Mathematical foundation of information theory

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Contents

Introduction: Probability Theory

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Definitions and Axioms

Definition (Probability space)**.** *A* probability space *is a triple* $(\Omega, \mathcal{A}, \mathbb{P})$ *where*

1. $\Omega \neq \emptyset$ *is a* set*,*
2. $\mathcal{A}$ *is a* σ-algebra *of subsets of* Ω*,*
3. $\mathbb{P}$ *is a* probability measure *on* $\mathcal{A}$*.*

Remarks. *(a) The set* Ω *is called the "sample space" and contains "all possible aspects/cases related to an experiment".*

(b) By definition, $\mathcal{A} \subset \mathcal{P}(\Omega)$ *where* $\mathcal{P}(\Omega)$ *is the collection of all subsets of* Ω*. The set* $\mathcal{P}(\Omega)$ *is called the power set of* Ω*.*

(c) The elements $A \in \mathcal{A}$ *are called "events". Events are observable (distinguishable) outcomes of an experiment.*

Examples. *(a) one coin toss:* $\Omega = \{0, 1\}$

(b) 500 *coin tosses:* $\Omega = \{0, 1\}^{500}$

(c) one coin toss + final position of coin (+ number of spins):

$$\Omega = \{0, 1\} \times \mathbb{R}^2 (\times \mathbb{N}_0)$$

Definition (Axioms)**.** *Let* $(\Omega, \mathcal{A}, \mathbb{P})$ *be a probability space with* $\mathbb{P} : \mathcal{A} \to [0, 1]$*. Then we define the following axioms:*

1. $\emptyset \in \mathcal{A}$
2. $A \in \mathcal{A} \implies A^C \in \mathcal{A}$
3. *If* $(A_n)_{n \in \mathbb{N}}$ *is a sequence in* $\mathcal{A}$*, then*
$$\bigcup_{i=1}^{\infty} A_n \in \mathcal{A}.$$
4. $\mathbb{P}(\Omega) = 1$
5. σ*-additivity: If* (A_n) *is a sequence in* $\mathcal{A}$ *such that* $A_k \cap A_l = \emptyset \, \forall k \neq l$ *then*
$$\mathbb{P}(\biguplus_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mathbb{P}(A_n)$$

Examples. *Let's consider the examples (b) and (c) from above.*

(b) 500 *coin tosses:*

$$\begin{aligned} A &= [\textit{even number of } 1s] \\ &= \{\omega = (\omega_1, \dots, \omega_{500}) \in \Omega : \omega_1 + \dots + \omega_{500} \textit{ is even}\}. \end{aligned}$$

(c) one coin toss + final position of coin + number of spins:

$$\begin{aligned} A &= [\textit{coin toss: } 1, \textit{ positon from origin: distance } < 2m, \textit{ spins: } 13] \\ &= \{1\} \times \mathcal{B}((0,0),2) \times \{13\} \end{aligned}$$

where $\mathcal{B}((0,0),2) = \{(x,y) \in \mathbb{R}^2 : \sqrt{x^2+y^2} < 2\}$.

Remark. fix this remark

1. *If Ω is at most countable: usually $\mathcal{A} = \mathcal{P}(\Omega)$*
2. *If Ω is uncountable, it is not good.*

Example. *Let $\Omega = \mathbb{R}$. Then $\mathcal{A} = \mathcal{B}_{\mathbb{R}}$, where $\mathcal{B}_{\mathbb{R}}$ is the collection of all Borel sets on $\mathbb{R}$. So $\mathcal{A}$ is the smallest σ-algebra over $\mathbb{R}$ which contains all intervals.*

Proposition (Consequences of the axioms)**.** *Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space with $\mathbb{P} : \mathcal{A} \to [0,1]$. Then we can deduce from the axioms the following statements:*

(a) If $A_1, \dots, A_N \in \mathcal{A}$, then $\bigcup_{n=1}^N A_n \in \mathcal{A}$ and since $\mathbb{P}(\emptyset) = 0$,

$$\mathbb{P}(\biguplus_{n=1}^N A_n) = \sum_{n=1}^N \mathbb{P}(A_n).$$

(b) If $(A_n)_{n\in\mathbb{N}}$ is a finite or countable sequence in $\mathcal{A}$, then

$$\bigcap_{n=1}^\infty A_n \in \mathcal{A}.$$

(c) Let $A, B \in \mathcal{A}$. Then

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

(d) Let $A, B \in \mathcal{A}$. If $A \subset B$, then

$$\mathbb{P}(A) \leq \mathbb{P}(B).$$

(e) "Continuity": If $(A_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{A}$ such that $\forall n$ $A_n \subset A_{n+1}$, then

$$\mathbb{P}(\bigcup_{n=1}^\infty A_n) = \lim_{n\to\infty} \mathbb{P}(A_n).$$

Proof. fix/complete proof

(c) Use $A \cup B = A \uplus (B \cap A^C)$. add pic

(e) add pic

$$
\begin{aligned}
A_N &= A_1 \cup (A_2 \setminus A_1) \cup \cdots \cup (A_N \setminus A_{N-1}) \\
&= A_1 \cup \bigcup_{n=2}^{N} (A_n \setminus A_{n-1}) \\
A &= A_1 \uplus \biguplus_{n=2}^{\infty} \underbrace{(A_n \setminus A_{n-1})}_{\in \mathcal{A}}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{P}(A) &= \mathbb{P}(A_1) + \sum_{n=2}^{\infty} \mathbb{P}(An \setminus A_{n-1}) \\
&= \lim_{N \to \infty} (\mathbb{P}(A_1) + \sum_{n=2}^{N} \mathbb{P}(A_n \setminus \mathbb{P}(A_{n-1})) \\
&= \lim_{N \to \infty} \mathbb{P}(A_N)
\end{aligned}
$$

□

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