results because manipulations for two agents are tractable for BALANCED LIKE. Our decision problems are similar to other such problems in stochastic game-theory (Ummels and Wojtczak, 2009).

5.2 Preliminaries

Let us recall that V denotes the matrix of bids of agents for items (i.e. *bidding profile*). For a given mechanism, we say that V is *envy-free ex ante (ex post)* if the expected allocation returned by the mechanism on V is envy-free ex ante (ex post). Similarly, we define *Pareto efficient ex post* and *ante* bidding profiles. Also, recall that V_{-i} denotes the bid sub-matrix of V without the vector $V_i = (v_{i1}, \ldots, v_{im})$ of bids of agent a_i for items o_1 to o_m, and $\overline{u}(i, V_i)$ denotes the ex ante utility of agent a_i for all items supposing the bids of other agents are fixed as in V_{-i}. We say that V_i^1 is a *strict best response* vector of agent a_i to V_{-i} if $\overline{u}(i, V_i^1) > \overline{u}(i, V_i^2)$ for each vector $V_i^2 \neq V_i^1$. We say that V_i^1 is a *weak best response* vector of agent a_i to V_{-i} if $\overline{u}(i, V_i^1) = \overline{u}(i, V_i^2)$ for some vector $V_i^2 \neq V_i^1$ and $\overline{u}(i, V_i^1, V_{-i}) \geq \overline{u}(i, V_i^3, V_{-i})$ for each vector $V_i^3 \neq V_i^1$. We next define pure Nash equilibria. We say that a bidding profile V is a *strict pure Nash equilibrium* if all bidding vectors in V are strict pure best responses, and V is a *weak pure Nash equilibrium* if some vectors in V are weak best responses and the other vectors in V are strict best responses.

We can similarly define *online pure Nash equilibria* that correspond to the repeated online games. However, at each round j, the past bids v_{i1} to v_{ij-1} of each agent a_i are now fixed and they can only misreport their bid v_{ij} for the current item. We omit however these formal definitions as most of our mechanisms are online strategy-proof and therefore their online dominant strategy is to bid sincerely. Such mechanisms are the ONLINE SERIAL DICTATORSHIP, ONLINE RANDOM PRIORITY, LIKE and BALANCED LIKE mechanisms. Interestingly, the MAXIMUM LIKE and PARETO LIKE mechanisms are not online strategy-proof in general. Moreover, for mechanisms such as LIKE, MAXIMUM LIKE, ONLINE SERIAL DICTATORSHIP and ONLINE RANDOM PRIORITY, the set of online pure Nash equilibria of the repeated game and the set of pure Nash equilibria of the "one-shot" game coincide because these mechanisms are memoryless. The online pure Nash equilium of the PARETO LIKE mechanism is the same as its "one-shot" equilibrium (i.e. every agent reports the maximum possible bid for an item if they like it or zero if they do not like it). We make the following simple observation.

Observation 6 *With general utilities and the* ONLINE SERIAL DICTATORSHIP, ONLINE RANDOM PRIORITY, MAXIMUM LIKE, PARETO LIKE, LIKE *or* BALANCED LIKE *mechanism, no agent a_i cannot strictly increase their expected utility if they bid $v_{ij_1} > 0, \ldots, v_{ij_k} > 0$ for some items $o_{j_1}, \ldots, o_{j_k}$ supposing $u_{ij_1} = 0, \ldots, u_{ij_k} = 0$ and the bids of other agents are fixed.*

By Observation 6, we only consider *monotone* pure Nash equilibria in which each agent bids positively or zero for each item they like, and bids zero for each item they dislike.

Observation 7 *With general utilities, the sincere profile is the unique pure Nash equilibrium of the* ONLINE SERIAL DICTATORSHIP, ONLINE RANDOM PRIORITY *and* LIKE *mechanisms.*

Observation 8 *With general utilities, the sincere profile is the unique online pure Nash equilibrium of the* BALANCED LIKE *mechanisms.*

In our work, we look at both agent and chair control simply because there could be an unique equilibrium or there could be multiple equilibria. In particular, we are interested in computing and counting complexity of monotone pure Nash equilibria (i.e. PURENASHEQUILIBRIUM or simply PNE and #PURENASHEQUILIBRIUM or simply #PNE). However, we also report results whether monotone pure Nash equilibria are always unique of exist (i.e. UNIQUEPNE and EXISTPNE). Interestingly, the MAXIMUM LIKE and PARETO LIKE mechanisms have unique equilibria that can be computed in polynomial time. This is somehow in contrast with the fact that manipulating the PARETO LIKE mechanism by a single agent is intractable as we argued in the previous chapter. For the BALANCED LIKE mechanism, both the computing and counting pure Nash equilibria are related to the matching graph problems we introduced in the previous chapter. For all mechanisms except for the BALANCED LIKE mechanism, it is not difficult to see that there is always a pure Nash equilibria. For the BALANCED LIKE mechanism, we conjecture that there is also always an unique pure Nash equilibrium.

Conjecture 1 *With general utilities, there is always a pure Nash equilibrium of the* BALANCED LIKE *mechanism.*

We show our hardness results using the same reductions and classes of problems from computational complexity from the previous chapter. Recall we use mappings such as *Karp*, *Turing*, *parsimonious* and *arithmetic* as well as complexity classes of decision and counting problems such as P, NP, coNP and #P (Bürgisser, 2000; Turing, 1936; Valiant, 1979a). We recall the graph problems that we introduced in the previous chapter because we use the same problems for our hardness results in this chapter. The #PERFECT MATCHINGS problem is in #P-hard on various bipartite graphs (Okamoto, Uehara, and Uno, 2009; Valiant, 1979b). The MINIMUM MAXIMAL MATCHING is in NP-hard on various bipartite graphs as well (Demange and Ekim, 2008; Sabán and Sethuraman, 2013b). We summarize our results in Table 5.1.

mechanism	PNE	#PNE	EXISTPNE	UNIQUEPNE
ON. SERIAL DICT.	–	–	✓	✓
ON. RAND. PRIO.	–	–	✓	✓
PARETO LIKE	P	P	✓	✓
MAXIMUM LIKE	P	P	✓	✓
LIKE	–	–	✓	✓
BALANCED LIKE	NP-hard	#P-hard	open	×

TABLE 5.1: Pure Nash Equilibria.

5.3 Unique and multiple pure nash equilibria

The MAXIMUM LIKE and PARETO LIKE mechanisms are not strategy-proof and therefore the sincere profile might not be a pure Nash equilibrium even when agents have identical utilities for items. The same holds for the BALANCED LIKE mechanism even with 0/1 utilities when both of these mechanisms are strategy-proof. We show some of these observations in the following simple example.

Example 10 *Consider the following instance and the* MAXIMUM LIKE *and* PARETO LIKE *mechanisms.*

o	o_1	o_2
a_1	2	2
a_2	1	1

Let us first run the MAXIMUM LIKE *mechanism and suppose agents bid sincerely. Agent a_1 recieves both items and agent a_2 no item. However, agent a_2 can increase their expected utility from 0 to 1 if they bid 2 for both items, or even from 0 to 2 if they bid more than 2 for each item. The mechanism is not strategy-proof and even online strategy-proof because agent a_2 can increase their expected utility by overbidding their sincere utility for just one of the items.*

Let us next run the PARETO LIKE *mechanism. Suppose agents bid sincerely. Each agent receives each item with the same probability of $\frac{1}{2}$ because the utilities are identical. Hence, agent a_2 receives expected utility of 1. Suppose next that agent a_2 bids strategically 3 for item o_2. The first item is again given to each agent with probability $\frac{1}{2}$. However, now item o_2 is given with probability $\frac{1}{2}$ to each agent only if item o_1 is allocated to agent a_1. Otherwise, it is given to agent a_2 with probability 1. Hence, their expected utility is $\frac{5}{4}$ which is strictly greater. The mechanism is not strategy-proof.* □

Interestingly, the MAXIMUM LIKE mechanism admits an unique pure Nash equilibrium. With the PARETO LIKE mechanism, there could be multiple pure Nash equilibria but all of them are identical with respect to the outcomes of the agents. In other words, the mechanism returns the same distribution of expected outcomes on each equilibrium profile. We therefore conclude that there is an unique pure Nash equilibrium of the PARETO LIKE mechanism modulo payoff equivalence. We say that two bidding profiles are *payoff equivalent* for a given mechanism if each agent receives the same expected utility in each of these profiles by running the mechanism on them. Surprisingly, one of these pure Nash equilibria of the PARETO LIKE mechanism is the unique Nash equilibrium of the MAXIMUM LIKE mechanism in which all agents bid the same positive value for items they like and 0 for items they dislike. Note that this pure Nash equilibrium is strict and therefore there is no weak pure Nash equilibria of the MAXIMUM LIKE and PARETO LIKE mechanisms. We next formalize this result.

Proposition 12 *With general utilities, there is an unique pure Nash equilibrium of the* MAXIMUM LIKE *and* PARETO LIKE *mechanisms.*

Proof. We supposed that there is some maximum utility value the agents can report. Let this be $\max$. With the MAXIMUM LIKE mechanisms, each agent a_i that does not receive item o_j has an incentive to report $\max$. But, now the same argument is valid for each other agents supposing that these are rational agents. Hence, the best response dynamics converge to the profile in which each agent reports $\max$ for each item they like and 0 for each item they dislike. With the PARETO LIKE mechanism, let us consider an allocation $\pi(j)$ of items o_1 to o_j. The key point is that only agents that are not feasible for items can manipulate the outcome. There are two cases. In the first one, let us suppose that there is an agent, say agent a_i, who is not feasible for item o_{j+1}. This agent could simulate high Pareto efficiency of allocating this item to them by reporting $\max$. By doing so, they could only possibly increase their outcome from 0 to $(1/f_{j+1}) \cdot u_{i(j+1)}$ given the number of feasible agents in $\pi(j)$ for item o_{j+1} is f_{j+1}. Each other agent that is feasible for item o_{j+1} in $\pi(j)$ remains feasible after agent a_i bids $\max$ for this item. They do not change the current outcome by reporting $\max$ because the value of $\max$ is at least as much as their utility for item o_{j+1}. In the second case, each agent that likes item o_{j+1} is feasible for it given $\pi(j)$. Hence, they can all bid $\max$ and remain feasible. □

In contrast, there could be weak or strict, unique or multiple pure Nash equilibria with the BALANCED LIKE mechanism. We demonstrate these results on a number of examples.

Example 11 (**Unique weak pure Nash equilibrium**) *Consider the next instance and the* BALANCED LIKE *mechanism.*

o	o_1	o_2
a_1	1	2
a_2	2	1

The profile in which the agents report their sincere utilities is not a pure Nash equilibrium. If both agents bid sincerely, each agent receives expected utility of $3/2$*. If agent* a_1 *bids strategically 0 for item* o_1*, each agent receives strictly higher expected utility of 2. This is the only pure Nash equilibrium which happens to be weak because the agents receive the same outcomes if* a_2 *reports 0 for* o_2 *in it.* $\square$

Example 12 (**Unique strict pure Nash equilibrium**) *Consider the following instance which we used in the proof of Theorem 2 from (Aleksandrov, Aziz, Gaspers, and Walsh, 2015) and the* BALANCED LIKE *mechanism.*

o	o_1	o_2	o_3
a_1	1	1	1
a_2	0	1	0
a_3	1	0	1

Supposing sincere play, agent a_1 *receives expected utility of* $\frac{9}{8}$*. Supposing they bid 0 for item* o_1*, agent* a_1 *receives expected utility of* $\frac{10}{8}$*. This is the only pure Nash equilibrium and the sincere profile is not pure Nash equilibrium. This equilibrium happens to be unique and strict.*$\square$

In general, there could be multiple pure Nash equilibria with the BALANCED LIKE mechanism. Some of these could be strict and some others could be weak. There also could be payoff equivalent or not payoff equivalent pure Nash equilibria. In fact, the number of weak pure Nash equilibria with equal outcomes could be exponential in the number of agents. Moreover, there could also be exponentially many strict pure Nash equilibria with unequal outcomes. We next clarify these observations in terms of the following two examples.

Example 13 (**Multiple weak pure Nash equilibria**) *We consider the fair division of items* o_1 *to* o_n *between agents* a_1 *to* a_n*. Suppose that* a_1 *likes each item with utility 1 and each other agent* a_i *likes only* o_i *with utility 1. With* BALANCED LIKE*, the sincere bidding strategy is dominant for each agent. However, agent* a_1 *receives the same expected outcome of* 1 *if they bid 1 for* o_1 *with probability 1 and, for each of the other* $n-1$ *items, they bid 1 with probability* 1 *or bid 0 with probability* 1*. Hence, there are* 2^{n-1} *weak pure Nash equilibria that give the same output as the sincere profile.* $\square$

Example 14 (**Multiple strict pure Nash equilibria**) *We use* n *gadgets, each of 4 agents and 4 items, to construct instance* $\mathcal{A}$ *with* $4n$ *agents and* $4n$ *items that have* 2^n *different strict pure Nash equilibria. The* i*th gadget has agents* a_{i1} *to* a_{i4} *and items* o_{i1} *to* o_{i4}*. Agent* a_{i1} *likes items* o_{i1} *and* o_{i4} *with 1, item* o_{i2} *with* $1+2i\cdot\epsilon$ *and dislikes item* o_{i3}*. Agent* a_{i2} *likes only item* o_{i2} *with 1. Agent* a_{i3} *likes items* o_{i1} *and* o_{i4} *with 1, item* o_{i3} *with* $1+(2i+1)\cdot\epsilon$ *and dislikes item* o_{i2}*. Agent* a_{i4} *likes only item* o_{i3} *with 1. The items are revealed in ordering* $o_i = (o_{i1}o_{i2}o_{i3}o_{i4})$*. Suppose that* $\epsilon < \frac{1}{4\cdot n}$ *and we use* BALANCED LIKE *mechanism. There are two strict pure Nash equilibria of this gadget* *if we consider it as an online instance by itself: (1) agent a_{i1} bids 0 for item o_{i1} and all other agents are sincere, and (2) agent a_{i3} bids 0 for item o_{i1} and all other agents are sincere. These two profiles are different as their expected allocations are different. The instance $\mathcal{A}$ contains n such gadgets: for $i \in [1, n]$, agents a_{i1} to a_{i4} and items o_{i1} to o_{i4}. The entire ordering of $\mathcal{A}$ is $(o_1 \ldots o_n)$. The two strict pure Nash equilibria of each gadget are independent and different than the two strict pure Nash equilibria of each other gadget. Hence, $\mathcal{A}$ admits 2^n different equilibria.* $\square$

5.4 Pareto efficient pure nash equilibria

We are interested in whether strategic play has an impact on Pareto effciency. For example, the deterministic ONLINE SERIAL DICTATORSHIP is Pareto efficient both ex post and ex ante and further it is strategy-proof. Hence, its unique equilibrium is the sincere profile which is Pareto efficient. This is not the case for other strategy-proof mechanisms. Say, we consider the ONLINE RANDOM PRIORITY mechanism. It is not Pareto efficient ex ante and so its unique equilibrium which is the sincere profile. Even worst, the sincere profile of the LIKE mechanism is neither Pareto efficient ex ante nor ex post. What about the manipulable mechanisms such as MAXIMUM LIKE and PARETO LIKE. Their unique pure Nash equilibrium is the profile in which all agents bid positively the maximum possible value for each item they like and zero for each item they dislike. Therefore, the outcome of this equilibrium coincides with the outcome of the LIKE mechanism which is not Pareto efficient as we already discussed. We are now left with the BALANCED LIKE mechanism. We next show that all agents that like a certain item may all bid zero for it in which case the item is discarded by the chair and not allocated by the mechanism. And, there could be plenty such items in a pure Nash equilibrium of this mechanism. This cannot happen with any of our other mechanisms. We demonstrate this in Example 15. Finally, with 0/1 utilities, all pure Nash equilibria of all these mechanisms are Pareto efficient both ex post and ex ante.

Example 15 (**inefficient pure Nash equilibrium**) *Consider the following instance and the* BALANCED LIKE *mechanism.*

o	o_1	o_2	$\ldots$	o_n
a_1	1	0	$\ldots$	u
a_2	0	1	$\ldots$	u
a_2	0	0	$\ldots$	u
a_n	0	0	$\ldots$	1

Basically, each agent a_i for $i \in [1, n-1]$ likes only items o_i with 1 and o_n with u, and agent a_n likes only item o_n with 1. If we suppose agents act sincerely, each agent a_i receives item o_i and therefore has an expected utility of 1. If we suppose that each agent a_i for $i \in [1, n-1]$ bids strategically only for item o_n, their expected utility is equal to $\frac{u}{n}$. We can clearly see that this value is strictly greater than 1 for $u > n$. There are $n-1$ discarded items. $\square$

We conclude that there might be pure Nash equilibria of the BALANCED LIKE mechanism which are very inefficient because items could be discarded. This cannot happen with any of our other mechanisms even though with some of them Pareto efficiency is also compromised.

5.5 Envy-free pure nash equilibria

We are also in whether strategic play has an impact on envy-freeness. For example, both the ONLINE RANDOM PRIORITY and LIKE mechanisms are envy-free ex ante and strategy-proof. In contrast, the ONLINE SERIAL DICTATORSHIP is strategy-proof but not envy-free ex ante simply because it is deterministic. Neither MAXIMUM LIKE nor PARETO LIKE is envy-free ex ante supposing agents act sincerely. Interestingly, by Proposition 12, they admit a unique pure Nash equilibrium and this equilibrium is payoff equivalent to the LIKE mechanism. Consequently, this equilibrium is envy-free ex ante with respect to both the declared and the sincere preferences of agents for items.

Observation 9 *With general utilities, the unique pure Nash equilibrium of the* MAXIMUM LIKE *and* PARETO LIKE *mechanisms is envy-free ex ante.*

This simple result is however not valid for the BALANCED LIKE mechanism. All pure Nash equilibria of this mechanism could be unfair in expectation even with 2 agents.

Example 16 (**unfair pure Nash equilibria**) *Consider the following instance and the* BALANCED LIKE *mechanism..*

o	o_1	o_2
a_1	0	1
a_2	1	$\frac{3}{2}$

The sincere profile is a pure Nash equilibrium in which agent a_1 receives item o_2 and their expected outcome is 1. There is another pure Nash equilibrium in which agent a_2 bids 0 for item o_2 and their expected outcome is again 1. In both of these profiles, agent a_2 envies ex post agent a_1 with $\frac{1}{2}$ because agent a_1 receives their most valued item o_2. This envy is also in ex ante sense.
□

5.6 Computing pure nash equilibria

We know that pure Nash equilibria exist and they could be unique or not. We continue with their computation. For the ONLINE SERIAL DICTATORSHIP, ONLINE RANDOM PRIORITY and LIKE mechanisms, the sincere profile is the unique pure Nash equilibrium. The BALANCED LIKE mechanism is online strategy-proof and therefore the sincere profile is its unique online pure Nash equilibrium. Moreover, online pure Nash equilibria of the MAXIMUM LIKE and PARETO LIKE mechanisms coincide with their pure Nash equilibria. We therefore focus on computing pure Nash equilibria of the MAXIMUM LIKE, PARETO LIKE and BALANCED LIKE mechanisms. Basically, we are interested in the following computational problem.

PURENASHEQUILIBRIUM
Input: instance $\mathcal{A}$ and $\mathcal{M}$
Output: a PNE of $\mathcal{A}$ with $\mathcal{M}$.

Recall that manipulations of the PARETO LIKE mechanism are intractable. Interestingly, by Proposition 12, we immediately conclude that it takes polynomial time to compute the unique pure Nash equilibrium of the PARETO LIKE mechanism. The two results do not conflict. The intractability of unilaterally manipulating the PARETO LIKE mechanism simply informs a given agent about their (possible, necessary and exact) utility gains supposing they play their best response strategy. In contrast, the tractability of computing such a strategy does not give the agent any information about how much additional gain they receive by following the strategy.

Proposition 13 *With general utilities and the* MAXIMUM LIKE *or* PARETO LIKE *mechanism, problem* PURENASHEQUILIBRIUM *is in* P.

We further study the pure Nash equilibria of the BALANCED LIKE mechanism. The fact that all other mechanisms admit a strict pure Nash equilibrium means that the best responses of agents cannot lead to a cycle. Interestingly, the BALANCED LIKE mechanism is strategy-proof on 0/1 instances and just 2 agents; see Theorem 3 from (Aleksandrov, Aziz, Gaspers, and Walsh, 2015). Consequently, the sincere profile is a pure Nash equilibrium in such instances. However, with 2 agents and general utilities, this mechanism is not strategy-proof and the best responses of the agents can form a cycle. We demonstrate one such cycle for the instance in Example 17.

Example 17 (**Best response cycles**) *Consider the pure Nash equilibrium of the* BALANCED LIKE *mechanism on the instance from Example 11.*

o	o_1	o_2
a_1	0	2
a_2	2	1

In this pure Nash equilibrium, agent a_1 bids 0 for item o_1 and receives expected utility of 2 *supposing that agent a_2 bids sincerely. If agent a_2 instead bids positively only for item o_1, then this new profile is no longer a pure Nash equilibrium as agent a_1 could report their sincere utility of 1 for item o_1 in it and thus would increase their expected utility from 2 to $\frac{5}{2}$. In this case, agent a_2 would receive expected outcome of 1. If this happens, agent a_2 can increase their outcome from 1 to $\frac{3}{2}$ by reporting their sincere utility of 1 for item o_2. This new profile is the sincere profile. The best responses converge to the sincere profile. Despite this fact, there is a unique pure Nash equilibrium. However, the presence of a cycle is caused by the fact the this equilibrium is a weak one.* □

We are next interest in the complexity of computing pure Nash equilibria of the BALANCED LIKE mechanism. If we can solve problem PURENASHEQUILIBRIUM in polynomial time, then we can solve its *decision* version in polynomial time. The decision version asks whether there is a pure Nash equilibrium of the given instance with the given mechanism. We show that the decision versions for weak and strict pure Nash equilibria of the BALANCED LIKE mechanism are intractable and therefore their corresponding computational problems are intractable as well. Our proof therefore implies that problem PURENASHEQUILIBRIUM is intractable for the BALANCED LIKE mechanism. We consider competitive pure Nash equilibria. Hence, each pure Nash equilibrium can be reached from the sincere profile via a sequence of finitely many best responses. Each of these best responses could be weak or strict. In our proof, we use an instance in which the dominant strategy of all agents except one is sincerety. The instance admits a strict pure Nash equilibrium if the ex ante probability of this agent for one of the items is positive, and admits exponentially many weak pure Nash equilibria if this ex ante probability is zero. Each of these exponentially many pure Nash equilibria is reachable from the sincere profile via a single weak best response. We therefore basically show that computing a weak (strict) best response of this possibly strategic agent is intractable. Our result further supposes 0/1 utilities and therefore we conclude that computing weak or strict Pareto efficient pure Nash equilibrium is intractable. The same holds for envy-freeness ex ante because each pure Nash equilibrium of the BALANCED LIKE mechanism is envy-free ex ante when agents have simple 0/1 utilities for items. For our hardness result, we give a reduction from the MINIMUM MAXIMAL MATCHING problem to the PURENASHEQUILIBRIUM problem.

Reduction 3 Let G be a 3-regular bipartite graph, $u_1, \ldots, u_N$ be the vertices from one of its partitions and $v_1, \ldots, v_N$ be the vertices from the other one of its partitions. Let v_{i1}, v_{i2}, v_{i3} denote the vertices and $e_{3\cdot(i-1)+1}$, $e_{3\cdot(i-1)+2}$ and $e_{3\cdot(i-1)+3}$ denote the edges in the graph that are incident with vertex u_i with $i \in [1, N]$. We can view each edge e_k as (u_i, v_j) for some $i, j \in [1, N]$. Further, let r be an integer. We next construct $\mathcal{PNE}_{G,r}$ given G and r as follows:

- **Agents:** 1 agent per edge e_k, agents $a_1, \ldots, a_{2\cdot(N-r)}$ and 2 special agents b, c
- **Items:** 1 item for each vertex v_j, items $x_1^k, \ldots, x_N^k, y_1^k, \ldots, y_N^k$ for $k \in \{1, 2\}$, $z_1, \ldots, z_{N-r}$ and 1 special item w
- **Non-zero utilities:**
  - **agent** $e_{3\cdot(i-1)+j}$ with $j \in \{1, 2, 3\}$ has utility 1 for items x_i^k, v_{ij}, y_i^k with $k \in \{1, 2\}$ and 1 for items $z_1, \ldots, z_{N-r}$
  - **agent** a_i has utility 1 for items $x_1^k, \ldots, x_N^k$ with $k \in \{1, 2\}$
  - **agent** b has each utility 1 for item w
  - **agent** c has utility 1 for items z_{N-r}, w
- **Ordering:** $o = (x_1^1 \ldots x_N^1 x_1^2 \ldots x_N^2 v_1 \ldots v_M y_1^1 \ldots y_N^1 y_1^2 \ldots y_N^2 z_1 \ldots z_{N-r} w)$

Recall that there is a maximal matching in G of cardinality at most r iff, with the BALANCED LIKE mechanism, agent b receives item w in $\mathcal{PNE}_{G,r}$ with probability greater than zero. For example, we can use this observation and show that the sincere profile in $\mathcal{PNE}_{G,r}$ is an unique pure Nash equilibrium of the BALANCED LIKE mechanism iff the probability of agent b for item w is greater than zero.

Lemma 2 *With the* BALANCED LIKE *mechanism, the sincere profile of* $\mathcal{PNE}_{G,r}$ *is a pure Nash equilibrium.*

Proof. Suppose the sincere profile is not a pure Nash equilibrium. Consequently, there is an agent that receives strictly higher expected utility by reporting zeros for items they sincerely like supposing the other agent strategies are the sincere ones. We consider four cases. In the first one, suppose this is agent a_1. There is a *symmetry* in the preferences of agents a_1, a_2, a_3, and each other triplet of a's agents. There is another *symmetry* in the preferences of agents a_i, a_j, a_k such that all like item v_h for each h. Based on these symmetries, we obtained that agent a_k is sincere iff agent a_i is sincere, and that agent a_1 is strictly sincere. In the second case, suppose this is agent b. They receive outcome of $p + (1-p) \cdot (1/2)$ if they bid 1 for x_{N-r}, w, $1/2$ if they bid 1 only for w, $p > 0$ if they bid 1 only for x_{N-r}, and 0 if they do not bid for items. Sincerity dominates. In the third case, suppose this is agent c. They receive $p + (1-p) \cdot (1/2)$ if they bid 1 for w, and 0 otherwise. Sincerity dominates. In the fourth case, suppose this is agent d_1. Note that the preferences of agents d_1 to $d_{2\cdot(N-r)}$ are identical. We immediately conclude that agent d_k is sincere iff agent d_i is sincere, and agent d_1 is strictly sincere.□

Lemma 3 *With the* BALANCED LIKE *mechanism, there are* $2^{3\cdot N+1}$ *weak pure Nash equilibria of* $\mathcal{PNE}_{G,r}$ *except the sincere profile iff agent* b *receives item* w *in* $\mathcal{PNE}_{G,r}$ *with probability zero.*

Proof. If agent b receives item w in $\mathcal{PNE}_{G,r}$ with probability 0, then agent c receives this item with probability 1. Hence, agent b receives item z_{N-r} with probability 1. Therefore, the bids of agents e_1 to $e_{3\cdot N}$ and agent b for items z_{N-r} and w could be either 0 or 1 and this will have not effect on their expected outcomes because agent c still receives item y with probability 1 for each combination of these bids. There are $2^{3\cdot N}$ such combinations and 2 combinations for the bid of agent b for item w. Recall that we consider only monotone pure Nash equilibria in which agents can possibly only bid zero for items they like. The profiles that correspond to these combinations of bids of agents for item y are weak pure Nash equilibria in which all agents receive exactly the same expected outcomes as in the sincere profile. In each such profile, each agent strictly decreases their expected utility if they report zeros for any other items beside item y. This follows from the proof of Lemma 2. If there are $2^{6\cdot N+1}$ weak pure Nash equilibrium except the sincere profile, then it must be the case that agent b receives item w with zero probability in the sincere profile because each agent agent strictly decreases their expected utility if they report zeros for any other item beside item z_{N-r}. This follows from the proof of Lemma 2.□

There are many hardness implications under Turing reductions from Lemmas 2 and 3 for the instance $\mathcal{PNE}_{G,r}$. For example, verifying if the sincere profile of this instance is a weak (strict) pure Nash equilibrium would require us to check if the probability of agent b for item w is zero (positive). Many other results follow as well. There is a strict pure Nash equilibrium iff the sincere profile is a strict pure Nash equilibrium. There is a weak pure Nash equilibrium iff the sincere profile is not a strict pure Nash equilibrium. Hence, there is a weak pure Nash equilibrium iff there is not a strict pure Nash equilibrium. Further, there is a pure Nash equilibrium iff there is a weak or a strict pure Nash equilibrium iff agent b receives item y with zero or positive probability. Also, the sincere profile is not unique pure Nash equilibrium iff the sincere profile is a weak pure Nash equilibrium. There is a pure Nash equilibrium with egalitarian welfare at least 1 (achieved by agent c for item y) iff the sincere profile is a weak pure Nash equilibrium. Even more, there is a (weak) strict pure Nash equilibrium in which agent b bids 1 for item y iff the sincere profile is a (weak) strict pure Nash equilibrium. These problems have indirect implications to the problem of finding a Nash equilibrium in general (Dickhaut and Kaplan, 1993; Porter, Nudelman, and Shoham, 2008; Sandholm, Gilpin, and Conitzer, 2005). We finally conclude with our result.

Theorem 9 *With 0/1 utilities and the* BALANCED LIKE *mechanism, problem* PURENASHEQUILIBRIUM *is* NP-*hard.*

In conclusion, we also consider group Nash equilibria in (Aleksandrov and Walsh, 2017c). In a group pure Nash equilibrium, multiple agents cooperate in order to achieve some common goal. Clearly, our hardness results imply that computing a group Nash equilibrium of the BALANCED LIKE mechanism is intractable when each agent is in a group alone. However, these equilibria become tractable when say all agents are in a single group. Interestingly, none of our mechanisms is not robust to group manipulations. This negative result should not discourage us from further studying the complexity of computing group pure Nash equilibria because *no* other mechanism is group strategy-proof in general. Finally, computing group pure Nash equilibria with our strategy-proof ONLINE SERIAL DICTATORSHIP, ONLINE RANDOM PRIORITY, LIKE mechanisms is tractable in general.

5.7 Counting pure nash equilibria

We focus in here on counting pure Nash equilibria. There is an unique pure Nash equilibrium for the ONLINE SERIAL DICTATORSHIP, ONLINE RANDOM PRIORITY, MAXIMUM LIKE, PARETO LIKE and LIKE mechanisms so selecting, enumerating and counting such profiles becomes irrelevant for these mechanisms. We conclude the same for the unique online pure Nash equilibria of the BALANCED LIKE mechanism. We hence focus on pure Nash equilibria of the BALANCED LIKE mechanism and show that counting such equilibria is intractable even when computing a single pure Nash equilibrium of the one-shot game is tractable. This is surprising as computing pure Nash equilibria of this mechanism is intractable in general as we showed in the previous section. Even if we compute a single pure Nash equilibrium efficiently, we cannot be sure that the agents will play according to it simply because there might be multiple pure Nash equilibria. As a result, the task of counting (or enumerating) pure Nash equilibria becomes more relevant than simply computing such profiles. In particular, we ask questions around counting pure Nash equilibria with some specific property. For example, how many profiles are there whose outcome is equal to the outcome of the *sincere* profile or to the one of some *equilibrium*? Or, how many equilibria are Pareto efficient or envy-free? Further, we aready know that strategic behavior might increase or decrease the welfare value; see Theorem 2 and Theorem 6 from (Aleksandrov, Aziz, Gaspers, and Walsh, 2015). For this reason, we find it natural to ask how many pure Nash equilibria maximize some of the social welfares. Recall that, for a given instance $\mathcal{I}$, the egalitarian welfare $\overline{e}(\mathcal{I})$ is equal to the minimum expected utility of an agent, the utilitarian welfare $\overline{u}(\mathcal{I})$ is equal to the sum of the expected utilities of the agents, and the Nash welfare $\overline{n}(\mathcal{I})$ is equal to the product of the expected utilities of the agents. Basically, we are interested in the next counting problem.

#PURENASHEQUILIBRIUM
Input: instance $\mathcal{A}$ and mechanism $\mathcal{M}$
Output: #PNE of $\mathcal{A}$ with $\mathcal{M}$.

We provide a reduction from the #PERFECT MATCHINGS problem to the #PURENASHEQUILIBRIUM problem. Recall that this problem is #P-hard on 3-regular bipartite graphs (Dagum and Luby, 1992). In fact, we construct a fair division instance that is pretty much similar to the instance we used in the reduction in the previous chapter from the #PERFECT MATCHINGS problem to the EXACTUTILITY problem.

Reduction 4 Let G be a 3-regular bipartite graph, $u_1, \ldots, u_N$ be the vertices from one of its partitions and $v_1, \ldots, v_N$ be the vertices from the other one of its partitions. Let v_{i1}, v_{i2}, v_{i3} denote the vertices in the graph that are incident with vertex u_i with $i \in [1, N]$. Further, let $e_{3\cdot(i-1)+1} = (u_i, v_{i1})$, $e_{3\cdot(i-1)+2} = (u_i, v_{i2})$, $e_{3\cdot(i-1)+3} = (u_i, v_{i3})$ denote the edges in the graph that are incident with vertex u_i with $i \in [1, N]$. Each edge e_k for $k \in [1, 3 \cdot N]$ can be represented as (u_i, v_j) for some $u_i \in \{u_1, \ldots, u_N\}$ and $v_j \in \{v_{i1}, v_{i2}, v_{i3}\}$. We next construct the online fair division instance $\mathcal{I}_G$ from the graph G as follows:

- **Agents:** 1 agent per edge e_k
- **Items:** 1 item per vertex v_j, 2 items u_{i1}, u_{i2} per vertex u_i
- **Non-zero utilities:**
  - **agent** $e_{3\cdot(i-1)+j}$ with $j \in \{1, 2, 3\}$ has utility 1 for items v_{ij}, u_{i1}, u_{i2}
- **Ordering:** $o = (v_1 \ldots v_N u_{11} u_{12} \ldots u_{N1} u_{N2})$

The key concept in our proofs is the one of *perfect* profile. This is a profile corresponds to a perfect matching in the graph G and each such profile has a number of nice properties that we exploit to derive our hardness results. For example, each such profile is envy-free ex post. We next show one such perfect profile for our instance $\mathcal{I}_G$.

Example 18 **(Instance** $\mathcal{I}_G$**)** *Let us consider the 3-regular bipartite graph G with vertices from $U = \{u_1, u_2, u_3, u_4\}$ of degree exactly 3 and $V = \{v_1, v_2, v_3, v_4\}$ of degree exactly 3. Consider the edges: $(u_1, v_1), (u_1, v_2), (u_1, v_4)$, $(u_2, v_2), (u_2, v_3), (u_2, v_4)$, $(u_3, v_1), (u_3, v_3), (u_3, v_4)$, $(u_4, v_1), (u_4, v_2), (u_4, v_3)$. The instance $\mathcal{I}_G$ that corresponds to the graph G has 12 agents e_1 to e_{12}, 12 items v_1 to v_4, u_{i1} and u_{i2} for $i \in [1, 4]$, and ordering $o = (v_1 v_2 v_3 v_4\ u_{11} u_{12}\ u_{21} u_{22}\ u_{31} u_{32}\ u_{41} u_{42})$.*

o	v_1	v_2	v_3	v_4	u_{11}	u_{12}	u_{21}	u_{22}	u_{31}	u_{32}	u_{41}	u_{42}
e_1	**1**	0	0	0	1	1	0	0	0	0	0	0
e_2	0	0	0	0	**1**	1	0	0	0	0	0	0
e_3	0	0	0	0	1	**1**	0	0	0	0	0	0
e_4	0	**1**	0	0	0	0	1	1	0	0	0	0
e_5	0	0	0	0	0	0	**1**	1	0	0	0	0
e_6	0	0	0	0	0	0	1	**1**	0	0	0	0
e_7	0	0	0	0	0	0	0	0	1	**1**	0	0
e_8	0	0	0	0	0	0	0	0	**1**	1	0	0
e_9	0	0	0	**1**	0	0	0	0	1	1	0	0
e_{10}	0	0	0	0	0	0	0	0	0	0	1	**1**
e_{11}	0	0	0	0	0	0	0	0	0	0	**1**	1
e_{12}	0	0	**1**	0	0	0	0	0	0	0	1	1

The profile we presented is a perfect profile for instance $\mathcal{I}_G$. For each v_i with $i \in [1, 4]$, there is exactly one agent from triplet $e_{3\cdot(i-1)+1}, e_{3\cdot(i-1)+2}, e_{3\cdot(i-1)+3}$ whose bid for v_i is marked in "red": agent e_1 for v_1, agent e_4 for v_2, agent e_9 for v_4 and agent e_{12} for v_3. This profile is envy-free ex ante as well as Pareto efficient ex ante. Each agent receives ex ante utility of 1 in such a profile. The allocation marked in "red" is an envy-free ex post allocation as well as Pareto efficient ex post. Each agent receives ex post utility of 1 in such an allocation.□

We are ready to proceed with our results. Basically, we show that counting perfect profiles is #P-hard and all of our other results follow from this result.

Theorem 10 *Even with 0/1 utilities and the* BALANCED LIKE *mechanism, counting pure Nash equilibria is in* #P*-hard.*

Proof. In the sincere profile, it is easy to show that the expected utility of each agent is equal to 1. In fact, the sincere profile in $\mathcal{I}_G$ is a pure Nash equilibrium. Let us next consider the following *perfect* profile for each perfect matching in G. Let us pick such a matching. WLOG, for each $i \in [1, N]$, suppose that the edge (u_i, v_j) is in this matching and v_j happens to be v_{i1}. The perfect profile that corresponds to this matching is then as follows: for $i \in [1, N]$, agent $e_{3\cdot(i-1)+1}$ bids 1 for items v_j, u_{i1}, u_{i2}, and agents $e_{3\cdot(i-1)+2}$ and $e_{3\cdot(i-1)+3}$ bid 1 only for items u_{i1}, u_{i2}. There is 1-to-1 correspondence between the perfect matchings in G and these perfect profiles in $\mathcal{I}_G$. Therefore, counting such profile is intractable. Note that computing one of them in $\mathcal{I}_G$ is tractable as it corresponds to a perfect matching in the grahp which can be computed in polynomial time (Hopcroft and Karp, 1973). Finally, it is easy to show that each perfect profile is a pure Nash equilibrium in which each agent receives expected utility of 1.□

Corollary 5 *With the* BALANCED LIKE *mechanism, counting pure Nash equilibria that maximize the egalitarian, utilitarian or Nash welfare is in* $\#$P*-hard.*

Proof. Let us consider the instance $\mathcal{I}_G$. For $\overline{e}(\mathcal{I}_G)$, each agent in each perfect profile of $\mathcal{I}_G$ receives expected utility of 1. Therefore, the value of $\overline{e}(\mathcal{I}_G)$ in this profile is 1. We argue that this is the maximum value of this welfare. To see this, consider any bidding profile in which some agents receive expected utility more than 1. Hence, there are some other agents that receive expected utility less than 1. This is guaranteed because there are as many items as agents in $\mathcal{I}_G$, the utilities are 0/1, each agent likes at least one item and each item is liked by at least one agent, and also because the mechanism is responsive. As a result, the sum of the expected utilities of the agents is equal to n and hence each perfect profile maximizes $\overline{e}(\mathcal{I}_G)$. For $\overline{u}(\mathcal{I}_G)$, each perfect profile maximizes it because at least one agent bids for each item. Note that the value of this welfare is n in each profile in which at least one agent bids for each item. For $\overline{n}(\mathcal{I}_G)$, we argue that the maximum value of this welfare is 1 and is achieved whenever each agent receives expected outcome of 1. This argument is based on the inequality between the arithmetic and geometric mean. In terms of fair division, this inequality says that n-root of the product of the expected utilities of the agents is at most $\frac{1}{n}$ times the sum of these expected utilities, i.e. $\overline{n}(\mathcal{I}_G) \leq (\frac{\overline{u}(\mathcal{I}_G)}{n})^n$. More importantly, equality is reached only when all expected utilities of agents are equal to 1. Therefore, each perfect profile maximizes the Nash welfare in our instance $\mathcal{I}_G$.□

Corollary 6 *With the* BALANCED LIKE *mechanism, counting pure Nash equilibria whose outcome is sincere is in* $\#$P*-hard.*

Proof. The result follows immediately because each agent in $\mathcal{I}_G$ receives expected outcome of 1 in the sincere profile and also in each perfect profile.□

Corollary 7 *With the* BALANCED LIKE *mechanism, counting bidding profiles whose outcome is equal to the outcome of some equilibrium is in* $\#$P*-hard.*

Proof. Consider the following profile for each perfect profile. For $i \in [1, N]$, agent $e_{3\cdot(i-1)+1}$ bids 1 for item v_j, agent $e_{3\cdot(i-1)+2}$ bids 1 for item u_{i1} and agent $e_{3\cdot(i-1)+3}$ bids 1 for item u_{i2}. We call this profile *deterministic* because a single agent bids for each item. Now, run the mechanism with this profile. Each agent receives (expected) utility of 1 in such a deterministic profile. This outcome is equal to their outcome in the sincere profile and also to their outcome in each perfect profile. However, the deterministic profile is not a pure Nash equilibria. Agent $e_{3\cdot(i-1)+3}$ increases their outcome from 1 to $\frac{3}{2}$ by bidding 1 instead of 0 for item u_{i1} supposing we keep the other bidding strategies fixed in such a deterministic profile.□

Corollary 8 *With the* BALANCED LIKE *mechanism, counting pure Nash equilibria whose outcome is Pareto efficient is in* $\#$P*-hard.*

Proof. Each perfect profile of $\mathcal{I}_G$ is Pareto efficient ex post and ex ante. These profiles are pure Nash equilibria.□

Corollary 9 *With the* BALANCED LIKE *mechanism, counting pure Nash equilibria whose outcome is envy-free is in* $\#$P*-hard.*

Proof. Note that the sincere profile is not envy-free ex post because the mechanism returns allocations in which one agent envies ex post another. Each perfect profile of $\mathcal{I}_G$ is envy-free ex ante because each agent receives expected utility of 1. Each such profile is also envy-free ex post because each agent receives exactly one item in each possible allocation. The perfect profiles of the instance are pure Nash equilibria.□

Counting pure Nash equilibria of the BALANCED LIKE mechanism that are Pareto efficient or envy-free or maximize the welfares is intractable. It immediately follows that counting the more general group pure Nash equilibria of this mechanism is intractable. Interestingly, we recently showed that our LIKE mechanism is not group strategy-proof and that counting its group equilibria is also intractable (Aleksandrov and Walsh, 2017c). Interestingly, it is not difficult to see that our ONLINE SERIAL DICTATORSHIP ONLINE RANDOM PRIORITY mechanisms admit a unique group pure Nash equilibrium (modulo payoff equivalence) so no counting is required. In this equilibrium, for each group and item, the agent from the group who values the item most bids for it.

5.8 Conclusions

We studied pure Nash equilibria of three allocations mechanisms: MAXIMUM LIKE, PARETO LIKE and BALANCED LIKE. For example, there is a unique pure Nash equilibrium of each of the MAXIMUM LIKE and PARETO LIKE mechanisms and these pure Nash equilibria are both payoff equivalent to the outcome of the LIKE mechanism. Computing these equilibria is tractable. The BALANCED LIKE mechanism is exactly on the opposite extreme. It could admit exponentially many different pure Nash equilibria. Also, even best response computations are intractable with this mechanism. Its equilibria as well. We submit that it is interesting whether computing and counting pure Nash equilibria becomes tractable for this mechanism in the case of 2 agents when there could also be exponentially many such profiles. We already observed such a transition in the complexity of computing its possible, necessary and exact outcomes (Aleksandrov and Walsh, 2017a). Even more, in future, we want to explore possible parameterizations and approximations of these problems. Finally, there is a sampling scheme might for our counting problems whose convergence is guaranteed by the Hoeffding's inequality (Sabán and Sethuraman, 2013b).

Chapter 6

Envy-free, Equitable and Pareto Efficient Allocations

In a world of increasing inequality, environmental and economic stresses, resource allocation remains a fundamental and challenging problem. Three central concepts in allocating resources between competing agents are efficiency, equity and fairness. We don't want outcomes which are inefficient, inequal or unfair. This is especially challenging when we take account of real world features like the fact that, in practice, resource allocation has often an online natural. Unfortunately the online nature of such resource allocation problems makes it even more difficult to achieve efficiency, equity and fairness. We must commit *now* to some allocation that we regret *later* when perhaps fewer resources arrive than expected.

Envy-freeness is a widely used fairness criterion. We do not want allocations where one agent envies another. Equitability is another fairness notion. It is very often the case that we want *equitable* allocations in which everybody receives the same utility. Pareto efficiency is widely used economic concept. We do not want allocations that could be improved. We focus in here on our simple and attractive online mechanisms - ONLINE SERIAL DICTATORSHIP, ONLINE RANDOM PRIORITY, MAXIMUM LIKE, PARETO LIKE, LIKE and BALANCED LIKE - and consider several natural questions for these mechanisms. Is an envy-free, equitable or Pareto efficient outcome *possible*? What is the probability of such an outcome? In particular, we consider the computational problems of verifying if the particular outcome returned by a mechanism is envy-free, equitable or Pareto efficient, or if a mechanism can *possibly* return an envy-free, equitable or Pareto efficient outcome. Such questions are interesting for many reasons. For example, the chair might obtain additional resources to allocate if the probability of an envy-free outcome is too low. As a second example, the chair might choose between two mechanisms depending on which is more likely to return a Pareto efficient outcome. As a third example, agents may only be willing to participate if they can possibly have an envy-free outcome. As a fourth example, a regulatory authority might wish to verify that a Pareto optimal allocation was returned, and to force agents to exchange items ex post when this is not the case. As a fifth example, an agent might seek compensation post hoc, or greater priority in later rounds if an envy-free allocation could have been returned but was not. As a sixth example, charities with the same capacities require the same amount of food.

We showed previously that no online mechanism is Pareto efficient and envy-free ex ante with general utilities. Moreover, envy-free ex post allocations might not even exist. Even if they do, we refer to an instance in which all envy-free ex post allocations are Pareto dominated by ex post allocations that are not envy-free. In general, envy-freeness does not imply equitability and the other way around. Also, envy-freeness and equitability might be *disjoint*. That is, there could be instances in which each envy-free allocation is not equitable and instances in which each equitable allocation is not envy-free. Similarly, Pareto efficiency and envy-freeness or equitability might be disjoint. We thus propose to consider these properties in isolation and use computation to identify which of the actual outcomes returned by our online mechanisms post the allocation are envy-free, equitable or Pareto efficient. In this way, we relate a number of existing computational results from (offline) fair division to online fair division. Many of these results are intractable despite the simplicity of the mechanisms considered. We also only discuss some new results for the BALANCED LIKE mechanism. In contrast, for the special and useful case of one item per agent (aka house allocation), we identify a number of tractability results. Our results are under the assumption that agents bid sincerely for items as we previously considered the impact of strategic play on envy-freeness and Pareto efficiency.

This chapter provides only a short overview of our results. We discuss related work in Section 6.1. We then summarize our computational results about envy-freeness, equitability and Pareto efficiency ex post in Sections 6.4, 6.2 and 6.3. Finally, we conclude in Section 6.5.

6.1 Related work

Our work is in-line with many existing works that analyze the computational complexity of computing envy-free and efficient ex post outcomes in offline fair divisions. Our results add to this research by reporting on this complexity for online fair divisions. Further computational complexity studies were performed for the case of incomplete preferences, achieving mostly intractability results. For example, (Bouveret and Lang, 2008; Bouveret, Endriss, and Lang, 2010) consider partial ordinal preferences and show that it is hard to compute (ex post) envy-free and efficient refinements of these preferences. On the other hand, (Aziz, Gaspers, Mackenzie, and Walsh, 2014; Aziz, Gaspers, Mackenzie, and Walsh, 2015) assume complete ordinal preferences and show that it is hard to compute (ex post) envy-free and efficient cardinal utility profiles consistent with these preferences. A further slightly different line of research softens the strict requirement of envy-freeness by investigating how to decrease the degree of envy, in this sense taking an "approximate view" on fairness (Lipton, Markakis, Mossel, and Saberi, 2004; Nguyen and Rothe, 2013). Recently, (Bliem, Bredereck, and Niedermeier, 2016) proved some fixed-parameter tractable results for computing envy-free and efficient outcomes. This is an interesting future direction for our research. Finally, (Keijzer, Bouveret, Klos, and Zhang, 2009) investigates the complexity of computing ex post allocations that are simultaneously envy-free and efficient. We do not present explicit results about computing such allocations, but some results of this type quickly follow from our results.

6.2 Envy-free allocations

We start with envy-freeness ex post of the actual allocations returned by online mechanisms. Interestingly, envy-freeness and Pareto efficiency ex post cannot be satisfied simultaneously even when envy-free ex post allocations exist. In online fair division, there are instances in which all Pareto efficient ex post allocations are not envy-free ex post. This follows from the fact that, in (offline) fair division, there are instances with envy-free allocations which are Pareto dominated only by allocations that are not envy-free (Keijzer, Bouveret, Klos, and Zhang, 2009). The opposite direction holds as well. There are instance in which none of the envy-free allocations is Pareto efficient. This follows immediately from the fact that envy-free ex post allocations may not exist. In fact, such allocations may not exist even with n agents and $m < n$ indivisible items.

Observation 10 *With n agents, $m < n$ items and even 0/1 utilities, there is no envy-free ex post allocation of any of our mechanisms.*

For any number $m \geq n$ of items, verifying envy-freeness ex post (i.e. ENVYFREENESS) can be done in $O(m)$ time. For each agent, we need to go through the allocation and compute the ex post utilities of each agent for each other agent's allocation. We then simply need to compute the envy of each agent for each other agents' allocation, accummulate the total envy and check if it is positive. We are therefore more interested in checking if our mechanisms return an envy-free ex post allocation (i.e. POSSIBLEENVYFREENESS). We relate some existing results from (offline) fair division to online fair division. For example, with $m = n$, computing envy-free ex post allocations is tractable because it is closely related to computing a maximum weight perfect matching in a bipartite graph; see Proposition 22 from (Bouveret and Lang, 2008).

Observation 11 *With n agents, $m = n$ items, general utilities and the* MAXIMUM LIKE*,* PARETO LIKE*,* LIKE *and* BALANCED LIKE *mechanisms, problem* POSSIBLEENVYFREENESS *is in* P*.*

Proof. It is not difficult to see that each of the PARETO LIKE, LIKE and BALANCED LIKE mechanisms returns an envy-free allocation if one such allocation exists. To see this, observe that each agent receives exactly one item in such an allocation, and this item is one of their most valued items. Let us pick one such envy-free ex post allocation and follow it according to the ordering. At each round j, the agent a_{i_j} who receives item o_j in this allocation has 0 items prior to round j and likes o_j. Consequently, the PARETO LIKE, LIKE and BALANCED LIKE mechanisms return this allocation with positive probability because agent a_{i_j} is feasible for o_j for each $j \in [1, m]$. For the MAXIMUM LIKE mechanism, suppose that, for each item, there is a different agent that likes it most. The allocation that gives each item to a different agents that like it most is envy-free ex post and returned by the MAXIMUM LIKE mechanism.□

The POSSIBLEENVYFREENESS problem becomes intractable when $m > n$. For example, deciding if an envy-free ex post allocation exists is in NP-hard even with 2 agents and identical preferences or multiple agents and 0/1 utilities; see Proposition 20 and Proposition 21 from (Bouveret and Lang, 2008). By comparison, in online fair division, the LIKE mechanism returns all possible ex post allocations with the same probabilities for each ordering of the items. Therefore, each related result from (offline) fair division holds for the LIKE mechanism when applied to an (offline) fair division problem with some fixed but arbitrary ordering of the items. Moreover, both the MAXIMUM LIKE and PARETO LIKE mechanisms return the same set of ex post allocations as the LIKE mechanism on instances with 0/1 utilities or 2 agents and identical preferences.

Observation 12 *With 2 agents, $m > 2$ items, general utilities and the* MAXIMUM LIKE*,* PARETO LIKE *and* LIKE *and mechanisms, problem* POSSIBLEENVYFREENESS *is in* NP*-hard.*

Observation 13 *With $n > 2$ agents, $m > n$ items, 0/1 utilities and the* MAXIMUM LIKE*,* PARETO LIKE *and* LIKE *and mechanisms, problem* POSSIBLEENVYFREENESS *is in* NP*-hard.*

Furthermore, the deterministic ONLINE SERIAL DICTATORSHIP mechanism returns a single ex post allocation.

Observation 14 *With $n \geq 2$ agents, $m \geq n$ items, general utilities and the* ONLINE SERIAL DICTATORSHIP *mechanism, problem* POSSIBLEENVYFREENESS *is in* P*.*

Finally, based on these observations, we mainly focus on the missing case when there are more items than agents and we use the BALANCED LIKE mechanism. We only summarize our results. We showed that deciding if this mechanism returns an envy-free ex post allocation is in NP-hard even with 2 agents and general utilities. Our reduction is related to the popular *partition problem* (Garey and Johnson, 1990). If we look at 0/1 instances only, we need more that 2 agents but the decision problem remains intractable. Our reduction is related to the *exact cardinality maximal matching problem* in bipartite graphs (O'Malley, 2007; Cseh and Manlove, 2016).

6.3 Equitable allocations

Equitable ex post allocations might also not exist similarly to envy-free ex post allocations. There is a relation between these two notions when the agents have simple binary utilities for items. With 0/1 utilities, each equitable allocation is envy-free because each agent in such an allocation receives the same number of items. The opposite direction does not hold, e.g. I have three items that I like with 1 but you like only one of them with 1 and you have one item that you and I like with 1. With general utilities, both concepts may not be related. We demonstrate this in Examples 19 and 20.

Example 19 *Consider the following fair division instance.*

o	o_1	o_2
a_1	1	2
a_2	3	1

There is only one allocation $\pi = \{(a_1, o_1), (a_2, o_2)\}$ that is envy-free. However, π is not equitable because the agents receive different utilities in it. □

Example 20 *Consider the following fair division instance.*

o	o_1	o_2
a_1	1	1
a_2	1	2

The allocation $\pi = \{(a_2, o_1), (a_1, o_2)\}$ is equitable ex post because each agent receives utility of 1*. However, agent a_1 assigns ex post utility of* 2 *to the allocation of agent a_1 in π and therefore they envy them.*□

The set of equitable allocations may also be disjoint with the set of Pareto efficient allocations. In Example 19, the equitable allocation is Pareto dominated by the allocation in which agents swap their items, and this allocation is not equitable. Despite these differences, there are also some similarities between equitability and envy-freeness as well. For example, equitable allocations can be envy-free in a special case only. To see this, consider an equitable allocation and suppose that each agent receives the maximum possible additive ex post utility they could. This allocation is envy-free ex post. Interestingly, such an equitable allocation further maximizes the utilitarian, egalitarian and Nash welfares. We are also interested in verifying and computing equitable allocations returned by the LIKE and BALANCED LIKE mechanisms (i.e. EQUITABILITY and POSSIBLEEQUITABILITY). In a similar fashion, we relate some existing results from (offline) fair division to online fair division. For example, with 0/1 utilities, non-existence of an equitable allocation is guaranteed if the number of items is not multiple to the number of agents.

Observation 15 *With n agents, $m \neq k \cdot n$ items and even 0/1 utilities, there is no equitable ex post allocation of any of our mechanisms.*

In the special case of as many items as agents, thse computational problems are tractable for some of our mechanisms.

Observation 16 *With n agents, $m = n$ items, general utilities and the* MAXIMUM LIKE*,* PARETO LIKE*,* LIKE *and* BALANCED LIKE *mechanisms, problem* POSSIBLEEQUITABILITY *is in* P.

Proof. It is not difficult to see that each of the PARETO LIKE, LIKE and BALANCED LIKE mechanisms returns an equitable allocation if one such allocation exists. To see this, observe that each agent receives exactly one item in such an allocation, and each agent values their item with the same utility. Let us pick one such equitable ex post allocation and follow it according to the ordering. At each round j, the agent a_{i_j} who receives item o_j in this allocation has 0 items prior to round j and likes o_j. Consequently, the PARETO LIKE, LIKE and BALANCED LIKE mechanisms return this allocation with positive probability because agent a_{i_j} is feasible for o_j for each $j \in [1, m]$. The argument for the MAXIMUM LIKE mechanism is different. Suppose that there are two items o_j and o_k and a single agent a_i such that agent a_i holds the greatest utilities for both of these items. The MAXIMUM LIKE mechanism then allocates these two items to agent a_i with probability 1 and thus some other agent receives no items due to the constraint $m = n$. Otherwise, suppose that, for each item, there is a different agent that likes it most and with the same value. The allocation that gives the items to agents that like them most is equitable ex post and returned by the MAXIMUM LIKE mechanism with positive probability.□

With 2 agents and general utilities, possible envy-freeness coincides with possible equitability in the context of the *partition problem* (Garey and Johnson, 1990). With 0/1 utilities and $m = k \cdot n$ items, we note similar relation. Possible envy-freeness coincides with possible equitability in terms of the reduction proposed in Proposition 21 from (Bouveret and Lang, 2008).

Observation 17 *With 2 agents, $m > 2$ items, general utilities and the* MAXIMUM LIKE*,* PARETO LIKE *and* LIKE *and mechanisms, problem* POSSIBLEEQUITABILITY *is in* NP*-hard.*

Observation 18 *With $n > 2$ agents, $m > n$ items, 0/1 utilities and the* MAXIMUM LIKE*,* PARETO LIKE *and* LIKE *and mechanisms, problem* POSSIBLEEQUITABILITY *is in* NP*-hard.*

The last observation we make is for the deterministic ONLINE SERIAL DICTATORSHIP mechanism. It relies on the fact that verifying equitability of a given allocation can be done in polynomial time.

Observation 19 *With $n \geq 2$ agents, $m \geq n$ items, general utilities and the* ONLINE SERIAL DICTATORSHIP *mechanism, problem* POSSIBLEEQUITABILITY *is in* P.

Finally, we again only report our results for the BALANCED LIKE mechanism. For example, possible equitability is in NP-hard and our reduction is from the *partition problem* (Garey and Johnson, 1990). With 0/1 utilities, our reduction is from the exact cover by 3 sets problem. An instance of this problem can be found in Proposition 21 from (Bouveret and Lang, 2008). In fact, we can show this result with just $m = 2 \cdot n$ which is the first natural case after $m = n$, whereas the reduction in Proposition 21 is for the case $m = 3 \cdot n$.

6.4 Pareto efficient allocations

We start with Pareto efficient ex post. Most of our mechanisms are guaranteed to return only Pareto efficient ex post allocations: MAXIMUM LIKE, ONLINE SERIAL DICTATORSHIP, ONLINE RANDOM PRIORITY and PARETO LIKE. However, this is not the case for the LIKE and BALANCED LIKE mechanisms. We show this in Example 21.

Example 21 *Consider the following fair division instance and the* LIKE *and* BALANCED LIKE *mechanisms.*

o	o_1	o_2
a_1	1	2
a_2	2	1

There is one allocation $\pi_1 = \{(a_1, o_1), (a_2, o_2)\}$ which is not Pareto efficient ex post because it is dominated by $\pi_2 = \{(a_1, o_2), (a_2, o_1)\}$. The LIKE *mechanism returns both π_1 and π_2 each with probability $\frac{1}{4}$. The* BALANCED LIKE *mechanism returns both π_1 and π_2 each with probability $\frac{1}{2}$.*$\square$

Since Pareto efficiency ex post cannot be guaranteed with the LIKE and BALANCED LIKE mechanisms, we want to discuss computational questions about deciding if an allocation returned by them is Pareto efficient ex post, or if they possibly return Pareto efficient ex post allocations at all. With 0/1 utilities, both of these mechanisms always returns allocations that are Pareto efficient ex post because they are responsive and allocate each item to an agent that likes it. With general utilities, we relate some existing results from (offline) fair division to our online setting of fair division. In (offline) fair division, verifying if a given ex post allocation is Pareto efficient is in coNP-hard with general utilities (i.e. EFFICIENCY) (Keijzer, Bouveret, Klos, and Zhang, 2009). Moreover, a Pareto efficient ex post allocation always exist (i.e. POSSIBLEEFFICIENCY). To see this, we can for example just allocate each item to an agent that most like it. Possible Pareto efficiency ex post is therefore tractable in regular (offline) fair division and most probably, for this reason, has not deserved enough attention in the research literature. In online fair division, our LIKE mechanism returns all possible allocations with the same positive probabilities for each ordering of the items. We therefore directly inherit some results from (offline) fair division to online fair division.

Observation 20 *With 0/1 utilities and the* LIKE *mechanism, problems* EFFICIENCY *and* POSSIBLEEFFICIENCY *are in* P.

Observation 21 *With general utilities and the* LIKE *mechanism, problem* EFFICIENCY *is in* coNP-*hard.*

Observation 22 *With general utilities and the* LIKE *mechanism, problem* POSSIBLEEFFICIENCY *is in* P.

We further obtained similar results to the BALANCED LIKE mechanism. For example, we showed that verifying Pareto efficiency ex post is in coNP-hard for the BALANCED LIKE mechanism even with utilities among 0, 1 or 2. Our reduction is related to a restricted version of the *3-unsatisfiability problem* in which each proposition occurs twice positively and once negatively

(Berman, Karpinski, and Scott, 2007; Schaefer, 1978; Tovey, 1984). More interestingly, checking if this mechanism returns a Pareto efficient ex post allocation is NP-hard under Turing reductions. This result contrasts the existing result in (offline) fair division. Our reduction is related to the *minimum cardinality maximal matching problem* in bipartite graphs (Demange and Ekim, 2008; Sabán and Sethuraman, 2013b). The key intuition behind this transition in the complexity compared to (offline) fair divsion is that the BALANCED LIKE mechanism is biased towards agents with fewest items and therefore tries to return allocations which are more fair than Pareto efficient. Finally, note that both the LIKE and BALANCED LIKE mechanisms return only Pareto efficient ex post allocations with identical utilities.

6.5 Conclusions

We showed that it is impossible to have an online mechanism for online fair division that always returns an envy-free and Pareto efficient ex ante outcome in general. We also highlighted that envy-free and equitable ex post allocations might not exist whereas Pareto efficient ex post allocations always exist. Moreover, even when envy-free and equitable allocations exist, these allocations could be Pareto dominated by allocations which are not envy-free or equitable ex post. Based on these incompatibilities, we decided to consider these three concepts in isolation. We thus studied computational questions about whether outcomes returned by online mechanisms might be envy-free, equitable or Pareto efficient. We summarize our results for the LIKE and BALANCED LIKE mechanisms in Table 6.1.

problem: n agents, m items	LIKE	BALANCED LIKE
EFFICIENCY$(m \geq n)$	coNP-h.	coNP-h.
POSSIBLEEFFICIENCY$(m \geq n)$	P	NP-h.
ENVYFREENESS$(m \geq n)$	P	P
POSSIBLEENVYFREENESS$(m = n)$	P	P
POSSIBLEENVYFREENESS$(m > n)$	NP-h.	NP-h.
EQUITABILITY$(m \geq n)$	P	P
POSSIBLEEQUITABILITY$(m = n)$	P	P
POSSIBLEEQUITABILITY$(m > n)$	NP-h.	NP-h.

TABLE 6.1: Envy-free, Equitable and Pareto Efficient Allocations.

Chapter 7

Competitive Analyses

The online nature requires that online algorithms make their decisions with only limited input. As a result, an online algorithm is less likely to perform as well as an offline algorithm. In this chapter, for the first time, we introduce techniques from competitive analysis *and* advice complexity into online fair division. Competitive analysis is a well-known technique to measure the quality of online versus offline decisions (e.g. the competitive ratio). Advice complexity is a rather novel framework in which online algorithms can consult an oracle in order to improve the quality of their decisions (e.g. the number of advice bits needed to improve online decisions). Online fair division is an important and challenging problem facing society today due to the uncertainty we may have about future resources, e.g. deceased organs to patients, donated food to charities, electric vehicles to charging stations, tenants to houses, even students to courses, etc. We often cannot wait until the end of the year, week or even day before starting to allocate incoming resources. For example, organs cannot be kept too long on ice or products cannot be stored in the warehouse before distributing to a food bank (Walsh, 2014b; Walsh, 2015). We extend past work by asking how many advice bits are needed by online mechanisms in order to increase the welfare.

Advice helps us understand how the competitive ratio depends on the uncertainty about the future. The advice can be based on information about past or future items. For example, consider the allocation of food donations to charities by a central decision maker. A number of contractors usually donate food on a regular basis and at specific times so the decision maker knows when some of the items will arrive. Also, each item could have a *type* that is the set of charities that like the item. The oracle might then keep track on the item types that have arrived in the past and thus bias the allocation of the new item type whenever possible. As another example, consider the allocation of deceased organs to patients. The administrator of a hospital might know what organs will arrive that can be exchanged with a neighboring hospital. They might use this offline information to improve significantly the best local online match for the current organ. Further, the oracle could keep track of how long patients are in the waiting list and thus bias the future organ matching decisions based on this information under various constraints, e.g. a patient should not wait for an organ more than 30 days or normally patients who have arrived earlier to the waiting list should receive organs before patients who have arrived later to this list, etc.

Our work is novel for several reasons. For example, we combine advice complexity and competitive analysis in the context of online fair division. As another example, we study multiple objectives at the same time: the *utilitarian*, *egalitarian* and *Nash* ex ante welfares. We propose three new online mechanisms: ADVICED LIKE, ADVICED BALANCED LIKE and ADVICED MAXIMUM LIKE. Each of these mechanisms makes an allocation decision for each item. An online adaptive adversary specifies each next item only after each of the mechanisms makes their decision for the current item. An online adaptive oracle with unlimited computational power can, for each item, specify on an oracle tape an agent who must receive the item in order to maximize the objective in interest at the end of the allocation. The oracle gives its advice for the next item after the adversary specifies the next item. And, each of these three mechanisms can read the oracle tape at each round. We consider three settings, namely online fair division setting with *complete advice*, *no advice* and *partial advice*. In each of these settings, we analyse the performance of these mechanisms and present a most competitive mechanism for each objective. With *complete* advice, the outcome of these online adviced mechanisms coincide with the outcome of the optimal offline mechanisms. We also report several inpossibility results in this setting. For example, *no* online mechanism can maximize the egalitarian or Nash welfare with less than a complete advice. With *no* advice, the online adviced mechanisms coincide with the popular LIKE, BALANCED LIKE and MAXIMUM LIKE mechanisms proposed earlier. We analyse these mechanisms and show that they are most competitive for our objectives. For example, the MAXIMUM LIKE mechanism is most competitive for the utilitarian welfare. With *partial* advice, we quantify the performance of the online adviced mechanisms with respect to the number of items for which they are adviced. We thus present a *most competitive* online adviced mechanism for each objective and a number of adviced items. Most of our results are under the assumption that agents would act sincerely. For this reason, we also discuss how these results change supposing agents would act strategically. The *price of anarchy* of an online mechanism describes its competitiveness supposing the worst strategic play of agents. We thus compare the prices of anarchy of these mechanisms with their competitive ratios.

Our work has limitations despite its novelty. The advice oracle is *deterministic*. This implies a restriction on the instances for which we use advice oracle. For example, the utilitarian welfare can always be maximized by a deterministic offline optimal mechanism. This is not the case for the egalitarian and Nash welfares. To see this, consider a single item and two or more agents that like it. There is no determinisitc optimal offline mechanism that maximizes these objectives on this instance. The optimal offline mechanism must be then randomized which means that the oracle needs, for each item, to specify on the oracle tape a probability for each agent which is then used by the online mechanism to allocate the item. The oracle must be *randomized* for such instances. Therefore, for the egalitarian and Nash welfares with advice, we consider only instances with general utilities in which there is always an ex post allocation that maximizes those objectives. This guarantees that there is always an optimal offline deterministic mechanism that maximizes these two objectives. Consequently, in such instances, a deterministic oracle can completely specify the behavior of such an optimal offline deterministic mechanism.

In Section 7.1, we discuss related work. We then propose competitive and comparison analyses as tools to measure the performance of online mechanisms with respect to common objectives such as the utilitarian, egalitarian and Nash *ex ante* welfares in Section 7.2. We further present the three *new* online adviced mechanisms. These mechanisms can read advice bits from the oracle tape in order to improve their current decisions. We assume that agents act sincerely in the next three sections. Section 7.3 contains our results for the setting with no advice. For the setting of complete advice, we report several impossibility results in Section 7.4. And, in Section 7.5, we quantify the performance of these online mechanisms with respect to the number of items for which they are adviced. We next discuss how strategic play can influence the competitiveness of online mechanisms in Section 7.6. Finally, in Section 7.7, we discuss how to maximizes in an online fashion three other objectives: the utilitarian, egalitarian and Nash *ex post* welfares. We conclude in Section 7.8.

7.1 Related work

Competitive analysis is a well-known technique to measure the quality of online versus offline decisions (Borodin and El-Yaniv, 1998; Sleator and Tarjan, 1985). *Online* decisions are irrevocable (i.e. we cannot change past decisions) and instantaneous (i.e. we cannot use future knowledge). *Offline* decisions are made supposing the entire problem information is available. Competitive analysis has been applied in various areas during the years, e.g. online bipartite matching, online stochastic matching, online sequential allocation, online sequential bin packing, online scheduling (Albers and Hellwig, 2012; György, Lugosi, and Ottucsák, 2010; Jaillet and Lu, 2014; Karp, Vazirani, and Vazirani, 1990; Khuller, Mitchell, and Vazirani, 1994). For some online problems, quite successful algorithms are already known under particular assumptions about the arriving input (Brubach, Sankararaman, Srinivasan, and Xu, 2016). For other problems, this is unfortunately not the case. For example, in the uniform knapsack problem, any deterministic online algorithm without advice has an unbounded competitive ratio. Interestingly, with just one bit of advice, it is possible to implement a 2-competitive algorithm for this problem (Marchetti-Spaccamela and Vercellis, 1995). In general, we can increase the competitive ratio of any online algorithm by giving it enough advice. This motivates the development of novel frameworks such as *advice complexity*.

An online algorithm has now an access to an *oracle tape* for the problem of interest and can request an *advice string* when making a decision. The oracle is normally assumed to have an unlimited computational power but the number of bits in the advice string must be polynomially bounded in the size of the input offline problem. For a detailed survey on advice complexity, we refer to (Boyar, Favrholdt, Kudahl, Larsen, and Mikkelsen, 2016). Advice complexity is also related to *semi-online* and *look-ahead* algorithms that suppose some of the input is available (Seiden, Sgall, and Woeginger, 2000). This raises a number of interesting questions. How many advice bits are sufficient to increase the competitive ratio of an online algorithm to a certain threshold? How many bits are needed to match an *optimal* offline algorithm? For example, in the popular paging problem, to achieve offline optimality with an online algorithm we need $\lceil \log_2 k \rceil$ bits of advice to specify which page to delete from the buffer of size k. This results in advice complexity of $n \cdot \lceil \log_2 k \rceil$ for instances with n requests, whereas it is shown that $n + k$ bits of advice suffice (Böckenhauer, Komm, Královic, Královic, and Mömke, 2009; Dobrev, Královic, and Pardubská, 2009). As another example, in online bipartite matching with a graph of size n (i.e. the number of vertices in a partition), a deterministic online algorithm is optimal (w.r.t. the expected matching size) whenever it has an access to $\lceil \log_2 n! \rceil$ but not less advice bits (Miyazaki, 2014).

We observed a 1-to-1 correspondence between online fair division and online bipartite matching where "girl" vertices arrive one by one over time and each such vertex is matched to one of possibly several eligible "boy" vertices. We thus could transfer and significantly extend objectives and algorithms from online bipartite matching to online fair division and vice versa. For example, a popular objective in online bipartite matching is the expected matching size (EMS). There are various algorithms proposed in the online bipartite matching literature that are most competitive for this objective: RANKING, RANDOM and BALANCE (Karp, Vazirani, and Vazirani, 1990). The RANKING and RANDOM mechanisms are randomized and normally applied in the special case when we want to allocate at most item per agent. The BALANCE mechanism is deterministic and is used for the more general setting when we want to allocate more items to agents. A good overview of online matching mechanisms for online bipartite matching problems can be found in (Mehta, 2013). As we mentioned, there are also very few results in regard to online bipartite matching and advice complexity (Miyazaki, 2014). For example, there is an online deterministic mechanism that uses $\lceil \log_2 n! \rceil$ advice bits and maximizes the matching objective (EMS). Also, there is *no* online randomized mechanism that uses less than $\lceil \log_2 n! \rceil$ advice bits and maximizes this objective even when we want to allocate one item per agent (Miyazaki, 2014). To the best of our knowledge, these matching mechanisms are never applied to online fair division problems. For this reason, we analysed the axiomatic properties of these three mechanisms. Unfortunately, they are not Pareto efficient because they can discard items and not envy-free because of they bias the future allocation towards agents with fewest items. Interestingly, the RANKING and RANDOM mechanisms are strategy-proof whereas the BALANCE mechanism is not.

Due to the bijection between online fair division and online bipartite matching instances, some of our results are related to online bipartite matching (Karp, Vazirani, and Vazirani, 1990) *but* we optimize the utilitarian, egalitarian and Nash welfares rather than just a single objective such as the expected matching size (EMS). In our work, we focus however on online fair division for several reasons. For example, agents in fair division have preferences and can be strategic which is an aspect not typically considered in bipartite matching. As a second example, allocations may be more difficult to compute than matchings especially if we want them to satisfy multiple fairness and efficiency criteria. As a third example, allocations are very often complete and give all items to agents whereas matchings are very often incomplete and give some items to agents. As a fourth example, we can view algorithms for bipartite matching as *mechanisms* for fair division and analyse these matching mechanisms with respect to our favourite axiomatic properties in fair division. At the same time, we also analysed the performance of mechanisms such as LIKE and BALANCED LIKE with respect to the expected matching size (Aleksandrov and Walsh, 2017a).

Finally, our results complement the ones in (Freeman, Zahedi, and Conitzer, 2017) where the authors optimize the Nash welfare in a dynamic fair division model and *not* an online fair division model as we do in here. In their setting, the items are fixed and, at each moment in time, agents submit preferences for the next moments that are possibly different that their preferences for the items at earlier moments in time. They propose a deterministic mechanism - GREEDY - that explicitly maximizes the Nash welfare by choosing a *single* agent from a set of agents. This mechanism is online in the sense that, at each round, it does not have access to information about future valuations of agents for items but it uses only the current valuations for these items. In online fair division, the items are not fixed, there is a single item at each round and future items are unknown. We can apply the GREEDY mechanism to our setting but is no longer competitive for the Nash ex ante welfare simply because the optimal offline allocation of the current item might be randomized whereas this mechanism is deterministic. In fact, we show that *no* (even deterministic) mechanism can maximize this welfare. The authors also propose a version of GREEDY that selects all agents that maximize the Nash ex ante welfare and they argue that there is no obvious way to compute the probabilities of the agents for the items at the current round in their setting. This is true even in our online setting. However, for the special case of 0/1 utilities, we give an explicit way how these probabilities could be computed with the BALANCED LIKE mechanism. We prove that BALANCED LIKE mechanism is most competitive for the Nash ex ante welfare on the 0/1 domain and provide theoretical guarantees for its competitive ratio. In contrast, no theoretical guarantees are provided for GREEDY. Further, GREEDY is not strategy-proof (even on 0/1 instances), envy-free or Pareto efficient (on general instances). Finally, we also discuss competitive performance of mechanisms supposing agents act strategically.

7.2 Preliminaries

Consider an instance $\mathcal{I}$ and a mechanism. The mechanism returns a number of expected outcomes to agents. Let us recall that $\overline{u}(i, \mathcal{I})$ denote the expected utility of agent a_i for all items in $\mathcal{I}$ supposing we allocate the items in this instance with the mechanism. We are interested in ex ante social objectives that depend on these expected utilities of agents for items. Three commonly studied objectives are the utilitarian $\overline{u}(\mathcal{I}) = \sum_{i=1}^{n} \overline{u}(i, \mathcal{I})$, egalitarian $\overline{e}(\mathcal{I}) = \min_{i=1}^{n} \overline{u}(i, \mathcal{I})$ and Nash $\overline{n}(\mathcal{I}) = \prod_{i=1}^{n} \overline{u}(i, \mathcal{I})$ ex ante welfares. These objectives could admit very different values even with 0/1 utilities. We show this in Example 22 on specific class of online fair division instances. We call such instances *upper-triangular* as we further use them to show some of our results.

Example 22 (**Upper-triangular instance**) *Consider $\mathcal{I}$ with n agents, n items and let each agent a_i has utilities equal to 1 for items o_1 to o_{n-i+1}. We call such an instance* upper-triagular. *We next present the upper-triangular instance for $n = 5$.*

o	o_1	o_2	o_3	o_4	o_5
a_1	1	1	1	1	1
a_2	1	1	1	1	0
a_3	1	1	1	0	0
a_4	1	1	0	0	0
a_5	1	0	0	0	0

Further, consider a mechanism that allocates each item to an agent that like it with the same probability. For the utilitarian welfare, we obtain that $\overline{u}(\mathcal{I}) = 5$. For the egalitarian welfare, we derive that $\overline{e}(\mathcal{I}) = \frac{1}{5}$ which is the expected utility of agent a_5. For the Nash welfare, we have that $\overline{n}(I) = \prod_{i=1}^{5}(\sum_{j=i}^{5} \frac{1}{i})$ as each agent that likes item o_j receives it with probability $\frac{1}{5-j+1}$. $\square$

We use these three objectives in order to define three performance measures of online mechanisms. These measures capture the worst-case performance of a given online mechanism because they take the minimum welfare value returned by the mechanism over all instances. In our work, we are interested in online mechanisms that maximize this minimum value.

$$\text{(UWA)} \ \min_{\mathcal{I}} \overline{u}(\mathcal{I}) \tag{7.1}$$

$$\text{(EWA)} \ \min_{\mathcal{I}} \overline{e}(\mathcal{I}) \tag{7.2}$$

$$\text{(NWA)} \ \min_{\mathcal{I}} \overline{n}(\mathcal{I}) \tag{7.3}$$

A common technique to analyse the performance of online mechanisms is *competitive analysis* (Sleator and Tarjan, 1985). One way to analyse the competitive perfomance of an online mechanism is compute its competitive ratio. In fair division, however, agents could be sincere or strategic about their preferences. Competitive ratio is calculated supposing agents act sincerely. The *price of anarchy* of an online mechanism is closely related to its competitive ratio but it also takes into account that agents can act strategically (Koutsoupias and Papadimitriou, 1999). From an utilitarian (egalitarian or Nash) perspective, the price of anarchy of an online mechanism is the ratio between the optimal utilitarian (egalitarian or Nash) social welfare, and the smallest utilitarian (egalitarian or Nash) social welfare of any equilibrium strategy. The price of anarchy is essentially the competitive ratio of a mechanism supposing the worst strategic play of agents. Competitive ratio and price of anarchy capture the worst-case performance of online mechanisms. However, this worst-case performance might not be that bad if all other mechanisms achieve the same performance. For this reason, we might want to compare online mechanisms on various instances. In other words, we might want to perform a *comparison analysis* of online mechanisms and decide which ones are better or worse on various instances. Competitive and comparison analyses are normally considered in isolation in the literature. We however consider them in combination.

Consider an online mechanism $\mathcal{M}$. The performance of $\mathcal{M}$ is normally compared with respect to an optimal offline mechanism. The outcome of such a mechanism could depend on the arriving sequence for some problems (). Interestingly, this is not the case in online fair division. More formally, we say that an online mechanism $\mathcal{M}$ has *competitive ratio* $c(k)$ with k advice bits w.r.t. welfare W if, for an instance $\mathcal{I}$ and an ordering o of m items, we have that $W(\text{OPT}) \leq c(k) \cdot W(M(\mathcal{I})) + b(k)$ holds where $b(k)$ is an additive constant and OPT is an optimal offline mechanism. The competitive ratio is measured supposing agents act sincerely as we mentioned earlier. The price of anarchy $c(k)$ of an online mechanism would the its competitive ratio supposing the worst strategic play of agents for items and with respect to welfare W. A mechanism $\mathcal{M}$ is *most* $c(k)$*-competitive* w.r.t. welfare W if $\mathcal{M}$ has a competitive ratio $c(k)$ w.r.t. W and each other mechanism has a competitive ratio that is at least $c(k)$. We say that $\mathcal{M}_1$ is *strictly better* than $\mathcal{M}_2$ on a set of instances if the welfare value of $\mathcal{M}_1$ is not lower than the one of $\mathcal{M}_2$ on all instances from the set, and greater than the one of $\mathcal{M}_2$ on some instances from the set. We say that$\mathcal{M}_1$ and $\mathcal{M}_2$ are *incomparable* if $\mathcal{M}_1$ is strictly better than $\mathcal{M}_2$ on some instances and $\mathcal{M}_2$ is strictly better than $\mathcal{M}_1$ on some other instances. Two online mechanisms are *equivalent* if their welfare values coincide on each instance.

We use an oracle tape to specify some of the behavior of the optimal offline mechanism. An *online* mechanism $\mathcal{M}$ supposes the past decisions are irrevocable and makes the current decision with no information about any future items, or even if any more items will arrive. An *online advice* mechanism ADVICED $\mathcal{M}$ supposes the past decisions are irrevocable and can consult the oracle or not for the allocation of the current item. If "yes", the oracle encodes on the oracle tape the identifier of the agent that should receive the current item when the mechanism reads the tape and allocates this item to the adviced agent. The encoding of the identifier of agent a_i requires $(\log_2 i)$ advice bids. If "no", the ADVICED $\mathcal{M}$ mechanism runs mechanism $\mathcal{M}$ in order to allocate the current item. With complete advice (i.e. the mechanism is adviced for each item), an online adviced mechanism returns the outcome of an optimal offline mechanism. Note however that an optimal offline mechanism supposes complete information about the problem whereas an online adviced mechanism supposes complete advice about the problem. With no advice (i.e. the mechanism is adviced for no item), an online adviced mechanism returns the outcome of mechanism $\mathcal{M}$.

We propose three online adviced mechanisms - the ADVICED MAXIMUM LIKE, ADVICED LIKE and ADVICED BALANCED LIKE mechanisms - that generalize the MAXIMUM LIKE, LIKE and BALANCED LIKE mechanisms respectively. We do not generalize the ONLINE SERIAL DICTATORSHIP, ONLINE RANDOM PRIORITY and PARETO LIKE mechanisms as (1) the ONLINE SERIAL DICTATORSHIP is deterministic and thus not competitive at all, and (2) the ONLINE RANDOM PRIORITY and PARETO LIKE mechanisms return the same expected outcomes as the LIKE mechanism with binary and identical utilities. The axiomatic properties of these online mechanisms are well-understood (Aleksandrov, Aziz, Gaspers, and Walsh, 2015). For this reason, we turn attention to their competitive properties.

7.3 Online fair division with no advice

The *no* advice occurs when the oracle is absent or simply does not specify anything on the oracle tape for any of the items. This may happen in practice. In our Foodbank problem, new unseen food items can be donated online without prior notification. Or, it might be simply too computationally hard to consult the oracle. Each online adviced mechanism in this setting uses its online "unadviced" mechanism to allocate each of the items. Hence, with no advice, each online adviced mechanism returns the outcome of its online "unadviced" mechanism. We therefore refer to our online adviced mechanisms in this section as MAXIMUM LIKE, LIKE and BALANCED LIKE. We firstly report our results for the utilitarian welfare, then for the egalitarian welfare and lastly for the Nash welfare.

7.3.1 Utilitarian welfare

We start with the competitive ratios of our mechanisms. The utilitarian welfare can be maximized offline in a straightforward and greedy way by allocating each item to an agent that likes it most. There are two interesting properties of this offline greedy procedure. First, the allocation of the items can be executed in any order, including in any online order in an online fair division problem. Second, the allocation of each item does not depend on the allocations of the items allocated before it. We note that both of these properties are satisfied by our MAXIMUM LIKE mechanism and make the following simple observation for any n and m.

Observation 23 *With general utilities, the* MAXIMUM LIKE *mechanism maximizes (UWA).*

Observation 23 is straightforward in our setting but there are fair division settings in which optimizing the utilitarian welfare is intractable even *offline* when information about the entire problem input is available (Nguyen, Nguyen, Roos, and Rothe, 2014). On the other hand, with binary utilities, note that each mechanism that gives all items to agents that like them maximizes the utilitarian welfare. Indeed, BALANCED LIKE and LIKE do maximize it.

Observation 24 *With 0/1 utilities, the* BALANCED LIKE *and* LIKE *mechanisms maximize (UWA).*

With general utilities, the LIKE mechanism is n-competitive; see Theorem 9 from (Aleksandrov, Aziz, Gaspers, and Walsh, 2015). By comparison, the BALANCED LIKE mechanism is not competitive from an utilitarian perspective even with just two agents and two items. We illustrate this result in Example 23. However, even with more items than agents, the BALANCED LIKE mechanism remains not competitive; see Theorem 10 from (Aleksandrov, Aziz, Gaspers, and Walsh, 2015).

Example 23 *Consider the following instance and the* BALANCED LIKE *mechanism.*

o	o_1	o_2
a_1	0	1
a_2	1	u

The optimal offline utilitarian mechanism gives both items to agent a_2 and therefore achieves a welfare that is equal to $u+1$. The BALANCED LIKE *mechanism gives item o_1 to agent a_2 and item o_2 to agent a_1, and the value of the welfare is equal to 2. Its ratio goes then to ∞ as u goes to ∞.* □

Our Example 23 is in-line with an impossibility example and an impossibility remark presented by (Khuller, Mitchell, and Vazirani, 1994) for online weighted bipartite matching. These results show that there is no deterministic or randomized online algorithm that maximizes (or minimizes) the *perfect utilitarian welfare* (the sum of the utilities in an allocation that gives to each agent exactly one item) where the competitive ratio of the algorithm depends only on the number of agents n. In contrast, our utilitarian welfare objective (UWA) is different because its maximum value could be achieved by allocating all items to a single agent. As a result, the competitive ratio of the MAXIMUM LIKE mechanism does not even depend on n and the ratio of the LIKE mechanism depends solely on n.

We conclude that that MAXIMUM LIKE is most competitive for (UWA) in any case. The LIKE is n-competitive whereas BALANCED LIKE might not be competitive at all. However, there are instances on which the BALANCED LIKE and LIKE mechanisms are equivalent, e.g. 0/1 utilities, positive utilities. With general utilities, LIKE has greater competitive ratio that BALANCED LIKE in the worst-case however BALANCED LIKE outperforms LIKE on some instances and even empirically in the average-case (Aleksandrov, Gaspers, and Walsh, 2015).

7.3.2 Egalitarian welfare

In this section, we study how we might optimize the egalitarian welfare. It is easy to see that there is no deterministic online mechanism that maximizes this welfare. For this reason, we therefore look at randomized mechanisms. For example, we show that our BALANCED LIKE mechanism maximizes this welfare with 0/1 utilities. Also, both the LIKE and BALANCED LIKE mechanisms are n-competitive from an egalitarian perspective. Moreover, in the 0/1 setting, the MAXIMUM LIKE mechanism is equivalent to the LIKE mechanism and is therefore also n-competitive. Interestingly, this ratio is optimal.

Theorem 11 *With 0/1 utilities, the* LIKE *and* BALANCED LIKE *mechanism are most n-competitive for (EWA).*

Proof. Consider an instance in which there is an ex post allocation in which each agent receives exactly k items. We know that a given agent, say a_i, receives at least one of these k items in each allocation returned by the mechanism. The worst case for agent a_i in such an instance is when the items arrive in such an ordering so that agent a_i receives exactly one of these k items in each of the allocations returned by the mechanism, and each other agent bids 1 for this item. One such instance is the 0/1 instance in the proof of Theorem 9 from (Aleksandrov, Aziz, Gaspers, and Walsh, 2015). The egalitarian welfare returned by the mechanism on such an instance is $\frac{1}{n}$. □

We next move to general utilities. The LIKE mechanism remains most n-competitive with any number of items; see Theorem 9 from (Aleksandrov, Aziz, Gaspers, and Walsh, 2015). The worst case for a given agent is when all other agents bid positively for their items. In contrast, the BALANCED LIKE mechanism becomes not competitive at all with $m > n$ items; see Theorem 10 from (Aleksandrov, Aziz, Gaspers, and Walsh, 2015). The reason for it is because this mechanism bias the allocation of future items and, as a result, agents might not be given chance for items they value greatly and the optimal offline mechanism would give them. A similar result holds for the MAXIMUM LIKE mechanism with even as many items as agents because this mechanism may give all items to a single agent thus leaving some other agent with no items. Interestingly, with as many items as agents, the BALANCED LIKE mechanism becomes most n-competitive in this general setting.

Theorem 12 *With general utilities and n items, the* BALANCED LIKE *mechanism is most n-competitive for (EWA).*

Proof. The mechanism have competitive ratio of n. Consider instance $\mathcal{I}$, agent a_i and the first item o_j in the ordering such that agent a_i has positive utility for it. We show that $\overline{e}(\mathcal{I})$ is at least $\frac{1}{n}$. The worst case for agent a_i is when they have been allocated 0 items prior round j and all agents together have positive utilities for item o_j. Therefore, we have $\overline{p}(i, j) \geq 1/n_j \geq 1/n$. Hence, agent a_i receives expected utility of at least $\frac{1}{n}$. This lower bound is achieved in Example 22.

Next, we confirm that every other mechanism has competitive ratio at least n. Consider the upper-triangular instance from Example 22 and a mechanism $\mathcal{M}$. If $\mathcal{M}$ shares the probability for the first item uniformly at random, then its competitive ratio is equal to $\frac{1}{n}$. If $\mathcal{M}$ shares the probability for the first item not uniformly at random, then its competitive ratio is lower than $\frac{1}{n}$. To see this, suppose that $\mathcal{M}$ gives the first item to agent a_n with probability $p > \frac{1}{n}$. The probability of some other agent must be smaller than $\frac{1}{n}$ as the probabilities of agents for the first item sum up to at most 1. WLOG, suppose that the probability q of agent a_1 for this first item is one such value smaller than $\frac{1}{n}$. The egalitarian welfare on the upper-triangular instance is then p. However, consider next the *lower-triangular* instance, i.e. agent a_i likes items o_1 to o_i. The mechanism $\mathcal{M}$ gives expected utility of $q < \frac{1}{n}$ to agent a_1. This value is also the welfare value of the mechanism on the lower-triangular instance. Consequently, $\mathcal{M}$ has competitive ratio of $\frac{1}{q}$ because the optimal offline welfare is 1. But, then its competitive ratio is strictly more than n.□

Let us next see how these mechanisms compare to one another. Surprisingly, there are instances on which MAXIMUM LIKE outperforms both the LIKE and BALANCED LIKE mechanisms even though it is not competitive in general.

Example 24 *Consider the following instance and all three mechanisms.*

o	o_1	o_2
a_1	2	1
a_2	1	2

Both LIKE *and* BALANCED LIKE *give expected utility of $\frac{3}{2}$ to each agent. However, the* MAXIMUM LIKE *mechanism gives item o_1 to agent a_1 and item o_2 to agent a_2. This means that each agent receives with this mechanism their most valued item. The egalitarian welfare of this mechanism is equal to 2 which is strictly more than $\frac{3}{2}$.*□

Proposition 14 *With 0/1 utilities, the* BALANCED LIKE *mechanism which is strictly better than the* LIKE *mechanism for (EWA).*

Proof. Pick any 0/1 instance, consider round j and any allocation $\pi(j-1)$ of past items o_1 to o_{j-1}. The ex post utility of agent a_i in this allocation is $u(i, \pi(j-1))$. WLOG, suppose that the egalitarian welfare given π is achieved by agent a_1, i.e. $u(1, \pi(j-1)) = \min_{i=1}^{n} u(i, \pi(j-1))$. We consider now two cases. In the first case, suppose we use the LIKE mechanism to allocate item o_j which happens to be liked by n_j agents. Each agent receives then $\frac{1}{n_j}$ of additional expected utility. The egalitarian welfare is then $u(1, \pi(j-1)) + \frac{1}{n_j}$. In the second case, suppose we use the BALANCED LIKE mechanism to allocate item o_j which happens to be feasible for $f_j < n_j$ agents. Note that each agent that is not feasible for item o_j has utility of at least 1 more than agent a_1. The egalitarian welfare is then $u(1, \pi(j-1)) + \frac{1}{f_j}$. But, this is strictly higher than the welfare of the LIKE mechanism in this $\pi(j-1)$. This holds for any such allocation.□

In fact, it is not difficult to see that the BALANCED LIKE mechanism maximizes the egalitarian welfare compared to any other online mechanism. To see this, just pick an online mechanism and assume that it maximizes the welfare and the BALANCED LIKE does not. Hence, there is an instance on which both mechanism achieve different welfare values. Pick one such instance and, by induction on the round number, show that the optimal choice for each item is to be allocated uniformly at random amongst the agents who like the item and are allocated fewest items previously. This means that the expected allocations of both mechanisms are the same for each instance which is in contradiction with our assumption that there is an instance on which the welfares returned by two mechanisms are different.

Observation 25 *With 0/1 utilities, the* BALANCED LIKE *mechanism maximizes the (EWA).*

We conclude that LIKE is most competitive with general utilities and any number of items. However, BALANCED LIKE is also most competitive with any number of agents and 0/1 utilities or as many items as agents and general utilities. With more items than agents, BALANCED LIKE has lower worst-case performance than LIKE although BALANCED LIKE outperforms LIKE on average (Aleksandrov, Gaspers, and Walsh, 2015). The MAXIMUM LIKE mechanism is not competitive but it might outperform both other mechanisms on some instances. Like for (UWA), both LIKE and BALANCED LIKE are equivalent for (EWA) on instances with positive utilities. Unlike for (UWA), BALANCED LIKE is strictly better than LIKE for (EWA) on instances with 0/1 utilities. Finally, the MAXIMUM LIKE mechanism is equivalent to these mechanisms on instances with identical preferences, and so is most competitive on such instances.

7.3.3 Nash welfare

We turn attention to the Nash welfare objective. We start with the 0/1 case and show that our BALANCED LIKE mechanism is equivalent to the GREEDY algorithm from (Freeman, Zahedi, and Conitzer, 2017).

Proposition 15 *With 0/1 utilities, the* BALANCED LIKE *and* GREEDY *mechanisms are equivalent.*

Proof. Given allocation $\pi(j-1)$ of past items, the GREEDY mechanism allocates the current item o_j so that $\prod_{i=1}^{n}(u(i,\pi(j-1))+p(i,j)\cdot u_{ij})$ is maximized. At round 1, this is maximized if we allocate the item uniformly at random to an agent that likes it. This follows by the inequality between the arithmetic and geometric means. The item is then allocated to some such agent. At round 2, this product is maximized when the item is allocated to an agent that likes the item and has fewest items amongst the agents that like the item. This again follows by the inequality between the arithmetic and geometric means applied at round 2 and given the allocation of the first item. In this way, it is not difficult to see that the decision made by GREEDY at each round j and given allocation $\pi(j-1)$ is to allocate item o_j with probability $\frac{1}{f_j}$ where f_j is the number of agents that like item o_j and have fewest items among those liking this item. But, this is the number of feasible agents of our BALANCED LIKE mechanism. □

We gave in the latter proof an *explicit* way how to compute the probabilities of agents for items that maximize the Nash welfare on 0/1 instance. Indeed, our BALANCED LIKE mechanism is most competitive for the Nash welfare.

Theorem 13 *With 0/1 utilities, the* BALANCED LIKE *is most competitive for (NWA).*

Proof. Suppose it is not and there is another mechanism $\mathcal{M}$ that on some instance achieves greater Nash welfare than the BALANCED LIKE mechanism. We can run the mechanism on such an instance and establish that the decisions it makes are the ones made by our BALANCED LIKE mechanism on the instance. We did this for GREEDY in Proposition 15. Therefore, the welfares returned by $\mathcal{M}$ and BALANCED LIKE cannot be different on any 0/1 instance. Hence, BALANCED LIKE is most competitive for this welfare. □

Even more, no other mechanism can perform better on any 0/1 instance than the BALANCED LIKE mechanism.

Observation 26 *With 0/1 utilities, the* BALANCED LIKE *mechanism maximizes the (NWA).*

As a result, the BALANCED LIKE mechanism is strictly better than the LIKE mechanism on 0/1 instances. This statement holds for the MAXIMUM LIKE mechanism as well because it is equivalent with the LIKE mechanism with 0/1 utilities. The latter observation is in-line with fact that BALANCED LIKE is strictly better than LIKE for the egalitarian welfare; see Proposition 14. In fact, a mechanism that maximizes both the egalitarian and utilitarian welfares on 0/1 instances also maximizes the Nash welfare.

With general utilities, as we discussed previously, it is not clear how to explicitly represent the probabilities of agents for items with the GREEDY mechanism. This is a not trivial task even in our online fair division model. However, we can use linear programming to compute these probabilities for this mechanism (Freeman, Zahedi, and Conitzer, 2017). At the same time, BALANCED LIKE interestingly becomes not competitive at all.

Example 25 *Consider the following instance and the* BALANCED LIKE *mechanism.*

o	o_1	o_2	o_3
a_1	0	1	1
a_2	1	u	0
a_3	0	0	1

The BALANCED LIKE *mechanism gives item o_1 to agent a_2, item o_2 to agent a_1 and item o_3 to agent a_3. The Nash welfare is equal to 1. Suppose that u is very large value. Clearly, the optimal offline mechanism then gives both items o_1 and o_2 to agent a_2 and shares the probability for item o_3 equally between agents a_1 and a_3. The mechanism returns Nash welfare of $\frac{(1+u)}{4}$. Consequently, the competitive ratio of our mechanism goes to ∞ as u goes to ∞.*□

Further, the LIKE mechanism is n^n-competitive from a Nash perspective. The worst case is when all agents bid positively and each agent receives expected utility $\frac{1}{n}$ with LIKE and 1 optimally offline. By comparison, the MAXIMUM LIKE mechanism is not competitive at all in this setting because it may give all items to a single agent. However, there are instances on which the MAXIMUM LIKE mechanism outperforms LIKE and BALANCED LIKE; see Example 25.

To conclude, we could use BALANCED LIKE mechanism to maximizes the Nash welfare with 0/1 utilities. With general utilities, the GREEDY mechanism might be a better choice as it performs well on average (Freeman, Zahedi, and Conitzer, 2017). We submit that indeed it is an interesting and challenging task to theoretically analyse the performance of GREEDY in general. Finally, the LIKE mechanism performs with ratio n^n at worst.

7.4 Online fair division with full advice

The *full* advice occurs when the oracle encodes on the oracle tape the identifier of an agent for each item. This situation might occur in practice. For example, in our Foodbank problem, the chair often has some information about items donated on a regular basis. The oracle can use this information to make better predictions especially if it is computational easy to consult it. Each online adviced mechanism in this setting reads the oracle tape at each round and allocates the current item to the adviced agent. Hence, with full advice, each online adviced mechanism returns the outcome of an optimal offline mechanism.

With start with the utilitarian welfare. With general utillities, n agents and m items, the utilitarian welfare can be maximized by a deterministic online adviced mechanism that allocates each item to the adviced agent. Such a mechanism would need $\lceil \log_2 n^m \rceil$ advice bits at worst which happens when agent a_n is adviced for each of the m items. However, note that the MAXIMUM LIKE mechanism also maximizes the utilitarian welfare and uses 0 advice bits. We therefore do not need an oracle for the utilitarian welfare. For this reason, we proceed with the egalitarian and Nash welfare and show that it is possible to maximizes them in an online fashion but only with a full advice. To show this, we use the next impossibility result from the literature in online bipartite matching with advice. We report this result in our terminology. The result is for online adviced deterministic mechanisms and the expected matching size objective (EMS).

Observation 27 *(Theorem 2 from (Miyazaki, 2014)) For a given undirected and unweighted bipartite graph, there is* **no** *online adviced deterministic mechanism that uses less than $\lceil \log_2 n! \rceil$ advice bits and maximizes the expected matching size objective (EMS).*

Interestingly, our egalitarian and Nash objectives can be maximized whenever an online adviced mechanism has access to a full advice. Otherwise, with a partial advice, this is impossible.

Theorem 14 *Even with 0/1 utilities, n agents and n items, there is* **no** *online adviced mechanism that uses less than $\lceil \log_2 n! \rceil$ advice bits and returns the optimal offline (EWA) or (NWA).*

Proof. Let $\mathcal{I}$ be a given online fair division instance with n agents, n items and 0/1 utilities. Suppose that there is an ex post allocation of all items in which each agent receives exactly one item. Within this assumption, we argue that the egalitarian and Nash welfares are maximized by any mechanism that gives expected utility of 1 to each agent. Let us now consider the egalitarian welfare. The maximum value of the egalitarian ex ante welfare is equal to 1 for each mechanism that allocates each item to an agent that likes it and shares the probability of 1 for this item only amongst agents that like the item. For the sake of contradiction, suppose that this is not true and the egalitarian ex ante welfare is strictly greater than 1. We have that the sum of the expected utilities of agents is equal to n because the utilities are 0/1 and there is no utility of probability loss for each item. Therefore, there is an agent whose expected utility is strictly lower than 1. This contradicts the fact that the egalitarian ex ante welfare is more than 1. Let us next consider the Nash welfare. It is easy to see that its value is maximized when each agent receives expected utility 1.

Suppose that there is an online adviced mechanism that maximizes each of these welfares on $\mathcal{I}$ and uses less than $\lceil \log_2 n! \rceil$ advice bits. We will reach a contradiction. For such a mechanism, each agent receives an expected utility of 1 and the probability of 1 for each item is shared completely between agents that like the item. These conditions on the expected utilities and probabilities are often called *bistochastic* in the literature. Also, the matrix of the probabilities of agents for items is called *bistochastic*. By the famous result of Birkhoff, every bistochastic matrix can be represented as a convex combination of permutation matrices whose weights sum up to 1 (Brualdi, 2006). Each permutation matrix in such a combination corresponds to an ex post allocation in $\mathcal{I}$ that gives exactly one item to each agent. We can associate an online adviced deterministic mechanism with such an allocation. Such a mechanism therefore returns the associated allocation, uses less than $\lceil \log_2 n! \rceil$ advice bits and maximizes not only the welfare objectives but the objective (EMS) as well. By Observation 27, we conclude that such a mechanism cannot exist which is a contradiction. We also conclude that there cannot be a randomized such online adviced mechanism that uses less than $\lceil \log_2 n! \rceil$ advice bits and maximizes the egalitarian or Nash welfares. Otherwise, such a randomized mechanism would have to return only ex post allocations in which agents receive one item each and whose probabilities sum up to 1. Now, pick one of its returned allocations and define an online adviced deterministic mechanism that follows this returned allocation. This again leads to another contradiction with the result in Observation 27. The result follows.□

Finally, we make two interesting observations of our impossibility results. First, we cannot use a deterministic matching mechanism such as BALANCE to maximizes the egalitarian of Nash welfares. We must use randomized mechanisms. Second, none of the matching mechanisms RANKING and RANDOM maximizes any of these two objectives.

7.5 Online fair division with partial advice

In this section, we consider that case when the oracle specifies agents for $k \in (1, m)$ out of the m items. For (UWA), (EWA), (NWA) and 0/1 utilities, the oracle specifies k agents for the first k items in the ordering for which the k agents are different. For (UWA) and general utilities, the oracle specifies an agent for each of k most valued items. The worst case for the ADVICED BALANCED LIKE mechanism is when the adviced allocation biases the allocation of future items towards agents who receive negligibly small utilities for these items. Instead, the ADVICED LIKE mechanism allocates each such unadviced item to an agent with probability at least $\frac{1}{n}$. The ADVICED MAXIMUM LIKE optimizes (UWA) for any k. For (EWA), (or (NWA)) and general utilities, the oracle computes an allocation of the k items to agents that maximizes the egalitarian (or Nash) welfare and then, for each adviced item in the ordering, it specifies on the oracle tape the agent in this computed allocation. The ADVICED BALANCED LIKE mechanism thus focus on agents with zero and fewest items whereas the ADVICED LIKE and ADVICED MAXIMUM LIKE mechanisms perform as the LIKE and MAXIMUM LIKE mechanisms. We summarize our results for this setting from (Aleksandrov and Walsh, 2017b) in Table 7.1.

mechanism	**(UWA)**	**(UWA)**	**(EWA)**	**(EWA)**	**(EWA)**	**(NWA)**	**(NWA)**
sincere	binary	general	binary	general	general	binary	general
	$m \geq n$	$m \geq n$	$m \geq n$	$m = n$	$m > n$	$m \geq n$	$m \geq n$
ADV.MAX.LIKE	1	1	$\frac{1}{n}$	0	0	$\frac{1}{n^{(n-k)}}$	0
ADV.BAL.LIKE	1	$\frac{k}{m}$	$\frac{1}{n-l}$	$\frac{1}{n-l}$	0	$\frac{1}{n^{(n-k)}}$	0
ADV.LIKE	1	$\frac{k}{m}+\frac{1}{n}-\frac{k}{nm}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n^{(n-k)}}$	$\frac{1}{n^{(n-k)}}$
strategic	binary	general	binary	general	general	binary	general
	$m \geq n$	$m \geq n$	$m \geq n$	$m = n$	$m > n$	$m \geq n$	$m \geq n$
ADV.MAX.LIKE	1	$\frac{k}{m}+\frac{1}{n}-\frac{k}{nm}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n^{(n-k)}}$	$\frac{1}{n^{(n-k)}}$
ADV.BAL.LIKE	1	$\frac{k}{m}+\frac{1}{m}$	$\frac{1}{n-l}$	$\frac{1}{n-l}$	$\frac{1}{n-l}$	$\frac{1}{n^{(n-k)}}$	$\frac{1}{n^{(n-k)}}$
ADV.LIKE	1	$\frac{k}{m}+\frac{1}{n}-\frac{k}{nm}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n^{(n-k)}}$	$\frac{1}{n^{(n-k)}}$

TABLE 7.1: Competitive Analyses.

To conclude, some of our results are promising. For example, the advice imporves the utilitarian welfare for ADVICED LIKE more than for ADVICED BALANCED LIKE. The chair could use these mechanisms for this welfare instead of ADVICED MAXIMUM LIKE because recall that the latter may give all items to a single agent supposing this agent likes all items most. But, its unique pure Nash equilibrium coincides with the sincere play of the ADVICED LIKE mechanism. We also observe an increase in the egalitarian performance of ADVICED BALANCED LIKE with 0/1 utilities or general utilities and as many items as agents. With more items than agents, the chair might prefer to use ADVICED LIKE instead. For the Nash welfare, we recommend ADVICED BALANCED LIKE because it is most competitive with 0/1 utilities and ADVICED LIKE with general utilities. By comparison, some of the other results are no so exciting. For example, we cannot use advice to increase the performance of mechanisms for egalitarian or Nash welfares with general utilities. This is perhaps because our oracle is deterministic. Finally, we submit that it is interesting to consider randomized oracles in future as well.

7.6 Price of anarchy

In this section, we next the prices of anarchy of the online adviced mechanisms. There is a unique pure Nash equilibrium with the MAXIMUM LIKE mechanism in which each agent bids the maximum utility for each item they like and zero for each item they dislike.

Proposition 16 *With general utilities, the price of anarchy of the* MAXIMUM LIKE *mechanism is equal to n for (UWA).*

Proof. In the unique pure Nash equilibrium, the mechanism returns the same outcomes as if we run the LIKE mechanism. The worst case is then when each agent bids the maximum possible utility value for each item they like and they like each item. Consider then the following instance with n agents and m items: agent a_i likes item o_i with 1 and each other item with some very small and positive ϵ that is close to 0. The optimal offline mechanism returns welfare that goes to n when ϵ goes to 0. In the pure Nash equilibrium, each agent bids 1 for each item when the MAXIMUM LIKE mechanism returns welfare that goes to 1 when ϵ goes to 0. □

The price of anarchy of the LIKE mechanism coincides with its competitive ratio because this mechanism is strategy-proof, i.e. it is equal to n. In contrast, the BALANCED LIKE mechanism is only strategy-proof with 0/1 utilities and 2 agents; see Theorem 3 from (Aleksandrov, Aziz, Gaspers, and Walsh, 2015). Interestingly, with 0/1 utilities and 3 or more of agents, this mechanism again maximizes the utilitarian welfare in each pure Nash equilibrium. The reason for this is because at least one agent bids 1 for each item in each of its pure Nash equilibrium. This is not the case with general utilities. It might happen in a given pure Nash equilibrium that all agents who sincerely like a given item bid 0 for it. This causes a decrease in the welfare. However, the BALANCED LIKE mechanism is most competitive from a utilitarian perspective in each of its pure Nash equilibria.

Observation 28 *The price of anarchy of the* BALANCED LIKE *mechanism is 1 with 0/1 utilities and n with general utilities for (UWA).*

Interestingly, the competitive ratios of both LIKE and BALANCED LIKE mechanisms coincide on each instance with positive utilities. This happens because each agent receive each item with probability $\frac{1}{n}$ on such instances. This is however not true for their prices of anarchy on all such instances.

Example 26 *Consider the following instance and the* LIKE *and* BALANCED LIKE *mechanisms.*

o	o_1	o_2
a_1	$\frac{3}{2}$	1
a_2	1	$\frac{3}{2}$

The price of anarchy of the LIKE *mechanism is equal to its competitive ratio which is* $\frac{5}{6}$. *The unique pure Nash equilibrium of the* BALANCED LIKE *mechanism is when agent* a_2 *bids 0 for item* o_1. *The price of anarchy of this mechanism is therefore* 1. $\square$

For (UWA), strategic play decreases the performance of the MAXIMUM LIKE mechanism which is then equivalent to the LIKE mechanism. Interestingly, we observe the opposite effect for the BALANCED LIKE mechanism. Strategic play always increases the welfare.

We next consider how strategic play influences (EWA). The first interesting observation is that the MAXIMUM LIKE mechanism has price of anarchy of n because its expected allocation coincides with the one returned by the LIKE mechanism. The price of anarchy of the latter is equal to n and coincides with its competitive ratio. The second interesting observation is that the BALANCED LIKE mechanism remains most competitive with 0/1 utilities because each agent bids for at least one item in each of its pure Nash equilibrium. However, strategic play may increase or decrease the egalitarian welfare; see Theorem 2 and Theorem 6 from (Aleksandrov, Aziz, Gaspers, and Walsh, 2015). However, the BALANCED LIKE mechanism is most competitive from an egalitarian welfare in each of its pure Nash equilibria.

Observation 29 *The price of anarchy of the* BALANCED LIKE *mechanism is* n *with 0/1 and general utilities for (EWA).*

For (EWA), strategic play increases the performance of the MAXIMUM LIKE mechanism which is then equivalent to the LIKE mechanism, and in fact the MAXIMUM LIKE mechanism becomes most competitive. For BALANCED LIKE, strategic play always increases or decreases the welfare in general.

Finally, we can also ask if strategic play can be used to support the performance of an online adviced mechanism that uses less than a full advice. However, we observe that our impossibility results hold on upper-triangular instances and, for a responsive mechanism on such instances, the sincere and strategic bidding of agents coincide. Therefore, our impossibility results further hold supposing agent act strategicaly. Also, it would be interesting in future to analyse the impact of strategic play on the Nash welfare.

7.7 Ex post welfares

We considered previously how we could possibly maximizes the utilitarian, egalitarian and Nash ex ante welfares in online fair division. We therefore discuss in here discuss how we could maximizes the utilitarian, egalitarian and Nash ex post welfase. The utilitarian ex post welfare is $u(\mathcal{I}) = \sum_{\pi\in\Pi}\sum_{i=1}^{n} u(i,\pi)$. The egalitarian ex post welfare is $e(\mathcal{I}) = \sum_{\pi\in\Pi}\min_{i=1}^{n} u(i,\pi)$. The Nash ex post welfare is $n(\mathcal{I}) = \sum_{\pi\in\Pi}\prod_{i=1}^{n} u(i,\pi)$. Similarly as for the ex ante welfares, we are interested in mechanisms that maximizes these welfares with respect to the next three performance measures.

$$\text{(UWP)} \min_{\mathcal{I}} u(\mathcal{I}) \tag{7.4}$$

$$\text{(EWP)} \min_{\mathcal{I}} e(\mathcal{I}) \tag{7.5}$$

$$\text{(NWP)} \min_{\mathcal{I}} n(\mathcal{I}) \tag{7.6}$$

We observe again some relations between these objectives. For example, for each instance $\mathcal{I}$ and responive mechanism, we have that $u(\mathcal{I}) > e(\mathcal{I})$ with $n \geq 2$. We also have that the Nash welfare in each ex post allocation is maximized whenever each agent receives the same ex post utility in this allocation and this utility is maximum, i.e. the inequality between arithmetic and geometric means applied to the ex post outcomes of the agents in a given actual allocation. If the utilities of agents for items are integers, we further have that $n(\mathcal{I}) \geq e(\mathcal{I})$. There are some relations of these objectives with the ex ante objectives that we previously considered; e.g. $u(\mathcal{I}) = \overline{u}(\mathcal{I})$, $e(\mathcal{I}) \leq \overline{e}(\mathcal{I})$. Based on these relations, we can immediately inherit some results from our analysis for the ex ante welfares.

Observation 30 *With general utilities, the* MAXIMUM LIKE *mechanism maximizes (UWP).*

Interestingly, the egalitarian and Nash ex post welfares are more difficult to maximize. In fact, our impossibility results still holds for these two objectives, i.e. *no* online mechanism can maximize them with less than a full advice. For both ex post welfares, we focus on instances in which there is an ex post allocation in which each agent receives some items. Otherwise, we obtained that there could be instances on which any online mechanism can return the outcome of any offline mechanism with no advice. We start with 0/1 utilities. The worst case is when there is a single allocation in which each agent receives exactly one item and there is an agent with zero items in each other allocation. Recall that in this setting both the MAXIMUM LIKE mechanism return the same expected allocation as the LIKE mechanism.

Proposition 17 *With 0/1 utilities, the* LIKE *and* BALANCED LIKE *mechanisms are most $n!$-competitive for (EWP) and (NWP).*

Proof. Consider an instance in which there are n agents and n items. Suppose that agent a_i likes items o_1 to o_i for each $i \in [1, n]$. There is just one allocation in which each agent receives one item. This allocation gives item o_i to agent a_i, and moreover it is returned by both mechanisms with probability $\frac{1}{n!}$. Hence, these mechanisms return egalitarian ex post welfare of $\frac{1}{n!}$ and Nash ex post welfare of $\frac{1}{n!}$. The optimal offline welfares are equal to 1. It is not hard to see that the instance is one of the worst instances for these mechanisms and these welfares. □

This is the worst case behavior of these two mechanisms on 0/1 instances. However, the BALANCED LIKE mechanism is strictly better than the LIKE for each of these objectives in isolation and for each instance. The reason is that LIKE returns allocations in which some agents receive zero items much more often than BALANCED LIKE. We demonstrate this in Example 27.

Example 27 *Consider an instance with n agents and n items, and suppose that each agent likes each item with 1. The* LIKE *mechanism returns $n!$ allocations in which each agent receives exactly one item and $n^n - n!$ allocations in which some agents receive zero items. Therefore, its egalitarian and Nash ex post welfares are both equal to $\frac{n!}{n^n}$. The* BALANCED LIKE *mechanism returns only the $n!$ allocations which maximize these objectives. Their values are then both equal to 1.*□

We next consider general utilities. Interestingly, the competitive ratio of the LIKE mechanism increases to n^n and the one of the BALANCED LIKE mechanism remains $n!$. We next proof that these bounds are optimal.

Proposition 18 *With general utilities, the* LIKE *mechanism is most n^n-competitive and the* BALANCED LIKE *mechanism is most $n!$-competitive for (EWP) and (NWP).*

Proof. The worst-case instance is when there is only one ex post allocation which maximizes these welfares and each other allocation does not. One such instance for the BALANCED LIKE mechanism is the one we used in the proof of Proposition 17. For the LIKE mechanism, we need another instance. Consider n agents and n items, and further suppose that, for each $i \in [1, n]$, agent a_i likes item o_i with 1 and each other item with ϵ for some very small $\epsilon > 0$. The optimal offline mechanism gives item o_i to agent a_i. The welfare values are both equal to 1. The LIKE mechanism returns the optimal offline allocation with probability $\frac{1}{n^n}$ as well as each other allocation with the same probability. However, both the egalitarian and Nash ex post welfares of this mechanism approach $\frac{1}{n^n}$ as ϵ goes to 0. Note that the MAXIMUM LIKE mechanism returns the optimal offline allocation on this instance but there are other instances on which it gives all items to a single agent and thus achieves (EWP) and (NWP) of 0. This cannot happen with our LIKE mechanism as this instance is the worst one for it and for each other mechanism $\mathcal{M}$. To see this, let us suppose that $\mathcal{M}$ returns the optimal offline allocation with probability p that is strictly greater than $\frac{1}{n^n}$. WLOG, we could assume that agent a_1 received item o_1 with probability higher than $\frac{1}{n}$. However, the online adaptive adversary could not limit this decision of $\mathcal{M}$ by releasing an item that agent a_1 value greatly and each other agent with ϵ close to 0. But then $\mathcal{M}$ could not return the optimal offline allocation with p greater than $\frac{1}{n^n}$. □

Finally, with general utilities, we suggest the MAXIMUM LIKE mechanism for (UWP), the LIKE mechanism for (EWP) and (NWP) (or probably GREEDY). With 0/1 utilities, the chair might use the BALANCED LIKE mechanism for each of the (UWP), (EWP) and (NWP).

7.8 Conclusions

We proposed a novel combination of competitive analysis, advice complexity and online fair division. Our results help us understand better the interface between matching and fair division problems. In conclusion, the chair might use ADVICED MAXIMUM LIKE for (UWA) and ADVICED LIKE for (EWA) and (NWA) with general utilities. And, with 0/1 instances, they might use ADVICED BALANCED LIKE for each objective. These are the recommentations based on our worst-case analyses. However, note that BALANCED LIKE outperforms LIKE on average and therefore ADVICED BALANCED LIKE outperforms ADVICED LIKE on average as well (Aleksandrov, Gaspers, and Walsh, 2015). We quantified the performance of these mechanisms with respect to the number of advice bits they can read from an oracle tape. We presented two impossibility results: *no* online mechanism can maximize the egalitarian or Nash welfare with less than a full advice. Moreover, we observed that it is not appropriate to use the currently available matching mechanisms for fair division objectives because they discard items. We also discussed results for the corresponding ex post welfares (UWP), (EWP) and (NWP). In summary, with 0/1 utilities, our BALANCED LIKE mechanism maximizes all three ex ante and three ex post objectives and remains most competitive in some of the settings with general utilities. Strategic play often increased the welfares with general utilities.

Despite its novelty, our work has several limitations. First, the oracle is deterministic. It would be interesting to consider in future randomized advice oracles as well. Second, we consider multiple *ex ante and ex post* welfares. However, we could also measure the performance of mechanisms with respect to other measures, e.g. inequality indices such as the Gini, Robin Hood and Atkinson indices (Endriss, 2013; Schneckenburger, Dorn, and Endriss, 2017). Some of our results already extend to these objectives. For example, our impossibility results hold for each of these objectives. Another future direction would be to analyse the price of fairness of these mechanisms (Bertsimas, Farias, and Trichakis, 2011; Dickerson, Procaccia, and Sandholm, 2014). Finally, our model of online fair division can be further generalized. For example, there are online bipartite matching models with weights attached to the "boy" vertices or "girl" vertices arriving from a known probability distribution (e.g. random order) (Mehta, 2013). Our impossibility results extend to these more general settings but it is still interesting to see whether our mechanisms remain most competitive.

Chapter 8

Future Work

We discuss in this chapter several possible future directions that could extend our work. For example, we could consider more mechanisms that say maximize the separate welfare notions in isolation. For example, one such mechanism is the EGALITARIAN LIKE mechanism that allocates each next item uniformly at random to an agent that maximizes the egalitarian welfare. Moreover, we could also consider mechanisms that minimize inequality indices such as Gini, Hoover or Atkinson indices (Gini, 1912; Hoover, 1936; Atkinson, 1970). For example, the GINI LIKE mechanism allocates each next item to arrive uniformly at random to an agent that likes the item and minimizes the Gini index. We submit that it would be interesting to analyse the worst-case axiomatic and competitive performance of such mechanisms.

In this thesis, we assumed that the ordering of items is fixed and considered two extreme cases of *complete* and *no* information in an online setting of fair division. However, there are *many* more settings in between these two extremes. For example, the items could arrive from a know probabilistic distribution or the ordering could be drawn with some probability. A recent work addresses issues related to efficiency and strategic behavior in such probabilistic settings (Gorokh, Banerjee, and Iyer, 2017). For example, in online fair division with probabilistic information, we lose some of our tractable results even for the LIKE mechanism (Aleksandrov and Walsh, 2017a). However, it is still interesting to see how the competitive performance of our online mechanisms changes in such probabilistic settings. A good starting point for this line of research is the survey of Mehta (Mehta, 2013).

We moreover argue that some of our results apply to more general utility models beside additive and monotone preferences. For example, some results hold for additive and non-monotone preferences, e.g. agents can have negative, zero or positive utilities for items. As another example, some results hold for non-additive preferences with complementarities or substitutabilities, e.g. I like both shoes and not just one of them, I have some value for one car but have no value for two cars, etc. These features are realistic but add to the complexity of our model and does not bring anything new and fruitful to our worst-case analyses. They can only make our tractable results intractable. Finally, for this reason, we believe that the simplicity of our model and mechanisms is a key feature of our work.

Chapter 9

Conclusions

In Chapter 1, I introduce offline fair division and online fair division. In (offline) fair division, there is a set of agents and a set of items, and agents have preferences for items. The problem is to decide how to allocate the items to the agents by taking into account their preferences supposing complete information about the problem. In online fair division, this assumption is relaxed. The items now arrive in some order and need to be allocated promptly to agents supposing no information about the future items, or even if any more items will arrive. Online fair division is a natural extension of (offline) fair division because most practical problems are online.

In Chapter 2, I present the model for online fair division. I further make the following assumptions: the agents are *known*, the items are *indivisible*, the order in which they arrive is *strict* and *deterministic* (i.e. a single item arrives with probability 1 in each moment in time), the information about the items is either *complete* or *none*, and the utilities of agents are *additive* and *monotone*. I also define three common (offline) and three new (online) axiomatic properties: (online) strategy-proofness, (online) envy-freeness and (online) Pareto efficiency. Strategy proofness implies online strategy proofness. Online envy-freeness is equivalent to envy-freeness. Online Pareto efficiency is equivalent to Pareto efficiency.

In Chapter 3, I propose two existing - LIKE and BALANCED LIKE - and four new - ONLINE SERIAL DICTATORSHIP, ONLINE RANDOM PRIORITY, PARETO LIKE and MAXIMUM LIKE mechanisms for online fair division. I report a number of axiomatic characterizations and show that some of these mechanisms indeed are characterizing. The LIKE mechanisms characterizes many envy-free mechanisms in general, and all such strategy-proof mechanisms. The BALANCED LIKE characterizes bounded envy-freeness ex post with binary utilities. I also show that the BALANCED LIKE mechanism is online strategy-proof. The PARETO LIKE characterizes completely Pareto efficiency ex post and partially Pareto efficiency ex ante. The ONLINE SERIAL DICTATORSHIP mechanism characterizes strategy-proofness, Pareto efficiency ex post and ex ante. The ONLINE RANDOM PRIORITY mechanism characterizes strategy-proofness, Pareto efficiency ex post and envy-free ex ante. In (offline) fair division, envy-freeness and Pareto efficiency ex ante can be achieved simultaneously. In online fair division, we show that this is impossible and no mechanism can be envy-free and Pareto efficiency ex ante at the same time.

In Chapter 4, I study computational problems related to these mechanisms. The LIKE mechanism is tractable in the sense that computing its outcomes can be done in polynomial time. The BALANCED LIKE and PARETO LIKE mechanisms are intractable in the general case: BALANCED LIKE even with simple binary utilities and PARETO LIKE with general utilities. Their outcomes can be computed in exponential time. There are however some tractable cases for these two mechanisms, e.g. 2 agents for BALANCED LIKE and binary utilities for ONLINE SERIAL DICTATORSHIP. Computing outcomes with both mechanisms is tractable with identical utilities.

In Chapter 5, I turn attention to the pure Nash equilibria of these mechanisms. The ONLINE SERIAL DICTATORSHIP, ONLINE RANDOM PRIORITY and LIKE mechanisms are strategy-proof. Hence, with these mechanisms, agents more likely will report their sincere utilities for items. The PARETO LIKE, MAXIMUM LIKE and BALANCED LIKE mechanisms are not strategy-proof. With these mechanisms, agents have an incentive to misreport their sincere utilities for items. The PARETO LIKE mechanism admits a unique and tractable pure Nash equilibrium whose outcome coincides with the outcome returned by the LIKE mechanism. The same holds for the MAXIMUM LIKE mechanisms. The BALANCED LIKE mechanism admits multiple and intractable pure Nash equilibria whose outcomes can be very different. Consequently, agents might give up manipulating it which is perhaps a good news because this mechanism bounds the envy between them.

In Chapter 6, I discuss fair and efficient outcomes returned by these online mechanisms. Each of the ONLINE SERIAL DICTATORSHIP, ONLINE RANDOM PRIORITY, PARETO LIKE and MAXIMUM LIKE mechanisms is guaranteed to return Pareto efficient outcomes. For this reason, I mainly look at how to use computation in order to identify such outcomes returned by the LIKE and BALANCED LIKE mechanisms. Most of my results are related to similar problems in (offline) fair division. However, there are some fundamentally new results. For example, in (offline) fair division, deciding if a Pareto efficient ex post allocation exists is tractable. By comparison, in online fair division, this decision problem is intractable in general.

In Chapter 7, I perform competitive analyses of these online mechanisms supposing that there is an oracle that supports their decisions by giving to mechanisms a fixed number of advice bits. Basically, I show that the MAXIMUM LIKE, LIKE and BALANCED LIKE mechanisms are most competitive for three common welfare notions, i.e. the utilitarian, egalitarian and Nash welfares. These mechanisms perform equally well in the case when agents have identical utilities for items.

In Chapter 8, I discuss how I could relax some of the initial assumptions. For example, I already have some results for the case when the ordering of items is not fixed or strict, or when the preferences are non-additive or non-monotone. I also look at the case with divisible items.

In conclusion, I extensively compared my results and mechanisms against existing results and mechanisms from the research literature. Based on my results, I can suggest to the chair to use the ONLINE SERIAL DICTATORSHIP, ONLINE RANDOM PRIORITY, PARETO LIKE or MAXIMUM LIKE mechanism if they want a Pareto efficient outcome, and the LIKE or BALANCED LIKE mechanism if they want a fair outcome. Strategic behavoir suggests that agents prefer envy-freeness to Pareto efficiency. For this reason, we believe that mechanisms such as LIKE and BALANCED LIKE are likely to be preferred to mechanisms such as ONLINE SERIAL DICTATORSHIP, ONLINE RANDOM PRIORITY, PARETO LIKE and MAXIMUM LIKE.

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