Proof. $(v) \Rightarrow (iii), (iv)$
$U_{x,x} \overset{(c)z=1}{=} \sum_y p_{x,y} \underbrace{F_{y,x}}_{1} = 1$ for all x

$(iv) \Rightarrow (v)$
fix y. Then $h(x) = F_{x,y}$. By (d), $h(x) = \sum_w p_{x,w} h(w)$ $x \neq y$. By (c), $F_{y,y} = 1 = U_{y,y} = \sum_w p_{y,w} F_{w,y}$ Then $h(y) = \sum_w p_{y,w} h(w)$ $Ph = h$ for $0 < h \leq 1 = h(y)$ Thus, h is constant and $h \equiv 1$

$\mathbb{E}_x(t^x) = U'_{x,x}(1-)$. Then there exists an x such that $U'_{x,x}(1-) < \infty \iff \forall x$. $M_{x,y} \geq \frac{G_{x,x}(z)}{G_{y,y}(z)} = \frac{1-U_{y,y}(z)}{1-U_{x,x}(z)}$

$\mathcal{X}$ finite P irreducible, then it is positive recurrent $\mathbb{E}_x(t^x) < \infty$ for all x

$$\sum_y G_{x,y}(z) = \frac{1}{1-z}$$

Since $z \to 1-$, there exists a y such that $G_{x,y} = \infty$ is recurrent

$$1 = \sum_y F_{x,y}(z) \frac{1-z}{1-U_{y,y}(z)}$$

since $z \to 1-$,

$$1 = \sum_y 1 \frac{1}{\mathbb{E}_y(t^y)}$$

There exists a y such that $\mathbb{E}_y(t^y) < \infty$, thus for all y.

$$\nu_y = \frac{1}{\mathbb{E}_y(t^y)} > 0$$

probability distribution on $\mathcal{X}$. □

Theorem 0.1 (Ergodensatz für endliche irreduzible MK)**.** *Let $f : \mathcal{X} \to \mathbb{R}$. Then for every starting distribution*

$$\frac{1}{n} \sum_{k=0}^{n-1} f(X_k(\omega)) \overset{a.s.}{\to} \int_{\mathcal{X}} f d\nu = \sum_x p(x) \nu_x$$

In particular, ν is the unique stationary probability measure.

Proof. Let μ be the starting distribution. Then define a sequence of stopping times as follows: Choose a fix $y \in \mathcal{X}$.

$$\tau_0 = 0$$
$$\tau_n = \inf\{k > \tau_{n-1} : X_k = y\}$$

Thus (Note that start distribution is μ),

$$\tau_1 = t^y < \infty \text{ a.s.} \tau_n \qquad < \infty \text{ a.s.}$$

Look at the increments: $\sigma_n = \tau_n - \tau_{n-1}$ for $n \geq 1$. □

Lemma 0.2. *Let $(\sigma_n)_{n\in\mathbb{N}}$ be independent. And for $n \geq 2$ let $(\sigma_n)_{n\in\mathbb{N}}$ be distributed like t^y at the start in y.*

Proof. By "Händewackeln" $\square$

Continue proof of theorem:

Proof. $\tau_n = \sigma_1 + \sum_{k=2}^n \sigma_k$ We know $\mathbb{E}(\sigma_k) = \mathbb{E}_y(t^y) < \infty$. Then by LLN,

$$\frac{1}{n}\tau_n = \frac{\sigma_1}{n} + \frac{n-1}{n}\frac{1}{n-1}\sum_{k=2}^n \sigma_k$$

Then this tends to $\mathbb{E}_y(t^y)$ almost surely. (for all $\omega \in \Omega_0 \in \mathcal{A}$: $\mathbb{P}_\mu(\Omega_0) = 1$)

Therefore, τ_n tends to ∞ for $n \in \mathbb{N}$. For any $n \in \mathbb{N}$ there exists a $k_n = k_n(\omega)$ such that $\underbrace{\tau_{k_n}(\omega)}_{\tau_{k_n(\omega)}(\omega)} \leq n < \tau_{k_n+1}(\omega)$. If $n \to \infty$, then $k_n(\omega) \to \infty$.

$$\frac{\tau_{k_n}}{k_n} \leq \frac{n}{k_n} \leq \frac{\tau_{k_n+1}}{k+1}\frac{k_n+1}{k_n}$$

For $n \to \infty$ $\frac{\tau_{k_n}}{k_n} \to \mathbb{E}_y(t^y)$ and $\frac{\tau_{k_n+1}}{k_n+1} \to \mathbb{E}_y(t^y)$ and $\frac{k_n+1}{k_n} \to 1$ Thus, $\frac{k_n}{n} \to \frac{1}{\mathbb{E}_y(t^y)}$.

$$V_k^y = \nVdash_{[X_k=y]}$$
$$\sum_{k=1}^n V_k^y = k_n$$
$$\implies \frac{1}{n}\sum_{k=1}^n V_k^y \to \frac{1}{\mathbb{E}_y(t^y)} \text{ a.s.}$$

Whether we consider $\sum_{k=1}^n V_k^y$ or $\sum_{k=0}^n V_k^y$ does not influence the Grenzwert

Let $f : \mathcal{X} \to \mathbb{R}$. Then

$$f = \sum_{y\in\mathcal{X}} f(y)\nVdash_y$$
$$f(X_k) = \sum_y f(y)V_k^y$$

$$\frac{1}{n}\sum_{k=0}^{n-1} f(X_k) = \frac{1}{n}\sum_{k=0}^{n-1}\sum_{y\in\mathcal{X}} f(y)V_k^y$$

Since we have finite sums we can exchange them

$$= \sum_{y\in\mathcal{X}} f(y)(\frac{1}{n}\sum_{k=0}^{n-1} V_k^y) \to \sum_{y\in\mathcal{X}} f(y)\frac{1}{\mathbb{E}_y(t^y)} \text{ a.s.}$$

$$\mathbb{E}_\mu(\frac{1}{n}\sum_{k=0}^{n-1} V_k^y) \overset{n\to\infty}{\to} \mathbb{E}_\mu(\nu_y) = \nu_y$$

is true by Lebesgue!

Let $\mu = \delta_x$.

$$\frac{1}{n}\sum_{k=0}^{n-1} p_{x,y}^{(k)} \to \nu_y$$

for all y

$$\frac{1}{n}\sum_{k=0}^{n-1} \delta_y P^k \to \nu$$

This results in

$$\frac{1}{n}\sum_{k=0}^{n-1} \mu P^k \to \nu$$

for all μ starting distribution

Let $\mu P = \mu$, then

$$\frac{1}{n}\sum_{k=0}^{n-1} \mu P^k = \mu \to \nu$$

$\mu = \nu$. □

Remark. *Wenn "Ergdoensatz für endl. pos rekurrent irred MK", dann bei GW und summen vertauschungen aufpassen $f \in L^1(\mathcal{X}, \nu)$ und*

$$\sum_x |f(x)|\nu_x < \infty$$

Schwieriger Fall: $\sum_{y\in\mathcal{X}} f(y)(\frac{1}{n}\sum_{k=0}^{n-1} V_k^y)$ wenn f keinen endlichen Träger hat (allg Ergodensatz)