The ergodic theorem for finite Markov chains

Let $(\mathcal{X}, P)$ where $P = (p_{x,y})_{x,y\in\mathcal{X}} = (p(y|x))_{x,y\in\mathcal{X}}$ is the transition matrix, $\mathcal{X}$. and $\mu = (\mu_x)_{x\in\mathcal{X}}$ start distribution. $\mathbb{P}_\mu$ $(X_n)_{n\in\mathbb{N}_0}$ Markov chain

Where is a suitable probability space? It is called trajectory space $\Omega = \mathcal{X}^{\mathbb{N}_0}$

$\mathcal{A}$ is generated by all "cylinder sets". $k \in \mathbb{N}_0$, $a_0, \ldots, a_k \in \mathcal{X}$: $C(a_0, \ldots, a_k) = \{\omega = (x_n)_{n\in\mathbb{N}_0} : x_0 = a_0, \ldots, x_k = a_k\}$ inclusive Ω

$X_n = n$-th projection

$$\mathbb{P}_\mu(C(a_0, \ldots, a_k)) = \mu_{a_0} p_{a_0,a_1} \cdots p_{a_{k-1},a_k}$$

unique continuation to (σ-algebra) probability measure on $(\Omega, \mathcal{A})$.

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Definition.

$$s^{\times}(\omega) := \inf\{n \geq 0 : X_n(\omega) = x\}$$
$$t^{\times}(\omega) := \inf\{n \geq 1 : X_n(\omega) = x\}$$

$s^{\times}, t^{\times} : (\Omega, \mathcal{A}) \to \mathbb{N}_0 \cup \{\infty\}$ *are called stopping times.*

$$f^{(n)}_{x,y} = \mathbb{P}_x[s^y = n]$$
$$u^{(n)}_{x,y} = \mathbb{P}_x[t^y = n]$$

$$f^{(0)}_{x,y} = \begin{cases} 1, & y = x \\ 0, & x \neq y \end{cases}$$

If $x \neq y$, *then* $u^{(n)}_{x,y} = f^{(n)}_{x,y}$ *for all* n. $u^{(n)}_{x,x}$ *is called the "first return probability" for* $n \geq 1$. $u^{(0)} \equiv 0$ *(always)*

$$V^y_n = \begin{cases} 1, & X_n = y \\ 0, & X_n \neq y \end{cases}$$

$$\mathbb{E}_x(V^y_n) = \mathbb{E}(\mathbb{1}_{[X_n = y]}) = p^{(n)}_{x,y}$$

Consider the generating function

$$G(x, y|z) = G_{x,y}(z) = \sum_{n=0}^{\infty} p^{(n)}_{x,y} z^n$$

$$F_{x,y}(z) = \sum_{n=0}^{\infty} f^{(n)}_{x,y} z^n$$

$$U_{x,y}(z) = \sum_{n=0}^{\infty} u^{(n)}_{x,y} z^n$$

for $0 \leq z < 1$. *since* $\underbrace{G_{x,y} = G_{x,y}(1)}_{\mathbb{E}_x(\sum_{n=0}^{\infty} V^y_n) \leq \infty}$, $\underbrace{F_{x,y} = F_{x,y}(1)}_{\mathbb{P}_x[\exists n \geq 0 : X_n = y] = \mathbb{P}_x[s^y < \infty]}$, $U_{x,y} = U_{x,y}(1)$ *and* $U_{x,x} = \mathbb{P}_x[t^x < \infty]$ *And remember that* $F_{x,x}(z) \equiv 1$.

Remark. *We use* inf *and not* min *since the infimum of the empty set is* $+\infty$.

Remark. *Actually, its for $0 \leq z \leq 1$ since $G_{x,y}$ could be ∞. But we can neglect this.*

Theorem 0.1. *($0 \leq z < 1$)*

(a) $G_{x,x}(z) = \frac{1}{1-U_{x,x}(z)}$

(b) $G_{x,y}(z) = F_{x,y}(z)G_{y,y}(z)$

(c) $U_{x,x}(z) = \sum_y p_{x,y} z F_{y,x}(z)$

(d) $x \neq y$*:* $F_{x,y}(z) = \sum_{w \in \mathcal{X}} p_{x,w} z F_{w,y}(z)$

(e) $G_{x,y}(z) = \delta_y(x) + \sum_w G_{x,w}(z) p_{w,y} z$*; you can think like the following (commutative matrix)* $G(z) = (G_{x,y}(z))_{x,y \in \mathcal{X}} = \sum_{n=0}^{\infty} P^n z^n = I + G(z)Pz = I + PzG(z)$ *If* $\mathcal{X}$ *is finite and* $0 < z < 1$ *or* $|z| < 1$ *for* $z \in \mathbb{C}$*:* $G/z) = I - zP^{-1}$

Proof. $n \geq 1$.

$$
\begin{aligned}
p_{x,x}^{(n)} = \mathbb{P}_x[X_n = x] &= \sum_{k=1}^{n} \mathbb{P}_x[X_n = x, t^1 = k] \\
&= \sum_{k=1}^{n} \underbrace{\mathbb{P}_x[t^x = k]}_{u_{x,x}^{(k)}} \cdot \underbrace{\mathbb{P}_x[X_n = x | X_k = x, X_j \neq k (j = 1, \ldots, k-1)]}_{p_{x,x}^{(n-k)} \text{ by Markov property}}
\end{aligned}
$$

$$
\begin{aligned}
n \geq 1 : & p_{x,x}^{(n)} = \sum_{k=0}^{n} u_{x,x}^{(k)} p_{x,x}^{(n-k)} \\
n = 0 : & p_{x,x}^{(0)} = 1 \text{ and } u_{x,x}^{(0)} = 0
\end{aligned}
$$

Thus,

$$
\begin{aligned}
G_{x,x}(z) &= 1 + \sum_{n=0}^{\infty} \sum_{k=0}^{n} u_{x,x}^{(k)} p_{x,x}^{(n-k)} z^n \\
&= 1 + \sum 1 + U_{x,x}(z) G_{x,x}(z)
\end{aligned}
$$

for $z < 1$: this proves (a).

Ad b: replace $p_{x,x}^{(n)}$ with $p_{x,y}^{(n)}$ and start sum at 0. Replace t^x by s^y. Do as an exercise. (Special case 0 is already in s included. Then

$$
p_{x,y}^{(n)} = \sum_{k=0}^{n} f_{x,y}^{(k)} p_{y,y}^{(n-k)}
$$

for all $n \geq 0$. $\square$

Definition. $x \in \mathcal{X}$ *is called* recurrent *if* $U_{x,x} = 1$, $\mathbb{P}_x[t^x < \infty] = 1$*. Otherwise* x *is called* transient.

Lemma 0.2. $x \in \mathcal{X}$ *recurrent if and only if* $U_{x,x} = 1$*.*

Remark. $\Rightarrow$ *x is recurrent* $\Rightarrow G_{x,x} = \infty \Rightarrow U_{x,x} = 1$

Proof. (a) $(0 < z < 1)$ z is monotone increasing. + blabla □

Lemma 0.3. *$(\mathcal{X}, P)$ irreducible then we distinguish two cases:*

rec. $G_{x,y} = \infty$ *for all* x, y

trans $G_{x,y} < \infty$ *for all* x, y.

Proof. $x, y, x', y' \in \mathcal{X}$ irreducible $\exists k, l : p^{(k)}_{x,x'} > 0, p^{(l)}_{y',y} > 0$ Now consider $p^{(k)}_{x,x'} p^{(n)}_{x',y'} p^{(l)}_{y',y} \leq p^{k+n+l}_{x,y}$ Use $p^k p^n p^l = p^{k+n+l}$.

$$\sum_{n=0}^{\infty} p^{(k)}_{x,x'} p^{(n)}_{x',y'} p^{(l)}_{y',y} \leq \sum_{n=0}^{\infty} p^{k+n+l}_{x,y}$$

$$\underbrace{p^{(k)}_{x,x'}}_{>0} G_{x',y'} \underbrace{p_{y',y}}_{>0} \leq G_{x,y}$$

□

Thus we have the following theorem.

Theorem 0.4. *Let $(\mathcal{X}, P)$ be irreducible. Then the following are equivalent*

1. *There exist x, y such that $G_{x,y} = \infty$*
2. *For all x, y: $G_{x,y} = \infty$*
3. *There exists an x such that $U_{x,x} = 1$*
4. *For all x: $U_{x,x} = 1$*
5. *$F_{x,y} = 1$ for all x, y.*

Proof. No proof (yet). □

Lemma 0.5. *If $(\mathcal{X}, P)$ is finite and irreducible, then it is recurrent.*

Proof.

$$\sum_{y} p^{(n)}_{x,y} = 1$$

$$\sum_{n=0}^{\infty} \sum_{y} p^{(n)}_{x,y} = \infty$$

Since we can use Fubini and get

$$\sum_{n=0}^{\infty} \sum_{y} p^{(n)}_{x,y} = \sum_{y \in \mathcal{X}} G_{x,y} \implies \exists y : G_{x,y} = \infty$$

□

$\mathbb{E}_x(t^x) = \sum_{n=1}^{\infty} n \underbrace{\mathbb{P}_x[t^x = n]}_{u_{x,x}^{(n)}} = U'_{x,x}(1-)$ Think about $u_{x,x}^{(n)} z^{n-1}$ to get the derivate of $U_{x,x}$

Recall and add z^n to the proof before:

$$\sum_y p_{x,y}^{(n)} z^n = 1 \cdot z^n$$

$$\sum_{n=0}^{\infty} \sum_y p_{x,y}^{(n)} z^n = \frac{1}{1-z}$$

Then for $0 < z < 1$:

$$\sum_{y \in \mathcal{X}} G_{x,y}(z) = \frac{1}{1-z}$$

$$G_{x,x}(z) = 1 + \sum_y G_{x,y}(z) p_{y,x} z$$

$$\sum_{y \in \mathcal{X}} F_{x,y}(z) \frac{1-z}{1 - U_{y,y(z)}} = 1$$

Now let z be monotone increasing towards 1:

$$\sum_{x \in \mathcal{X}} F_{x,y} \frac{1}{U'_{y,y}(1-)} = 1$$

Since $\mathcal{X}$ is finite.

Assume: $U'_{y,y}(1-) = \infty$ for all y. Thus, there exists a y such that $U'_{y,y}(1-) < \infty$.

Variante 1: Like right before the theorem of $(\mathcal{X}, P)$ is irreducble,..., you can find a $C_{x,y}$ such that

$$C_{x,y} \geq \frac{G_{x,x}(z)}{G_{y,y}(z)} = \frac{1 - U_{y,y}(z)}{1 - U_{x,x}(z)}$$

Definition. *$(\mathcal{X}, P)$ is called positive recurrent if $\mathbb{E}_x(t^x) < \infty$ for an (all) x. It is called null-recurrent if $\mathbb{E}_x(t^x) = \infty$ for an (all) x.*

Lemma 0.6. *$(\mathcal{X}, P)$ finite and irreducible. then it is positive recurrent.*

Our goal:

$$\frac{1}{n} \sum_{k=0}^{n-1} f(X_k)$$

where $f : \mathcal{X} \to \mathbb{R}$.