Advanced and algorithmic graph theory

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1 Introduction and notations

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Let $G = (V, E)$ be a graph. Then V is the *vertex set* of G and E is the *edge set* of G. If

$$E \subseteq V \times V$$

then G is called a *directed graph*. And if

$$E \subseteq \{\{a, b\} : a \neq b, a, b \in V\}$$

then G is called an *undirected graph*.
A *trivial graph* is the empty graph $G = (\emptyset, \emptyset)$.

We will always consider the case of undirected graphs if not specified otherwise.

Definition (Order of G)**.** The *order of* G is denoted by $|G| := |V|$. We assume that V is finite if not otherwise specified. And we denote by $\|G\| := |E|$.

Notation (Edges)**.** Edges are denoted by $\{i, j\}$, (i, j), or ij. If $e = \{i, j\} \in E$, then

(a) i and j are *adjacent*,

(b) i is *incident* to e (or i and e are incident),

(c) i and j are *neighbours*.

Definition (Complete graph)**.** A graph $G = (V, E)$ is called a *complete graph* if and only if

$$E = \{\{a, b\} : a \neq b, a, b \in V\}.$$

It is called K_n if $|V| = n$.

Definition (Independet or stable set)**.** A *set of vertices* $A \subseteq V$ is called *independent* or *stable* if and only if

$$\forall a, b \in A : \{a, b\} \notin E$$

Definition (Isomorphic)**.** Two graphs $G = (V, E)$ and $G' = (V', E')$ are *isomorphic* if and only if there exists a bijective map $\varphi : V \to V'$ such that for all $a, b \in V$

$$\{a, b\} \in E \iff \{\varphi(a), \varphi(b)\} \in E'.$$

Then φ is called an *isomorphism* and we write $G \equiv G'$.

Definition (Graph property)**.** A class of graphs that is closed under isomorphisms is called a *graph property*.

Example (Triangle)**.** Let $G = K_3$. Then $G' \equiv G$ implies that G' is a triangle. Another example would be K_4.

add pic

Definition (Graph invariant)**.** A mapping taking graphs as arguments is called a graph invariant if and only if it assigns equal images (values) to isomorphic graphs.

Examples.

1. Number of vertices,
2. Number of edges,
3. Longest number (cardinality of longest clique) of pairwise adjacent vertices.

Definition (Clique)**.** A subset $A \subseteq V$ is called a *clique* if and only if

$$\forall a, b \in A,\ a \neq b \implies \{a, b\} \in E.$$

Definition (Union and intersection of graphs)**.** Let G and G' be two graphs. Then we define

1. the *union* of two graphs as

$$G \cup G' \coloneqq (V \cup V', E \cup E')$$

2. the *intersectoin* of two graphs as

$$G \cap G' \coloneqq (V \cap V', E \cap E')$$

If $G \cap G' = (\emptyset, \emptyset)$, we say G and G' are *disjoint*.

Definition (Subgraphs)**.**

1. If $V \subseteq V'$ and $E \subseteq E'$, we say G is *subgraph* of G' and write $G \leq G'$.
2. If $G \leq G'$ and $G \neq G'$, we say G is a *proper subgraph* of G'.
3. If $G \subseteq G'$ such that

$$\forall a, b \in V(G) : \{a, b\} \in E' \implies \{a, b\} \in E,$$

then G is an *induced subgraph.* We say $V \coloneqq V(G)$ *induces or spans* G in G' and denote it by $G'[V]$. add pic

Definition (Adding/removing vertices or edges in/from graphs)**.** Let $G = (V, E)$ and $G' = (V', E')$ be graphs.

(a) If $U \subseteq V(G)$, we write

$$G - U \coloneqq G[V \setminus U].$$

If $U = \{v\}$, we write $G - v$ instead of $G - \{v\}$. add pic

(b) If $G' \subseteq G$, we write $G - G' \coloneqq G - V(G')$ add pic

(c) If $F \subseteq E$, we write

$$G + F \coloneqq (V, E \cup F)$$

and

$$G - F \coloneqq (V, E \setminus F).$$

If $F = \{e\}$, we write $G + e$ instead of $G + \{e\}$ and $G - e$ instead of $G - \{e\}$.

Definition (Edge maximal with respect to a given graph property)**.** A graph G is called *edge maximal with respect to a given graph property* if and only if G itself has the property, but no graph

$$G + \{x, y\}$$

has the property for some $x, y \in V(G)$, $x \neq y$ with $\{x, y\} \notin E(G)$.

Example. Let G be a graph with property P, where P = "triangle free".

(a) add pic

(b) add same pic

Both graphs are maximal with respect to P.

Remark. If we call a graph *minimal or maximal with respect to some property* without any other specification of the order, then it is meant to be according to the subgraph relation.

Definition (Product of graphs)**.** If G and G' are disjoint, define $G * G'$ as a graph obtained from

$$G \cup G' = (V(G) \cup V(G'), E(G) \cup E(G'))$$

by adding all edges $\{x, y\}$ with $x \in V(G)$ and $y \in V(G')$. add pic

Definition (Complement graph)**.** The *complement* of G is denoted by G^C or $\overline{G}$ and is defined as

$$\overline{G} := (V(G), \{\{a, b\} : a \neq b,\ a, b \in V(G)\} \setminus E(G))$$

Definition (Line graph)**.** The line graph of G is denoted by

$$L(G) = (E(G), \{\{e, f\} : e, f \in E, e \neq f, e \cap f \neq \emptyset\})$$

add pic 7

Definition (Degree of G)**.** Denote the set of neighbours of a vertex $v \in V$ by $N_G(v)$. Then we define $\deg_G(v) \equiv d_G(v) := |N_G(v)|$ as the *degree* of v in G. If $d_G(v) = 0$ we say that v is *isolated in* G.
We define

1. the *minimum degree of* G as

$$\delta(G) = \min_{v \in V(G)} d_G(v)$$

2. the *maximum degree of* G as

$$\Delta(G) = \max_{v \in V(G)} d_G(v)$$

3. the *average degree of* G as

$$d(G) = \frac{1}{|V(G)|} \sum d_G(v)$$

Definition (k-regular graph)**.** A graph G is k-regular if and only if

$$\deg_G(v) = k$$

for all $v \in V$ and for some $k \in \mathbb{N}_*$.
If $k = 3$, we call G cubic.
We define

$$\varepsilon(G) \coloneqq \frac{|E|}{|V|}.$$

Definition (Path)**.** A path is a nonempty graph $P = (V, E)$ of the form

$$V = \{x_0, x_1, \ldots, x_k\}$$

and

$$E = \{x_0x_1, x_1x_2, \ldots, x_{k-1}x_k\}$$

where all edges are all pairwise distinct. The vertices x_0 and x_k are the *end vertices* of P. And the vertices x_i for $1 \leq i \leq k-1$ are the inner vertices of P.

Definition (Length of path)**.** Let $P = (V, E)$ be a path. The *length of the path* is defined as the number of edges $|E|$. A path of length k is denoted by P^k. (Notice that $k = 0$ is possible) add pic

Remark. We often refer to a path P^k as $x_0x_1 \ldots x_k = P^k$.

Notation. Let $P = x_0x_1 \ldots x_k$. We write

$$\begin{aligned} Px_i &\coloneqq x_0 \ldots x_i \\ x_iP &\coloneqq x_i \ldots x_k \\ x_iPx_j &\coloneqq x_i \ldots x_j \end{aligned}$$

Let $\mathring{P} \coloneqq x_1x_2 \ldots x_{k-1}$. Then we write

$$\begin{aligned} \mathring{P}x_i &\coloneqq x_0 \ldots x_{i-1} \\ x_i\mathring{P} &\coloneqq x_{i+1} \ldots x_k \\ x_i\mathring{P}x_j &\coloneqq x_{i+1} \ldots x_{j-1} \equiv x_{i+1}Px_{j-1} \text{ for } i+1 \leq j \end{aligned}$$

add pic

Definition (A-B-path)**.** Let $A, B \subseteq V(G)$. A path $P = x_0x_1 \ldots x_k$ is callled an *A-B-path* if

$$V(P) \cap A = \{x_0\}$$

and

$$V(P) \cap B = \{x_k\}.$$

If $A = \{a\}$ and $B = \{b\}$ write *a-b-path* instead of $\{a\}$-$\{b\}$-path. add pic

Definition (Independent path)**.** Two or more paths are *independent* if and only if none of them contains as inner vertex an inner vertex of some other path.

Example. The paths $P_1 = x_0x_1x_2x_3$ and $P_2 = y_0y_1y_2y_3$ are independent. add pic
If the path $P_3 = x_0x_2y_2$ is added, they are not an independent set of paths anymore.

Definition (H-path)**.** Let H be a given graph. We call a path P an *H-path* if P is non-trivial and

$$V(P) \cap V(H) = \{x_0, x_k\}$$

where x_0 and x_k are the end vertices of P.

Definition (Cycle)**.** If $P = (x_0, x_1, \ldots, x_{k-1})$ is a path and $k \geq 3$, then $C = P + \{x_{k-1}, x_0\}$ is called a cycle. Its length is k and we denoted it by C^k.

Definition (Girth and circumference)**.** Let G be a graph.

(a) The minimal length of a cycle in G is the *girth* (german: Taillenweite) $\mathrm{g}(G)$ of G.

(b) The maximal length of a cycle in G is the *circumference* $\mathrm{c}(G)$ of G.

If G has no cycle at all then $\mathrm{g}(G) = \infty$ and $\mathrm{c}(G) = 0$.

Definition (Chord)**.** Let $C^k = x_0x_1 \ldots x_{k-1}$ be a cycle in a graph G. An edge $\{x_i, x_j\}$ with $1 \leq i \neq j \leq k-1$ joining two vertices of C^k such that $\{x_i, x_j\} \notin E(C^k)$ is called a *chord.* An induced cycle in G is a cycle without chords. add pic

Proposition 1.1. *Every graph contains a path of length $\delta(G)$ and a cycle of length $\delta(G) + 1$, provided that $\delta(G) \geq 2$.*

Proof. Homework: Consider longest path $\square$ add pic

Definition (Distance and diameter)**.** Let G be a graph.

(a) The *distance of two vertices* $x, y \in V(G)$ is the length of the shortest x-y-path denoted by $\mathrm{dist}_G(x, y)$. Set $\mathrm{dist}_G(x, y) = \infty$ if there is no x-y-path in G.

(b) The diameter of G is defined as

$$\mathrm{diam}(G) \coloneqq \max_{x,y \in V(G)} \mathrm{dist}_G(x, y).$$

Proposition 1.2. *Every graph containing a cycle satisfies*

$$\mathrm{g}(G) \leq 2\,\mathrm{diam}(G) + 1.$$

Proof. Let C be a shortest cycle in G. If add pic

$$\mathrm{g}(G) \geq 2\,\mathrm{diam}(G) + 2,$$

then there exist $x, y \in V(C)$ such that

$$\mathrm{dist}_C(x, y) \geq \mathrm{diam}(G) + 1.$$

In G the condition $\mathrm{dist}_G(x, y) \leq \mathrm{diam}(G)$ holds, so any shortest path P between x, y is not a subgraph of C. Thus P contains a C-path $x'Py'$. Use $x'Py'$ and the shortest x'-y'-path in C to construct a cycle C' strictly shorter than C `. $\square$

03.04.2016

Definition. A vertex v of G is called central if and only if the greatest distance of v from any other vertex in G is as small as possible

$$\min_{x\in V(G)} \max_{y\in V(G)} \operatorname{dist}(x,y) = \max_{y\in V(G)} \operatorname{dist}(v,y).$$

Radius of G

$$\operatorname{rad}(G) \coloneqq \min_{x\in V(G)} \max_{y\in V(G)} \operatorname{dist}_G(x,y)$$

Observe:

$$\operatorname{rad}(G) \le \operatorname{diam}(G) \le 2\cdot \operatorname{rad}(G)$$

(Exercise or homework)

Proposition 1.3. *If* $\operatorname{rad}(G) \le k$ *and* $\Delta(G) \le d$ *with* $d \ge 3$*, then*

$$|V(G)| \le \frac{d}{d-2}(d-1)^k.$$

Proof. Let z be a central vertex in G and let D_i be the set of vertices of distance i from z.

$$V(G) = \bigcap_{i=0}^{k} D_i \quad |V(G)| = \sum_{i=0}^{k} |D_i| \tag{1}$$

$$|D_0| = 1 \quad D_0 = \{z\} \tag{2}$$

$$|D_1| \le d \tag{3}$$

$$|D_{i+1}| \le (d-1)|D_i| \, \forall 1 \le i \le k-1 \tag{4}$$

$$[|D_i| \le (d-1)^{i-1} d] \tag{5}$$

(4) because every $x \in D_{i+1}$ is a neighbour of some $y \in D_i$ and $\deg(y) \le d$ and there exists a neighbour of y in D_{i+1}, therefore add pic

$$\deg_{D_{i+1}}(y) \le d-1$$

Plug (4) in (1):

$$\begin{aligned} |V(G)| &= 1 + \sum_{i=1}^{k} |D_i| \\ &\le 1 + \sum_{i=1}^{k} d(d-1)^i = 1 + d\sum_{i=0}^{k-1}(d-1)^i \\ &= 1 + d\frac{(d-1)^k - 1}{d-2} = 1 + \frac{d(d-1)^k}{d-2} - \frac{d}{d-2} \\ &\le \frac{d(d-1)^k}{d-2} \end{aligned}$$

Analogously, one gets lower bounds for $|V(G)|$ if $\delta(G)$ and $\operatorname{rad}(G)$ or $\operatorname{g}(G)$ are large. Similar results also if $\delta(G)$ is replaced by $\operatorname{d}(G)$ $\square$

Theorem 1.4 (Alon, Hoory and Linial 2002)**.** *Let G be a graph with* $\mathrm{d}(G) \geq d \geq 2$ *and* $\mathrm{g}(G) \geq g$, $g \in \mathbb{N}$, *then*

$$|G| = |V(G)| \geq n(d,g),$$

where $n(d,g) := 1 + d\sum_{i=0}^{r-1}(d-1)^i$ *if* $g = 2r+1$ *(odd) and* $n(d,g) := 2\sum_{i=0}^{r-1}(d-1)^i$ *if* $g = 2r$ *(even).*

Proof. No proof. □

A *walk of length k* in a graph G is a non-empty alternating sequence of vertices and edges $v_0, e_0, v_1, e_1, \ldots, e_{k-1}, v_k$ where $e_i = \{v_i, v_{i+1}\} \in E(G)$ for all $0 \leq i \leq k-1$ if $v_0 = v_k$ the *walk is closed.* A walk with parwise distinct vertices is a path.

1.1 Connectivity

A non-empty graph is *connected* if any two vertices x and y are joined by a x-y-path in G. If $U \subseteq V(G)$ and $G[U]$ is connected, we say *"U is connected in G".*

Proposition 1.5. *The vertices of a connected graph G can always be enumerated, say as $v_1, v_2, \ldots, v_n$ where $n := |G|$ such that*

$$G_i = G[v_1, v_2, \ldots, v_i]$$

is connected for all $1 \leq i \leq n$.

Proof. No proof. □

add pic

Definition. A maximal connected subgraph is called a (*connected*) *component* of G.

Notice: A component is always non-empty.

Definition. If $A, B \subseteq V$ and $X \subseteq V \cup E$ are such that ever A-B-path contains an element from X, then we say *"X separates A and B in G".* If $X \subseteq V(G)$ and X seperates A and B, we call X a *separator.* add pic

Definition. A vertex v which separates two other vertices of the same component is called a *cut vertex.* An edge separating its end vertices is called a *bridge.*

add pic 2 with color

Definition. A graph G is called *k-connected* if $|G| > k$ and $G - X$ is connected for all $X \subset V(G)$ with $|X| \leq k-1$.

(a) 0-connected: singleton, actually ever graph but the $(\emptyset, \emptyset)$.

(b) 1-connected: a connected graph G with $|V(G)| > 1$.

The *connectivity* of G, $\kappa(G)$, is the largest $k \in \mathbb{N}_*$ such that G is k-connected. A graph G is called 1-*edge-connected* if $|G| > 1$ and $G - F$ is connected, for all $F \subseteq E$ with $|F| \leq l-1$. The *edge connectivity* of G, $\lambda(G)$, is the largest $l \in \mathbb{N}_*$ such that G is l-edge-connected.

Theorem 1.6 (1.6 Whitney, 1932)**.** *If G is non-trivial, $|G| > 1$, then $\kappa(G) \leq \lambda(G) \leq \delta(G)$.*

Proof. The inequality $\lambda(G) \leq \delta(G)$ is trivial. add pic
We prove $\kappa(G) \leq \lambda(G)$.
Let $w \in V(G)$ be such that $\mathrm{d}(w) = \delta(G)$. Let F be any minimal subset of $E(G)$ such that $G - F$ is disconnected.
We show $\kappa(G) \leq |F|$ (this implies $\kappa(G) \leq \lambda(G)$)

Case 1: There exists a vertex v in G which is not incident with some edge in F. Let C be the component of $G - F$ containing v. Consider vertices of C add pic which are incident with an edge in F. They separate v from $G - C$ and are at most $|F|$. $\implies \kappa(G) \leq |F|$.

Case 2: Every vertex $v \in V(G)$ is incident to some $e \in F$. neighbours w of v with add pic $\{v, w\} \notin F$ belong all to C. The number of such neighbours is $\leq |F|$. Thus $\mathrm{d}_C(v) \leq |F|$. $N(v)$ is a separator of v from $G - \{v\}$ and therefore $\kappa(G) \leq |N(v)| \leq |F|$.

□

High connectivity need high minimum degree! The converse is in general not true, i.e. there are graphs G with "large" $\delta(G)$ but "small" $\kappa(G)$, $\lambda(G) \leftarrow$ exercises
High $\delta(G)$ implies however the existence of a *subgraph* of high connectivity.

Proposition 1.7 (Mooler 1972)**.** *Let $k \in \mathbb{N}$, $k \neq 0$. Ever graph G with $\mathrm{d}(G) \geq 4k$ has a $(k+1)$-connected subgraph H such that*

$$\varepsilon(H) = \frac{|E(H)|}{|V(H)|} > \varepsilon(G) - k.$$

(where ε is the density of G).

2 Connectivity

2.1 The theorem of Menger and Consequences

Duality between *connecting and dividing.*

Theorem 2.1 (Menger 1927)**.** *1. edge version:*
Let s and t be two vertices in graph G. Then the maximum number of s-t-paths in G which share no (pairwise) edges equal the minimal cardinality of a set of edges which separates s and t.

2. vertex version:
Let s and t be two vertices in G such that $\{s, t\} \notin E(G)$. Then the maximal number of independent s-t-paths in G equals the minimal cardinality of a set of vertices which separates s and t in G.

Proof. The proof is based on the proof of the max-flow-min-cut-duality.
(edge version)
max ... = min... is trivial
max ... $\geq$ min ... will be proven trivial case: $\implies$ $|F| \geq$ number of paths, for al F which separates s and t. Therefore, min($|F|$) $\geq$ max(number of paths).
We show $\geq$: Transform G in network $N = (V, A, s, t, c)$ with $V := V(G)$ and A set of arcs, c capacity.

add pic

$$\{x,y\} \in E(G) \begin{cases} \leftarrow \{x,y\} \cap \{s,t\} = \emptyset & \text{pic} \\ \leftarrow \{s,y\} & \text{pic} \\ \leftarrow \{x,t\} & \text{pic} \end{cases}$$

In particular
Every s-t-cut in N defines a set of edges separating s and t such that the capacity of the s-t-cut equals the cardinality of the separator set of edges. capacity cut = number of arcs from s part to t part = number of edges in separator set F.
min cap s-t-cut = max value of s-t-flow which is integral (wlog)
Recall: 1)$c \in \mathbb{N}_* \implies \exists$ integral max flow
2) flow decomposition theorem: Every s-t-flow can be represented as follows

add pic for each case

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add pic

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where $\mathcal{P}$ is a set of s-t-paths, $\mathcal{C}$ is a set of cycles in (V, A). $w : \mathcal{P} \cup \mathcal{C} \to \mathbb{R}_+$ and $|\mathcal{P}| + |\mathcal{C}| \leq |A|$

$$f(e) =$$

consider set of s-t-paths P from $\mathcal{P}$ for which $w(P) = 1$. Number of this paths equals value of f.

add pic

□

07.03.2016

Proof. vertex version:
For all $s, t \in V(G)$ $\{s,t\} \notin E(G)$, max number of independent s-t-paths = min cardinality of an s-t-separator

"$\leq$" trivial implies every s-t-separator contains at least one vertex per path implies cardinality of every s-t-sep $\geq k$.
"$\geq$" Assume there exists an s-t- sep with card k. Show that there are at least k independetn s-t-paths.
construct a network $N' = (V', A', s, t, c = 1)$.

add pic

$$V' = \bigcup_{v \in V(G)} \{v^-, v^+\} \cup \{s,t\}$$

add pic

$$\begin{aligned} A' = &\{(v^-, v^+) : v \in V(G)\} \\ &\cup \{(x^+, y^-), (y^+, x^-) : e = \{x,y\} \in E(G), \{x,y\} \cap \{s,t\} = \emptyset\} \\ &\cup \{(s, x^-) : \{s,x\} \in E(G)\} \cup \{(y^+, t) : \{y,t\} \in E(G)\} \end{aligned}$$

Observe: a min s-t-cut $\delta(X)$ in N' contains w.l.o.g. just arcs of the from (v^-, v^+). $\delta(X')$ is again a min cut. add pic add pic

Observe 2: A min s-t-cut of the above type in N' corresponds to an s-t-separator S in G, precisely

$$\mathcal{S} = \{y \in V(G) : (y^-, y^+) \in \delta(X)\} \implies |S| = |\delta(X)| = |c(\delta(X))|$$

This implies min card of an s-t-separator in $G \leq$ min capacity of an s-t-cut in N'. By max-flow-min-cut theorem, min capacity of an s-t-cut in N' = max value of s-t-flow in N' = max value of an integral s-t-flow in $N' \overset{\text{flow decomposition}}{=}$ number of s-t-paths (explanation below) carrying integral flow (= 1 unit of flow) in N' = number of independent s-t-paths in $G \leq$ max number of independent s-t-paths.

different s-t-paths cannot share an edge or a vertex implies independent s-t-paths add pic $\square$

Corollary 1. *The edge connectivity and the (vertex) connectivity can be defined equivalently as follows:*
A graph G with $|G| > k$ is k-connected if and only if

(a) every separator has cardinality $\geq k$, or equivalently

(b) $\forall s, t \in V(G)$ there exist k independent s-t-paths.

A graph G with $|G| > 1$ is l-edge connected if and only if

(a) every set of edges which separates G has cardinality $\geq l$, or equivalently,

(b) $\forall s, t \in V(G)$ there exists at least l edge disjoint s-t-paths.

Proof. edge connect. equiv. def. follows directly from Menger. (vertex) connect. We still need to show existence of $\geq k$ independent s-t-paths for $s, t \in V(G)$ such that $\{s, t\} \in E(G)$. If $\{s, t\} \notin E(G)$, apply Menger.
Assume G is k-connected ($|G| > k$). Let $\{s, t\} \in E(G)$. Show that there exist $\geq k$ independent s-t-paths in G. wlog $k \geq 2$. (If $k = 1$, need to show existence of one s-t-path and this is e.g. the edge.)
Claim: Consider $G - \underbrace{\{s, t\}}_{=:e}$. Every s-t-separator has cardinality $\geq k - 1$. (to prove).
Apply Menger in $G - \underbrace{\{s, t\}}_{e}$ and find $k - 1$ independent s-t-paths in $G - e$. Add edge $\{s, t\}$ as the k-th path add pic add pic
Proof the claim: By contradiction.
Let A be an s-t-separator in $G - e$ with $|A| \leq k - 2$. A does not separate s and t in G. Therefore, $G - A - \{e\}$ has (exactly) 2 connected components X, Y with $s \in X$ and $t \in Y$. $|G - e - A| \geq 3$ implies that there exists a vertex $v \in \{s, t\}$ in $G - e - A$ wlog let v be on the same comp as s. Consider $A \cup \{s\}$. H is a separator of v and t in G. ` because $|A \cup \{s\}| \leq k - 1$ and G is k-connected. $\square$ add pic add pic

Corollary 2 (fan theorem)**.** *Let G be a graph with $|G| \geq k+1$. G is k-connected if and only if $\forall A = \{a_1, a_2, \ldots, a_k\} \subset V, \forall s \in V \setminus A$, there exist a system of independent s-A-paths in G.*

Proof. add pic

$\Leftarrow$) property holds. Show that G is k-connected.
$d_G(v) \geq k$ for all $v \in V(G)$. We show that G is k-connected by showing the existence of at least k independent s-t-paths for all $s, t \in V(G)$ with $s \neq t$. Consider $, t \in V(G)$, $s \neq t$, $|N(g)| \geq k$. Let $A \subseteq N(t)$ with $|A| = k$. add pic
Case1: $s \notin A$ Apply property for s and A and extending the paths by edges from t the end vertices of paths in A. add pic
Case2: $s \in A$ add pic
Case2a: $|N(t)| \geq k+1$ implies existence of $v \in N(t) \setminus A$ Consider s, $(A \setminus \{s\}) \cup \{v\} =: A'$. Thus, there exist k independent s-A-pathsi in G. Extend them to s-t-paths as in Case1. Case2b Apply property for s and $(A \setminus \{s\} \cup \{t\})$ and extend to t those which do not of t just as in Case1 or Case2a add pic

$\Rightarrow$) Assume G is k-connected. show property.
wlog G is not a complete graph (in a complete graph the property holds anyway). Let $A \subset V$, $|A| = k$ and let $s \in V \setminus A$. Add a new vertex t to G which is connected to all vertices in A. Let G' be the resulting graph. add pic
Claim: G' is k-connected.
If the Claim holds, find k independent s-t-paths in G' and remove from those the edges incident to t.
Proof of the claim: Assume G' is not k-connected. Then there exists a separator S with $|S| \leq k-1$.
Case1: $t \in S \implies S \setminus \{t\}$ is separator in G with $|S \setminus \{t\}| \leq k - 2$`$G$ $k$-connected Case2: $t \notin S$ $|S| \leq k-1 \implies S$ is not a separator in $G$ *implies* $G - S$ is connected. $N(t) = A$, $|A| = k$. So $t$ is connected to $G - S$. $S$ is not a separator of $G'$ (does not separate $G$ and does not separate $t$) `. add pic add pic

□

Definition. A set of a-B-paths in a graph G is called a *fan* if the paths have only one vertex a in common. add pic

Theorem 2.2 (Dirac 1960)**.** *Any k-vertices of a k-connected graph, $k \geq 2$, lie on a common cycle.*

Proof. Exercises □

10.03.2016

2.2 2.2 2-connected and 3-connected graphs

10.03.2016

Theorem 2.3 (ear decomposition)**.** *A graph G is 2-connected if and only if the exist a so called ear decomposition, i.e.*

$$G = C \cup P_1 \cup \cdots \cup P_k$$

with $k \in \mathbb{N}$ where C is a cycle and P_i is a path having just its end-vertices in common with

$$C \cup P_1 \cup \cdots \cup P_{i-1}$$

for all $1 \leq i \leq k$. add pic

Proof. $\Leftarrow$) If $G = C \cup P_1 \cup \cdots \cup P_k$ with ... $\implies$ G is 2-connected.
Induction on k. $k = 0$. Then $G = C$ is 2-connected.
Assume it holds for $k = i$. Now for $k = i + 1$: By Menger need to show: $\forall$ such that there exists 4 indep. s-t-paths.

$$s, t \in V(C \cup P_1 \cup \cdots \cup P_i) \checkmark$$

$$\begin{aligned} s, t \in P_{i+1} &\implies \text{let } a, t \text{ be end points of } P_{i+1} \\ &\implies a, b \in V(C \cup P_1 \cup \cdots \cup P_i) \end{aligned}$$

Then extend one of a-b-paths in $C \cup P_1 \cup \cdots \cup P_i$ to an s-t path by adding the ubpaths from a to s and from b to t in P_{i+1} path the s-t-subpath of P_{i+1}. add pic orange 1
$s \in P_{i+1}$, $t \in V(C \cup P_1 \cup \cdots \cup P_i)$ Apply fan theorem in $C \cup P_1 \cup \cdots \cup P_i$ with $t \notin \{a, b\} =: A$ $\implies$ there exists an t-a-path P_1 and a t-b-path P_2 independenin $(C \cup P_1 \cup \cdots \cup P_i)$ Extend then tow independent s-t-paths by adding suitable subpaths of P_{i+1} add pic

$\Rightarrow$) Assume G is a 2-connected graph $\implies$ ther exists ear decompostion $C \cup P_1 \cup \cdots \cup P_i$.
$|G| \geq 3$ $\implies$ $\exists s, t \in V(G), s \neq t$ $\implies$ $\exists 2$ independent s-t-paths in G, they build a cycle C.
Set $G_0 \coloneqq C$ and $i = 1$. If $G_0 \neq G$, there exists an edge $e = \{x, y\}$ / $inE(G_O)$, such that

$$\{x, y\} \cap V(G_0) \neq \emptyset.$$

Case1: $\{x, y\} \subseteq V(G_0)$. Set $P_1 = x, y$ and $G_1 \coloneqq G_0 \cup P_1$.
Case2: wlog $x \in V(G_0), y \notin V(G_0)$ There exist 2-indep x-y-paths in G $\implies$ there exist x-y-path which does not use $\{x, y\} = e$ Then there exists a path P connecting y to $G_0 \setminus \{x\}$. Construct $P_1 \coloneqq P \cup \{x, y\}$. Set $G_1 \coloneqq G_0 \cup P_1$. Ask again is $G_1 = G$. If yes $\checkmark$. If not find again $\{x, y\} \in E(G_1)$ and so on. Process end because G is finite and so does $E(G)$. add pic

$\square$

Definition. A maximal connected subgaph without cut-vertex is called a *block*. add pic

Observe:
1)A block B is 2-connected if $|B| > 2$
$|B| = 2$ vertex-edge-vertex-pic bridge
$|B| = 1$ vertex-pic singleton

Example. add pic

2) If two blocks overlap, then on a cut vertex.
3) Edges of blocks build a partition of $E(G)$.

Definition. The block-cut-vertex-graph $bc(G)$ of a given graph G is a graph with vertex set $\mathcal{B} \cup \mathcal{C}$ where $\mathcal{B}$ is the set of the blocks in G and $\mathcal{C}$ is the set of the cut-verices in G_0. there exists an edge $\{x, y\} \in E(bc(G))$ if and only if $x \in \mathcal{B}$ and $y \in \mathcal{C}$ and $y \in x$ or $y \in \mathcal{B}$, $x \in \mathcal{C}$, $x \in y$.

Theorem 2.4 (Gallai 1964, Horary and Puns 1966)**.** *The $bc(G)$ of a connected graph G is a tree.*

Proof. 2 things need: a) $bc(G)$ is connedted; b) $bc(G)$ is cycle-free

(a) trivial: Let $x, y \in V(bc(G))$

$$x, y \in \mathcal{B}$$
$$x, y \in \mathcal{C}$$
$$x \in \mathcal{B}, y \in \mathcal{C} \text{ or } y \in \mathcal{B}, x \in \mathcal{C}$$

Since G connedted $V(X), V(y)$ are connected in G in a path P. since for all $e \in E(G)$ belongs to some block the block contaionin the edges of P build a path in $bc(G)$.

add pic

(b) Assume by contradiction there exisist a cycle in $bc(G)$

$$B_0 C_0 B_1 C_1 \dots B_k C_k$$

Consider B_1. It is not maximal 2 connected because $B_0 \cup B_1 \cup \dots B_k (\subseteq G)$ is 2-connected and $B_0 \subsetneqq B_0 \cup \dots \cup B_k$ `

add pic

$\square$

Definition. Let $e = \{x, y\}$ be an edge of graph $G = (V, E)$. Denote by $G|e$ the graph obtained from G by contracting edge e, i.e.

$$V(G|e) = (V(G) \setminus \{x, y\}) \cup \{v_e\}$$

where $v_e \notin V(G)$ and

$$\begin{aligned} E(G|e) = &\{\{u, v\} \in E(G) : \{u, v\} \cap \{x, y\} = \emptyset\} \\ &\cup \{\{w, v_e\} : w \in V(G) \setminus \{x, y\} \text{ and } [\{w, x\} \in E(G) \text{ or } \{w, y\} \in E(G)]\} \end{aligned}$$

More generally if X is another graph and $\{V_x : x \in V(X)\}$ is a partition of $V(G)$ into conected subsets (i.e. $G[V_x]$ is connected) such that for all $x, y \in X$ there exists a V_x-V_y-edge in G if and only if $\{x, y\} \in E(X)$, we define $G = MX$. The set V_x are called *branch sets* of G. If $V_x = U \subseteq V$ and all other branch sets are singletons we denote $X = G|U$.

add pic

add pic orange 8

add pic

Proposition 2.5 (2.5)**.** *G is an MX if and only if X can be obtained from G by applying a sequence of edge contractions, i.e., there are graphs $G_0, G_1, \dots, G_k$ and edges $e_i \in G_i$ such that $G_0 := G$, $G_k :\cong X$ and $G_{i+1} := G_i|e_i$, for all $0 \leq i \leq k-1$.*

Proof. Induction on $|G| - |X|$ (Homework!) $\square$

Definition. If $G = MX$ and $G$is a subgaph of Y, $G \subseteq Y$, then we say *X is a minor of Y*.

Example. Since $G \subsetneq Y$ and $G = MX$, then X is a minor of Y. Notation: $X \leq Y$.

add pic

Lemma 2.6 (3.6)**.** *If G is a 3-connected graph and $|G| > 4$, then there exists an edge $e \in E(G)$ such that $G|e$ is 3-connected.*

Proof. By contradiction, assume there exists no such an edge in G. Then for all $\{x, y\} \in E(G)$ $G(\{x, y\}$ has a separating set S with at most 2 vertices.Since G is 3-connected S is not a separator in G. $v_{\{x,y\}} \in S$. $|S| = 2$. Then there exists $z \in S|\{v_{xy}\}$.

add pic

$$T := \{x, y, z\}$$

Every 2 vertices separated by S in $G|\{x, y\}$, are separated by T in G. Thus, every vertice og T hast a neighbour in every component of $G - T$ because otherwise if e.g. z has no neighbour in some componennt C then $T \setminus \{z\}$ would still be a separator of cardinality 2(!!)

add pic

Choose an $\{x, y\}$, a z and a component C such that $|C|$ is as small as possible. Pick $v \in N(z) \cap C$. By assumption $G|\{z.v\}$ is not 3-connected. So we find a separator

$$S_1 = \{v_{zv}, w\},\ T_1 = \{z, v, w\}$$

such that T_1 separates G and as bfore every vertex in T_1 as a neighbour in every component of $G - T_1$. Since x and y are dependent in $G - T_1$. There exists a component D in $G - T_1$ such that $D \cap \{x, y\} = \emptyset$. Every neighbour of v in D lies in C (because v in C and the neighbour cannot be x, y or z) $\implies D \cap C \neq \emptyset \implies D \subsetneq C$

add pic

$D \subset C$: If not, let $t' \in N(v) \cap D$ and let $t \in D$. t and t' are connected in D so also connected in $G - T$. So t, t' belong to the same component in $G - T$ and since $t' \in C$ also $t \in C$.
$D \neq C$: $(v \in C,\ v \notin D)$

$|D| < |C|$` to minimality of C. $\square$

Theorem 2.7 (Tutte 1061)**.** *A graph G is 3-connected if and only if there exists a sequence of graphs $G_0, G_1, \ldots, G_n$ with the properties:*

(a) $G_0 := K_4$, $G_n := G$ *and*

(b) G_{i+1} *has an edge* $e_{i+1} = \{x, y\}$ *with* $\mathrm{d}_{G_{i+1}}(x) \geq 3$, $\mathrm{d}_{G_{i+1}}(y) \geq 3$ *such that* $G_i = G_{i+1}|e_{i+1}$ *for all* $0 \leq i \leq n - 1$

17.03.2016

Theorem 2.8 (Tutte)**.** *G is 3-connected if and only if there exists a sequence $G_0, G_1, \ldots, G_n$ such that $G_0 = K_4$ and $G_n = G$ and for all $0 \leq i \leq n - 1$ exists $\{x, y\} \in E(G_{i+1})$ such that*

$$G_i = G_{i+1}|\{x, y\}$$

with $\mathrm{d}_{G_{i-1}}(x) \geq 3$ *and* $\mathrm{d}_{G_{i+1}}(y) \geq 3$.

Proof. $\Rightarrow$) trivial follows from the lemma and observe that K_4 is the only 3-connected graph on 4 vertices $(3 = k(G) \leq \lambda(G) \leq \delta(G))$

⇐) Given the sequences as above. Prove that G is 3-connected. We show: If $G_9 = G_{i+1}|\{x,y\}$ is 3-connected, then G_{i+1} is 3-connected.
Assume this is not true. Then there exists a separator S in G_{i+1} with $|S| \leq 2$. Let C_1, C_2 be two connected components in $G_{i+1} - S$ and $\{x,y\} \in E(G_i)$. The only possible inclusions for x, y wrt C_1, S, C_2 is $x \in C_1$, $y \in S$ (or $x \in S, y \in C_1$, analogous). (The other caes are excluded because otherwise S with $|S| \leq 2$ would be a separator of G_i (!!), or S would not be a separator in G_{i+1}!!
Then $\forall v \in C_1 \setminus \{y\}$, the S is a separator of v from C_2 in G_i. (`) Thus, $C_1$is a singleton $C_1 = \{y\}$. This implies that $\mathrm{d}_{G_{i+1}}(y) \leq 2$ ` So every G_i in the sequence is 3-connected and so does $G_n = G$.

add pic

□

2.3 2.3 The Computation of $k(G)$ and $\lambda(G)$

Question: $k(G) \geq 1$? is answered e.g. by depth first seqrch (DFS) in $\mathcal{O}(m+n)$ time.
Question: $k(G) \geq 2$? is answered e.g. ....

Main Code DFS(s) for a graph $G = (V, E)$, $s \in V$ Begin
for $v \in V$ do DFS Num(r) =: 0
for $v \in V$ do SEEN(v) =: FALSE
for $v \in V$ do PREC(v) =: v
$t := 0$
SEEN(s) := TRUE
DFS(G,s)
end (main)

Procedure DFS(G,v)
$t := t + 1$
DFS Num(v) := t
for $w \in N(v)$ do
if NOT SEEN(w) then
SEEN(w) = TRUE; PREC(w) = v; DFS(G,w)
end (for)

DFSNum(v): order in which vetices are visisted
SEEN(v): has v been visited up to the current moment
PREC(v): perdedent of v, i.e. vertex from where v was visited

add pic

Proposition 2.9 (2.8?). *Let G be a connected graph and $s \in V(G)$. Apply DFS(G,s).*

(a) The edges of the form (prec(v),v) form a spanning tree T in G.

(b) DFS(G,s) runs in $\mathcal{O}(m+n)$ time.

Call edges (prec(v),v) tree-edges.

The other edges in G are classified into *forward edges* and *backward edges*. An edge $\{v,w\} \in E(G) \setminus E(T)$ is called a forward edge if and only if DFSNum(w)

¿ DFSNum(v) at the moment when DFS passes the edge $\{v, w\}$ for the first time starting at v.
backward edge DFSNum(w) ¡ DFSNum(v)

Lemma 2.10 (2.9?). *(a) There are no forward edges.*

(b) For all backward edges $\{v, w\}$ the path joining s and v in T contains w.

Definition. Let $v \in V(G)$ and T be the spanning tree produced by DFS(G,s). A descendent of v is any $w \in V(G)$ such that v is in the unique s-w-path in T. In this case we call v an ancestor of w. Thus there is an order relation defiinied in $V(G)$ by ancestors and decendents. Choose s as a root in T. Then the root s is the smallest element in this order. For all $v \in V(G)$ its descendents form a tree, which we think of as being rooted at v and denote by T_v.

Definition. For all $v \in V(G)$ set LowPoint(v) $= min\{DFSNum(v) : \{DFSNum(z) : \exists descendant x of v in T, such that \{x, z\} a backward edge\}\}$. It could be that $x = v$. A directed tree edge (v, w) is called a leading edge if and only if LowPoint(w) $\geq DFSNum(v)$.

add pic 3

Lemma 2.11. *Let $\{v, w\}$ be a directed leading edge. Consider the subtree $T_w + \{v, w\}$ of T. Then all backward edges starting in vertices from $T_w \cup \{v, w\}$ lead to $T_w + \{v, w\}$.*

add pic - not necessarily

Theorem 2.12. *For all $v \in V(G) \setminus \{s\}$ and for all directed according to our order relation edges $\{v, w\} \in E(G)$ the following holds: $\{v, w\}$ and the tree edge ending at v belong to the same block of G if and only if either*

(a) $\{v, w\}$ is a backward edge or

(b) $\{v, w\} \in E(T)$ but not a leading edge.

Let us bould the blocks of a connected graph G with theorem 2.11 (or 2.12???)

Theorem 2.13 (2.12). *(a) The root s is a cut-vertex if and only if there exist more than 1 leading edge starting at s*

(b) A vertex $v \in V(G) \setminus \{s\}$ is a cut-vertes if aond only if there exists at least one leading edge starting at v.

Proof. No proof. □

Theorem 2.14 (Even, Tarjan 1972). *The blocks and the cut-vertices as well as the bc(G) of (a connected) graph G can be determined in linear time, i.e. in $\mathcal{O}(n+m)$ time, where $n := |V(G)|$ and $m := |E(G)|$. As a sequence the question "$k(G) \geq 2$?" can be answered in $\mathcal{O}(m + n)$ time.*

Proof. no proof □

Remark. Also "$k(G) \geq 3$?" can be decided in linear time.
Reference: J.E. Hopcroft and R.E. Tarjan, Dividing a graph into 3-connected components, SIAM Journal on computing 2, 1972, 135-158

$k(G) = ?$
From Menger $k(G) = \min\{u_{s,t} : s,t \in V(G), s \neq t\}$, where $u_{s,t} :=$ maximal number of independent a-t-paths in G. $u_{s,t}$ can be determined by solving a max flow problem as in the proof of Mengers's Theorem.

So time complexity: $\mathcal{O}(n^2 \cdot$ time complesxity(max flow)$) = \mathcal{O}(n^2 \cdot \underbrace{n^2\sqrt{m}}_{\text{push-relabel}}) = \mathcal{O}(n^4\sqrt{m})$ But unit capacity max flow solvable in $\mathcal{O}(m \cdot \min\{2n^{\frac{2}{3}}, \sqrt{m}\}) \implies \mathcal{O}(m \cdot n^{\frac{2}{3}})$ with results in a complex time of $\mathcal{O}(n^2 n^{\frac{2}{3}}\sqrt{m})$.

Theorem 2.15 (Even, Tarjan 1975)**.** *$k(G)$ can be computed in $\mathcal{O}(\sqrt{n}m^2)$ time with $n := |V(G)|$ and $m := |E(G)|$.*

No proof

Remark. Just a linear number of $u_{s,t}$ need to be computed)
Reference: S Even and R.E. Tarjan, Network flows and testing graph connectivity, SIAM Journaml Computing, 1975, 507-518.

$\lambda(G) =?$
efficiently computable by flow arguments
Menger $\lambda(G) = \min\{\overline{u_{st}} : s,t \in V(G), s \neq t\}$ where $\overline{u_{st}} =$ maximal number of edge disjoint s-t-paths in G. $\overline{u_{st}}$ can be again compared by solving a max flow problem as in proof of Menger's Theorem.

$\min\{\overline{u_{st}} : s,t \in V(G), s \neq t\}$ is the same as min (overall) int in the anxilary graph; can be also solved without using any flow argument in a time better than $\mathcal{O}(n^2)\times$(complexity of max flow)

Theorem 2.16 (2.15)**.** *$\lambda(G)$ can be computed in $\mathcal{O}(m \cdot n + n^2 \log n)$, where $n = |V(G)|$ and $m = |E(G)|$*

Proof. Proven by: Nagamchi and Ibaraki; 1992, Stoer and Wagner 1997, Frank 1994
Reference: Stoer and Wagner: A simple min cut algorithm, Journal of the ACM 44, 1997, 585-591. □

3 Chapter3: Hamiltonian cycles and Hamiltonian graphs

11.04.2016

Definition. A Hamiltonian cycle in $G = (V,E)$ is a cycle C which contains all vertices in V. (HC) A graph $G = (V,E)$ is called Hamiltonian if and only if it contains an HC.

Proposition 3.1. *The decision problem Hamiltonian Cycle (HC) is NP-complete*

$$HC \begin{cases} \textit{Input Graph } G = (V,E) \\ \textit{Question: Is there a HC in } G? \end{cases}$$

Thus, no hope for nice characterizations of existence of HC. Will give some non-trivial sufficient conditions and some rather simple necessary conditions.

Lemma 3.2. *Let G be Hamiltonian graph. Then for all $0 \neq S \subsetneq V$ the graph $G - S$ contains at most $|S|$ connnected components.*

Proof. (trivial) Is it sufficient? NO! Counterexample → pic Not Hamiltonian because every HC should contain the red edges. Thus, central vertx would have degree ≥ 3 in the cycle `. add pic add pic

Check that condition of Lemma 3.2 is fulfilled. □

Lemma 3.3 (Bondy, Chvatal 1976)**.** *Let $G = (V, E)$ with $n := |V|$ be an undirected graph and let $u, v \in V$, $\{u, v\} \notin E$, such that*

$$\mathrm{d}(u) + \mathrm{d}(v) \geq n.$$

Then the following holds, G is Hamiltonian if and only if $G + \{u, v\}$ is Hamiltonian.

Proof. G Hamiltonian $\implies$ $G + \{u, v\}$ is Hamiltonian (trivial)
Let $G + \{u, v\}$ be Hamiltonian $\implies$ $\exists$ HC C in $G + \{u, v\}$. If $e = \{u, v\} \notin E(C)$ then C is a HC in G. ✓
Assume w.l.o.g. $e \in E(C)$. Let $C = (u =: v_1, v_2, \ldots, v_n := v)$. Claim: It is enough to find an index $i \in \{2, \ldots, n-2\}$ such that $e_R := \{u, v_{i+1}\} \in E$ and $e_L := \{v_i, v\} \in E$. Instead let C' such that $E(C') := (E(C) \setminus \{\{u, v\}, \{v_i, v_{i+1}\}\}) \cup \{e_R, e_L\}$ C' is then a cycle of length n, i.e. a HC In G. add pic
Now show that such index i always exists. Let $R := \{i : 2 \leq i \leq n-2, \{u, v_{i+1}\} \in E\}$, $L := \{i : 2 \leq i \leq n-2, \{v_i, v\} \in E\}$.

$$L \cup R \subseteq \{2, \ldots, n-2\} \implies |L \cup R| \leq n-3$$

$$|R| := |N(u) \setminus \{v_2\}| = \mathrm{d}(u) - 1$$

$$|L| := |N(v) \setminus \{v_{n-2}\}| = \mathrm{d}(v) - 1$$

Then

$$|R| + |L| = \mathrm{d}(u) - 1 + \mathrm{d}(v) - 1 \geq n - 2$$

Thus $|R| + |L| > |R \cup L| \implies R \cap L \neq \emptyset$. Let $i \in R \cap L$ there the edges e_R and e_l as above exist and the pair-exchange argument applies. □

Definition. The k-th *Hamiltonian hull* $H_k(G)$ of a graph G is recursively defined as follows:

- If there are non-adjecent vertices $u, v \in V$ with $\mathrm{d}(u) + \mathrm{d}(v) \geq k$, then let $H_k(G) := H_k(G \cup \{u, v\})$, otherwise $H_k(G) = G$.

Lemma 3.4. *The k-th Hamiltonian Hull of graph G is well defined.*

Proof. "well defined" means the result $H_k(G)$ does not depend on the order in which the edges are throw in. Assume the contrary, i.e.

$$G_1 := (V, E \uplus \{f, \ldots, f_{l_1}\})$$

and

$$G_2 := (V, E \uplus \{g_1, g_2, \ldots, g_{l_2}\})$$

be two k-th Hamiltonian hulls of G obtained by adding edges $f_1, \ldots, f_{l_1}$ and $g_1, \ldots, g_{l_2}$, respectively. Assume wlog $l_1 \geq l_2$. Assume $G_1 \neq G_2$. Let $f_i = \{x, y\}$

be the first edge in the sequence $f_1, \dots, f_{l_1}$ such that $f_i \not{in} E(G_2)$ (it exists because $G_1 \neq G_2$, $l_1 \geq l_2$).
Consider $H := (V, E \cup \{f_1, \dots, f_{i-1}\})$. According to definition of G_1: $f_i \notin E(H)$, $\mathrm{d}(x) + \mathrm{d}(y) \geq k$. Since H is a subgraph of G_2 and $\mathrm{d}_H(x) + \mathrm{d}_H(y) \geq k \implies \mathrm{d}_{G_2}(x) + \mathrm{d}_{G_2}(y) \geq k$. Since, moreover, $f_2 \in E(G_2)$, we would have to still add f_i to G_2 so $G_2 \neq H_k(G)$ according to the definition. ` ☐

Theorem 3.5 (Bondy, Chvatal 1976)**.** *A graph is Hamiltonian if and only if its n-th Hamiltonian hull is Hamiltonian.*

Since a complete graph K_n is Hamiltonian if $n \geq 3$ we get

Corollary 3 (of Theorem 3.5; Ore 1960)**.** *If in a graph G with $n := |V(G)|$, $n \geq 3$, $\mathrm{d}(v) + \mathrm{d}(u) \geq n$ holds for every $u, v \in V$, $\{u, v\} \notin E(G)$, then G is Hamiltonian.*

Corollary 4 (of Theorem 3.5; Dirac 1952)**.** *A graph G with $n := |V(G)|$ and $\delta(G) \geq \frac{n}{2}$ is Hamiltonian.*

This bound is best possible.

Examples.

1. E.g. there exists a graph with $\delta(G) = \frac{n}{2} - 1$ which are not Hamiltonian, which are even not connected. add pic
2. $K_{r,r+1}$ with $n = 2r + 1$ $\delta(K_{r,r+1}) := r = \frac{n-1}{2}$. add pic
   Is $K_{r,r+1}$ Hamiltonian: no cycle of odd length, in particular no cycle of length $2r + 1$ exists in G implies H not Hamiltonian.

Theorem 3.6 (Chvatas, Erdös 1972)**.** *For a graph G with $n = |V(G)| \geq 3$ the implication "$k(G) \geq \alpha(G) \implies G$ is Hamiltonian" holds, where $\alpha(G)$ is in the stability number of G, i.e.*

$$\alpha(G) := \max\{|S| : S \subseteq V(G), G[S] \text{ has no edges}\}.$$

Proof. $\alpha(G) = 1 \implies G = K_{|G|}$ is Hamiltonian ✓
wlog $\alpha(G) \geq 2$ and let C be a cycle in G. If $|V(C)| < n$, consider $G[V \setminus V(C)] = G - C$ and let S be a connected component of $G - C$. Let

$$T := C \cap \left(\bigcup_{s \in S} N(s) \right)$$

Since $k(G) \geq \alpha(G) \geq 2$, $|T| \geq 2$ because T is a separator. Since $G[S]$ is connected we have: for all $v_1, v_2 \in T$ there exists a v_1-v_2-path which only uses inner vertices from S. Let $C = (v_1, v_2, \dots, v_l, v_1)$ and denote by v_i^* the direct successor of v_i in C. If for some $v_i \in T$ also $v_i^* \in T$, then extend the cycle C (as in the picture, add at least two green edges and remove one edge $\{v_i, v_i^*\}$.) The cycle can also be extended if $\{v_i^*, v_j^*\} \in E$ for some $v_i, v_j \in T$. Indeed consider $C' := (C \setminus \{\{v_i, v_i^*\}, \{v_j, v_j^*\}\}) \cup \{v_i^*, v_j^*\} \cup v_i$-$v_j$-path. C' is longer because we add at least 3 green edges and remove exactly two.

add pic

add pic

add pic

We show: $k(G) \geq \alpha(G)$ implies the existence of two vertices v_i^*, v_j^* as above as long as $|V(G)| < n$. Indeed, let $T^* := \{t^* : t \in T\}$ and let $|V(C)| < n$. Assume by contradiction that C cannot be extended further as above. Then,

$$\{s, t^*\} \notin E \text{ for all } s \in S \text{ and for all } t \in T. \tag{6}$$

t^* is not connected to S otherwise we it can be extended. Thus, add pic

$$|T^*| = |T| \text{ and } T^* \text{ is a stable set} \tag{7}$$

because if $\{v_i^*, v_j^*\} \in E$, $v_i^*, v_j^* \in T$ can extend above. And 6 and 7 implies $T^* \cup \{s\}$ stable set

But T is a separator in $G \implies k(G) \le |T| = |T^*| < |T^*| + 1 = |T^* \cup \{s\}| < \alpha(G)$. □

Remark. Thus result is best possible.

There are graphs with $k(G) = \alpha(G) - 1$ which are not Ham. e.g. the Petersen graph (details in exercises) $K_{r,r+1}$ is a "best possible" example.

If $\lambda(G) \ge \alpha(G)$ holds, G does not have to be Hamiltonian.

Example. $\lambda(G) = \alpha(G) = 2$ and G not Hamiltonian. add pic

Corollary 5. *There exists an $\mathcal{O}(nm)$ algorithm which either want a Hamiltonian cycle in G for findes a stabel set S and a separator T in G with $|T| \le |S|$. ($k(G) \le |T| \le |S| \le \alpha(G)$) (exercises)*

Proposition 3.7 (Bondy 1978)**.** *Let G be a graph, $|V(G)| \ge 3$, and $\mathrm{d}(u) + \mathrm{d}(v) \ge n$ for all $u, v \in V(G)$ with $\{u, v\} \notin E$. Then $k(G) \ge \alpha(G)$ holds.*

Proof. Show $|S| \le |A|$ for all stable sets S and every separator A. Let S, A be an arbitrary stable set and an arbitrary separator, respectively. If $S \subseteq A$ ✓ Assume wlog $S \nsubseteq A$. Let X be a connected component of $G - A$ with $X \cap S \neq \emptyset$ Let $x \in S \cup X$ and let $Y := V \setminus (A \cup X)$. $Y \neq \emptyset$ because A separator, so there exists at least one connected component different from X. 2 Cases: add pic

1. $S \cup Y = \emptyset$. Let $y \in Y$ arbitrary. $\{x, y\} \notin E(G) \implies \mathrm{d}(x) + \mathrm{d}(y) \ge n$

   $$n \le \mathrm{d}(x) + \mathrm{d}(y) = |N_G(x)| + |N_G(y)| \le \underbrace{|(X \cup A) \setminus S|}_{\ge \mathrm{d}_G(x)} + \underbrace{|Y| + |A| - 1}_{\ge \mathrm{d}_G(y)} = n - |S| + |A| - 1$$

   Thus, $|S| \le |A| - 1$.

□

21.04.2016

Proof. ($\Rightarrow$) $0 < a_1 \le a_2 \le \dots \le a_n < n$, $a_i \in \mathbb{N}$ for all $1 \le i \le n$ and $a_i \le i \iff a_{n-i} \ge n - i$ for all $i < \frac{n}{2}$ Thus, $(a_i)_{1 \le i \le n}$ ham seq By condtradiction there exists a G with degree sequence $(d_i)_{1 \le i \le n}$, $d_i \ge a_i$ for all $1 \le i \le n$ and G non ham

We showed $d_i \le i \implies d_{n-i} \ge n - i$ for all $i < \frac{n}{2}$ (**??**)

Let $x, y \in V(G)$ such that $\{x, y\} \notin E(G)$ with max $\mathrm{d}(x) + \mathrm{d}(y)$, wlog $\mathrm{d}(x) \le \mathrm{d}(y)$. Thus, $G \cup \{x, y\}$ is ham and every cycle H in $G \cup \{x, y\}$ contains $\{x, y\}$. $H - \{x, y\}$ is a ham path in G. Denote it by $x_1 := x, x_2, \dots, x_n := y$) As in proof of Dirac's theorem add pic add pic

$$I := \{i : 1 \le i \le n - 1, \{x_1, x_{i-1}\} \in E\}$$
$$J := \{i : 1 \le i \le n - 1, \{x_i, x_n\} \in E\}$$

$I \cup J \subseteq \{1, 2, \ldots, n-1\}$, $I \cap J = \emptyset$ (as in Dirac's theorem)

$$\mathrm{d}_G(x) + \mathrm{d}_G(y) = |I| + |J| = |I \cup J| \leq n - 1 \tag{8}$$

Recall

$$\mathrm{d}_G(x) = \mathrm{d}(x) < \frac{n}{2} \tag{9}$$

For all $i \in I$

$$\mathrm{d}(x_i) \leq \mathrm{d}(x)$$

because otherwise $\mathrm{d}(x_i) + \mathrm{d}(y) > \mathrm{d}(x) + \mathrm{d}(y)$ and $\{x_i, y\} \notin E(G)$, so instead of x and y we would have chosen x_i and y `.

So G contains at least $|I|$ vertices with degree $\leq \mathrm{d}(x) =: h = |I|$. Thus, $d_h \leq h \overset{??}{\implies} d_{n-h} \geq n - h$ add pic

There exists at least one vertex among these $h+1$ ($d_{n-h} \leq d_{n-h+1} \leq \cdots \leq d_n$) which is non-adjacent to x. Let this vertex be z, so $\{x, z\} \notin E(G)$.

$$\mathrm{d}(x) + \mathrm{d}(z) \geq h + n - h = n$$

which contradicts 8.

So $(a_i)_{1 \leq i \leq n}$ has to be a hamilt seq

($\Leftarrow$) $(a_i)_{1 \leq i \leq n}$ is hamilt $\implies$ condition **??** is fulfilled or equivalently, condition **??** is not fulfilled $\implies$ $(a_i)_{1 \leq i \leq n}$ is not hamilt, i.e. there exists a graph G with degree sequence $d_i \geq a_i$ for all i which is not hamilt.

Construct such a graph for the sequence a_i which violated the condition **??** at index h, $a_h \leq h$ and $a_{n-h} \leq n - h - 1$. Consider

$$(\underbrace{h, h, \ldots, h}_{h \text{ times}}, \underbrace{n-h-1, \ldots, n-h-1}_{n-2h \text{ times}}, \underbrace{n-1, \ldots, n-1}_{h \text{ times}}) \tag{10}$$

$$k \leq n - h - 1$$
$$2h \leq n - 1 (\text{ because } h < \frac{n}{2}$$

So the sequence is non decreasing and of length n.

Construct G having the above mentioned sequence as the degree sequence add pic

$$E = \{\{v_i, v_j\} : i, j > h\} \cup \{\{v_i, v_j\} : 1 \leq i \leq h, j > n - h\}$$

"G is complete over red and green vertices. G is complete bipartite over blue on on one side and green on the other siede."

Show degree seq of is First,

$\mathrm{d}(v_i) = h \ \ \forall 1 \leq i \leq k$ (blue vertices)

$\mathrm{d}(v_i) = n - h - 1 \ \ \forall h + 1 \leq i \leq n - h$ (red vertices)

$\mathrm{d}(v_i) = h + n - h - 1 = n - 1 \ \ \forall n - h + 1 \leq i \leq n$ (green vertices)

Second, pic Red vertices are missing because the successor of every blue vertex is a green vertex, and the are exactly as many green (h) vertices as blue vertices, and also the predecessor of every blue vertex is a green vertex add pic

$$d_{n-h} \geq n - h \text{ and}$$
$$d_h > h$$

Remark (Exercise). Insert exercise here. Analogous construction like above

□

4 4. Topological graph theory; planar graph

History of this theory:

1. Euler formula
2. Theorem of Kuratowski
3. four colours conjecture and then theorem
4. algorithmic aspects: a number of NP-hard opt. problems in general graphs are efficiently solvable or approximable in planar graphs (max cut, k-multi cut, colouring problem, stable set,...)

4.1 4.1 Definitions and elementary concepts

Definition. A graph G is called *planar* if there exists an embedding (or representation) of G in the plane such that

1. the vertices of G are points in the plane
2. the edges of G are Jordan curves, such that
   (a) every Jordan curve intersects the set of points representing vertices only on its endpoint, and
   (b) any two different Jordan curves can intersect only at some endpoints.

A *plane graph* is just a representation of a planar graph as described above in the plane.

add pic

Definition. A *Jordan curve* is (the image of) a homeomorphic mapping f of $I_1 = [0, 1]$ into a topological space, i.e. $f : [0, 1] \to A$ (top. space) such that f is bijective, continuous and f^{-1} is also continuous.

We will prove all results in this chapter for the topological space $(\mathbb{R}^2, O)$ where the basis O of the open sets is

$$O := \{B(x, r) : x \in \mathbb{R}^2, r > 0\}$$

with $B(x, r) := \{y \in R^2 : l_2(x, y) < r\}$.

Remark. Dimension is 1. S_1 is cycle of radius 1, P the North Pole $f : S_1 \setminus \{P\} \to \mathbb{R}$, $x \mapsto f(x)$ Stereographic projection $S_1 \setminus \{P\} \ni \mathbb{R}$, $S_2 \setminus \{P\} \ni \mathbb{R}^2$

add pic

add pic

Definition. If the edges are removed from a planar embedding of a planar graph, then the plane would decompose in connected areas which are called *regions* (or *face*) of which exactly one is unbounded. This unbounded region is called *outer region*. The *boarder of a region* consists of all edges which are contained in the (topol.) closure of the region.

Remark. Consider an arbitrary plane embedding of a planar graph G and a face F which is not the outer face. There exists another plane embedding H_2 of G in which the image of f is the outer face. By stereographic projection place a sphere "on the plane" such that it touces the plane where H_1 lies and do the projection H_1 on this sphere. Rotate sphere s. t. the nothe pole becomes the touching point that lies on F, project the sphere embedding back to the plane. This is the required H_2. add pic

Proposition 4.1 (4.1)**.** *A graph is planar if and only if all its blocks are planar.*

Proof. wlog assume that G is connected, otherwise ... with connected components am embedded there s.t. the embeddings are "far apart".

Consider $\mathrm{bc}(G)$, we know it is a face.

$\Rightarrow$ trivial

$\Leftarrow$ Assume every block is planar, Show that G is planar. Pick up an arbitrary block, say b_1, and construct one plane embedding of it. Do a DFS (or BFS...) on $\mathrm{bc}(G)$ starting at b_1. Visit the other blocks of G in the order implied by this search. and every time a new block is encountered construct a planar embedding oth this block such that the cur vertex from which th block is entered has on the boarder of the outer face of the embedding construction so far. (Maybe the embedding constructed so far needs to be transformed so that the cut vertex mentioned above lies on its outer face.) add pic

□

Proposition 4.2. *A planar graph is 2-connected if and only if the boarder of each face of any plane embedding of G is a cycle.*

Proof. $\Leftarrow$ add pic

$\Rightarrow$ Exercise

□

25.04.2016

Definition. Let φ and φ' be two *embeddings* of a planar graph G in the plane, i.e. there exist two plane graphs H and H' such that $\varphi(\varphi')$ is an isomorphism of G and $H(H')$. φ and φ' are called *equivalent* (we also say H and H' are equivalent) if $\varphi' \circ \varphi^{-1}$ is an isomorphism of H to H', such that for all sequences R of edges in $E(G)$ the following holds: $\varphi(R)$ is a border of a region in H if and only if $\varphi'(R)$ is a border of a region in H' in the same order or in the reversed order or composed to the order in $\varphi(R)$.

Remark.

$$\begin{aligned} \varphi &: G \to H \\ \varphi^{-1} &: H \to G \\ \varphi' &: G \to H' \end{aligned}$$

Example. add pic

Remark. $\varphi \sim \varphi' \overset{\text{Def}}{\iff} \varphi, \varphi'$ are equivalent plane embedding of a planar graph G. Equivalence relation (check it as a homework!)

Proposition 4.3 (Wagner 1936, Färy 1948)**.** *For every plane embedding of a planar graph, there exists an equivalent planar embedding in which edges are drawn as segments of straight lines, i.e. the Jordan curves in the definition of the embedding are piecewise linear functions.*

Proof. No proof. □

Definition. A planar graph G is called *uniquely embeddable* if all its plane embeddings are equivalent

Example. add pic

Theorem 4.4 (4.4 Whitney year?)**.** *A 3-connected planar grap his uniquely embeddable.*

Proof. No proof. □

Theorem 4.5 (Euler formula)**.** *Let G be a planar and connected graph. Then*

$$|V| - |E| + |F| = 2.$$

Corollary 6. *If G is planar and $|V| \geq 3$, then*

$$|E(G)| \leq 3|V(G)| - 6$$

Corollary 7. *If G is planar, bipartite and $|V| \geq 3$, then*

$$|E(G)| \leq 2|V(G)| - 4$$

Definition. A planar graph G is called *maximal planar* if $G \cup \{x, y\}$ is not planar, for all $x, y \in V(G)$ such that $\{x, y\} \notin E(G)$. A *triangulation of the plane* (or a *triangle graph*) is a plane graph such that all of its faces have exactly 3 edges on the border (including the outer face) add pic

Proposition 4.6 (4.6)**.** *For a planar graph G the following statements are equivalent:*

1. *G is maximal planar*
2. *Every plane embedding of G is a triangulation of the plane.*
3. *$|E| = 3|V| - 6$*

Proof. (1) $\Rightarrow$ (2)
face F with $|\delta(F)| > 3$ "divide it" by edges and obtain again a planar graph ` add pic
(G was max planar)
(2) $\Rightarrow$ (1)
trivial because adding an edge would violate the upper bound for $|E(G)|$ in a planar graph G.

(2) ⇒ (3)
Count edge-region incident in two ways

$$\#(R,e) \text{ such that } e \in \delta(R) = \sum_{e\in E(G), R \text{ region}: e\in\delta(R)} 1 \overset{(\leq)}{=} 2|E|$$

$$\delta(R) = \sum_{R \text{ region}} \sum_{e\in E(G): e\in\delta(R)} 1 = 3\left|\underbrace{\{R : R \text{ is a region}\}}_{\mathcal{R}}\right| = 3|\mathcal{R}|$$

Put $2|E| = 3|\mathcal{R}|$ into Euler formula. $F = \mathcal{R}$.

$$|V| - |E| + |\mathcal{R}| = 2$$
$$|V| - |E| + \frac{2}{3}|E| = 2$$
$$|V| - \frac{|E|}{3} = 2 \implies |E| = 3(|V| - 2)$$

□

4.2 The theorems of Kuratowski and Wagner

Definition. Let G be a graph obtained from another graph H through iterative edge divisions, i.e. insertions of a neq vertex of degree two in some edge of the graph. Then G is called a *subdivision of* H. Equivalently we get G from H through substituting every edge of H by a path such that paths corespond to different edges can only touch each other at their end points. We say G *contains* H *as a subdivision* if and only if G contains a subgraph H' such that H' is a subdivision of H. EGE contains H as a subdivision bcause $H' = G$ is obtained by subdividing H and $H < G$.

add pic

add pic

Observation: If G is planar, then every subdivision of G is planar.

Theorem 4.7 (4.7 Kuratowski 1930)**.** *A graph G is planar if and only if it does not contain K_5 and it does not contain $K_{3,3}$ as a subdivision.*

Proof. based on Thomassen 198?
Use 2 Lemmas.

Lemma 4.8 (4.8)**.** *Every 3-connected graph F with $|V(G)| \geq 5$ contains an $e \in E(G)$ such that $G|e$ is 3-connected.*

Proof. already proved) □

Lemma 4.9. *Let $e \in E(G)$ be an edge of some arbitrary graph G. If $G|e$ contains a subdivision of K_5 or $K_{3,3}$, then also G contains a subdivision of K_5 or $K_{3,3}$, respectively.*

Proof. Let $e = \{x,y\}$ and let $z \in V(G|e)$ representing e. Let S be the subdivision of K_5 or $K_{3,3}$ in $G|e$. For every edge $\{z,v\}$ in $G|e$ there exists a vertex $v \in V(G) \setminus \{x,y\}$ such that $\{x,v\} \in E(G)$ or $\{y,v\} \in E(G)$. Collect in T all edges of S which do not have z as an endpoint, as well as one of $\{x,v\}$ or $\{y,v\}$ (arbitrarily chosen) for all $v \in V(G) \setminus \{x,y\}$ such that $\{z,v\} \in E(G|e)$, and finally the edge $\{x,y\}$. We show T is the required subdivision of K_5 or $K_{3,3}$.

add pic

Two cases: Notice that $\det_T(x) = 1$ and $\deg_T(y) = 1$ cannot happen because then z is an isolated vertex in S ` add pic

1a $\deg_S(z) = 2$: add pic

1b $\deg_T(x) = 2$ or $\deg_T(y) = 2$ (symmetric): add pic

2 $\deg_T(x) \geq 3$ and $\deg_T(y) \geq 3$: T contains a subdivision of $K_{3,3}$ with the blue vertices on one side and the green vertices on the other. This case cannot happen because for $\{z, v_1\} \in S$ we choose just one "red" representative, either $\{x, v_1\}$ or $\{y, v_1\}$ in T. add pic add pic

□

Induction on $|V|$.

All graphs with $|V| \leq 4$ are planar. If $|V(G)| = 5$, then $G|e$ is planar. Hence if such a G does not contain a subdivision of K_5 (which is K_5 itself in this case) is planar $\Rightarrow$ statement holds. Assume wlog $|V(G)| \geq 6$. We show if G has no subdivision of K_5 or $K_{3,3}$ then G planar. Cases: add pic

1. G is not 3-connected. G is planar if and only if all its blocks are planar Proposition **??**). Show that every block is planar. Consider a block, it is 2-connected. Let $\{x, y\}$ be a separator. Let H be a connected component of $G \setminus \{x, y\}$. Denote $G_1 := G[V(H) \cup \{x, y\}]$ and $G_2 := G[V \setminus H]$. If $\{x, y\} \notin E(G)$ we add to G_1 and G_2. Since G is 2-connected, each of the vertices x and y is connected to all connected components in $G \setminus \{x, y\}$. (otherwise, if e.g. y was not connected to some component, then $\{x\}$ would be a separator `). So the introduction of $\{x, y\}$ does not create a subdivision of $K - 5$ or $K_{3,3}$ in G_1 or G_2 because otherwise the subdivision say in G_1 would yield a subdivision in G by subdivision $\{x, y\}$ through the x-y-path in G_2 (with inner vertices in $V \setminus (H \cup \{x, y\})$. So G_1, G_2 do not contains subdivisions of K_5 or $K_{3,3}$ and $|V(G)_1| < |V(G)|$, $|V(G_2)| < |V(G)|$ hold. Thus by Induction assumption, G_1, G_2 are planar. Then, Embedding their on the plane s.t. $\{x, y\}$ lies on the outer face, respectively merge this embedding so that they touch only on $\{x, y\}$ this yields a planear embedding of G. add pic add pic
2. So

□

02.05.2016

2. G is 3-connected
   By Lemma **??**, there exists an $e \in E(G)$ such that $G|e$ is 3-connected. Let z be the vertex representing $e = \{x, y\}$ in $G|e$.
   By Lemma **??**, $G|e$ has no subdivision of $K_{3,3}$ and no subdivision of K_5 (otherwise also G would have such a subdivision).
   By induction assumption, $G|e$ is planar. Consider a planar embedding of $G \gg e$ with z not in the interior of the outer face. Let C be a cycle which is the border of smallest union of faces containing z. Let $x_1, \ldots, x_k$ be the neighbours of x in C. (all neighbours of x which are also neighbours of z add pic

in $G|e$ have to lie in C) in cyclical order. Let P_i be the unique subpath joining x_i and x_{i+1} in C, where $x_{k+1} := x_1$. add pic

$$\deg_{G|e}(z) \geq 3.$$

and $\deg_G(y) \geq 3$. Assume all neighbours of y different from x lie in one of theses paths P_i. Then we can extedn planar embedding of $G|e$ to a planar embedding of G ✓ add pic 2 -red

Now we show that the assumptions of the theorem exclude all other possibilities for neighbours of y. What are other theoretical possibilities? The negation of what we have: For all i there exists a neighbour of y different from x in $V(P_i)$ and there exists a neighbour of y (different form x) which is not in $V(P_i)$.

(a) $\deg_G(y) = 3$ Let end vertices of P_i be u and v. The blue-green vertices lead to a subdivision of $K_{3,3}$ in G ` add pic

(b) $\deg_G(y) \geq 4$ Let u, v, w be three neighbours of y in C. If u, v, w are also neighbours of x then subdivision of K_5 `. If $N_G(y) \setminus \{x\} \not\subseteq \{x_1, \dots, x_k\}$, there exists a $u \in N_G(y) \setminus \{x\}$ wich is an inner point of same P_i there exists a $v \in \mathbb{N}_G(y) \setminus \{x\}$ with $v \notin V(P_i)$ because o.w. We would be in the same case as before (there exists an i s.t. $\mathrm{d}_G(y) \setminus \{x\} \subseteq V(P_i)$) blue-green vertices yield a subdivision of $K_{3,3}$ in G. add pic pic 4 pencil drawing add pic

Proof end ■

Definition. A graph G contains a graph H as a minor, $G \succ H$ if G contains a sub-graph H' which can be contracted to H, i.e. it is possible to obtains H starting with H' and applying a consecutive contraction of edges. add pic

Remark. 1. Relation "$\succ$" in the set of graphs is transitive. $H_1 \prec H_2$ and $H_2 \prec H_3$ imply $H_1 \prec H_3$ (convince yourself, homework!)

2. If G contains a subdivision of H, then G contains H as a minor. The converse is not true (in general). $G \succ K_5$ (minor) but not as a subdivision Why no subdivision? Every subdivision of K_5 has at least 6 vertices and 5 vertices are of degree 4. But there are only 4 vertices of G of degree 4 ` add pic add pic

Theorem 4.10 (4.10?). *(a) Let G and H be connected graphs. Then $G \succ H$ if and only if there exists a mapping $\phi : V(G) \to V(H)$ such that for the pre-images $\phi^{-1}(v)$ for $v \in V(H)$ the following hold*

1. $G[\phi^{-1}(v)]$ is connected for all $v \in V(H)$

2. For all $\{x, y\} \in E(H)$ there the cut $\delta(\phi^{-1}(u), \phi^{-1}(v))$ is non empty.

(b) If $\Delta(H) \leq 3$ then $G \succ H$ if and only if G contains H as a subdivision.

Proof. As an exercise. □

Corollary 8. *A graph G is planar if and only if G contains neither K_5 nor $K_{3,3}$ as a minor.*

Proof. Since every minor of a planar graph is planar, direction $\Rightarrow$ is trivial.
$\Leftarrow$ G has neither K_5 or $K_{3,3}$ as a minor. Then G has neither a subdivision of K_5 nor a subdivision of $K_{3,3}$ (Remark 2). Thus by Kuratowski, G is planar. □

4.3 4.3 Duality in planar graphs

Definition. Let $G = (V, E)$ be a planar graph and $G = (V, E, \mathcal{R})$ be a planar embedding of G with set of regions (faces) $\mathcal{R}$. The *geometric dual* G^* of G is a triple $G^* = (V^* = \mathcal{R}, E^*, \mathcal{R}^*)$ obtained as follows. Choose one vertex in the interior of every region of G. For every $e \in E$ connect by a Jordan curve e^* the vertices chosen in the regions having e in their borders such that

a) e^* intersects e in an inner point and

b) e^* is contained in the union of the two regions mentioned above.

If e is a bride, then e^* is a loop.

Example. add pic

Remark.

1. There exists a bijection between E and E^*.
2. G^* is (in general) a multi-graph with loops and multiple edges.
3. $e \in E$ is a bridge if and only if e^* is a loop
   there are multiple edges between two vertices R_1 and R_2 in G^* if and only if R_1 and R_2 as regions in G have more than 1 common edge in their borders.
4. G^* depends on the embedding of G.
   But the geometric duals correspond to equivalent embeddings are isomorphic (exercises)
5. geometric dual is a connected multi-graph

Definition. A planar graph G is called *self dual* if it is isomorphic to any of its geometric duals.

Example. add pic

4.3.1 4.3.1 connectivity and polyhedral graphs

Observation: A 2-edge connected planar graph has a geometric dual without loops. The geometric dual of a graph is not necessarily 2-connected, just connected for sure.

Example. G^* is not 2-connected (because empty graph is by def not connected) add pic

Proposition 4.11 (4.11)**.** *A 2-connected planar graph $G = (V, E, R)$ with at least 3 faces has a 2-connected geometric dual G^*.*

Proof. If G has exactly 3 regions, then G^* is isomorphic to K_3 with some multiple edge possible, so it is 3-connected.
Assume $|\mathcal{R}| \geq 4$.
We show every 2 edges e^* and f^* which are not parallel (parallel edges have the same end points) lie in a common cycle. (enough because there exist equivalence relation... see exercise 9).

a) Let $e^* \cap f^* = \{x\}$, $x \in \mathcal{R}$. Consider the border C of x and run over it in clockwise order starting at e. Since G is 2-connected C is a cycle. Let $C = (e_1 = e, e_2, \dots, e_k = f, ...)$. Let $R_1, R_2, \dots, R_k$ be the regions of G having the edges $e_1, \dots, e_k$ in their common border with x. Any two consecutive regions in this sequence share a border edge; let $e_1^*, e_2^*, \dots, e_k^*$ be the sequence of the corresponding edges in G^*. $e_1^*, e_2^*, \dots, e_k^*$ is a cycle. add pic

b) $e^* \cap f^* = \emptyset$ e, f lie in a common cycle in G (because G is 2-connected) This cycle divides the plane in which G is drawn in an inner area and an outer area. Both in the inner area and in the outer area there is a chain of regions of G which "connect" e^* and f^*. These two chains form the required cycle in G. add pic

□

09.05.2016
The last theorem is not a characterization but a sufficient condition.

09.05.2016 Assume the geometrical dual G^* of some graph G has multiple edges. Thus, there exist at least 2 regions in G with more than 1 common edge in their borders. add pic

Any pair of parallel edges in G^* encloses a subgraph of G which is connected to the rest exactly through the edges e, f in G which are the counter parts of e^*, f^*. Thus, the remaining e, f from G disconnects that subgraph from the set of F. Thus, G is not 3-edge-connected. Thus, G is not 3-connected. Moreover, if G is 3-connected (even if G is 2-connected) , then G^* has no loops either. add pic
Thus, G^* is simple.

Example. platonic solids add pic

Definition. A *graph* is called *polyhedral* if it is isomorphic to a 3-dimensional, convex, bounded polyhedron (bounded polyhedron = polytope) where the vertices, edges, facets of th polytope are the vertices, edges and region of the graph, respectively.

Homework: Polyhedral graphs are 3-connected (2-connectivity is trivial since every face is bounded by a cycle). Polyhedral graphs are planar (construct a sphere in the interior of the polytope project the polytope on the surface of the sphere, and then from the sphere to the plane)

Theorem 4.12 (4.12 Steinitz 1922)**.** *A graph is polyhedral if and only if it is planar and 3-connected.*

Proof. No proof. □

Proposition 4.13. *If a graph is 3-connected, then its geometric dual is 3-connected.*

Sketch of the proof. Let S, T be two vertices in G^* (i.e. two regions in G), $S \neq T$. Show that there exist 3 independent S-T-paths in G^*. Take two points such that in the interior of S, T in G, respectively. Since G is 3-connected: for all $v \in V(G)$ and for all 3 vertices on the border of S (T), there exist 3 independent add pic paths joining v to the vertices on the border (fan theorem). connect s (t) to 3 (arbitrarily chosen and then fixed) vertices on the border of S (T). Let G' be the resulting graph. G' is 3 connected (e.g. 3 independent 1-v-paths (s-v-paths?) would be 3 independent paths from v to the 3 vertices in the border (fan thm) extended by the three edges connecting s to the border), Thus, there exist 3 independent s-t-paths in G. Consider a pair of these 3 paths and the area enclosed by them. There is a chain of regions which starts with S, ends with T, intersect this area such that any two consecutive regions in the chain are neighbouring. Such a chain in G is an S-T-path in G^*. Any pair of path defines such a chain of regions, and hence an s-t-paths. The 3 chains defined by the 3 pairs of paths have no inner region in common. Therefore, the 3 corresponding s-t-paths are independent. (should be rigorously shown!) □

add pic

add pic

Definition. A polyhedral is called *regular* (or *platonic solid*) if aond only if it has conruent, regular, polygonal faces with the same number of faces meeting at each vertex.

Thus, a polyhedral graph corresponding to a platonic solid is regular.

Theorem 4.14 (4.14)**.** *There are exactly 5 platonic solids.*

Proof. Consider the polyhedral graph coresopnding to a platonic solid. It is regular; denote by k the common degree of its vertices and by h the number of vertices in each region. Apply Euler

$$|V| - |E| + |\mathcal{R}| = 2$$

together with $|V|k = 2|E|$, $|\mathcal{R}|k = 2\mathcal{E}$. Then we get

$$\frac{2|E|}{k} - |E| + \frac{2|E|}{h} = 2$$

Thus,

$$\frac{1}{k} + \frac{1}{h} - \frac{2 + |E|}{2|E|} > \frac{1}{2}$$

Furthermore, $k = \delta(G) \geq 3$ since G is 3-connected, and $k \geq 3$ (number of edges on the border of a face (cycle)). Thus,

$$\frac{1}{k} > \frac{1}{2} - \frac{1}{h} \geq \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \implies k \leq 5$$

and

$$\frac{1}{h} > \frac{1}{2} - \frac{1}{k} \geq \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \implies h \leq 5$$

So we have $3 \leq h, k \leq 5$. Check all 9 pairs (h,k) and observe that only 5 of them correspond to the corresponding quantities in graphs. □

Corollary 9. *The graphs of the platonic solids are exactly those polyhedral graphs which are regular toghether with their corresponding geometric duals.*

Proof. Exercises. □

Theorem 4.15 (4.15 Whitney 1934)**.** *Let G be a planar embedding of G and G^* its geometric dual. The set $C \subseteq E(G)$ is a cycle in G if and only if the corresponding set of edges $C^* \subseteq E(G^*)$ is an inclusion-minimal separating set of edges.*

Proof. $\Rightarrow$ Let C be a cycle in G. Consider the two areas of the plane in which G is drawn, the area inside C and the area outside C. G^* has at least one vertex inside C and at least one vertex outside C. Every path connecting square vertex inside C to some vertex outside C in G^* has to intersect C and contains hence an edge e^* which the counter part of some edge $e \in E(C)$. Thus, C^* is a separating set of edges in G^*. Is it inclusion-minimal? Yes, because the subgraph of G^* inside C and the subgraph of G^* outside are both connected. add pic

$\Leftarrow$ Let C^* be an inclusion-minimal separating set of edges in G^*. Show that C is a cycle. $G^* - C^*$ is disconnected. Thus, it has at least 2 connected components, and it cannot have more than 2, because otherwise the separating set was not inclusion minimal. Contract each connected component in one vertex and consider the dual of the contraction. The edges corresponding to C^* underlinein this dual build a cycle. "Remove" the contraction. The cycle C is also a subgraph of the dual of the non-contracted G^*, so it is a subgraph of G'. add pic add pic

$\square$

Definition. Let $G = (V, E)$ be a graph. $G^* = (V^*, E^*)$ is called *a combinatorial dual* to G if and only if there exists a bijection $\varphi : E \to E^*$ such that $C \subseteq E(G)$ is a cycle in G if and only if $\varphi(C)$ is an inclusion minimal separating set of edges in G^*.

Does every graph in G have a combinatorial dual? Planar graphs do! (Whitney) Only planar graphs do.

Theorem 4.16 (Whitney)**.** *The only graphs which have a combinatorial dual are the planar graphs.*

Sketch of the proof idea. 4 Lemmas: Then the proof follows directly from Kuratowski

Lemma 4.17. *If G has a combinatorial dual, then every subgraph of G has a combinatorial dual.*

Lemma 4.18. *Let G be a graph and G^* a combinatorial dual of G. Let $e_0, e_1 \in E(G)$. Let e_0^*, e_1^* be the edges corresponding to e_0, e_1 in G^*, respectively. e_0^*, e_1^* are parallel if and only if every cycle containing one of e_0, e_1 contains also the other.*

Lemma 4.19. *Let H be a subdivision of G. If H has a combinatorial dual, then also G has one.*

Lemma 4.20. *Neither $K_{3,3}$ nor K_5 have a combinatorial dual.*

$\square$

Recognition Problem
Instance: A graph $G = (V, E)$
Question: Is G planar?

Kuratowski gives a characterisation which would lead to an exponential algorithm. Solvale in linear time if in $\mathcal{O}(n + m)$.

J.Hopcraft and R. Tarjan, Efficient planarity testing, Journal of ACM (Association of computer machinery) 21 (4), 549-568, 1974.

State of the are algorithm

which goes back to J.M. Voyer and W.J. Myrvold, On the cutting edge simplified *bigO*(n) planarity by edge addition, Journal of Graph Algorithm and Applications 8(3), 241-273, 2004. (volume 8, issue 3)

5 Colouring problems in graphs

12.05.2016

5.1 Definitions, bounds and other basics

Definition (vertex k-colouring)**.** Let $G = (V, E)$ be a graph. A proper *vertex k-colouring* in G is a mapping $c : V(G) \to \{1, 2, \dots, k\}$ such that for all $u, v \in V(G)$:

$$\{u, v\} \in E(G) \implies c(u) \neq c(v).$$

Definition (k-edge-colouring)**.** Let $G = (V, E)$ be a graph. A proper *k-edge-colouring* in G is a mapping $c : E(G) \to \{1, 2, \dots, k\}$ such that for all $e_1, e_2 \in E(G)$:

$$e_1 \cap e_2 \neq \emptyset \implies c(e_1) \neq c(e_2).$$

Definition (k-colourable, chromatic number, k-chromatic)**.** G is called *k-colourable* if a proper k-colouring exists. The *chromatic number* of G, denoted by $\chi(G)$, is the smallest $k \in \mathbb{N}$ such that G is k-colourable. If $\chi(G) = k$, then G is called *k-chromatic.*

Definition (k-edge-colourable, chromatic index, k-edge-chromatic)**.** G is called *k-edge-colourable* if a proper k-edge-colouring exists. The *chromatic index* of G, denoted by $\chi'(G)$, is the smallest $k \in \mathbb{N}$ such that G is k-edge-colourable. If $\chi'(G) = k$, then G is called *k-edge-chromatic.*

Example. (a) $\chi(K_n) = n$,

(b) $\chi(K_{n,n}) = 2$,

Remark. 1. $\chi(G) = 1 \implies E(G) = \emptyset$,

2. $\chi(G) = 2 \iff G$ bipartite $\iff$ there exists no odd cycle in G

3. Let $L(G)$ be the line graph, then $\chi(L(G)) = \chi'(G)$.[apply definition to check this statement]

Definition (colour class)**.** Let c be a feasible k-colouring of a graph G. A *colour class* is a maximal set of vertices (with respect to set inclusion), where all of them have the same colour, i.e. $c^{-1}(\{i\})$, $i \in \{1, 2, \dots, k\}$, is the colour class related to colour i.

Definition (clique partition number)**.** Let G be a graph. The smallest number of cliques in which G can be partitioned is called *clique partition number* of G and is denoted by $\theta(G)$.

Remark. Clearly, $\{c^{-1}(\{i\}) : 1 \leq i \leq k\}$ builds a partition of $V(G)$. Every colour class is a stable set in G and a clique in G^C.

1. Therefore, G is k-colourable if and only if G^C can be partitioned into k cliques. Furthermore, $\chi(G)$ is the smallest number of cliques in which G^C can be partitioned. So $\chi(G) = \theta(G^C)$.

2. We get a first *lower bound*: Let $k := \chi(G)$ and c a k-colouring. We can write the vertex set as disjoint union of all partitions,

$$V(G) = \uplus_{1 \leq i \leq k} c^{-1}(\{i\}),$$

then

$$|V(G)| = \sum_{i=1}^{k} \left|c^{-1}(\{i\})\right| \leq k\alpha(G),$$

since $\left|c^{-1}(\{i\})\right| \leq \alpha(G)$ holds for all $1 \leq i \leq k$. It follows that

$$\chi(G) = k \geq \frac{|V(G)|}{\alpha(G)}.$$

How good is this bound?

It is tight for

(a) K_n (since $\chi(K_n) = n$ and $\alpha(G) = 1$),

(b) $K_{n,n}$ (since $\chi(K_{n,n}) = 2$ and $\alpha(K_{n,n}) = n$).

In general, the bound can be arbitrarily bad (see exercises).

Theorem 5.1 (The four-colour theorem)**.** *Every planar graph is four-colourable.*

Proof. No proof.

K. Appel and W. Haken, Every planar map is 4-colourable, J. AMS, 1989.

N. Robertseon, D. Sanders, P.D. Seymour, R. Thomas, The four-colouring theorem, J. Combinatorial Theory B 70, 1997, 2-44. □

The graph k-colouring problem
Instance: $G = (V, E)$ graph, $k \in \mathbb{N}$
Question: Is G k-colourable?

create a suited problem environment

Remark. The graph k-colouring problem is

1. polynomial for $k = 1 \iff E(G) \neq \emptyset$

2. polynomial for $k = 2 \iff$ there exist no odd cycles

3. NP-complete $\forall k \geq 3$

4. NP-complete for G planar and $\Delta(G) \leq 4$

5. polynomial for $\Delta(G) \leq 3$ (Brooks 1941)

6. polynomially solvable for a number of graph classes like interval graphs, perfect graphs, comparability graphs.

M.R. Garey and D.S. Johnson, computers and Interactability. A Guide to the Theory of NP-Comleteness, W.H. Freeman and Campany, NY, 1979.

Theorem 5.2 (Grötzsch 1959)**.** *Every planar graph which does not contain any triangle (triangle-free planar graph) is 3-colourable.*

Proof. No proof.

C. Thomassen, A short list colour proof of Grötzsch' theorem, J. Comb Theory B 88, 2003, 189-192. □

5.2 Colouring vertices

12.05.2016

Proposition 5.3 (5.3 An upper bound)**.** *Every graph G with m edges satisfies*

$$\chi(G) \le \frac{1}{2} + \sqrt{2m + \frac{1}{4}}.$$

Proof. Let c be a colouring with $k := \chi(G)$ colours. Then there exists one edge between any two colour classes with respect to c. (Otherwise, merge 2 colour classes and obtain a $(k-1)$-colouring ↯). Thus, add pic

$$|E(G)| \ge \frac{1}{2}k(k-1).$$

Solve this inequality with respect to k and obtain the wanted bound. (Bound quality will be given in exercises!) □

Algorithm 1 A greedy colouring algorithm for an input graph $G = (V, E)$

Fix an arbitrary ordering of vertices $v_1, v_2, \dots, v_n$ $(n := |V(G)|)$
$i = 1$
Colour v_i with the smallest colour $k \in \mathbb{N}$ which has not been used to colour $v_j \in N(v_i)$, $1 \le j < i$.
$i := i + 1$

Remark.

1. Greedy uses $\le \Delta(G) + 1$ colours. add pic

$$\#\{\text{coloured neighbours of } v_i\} = \deg_{G[v_1,\dots,v_{i-1}]}(v_i) \le \deg(v_i) \le \Delta(G)$$

$$\implies \text{colour } \Delta(G)+1 \text{ is free and can be used for } v_i.$$

2. If $G = K_n$ or $G = C_{2n+1}$, then greedy is optimal.

   For pic: add pic

$$\chi(C_{2n+1}) = 3 = \Delta(C_{2n+1}) + 1$$

   Hence, greedy has to use at least $\Delta(C_{2n+1}) + 1$ colours and by 1, it uses exactly $\Delta(C_{2n+1}) + 1$ colours.

3. $\Delta(G)+1$ is in general a loose bound. (We can colour v_i by one of the colours $1, 2, \ldots, \deg_{G[\{v_1,\ldots,v_{i-1}\}]}(v_i)+1$ and still are bounded by $\Delta(G)+1$!)

Question: Can the bound $\Delta(G)+1$ be improved? YES

Definition (colouring number)**.** The smallest number $k \in \mathbb{N}$ such that there exists a vertex ordering in $V(G)$ where every vertex is preceded by fewer than k of its neighbours is called *colouring number* $\mathrm{col}(G)$ of G.

$$\mathrm{col}(G) = \min\{k \in \mathbb{N} : \exists \pi \in S_n, \deg_{G[\{v_{\pi(1)},\ldots,v_{\pi(i-1)}\}]}(v_{\pi(i)}) < k, \forall 1 \leq i \leq n\}$$

Then

$$\mathrm{col}(G) \leq \max_{H \subseteq G} \delta(H) + 1. \tag{11}$$

Argument: Use a particular ordering, namely: Build the ordering from the end, choose v_n such that $\delta(G) = \deg(v_n)$. Consider $G - v_n$. Choose as v_{n-1} a vertex with smallest degree in $G - v_n$

$$\deg_{G-v_n}(v_{n-1}) = \delta(G - v_n)$$

Consider $G - \{v_{n-1}, v_n\}$ and repeat the procedure until a complete ordering of vertices arises. Let $k = max_{H \subseteq G} \delta(H) + 1$. Check whether k has the property in Definition of $\mathrm{col}(G)$. Yes it does, so $\mathrm{col}(G) \leq \min\{\ldots, \max_{H \subseteq G} \delta(H)+1, \ldots\} \leq \max_{H \subseteq G} \delta(H) + 1$

add pic

On the other hand

$$\mathrm{col}(G) \geq \mathrm{col}(H) \text{ for all } H \subseteq G (\text{trivial}) \tag{12}$$

and also

$$\mathrm{col}(H) \geq \delta(H) + 1 \tag{13}$$

("back" degree of the last vertex $v_i \leq \deg(v_i)$ and $\deg(v_i) \geq \delta(H)$).

Putting things together:

Proposition 5.4 (5.4)**.** *For every graph $\chi(G) \leq \mathrm{col}(G) = \max\{\delta(H) : H \subseteq G\} + 1$ holds.*

Proof. By 12 and 13, $\mathrm{col}(G) \geq \mathrm{col}(H) \geq \delta(H) + 1 \forall H \subseteq G \implies \mathrm{col}(G) \geq \max_{H \subseteq G} \delta(H) + 1$. Thus, together with 11 implies $\mathrm{col}(G) = \max_{H \subseteq G} \delta(H) + 1$.

Show that $\chi(G) \leq \mathrm{col}(G)$. Let $k = \mathrm{col}(G)$. Then there exists an ordering π as in defintion of $\mathrm{col}(G)$. Let $v_1, \ldots, v_n$ be the orderin of vertices according to π ($v_i := v_{\pi(i)}$) Apply greed with this ordering. In every step i greedy will colour v_i with a colour among $1, 2, \ldots, \underbrace{\deg_{G[\{v_1,\ldots,v_{i-1}\}]}(v_i)}_{<k} + 1 \leq k = \mathrm{col}(G)$

So G is $\mathrm{col}(G)$-colourable. Thus, $\chi(G) \leq \mathrm{col}(G)$. $\square$

Corollary. *Every graph G has a subgraph H with $\delta(H) \geq \chi(G) - 1$.*

Theorem 5.5 (5.5 Brooks 1941)**.** *Let G be a connected graph. If G is neither complete nor an odd cycle, then $\chi(G) \leq \Delta(G)$.*

Proof. By induction on $|G|$. If $\Delta(G) \leq 2$, then G is a disjoint union of paths and cycles if it is not connected (general case). Since G is connected, then G is a cycle or a path. If cycle, then it is an even cycle $C_{2k} \implies \chi(C_{2k}) = \Delta(C_{2k}) = 2$. If path, then $P_n \implies \chi(P_n) = 2 = \Delta(P_n)$ for $n > 2$ and $n = 2 \implies K_2$ excluded from the theorem.

Assume wlog $\Delta(G) \geq 3$ in the rest of the proof. □

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Proof. (Brooks)
Claim: $\lambda(G) \leq \Delta(G)$ ($G \not\simeq K_n$, $G \not\simeq C_{2k+1}$, G connected) Assume wlog $\Delta(G) \geq 3 \implies |V(G)| \geq 4$. For $|V(G)| = 4$ Statement fulfilled for G_1, G_2, G_3, induction basis ✓ add pic

Induction on $|V(G)|$.

Induction step: consider G with $\Delta(G) \geq 3$ adn assume for all graphs with smaller number of vertices (fulfilling the condition of thm) the statement holds. Show that statement holds also for G.

Assume by contradiction that $\chi(G) > \Delta(G) =: \Delta$ Let $v \in V(G)$ arbitrarily and $H := G - \{v\}$. Check connected component of H whether condtion of thm are fulfilled. Let H' be such a connected component of H. If H' is not K_n, C_{2k+1}, then $\chi(H') \leq \Delta(H') \leq \Delta(G) = \Delta$. If H' is complete, then $\chi(H') = |V(H')| = \Delta(H') + 1 \leq \Delta(G)$. If H' is C_{2k+1}, then $\chi(H') = 3 \leq \Delta(G)$. add pic add pic

Thus $\chi(H) \leq \Delta$. Let us consider a Δ-colouring for H (notice that G cannot be colourd by Δ colours). Thus

$$\text{Every } \Delta \text{ colouring of } H \text{ uses all colours } 1, 2, \ldots, \Delta \text{ on the neighbours of } v \text{, in paritcular } \deg_G(v) = \Delta' \tag{14}$$

Let us denote by $v_i k \in N(v)$ the negihbours of v coloured by i, $1 \leq i \leq \Delta$. For all $i \neq j$ let $H_{i,j}$ be the subgraph of H induced by vertices coloured i or j. add pic

We show some properties of $H_{i,j}$:

$$\forall i \neq j : v_i \text{ and } v_j \text{ belong to the same connecected component of } H_{i,j} \tag{15}$$

Otherwise exchange colours in the connected component containing v_i; a proper colouring of H in which v_i and v_j are both coloured by j arises ` (14) Let $C_{i,j}$ be the connected component of $H_{i,j}$ containing both v_i and v_j. add pic

$$\text{We show that } C_{i,j} \text{ is a } v_i\text{-}v_j\text{-path} \tag{16}$$

Indeed, let P be a v_i-v_j-path in $C_{i,j}$. Since $\mathrm{d}_H(v_i) \leq \Delta - 1$, the neighbours of v_i have pairwise different colours; otherwise we could recolour v_i in contradiction to (14) add pic add pic

$\implies \deg_{C_{i,j}}(v_i) = 1$ and analogously $\deg_{C_{i,j}}(v_j) = 1$ Then if $C_{i,j} \neq P$, then P has an innner vertex with at least 3 neighbours (in $C_{i,j}$) coloured by the same colour. Let u be the first such vertex in the path P from v_i to v_j. There are at most $\Delta - 2$ colours used by neighbours of u. So recolour u with a colour different from i and j such that the colouring remains feasible. But then v_i and v_j would not belong to the same connected component of $H_{i,j}$ (wrt the new colouring) ` (15) add pic add pic

Last property:

For pairwise distinct colours i, j, k the path $C_{i,j}$ and $C_{i,k}$ meet only at v_i.
$$(17)$$

because otherwise there is a vertex u with 2 neighbours coloured j and k, respectively. add pic $\implies$ $\Delta - 2$ colours are used for neighbours of u, so u could be add pic recoloured and v_i, v_j would belong to different connected components of $H_{i,j}$ wrt new colouring ` (15).

Distinguish 2 cases:

a) $G[N(v)]$ is a clique:
$N(v) \cup v$ induces a clique $K_{\Delta+1}$ in G. add pic Then $G = K_{\Delta+1}$, otherwise if $x \notin N(v) \cup \{v\}$, then $\exists w \in N(v) \cup \{v\}$ such that $\{x, w\} \in E(G)$ and $\deg(w) \geq \deg_{K_{\Delta+1}}(w) + 1 = \Delta + 1$ ` $\Delta(G) = \Delta$ So $G = K_{\Delta+1} \implies$ this case cannot happen!

b) $G[N(v)]$ is not a clique $\exists v_1, v_2 \in N(v)$, $\{v_1, v_2\} \notin E(G)$. Consider path $C_{1,2}$ and let $u \neq v_2$ be the neighbours of v_1 in this path. So colour of add pic u, $c(u)$, is 2; $c(u) = 2$. Interchange colours 1 and 3 in $C_{1,3}$ and obtain a new colouring c' of H. Let v_i', $H_{i,j}'$, $C_{i,j}'$ etc be defined as described add pic above but wrt to c'. v_1 is coloured now by 3, $c'(v_1) = 3$ and u is a neighbour of v_1 $\implies$ $u \in C_{2,3}'$ $C_{1,2}^{\circ}$ retains its colouring (due to (17)). add pic $\implies u \in C_{1,2}^{\circ} \subseteq C_{1,2}'$ and $u \in C_{2,3}'$ ` (17) for colouring c'.

So neither a) nor b) can happen ` coming from assumption $\chi(G) > \Delta(G)$. Thus $\chi(G) \leq \Delta(G)$ $\square$

Theorem 5.6 (5.6 Erdös 1959)**.** *For every $k \in \mathbb{N}$ there exists a graph G with girth $\mathrm{g}(G) > k$ and chromatic number $\chi(G) > k$.*

Proof. No proof. $\square$

5.3 5.3 Colouring edges

Trivial lower bounds:

a) $\chi(G) \geq \Delta(G)$ add pic

b) $\chi'(G) \geq \lceil \frac{m}{\nu(G)} \rceil$ where $m := |E(G)|$ and $\nu(G)$ is the matching number (cardinality of largest matching) (Colour classes $E_1, E_2, \ldots, E_{\chi'(G)}$ $m = \sum_{i=1}^{\chi'(G)} |E_i| \leq \nu(G)\chi'(G)$)

Remark. In this subsection, when we talk about "colouring" we always refer to an "edge-colouring".

Proposition 5.7 (5.7 König 1916)**.** *Every bipartite graph $G = (U, V, E)$ ($V(G) = U \biguplus V$) satisfies $\chi'(G) = \Delta(G)$. An $\Delta(G)$-edge-colouring can be found in $\mathcal{O}\, n \cdot m$ time.*

Proof. Wlog G is connected (otherwise show it for every connected component and the result would follow for the whole graphs - convince yourself!) This implies, $m = \Omega(n)$. $n := |V(G)|$. Start with an empty (edge) colouring extend it by colouring one edge at each step, such a proper partial colouring arises (i.e.

colouring which assigns coloours to some edges without conflicts). If we show that every step (colouring of one additional edges) can be done in $\mathcal{O}\,n$ time while using $\Delta(G)$ colours, then we are done. Let $e = \{x, y\} \in E(G)$ not coloured yet. Let $x \in U$, $y \in V$. Consider $G - e$ and $\deg_{G-e}(x) \leq \Delta - 1$, $\deg_{G-e}(y) \leq \Delta - 1$ $\implies$ there is at least one colour in $\{1, 2, \dots, \Delta\}$ which is not used for coloured edges of $G - e$. Call such a colour free colour at x. Analogously there exists a free colour at y. add pic

If there is a colour f_e free in x and also in y, then colour e by f_e. Otherwise, for every colour α which is free in x there exists an edge incident to y coloured by α. Analogously, for every colour β which is free in y there exists an edge incident to x coloured by β.

Consider $A(\alpha, \beta)$ the subgraph of $G - e$ consisting of edges coloured by α or β. $\deg_{H_{\alpha,\beta}}(v) \leq 2 \implies$ connected components of $H_{\alpha,\beta}$ are paths or even cycles. $\deg_{H_{\alpha,\beta}}(x) = 1 = \deg_{H_{\alpha,\beta}}(y) \implies x, y$ are endpoints of paths.

Can x, y be endpoints of the same path No because otherwise add cycle in G ` add pic 18

Exchange colours either in the path containing x as indepenten or in the path containing y as indepenedent $\implies$ β or α will become free for both x and y, , respectiviely. Colour e by β or α. add pic $\square$

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- G bipartite $\implies \chi'(G) = \Delta(G)$ ✓
- a $\Delta(G)$-edge coloring can be constucted in $\mathcal{O}(nm)$ time ($n := |V(G)|, m := |E(G)|$)

missing colour at a vertex v is a color which is not used for edges incident to v and already coloured. Let a missing in x, let b missing in y; $H_{a,b}$ consists of edges with colour a or b, is a disjoint union of paths and cycles. Both x, y are endpoints of path in $H_{a,b}$, not of the same path, consider add pic add pic

The colours f (the one missing in both x and y, if any), a, b can be determined in $\mathcal{O}(\deg(x) + \deg(y))$ $P_{a,b}$ can be computed in $\mathcal{O}(n)$ if we use array

$$Neighbour(v, f) = \begin{cases} w \in V(G) \text{ s.t. } c(\{v, w\}) = f & v \in V(G) \\ 0 & \forall v \in V(G) \end{cases}$$

Also the minalisation by $Neigbour(v, f) := 0$ for all $v_i \simeq f$ is done in $\mathcal{O}(nm)$ time Update of $Neigbour(v, f)$ can also be done in $\mathcal{O}(n)$ time per edge newly colourd. Need to update just for $f = a$ and $f = b$: so compare effort per edge is $\mathcal{O}(n)$ with m edges results in $\mathcal{O}(nm)$

Theorem 5.8 (5.8 Vizing 1964). *For every graph G, $\Delta(G) \leq \chi'(G) \leq \Delta(G)+1$ holds.*

Proof. (constuctive)

Need to prove upper bound: Induction on $|E(G)|$. Induction basis $|E(G)| = 0$: $\chi'(G) = 0 \leq \underbrace{\Delta(G)}_{=0} + 1$. Induction step: Consider G with $|E(G)| \geq 1$ and assume statement holds for graph with $\leq |E(G)| - 1$ edges. Colouring $\equiv (\Delta(G)+1)$-edge-colouring; α-edge $\equiv$ edge coloured by colour α

Let $e \in E(G)$. $G - e$ is edge colourable by $\Delta(G-e)+1 \leq \Delta(G)+1$ colours. Consider such a colouring c of $G - e$. For all $v \in V(G)$ we have $\deg(v) \leq \Delta(G) < \Delta(G)+1 \implies$ there esists a colouring missing in v. Denote F_v set of missing colours in v for all $v \in V(G)$ If $F_x \cap F_y \neq \emptyset$, then take $f \in F_x \cap F_y$ and colour e by f. It results a colouring of G. Assume $F_x \cap F_y = \emptyset$. Let $b \in F_y$. since $b \notin F_x$, there exists $y' \in V(G)$ such that $\{x, y\prime\} \in E(G)$, $c(x, y') = b$. Let $b' \in F_{y'}$. If $b' \in F_x$, change colour of $\{x, y'\}$ to b' and use b to colour $\{x, y\}$ ✓ If not, repeat everything at y''.

add pic

add pic

In general, construct a sequence $\{y_j, 1 \leq j \leq k\} \subseteq T(x)$ of neighbours of x and a sequence $\{b_j, 1 \leq j \leq k\} \subseteq \{1, 2, \ldots, \Delta(G)+1\}$ of colour, such that

1. $y_1 = y$, $b_1 = b$
2. For every $1 \leq j \leq k-1$ $c(\{x, y_{j+1}\}) = b_j$
3. $y_1, y_2, \ldots, y_k$ are pairwise different.

When will the construction of such a sequence stop? If in step k we see that $\{x, y_k\}$ can be recoloured by f $(F_x \cap F_y \neq \emptyset$, $f \in F_x \cap F_y)$ Then we recolour als the other edges $\{x, y_j\}$, $1 \leq j \leq k-1$, as follows

ColourShift(k,f) begin

algorithm start

$$c(\{x, y_k\}) = f$$

For $j = k-1$ downto 1 do

$$c(\{x, y_j\}) = b_j$$

end

algorithm end

Afterwards colour b because a missing colour in x and we can colour $\{x, y\}$ by b ✓

The construction also stops if

$$y_k \in \{y_2, \ldots, y_{k-2}\}$$

Construct the sequences as follows $k = 0$, $y_1 = y$ Repeat

algorithm start

$$k := k+1$$

choose $b_k \in F_{y_k}$ If $b_k \in F_x$ then begin ColourShift(k, b_k) exit end else begin

$$y_{k+1} := \text{ the neighbour of } x \text{ such that } c(\{x, y_{k+1}\}) = b_k$$

end until $y_{k+1} \in \{y_1, \ldots, y_k\}$

algorithm end

Note that "repeat" will perform at most $\deg(x)$ times.

Assume $\backslash repeate$" is not terminated with exit. How to colour $\{x, y\}$ in this case? There exists an $i \in \{2, \ldots, k-1\}$ such that

$$y_{k+1} = y_i \iff b_k = b_{i-1}$$

where b_k is colour of $\{x, y_{k+1}\}$ and b_{i-1} colour of $\{x, y_i\}$.

Let $a \in F_x$. Denote by H_{a,b_k} the subgraph of G consitsing of 4dges coloured by a or b_k. H_{a,b_k} is disjiont union of paths and cycles. Since $b_k \in F_{y_k}$, degree of y_k in subgraph is ≤ 1. Thus, y_k is an endpoint of a path. (Could be empty)

Let z be the other endpoint of P. Distinguish 3 cases.

1. $z = y_{i-1}$. $b_k = b_{i-1} \in F_{y-1}$ Complete colouring by applying ColorShift$(i-1, a)$. add pic

2. $z = x$. Switch colour in P and ColorShift(k, a) add pic

3. $z \notin \{x, y_{i-1}\}$. Switch colours in P, ColorShift(k, a) add pic

If $P = \{y_k\}$, then $a \in F_{y_k}$ and do Color Shift(k, a). □

Corollary 10. *Every graph G can be edge-coloured by $\Delta(G) + 1$ colours in $\mathcal{O}(nm)$ time.*

Proof. Need to show that after starting with empty colouring the edges can be coloured one edge at a time in $\mathcal{O}(n)$ time per edge.

Colour an edge by applying ColorShift after having constructed the sequence. Show that both can be done in $\mathcal{O}(n)$ time.

The arrays:

1. MissingColour$(v) \in F_v$ for all $v \in V(G)$

2. Neighbour$(v, f) = \begin{cases} 0 & f \in F_v \\ u & \text{if } \{v, u\} \in E(G), c(\{v, u\}) = f \end{cases}$ for all $v \in V(G)$, for all $f \in \{1, \ldots, \Delta(G) + 1\}$

3. SequenceEntry$(v) = \begin{cases} 0 & v \notin \{y_1, \ldots, y_k\} \\ i & v = y_i \end{cases}$ for all $v \in V(G)$

Initialisation: MissingColor$(v) = 1$ for all $v \in V(G)$, Neighbour $\equiv 0$, SequenceEntry $\equiv 0$

For every edge e (to be coloured): All operations in Repeat, excluding ColorShift need at most $\mathcal{O}(\Delta(G)) = \mathcal{O}(n)$ times. ColorShift: $\mathcal{O}(\Delta(G)) = \mathcal{O}(n)$ because sequence has length $\leq \deg(x) \leq \Delta(G)$.

It remains to show that the arrays can be updated in $\mathcal{O}(n)$ after the colouring of each edge. Neighbour(x, f) has to be updated, $\mathcal{O}(\Delta(G)) = \mathcal{O}(n)$ time. add pic
Neighbour(y, b), Neighbour(y_2, b_2), Neighbour(y_2, b_1),...,Neighbour(y_k, b_k),Neighbour(y_k, b_{k-1}) $\mathcal{O}(2 \cdot \deg(x)) = \mathcal{O}(\Delta(G)) = \mathcal{O}(n)$

MissingColour$(y_k) = b_{k-1}$,...,MissingColour$(y_2) = b_1$; $\mathcal{O}(\deg(x))$ vertices constant time each thus $\mathcal{O}(n)$ MissingColour(x), MissingColour(y); just 2 vertices, check every colour, $\mathcal{O}(\Delta(G) + 1) = \mathcal{O}(n)$

SequenceEntry$(y_{k+1}) = k + 1$ in repeat $\mathcal{O}(\Delta(G))$ time $= \mathcal{O}(n)$

(to be reinitialized by 0 after the colouring of every edge, else $\mathcal{O}(n)$ time). □

Definition. A graph G belongs to class 1 ("is a class-1-graph") if $\chi'(G) = \Delta(G)$. A graph G belongs to class 2 ("is a class-2-graph") if $\chi'(G) = \Delta(G) + 1$.

Example. bipartiete graphs are class 1, C_{2n} is class 1, C_{2n+1} is class 2, K_n can be also classified depending on parity of n (Exercise)

Lemma 5.9. *A regular graph G belongs to class 1 if and only if it is* one-factisable, *i.e. $E(G) = \biguplus_{i \in I} M_i$ where M_i is a p. Matching of G, for all $i \in I$.*

Proof. Exercise □

Lemma 5.10. *A graph G with $m = |E(G)| > \Delta(G)\nu(G)$ is a class 2 graph.*

Proof.

$$\chi'(G) \geq \lceil \frac{m}{\nu(G)} \rceil \geq \frac{m}{\nu(G)} > \frac{\Delta(G)\nu(G)}{\nu(G)} = \Delta(G)$$
$$\stackrel{\text{Vizing}}{\Longrightarrow} \chi'(G) = \Delta(G) + 1 \text{ (class 2)}$$

□

Theorem 5.11 (5.9 Frieze, Jackson, McDiarmid, Reed 1988)**.** *Let p_n be the percentage of class 2 graphs on n nodes where the nodes are numbered and the numbering is fixed. Then for every $\varepsilon > 0$ and n large enough the inequalities*

$$n^{-(\frac{1}{2}+\varepsilon)n} < p_n < n^{-(\frac{1}{8}-\varepsilon)n}$$

holds.

Proof. No proof. □

Means class 2 graphs are a rare phenomenon. It is based on a result of Erdös and Wilson (1977) who have shown:

The percentage of graphs with n nodes (fixed numbering) which have more than one node of degree $\Delta(G)$ tends to 0 as $n \to \infty$.

02.06.2016

Corollary 11 (Vizing, Fournier 1973)**.** *(Corollary of Vizing's Theorem) Let $G = (V, E)$ be a graph with $S := \{v \in V(G) : \deg(v) = \Delta(G)\}$. If $G[S]$ is cycle-free (i.e. $G[S]$ is a forest), then G is a class 1 graph. A corresponding $\Delta(G)$-edge-colouring can be constructed in $\mathcal{O}(mn)$ time ($m := |E(G)|$, $n := |V(G)|$).*

Proof. Proof idea analogous to proof of Vizing's theorem: we willshow that under the conditions of corollary one of the colours $\{1, 2, \dots, \Delta(G)\}$ is missing in every vertex. Having shown that, the proof will be the same as in Vizing's theorem.

If $\Delta(G) = 0$, then nothing to show. Assume $\Delta(G) \geq 1$. Let F be the set of edges of $G[S]$, $F := E(G[S])$. Induction on $|F|$. Induction basis: $F = \emptyset$, so S is a stable set in G. For every $x \in S$ choose one edge $\{x, y\} \in E(G)$ (arbitrarily but fixed). Let M be the set of all these edges. Consider $G - M$ and apply Vizing:

$$\chi'(G - M) \leq \Delta(G - M) + 1 < \Delta(G) - 1 + 1$$

Thus,

$$\chi'(G - M) \leq \Delta(G)$$

Colour $G - M$ by colours $\{1, 2, \dots, \Delta(G)\}$. Colour iteratively the edges from M (one edge at a time) with colours $\{1, \dots, \Delta(G)\}$ like in the proof of Vizing's theorem, it works because one colour is missing for every vertex all the time. Assume we want to colour $\{x, y\}$ add pic

Induction hypothesis: Works for F with $|F| \leq k$. Induction step: Works for F with $|F| = k + 1$.

Let $x \in S$ be a leaf in $G[S]$ and let $\{x, y\} \in F$. Consider $G - \{x, y\}$. Set of vertices of max degree in $G - \{x, y\}$ is $S \setminus \{x\}$. add pic

$$|E((G-\{x,y\})[S\setminus\{x,y\}])| < |F|$$

($S\setminus\{x,y\}$ are the vertices of max degree in $G-\{x,y\}$) Apply Induction hypothesis for $G-\{x,y\} \implies G-\{x,y\}$ is class 1 coloured by $\Delta(G-\{x,y\}) \leq \Delta(G)$ colours. Can we colour $\{x,y\}$ by a colour in $\{1,2,\ldots,\Delta(G)\}$? Yes, because there is a missing colour both in x and in y ($\deg_{G-\{x,y\}}(x) = \deg_{G_{\{x,y\}}}(y) = \Delta(G)-1$) so we can colour like in Vizing's proof.

Time complexity: Colouring like in Vizing's is possible in $\mathcal{O}(mn)$ time. More than in Vizing: identify S (some search algorithm over G (DFS); $\mathcal{O}(m+n)$), check that $G[S]$ is a forest (DFS in $G[S]$; $\mathcal{O}(m+n)$; there exists a cycle if and only if there exists a backward edge)

special case □ add pic

5.4 5.4 Edge choosability and list colouring

Definition. Let $G=(V,E)$ and a list $L(e)$ of colours for all $e\in E(G)$. A *list colouring* is a mapping $c: E\to\bigcup_{e\in E(G)} L(e)$, such that $c(e)\in L(e)$, for all $e\in E(G)$ and $c(e)\neq c(f)$ for all $e,f\in E(G)$, $e\cap f\neq\emptyset$. Analogou list vertex colouring.

Example. add pic

Remark. 1. Example cases all $L(e)$ coincide classic colouring problem add pic

2. lists are disjoint; trivial problem

Definition. G is called *k-edge-chooseable* if for any given lists $L(e)$, $e\in E(G)$, which constains at least k different elements each, there exists an edge list colouring. The smallest number k, such that G is k-edge-chooseable is called *list chromatic index of G, $\chi'_e(G)$*.

$$\chi'(G)\leq\chi'_e(G)$$

because G is not $\chi'(G)-1$) edge choosable, just take $L(e)=\{1,2,\ldots,\chi'(G)-1\}$ for all $e\in E(G)$.

Conjecture 5.12 (Vizing)**.** *$\chi'(G)=\chi'_e(G)$ for all graphs (196? probably)*

Still open:

Theorem 5.13 (5.10 Galvin 1995)**.** *Bipartite graphs are $\chi'(G)$-edge-choosable.*

Proof. Based on Slivnik (1996).

□

Definition. Let $G=(L\uplus R,E)$ be a bipartite graph and $f: E\to\mathbb{N}$. We say G is f-edge-choosable if and only if for any given lists of colours $L(e)$, $e\in E(G)$, with $|L(e)|\geq f(e)$, there exists a list-edge-colouring.

Notation. Let $G=(L\uplus R,E)$. For all $e\in E$ denote by e_L, e_R its endpoints in the sets L and R, respectively ($e=\{\underbrace{e_L}_{\in L},\underbrace{e_R}_{\in R}\}$).

Lemma 5.14. *Let $G = (L \uplus R, E)$ be a bipartite graph and let $c : E \to \mathbb{N}$ be a proper edge-colouring of G. Define a function $T_{G,c} : E \to \mathbb{N}$,*

$$T_{G,c}(e) := \underbrace{|\{f \in E : (f_L = e_L) \wedge (c(f) > c(e))\}|}_{f \text{ dominates } e \text{ from the left}} + \underbrace{|\{f \in E : (f_R = e_R) \wedge (c(f) < c(e))\}|}_{f \text{ dominates } e \text{ from rht right}}$$

Then G is $(T_{G,c} + 1)$-edge-choosable. add pic

Proof. No proof. □

Proof of Theorem 5.13 assuming Lemma 5.14 holds. Let c be an optimal edge colouring of $G = (L \uplus R, E)$, i.e. c uses colour $\{1, 2, \ldots, \chi'(G)\}$.

$$T_{G,c}(e) \leq (\chi'(G) - c(e)) + (c(e) - 1) = \chi'(G) - 1$$

Apply Lemma 5.14: G is $(T_{G,c} + 1)$-edge-choosable, i.e. $\chi'(G)$-edge-chooseable. □

6 Perfect graphs

Definition. A graph G is callec *perfect* if and only if for all induced subgarphs H of G, $\chi(H) = \omega(H)$ holds, i.e. the clique number $\omega(H)$ which generarlly is a trivial but not tight lower bound on $Xi(H)$, is in this case a tight lower bound.

Example.

1.)$\chi(G) = \omega(G)$ is fullfilled for K_n but not for its subgraphs $H \subseteq K_n$ (induced!): $\chi(H) = \omega(H)$ because induced subgarphs of K_n are complete; thus, perfect
2. bipartite graphs G: $\chi(G) = \omega(G) = 2$, $H \subseteq G \implies H$ bipartite $\implies \chi(H) = \omega(H)$ (all induced subgraphs are again bipartite); thus, bipartite graphs are perfect
3. comparability graphs are perfect (exercise)
4. Interval graphs are perfect (exercises)
5. chordal graphs

Definition. A graph $G = (V, E)$ is a comparability graph if and only if there exists partial ordering of the vertices usch that only comparable vertices wrt ordering are joined by an edge. (e.g. trees)

Definition. A graph $G = (V, E)$ is called an *interval graph* if and only if there exists a set $\{I_v : v \in V(G)\}$ of intervals in the real line such that $\{u, v\} \in E(G)$ is an edge if and only if $I_v \cap I_u \neq \emptyset$. add pic

Definition. A graph $G = (V, E)$ is called a *chordal graph* if and only if every cycle of length more than 3 has a chord, or equivalently there are no induced cycles of length more than 3. (also called "triangulated graph") add pic

06.06.2016

Definition. If a graph G has induced subgraphs G_1, G_2, S such that $G = G_1 \cup G_2$ ($V(G) = V(G_1) \cup V(G_2)$, $E(G) = E(G_1) \cup E(G_2)$) and $S = G_1 \cap G_2$ we say that G *is obtained from G_1 and G_2 by pasting them along S.* add pic

Proposition 6.1. *A graph is chordal if and only if it can be constructed recursively by pasting chordal graphs along complete subgraphs starting from complete graphs.*

Proof. $\Leftarrow$ If G is obtained from two chordal graphs G_1, G_2 pasted along a complete subgraph S, then it is chordal. Consider an induced cycle C in G. If C induced cycle of $G_i \implies |C| = 3$ G_i chordal, $i = 1, 2$. Otherwise C "jumps" from G_1 to G_2 and back to G_1. If length of $C \geq 4$, then there exist $s_1, s_2 \in V(C)$ such that $s_1, s_2 \in S$, and therefore $\{s_1, s_2\} \in E(G)$. So C is not chordal. add pic

$\Rightarrow$ Assume that G is chordal, show that it has the required structure. Induction on $|V(G)|$. If G is complete ✓ (Set $G_1 = G$, $G_2 = G$, $S = G$) Assume wlog G is not complete, in particular $|V(G)| > 1$. Let $a, b \in V(G)$ such that $\{a, b\} \notin E(G)$. Consider minimal separator X of a and b (minimal wrt inclusion). Let C be the connected component of $G \setminus X$ which contains a.

$$G_1 := G[V(C) \cup X], G_2 := G[V(G) \setminus V(C)], S = G_1 \cap G_2$$

Need to show G_1, G_2 chordal and S complete. G_1, G_2 induced subgraphs of chordal graph $G \implies G_1, G_2$ are chordal. It remains to show S is complete. Assume it is not, let $u, v \in V(S)$ such that $\{u, v\} \in E(G)$. Since X is a minimal separator both u and v have neighbours in C and also in $G \setminus (V(C) \cup X)$. So there exists a u-v-path P_1 with inner vertices lying in C. Analogously there exists u-v-path P_2 with inner vertices lying in $G \setminus (V(C) \cup X)$. Choose P_1 and P_2 to be shortest such paths, respectively. Then there exists a cycle P_1, P_2 with length ≥ 4 and no chord. ` chardinality of G. $\implies$ S is complete. add pic

□

Proposition 6.2 (6.2)**.** *Every chordal graph is perfect.*

Proof. Since complete graphs are perfect, it is enough to show it for non complete chordal graphs. Let G be such a graph and let G_1, G_2 be such a graph and let G_1, G_2 and S like in Proposition 6.1. Let H be an induced subgraph of G. Want to show $\chi(H) \leq \omega(H)$. Let $H_i = H \cap G_i$, $i = 1, 2$-, $T := H \cap S$ complete because S complete. H arises from H_1 and H_2 by pasting them along T. As an induced subgraph of G_i each H_i can be coloured by $\omega(H_i)$ colours, because G_i is a chorda graph and $|V(G_i)| < |V(G)|$. So by induction, assumption G_i is perfect and $\chi(H_i) \leq \omega(H_i)$ for its induced subgraph H_i. Paste a colouring of H from colourings of H_i (with $\leq \omega(H_i)$ colours each) by eventually permuting the colours in H_2. We will use $\leq \max\{\omega(H_1), \omega(H_2)\}$ colours. But $\max\{\omega(H_1), \omega(H_2)\} \leq \omega(H)$. add pic □

Idea: Characterization of perfect graphs by forbidden induced subgraph (e.g. all imperfect induced subgraphs should be forbidden!!!) such that the class of forbidden induced subgraphs is as "small" as possible. Strong perfect graph conjecture of Berge (1963) was proven in 2002.

Theorem 6.3 (6.3 Chudnovsky, Robertson, Seymour, Thomas 2002)**.** *A graph G is perfect if and only if neither G nor $\overline{G}$ (complement of G) contains an odd cycle of length at least 5 as an induced subgraph*

Proof. No proof. $\square$

We prove the former weak perfect graph conjecture (Berge 1963).

Theorem 6.4 (6.4 Lovesz 1972)**.** *G is perfect $\iff$ $\overline{G}$ is perfect.*

Definition. Consider graph G and $x \in V(G)$. Let G' be obtained from G by adding a new vertex x' and connecting it only to $\{x\} \cup N(x)$. We say G' *is obtained from G by expanding x to x'.* add pic

Lemma 6.5. *Any graph obrained from a perfect graph by expanding a vertex is again perfect.*

Proof. Induction on $|V(G)|$ where G is the considered graph for expansion. Observe: Expanding some vertex of K_n yields K_{n+1}. (Induction basis $|v(G)| = 1 \implies G = K_1 \checkmark$) So assume G perfect but is not complete, i.e. $|V(G)| > 1$. Let G' be obtained from G by expanding x to x'. We show that G' is perfect, i.e.

$$\chi(H) \leq \omega(H),\ H \text{ induced subgraph of } G'. \tag{18}$$

Observe: 18 is immediately fulfilled if H is a proper induced subgraph of G: Indeed if $x' \notin V(H)$, then it is not subgraph of G then 18 follows because G add pic perfect. Otherwise, $x' \in V(H)$, $H_1 := H \cap G$ is a proper induced subgarph of G, and H is obtained from H_1 by extending it on x. OR H if $x \notin V(H)$ then H is an induced subgraph of $G'' = G' \setminus \{x\} \simeq G$ which is perfect.

So it remains to show $\chi(G') \leq \omega(G')$. Let $\omega := \omega(G)$. Then $\omega(G') \in \{\omega, \omega + 1\}$. If $\omega(G') = \omega + 1$, $\checkmark$ because $\chi(G') \leq \chi(G) + 1 \leq \omega(G) + 1 = \omega + 1 = \omega(G')$. Assume $\omega(G') = \omega$. Then $x \notin K$ *for any clique K with ω vertices in G.* because add pic $K \cup \{x'\}$ would yield a clique with $\omega + 1$ elements in G' \`. Let us colour G with ω colours. Every $K_\omega \subseteq G$ meets the colour class of x but not x itself. Thus, $H := G - (X \setminus \{x\})$ has clique number $\omega(H) < \omega$ where X is the colour class of x. (all maximal cliques of G "loose" one vertex in H, namely the one coloured add pic like x)

Observe that $X \setminus \{x\} \cup \{x'\}$ is a stable set in G', so extend a coluring of H which use $< \omega$ colours to a colouring of G' by adding a new colour on $X \setminus \{x\} \cup \{u'\}$. This yields a colouring with $\leq \omega$ colours for G'. $\checkmark$ $\square$

Proof of Theorem 6.4. Induction on $|V(G)|$. We show G is perfect $\implies$ $\overline{G}$ perfect. For $|V(G)| = 1$ we have $\overline{G} = G\checkmark$. Let $|V(G)| \geq 2$. Let $\mathcal{K}$ denote the set of all vertex set of complete subgraphs of G,

$$\mathcal{K} := \{K \subseteq V(G) : G[K] \text{ is complete}\}$$
$$\mathcal{A} := \{S \subseteq V(G) : G[S] \text{ is empty graph}, |S| = \alpha\}$$

where $\alpha := \alpha(G)$ (cardinality of max stable set in G). Observe: Every proper unduced subgraph of $\overline{G}$ is the complement of a proper induced subgraph of G and hence it is perfect by induced.(induction??) It remains to show $\chi(\overline{G}) \leq \omega(\overline{G}) = \alpha(G)$.

To this end we shall find a set $K \in \mathcal{K}$ such that $K \cap A \neq \emptyset$, for all $A \in \mathcal{A}$, then $\omega(\overline{G} - K) = \alpha(\overline{G} - K) \overset{18}{<} \alpha = \omega(\overline{G})$, so by induction hypothesis

$$\chi(\overline{G}) \leq \chi(\overline{G} - K) + 1 \leq \omega(\overline{G} - K) + 1 \overset{18}{\leq} \omega(\overline{G})$$

Assume there exists no such a $K \in \mathcal{K}$. Then for all $K \in \mathcal{K}$ there exists a $A_K \in \mathcal{A}$ such that $K \cap A_K = \emptyset$. Let $k(x) := |\{K \in \mathcal{K} : x \in A_K\}|$. Construct G' via substituting every vertex x by a complete graph $K_{k(x)}$ and for all $\{x, y\} \in E(G)$ cannect all vertices of $K_{k(x)}$ to all vertices of $K_{k(y)}$. then add pic $V(G') = \bigcup_{x \in V(G)} V(G_x)$ and $u, v \in V(G')$ are connected by an edge if either $u, v \in G_x$ for some $x \in V(G)$ or $u \in G_x$, $v \in G_y$ with $\{x, y\} \in E(G)$.

Notice that G' can be obtained by vertex expansion from $G[\{x \in v : k(x) > 0\}]$ So G' is perfect because $G[\{x \in v : k(x) > 0\}]$ is perfect as induced subgraph of G and because of Lemma 6.5.

$$\chi(G') \leq \omega(G')$$

□

13.06.2016

Lemma 6.6. *G perfect, G' extension of G on x implies G' is perfect.*

Proof of Lovasz. Induction on $|V(G)|$
Notation $\alpha := \alpha(G)$, $\mathcal{K}$ the set of vertex sets of complete subgraphs of G, $\mathcal{A}$ set of stable set with cardinality α in G.

Claim: If there exists $k \in \mathcal{K}$, such that $k \cap A \neq \emptyset$ for all $A \in \mathcal{A}$, then $\chi(\overline{G}) = \omega(\overline{G})$. ✓

Prove that there exists a $K \in \mathcal{K}$ such that $K \cap A \neq \emptyset$ for all $A \in \mathcal{A}$.

Assume this does not hold: For all $K \in \mathcal{K}$ exists an $A \in \mathcal{A}$ $K \cap A = \emptyset$. Denote by A_K such a set A in $\mathcal{A}$, so $K \cap A_K = \emptyset$. Let $k(x) := |\{K \in \mathcal{K} : x \in A_K\}|$, for all $x \in V(G)$.

Construct G'. pic add pic

Observe: G' can be obtained from $G[\{x \in V(G) : k(x) > 0\}]$ by repeated extension on some vertex. G perfect $\implies$ $G[...]$ perfect $\overset{\text{Lemma}}{\implies}$ G' perfect $\implies$

$$\chi(G') \underset{\leq}{=} \omega(G') \tag{19}$$

Compute $\chi(G')$, $\omega(G')$ Consider max complete subgraph of G' (max wrt inclusion). It has the form $G'[\bigcup_{x \in X} G_x]$ where $x \in \mathcal{K}$. So there exists an $X \in \mathcal{K}$

$$\begin{aligned}
\omega(G') &= \left| G'[\bigcup_{x \in X} G_x] \right| = \sum_{x \in X} k(x) \\
&= \underbrace{|\{(x, k) : x \in X, K \in \mathcal{K}, x \in A_K\}|}_{=K(x) \text{ for every fixed } x} = \sum_{K \in \mathcal{K}} \underbrace{|X \cap A_K|}_{\leq 1} \\
&\leq |\mathcal{K}| - 1
\end{aligned}$$

Then

$$\omega(G') \leq |\mathcal{K}| - 1 \tag{20}$$

$$|V(G')| = \sum_{x \in V} k(x) = |\{(x, K) : x \in V(G), K \in \mathcal{K}, x \in A_K\}| = \sum_{K \in \mathcal{K}} |V \cap A_K| = \sum_{K \in \mathcal{K}} |A_K| = |K|\alpha$$

since $|A_K| = \alpha$.

$$\chi(G') \geq \frac{|V(G')|}{\alpha(G')} = \frac{|\mathcal{K}|\alpha}{\alpha} = |\mathcal{K}|$$

So

$$\chi(G') \geq |\mathcal{K}| \tag{21}$$

20 and 21 imply $\chi(G' \geq |\mathcal{K}| > |\mathcal{K}| - 1 \geq \omega(G')$ which gives a contradiction ` to 19

Thus, $\alpha(G') \geq \alpha$ and $\alpha(G') \leq \alpha$ is trivial. □

Theorem 6.7 (6.6 Lovasz 1972)**.** *A graph G is perfect if and only if $|V(H)| \leq \alpha(H)\omega(H)$ for all induced subgraphs H.*

Notice that Theorem 6.7 implies Theorem **??**; need $V(\overline{H}) \leq \alpha(\overline{H})\omega(\overline{H})$ for all induced subgraph $\overline{H}$ of $\overline{G}$. $\overline{H}$ induced subgraph of $\overline{G}$ $\iff$ H induces subgraph of G. For H Theorem 6.7 implies $\left|V(\overline{H})\right| = |V(H)| \leq \alpha(H)\omega(H) = \omega(\overline{H})\alpha(\overline{H})$. Note that $\alpha(H) = \omega(\overline{H})$ and $\omega(H) = \alpha(\overline{H})$.

Proof. Let $V(G) = \{v_1, \ldots, v_n\}$. Set $\alpha := \alpha(G), \omega := \omega(G)$

- ⇒ (trivial) Assume G is perfect. Let H be an induced subgraph. $\chi(H) = \omega(H)$. Consider optimal colouring of H. $V(H) = \biguplus$ colour classes $\implies$ $|V(G)| = \sum_{\text{colour classes}} |\text{colour class}| \leq \chi(H)\alpha(H)$ Where $\chi(H)$ is the nr of summants, $\alpha(H)$ is bound cardinality of every summand

- ⇐ Inequ holds for all induced subgarph H. Show G is perfect.

  Induction on $n = |V(G)|$. Induction basis $n = 1$: trivial. Assume the statement holds for graphs with less than n vertices. Show if for G with $|V(G)| = n$.

  Assume by contradiction that G is not perfect. By induction hypothesis, every prober induced subgraph of G is perfect. Thus, $\chi(G) > \omega(G)$.

  Let $U \subseteq V$ be a non-empty independent set in G.

  $$\chi(G - U) = \omega(G - U) = \omega \tag{22}$$

  (induction hyp)

  Indeed $\omega(G - U) \leq \omega(G)$. If $\omega(G - U) < \omega$, then $\chi(G - U) < \omega \implies \chi(G) \leq \omega$ (colour the whole U by one optimal colour) `

  Apply 22 to $U = \{u\}$ (singleton) and consider an ω-colouring of $G - U$. Let K be the vertex set of a clique K_ω in G.

(a) If $u \notin K$ then K meets every colour class of $G - U$.

(b) If $u \in K$ then K meets all but exactly one colour class of $G - U$ because $K - U$ is $K_{\omega-1}$ in G_U.

Let $A_0 = \{u_1, u_2, \dots, u_\alpha\}$ be a stable set of cardinality α in G. Let $A_1, \dots, A_\omega$ be the colour classes of an ω-colouring in $G - U_1$. Let $A_{\omega+1}, \dots, A_{2\omega}$ be the colour classes of an ω-colouring in $G - U_2$. and so on $A_{(\alpha-1)\omega+1}, \dots, A_{\alpha\omega}$ be the colour classes of an ω-colouring in $G - U_\alpha$.

Altogether consider $A_0, A_1, \dots, A_{\alpha\omega}$ independent sets. For all $0 \leq i \leq \alpha\omega$ there exists a $K_\omega \subseteq G - A_i$ by 22. Denote by k_i the vertex of such a clique.

Notice that if K is the vertex set of a K_ω in G then $K \cap A_i = \emptyset$ for exactly one $i \in \{0, 1, \dots, \alpha\omega\}$ (4) add ref

Indeed if $K \cap A_0 = \emptyset$ then $K \cap A_i \neq \emptyset$ for all $i \neq 0$ by definition of A_i and 0a. (for all i $u_i \notin K$???) If $K \cap A_0 \neq \emptyset$, then $|K \cap A_0| = 1$. Let $\{U_{i_0}\} = K \cap A_0$. Then $U_{i_0} \in K$, apply 0b and get only one empty intersection with colour classes of $G - U_{i_0}$. For $i \neq i_0$ $u_i \notin K$, apply 0a, and obtain non-empty intersection with all corresponding colour classes.

Let J be a real $(\alpha\omega + 1) \times (\alpha\omega + 1)$-matrix with 0 on diagonal and ones everywhere else. matrix Let $A = (a_{ij})$ be an $(\alpha\omega+1) \times n$-matrix such that add pic each row i the incidence vector of A_i: $a_{ij} = \begin{cases} 1 & V_j \in A_i \\ 0 & \text{otherwise} \end{cases}$ Let $B = (b_{ij})$ be an $n \times (\alpha\omega + 1$-matrix such that each column j is the incidence vector of K_j; $b_{ij} = \begin{cases} 1 & v_i \in K_j \\ 0 & \text{otherwise} \end{cases}$

By property 4, add ref

$$|K_J \cap A_i| = 1$$

for all i, j with $i \neq j$ and of course by definition of K_i

$$|A_i \cap K_i| = 0$$

Thus, $A \cdot B = J$

$$\det(J) \neq 0 \implies \operatorname{rank}(A) = \alpha\omega + 1 \leq n \quad `n \leq \alpha\omega$$

□

6.1 6.2 Recognition of chordal graphs and computation of ω, χ, α and θ in chordal graphs

Recognition problem:
Input: $G = (V, E)$, $n = |V(G)|$, $m = |E(G)|$
Question: Is G chordal?

Solvable in linear time $\mathcal{O}(n + m)$.

Lemma 6.8. *A graph G is chordal if and only if every minimum separating set of vertices induces a clique.*

Proof. $\Rightarrow$ Let S be a minimum set of vertices. If $|S| = 1$, nothing to show. Assume $|S| \geq 2$. Let H_1, H_2 be two connected components of $G - S$, for all $s \in S$ has neighbours in H_1 and H_2 (otherwise $S \setminus \{s\}$ is also separating). pic Let u, v be two arbitrary vertices in S. Let P_i be a shortest u-v-path (add pic) using only inner vertices of H_i, $i = 1, 2$. P_1 and P_2 build a cycle of length ≥ 4. It has to have a chord. Since P_i are shortest paths the only possible chord is $\{u, v\}$, so $\{u, v\} \in E$

$\Leftarrow$ (homework)

□

Definition. A vertex v is called *simplicial vertex* in G if its neighbourhood is a clique in G.

Example. graph (add pic)

Lemma 6.9 (6.8 Dirac 1961)**.** *Every chordal has at least one simplicial vertex and if G is not complete, then it has at least 2 simplicial vertices which are not connected by an edge.*

Proof. Induction on $n = |V(G)|$. Homework: check it for $i = 1, 2, 3$ or for $G = K_n$.

Assume there exist $u, v \in V(G)$ such that $\{u, v\} \notin E(G)$. consider an inclusion minimal set in $V(G)$ which separates u and v. Then $G - S$ has connected components H_i, $i \in I$, $|I| \geq 2$. S induces a clique in G (Lemma 6.8). Moreover, $G[H_i \cup S]$ is chordal, for every $i \in I$ and by induction hypothesis $G[H_i \cup S]$ has at least on simplicial vertex.

If $v \in H_i$ is a simplicial vertex [Pic: component graph] $\Gamma_{G[H_i \cup S]}(v) = \Gamma_G(v)$ (add pic) and thus $\Gamma_G(v)$ is a complete graph so v is simplicial vertex in G.

If $G[H_i \cup S]$ is not complete, there are at least 2 simplicial vertices not connected by an edge in $G[H_i \cup S]$ (induction hypothesis). So one of them has to lie in H_i. If $G[H_i \cup S]$ complete, every vertex is simplicial vertex, so choose $v \in H_i$. A simplicial vertex in $G[H_i \cup S]$ and also in G.

Since $|I| \geq 2$, we can find at least 2 simplicial vertices, at least one in each connected component H_i in G. □

Definition. Let $G = (V, E)$ a graph on n vertices. A total ordering $\sigma : V(G) \to \{1, 2, \dots, n\}$ is called *perfect (vertex) elimination scheme* (PES), if $v \in V$ is simplicial vertex in $G[\{u \in V : \sigma(u) \geq \sigma(v)\}]$, i.e. $U\Gamma(v) = \{u \in V : \{v, u\} \in E(G); \sigma(u) > \sigma(v)\}$ (upper neighbourhood) is a clique.

16.06.2016

Definition. *Upper neighbourhood*

$$U\Gamma(v) \coloneqq \{w \in \Gamma(v) : \sigma(w) > \sigma(v)\}$$

where $\sigma : V(G) \to \{1, 2, \dots, n\}$ (bijective or ordering), $n \coloneqq |V(G)|$.

Proposition 6.10 (6.9 Fulkerson, Gross 1965)**.** *$G = (V, E)$ is chordal if and only if there exists a $\sigma : V(G) \to \{1, 2, \ldots, n\}$ bijective, which is PES.*

Proof. $\Rightarrow$ trivial by respected application of Lemma **??**. There exists v_1 simplical; $G' = G \setminus \{v_1\} \subseteq G$ is chordal, There exists a simplicial vertex v_2, $G' = G \setminus \{v_1, v_2\} \subseteq G$ chordal, There exists a s. v. v_2 and so on. Claim: $v_1, v_2, \ldots, v_n$ is PES. Indeed because v_i is s. v. in $G \setminus \{v_1, \ldots, v_{i-1}\}$, $1 \leq i \leq n$ and therefore $\Gamma_{G \setminus \{v_1,\ldots,v_{i-1}\}}(v_i) = U\Gamma_G(v_i)$ is a clique.

$\Leftarrow$ Assume G has a PES. Show G is chordal. Let C be a cycle with ≥ 4 vertices. Let $u = arg \min\{\sigma^{-1}(v) : v \in V(C)\}$ add pic $\implies u^-, u^+ \in U\Gamma(u) \implies \{u^-, u^+\} \in E(G)$ which is a clique

pic

□

Algorithmic idea for recognition of chordal graphs

Set $i = 0$ check whether G a simplical vertex. If not, stop "G is not chordal". If yes, let v_i be a s.v. Set $G = G \setminus \{v_i\}$, $i = i + 1$. Repeat until $i = n$.

Possible implementation in $\mathcal{O}(n^2 m)$ because for-loop is repeated $\leq n$ times.

Each repetition in $\mathcal{O}(nm)$: $G := G \setminus \{v_i\}$ in $\mathcal{O}(\deg(v_i))$, check for s.v. in G; there are at most n candidates for a s.v. The simplicity could be checked in $\mathcal{O}(m)$ times (need to check whether $\Gamma_G(v)$ contains all edges in $\mathcal{O}(m)$ time.

Speed up:

R.E. Tarjan and M. Vannekakis, Simple linear time algorithms to test chordalty of graphs, but acyclicity of hypergraphs and selectivly reduce ancyclic hypergraphs, SIAM J. on Computing 13, 1984, 566-579; Addendum 14, 1985, 254-255

2 step procedure

1. Construction of total ordering σ in $V(G)$ such that G is PES $\iff$ G chordal
2. check whether σ is PES.

Ad 1) This "special" σ is the reverse *maximum adjecency order*

Definition. MA Order: $\sigma : V(G) \to \{1, 2, \ldots, n\}$ bijective such that for all $1 \leq i \leq n$:

$$|\{v \in \{v_1, \ldots, v_{i-1}\} : \{v_i, v\} \in E(G)\}| = \max_{u \notin \{v_1,\ldots,v_{i-1}\}} |\{v \in \{v_1, \ldots, v_{i-1}\} : \{u, v\} \in E(G)\}|$$

text missing

Example. add pic

pic

Result from CombOpt 1 (or Korte and Vygens, Conbinatonal Optimisation Theory and Applications, 2010, Springer)

MA can be cmputed $\mathcal{O}(m + n \log n)$ time by Fibonacci heaps.

"Special" σ is the reversed MA order.

Proposition 6.11 (6.10)**.** *A total ordering σ in $V(G)$ is a PES $\iff$ If $v_i, v_j \in V(G)$ are connected by a path P with inner vertices n with $\sigma(n) < \min\{\sigma(v_i), \sigma(v_j)\}$, then $\{v_i, v_j\} \in E(G)$.*

Proof. $\Leftarrow$ trivial: Consider $U\Gamma(v)$ add pic Path P fulfills the property mentioned in the implication $\implies \{u_1, u_2\} \in E(G)$ pic

$\Rightarrow$ Assume σ is PES. Show the implication. Let $v_i, v_j \in V(G)$ be connected by path P as in the implication. Want to show $\{v_i, v_j\} \in E(G)$. Assume wlog P is the shortest such path. If $\{v_i, v_j\} \in E(G)$ ✓ If not, there exists an inner vertex in P. Choose the inner vertex u of P with smallest σ-label. add pic $u^-, u^+ \in U\Gamma(v) \overset{\sigma \text{ is PES}}{implies} \{u^-, u^+\} \in E(G) \implies P$ could be shortened ` pic

□

Proposition 6.12 (6.11)**.** *Let G be a chordal graph. Let MA order yield $v_1, \dots, v_n$. Then the vertices of G in reversed order, $v_n, v_{n-1}, \dots, v_1$ build a PES.*

Proof. $\sigma : V(G) \to \{1, 2, \dots, n\}$ bijective. $\sigma(v_i) = n - i + 1$

We claim that property 23 holds:

$$\text{If } \sigma(u) < \sigma(r) < \sigma(w) \text{ and } \{u, w\} \in E(G), \{v, w\} \notin E(G), \text{ then } \exists z \in V(G), \tag{23}$$

$$\text{such that } \sigma(v) < \sigma(z), \{v, z\} \in E(G), \{u, u\} \notin E(G) \tag{24}$$

Indeed: When MA labels v (which is not connected to w while u is), there must be a labelled vertex which is a neighbour of v but not a neighbour of u; call it z and check the properties in 23.

Assume σ is not a PES. By Propositon 6.11, there exist $v_i, v_j \notin E(G)$ such that there exists a v_i-v_j-path P with

$$\text{inner vertices } u \text{ with } \sigma(u) < \min\{\sigma(v_i), \sigma(v_j)\}. \tag{25}$$

Select such a pair v_i, v_j where v_j has the largest label $\sigma(v_j)$. If there are many possible vertices v_i select the one with largest σ. Let P be a shortest v_i-v_j-path with property 25. add pic Since $\{v_i, v_j\} \in E(G) \implies \exists$ inner vertex pic of P, let u be inner vertex adjacent to v_j. Apply property 23 to u, v_i, v_j with $\sigma(u) < \sigma(v_i) < \sigma(v_j)$. $\implies \exists z : \sigma(z) > \sigma(v_i)$, $\{z, v_i\} \in E(G), \{u, z\} \notin E(G)$. $\{z, v_i\} \in E(G)$; otherwise path z-v_j would fulfill condition of 23 and contradict the selection of v_i.

$P \cup \{z, v_j\}$ is a cycle C with ≥ 4 vertices $\overset{G \text{ chordal}}{\implies} \exists$ chordal in C, chord has to pass through z otherwise P was not shortest.

Let x be the first vertex in P (starting at v_j) which is connected z, x has to the left of n ($\{z, u\} \notin E(G)$) But then there is new cycle C' (length ≥ 4) without a chord `. □

Theorem 6.13. *It can be checkd in $\mathcal{O}(n + m)$ time if the reversed MA order is a PES.*

Proof. No proof. □

Corollary 12. *This leads to an $\mathcal{O}(m + n \log n)$ algorithm for recognition of chordal graphs.*

Remark. The number $\alpha(G)$, $\omega(G)$, $\chi(G)$, $\theta(G)$ can be computed in $\mathcal{O}(m + n\log n)$ time in chordal graphs (and in efficient time n perfect graphs).

Consider a PES σ (reversed MA order) Colouring: Greedy with order σ leads to an optimal colouring (exercises). Clique: All max cliques (wrt inclusion) are among $\{v\} \cup U\Gamma(v)$ (exercises). Stability: Set $s_1 = \sigma^{-1}(1)$; inductively set s_k to be the first vertex after s_{k-1} which has no neighbours among $s_1, \dots, s_{k-1}$. Consider $S \coloneqq \{s_1, \dots, s_\ell\}$, it is a stable set and also a max stable set because $(\{s_k\} \cup U\Gamma(s_k))_{1\le k\le \ell}$ is a family of cliques such that

$$V(G) = \bigcup_{k=1}^{\ell} \{\{s_k\} \cup \Gamma(s_k)\}.$$

So we found a stable set S with $|S| = \ell$ and a covering by cliques with ℓ cliques. $\max|S| \le$ min number of cliques in clique covering

$\max_{S \text{ stable}}|S| \le$ min cardinality of a clique covering, i.e.

$$\alpha(G) \le \theta(G)$$

Implies $S = \{s_1, \dots, s_\ell\}$ and $(\{s_k\} \cup U\Gamma(s_k))_{1\le k\le \ell}$ is a certificate of optimality.

7 Random graphs

20.06.2016

7.1 The notion of a random graph

$V = \{0, 1, \dots, n-1\}$. Let G be the set of all graphs on V. Introduce a probability space on G, i.e. a triple $(\Omega, \mathcal{F}, P)$ where Ω is the sample space, $\mathcal{F}$ is a σ-algebra of events, P is the probability measure. Let $p \in [0,1]$, $p \in \mathbb{R}$. For every $e \in V^2$ (non-ordered pair of different vertices), decide by some random experiment whether e is an edge or not, e.g. biased coin with probability p for "head" and $1-p$ for "tail". Then e is an edge with probability p and no edge with probability $1-p$. Repeat this experiment independently for each e. At the end we have a graph G on n vertices in V. Consider a given graph G_0 on V. $\text{Prob}(G = G_0) = p^{|E(G_0)|}(1-p)^{\binom{n}{2}|E(G_0)|}$.

Need to check whether this prop form a probability measure.

Alternative way of formally defining $(\Omega, \mathcal{F}, P)$:

Construct $(\Omega_e, \mathcal{F}_e, P_e)$ a probability space for all pair of different vertices e.

$$\begin{aligned} \Omega_e &= \{0_e, 1_e\} \\ \mathcal{F}_e &= 2^{\Omega_e} \\ P_e(\{1_e\}) &= p \\ P_e(\{0_e\}) &= 1 - p \coloneqq q \end{aligned}$$

Set $(\Omega, \mathcal{F}, P) \coloneqq \prod_{e\in V^2}(\Omega_e, \mathcal{F}_e, P_e)$.

(Note: alternative notation: $V^2 = \binom{V}{2}$)

An element $\omega \in \Omega$ is given as $(x_e)_{e\in V^2}$ where $x_e \in \{0_e, 1_e\}$. Identify ω with a graph on V such that for all e with $x_e = 1_e$ the edge e is present and otherwise not.

The probability space $(\Omega, \mathcal{F}, P)$ will be denoted shortly $G(n,p)$. Any set of graphs on V can be considered as an event in $G(n,p)$ and we can talk about probabilities of such events.

Example. 1. Consider event $A_e := \{\omega = (x_e)_{e\in V^2} : x_e = 1_e\}$ which is the class of all graphs on V which contain edge e. One can prove formally:

Proposition 7.1. *The events A_e, $e \in V^2$, are independent of each other and $P(A_e) = p$, for every $e \in V^2$.*

Proof. No proof. $\square$

2. Compute probability of the event that G (random graph from $G(n,p)$) contains some given graph H on a subset of V as a subgraph. Let $|V(H)| =: k$ and $|E(H)| =: l$:

$$P(H \subseteq G) = p^l$$

$$P[H \text{ is induced subgraph of } G] = p^l q^{\binom{k}{2}-l}$$

Notice: The probability that G contains an (induced) subgraph which is isomorphic to H is larger and it is not so easy to compute because different isomorphic "copies" may contains common edges, which makes the corresponding events dependent.

$$P[H \simeq G[U] \text{ over all subsets } U \subseteq V \text{ with } |U| = k] = P(\bigcup_{U\in\binom{V}{k}} A_U)$$

where

$$A_U = \{G' \in G(n-p) : G' = G[U]\}$$

$$P(A_U) = p^l q^{\binom{k}{2}-l}$$

Thus,

$$P[H \simeq G[U] \text{ over all subsets } U \subseteq V \text{ with } |U| = k] \leq \sum_{U\in\binom{V}{k}} P(A_U) = \binom{n}{k} p^l q^{\binom{k}{2}-l} \tag{26}$$

Lemma 7.2 (7.2)**.** *For every natural number n, k, $n \geq k \geq 2$, the probability that $G \in G(n,p)$ has a set of k independent vertices is at most $P(\alpha(G) \geq k) \leq \binom{n}{k} q^{\binom{k}{2}}$. Analogously $P(\omega(G) \geq k) \leq \binom{n}{k} p^{\binom{k}{2}}$.*

Proof. Apply 26 with $l = 0$. for the first statement and 26 with $l = \binom{k}{2}$ for the second statement. $\square$

Assume $P(\alpha(G) \geq k) < \frac{1}{2}$ for some (small) k and $P(\omega(G) \geq k) < \frac{1}{2}$ for some (small) k. Then $P(\alpha(G) \geq k \vee \omega(G) \geq k) < \frac{1}{2} + \frac{1}{2} = 1$. Thus,

$$P(\alpha(G) < k \wedge \omega(G) < k) > 0$$

$\implies$ such a graph with those particular properties exists.

Definition. Let H be a graph. Then Ramsey number $R(H)$ is the smallest $n \in \mathbb{N}$, such that every graph G on at least n vertices has the property "G or $\overline{G}$ contains an induced subgraph isomorphic to H" (= Ramsey property). If $H = K_r$ then denote $R(n) \coloneqq R(K_r)$. More generally, let H_1, H_2 be two graphs. Denote $R(H_1, H_2)$ the smallest number $n \in \mathbb{N}$, such that for every graph G on at least n vertices the property "G contains an induced subgraph isomorphic to H_1 or $\overline{G}$ contains an induced subgraph isomorphic to H_2".

Proposition 7.3 (7.3)**.** *Let $s, t \in \mathbb{N}$. Let T be a tree with t vertices. Then $R(T, K_s) = (s-1)(t-1)+1$.*

Proof. No proof. □

Theorem 7.4. *For every $r \in \mathbb{N}$ $R(r) \le 2^{2r-3}$ holds*

Proof. No proof. □

Theorem 7.5 (Erdös 1947)**.** *For every $k \in \mathbb{N}$, $k \ge 3$, $R(k) > 2^{\frac{k}{2}}$ holds.*

Proof. For $k = 3$: $n \ge 3$ for every graph fulfilling Ramsey property for K_3 $\implies$ $R(k) \ge 3 > 2^{\frac{3}{2}}$.

Let $k \ge 4$. Show that for all $n \le 2^{\frac{k}{2}}$ and for every $G \in \mathcal{G}(n, \frac{1}{2})$ the probability $P(\alpha(G) \ge k)$ and $P(\omega(G) \ge k)$ are both $< \frac{1}{2}$.

Lemma 7.6.

$$P(\alpha(G) \ge k) \le \binom{n}{k}\left(\frac{1}{2}\right)^{\binom{k}{2}} < \frac{n^k}{2^k} 2^{-\frac{k(k-1)}{2}} \le \frac{2^{\frac{k^2}{2}}}{2^k} 2^{-\frac{k^2-k}{2}} = 2^{\frac{k^2}{2}-k-\frac{k^2}{2}+\frac{k}{2}} = 2^{-\frac{k}{2}} < \frac{1}{2} \textit{ for } k \ge 4$$

$$P(\omega(G) \ge k) \le \binom{n}{k}\left(\frac{1}{2}\right)^{\binom{k}{2}} < \frac{1}{2}$$

Therefore, Prob(G has no stable set on k vertices and no clieque on k vertices) $>$ $1 - \frac{1}{2} - \frac{1}{2}$ for $n \le 2\frac{k}{2}$ $\implies$ Ramsey property does not hold for $n \le 2^{\frac{k}{2}}$ $\implies$ $R(k) > 2^{\frac{k}{2}}$ □

In the context of random graphs *graph invariants* like average degree, min degree, connectivity number, girth, chromatic number,... may be interpreted as non-negative random variables in $\mathcal{G}(n,p)$, i.e. a function

$$\begin{aligned} X : \mathcal{G}(n-p) &\to [0, +\infty) \\ \text{e.g. } G &\mapsto \chi(G) \\ G &\mapsto \delta(G) \\ G &\mapsto \mathrm{g}(G) \\ G &\mapsto \kappa(G) \end{aligned}$$

Let $E(X)$ the expected value of X. If $E(X)$ is "small" then X is "large" part for a "small" portion of elements in $\mathcal{G}(n,p)$. So proving $E(X)$ for a class of graphs in $\mathcal{G}(n,p)$ with a certain property woould lead to existence proofs for graphs with the above mentioned properties and a small invariant X.

Lemma 7.7 (Markov Inequality)**.** *Let $X \geq 0$ on $\mathcal{G}(n,p)$ and $a > 0$, $a \in \mathbb{R}$. Then*

$$P[X \geq a] \leq \frac{E(X)}{a}.$$

Proof.

$$\begin{aligned} E(X) &= \sum_{G \in \mathcal{G}(n,p)} P(\{G\})X(G) \\ &\geq \sum_{G \in \mathcal{G}(n,p): X(G) \geq a} P(\{G\})X(G) \\ &\geq a \sum_{G \in \mathcal{G}(n,p): X(G) \geq a} P(\{G\}) = aP(X \geq a) \end{aligned}$$

□

Notice: $E(X)$ can be computed by double counting since $\mathcal{G}(n,p)$ is finite. E.g. if X counts the number of subgraphs of G from a family of graphs $\mathcal{H}$ (on V), then by definition $E(X) = \sum_{G \in \mathcal{G}(n,p)} P(\{G\}) \#\{\text{subgraphs from } \mathcal{H} \text{ in } G\} = \sum_{G \in \mathcal{G}(n,p)} P(\{G\}) \sum_{H \subset \mathcal{H}} P(\{H \subseteq G\}) = \sum_{G \in \mathcal{G}(n,p)} \sum_{H \in \mathcal{H}, H \subseteq G} P(\{G\}) = \sum_{H \in \mathcal{H}} P(H \subseteq G)$

Illustrate double counting with Lemma 7.8. Denote $(n)_k := n(n-1)\dots(n-k+1)$.

Lemma 7.8. *The expected number of k-cycles in $G \in \mathcal{G}(n,p)$ is $E(X) = \frac{(n)_k}{2k}p^k$ where $X : \mathcal{G}(n,p) \to \mathbb{N}$, $G \mapsto \#\{k\text{-cycles contained in } G \text{ as a subgraph}\}$.*

Proof. For every cycle C with k vertices on V introduce $X_C : \mathcal{G}(n,p) \to \{0,1\}$, $G \mapsto \begin{cases} 1 & C \subseteq G \\ 0 & \text{otherwise} \end{cases}$

$$E(X_C) = 1P(X_C = 1) + 0P(X_C = 0) = P(X_C = 1) = P(C \subseteq G) = p^k$$

pic can be run over in $2k$ ways (2 directions, k possibilities for start vertex). add pic

$$X = \sum_{C \text{ is } k\text{-cycle}} X_C \implies E(X) = \sum_{C \text{ is } k\text{-cycle}} E(X_C) = \frac{(n)_k}{2k}p^k$$

□

7.2 The probabilistic method; an application

Let $k \in \mathbb{N}$. We will show for $n \to \infty$ there exist graphs G with $\mathrm{g}(G) > k$ and $\chi(G) > k$.

Call cycles on at most k vertices *small cycles*; $\mathrm{g}(G) > k \iff$ there are no small cycles in G. Call sets with at least $\frac{|V|}{k}$ vertices big sets; $\chi(G) > k \impliedby$ there are no big independent sets in G.

Intuition: p smaller $\implies$ probability of existence of small cycles gets smaller. p larger $\implies$ probability of existence of big independent sets gets smaller.

Assume there is a p_C such for $p < p_C$ the probability of existence of small cycles $< \frac{1}{2}$ (unlikely event) Assume p_S such that for $p > p_S$ the probability of existence of big independent set $< \frac{1}{2}$ (unlikely event).

If $p_S < p_C$ happens to be true then take $p \in (P_S, p_C)$ and for sure $P(\exists G \text{ with } \mathrm{g}(G) > k, \chi(G) > k \text{ in } \mathcal{G}(n,p)) > 0$ ✓

It is not the case because $p_C < \frac{1}{n}$ and $p_S > n^{\varepsilon - 1}$ for some $\varepsilon > 0$.

27.06.2016

Lemma 7.9. *Let $k > 0$, $k \in \mathbb{N}$, and let $p := p(n) \geq \frac{6k \ln n}{n}$ for n large. Then $P[\alpha \geq \frac{1}{2}\frac{n}{k}] \to 0$ for $n \to \infty$.*

Proof. $\forall n, r \in \mathbb{N}$, $n \geq r \geq 2$, $\forall G \in \mathcal{G}(n-p) : P[\alpha \geq r] \leq \binom{n}{r} q^{\binom{r}{2}}$ (Lemma **??**).
$\implies P[\alpha \geq r] \leq n^r q^{\binom{r}{2}} = (nq^{\frac{r-1}{2}})^r \overset{1-p \leq e^{-p}}{\leq} (ne^{-p(r-1)/2})^r$
For $r \geq \frac{1}{2}\frac{n}{k}$: $ne^{\frac{-p(r-1)}{2}} = ne^{\frac{-pr}{2} + \frac{p}{2}} \leq ne^{-\frac{3}{2}\ln n + \frac{p}{2}} \leq nn^{-\frac{3}{2}} e^{\frac{1}{2}} = \frac{\sqrt{e}}{\sqrt{n}}$. Thus the above tends to 0, i.e.

$$\leq (ne^{-p(r-1)/2})^r \overset{n \to \infty}{\to} 0$$

So

$$\lim_{n \to \infty} P[\alpha \geq \frac{1}{2}\frac{n}{k}] = \lim_{n \to \infty} P[\alpha \geq r] = 0$$

□

Theorem 7.10 (Erdös 1959)**.** *For all natural numbers k there exists a graph H with $\mathrm{g}(H) \geq k$ and $\chi(H) \geq k$.*

Proof. Assume wlog $k \geq 3$ and fix ε such that $0 < \varepsilon < \frac{1}{k}$. Set $p := n^{\varepsilon - 1} = \frac{n^\varepsilon}{n}$. Let $X(G)$ be the number of short cycles (length $\leq k$) in $G \in \mathcal{G}(n,p)$. By Lemma 7.8

$$E(X) = \sum_{i=3}^{k} E(X_i) = \sum_{i=3}^{k} \frac{(n)_i}{2i} p^i \leq \frac{1}{2} \sum_{i=3}^{k} n^i p^i = \frac{k-2}{2} n^k p^k$$

(because the expectation of number cycles of length i $E(X_i) = \frac{(n)_i}{2i} p^i$) Note that $(np)^i \leq (np)^k$ because $np = nn^{\varepsilon - 1} = n^\varepsilon \geq 1$. So $E(X) \leq \frac{k-2}{2} n^k p^k$.

Markov's inequality

$$P(X \geq \frac{n}{2}) \leq \frac{E(X)}{\frac{n}{2}} \leq (k-2)nk - 1p^k = (k-2)n^{k-1} n^{\varepsilon k - k} = (k-2)n^{\varepsilon k - 1} \overset{n \to \infty}{\to} 0$$

So $P(X \geq \frac{n}{2}) \overset{n \to \infty}{\to} 0$.

(we can conclude from this that) There exists a graph $G \in \mathcal{G}(n-p)$ with fewer that $\frac{n}{2}$ short cycles. Delete one vertex from each short cycle in this graph G. Let us call the resulting graph H. $|H| \geq \frac{n}{2}$. Then $\mathrm{g}(H) > k$. By definition of G: $\chi(H) \geq \frac{|H|}{\alpha(H)} \geq \frac{|H|}{\alpha(G)} \geq \frac{n/2}{\alpha(G)} > k$ From Lemma 7.9 we found a $G \in \mathcal{G}(n,p)$ with $\alpha < \frac{n/2}{k} \implies \frac{n/2}{\alpha}$. □

Corollary 13. *There are graphs with arbitrarily large girth and arbitrarily large values of the invariants κ (connectivity number), $\varepsilon = \frac{|E(G)|}{|V(G)|}$ and δ (minimum degree).*

Proof. (Sketch) Uses the following two statements

- Apply Corollary "Every graph G has a subgraph of minimum degree $\chi(G) - 1$".

- Theorem of Mader "Let $k \in \mathbb{N}$ ($k \neq 0$). Every graph with $\mathrm{d}(G) \geq 4k$ has a $(k+1)$- connected subgraph H such that $\varepsilon(H) > \varepsilon(G) - k$."

$\square$

7.3 Properties of almost all graphs

Let $p = p(n)$ be a given fixed function (possibly constant) $P[G \in \mathcal{P}]$ where $\mathcal{P}$ is a graph property (class of graphs closed wrt isomorphism)

If $P[G \in \mathcal{P}] \overset{n \to \infty}{\to} 1$ we say "almost every $G \in \mathcal{G}(n,p)$ has property $\mathcal{P}$". If $P[G \in \mathcal{P}] \overset{n \to \infty}{\to} 0$ we say "almost no $G \in \mathcal{G}(n,p)$ has property $\mathcal{P}$".

Example. Lemma 7.9

Another example is

Example. Proposition **??**

Proposition 7.11. *For all given $p \in (0,1)$ and for every given graph H almost every graph $G \in \mathcal{G}(n,p)$ contains an induced subgraph isomorphic to H.*

Proof. $k := |H|$, $l := |E(H)|$. Let $n \geq k$. Let $U \subseteq \{0, \dots, n-1\}$ be a fixed subset with $|U| = k$. Then $P[G[U] \simeq H] =: r = p^l q^{\binom{k}{2}-l} = p^l(1-p)^{\binom{k}{2}-l} > 0$. U can be specified in $\binom{n}{k}$ ways; Among them there are at least $\lfloor \frac{n}{k} \rfloor$ which are pairwise disjoint. Prob[No $G[U]$ (among the disjoint ones) is isomorphic to H] $\leq (1-r)^{\lfloor \frac{n}{k} \rfloor}$ Prob[H is not an induced subgraph of G] $\leq (1-r)^{\lfloor \frac{n}{k} \rfloor} \overset{n \to \infty}{\to} 0$. $\square$

Given $i, j \in \mathbb{N}$ let $P_{i,j}$ denote the property "considered graph contains for all disjoint pair of vertex sets U, W, with $|U| \leq i$, $|W| \leq j$, a vertex $v \notin U \cup W$ that is adjacent to all vertices in U and to none of the vertices in W". add pic

Lemma 7.12. *For all constant $p \in (0,1)$ and for all $i, j \in \mathbb{N}$ almost every graph in $\mathcal{G}(n,p)$ has the property $P_{i,j}$.*

Proof. For fixed U, W ($|U| \leq i$, $|W| \leq j$) and fixed $v \notin U \cup W$ with probability that v is connected by an edge to all $n \in U$ and to no $w \to W$ is $\geq p^i(1-p)^j$. Prob[no such $v \notin U \cup W$ can be found] $\leq (1 - p^i(1-p)^j)^{n-i-j}$ (assuming $n > i + j$). Notice that the corresponding events are independent for different vertices v.

$\lim_{n \to \infty}$Prob[no such $v \notin U \cup W$ can be found] $\leq \lim_{n \to \infty}(1 - p^i(1 - p)^j)^{n-i-j} = 0$ $\square$

Corollary 14. *For every $p \in (0,1)$ constant and for every $k \in \mathbb{N}$ almost every graph in $\mathcal{G}(n,p)$ is k-connected.*

Proof. We show that every graph in $\mathcal{P}_{2,k-1}$ is k-connected. Every graph in $P_{2,k-1}$ has at least $k + 2 = 2 + (k-1) + 1$ vertices. If W is a set of fewer than k vertices ($\leq k-1$): Assume $G \setminus W$ is disconnected $\implies \exists$ two vertices v_1, v_2 which were connected by a path and are not connected by a path any more in $G \setminus W$. Apply property $P_{2,k-1}$ with $U = \{v_1, v_2\}$ and $W = W$. Then there exists a v with $\{v, v_1\} \in E(G)$, $\{v, v_2\} \in E(G)$ ` $\square$ add pic

For constant p we had the situation that certain properties were almost surely present (independent on the value of p) and other properties were almost surely non-present (independent on value of p). The situation changes if p is not constant but a function of n.

$p < \frac{1}{n^2}$: almost no edges in $G \in \mathcal{G}(n,p)$
$p = \frac{\sqrt{n}}{n^2}$: there exists a connected component with more than 2 vertices, almost surely.
$p \approx \frac{1}{n}$: there exists a cycle in almost every $G \in \mathcal{G}(n,p)$
$p \approx \frac{\log n}{n}$: $G \in \mathcal{G}(n,p)$ almost surely connected
$p = \frac{(1+\varepsilon)\log n}{n}, \varepsilon > 0$: graph $G \in \mathcal{G}(n,p)$ has almost surely a hamiltonian cycle.

Evolution of random graphs (sub area of random graphs)
Main question is: Given a property, what is the value of p which makes the property almost sure when for smaller p the property was not almost surely there. Threshold p.

Definition. Call a real function $t : \mathbb{N} \to \mathbb{R}$ with $t(n) \neq 0$ for all $n \in \mathbb{N}$ a *threshold function* for a graph property $\mathcal{P}$ if the following holds for every $p = p(n)$ and for every $G \in \mathcal{G}(n,p)$

$$\lim_{n\to\infty} P[G \in \mathcal{P}] = \begin{cases} 0 & \text{if } \frac{p}{t} \overset{n\to\infty}{\to} 0 \\ 1 & \text{if } \frac{p}{t} \overset{n\to\infty}{\to} \infty \end{cases}$$

$\mathcal{P}$ could be for example class of graphs containing a particular given graph as an (induced) subgraph (an isomorphic copy of it).

Definition. A graph H is called balanced if $\varepsilon(H') = \frac{|E(H'|}{|V(H')|} \leq \varepsilon(H) = \frac{|E(H)|}{|V(H)|}$ for all subgraphs H' of H.

Example.

1. complete graph
2. cycle
3. not balanced: clique + (large enough) stable set, connect every vertex of the stable set with every vertex of the clique.

Theorem 7.13 (Erdös, Renyi 1960)**.** *If H is a balanced graph with k vertices and $l \geq 1$ edges, then $t(n) := n^{-k/l}$ is a threshold function for property P_H, where P_H is the class of graphs containing an induced subgraph isomorphic to H.*

Corollary 15 (1 of 7.13)**.** *If $k \geq 3$ then $t = \frac{1}{n}$ is a threshold function for the property $\mathcal{P}_{C_k}$, i.e. the property of containing a k-cycle.*

Corollary 16 (2 fo 7.13)**.** *If T is a tree with $k \geq 2$ vertices, then $t(n) = n^{-k(k-1)}$ is a threshold function for the property $\mathcal{P}_T$, i.e. property of containing a subgraph which is isomorphic to T.*