02.03.2016

If Ω is at most countable, $\mathcal{A} = \mathcal{P}(\Omega)$. $\mathbb{P}(A) = \sum_{\omega \in A} \underbrace{\mathcal{P}(\{\omega\})}_{p(\omega)}$.

If we have ∞ many coin tosses, $\Omega = \{0,1\}^{\mathbb{N}}$ is uncountable.

Definition (Conditional probability)**.** $A, B \in \mathcal{A}$

$$\mathbb{P}(A|B) < \begin{cases} \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)} & \mathbb{P}(B) > 0 \\ 0 & \mathbb{P}(B) = 0 \end{cases}$$

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Lemma (Rule of total probability)**.** *$\Omega = \biguplus_{i\in I} B_i$ for $B_i \in \mathcal{A}$, I finite or countable. Then*

$$\mathbb{P}(A) = \sum_{i\in I} \mathbb{P}(A|B_i)\mathbb{P}(B_i)$$

Proof. Let $A \in \mathcal{A}$.

$$A = \biguplus_{i\in I} A \cap B_i$$

$$\begin{aligned} \mathbb{P}(A) &= \sum_{i\in I} \mathbb{P}(A \cap B_i) \\ &= \end{aligned}$$

$\square$

Example (very simple example on this kind of proof)**.** *There are 2 urns. Experiment:*

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1. *Transfer a random ball form urn I to urn II*
2. *extract a ball from urn II*

We want to compute $\mathbb{P}$[ball extracted from II is red]. Now consider:

$$\begin{aligned} A &= [\textit{ball extracted from II is red}] \\ B_1 &= [\textit{transfered ball is red}] \\ B_2 &= [\textit{transfered ball is yellow}] \end{aligned}$$

So we can write

$$\begin{aligned} \mathbb{P}(A) &= \underbrace{\mathbb{P}(A|B_1)}_{\frac{3}{10}}\underbrace{\mathbb{P}(B_1)}_{\frac{6}{10}} + \underbrace{\mathbb{P}(A|B_2)}_{\frac{2}{10}}\underbrace{\mathbb{P}(B_2)}_{\frac{4}{10}} \\ &= \frac{26}{100} \end{aligned}$$

Another question:
Suppose the ball from I is red. What is the probability that the transferred ball was red?

$$\mathbb{P}[\textit{transferred ball was red}] = ?$$

Use the formula of Bayes:

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \frac{\mathbb{P}(B)}{\P(A)}$$
$$= \mathbb{P}(A|B) \frac{\mathbb{P}(B)}{\mathbb{P}(A)}$$

Combination of total probability and Bayes (sometimes also called formula of Bayes):

$$\mathbb{P}(B_j|A) = \frac{\mathbb{P}(A|B_j)\mathbb{P}(B_j)}{\sum_i \mathbb{P}(A|B_j)\mathbb{P}(B_j)}$$

Now we can use this formula:

$$\mathbb{P}(B_1|A) = \frac{\mathbb{P}(A|B_1)\mathbb{P}(B_1)}{\mathbb{P}(A)}$$
$$= \frac{\frac{3}{10} \cdot \frac{6}{10}}{\frac{26}{100}} = \frac{18}{26}$$

Definition. $A, B \in \mathcal{A}$ *are* independent *if*

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$

Definition. *Let* $k \geq 2$. $A_1, A_2, \ldots, A_n \in \mathcal{A}$ *are independent, if for all* $\{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$

$$\mathbb{P}(\bigcap_{j=1}^{\infty} A_{i_j}) = \prod_{j=1}^{k} \mathbb{P}(A_{i_j})$$

Example. *Experiment: Extract a random ball. For* $i = 1, 2, 3$: add pic

$$A_i = [\textit{extracted ball carries digit } i]$$

Then

$$\mathbb{P}(A_i) = \frac{2}{4} = \frac{1}{2}$$
$$\mathbb{P}(A_i \cap A_j) = \frac{1}{4} = \mathbb{P}(A_i) \cdot \mathbb{P}(A_j) \text{ for } i \neq j$$
$$\mathbb{P}(A_1 \cap A_2 \cap A_3) = \frac{1}{4} + \mathbb{P}(A_1) \cdot \mathbb{P}(A_2) \cdot \mathbb{P}(A_3)$$

Observations. 1. *If A and B are independent, then*

(a) A^C *and* B

(b) A *and* B^C

(c) A^C *and* B^C

are independent.

2. *Let* $A_i^1 :- A_i$ *and* $A_i^{-1} :- A_i^C$. *Then* $A_1, \ldots, A_n$ *are independent if and only if for all* $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$

$$\mathbb{P}(A_1^{\varepsilon_1} \cap \cdots \cap A_n^{\varepsilon_n}) = \prod_{i=1}^{n} \mathbb{P}(A_i^{\varepsilon_i})$$

Definition. *Let A_i where $(i \in I)$ be a collection of events (in $\mathcal{A}$) are independent, if any finite subcollection is independent:*

$$\forall J \subset I,\ J \text{ finite}$$
$$\mathbb{P}(\bigcap_{j \in J} A_j) = \prod_{j \in J} \mathbb{P}(A_j)$$

0.1 Random variables

Definition. *A random variable is a function*

$$X : (\Omega, \mathcal{A}) \to \mathbb{R}$$

which is measurable

$$X^{-1}(I) \in \mathcal{A}$$

for all interval $I \subset \mathbb{R}$. Equivalent:

$$X^{-1}(B) \in \mathcal{A}$$

for all $b \in \mathcal{B}_{\mathbb{R}}$.

Example. *Coin toss: $\Omega = \{0,1\} \times \mathbb{R}^2 \times \mathbb{N}_0$ X_1: 0 or 1 and $\omega = (\varepsilon, x, y, n)$ $\varepsilon \in \{0,1\}$, $x, y \in \mathbb{R}$, $n \in \mathbb{N}_0$.*

$$X_1(\omega) = \varepsilon$$
$$X_2(\omega) = \sqrt{x^2 + y^2}$$

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The distribution of X is the probability measure on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ given by

$$\begin{aligned} P_X(B) &= \mathbb{P}(X^{-1}(B)) \\ &= \mathbb{P}[X \in B] \end{aligned}$$

for $B \in \mathcal{B}_{\mathbb{R}}$.

Typical classes of Random variables and distributions

1. value set of X is at most countable (finite or countable).

$$X(\Omega) = \{x_i : i \in I\}$$

where I is finite or countable.

$$\begin{aligned} P_X(B) &= \sum_{i : x_i \in B} P_X(\{x_i\}) \\ &= \sum_{i : x_i \in B} \underbrace{\mathbb{P}[X = x_i]}_{p(x_i) = p_X(x_i)} \end{aligned}$$

$$\sum_i p_X(x_i) = 1$$

only in discrete case, you write

$$p_X(x) = \mathbb{P}[X = x] = \begin{cases} 0 & x \notin \{x_i : i \in I\} \\ p_X(x_i) & \end{cases}$$

$$\sum_{x \in \mathbb{R}} p_X(x) = 1$$

2. *continuous*
   There is a density function $f_X : \mathbb{R} \to [0, \infty)$ such that

$$\mathbb{P}_X(B) = \int_B f_X(x) dx$$

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$$\int_{\mathbb{R}} f_X(x) dx = 1$$
$$\mathbb{P}[X = x] = 0$$

Note: for us, two random variables X, X' are the same, if

$$\mathbb{P}[X \neq X'] = 0$$

which is the same as

$$\mathbb{P}(\{\omega \in \Omega : X(\omega) \neq X'(\omega)\}) = 0$$

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