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Definition (Random variable (RV))**.** $X : (\Omega, \mathcal{A}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ *where* $\mathcal{B}_{\mathbb{R}}$ *are Borel sets (think of intervals).*

$$X^{-1}(I) \in \mathcal{A}$$

for all interval I *(Borel set* I*).*

Definition (Distribution of X)**.** P_X *on* $(\mathbb{R}, \mathcal{B})$*:*

$$P_X(I) = \mathbb{P}[X \in I] = \mathbb{P}(X^{-1}(I))$$

Definition (Discrete/continuous RVs (and more!))**.**

1. *discrete density:*

   $$p_X(x) = \mathbb{P}[X = x] > 0$$

   only in the (finite or countable) value set of X*.*

   $$\mathbb{P}[X \in I] = \sum_{x \in I} p_X(x)$$

2. *density:*
   $f_X : \mathbb{R} \to [0, \infty)$ *such that*

   $$\mathbb{P}[X \in I] = \int_I f_X(x) dx$$

add pic

Definition. *The expected value of a RV* X *is*

$$\mathbb{E}(X) = \int_\Omega X d\,\mathbb{P}$$

if the integral exists. In particular, if X *is* discrete*, then*

$$\mathbb{E}(X) = \sum_{x \in \mathbb{R}} x p_X(x)$$

If X *is* continuous*, then*

$$\mathbb{E}(X) = \int_\mathbb{R} x f_X(x) dx$$

We are interested in the situation where

$$\mathbb{E}(|X|) < \infty,$$

that is:

$$\sum |x| p_X(x) < \infty \text{ (discrete case)}$$

$$\int_\mathbb{R} f_X(x) dx < \infty \text{ (continuous case)}$$

Remark (for Math)**.** $X = X^+ - X^-$*: at least one of* $\int X^\pm$ *is finite*

Example. *Let* $p(n) = \frac{6}{\pi^2} \frac{1}{n^2}$ *for* $n \in \mathbb{N}$*.*

$$\sum np(n) = \frac{6}{\pi^2} \sum \frac{1}{n} = \infty$$

0.1 Convergence of sequences of RVs

Definition. *Let (X_n) be a sequence of random variables and X another random variable. Then*

1. *$X_n \to X$ (X_n converges to X) almost surely*

$$\begin{aligned}\mathbb{P}[\lim_{n\to\infty} X_n = X] &= \mathbb{P}(\{\omega \in \Omega : (X_n(\omega))_{n\in\mathbb{N}} \text{ converges and the limit is } X(\omega)\}) \\ &= 1\end{aligned}$$

2. *$X_n \to X$ in probability*
   $\forall \varepsilon > 0$

$$\mathbb{P}[|X_n - X| \geq \varepsilon] = \mathbb{P}(\{\omega : |X_n(\omega) - X(\omega)| \geq \varepsilon\}) \to 0$$

Question: Is the statement $\mathbb{E}(X_n) \overset{?}{\to} \mathbb{E}(X)$ true?

Theorem 0.1 (Monotone convergence)**.** *If $0 \leq X_n \leq X_{n+1}$ and $X = \lim_{n\to\infty} X_n$ almost surely, then*

$$\mathbb{E}(X) = \lim_{n\to\infty} \mathbb{E}(X_n)$$

Theorem 0.2 (Lemma of Fatou)**.** *If $X_n \geq 0$ for all n, then*

$$\mathbb{E}(\liminf X_n) \leq \liminf \mathbb{E}(X_n)$$

Theorem 0.3. *If $X_n \to X$ almost surely and there is a random variable Y such that $|X_n| \leq Y$ a.s. for all n and $\mathbb{E}(Y) < \infty$, then*

$$\lim \mathbb{E}(X_n) = \mathbb{E}(X).$$

Example. *A counter example for the case "If $X_n \to X$ a.s., then $\lim \mathbb{E}(X_n) = \mathbb{E}(X)$.":*
$\Omega = [0,1]$, $\mathcal{A} = \mathcal{B}_{[0,1]}$, $\mathbb{P} =$ lebesgue measure. Choose the curve such that the triangle has a small base and a high height. Then $\mathbb{E}(X_n) = 1$, $\lim X_n(\omega) = 0 = X(\omega)$, $\mathbb{E}(X) = 0$. add pic

Remark. *Let f map from $X_n \to X$ a.s. on $X_n \to X$ in probability, then*

$$\mathbb{P}[X \neq X'] = 0 : \; X = X' a.s.$$

Proof.

$$\begin{aligned}
|X - X'| &\leq |X - X_n| + |X_n - X'| \\
\implies [|X - X'| \geq \varepsilon] &\subseteq [|X - X_n| \geq \frac{\varepsilon}{2}] \cup [|X_n - X'| \geq \frac{\varepsilon}{2}] \\
\mathbb{P}[|X - X'| \geq \varepsilon] &\leq \mathbb{P}[|X_n - X| \geq \frac{\varepsilon}{2}] + \mathbb{P}[|X_n - X'| \geq \frac{\varepsilon}{2}] \\
&\overset{(n\to\infty)}{\to} 0 \\
\text{so } \mathbb{P}[|X - X'| \geq \frac{1}{r}] &= 0 \text{ for all } r \in \mathbb{N} \\
\implies \mathbb{P}(\bigcup_{r=1}^{\infty} [|X - X'| \geq \frac{1}{r}]) &= \mathbb{P}[|X - X'| > 0] = 0
\end{aligned}$$

□

Theorem 0.4. *$X_n \to X$ a.s. if and only if $U_k \to 0$ in probability where*

$$U_k = \sup_{n \geq k} |X_n - X|$$

In particular, $X_n \to X$ in prob. (because $U_k \geq |X_k - X|$)

Proof. Let $\varepsilon = \frac{1}{r}$.

$$\begin{aligned}
&\mathbb{P}[\forall r \exists k \forall n \geq k : |X_n - X| < \frac{1}{r}] = 1 \\
&\qquad\qquad = \mathbb{P}(\bigcap_r \underbrace{\bigcup_k \bigcap_{n \geq k} [|X_n - X| < \frac{1}{r}]}_{A_{r+1} \subset A_r \in \mathcal{A}}) \\
&\implies \mathbb{P}(A_r) = 1 \, \forall r \in \mathbb{N}
\end{aligned}$$

Since A_r are decreasing and $\mathbb{P}(A_r) = 1 \forall r \in \mathbb{N}$, the other direction holds as well.
$\bigcap_{n \geq k} [|X_n - X| < \frac{1}{r}] \subset [U_k \leq \frac{1}{r}]$
Conversely, $[U_k \leq \frac{1}{r}] \subset \bigcap_{n \geq k} [|X_n - X| < \frac{1}{r-1}]$
$A_r \subset \bigcup_k [U_k \leq \frac{1}{r}]$
So, $1 = \mathbb{P}(A_r) \leq \mathbb{P}(\bigcup_k [U_k \leq \frac{1}{r}]) \leq 1$
Since $U_k \geq U_{k+1}$, the events $U_k \leq \frac{1}{r}$ are increasing in k.

$$\begin{aligned}
&\iff \forall r \lim_{k \to \infty} \mathbb{P}[U_k \leq \frac{1}{r}] = 1 \\
&\iff \forall r \lim \mathbb{P}[U_k > \frac{1}{r}] = 0
\end{aligned}$$

This means, $U_k \to 0$ in prob. This concludes the entire proof. $\square$