

Number Theory

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November 12, 2015

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Organizatorial stuff

Dates (in TUGrazOnline):

Mon	14:15–15:45	C208	Exercises (starting 19.10. first exercise class)
Tue	14:15–15:45	C307	Lecture (starting 20.10. first (real) lecture)
Wed	08:15–09:45	C208	Lecture

From now until 15.12. lectures by Martin Widmer. Then C. Frei.

End: oral exams

Exercises: Find details on website of the instructor Dijana Kreso. math.tugraz.at/~kreso

0 Basics

$$\mathbb{N} = \{1, 2, \dots\} \quad (1)$$

$$\mathbb{N}_0 = \mathbb{N} \cup \{0\} \quad (2)$$

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\} \quad (3)$$

0.1 Divisibility

Definition 0.1.1. Let $a, b \in \mathbb{Z}$. a divides b (written $a \mid b$) if $\exists q \in \mathbb{Z} : b = qa$.
Some properties: Let $a, b, c \in \mathbb{Z}$. Then the following statements hold:

$$a \mid b \Rightarrow ac \mid bc \quad (4)$$

$$a \mid b \wedge b \mid c \Rightarrow a \mid c \quad (5)$$

$$a \mid b \wedge b \mid a \Leftrightarrow a = b \quad (6)$$

$$a \mid b \wedge a \mid c \Rightarrow a \mid (b + c) \quad (7)$$

Definition 0.1.2 (Remainder). Let $a \in \mathbb{Z}$, $b \in \mathbb{N}$. Then there are unique $q, r \in \mathbb{Z}$ such that

$$a = qb + r \text{ and } 0 \leq r < b.$$

Remark. 1. $b \mid a \Leftrightarrow r = 0$

2. $q = \lfloor \frac{a}{b} \rfloor$ (largest integer $\leq \frac{a}{b}$)

3. we will sometimes write: $a \bmod b := c$

Definition 0.1.3. Let $a_1, a_2, \dots, a_n, d \in \mathbb{Z}$. d is a greatest common divisor (gcd) of a_1, \dots, a_n if $d \mid a_i \forall 1 \leq i \leq n$, and for every $e \in \mathbb{Z}$ with $e \mid a_i \forall 1 \leq i \leq n$, $e \mid d$.

Remark. 1. a gcd of a_1, \dots, a_n is unique up to sign

2. we write $d = \gcd(a_1, \dots, a_n)$ if d is a gcd of a_1, \dots, a_n

3. for $a_1, \dots, a_n \in \mathbb{Z}$, a gcd exists and can be written as a linear combination of a_1, \dots, a_n , i.e., $\exists x_1, \dots, x_n \in \mathbb{Z}$ such that

$$\gcd(a_1, \dots, a_n) = x_1 a_1 + \dots + x_n a_n$$

4. $\gcd(a_1, \dots, a_n) = \gcd(\gcd(a_1, \dots, a_{n-1}), a_n)$

5. if $a \mid bc$ and $\gcd(a, b) = 1$ then $a \mid c$.

6. let $a' := \frac{a}{\gcd(a, b)}$, $b' = \frac{b}{\gcd(a, b)}$. Then $\gcd(a', b') = 1$

The algorithm is correct, since $\gcd(a, b) = \gcd(b, a \bmod b)$.
The algorithm terminates because b decreases in each step.
The algorithm is fast: ($\mathcal{O}(\log b)$)

The Euclidean algorithm also allows us to find x, y such that $\gcd(a, b) = ax + by$ by doing all computations backwards.

Hier verwendest du $:=$, sonst aber nur $=$, evtl. einheitlich machen für alle Definitionen?

sollte ausgebaut werden, 1. $\mathcal{O}(\log n)$ steps, 2. stimmt nur wenn $|r| \leq b/2$

Algorithm 1 Compute the gcd of two integers: Euclidean algorithm

Given: $a, b \in \mathbb{Z}$. $|a| \geq |b|$ **Find:** $a := \gcd(a, b)$ replace a by $|a|$, b by $|b|$ **while** $b \neq 0$ **do**write $a = qb + r$, $0 \leq r < b$ $a := b$ $b := r$ **end while****return** a

Example. $\gcd(56, 22) = ?$

$$a = 56, b = 22$$

$$56 = 2 \cdot 22 + 12$$

$$a = 22, b = 12 \neq 0$$

$$22 = 1 \cdot 12 + 10$$

$$a = 12, b = 10 \neq 0$$

$$12 = 1 \cdot 10 + 2$$

$$a = 10, b = 2 \neq 0$$

$$10 = 5 \cdot 2 + 0$$

$$a = 2, b = 0$$

$$\Rightarrow \gcd(56, 22) = 2$$

Doing the computations backwards:

$$2 = 12 - 10 = 12 - (22 - 12) = -22 + 2 \cdot 12 = -22 + 2(56 - 2 \cdot 22) = 2 \cdot 56 - 5 \cdot 22$$

$$x = 2, y = -5$$

Application (linear diophantine equations). Let $a, b, c \in \mathbb{Z}$, $a, b, c \neq 0$. Find all $(x, y) \in \mathbb{Z}^2$ which satisfy

$$ax + by = c. \quad (8)$$

Existence of solution let $d = \gcd(a, b)$.

$$(d \mid a \Rightarrow d \mid xa) \wedge (d \mid b \Rightarrow d \mid yb)$$

$$\Rightarrow d \mid xa + yb = c$$

$$\Rightarrow \text{eq. (8)}$$

can have solutions only if $d \mid c$.**Solution in case $d = 1$** Let $x_0, y_0 \in \mathbb{Z}$ such that $ax_0 + by_0 = 1$ using the Euclidean algorithm. Then from $acx_0 + bcy_0 = c$ the solution (cx_0, cy_0) of (eq. (8)) follows: for all $n \in \mathbb{Z}$: $(x, y) := (cx_0 + nb, cy_0 + na)$ is a solution.

Indeed,

$$ax + by = acx_0 + anb + bcy_0 - bna = c \quad \checkmark$$

These (x, y) are all solutions: let (x, y) be a solution. Then

$$\begin{aligned} ax + by &= c \\ acx_0 + bcy_0 &= c \\ \Rightarrow a(x - cx_0) &= b(cy_0 - y) \\ \gcd(a, b) = 1 &\Rightarrow b \mid x - cx_0 \Rightarrow x = cx_0 + nb, n \in \mathbb{Z} \\ \Rightarrow a \mid cy_0 - y &\Rightarrow y = cy_0 + ma, m \in \mathbb{Z} \\ c = ax + by &= acx_0 + anb + bcy_0 + bma \\ &= c + (n + m)ab \Rightarrow (n + m)ab = 0 \Rightarrow m = -n \end{aligned}$$

Solutions in the general case Assume $d = \gcd(a, b)$ and $d \mid c$, let

$$a' = \frac{a}{d} \quad b' = \frac{b}{d} \quad c' := \frac{c}{d}$$

Then $\gcd(a', b') = 1$ and the solution to (eq. (8)) is exactly the solution of $a'x + b'y = c'$.

0.2 Primes

Definition 0.2.1. $p \in \mathbb{N}$, $p > 1$ is a prime number if the only positive divisors of p are 1 and p , i.e., $a \in \mathbb{N}$, $a \mid p \Rightarrow a \in \{1, p\}$. $\mathbb{P} := \{\text{primes}\} \subset \mathbb{N}$, $\mathbb{P} = \{2, 3, 5, 7, 11, 13, \dots\}$. p prime and $p \mid ab \Rightarrow p \mid a$ or $p \mid b$

Theorem 0.2.2 (Fundamental theorem of arithmetic). Every $n \in \mathbb{N}$ can be written uniquely (up to reordering) as a product of primes. i.e. there are distinct primes p_1, \dots, p_l , and $\alpha_1, \dots, \alpha_l \in \mathbb{N}$ such that $n = p_1^{\alpha_1} \dots p_l^{\alpha_l}$

Sketch.

Existence let $p_0 > 1$ be the smallest divisor > 1 of n . Then p_0 is prime. $n = p_0 n_0$, induction \checkmark

Uniqueness let $p_1 \dots p_m = q_1 \dots q_l = n$, p_i, q_j primes. $p_1 \mid q_1 \dots q_l \Rightarrow \exists i : p_1 \mid q_i$, both prime $\Rightarrow p_1 = q_i$, wlog: $i = 1$. $p_1 \dots p_m = q_1 \dots q_l$, induction \checkmark

□

Theorem 0.2.3 (Euclid). There are ∞ -many primes.

Proof. Given primes $p_1, \dots, p_n \in \mathbb{P}$. We construct one more prime

$$N := p_1 \dots p_n + 1.$$

Assume P is a prime factor of N . If $P \in \{p_1, \dots, p_n\}$ then $P \mid N$ and $P \mid p_1 \dots p_n \Rightarrow P \mid 1$ \nmid

□

Remark (prime factors and gcds). Let $a_1, \dots, a_n \in \mathbb{Z}$, write

$$a_i = \prod_{p \in \mathbb{P}} p^{\alpha_{p,i}}, \quad \alpha_{p,i} \in \mathbb{N}_0,$$

almost all $a_i = 0$, then

$$\gcd(a_1, \dots, a_n) = \prod_{p \in \mathbb{P}} p^{\min_{1 \leq i \leq n} \{\alpha_{p,i}\}}$$

1. Beistriche für bessere Lesbarkeit
2. faustregel, vor und nach "i.e." gehört eigentlich beistrich

0.3 Congruences

All rings are commutative with 1.

Definition 0.3.1. Let $a, b \in \mathbb{Z}$, $n \in \mathbb{N}$. Then a is congruent to $b \pmod{n}$, $a \equiv b \pmod{n}$, if $n \mid a - b$. We write $\bar{a} = [a]_n := \{b \in \mathbb{Z} : b \equiv a \pmod{n}\}$

Remark. 1. Congruence $\text{mod } n$ is an equivalence relation

2. $\bar{0}, \bar{1}, \dots, \overline{n-1}$ is a partition of \mathbb{Z} .

3. if $a \equiv b \pmod{n}$, $c \equiv d \pmod{n}$, then $-a \equiv -b \pmod{n}$, $a \pm d \pmod{n}$.

Definition 0.3.2. $\mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n := \{[a]_n : a \in \mathbb{Z}\} = \{\bar{0}, \bar{1}, \dots, \overline{n-1}\}$ residue class ring modulo n

Remark. \mathbb{Z}_n is a ring with operation $\bar{a} \pm \bar{b} := \overline{a \pm b}$ (well defined due to item 3 of section 0.3) $\mathbb{Z}_n^\times = \{\bar{a} \in \mathbb{Z}_n : \exists \bar{b} \in \mathbb{Z}_n : \bar{a}\bar{b} = \bar{1}\}$... group of units \pmod{n}

Lemma 0.3.3. Let $a \in \mathbb{Z}$. Then $\bar{a} \in \mathbb{Z}_n^\times \Leftrightarrow \gcd(a, n) = 1$.

Proof.

“ \Rightarrow ” $\bar{a}\bar{b} = \bar{1} \Leftrightarrow ab \equiv 1 \pmod{n} \Leftrightarrow n \mid ab - 1$
 \Rightarrow no prime factor of n divides a
 $\Rightarrow \gcd(a, n) = 1$.

“ \Leftarrow ” $1 = \gcd(a, n) = ax + ny \Rightarrow \bar{1} = \bar{a}\bar{x}$

□

Remark. The inverse of \bar{a} can be computed by the Euclidean algorithm.

Example (Simultaneous congruences). Find $x \in \mathbb{Z}$ such that

$$x \equiv 2 \pmod{3} \tag{9}$$

$$x \equiv 1 \pmod{5} \tag{10}$$

$$x \equiv 0 \pmod{7} \tag{11}$$

Theorem 0.3.4 (Chinese remainder theorem (CRT)). Let

$$n_1, \dots, n_l \in \mathbb{N} \text{ subject to } \gcd(n_i, n_j) = 1 \ \forall i \neq j$$

$$x_1, \dots, x_l \in \mathbb{Z}.$$

Then

$$\exists x \in \mathbb{Z} \text{ such that } x \equiv x_i \pmod{n_i} \ \forall 1 \leq i \leq l$$

where x is unique modulo $n_1 \cdots n_l$.

Proof. How to compute x ? For $i \in \{1, \dots, l\}$, let

$$N_i := \prod_{j \neq i} n_j = n_1 \dots n_{i-1} n_{i+1} \dots n_l$$

and let

$$N := \prod_i n_i = n_1 N_1 = n_2 N_2 = \dots = n_l N_l$$

because $\gcd(n_i, N_i) = 1 \Rightarrow N_i$ is invertible mod n_i . Let

$$m_i N_i \equiv 1 \pmod{n_i}$$

and let

$$x := N_1 m_1 x_1 + \cdots + N_l m_l x_l.$$

We have $N_i m_i x_i \equiv 0 \pmod{n_j, j \neq i}$ □

Example.

$$\begin{aligned} n_1 &= 3, & n_2 &= 5, & n_3 &= 7 \\ x_1 &= 2, & x_2 &= 1, & x_3 &= 0 \\ N_1 &= 35, & N_2 &= 21, & N_3 &=? \\ \bar{m}_1 &= \overline{35}^{-1} \pmod{3} = \bar{2}^{-1} \pmod{3} = \bar{2} \pmod{3} \Rightarrow m_1 = 2 \\ \bar{m}_2 &= \overline{21}^{-1} \pmod{5} = \bar{1}^{-1} \pmod{5} = \bar{1} \pmod{5} \Rightarrow m_2 = 1 \end{aligned}$$

$$\begin{aligned} x &= 35 \cdot 2 \cdot 2 + 21 \cdot 1 \cdot 1 + 0 \\ &= 140 + 21 \\ &= 161 \\ &\equiv 56 \pmod{105} \end{aligned}$$

Example (more abstract CRT). Let $n_1, \dots, n_l \in \mathbb{N}$, with $\gcd(n_i, n_j) = 1 \ \forall i \neq j$. There is a ring isomorphism $f : \mathbb{Z}_{n_1 \dots n_l} \xrightarrow{\cong} \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_l}$ that satisfies $f([a]_{n_1 \dots n_l}) = ([a]_{n_1}, \dots, [a]_{n_l}) \ \forall a \in \mathbb{Z}$. In particular: $\mathbb{Z}_{n_1 \dots n_l}^\times \cong \mathbb{Z}_{n_1}^\times \times \cdots \times \mathbb{Z}_{n_l}^\times$ (restrict f to $\mathbb{Z}_{n_1 \dots n_l}^\times$)

0.4 Arithmetic functions

Definition 0.4.1. $f : \mathbb{N} \rightarrow \mathbb{C}$ is an arithmetic function. f is multiplicative if $\forall m, n$ it holds that $\gcd(m, n) = 1$. We have $f(mn) = f(m)f(n)$. f is completely multiplicative if $\forall m, n : f(mn) = f(m)f(n)$. Let $f : \mathbb{N} \rightarrow \mathbb{C}$. Its summatory function is $S_f(n) := \sum_{d|n} f(d)$.

Proof. If $\gcd(m, n) = 1$ and $d \mid mn$, then \exists unique d_1, d_2 such that $d = d_1 \cdot d_2$ with $d_1 \mid m, d_2 \mid n$.

$$S_f(mn) = \sum_{d|mn} f(d) = \sum_{d_1|m} \sum_{d_2|n} f(d_1 d_2) = \sum_{d_1|m} f(d_1) \sum_{d_2|n} f(d_2) = S_f(m) S_f(n)$$

□

Example.

$$\begin{aligned} \tau(n) &:= S_1(n) = \sum_{d|n} 1 && \dots \text{number of divisors of } n \\ \sigma(n) &:= S_{id}(n) = \sum_{d|n} d && \dots \text{divisor sum of } n \end{aligned}$$

Definition 0.4.2. The function $\phi(n) := |\mathbb{Z}_n^\times|$ is called Euler's ϕ -function.

Remark. 1. $\phi(n) = |\{0 \leq a < n : \gcd(a, n) = 1\}|$

2. ϕ is multiplicative (CRT: $\gcd(m, n) = 1$. $\mathbb{Z}_{nm}^\times \cong \mathbb{Z}_n^\times \times \mathbb{Z}_m^\times$)

3. $\phi(p) = p - 1$ (\mathbb{Z}_p is a field)

Lemma 0.4.3. $\phi(p^n) = p^n - p^{n-1}$

Proof.

$$\begin{aligned}\phi(p^n) &= |\{0 \leq a < p^n\}| - |\{0 \leq a < p^n : \gcd(a, p^n) \neq 1\}| \\ &= p^n - |\{0 \leq a < p^n : p|a\}| \\ &= p^n - p^{n-1}\end{aligned}$$

□

Proposition 0.4.4. If $n = p_1^{\alpha_1} \dots p_l^{\alpha_l}$ with $p_i \neq p_j$ primes, $\alpha_i \in \mathbb{N}$. Then

$$\phi(n) = \prod_{i=1}^l p_i^{\alpha_i} \left(1 - \frac{1}{p_i}\right) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$$

Theorem 0.4.5 (Euler-Fermat). Then $a^{\phi(n)} \equiv 1 \pmod n$. In particular: $a^{p-1} \equiv 1 \pmod p \ \forall p \nmid a$ (little Fermat).

Proof 1. Lagrange's Theorem, $G = \mathbb{Z}_n^\times$, $\bar{a} \in G \Rightarrow \bar{a}^{|G|} = \bar{1}$, $|G| = \phi(n)$. □

Proof 2. $\prod_{x \in \mathbb{Z}_n^\times} x = \prod_{x \in \mathbb{Z}_n^\times} (\bar{a}x) = \bar{a}^{\phi(n)} \prod_{x \in \mathbb{Z}_n^\times} x \Rightarrow a^{\phi(n)} \equiv 1 \pmod n$ □

Definition 0.4.6. The Möbius function $\mu : \mathbb{N} \rightarrow \{-1, 0, +1\}$ is defined as

$$\mu(n) = \begin{cases} (-1)^l & n = p_1 \dots p_l, p_i \neq p_j, i \neq j, p_i \text{ primes} \\ 0 & \text{otherwise i.e. if } \exists p : p^2 \mid n \end{cases}$$

Remark.

1. $\mu(1) = 1$, $\mu(2) = -1$, $\mu(3) = -1$, $\mu(4) = 0$, $\mu(5) = -1$, $\mu(6) = 1$, ...

2. μ is multiplicative

Lemma 0.4.7.

$$S_\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$$

Proof.

$$S_\mu(1) = \sum_{d|1} \mu(d) = \mu(1) = 1$$

By multiplicativity, it suffices to prove $S_\mu(p^n) = 0 \ \forall p, n$.

$$\begin{aligned}S_\mu(p^n) &= \sum_{d|p^n} \mu(d) \\ &= \sum_{i=0}^n \mu(p^i) \\ &= \mu(1) + \mu(p) + 0 + \dots + 0 \\ &= 0\end{aligned}$$

□

Theorem 0.4.8 (Möbius inversion formula). *Let $f : \mathbb{N} \rightarrow \mathbb{C}$. Then*

$$f(n) = \sum_{d|n} \mu(d) S_f\left(\frac{n}{d}\right).$$

Proof.

$$\begin{aligned} \sum_{d|n} \mu(d) S_f\left(\frac{n}{d}\right) &= \sum_{d|n} \mu(d) \sum_{e|\frac{n}{d}} f(e) \\ &= \sum_{e|n} f(e) \sum_{\substack{d|n \\ s.t. e|\frac{n}{d}}} \mu(d) \end{aligned}$$

$$\begin{aligned} \text{For the next step we use } d | n \wedge e | \frac{n}{d} &\Leftrightarrow ed | n \Leftrightarrow e | n \wedge d | \frac{n}{e} \\ &= \sum_{e|n} f(e) \sum_{d|\frac{n}{e}} \mu(d) \\ &= f(n) \end{aligned}$$

$$\text{since } \sum_{d|\frac{n}{e}} \mu(d) = \begin{cases} 1 & \frac{n}{e} = 1 \\ 0 & \text{otherwise} \end{cases}$$

□

0.5 Structure of \mathbb{Z}_n^\times

$n = p_1^{\alpha_1} \dots p_l^{\alpha_l}$ with $p_i \neq p_j, i \neq j, \alpha_i \in \mathbb{N}$ where p_i are primes

From the CRT it follows that $\mathbb{Z}_n^\times \cong \mathbb{Z}_{p_1^{\alpha_1}}^\times \times \dots \times \mathbb{Z}_{p_l^{\alpha_l}}^\times$. So we only consider prime powers $p^\alpha, p \in \mathbb{P}, \alpha \in \mathbb{N}$

0.5.1 Case 1: $\alpha = 1$

Theorem 0.5.1. \mathbb{Z}_p^\times is cyclic, i.e. $\mathbb{Z}_p^\times \cong \mathbb{Z}_{(p-1)}$

Proof. Use structure theorem for finite abelian groups. If G is a finite abelian group then $\exists d_1, \dots, d_l \in \mathbb{N}$ such that $1 < d_1 | d_2 | d_3 | \dots | d_l$, and $G \cong \mathbb{Z}_{d_1}^\times \times \dots \times \mathbb{Z}_{d_l}^\times$ thus, $\mathbb{Z}_p^\times \cong \mathbb{Z}_{d_1}^\times \times \dots \times \mathbb{Z}_{d_l}^\times$ (every element $x \in \mathbb{Z}_{d_1}^\times \times \dots \times \mathbb{Z}_{d_l}^\times$ satisfies $d_l x = 0 \Rightarrow$ every $x \in \mathbb{Z}_p^\times$ satisfies $x^{d_l} = 1$). $x^{d_l} - 1$ is a polynomial of degree d_l over the field $\mathbb{Z}_p \Rightarrow x^{d_l} - 1$ has $\leq d_l$ roots $\Rightarrow p-1 \leq d_l$, but $p-1 = d_1 \dots d_l \Rightarrow l = 1, p-1 = d_l$ □

Remark. The same proof shows: Let F be a field, $G \leq F^\times, |G| < \infty$. Then G is cyclic.

0.5.2 Case 2: $\alpha \geq 2; p \geq 3$

Denote $|x|$ as the order of x in $\mathbb{Z}_{p^\alpha}^\times$; i.e. $|x| = \min \{l \in \mathbb{N} : x^l \equiv 1 \pmod{p^\alpha}\}$

$|\mathbb{Z}_{p^\alpha}^\times| = \phi(p^\alpha) = p^{\alpha-1}(p-1)$, find $x, y \in \mathbb{Z}_{p^\alpha}^\times$ such that $|x| = p^{\alpha-1}, |y| = p-1$ then $|xy| = |x||y| = p^{\alpha-1}(p-1)$, since $\gcd(|x|, |y|) = 1$

Lemma 0.5.2.

$$(1+p)^{p^{n-1}} \begin{cases} \equiv 1 & \pmod{p^n} \\ \not\equiv 1 & \pmod{p^{n+1}} \end{cases}$$

Proof. Proof by induction

$n = 1$ ✓

$n \rightarrow n + 1$

$$\begin{aligned}(1+p)^{p^{n-1}} &= 1 + ap^n, p \nmid a \\ (1+p)^{p^n} &= (1+ap^n)^p \\ &= 1 + pap^n + \sum_{i=2}^{p-1} \binom{p}{i} (ap^n)^i + (ap^n)^p\end{aligned}$$

$$\begin{aligned}p^{np} \mid \bullet, \quad np \geq n+2, \quad (\text{or } p \geq 3), \quad p^{2n+1} \mid \bullet, \quad 2n+1 \geq n+2 \\ p \mid \binom{p}{i} = \frac{p!}{i!(p-i)!}, 1 \leq i < p \Rightarrow (1+p)^{p^n} \equiv 1 + ap^{n+1} \pmod{p^{n+2}}, p \nmid a\end{aligned}$$

□

2× Lemma: $x = 1 + p$ satisfies $|x| = p^{\alpha-1}$, now find y .

1. $\exists z \in \mathbb{Z} : |\bar{z}| = p - 1$ is \mathbb{Z}_p^\times
2. let $l := |E|$ is $\mathbb{Z}_{p^\alpha}^\times$
3. Then $p^\alpha \mid z^l - 1 \Rightarrow z^l \equiv 1 \pmod{p}$
4. $\Rightarrow p - 1 \mid l$.
5. Let $y := z^{\frac{l}{p-1}}$, then $|\bar{y}| = p - 1$.

We have proven: Theorem: $\mathbb{Z}_{p^\alpha}^\times$ is cyclic, i.e. $\mathbb{Z}_{p^\alpha}^\times \cong \mathbb{Z}_{p^{\alpha-1}(p-1)}$, if $p \geq 3, \alpha \geq 1$.
 $p = 2$: $\mathbb{Z}_{2^\alpha}^\times \cong \{0, \alpha = 1 \quad \mathbb{Z}_2, \alpha = 2 \quad \mathbb{Z}_2 \times \mathbb{Z}_{p^{\alpha-2}}, \alpha \geq 3\}$

Corollary 0.5.3. Let $m \in \mathbb{N}$. Then \mathbb{Z}_m^\times is cyclic iff m has one of the following forms:

- $m = 2$
- $m = 4$
- $m = p^\alpha, p \geq 3, \alpha \in \mathbb{N}$
- $m = 2p^\alpha, p \geq 3, \alpha \in \mathbb{N}$

In these cases a generator of \mathbb{Z}_m^\times is called a *primitive root modulo m* .

New Lecturer

Chapter 1:

1. Approximation to algebraic numbers; Wolfgang M. Schmidt, 1972 L'Ehseignement Mathématique
2. Lectures Notes in Mathematics 785; W.M.Schmidt, Springer
3. LNM 1467, W.M.S., Springer
4. For section 2 (continued fractions) he will strictly follow the lecture notes of MT421 of Professor James McKee

1 Diophantine Approximation

1.1 Dirichlet's Theorem

Let $\alpha \in \mathbb{R}$. As \mathbb{Q} is dense in \mathbb{R} any $\alpha \in \mathbb{R}$ can be approximated arbitrarily well, by rational numbers p/q ($p \in \mathbb{Z}, q \in \mathbb{N} = \{1, 2, 3, \dots\}$).

The question is how well can we approximate α in terms of the denominator q , e.g., is it true that for every $\alpha \in \mathbb{R}$ there exist infinitely many $p/q \in \mathbb{Q}$ ($q \in \mathbb{N}$) such that $|\alpha - \frac{p}{q}| < \frac{1}{q^2}$?

The answer is no!

Take $\alpha = r/s$ ($s \in \mathbb{N}$) a rational number. Then

$$|\alpha - \frac{p}{q}| = |\frac{r}{s} - \frac{p}{q}| = |\frac{qr - ps}{sq}| \stackrel{\geq}{\geq} \frac{p}{q} \frac{1}{sq} > \frac{1}{q^2} \text{ provided } q > s.$$

This shows that we have only finitely many solutions $p/q \in \mathbb{Q}$ for $|\alpha - \frac{p}{q}| < \frac{1}{q^2}$.

Theorem 1.1.1 (Dirichlet's Theorem). *Suppose $\alpha, Q \in \mathbb{R}$ and $Q > 1$. Then $\exists p, q \in \mathbb{Z}$ s.t. $0 < q < Q$ and $|q\alpha - p| \leq \frac{1}{Q}$.*

Proof. for $\xi \in \mathbb{R}$ put $\{\xi\} = \xi - \lfloor \xi \rfloor$. so $0 \leq \{\xi\} < 1$. First suppose $Q \in \mathbb{Z}$. Consider the $Q+1$ numbers $0, 1, \{\alpha\}, \{2\alpha\}, \dots, \{(Q-1)\alpha\}$.

They all lie in $[0, 1]$. We split it up in Q subintervals:

$$[0, 1] = [0, \frac{1}{Q}] \cup [\frac{1}{Q}, \frac{2}{Q}] \cup \dots \cup [\frac{Q-1}{Q}, 1]$$

By the pigeon hole principle two of the previous numbers lie in the same subinterval. Thus $\exists r_1, r_2, s_1, s_2 \in \mathbb{Z}$ with $0 \leq r_1 < r_2 \leq Q-1$ such that $|(r_1\alpha - s_1) - (r_2\alpha - s_2)| \leq \frac{1}{Q}$. Then with $q = r_2 - r_1$ and $p = s_2 - s_1$ we get $|q\alpha - p| \leq \frac{1}{Q}$ and $0 < q < Q$. This proves the Theorem when $Q \in \mathbb{Z}$. Now suppose $Q \notin \mathbb{Z}$. We apply the previous with $Q' = \lfloor Q \rfloor + 1 > 1$. Hence, $\exists p, q \in \mathbb{Z}$ with $|q\alpha - p| \leq \frac{1}{Q'}$ and $0 < q < Q'$, and so $|q\alpha - p| \leq \frac{1}{Q}$ and $0 < q < Q$. \square

Corollary 1.1.2. *Suppose $\alpha \in \mathbb{R}/\mathbb{Q}$. Then there exist infinitely many solutions $p/q \in \mathbb{Q}$ ($q \in \mathbb{N}$) of $|\alpha - \frac{p}{q}| < \frac{1}{q^2}$.*

Proof. Take $Q_1 > 1$. By Theorem 1.1.1 we get $(p_1, q_1) \in \mathbb{Z}^2$ with $0 < q_1 < Q_1$, and $|q_1\alpha - p_1| \leq \frac{1}{Q_1}$. Thus $|\alpha - \frac{p_1}{q_1}| \leq \frac{1}{q_1 Q_1} < \frac{1}{q_1^2}$.

Next take $Q_2 = |\alpha - \frac{p_1}{q_1}|^{-1} + 1$. Then Thm 1.1.1 again yields $\frac{p_2}{q_2} \in \mathbb{Q}$ with $|\alpha - \frac{p_2}{q_2}| < \frac{1}{q_2^2}$ and $|\alpha - \frac{p_2}{q_2}| \leq \frac{1}{q_1 Q_2} \leq \frac{1}{Q_2} < |\alpha - \frac{p_1}{q_1}|$. So $\frac{p_2}{q_2}$ is a better approx than $\frac{p_1}{q_1}$. Repeating this process indefinitely proves the claim. \square

Theorem 1.1.3 (Pell-equation). *Suppose $m \in \mathbb{N}$ is not a square (i.e., $m \neq n^2 \forall n \in \mathbb{Z}$).*

Then

$$x^2 - my^2 = 1$$

has infinitely many solutions $(x, y) \in \mathbb{Z}^2$.

Proof. Apply Corollary 1.1.2 with $\alpha = \sqrt{m}$. So $\alpha \in \mathbb{R}/\mathbb{Q}$. We get $|\alpha - \frac{p}{q}| < \frac{1}{q^2}$ and $|\alpha + \frac{p}{q}| \leq 1 + 2\alpha$. Thus

$$|p^2 - mq^2| = q^2 \left| \alpha - \frac{p}{q} \right| \cdot \left| \alpha + \frac{p}{q} \right| < 1 + 2\sqrt{m}.$$

Hence, there exists $k \in \mathbb{Z}$ with $|k| < 1 + 2\sqrt{m}$ such that $p^2 - mq^2 = k$ for infinitely many $(p, q) \in \mathbb{Z}^2$ and p/q all distinct. As m is not a square we have $k \neq 0$.

Let S be the set of solutions $(p, q) \in \mathbb{Z}^2$ of $p^2 - mq^2 = k$. The map $S \rightarrow (\mathbb{Z}/k\mathbb{Z}) \times (\mathbb{Z}/k\mathbb{Z})$. This map is not injective ($S = \infty$) hence, $\exists (p_1, q_1) \neq (p_2, q_2)$ both in S such that $p_1 \equiv p_2, q_1 \equiv q_2 \pmod{k}$. (MOD)
Now we compute

$$k^2 = (p_1^2 - mq_1^2)(p_2^2 - mq_2^2) \quad (12)$$

$$= (p_1 + \sqrt{m}q_1)(p_2 - \sqrt{m}q_2) \quad (13)$$

$$= (r - \sqrt{m}s)(r + \sqrt{m}s) = r^2 - ms^2 \quad (14)$$

$$\text{where } r = p_1p_2 - mq_1q_2 \quad (15)$$

$$s = p_1q_2 - q_1p_2 = \frac{1}{q_1q_2} \left(\frac{p_1}{q_1} - \frac{p_2}{q_2} \right) \neq 0. \quad (16)$$

because of (MOD) $k \mid s$. Hence, $k^2 \mid s^2$. Thus $k^2 \mid r^2$. Hence $k \mid r$. Then $x = \frac{r}{k}$ and $y = \frac{s}{k}$ are both integers and

$$x^2 - my^2 = 1.$$

We have one solution but we need infinitely many! To this end we replace m by md^2 ($d \in \mathbb{N}$). The above argument yields a solution $(x', y') \in \mathbb{Z}^2$ of $x'^2 - md^2y'^2 = 1$. Thus, $(x, y) = (x', dy')$ is a new solution of $x^2 - my^2 = 1$.
(Critical: $s \neq 0$) □

1.2 Continued fractions

Let $\theta \in \mathbb{R}$. Put $a_0 = \lfloor \theta \rfloor$. If $a_0 \neq \theta$ then we find $\theta_1 > 1$ such that

$$\theta = a_0 + \frac{1}{\theta_1}$$

and we put $a_1 = \lfloor \theta_1 \rfloor$. If $a_1 \neq \theta_1$ then we can find $\theta_2 > 1$ such that

$$\theta_1 = a_1 + \frac{1}{\theta_2}$$

and we put $a_2 = \lfloor \theta_2 \rfloor$. This process can be continued indefinitely, unless $a_n = \theta_n$ for some n . Note that a_0 can be zero or negative but a_1, a_2, a_3, \dots are all positive integers.

We call this process the *continued fraction process*. The a_i are called *partial quotients* of θ .

Example.

$$\theta = \frac{19}{11}$$

Then $a_0 = \lfloor \theta \rfloor = 1$

Now $\theta = \frac{19}{11} = a_0 + \frac{1}{\theta_1} = 1 + \frac{8}{11} = 1 + \frac{1}{\frac{11}{8}}$

So $\theta_1 = \frac{11}{8}$.

Thus $a_1 = \lfloor \theta_1 \rfloor = 1$.

Now

$$\theta_1 = \frac{11}{8} = a_1 + \frac{1}{\theta_2} = 1 + \frac{3}{8} = 1 + \frac{1}{\frac{8}{3}}$$

Thus $\theta_2 = \frac{8}{3}$ and $a_2 = \lfloor \theta_2 \rfloor = 2$

and so on...

If the continued fraction process terminates then we have

$$\theta = a_0 + \frac{1}{\theta_1} \quad (17)$$

$$= a_0 + \frac{1}{a_2 + \frac{1}{\theta_2}} \quad (18)$$

$$= a_0 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{\theta_3}}} \quad \dots = a_0 + \frac{1}{a_1 + \frac{1}{\dots}} \quad (19)$$

In this case we write $\theta = [a_0, \dots, a_n]$.

We use the same notation when the a_i are any real numbers, not necessarily integers.

In particular

$$\theta = [a_0, \dots, a_i, \theta_{i+1}]$$

where $a \leq i < n$.

If the continued fraction process does not terminate then we write $\theta = [a_0, a_1, a_2, \dots]$.

Note that in this case, for every $n \geq 0$, we have

$$\theta = [a_0, \dots, a_n, \theta_{n+1}]$$

where a_0, \dots, a_n are integers but θ_{n+1} is not! For $n \geq 0$ we set

$$\frac{p_n}{q_n} = [a_0, \dots, a_n]$$

where $\gcd(p_n, q_n) = 1$. We shall say that $\frac{p_n}{q_n}$ is the n -th convergent of θ . We will prove that $\frac{p_n}{q_n} \rightarrow \theta$ as $n \rightarrow \infty$. Next we shall see that $p_n, q_n > 0$ both satisfy the same simple recurrence relation $x_n = a_n x_{n-1} + x_{n-2}$ with different starting values.

Lemma 1.2.1. *Let a_0, a_1, a_2, \dots be a sequence of integers with $a_i > 0$ ($i > 0$).*

Define p_n, q_n :

$$p_0 = a_0 \quad (20)$$

$$q_0 = 1 \quad (21)$$

$$p_1 = a_0 a_1 + 1 \quad (22)$$

$$q_1 = a_1 \quad (23)$$

$$p_n = a_n p_{n-1} + p_{n-2} \text{ for } n \geq 2 \quad (24)$$

$$q_n = a_n q_{n-1} + q_{n-2} \text{ for } n \geq 2. \quad (25)$$

Then:

$$1. \quad p_n q_{n+1} - p_{n+1} q_n = (-1)^{n+1}$$

$$2. \quad \gcd(p_n, q_n) = 1$$

$$3. \quad p_n/q_n = [a_0, \dots, a_n]$$

4. If the a_i are produced by the continued fraction process for θ , then, for every $n \geq 1$, $\frac{p_n}{q_n}$ is the n -th convergent of θ and

$$\theta = \frac{p_n \theta_{n+1} + p_{n-1}}{q_n \theta_{n+1} + q_{n-1}}$$

Proof. 1. We use induction on n . For $n = 0$ we note that

$$p_0 q_1 - p_1 q_0 = a_0 a_1 - a_0 a_1 - 1 = -1.$$

So the result holds for $n = 0$.

Now suppose result holds for $n = m - 1$.

consider case $n = m$. Using the recurrence relation, we set

$$p_m q_{m+1} - p_{m+1} q_m = p_m (a_m q_m + q_{m-1}) - q_m (a_m p_m + p_{m-1}) \quad (26)$$

$$= p_m q_{m-1} - p_{m-1} q_m = -(-1)^m = (-1)^{m+1}. \quad (27)$$

This proves claim for $n = m$.

2. Immediate from (a)

3. (c) + (d):

Remark about $\frac{p_n}{q_n}$ in (d) follows directly from (c). We prove the rest of (d), along with (c), using induction on n . Remember that (c) a priori does not require that the a_i are produced by the continued fraction process. Consider base case $n = 1$. For (c) note that $\frac{p_1}{q_1} = a_0 + \frac{1}{a_1} = [a_0, a_1]$. For (d) we note that

$$\frac{p_1 \theta_2 + p_0}{q_1 \theta_2 + q_0} = \frac{(a_0 a_1 + 1) \theta_2 + a_0}{a_1 \theta_2 + 1} = a_0 + \frac{\theta_2}{a_1 \theta_2 + 1} = a_0 + \frac{1}{a_1 + \frac{1}{\theta_2}} = \theta$$

Next suppose (c) and (d) both hold for $n = m - 1$, and consider $n = m$.

Using (d) with $n = m - 1$ we get

$$[a_0, \dots, a_m] = \frac{p_{m-1} a_m + p_{m-2}}{q_{m-1} a_m + q_{m-2}} = \frac{p_m}{q_m} \text{ by recurrence relation.}$$

This proves (c) for $n = m$.

To prove (d) with $n = m$ we observe that

$$\theta = [a_0, \dots, a_m, \theta m + 1] \quad (28)$$

$$= [a_0, \dots, a_m + \frac{1}{\theta_{m+1}}] \quad (29)$$

$$(d) \text{ for } n = m - 1 \frac{p_{m-1}(a_m + \frac{1}{\theta_{m+1}}) + p_{m-2}}{q_{m-1}(a_m \frac{1}{\theta_{m-1}}) + q_{m-2}} \quad (30)$$

$$= \text{rec.rel} \frac{p_m + p_{m-1}(\frac{1}{\theta_{m+1}})}{q_m + q_{m-1}(\frac{1}{\theta_{m+1}})} \quad (31)$$

$$= \frac{p_m \theta_{m+1} + p_{m-1}}{q_m \theta_{m+1} + q_{m-1}} \quad (32)$$

which is (d) for $n = m$.

□

Next we deduce some properties of continued fraction convergents.

Theorem 1.2.2. *Let $\theta = [a_0, a_1, a_2, \dots]$ with convergents $\frac{p_n}{q_n}$. For (a) - (d) we assume that the continued fraction process does not terminate*

1. For all $n \in \mathbb{N}_0$, θ lies between $\frac{p_n}{q_n}$ and $\frac{p_{n+1}}{q_{n+1}}$.
2. For all $n \in \mathbb{N}_0 : |\theta - \frac{p_n}{q_n}| \leq \frac{1}{q_n q_{n+1}}$
3. For $n \geq 1$ we have $q_{n+2} \geq 2 \cdot q_n$
4. $\frac{p_n}{q_n} \rightarrow \theta$ as $n \rightarrow \infty$
5. The continued fraction process terminates if and only if θ is rational.

Proof. 1. Note $\theta = [a_0, \dots, a_n, \theta_{n+1}] = [a_0, \dots, a_n + \frac{1}{\theta_{n+1}}]$ where $0 < \frac{1}{\theta_{n+1}} < \frac{1}{a_{n+1}}$. So that θ lies between $[a_0, \dots, a_n]$ and $[a_0, \dots, a_n + \frac{1}{a_{n+1}}]$. But $[a_0, \dots, a_n + \frac{1}{a_{n+1}}] = [a_0, \dots, a_{n+1}]$. This shows (a).

2. By (a) we have $|\theta - \frac{p_n}{q_n}| \leq |\frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}}| = |\frac{p_n q_{n+1} - p_{n+1} q_n}{q_n q_{n+1}}| \stackrel{\text{Lemma 1.2.1(a)}}{=} \frac{1}{q_n q_{n+1}}$

3. Follows from the fact that $a_i > 0 (i > 0)$ using Lemma 1.2.1.

4. Follows from (b) and (c)

5. Only if part is obvious.

Conversely suppose $\theta = \frac{a}{b} \in \mathbb{Q}$ but the process does *not* terminate. Taking n such that $q_n > b$ yields

$$|\theta - \frac{p_n}{q_n}| \geq \frac{a}{b} - \frac{p_n}{q_n} \geq \frac{p_n}{q_n} \text{ as } q_n > b \text{ and } \gcd(p_n, q_n) = 1 \geq \frac{1}{b q_n} > \frac{1}{q_n q_{n+1}}$$

contradicting (b).

□

Example. Take $\theta = \frac{16}{9}$. We have $a_0 = 1$. Then $\theta = 1 + \frac{7}{9}$ so $\theta_1 = \frac{9}{7}$ and $a_1 = 1$. From $\theta_1 = \frac{9}{7} = 1 + \frac{2}{7}$ we get $\theta_2 = \frac{7}{2}$ and $a_2 = 3$. From $\theta_2 = \frac{7}{2} = 3 + \frac{1}{2}$ we get $\theta_3 = 2$ and $a_3 = 2$. Thus $\theta = \frac{16}{9} = [1, 1, 3, 2]$ and the convergents are $\frac{p_0}{q_0} = \frac{1}{1}$, $\frac{p_1}{q_1} = 1 + \frac{1}{1} = \frac{2}{1}$, $\frac{p_2}{q_2} = 1 + \frac{1}{1+\frac{1}{3}} = 1 + \frac{1}{\frac{4}{3}} = \frac{7}{4}$ and $\frac{p_3}{q_3} = \frac{16}{9}$.

Let's check some of the properties claimed.

$$p_1q_2 + p_2q_1 = 2 \cdot 4 - 7 \cdot 1 = 1\checkmark, p_2q_3 - p_3q_2 = 7 \cdot 9 - 16 \cdot 4 = -1\checkmark, \frac{p_2\theta_3 + p_1}{q_2\theta_3 + q_1} = \frac{7 \cdot 2 + 2}{4 \cdot 2 + 1} = \frac{16}{9} = \theta\checkmark$$

We now show that convergents give best-possible rational approximations.

Theorem 1.2.3. Let θ be an irrational real number, and let $\frac{p_n}{q_n}$ be the convergents ($n \geq 0$) with partial quotients a_n ($n \geq 0$).

Then

1. $|\theta - \frac{p_n}{q_n}|$ strictly decreases as n increases.
2. the convergents give successively closer approximations to θ .
3. $\frac{1}{(a_{n+1}+2)q_n^2} < |\theta - \frac{p_n}{q_n}| < \frac{1}{a_{n+1}q_n^2} \leq \frac{1}{q_n^2}$
4. If $p, q \in \mathbb{Z}$ with $0 < q < q_{n+1}$ then

$$|q\theta - p| \geq |q_n\theta - p_n|$$

Moreover, "=" only if $(p, q) = (p_n, q_n)$.

(In this sense convergents are best-possible approximations.)

5. If $(p, q) \in \mathbb{Z} \times \mathbb{N}$ and $|\theta - \frac{p}{q}| < \frac{1}{2 \cdot q^2}$ then $\frac{p}{q}$ is a convergent to θ .

Proof. 1. From Lemma 1.2.1(d) we have $\theta = \frac{p_n\theta_{n+1} + p_{n-1}}{q_n\theta_{n+1} + q_{n-1}}$. Using Lemma 1.2.1(a) we get

$$|q_n\theta - p_n| = \left| \frac{q_n p_n \theta_{n+1} + q_n p_{n-1} - p_n q_n \theta_{n+1} - p_n q_{n-1}}{q_n \theta_{n+1} + q_{n-1}} \right| \quad (33)$$

$$= \frac{1}{q_n \theta_{n+1} + q_{n-1}} \quad (34)$$

$$< \frac{1}{q_n + q_{n-1}} \quad (35)$$

$$= \frac{1}{(a_n + 1)q_{n-1} + q_{n-2}} \quad (36)$$

$$< \frac{1}{\theta_n q_{n-1} + q_{n-2}} \quad (37)$$

$$= |q_{n-1}\theta - p_{n-1}| \quad (38)$$

This shows (a) and (b) because the q_n are increasing.

- c We use $a_{n+1}q_n^2 < \theta_{n+1}q_n^2 + q_n q_{n-1} < (a_{n+1} + 2)q_n^2$ and combine it with the equation (proof part (a)),

$$|\theta - \frac{p}{q}| = \frac{1}{q_n^2 \theta_{n+1} + q_n q_{n-1}}$$

d) By Lemma 1.2.1(a) we can find $\begin{pmatrix} u \\ v \end{pmatrix} \in n\mathbb{Z}^2$ such that

$$\begin{pmatrix} p_n & p_{n+1} \\ q_n & q_{n+1} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}.$$

As $0 < q < q_{n+1}$ we have $u \neq 0$. If $v = 0$ then $(p, q) = u \cdot (p_n, q_n)$ and the claim is trivial. ($u = 1 \Rightarrow$ equality, $u > 1 \Rightarrow$ strictly \downarrow)

So let's assume $v \neq 0$. Then u and v cannot both be negative (as $q > 0$) nor both be positive (as $q < q_{n+1}$). So they have opposite signs.

By Theorem 1.2.2(a) also $q_n\theta - p_n$ and $q_{n+1}\theta - p_{n+1}$ have opposite signs. Hence, $|q\theta - p| = |u(q_n\theta - p_n) + v(q_{n+1}\theta - p_{n+1})| > |q_n\theta - p_n|$.

e) Take n with $q_n \leq q < q_{n+1}$. Then

$$\begin{aligned} \left| \frac{p}{q} - \frac{p_n}{q_n} \right| &\leq \left| \theta - \frac{p}{q} \right| + \left| \theta - \frac{p_n}{q_n} \right| \\ &= \frac{|q\theta - p|}{q} + \frac{|q_n\theta - p_n|}{q_n} \\ &\stackrel{(d)}{\leq} \left(\frac{1}{q} + \frac{1}{q_n} \right) |q\theta - p| \\ &\leq \frac{2}{q_n} \frac{1}{2q} \\ &= \frac{1}{qq_n} \end{aligned}$$

Hence, $\frac{p}{q} = \frac{p_n}{q_n}$.

□

Remark. • (d) implies that if $p, q \in \mathbb{Z}$, $0 < q \leq p_n$ then

$$\begin{aligned} \left| \theta - \frac{p}{q} \right| &\geq \left| \theta - \frac{p_n}{q_n} \right| \cdot \frac{p_n}{q} \\ &\geq \left| \theta - \frac{p_n}{q_n} \right| \end{aligned}$$

with "=" only if $\frac{p}{q} = \frac{p_n}{q_n}$.

• We say $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is badly approximable if

$$\exists c > 0 \text{ such that } \left| \alpha - \frac{p}{q} \right| > \frac{c}{q^2} \forall (p, q) \in \mathbb{Z} \times \mathbb{N}$$

• By (c) and (d) we see that $\theta = [a_0, a_1, a_2, \dots]$ is badly approximable if and only if the partial quotients a_i are uniformly bounded, i.e., $\exists M > 0$ such that $a_i < M \forall i$.

• (c) suggests that the "worst-approximable" number is $\theta = [1, 1, 1, \dots]$. That's indeed the case c.f Exercise sheet 2 # 5, 6 (using that $\theta = 1 + \frac{1}{1+\dots} = 1 + \frac{1}{\theta}$. So $\theta^2 - \theta - 1 = 0$. So $\theta = \frac{1 \pm \sqrt{5}}{2}$ but $a_0 = 1$ so $\theta = \frac{1 + \sqrt{5}}{2}$).

Counting Diophantine approximations 1:

Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and let $\phi : [1, \infty) \rightarrow (0, \infty)$ be decreasing. Consider the number of " ϕ -good" approximations:

$$N_\alpha(\phi, Q) = \#\left\{\frac{p}{q} \in \mathbb{Q}; \left|\alpha - \frac{p}{q}\right| < \phi(q), 1 \leq q \leq Q\right\}$$

We put $S_\alpha(\phi, Q) = \{(x, y) \in \mathbb{R}^2 : |\alpha - \frac{x}{y}| < \phi(y), 1 \leq y \leq Q\}$. Then

$$N_\alpha(\phi, Q) = \#\{(p, q) \in \mathbb{Z} \times \mathbb{N} : \gcd(p, q) = 1\} \cap S_\alpha(\phi, Q)$$

Note that by Corollary 1.1.2 we have $N_\alpha(\phi, Q) \rightarrow \infty$ as $Q \rightarrow \infty$ provided $\phi(y) \geq \frac{1}{y^2}$, and by Exercise sheet 2, even when $\phi(y) \geq \frac{1}{\sqrt{5}y^2}$. If ϕ decays slowly enough then one can easily show that

$$N_\alpha(\phi, Q) = 2 \cdot \underbrace{\int_1^Q y\phi(y)dy}_{\text{Vol } S_\alpha(\phi, Q)} \underbrace{(\text{It } o(1) \text{ tends to } 0 \text{ as } Q \rightarrow \infty)}_{\text{as } Q \rightarrow \infty}$$

More specifically, using tools we develop in Chapter 3, one can easily show that

$$\#\mathbb{Z}^2 \cap S_\alpha(\phi, Q) = 2 \cdot \int_1^Q y\phi(y)dy + \mathcal{O}(Q),$$

using Möbius-inversion, one can show that

$$N_\alpha(\phi, Q) = \frac{2}{S(2)} \cdot \int_1^Q y\phi(y)dy + \mathcal{O}(Q \log Q).$$

So we get an asymptotic formula

$$N_\alpha(\phi, Q) \sim \frac{2}{S(2)} \text{Vol } S_\alpha(\phi, Q)$$

provided

$$\frac{Q \log Q}{\int_1^Q y\phi(y)dy} \rightarrow 0 \text{ as } Q \rightarrow \infty.$$

So, e.g., if $\phi(y) \geq \frac{(\log y)^2}{y}$.

However, the case when $\phi(y)$ decays much quicker is more interesting. Serge Lang in 1967 proved that if α is reals quadratic then

$$N_\alpha\left(\frac{1}{x^2}, Q\right) = c_\alpha \cdot \log(Q) + \mathcal{O}(1). \quad (c_\alpha > 0).$$

He mentioned that it would seem quite difficult to prove an asymptotic result for algebraic α , let alone transcendental.

Adams showed

$$N_e\left(\frac{1}{x^2}, Q\right) = c_e \cdot \frac{\log Q}{\log \log Q} + \mathcal{O}(1) \quad (c_e > 0)$$

where $e = 2.7122 \dots$

Lang and Adams both used continuous fractions expansion. How can one prove asymptotics for $N_\alpha(\phi, Q)$? Here is an example.

Example. Suppose $\phi(x) = \frac{1}{2x^2}$. Consider the continuous fraction expansion $\alpha = [a_0, a_1, a_2, \dots]$. By Theorem 1.2.3 we know $|\alpha - \frac{p}{q}| < \phi(q) \Rightarrow \frac{p}{q}$ is a convergent. Moreover, if $\frac{p}{q} = \frac{p_n}{q_n}$ is the n -th convergent then $|\alpha - \frac{p}{q}| < \frac{1}{a_{n+1}q^2}$. So if all $a_i > 1$ then $|\alpha - \frac{p}{q}| < \phi(q) \forall$ convergent $\frac{p}{q}$. Hence, $N_\alpha(\phi, Q) = \#\{n : q_n \leq Q\}$. So, we need to compute the number of convergents $\frac{p_n}{q_n}$ with $q_n \leq Q$. We shall soon see that this is rather simple if $\alpha = [b, a, b, a, b, a, \dots]$ with $a \mid b$. We will get back to this after Theorem 1.2.5.

A continued fraction $[a_0, a_1, a_2, \dots]$ is called *periodic* if

$$\exists k \in \mathbb{N} \text{ and } L \in \mathbb{N}_0 \text{ such that } a_{k+l} a_l \forall l \geq L.$$

In this case we write $[a_0, a_1, a_2, \dots] = [a_0, \dots, a_L, a_{L+1}, \bar{}, a_{L+k-1}]$.

Theorem 1.2.4. $\theta = [a_0, a_1, a_2, \dots]$ is periodic $\iff \theta$ is real quadratic (θ is real quadratic means $\exists D \in \mathbb{Z}[x] \setminus 0$ with $D(\theta) = 0$, but $\theta \notin \mathbb{Q}$ and $\theta \in \mathbb{R}$)

See Ex Sheet 2 #3 for a special instance.

A proof can be found, e.g., in Hardy & Wright "The Theory of numbers", Oxford University press

Let's go back to the problem of computing p_n, q_n of the n -th convergent. The general recursion formula is unhandy. But in certain cases there is a simple explicit formula. Consider $\theta = [b, a, b, a, \dots] = [b, \bar{a}]$ and suppose $b = a \cdot c$ for some $c \in \mathbb{N}$. Now $\theta = b + \frac{1}{a + \frac{1}{\bar{\theta}}} = b + \frac{1}{a + \frac{1}{\bar{\theta}}}$. Thus $\underbrace{a\theta^2 - ab\theta - b\theta^2 - b\theta - c}_{=0} = 0$, so $\theta = \frac{b + \sqrt{b^2 + 4c}}{2}$ and we put $\bar{\theta} = \frac{b - \sqrt{b^2 + 4c}}{2}$.

Theorem 1.2.5. The p_n and q_n of the n -th convergent $\frac{p_n}{q_n}$ of $\theta = [b, \bar{a}]$ ($b = ac$) are give by

$$p_n = c^{-\lfloor \frac{n+1}{2} \rfloor} \cdot U_{n+2}, q_n = c^{-\lfloor \frac{n+q}{2} \rfloor} \cdot u_{n+1}$$

where

$$u_n = \frac{\theta^n - \bar{\theta}^n}{\theta - \bar{\theta}}.$$

(Recall: $\theta = \frac{b + \sqrt{b^2 + 4c}}{2}, \bar{\theta} = \frac{b - \sqrt{b^2 + 4c}}{2}$, so $\theta - b\theta - c = 0, \bar{\theta}^2 - b\bar{\theta} - c = 0$)

Proof. For $n = 0, 1$ we note that

$$q_0 = q = u_1 \tag{39}$$

$$q_1 = a = \frac{b}{c} = \frac{u_2}{c} \tag{40}$$

$$p_0 = b = \theta + \bar{\theta} = u_2 \tag{41}$$

$$p_1 = ab + 1 = \frac{b^2 + c}{c} = \frac{(\theta + \bar{\theta})^2 - \theta\bar{\theta}}{c} = \frac{u_3}{c} \tag{42}$$

Put $\omega_{n+2} = c^{-\lfloor \frac{n+1}{2} \rfloor} u_{n+2}$.

So we need to show that $p_n = \omega_{n+2}$.

Using that $\theta^{n+2} = b\theta^{n+1} + c\theta^n$ and $\bar{\theta}^{n+2} = b\bar{\theta}^{n+1} + c\bar{\theta}^n$ and hence $u_{n+2} = \frac{\theta^{n+2} - \bar{\theta}^{n+2}}{\theta - \bar{\theta}} = bu_{n+1} + cu_n$.

Moreover, $u_{2m+2} = c^m \omega_{2m} + 2$, $u_{2m+1} = c^m \omega_{2m+1}$. Inserting this into the above, distinguishing n even or odd yields:

$$\omega_{2m+2} = b\omega_{2m+1} + \omega_{2m} \quad (43)$$

$$\omega_{2m+1} = a\omega_{2m} + \omega_{2m-1} \quad (44)$$

Hence, p_n and ω_{n+2} satisfy the same recurrence relation. and here the same two starting values, so $p_n = \omega_{n+2}$.

Similar for q_n . □

Counting Diophantine Approximation 2:

We can use Theorem 1.2.5 to show that if $\theta = [\bar{b}, a]$ with $b = ac, a > 1$ then

$$N_\theta\left(\frac{1}{2x^2}, Q\right) = \frac{\log Q}{\log\left(\frac{Q}{\sqrt{c}}\right)} + \mathcal{O}(1)$$

Indeed, we have already seen, that

$$N_\theta\left(\frac{1}{2x^2}, Q\right) = \#\{n : q_n \leq Q\}$$

By Theorem 1.2.5 we know

$$q_n \leq Q \iff c^{-\lfloor \frac{n+1}{2} \rfloor} \frac{\theta^n - \bar{\theta}^n}{\theta - \bar{\theta}} = \left(\frac{\theta}{\sqrt{c}}\right)^n \left(1 - \left(\frac{\bar{\theta}}{\theta}\right)^n\right) \epsilon \leq Q$$

$$\text{where } \epsilon = \begin{cases} \frac{1}{\theta - \bar{\theta}} & 2 \mid n \\ \frac{1}{\sqrt{c}(\theta - \bar{\theta})} & 2 \nmid n \end{cases}$$

$$\iff n \log\left(\frac{\theta}{\sqrt{c}}\right) + \log\left(1 - \left(\frac{\bar{\theta}}{\theta}\right)^n\right) + \log \epsilon \leq \log Q$$

Using Taylor series expansion we see that

$$\left|\log\left(1 - \left(\frac{\bar{\theta}}{\theta}\right)^n\right)\right| \leq \left|\frac{\bar{\theta}}{\theta - \bar{\theta}}\right|$$

This proves the claim.

1.3 Liouville's Theorem

Let $\alpha \in \mathbb{C}$. If $\exists D(x) \in \mathbb{Z}[x]$, $D \neq 0$ and $D(\alpha) = 0$ then we say α is *algebraic*. In this case $\exists D(x) = a_0 x^d + \dots + a_d \in \mathbb{Z}[x]$ with

- $D(\alpha) = 0$
- $a_0 > 0$
- $\gcd(a_0, \dots, a_d) = 1$
- $\deg D(x)$ minimal

Imposing all these condition renders D unique; We write $D_\alpha(x)$ and call this the *minimal polynomial* of α . If α is algebraic then we say $\deg D_\alpha$ is the *degree* of α .

Example. • $\alpha = 0, D_\alpha(x) = x$

• $\alpha = \sqrt{2} + 1, D_\alpha(x) = (x - 1)^2 - 2 = x^2 - 2x - 1$

• $\alpha = \frac{1}{\sqrt{2}}, D_\alpha(x) = 2x^2 - 1$

Theorem 1.3.1 (1.3.1 Liouville's Theorem). *Suppose α is a real, algebraic number of degree d . Then $\exists c(\alpha) > 0$ such that*

$$|\alpha - \frac{p}{q}| > \frac{c(\alpha)}{q^d}$$

for every $(p, q) \in \mathbb{Z} \times \mathbb{N}$ with $\alpha \neq \frac{p}{q}$.

Proof. Suppose $|\alpha - \frac{p}{q}| > 1$ then the claim holds for every $c(\alpha) > 1$. Now suppose $|\alpha - \frac{p}{q}| \leq 1$. Taylor series expansion at D_α about α gives:

$$D_\alpha(x) = \sum_{i=1}^d (x - \alpha)^i \frac{1}{i!} D_\alpha^{(i)}(\alpha)$$

Hence,

$$|D_\alpha\left(\frac{p}{q}\right)| = \left| \sum_{i=1}^d \left(\frac{p}{q} - \alpha\right)^i \frac{1}{i!} D_\alpha^{(i)}(\alpha) \right| \leq |D_\alpha^{(d)}(\alpha)| \left| \frac{p}{q} - \alpha \right| \frac{1}{c(\alpha)}$$

where

$$c(\alpha) = \left(1 + \sum_{i=1}^d \frac{1}{i!} |D_\alpha^{(i)}(\alpha)| \right)^{-1}$$

Now if D_α has a rational root then it must have degree one, so have only *one* root. Thus $D_\alpha\left(\frac{p}{q}\right) \neq 0$ unless $\alpha = \frac{p}{q}$. Hence, if $\alpha \neq \frac{p}{q}$ we get

$$|D_\alpha\left(\frac{p}{q}\right)| = \left| \frac{\text{non-zero integer}}{q^d} \right| \geq \frac{1}{q^d}.$$

Combing this with (D)label yields

$$|\alpha - \frac{p}{q}| > \frac{c(\alpha)}{q^d}.$$

□

We say a real number α is a *Liouville number* if for every $n \in \mathbb{N}$

$$0 < |\alpha - \frac{p}{q}| < \frac{1}{q^n}$$

has a solution. $p, q \in \mathbb{Z}$ with $q > 1$.

Example. $\alpha = \sum_{k=1}^{\infty} 10^{-k^k}$ is a *Liouville number*. Let $n \in \mathbb{N}$ and put $p = \sum_{k=1}^n 10^{n-k^k}$ and $q = 10^{n^n}$. Then $0 < |\alpha - \frac{p}{q}| = \sum_{k>n} 10^{-k} \leq 2 \cdot 10^{-(n+1)^{(n+1)}} < 10^{-n^{(n+1)}} = q^{-n}$

Corollary 1.3.2 (1.3.2). *Every Liouville number is transcendental (i.e., not algebraic).*

Proof. Immediate from Theorem 1.3.1 (Liouville's Theorem). \square

Algebraic numbers are enumerable and thus have Lebesgue measure zero. It's not difficult to show that the set of Liouville numbers, while *not* enumerable, also has measure zero. In fact "most" real numbers are "not very far" from badly approximable as the following theorem shows.

Theorem 1.3.3 (Khinchine). *Suppose $\psi : \mathbb{N} \rightarrow (0, \infty)$ is monotone decreasing (not necessarily strictly). The set*

$$A_\psi = \left\{ \alpha \in \mathbb{R} : \left| \alpha - \frac{p}{q} \right| < \frac{\psi(q)}{q} \text{ has } \infty\text{-many solutions } (p, q) \in \mathbb{Z} \times \mathbb{N} \right\}$$

has a Lebesgue measure zero if $\sum_{q=1}^{\infty} \psi(q)$ converges and has full Lebesgue measure (i.e. the complement has measure zero) if $\sum_{q=1}^{\infty} \psi(q)$ diverges.

We will not prove this Theorem. (For a proof see e.g. Glyn Harman "Metric number theory".)

Example. • Take $\psi(q) = \frac{1}{q}$. We already know that $A_\psi = \mathbb{R} \setminus \mathbb{Q}$. And indeed $\sum \psi(q)$ diverges...

- $\psi(q) = \frac{1}{q \log(q-1)}$. Then $\sum \psi(q)$ diverges and thus A_ψ has full measure.
- $\psi(q) = \frac{1}{q(\log(q+1))^{1+\epsilon}}$ ($\epsilon > 0$) then $\sum \psi(q)$ converges, so A_ψ has measure zero.

1.4 4 Theorems of Thue- Siegel and Poth

In Section 1 we have seen that ∞ -many solutions $\frac{p}{q}$ to $\left| \sqrt{2} - \frac{p}{q} \right| < \frac{1}{q^2}$ leads to ∞ -many solutions $(x, y) \in \mathbb{Z}^2$ of $x^2 - 2y^2 = 1$. What about $x^3 - 2y^3 = 1$? Starting as for $x^2 - 2y^2$ we get

$$y^3 \left| \frac{x}{y} - 2^{1/3} \right| \underbrace{\left| \frac{x}{y} - 2^{1/3} \omega \right|}_{\geq \text{Im } \omega} \underbrace{\left| \frac{x}{y} - 2^{1/3} \omega^2 \right|}_{\geq (\text{Im } \omega)^2}$$

where $\omega = e^{\frac{2\pi i}{3}}$.

So to get boundedness of $x^3 - 2y^3$ for ∞ -many (x, y) we need $\exists c > 0$ such that

$$\left| \frac{x}{y} - 2^{1/3} \right| < \frac{c}{y^3}$$

has ∞ -many solutions $(x, y) \in \mathbb{Z} \times \mathbb{N}$.

Theorem 1.3.3 tells us that we would be extremely lucky if that were the case. And even if so, we still would lack the group structure for $\mathbb{Z} + \sqrt{2}\mathbb{Z}$ (closed under multiplication but $\mathbb{Z} + 2^{1/3}\mathbb{Z}$ is not). On the other hand, suppose we could show that

$$\left| \frac{x}{y} - 2^{1/3} \right| < 1/y^\lambda$$

has only finitely many solutions $(x, y) \in \mathbb{Z} \times \mathbb{N}$ for some fixed $\lambda < 3$. As $x^3 - 2y^3 = 1$, and $y \neq 0$ yields:

$$\left| \frac{x}{y} - 2^{1/3} \right| < \frac{1}{2^{1/3}(\text{Im } \omega)^2 y^3}$$

We would conclude that $x^3 - 2y^3 = 1$ has only finitely many solutions $(x, y) \in \mathbb{Z}^2$. Note that "deg" $2^{1/3} = 3(D(x) = x^3 - 2)$ and so Liouville's Theorem yields only $\lambda = 3$ not $\lambda < 3$. So the big challenge is to improve Liouville's Theorem. After Liouville it has taken 65 years until the first breakthrough was obtained by Axel Thue in 1909.

Theorem 1.4.1 (1.4.1 Thue). *Let α be a real algebraic number of degree $d \geq 2$, and let $\lambda > \frac{d}{2} + 1$. Then $\exists c = c(\alpha, \lambda) > 0$ such that*

$$\left| \alpha - \frac{p}{q} \right| > \frac{c}{q^\lambda}, \quad \forall (p, q) \in \mathbb{Z} \times \mathbb{N}.$$

- Note that for $d = 2$ Liouville is stronger
- Given α and λ there is no method to determine a feasible value for c . This is in stark contrast to Liouville's Theorem.

Just as for $x^3 - 2y^2 = 1$ one can now very easily show that if $f(X, Y) = a_0(X - \alpha_1 Y) \cdots (X - \alpha_d Y) \in \mathbb{Q}[X, Y]$ with $a_0 \neq 0, d \geq 3$, and $\alpha_1, \dots, \alpha_d$ pairwise distinct, and $b \in \mathbb{Q} \setminus \{0\}$, then

$$f(x, y) = b$$

has only finitely many solutions $(x, y) \in \mathbb{Z}^2$.

Wrong if $d = 2$:

$$X^2 - 2Y^2 = 1$$

or $b = 0$:

$$X^3 - Y^3 = 0$$

or $\alpha_1, \dots, \alpha_d$ not pairwise distinct:

$$(X - Y)^5 = 1$$

We will show that Theorem 1.4.1 implies even the following stronger result.

Theorem 1.4.2 (1.4.2 Generalized Thue equations). *Let $f(X, Y) = a_0(X - \alpha_1 Y) \cdots (X - \alpha_d Y) \in \mathbb{Q}[X, Y]$ with $a_0 \neq 0, d \geq 3$ and $\alpha_1, \dots, \alpha_d$ pairwise distinct. Let $g(X, Y) \in \mathbb{Q}[X, Y]$ of total degree $< \frac{d}{2} - 1$. Then there are only finitely many $(X, Y) \in \mathbb{Z}^2$ with*

$$f(x, y) = g(x, y)$$

and $g(x, y) \neq 0$.

Example.

$$x^5 - 2y^5 = x - y$$

has only finitely many solutions $(x, y) \in \mathbb{Z}^2$. Indeed if $x - y = 0$ then $x^5 - 2y^5 = 0$ thus $x = y = 0$. Note Theorem can go wrong if $\alpha_1 = \alpha_2$:

$$(X^2 - 2Y^2)^2 = 1.$$

(assuming Theorem 1.4.1). If $y = 0$ then we have at most d possibilities for x . So we can assume $y \neq 0$. We claim that

$$|x| \leq c_1 |y|$$

for some $c_1 = c_1(f, g)$. Clearly true when $|x| \leq |y|$, so let's assume $|x| > |y|$. Then we write

$$f(x, y) = \sum_{i=0}^d a_i x^{d-i} y^i = \sum_{j+k \leq d-1} b_{jk} x^j y^k = g(x, y)$$

Dividing by x^{d-i} yields

$$a_0 x = - \sum_{i=0}^d a_i \frac{y^i}{x^{i-1}} + \sum_{j+k \leq d-1} b_{jk} x^{j-d+1} y^k$$

We have

$$\left| \frac{y^i}{x^{i-1}} \right| \leq |y|$$

and

$$\left| \frac{y^k}{x^{d-1-j}} \right| \leq |y|^{j+k-(d-1)} \leq 1$$

Therefore $|x| \leq c_1 |y|$, e.g. with $c_1 = \frac{1}{|a_0|} (\sum |a_i| + \sum |b_{jk}|) + 1$. From

$$f(x, y) = g(x, y), (*)$$

we get

$$|\alpha_0| \prod_{j=1}^d \left| \frac{x}{y} - \alpha_j \right| \leq c_2 |y|^{e-d}$$

where $c_2 = c_2(c_1, g)$ and $e < \frac{d}{2} - 1$. So assume $(*)$ has ∞ -many solutions $(x, y) \in \mathbb{Z}^2$. Then $\exists i$, say $i = 1$, such that $\left| \frac{x}{y} - \alpha_1 \right| \leq \mu := \frac{1}{2} \min_{j \neq i} \{|\alpha_j - \alpha_1|\} > 0$ for ∞ -many (x, y) of these solutions of $(*)$. Now

$$\left| \frac{x}{y} - \alpha_j \right| \geq \left| |\alpha_j - \alpha_i| - \left| \frac{x}{y} - \alpha_1 \right| \right| \geq 2\mu - \mu = \mu > 0$$

Hence, we conclude

$$\left| \frac{x}{y} - \alpha_1 \right| \leq \frac{c_2}{|a_0|} \mu^{1-d} |y|^{e-d}, (**)$$

for these solutions (x, y) . Here we can assume $y > 0$ (just replace x by $-x$). Now let d_1 be the degree of α_1 . As $f(x, 1) \in \mathbb{Q}[x]$, $f(x, 1) \neq 0$ and $f(\alpha_1, 1) = 0$. Thus $d_1 \leq d$. Moreover, $d - e > \frac{d}{2} + 1$ and this $\exists \lambda$ such that

$$d - e > \lambda > \frac{d_1}{2} + 1.$$

If $d_1 \geq 2$ then Theorem 1.4.1 implies that $(*)$ has only finitely many solutions $(x, y) \in \mathbb{Z}^2$. Finally suppose $d_1 = 1$. Then $\alpha_1 = \frac{p}{q}$, and $(**)$ yields:

$$\left| x - \frac{p}{q} y \right| \leq c_3 y^{e-d+1} \leq c_3 y^{-\frac{d}{2}}.$$

Thus $x = \frac{p}{q} y = \alpha_1 y$ for y large enough. But then $0 = f(x, y) = g(x, y)$ a contradiction. \square

After Thue came Siegel (1921) who improved the exponent $\frac{d}{2} + 1$ to $2\sqrt{d}$. This was slightly improved by Dyson and Gelfand (1947) to $\sqrt{2d}$. Finally in 1955 came Roth:

Theorem 1.4.3 (1.4.3 (Roth)). *Let α be a real, algebraic irrational number, and $\lambda > 2$. Then $\exists c = c(\alpha, \lambda) > 0$ such that*

$$\left| \alpha - \frac{p}{q} \right| \geq \frac{c}{q^\lambda}, \quad \forall (p, q) \in \mathbb{Z} \times \mathbb{N}.$$

By Corollary 1.1.2 $\lambda > 2$ is best-possible. But if we allow more general functions $\phi(q)$, not only powers of q , then an improvement might be possible. However, since 1955 nobody was able to replace $q^{-\lambda}$ by a function $\phi(q)$ that decays more slowly, e.g. $\phi(q) = q^{-2}(\log q)^{-1}$. However, back to the case where $\phi(q)$ is a power of q . From Theorem 1.3.3. we know that for a generic real α

$$\left| \frac{p}{q} - \alpha \right| < q^{-\lambda}$$

has only finitely many solutions $p, q \in \mathbb{Z} \times \mathbb{N}$ provided $\lambda > 2$. Any by Corollary 1.1.2 every irrational real number has ∞ -many solutions when $\lambda = 2$. And so from Roth's Theorem we see an algebraic irrational behaves "essentially" like a generic number.

Roth's Theorem has various new applications to, e.g., Diophantine equations and transcendence. Let's consider just one now transcendence result: Take $\alpha = \sum_{k=1}^{\infty} 2^{-3^k}$; put $q_n = 2^{3^n}$ and $p_n = q_n \sum_{k=1}^n 2^{-3^k}$. Then $0 < \left| \alpha - \frac{p_n}{q_n} \right| = \sum_{k=n+1}^{\infty} 2^{-3^k} < 2 \cdot 2^{-3^{n+1}} = 2 \cdot 2 \cdot q_n^{-1}$ so by Roth's Theorem α is transcendental.

How does one prove results like Roth's Theorem of the kind

$$\left| \alpha - \frac{p}{q} \right| \geq \phi(q)?$$

The idea is to find good rational approximations.

$$\left| \alpha - \frac{p_n}{q_n} \right| \leq \delta_n$$

with δ_n "pretty small". Then

$$\left| \alpha - \frac{p}{q} \right| \geq \left| \frac{p_n}{q_n} - \frac{p}{q} \right| - \left| \alpha - \frac{p_n}{q_n} \right|$$

If

$$\frac{p_n}{q_n} \neq \frac{p}{q} \tag{45}$$

then

$$\left| \alpha - \frac{p}{q} \right| \geq \frac{1}{qq_n} - \delta_n.$$

If we are lucky then $\delta_n < \frac{1}{qq_n}$ and we get a positive lower bound. How do we find these $\frac{p_n}{q_n}$?

Usually this is a difficult task, but sometimes one can easily see these approximations $\frac{p_n}{q_n}$. Here is an example.

Take again $\alpha = \sum_{k=1}^{\infty} 2^{-3^k}$. Then we can take again $q_n = 2^{3^n}$, $p_n = q_n \sum_{k=1}^n 2^{-3^k}$; so $\left| \alpha - \frac{p_n}{q_n} \right| < 2 \cdot q_n^{-3}$. Hence, if

$$\frac{p_n}{q_n} \neq \frac{p}{q}$$

then

$$\left| \alpha - \frac{p}{q} \right| \geq \frac{1}{qq_n} - \frac{2}{q_n^3}$$

If $q_n^2 > 4 \cdot q$ then

$$\frac{1}{qq_n} - \frac{2}{q_n^3} \geq \frac{q}{2 \cdot qq_n}$$

As $\frac{p_n}{q_n}$ tends strictly monotonously to α , we have $\frac{p_n}{q_n} \neq \frac{p}{q}$ or $\frac{p_{n+1}}{q_{n+1}} \neq \frac{p}{q}$. Let m be minimal with $q_m > 4 \cdot q$. Hence

$$q_m^{\frac{2}{3}} = q_{m-1}^2 \leq 4 \cdot q < q_m^2$$

If $\frac{p_m}{q_m} \neq \frac{p}{q}$ we take $n = m$ and $n = m + 1$ else. We conclude

$$\left| \alpha - \frac{p}{q} \right| \geq \frac{1}{2qq_n} \geq \frac{1}{2qq_{m+1}} \geq \frac{1}{2q} \frac{1}{q_m^3} \geq \frac{1}{2q} \frac{1}{(4q)^{\frac{9}{2}}} = 2^{-10} q^{-\frac{11}{2}}$$

In this example everything works out nicely, e.g., (ref*) could easily be guaranteed by using $\frac{p_n}{q_n}$ tending strictly monotonously to α . However, in Roth's Theorem (ref*) becomes the major-problem.

1.5 5 Simultaneous Diophantine approximation and the Subset Theorem

Suppose $\alpha_1, \dots, \alpha_n$ are real numbers. Theorem 1.1.1 can be generated to yield a solution $(x_1, \dots, x_n, y) \in \mathbb{Z}^n \times \mathbb{N}$ at the system

$$\left| \frac{x_i}{y} - \alpha_i \right| \leq \frac{1}{y \cdot Q} (1 \leq i \leq n), 0 < y < Q.$$

(c.f. Exercise sheet 4). This in turn yields ∞ -many solutions $(x_1, \dots, x_n, y) \in \mathbb{Z}^n \times \mathbb{N}$ of the system

$$\left| \frac{x_i}{y} - \alpha_i \right| < \frac{1}{y^{1+\frac{1}{n}}} (1 \leq i \leq n).$$

provided at least one of the α_i 's is irrational. So Corollary 1.1.2 extends to simultaneous approximation. A much deeper fact is that Roth's Theorem also extends to simultaneous approximation.

For $\underline{x} \in \mathbb{R}^n$ we write $\|\underline{x}\| = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$ for the Euclidean length.

Theorem 1.5.1 (Subspace Theorem, Schmidt). *Suppose $L_i(\underline{x}) = \sum_{j=1}^n a_{ij}x_j$ ($1 \leq i \leq n$) are linearly independent linear forms with algebraic coefficients a_{ij} . Let $\delta > 0$. Then the solutions $\underline{x} \in \mathbb{Z}^n \setminus \underline{0}$ of*

$$|L_1(\underline{x}) \dots L_n(\underline{x})| < \|\underline{x}\|^{-\delta}$$

lie in finitely many proper subspaces of \mathbb{Q}^n .

Remark. *linearly independent linear forms means the coefficient vectors (a_{i1}, \dots, a_{in}) are linearly independent over \mathbb{C} .*

Corollary 1.5.2 (1.5.2). *Let $\delta > 0$, suppose $\alpha_1, \dots, \alpha_n$ are algebraic and $1, \alpha_1, \dots, \alpha_n$ are linearly independent over \mathbb{Q} . Then there are only finitely many $(x_1, \dots, x_n, y) \in \mathbb{Z}^n \times \mathbb{N}$ with*

$$(5.1) \left| \frac{x_i}{y} - \alpha_i \right| < \frac{1}{y^{1+\frac{1}{n}+\delta}} \quad (1 \leq i \leq n) \quad (46)$$

Proof. (assuming Theorem 1.5.1) Put $\underline{X} = (X_1, \dots, X_n, Y)$, $L_i(\underline{X}) = \alpha_i Y - X_i$ ($1 \leq i \leq n$), $L_n(\underline{X}) = Y$. These $n+1$ linear forms in $n+1$ unknowns are linearly independent. With $\underline{x} = (x_1, \dots, x_n, y)$ the solutions of (5.1) yield

$$|L_1(\underline{x}) \dots L_{n+1}(\underline{x})| < \frac{1}{y^\delta} < \frac{1}{\|\underline{x}\|^{\frac{\delta}{2}}}$$

if y is large enough. so by Theorem 1.5.1 (in $n+1$ dimensions), we set that the solutions lie in finitely many proper subspaces of \mathbb{Q}^{n+1} . Pick one of these (of codimension I say). It is given by an equation $c_1 x_1 + \dots + c_n x_n + c_{n+1} y = 0$ where $c_i \in \mathbb{Q}$ not all zero. On this subspace we have

$$(c_1 \alpha_1 + \dots + c_n \alpha_n + c_{n+1})y = c_1(\alpha_1 y - x_1) + \dots + c_n(\alpha_n y - x_n).$$

Put $\gamma = c_1 \alpha_1 + \dots + c_n \alpha_n + c_{n+1}$. By \mathbb{Q} -linearly independence of $1, \alpha_1, \dots, \alpha_n$ we have $\gamma \neq 0$. Hence,

$$|\gamma||y| \leq |c_1||\alpha_1 y - x_1| + \dots + |c_n||\alpha_n y - x_n| \leq (|c_1| + \dots + |c_n|) \frac{1}{y^{1+\frac{1}{n}+\delta}} \leq |c_1| + \dots + |c_n|$$

So $|y|$ is bounded and we are done. \square

In applications one sometimes needs a "p-adic" version of the subspace Theorem in which one approximates with respect to also the so called p-adic absolute values.

Definition (Absolute values). *An absolute value on a field K is a map $|\bullet| : K \rightarrow [0, \infty)$ such that*

- $|x| = 0 \iff x = 0$
- $|x \cdot y| = |x| \cdot |y|$
- $|x + y| \leq |x| + |y|$

Example. • K arbitrary. $|x| = \begin{cases} 0 & x = 0 \\ 1 & x \neq 0 \end{cases}$ the trivial absolute value.

- $K = \mathbb{Q}$, $|\bullet|$ = standard absolute value on \mathbb{Q} . To distinguish it from other absolute values let's write it as $|\bullet| = |\bullet|_\infty$.
- $K = \mathbb{Q}$ and let $p \in \mathbb{N}$ be a prime number. If $x \in \mathbb{Q}, x \neq 0, \pm 1$, then \exists a unique prime factorisation $x = \pm p_1^{a_1} \dots p_s^{a_s}$ where p_1, \dots, p_s primes and $a_i \in \mathbb{Z} \setminus 0$. For any prime $p \in \mathbb{N}$ write $\text{ord}_p(x)$ for the exponent of p in the

primfractorisation of x (e.g. $\text{ord}_{p_i} x = a_i$). For $x = \pm 1$ we put $\text{ord}_p x = 0 \forall p_i$.
The p -adic absolute vlaue $1 \cdot 1_p$ on \mathbb{Q} is defined by

$$|x|_p = \begin{cases} 0 & : x = 0 \\ p^{-\text{ord}_p(x)} & : x \neq 0 \end{cases}$$

The multiplicativity is clear. Note that $\text{ord}_p(x_1 + x_2) \geq \min\{\text{ord}_p(x_1), \text{ord}_p(x_2)\}$.

Hence, $|x_1 + x_2|_p = p^{-\text{ord}_p(x_1 + x_2)} \leq p^{-\min\{\text{ord}_p(x_1), \text{ord}_p(x_2)\}} = \underbrace{\max\{|x_1|_p, |x_2|_p\}}_{\text{strongtriangleinequality}}$

$|x_1|_p + |x_2|_p$ An absolute value that satisfies the strtong triange inequality is called non-Archimedean.

Definition 1.5.3. We set $M_{\mathbb{Q}} = \{\text{primes in } \mathbb{N}\} \cup \{\infty\}$. Then for each $v \in M_{\mathbb{Q}}$ we get an absolute value $|\cdot|_v$. Note that if $v \in M_{\mathbb{Q}}$ and p a prime, $a \in \mathbb{Z}$, then

$$|\pm p^a|_v = \begin{cases} p & : v = p \\ p^a & : v = \infty \\ 1 & : v \neq p, v \neq \infty \end{cases}$$

Hence

$$\prod_{v \in M_{\mathbb{Q}}} |1 \pm p^a|_v = 1$$

and so vy multiplicativity we conclude

$$\prod_{v \in M_{\mathbb{Q}}} |x|_v = 1$$

for all $x \in \mathbb{Q}$, $x \neq 0$. (PF) Thsi is the so-called producct formula (PF) on \mathbb{Q} .

Next, we want to introduce a notion of "arithmetic complexity" on elements in \mathbb{Q}^{n+1} , the so-called projective height:

$$H_{\mathbb{P}^n} : \mathbb{Q}^{n+1} \rightarrow [1, \infty)$$

defined by

$$H_{\mathbb{P}^n}(\underline{x}) = \prod_{v \in M_{\mathbb{Q}}} |\underline{x}|_v$$

where $|\underline{x}|_v = \max\{|x_0|_v, \dots, |x_n|_v\}$.

Example. If $\underline{x} = (x_0, \dots, x_n)$ where $x_0, \dots, x_n \in \mathbb{Z}$ and $\gcd(x_0, \dots, x_n) = 1$. Then $|x|_p = 1$ for all primes p . Hence,

$$H_{\mathbb{P}^n}(\underline{x}) = \max\{|x_0|_{\infty}, \dots, |x_n|_{\infty}\}.$$

Note that $H_{\mathbb{P}^n}(\lambda \cdot \underline{x}) = H_{\mathbb{P}^n}(\underline{x}) \forall \lambda \in \mathbb{Q} \setminus 0$.

Theorem 1.5.4 (1.5.3 p-adic Subspace Theorem, Schlickewei and Schnidt).
Let $\delta > 0$ and let $S \subset M_{\mathbb{Q}}$ be finite and with $\infty \in S$. For $v \in S$ let L_{v_1}, \dots, L_{v_n} be n linearly independent linear forms in n variables with coefficients in \mathbb{Q} . Then the set of solutions $\underline{x} \in \mathbb{Q}^{n+1} \setminus 0$ of

$$\prod_{v \in S} \prod_{i=1}^n \frac{|L_{v_i}(\underline{x})_v|}{|\underline{x}|_v} < H_{\mathbb{P}^{n-1}}(\underline{x})^{-n-\delta}$$

lie in finitely many proper subsaced of \mathbb{Q}^n .

An interesting consequence is a finiteness result for S -unit equations.
 S -integers and S -units:
Let v be a non-Archimedean absolute value on a field K . Then

$$O_v = \{x \in K : |x|_v \leq 1\}$$

is called the valuation ring of v . It is indeed a ring, e.g., $|x|_v, |y|_v \leq 1$ then

$$|x + y|_v \leq \max\{|x|_v, |y|_v\} \leq 1.$$

In particular, if $K = \mathbb{Q}$ and $v = p$ then O_v is a subring of \mathbb{Q} .

Now let $S \subset M_{\mathbb{Q}}$ be finite and $\infty \in S$. We define the set of S -integers O_S to be

$$O_S = \cap_{v \notin S} O_v.$$

As $\infty \in S$, this is an intersection of rings, hence a ring.

Example. If $S = \{\infty\}$, then $O_S = \mathbb{Z}$. If $S = \{\infty, p_1, \dots, p_s\}$ then

$$O_S = \left\{ \frac{m}{p_1^{a_1} \dots p_s^{a_s}} : m \in \mathbb{Z}, a_1, \dots, a_s \in \mathbb{N}_0 \right\}$$

We say $x \in \mathbb{Q}$ is an S -unit if $x \neq 0$ and x, x^{-1} are both in O_S . So if $S = \{\infty\}$, then ± 1 are the only S -units. If $S = \{\infty, p_1, \dots, p_s\}$ then x is an S -unit $\iff x = \pm \prod_{p \in S \setminus \infty} p_p^a$ (and $ap \in \mathbb{Z}$).

Theorem 1.5.5 (1.5.4 S -unit equation). *Let $S \subset M_{\mathbb{Q}}$ be finite, and $\infty \in S$. Let $\alpha_0, \dots, \alpha_n$ be non-zero and in \mathbb{Q} . Then*

$$\alpha_0 x_0 + \dots + \alpha_n x_n = 0$$

has only finitely many solutions $\underline{x} = (x_0, \dots, x_n)$ if:

- x_0, \dots, x_n are S -units
- we identify proportional solutions (i.e., $\underline{x} = \lambda \underline{x}$ for $\lambda \in \mathbb{Q} \setminus 0$).
- no proper subsum vanishes, i.e., $\sum_I \alpha_i x_i \neq 0$ for all $\emptyset \subsetneq I \subsetneq \{0, 1, \dots, n\}$.

Remark. $S = \{\infty, p\}$ $x_0 + x_1 + x_2 + x_3 = 0$ then $x_0 = -x_1 = 1$, $x_2 = -x_3 = p^a$ ($a \in \mathbb{Z}$) are solutions in S -units. So non-vanishing condition is needed!

Example. The exponential Diophantine equation

$$3^x + 5^y - 7^z = 1$$

has solutions, e.g., $(x, y, z) = (0, 0, 0)$ or $(x, y, z) = (1, 1, 1)$. However, with $S = \{\infty, 3, 5, 7\}$ each solution (x, y, z) yields a solution $u_0 = 3x$, $u_1 = 5y$, $u_2 = -7z$, $u_3 = -1$ of the S -unit equation $u_0 + u_1 + u_2 + u_3 = 0$. These solutions are all non-proportional. Moreover, no subsum vanishes unless $xyz = 0$ but then we easily see that $x = y = z = 0$. Hence, Theorem 1.5.4 yields finiteness.

assuming Theorem 1.5.3. Induction on n . If $n = 1$ then $\alpha_0 x_0 + \alpha_1 x_1 = 0$, so all solutions are proportional to $(1, -\frac{\alpha_0}{\alpha_1})$. Now suppose the claim holds for all S -unit equations in $\leq n$ variables. As x_i are S -units we have

$$|x_i|_v = 1 \forall v \notin S.$$

By the product formula (PF)

$$1 = \prod_{v \in M_{\mathbb{Q}}} |x_i|_v = \prod_{v \in S} |x_i|_v,$$

and thus

$$\prod_{v \in S} \prod_{i=0}^n |x_i|_v = 1.$$

Let $\tilde{x} = (x_0, \dots, x_{n-1})$. For each $v \in S$ pick $i(v)$ with $0 \leq i(v) \leq n-1$. So we get $n^{\#S}$ such tuples $(i(v))_{v \in S}$. Choose one of those tuples and consider all solutions of $\alpha_0 x_0 + \dots + \alpha_n x_n = 0$ with

$$|\tilde{x}|_v = |x_{i(v)}|_v$$

Choose the set of linear forms

$$\{L_{v_j} : 1 \leq j \leq n\} = \{X_0, X_1, \dots, X_{n-1}, \frac{\alpha_0}{\alpha_n} X_0 + \dots + \frac{\alpha_{n-1}}{\alpha_n} X_{n-1}\} \setminus \{X_{i(v)}\}$$

Then

$$\prod_{v \in S} \prod_{j=1}^n \frac{|L_{v_j}(\tilde{x})|_v}{|\tilde{x}|_v} = \frac{1}{H_{\mathbb{P}^{n+1}}(\tilde{x})n+1}$$

By Theorem 1.5.3 the solutions \tilde{x} lie in finitely many proper subspaces. Take one of these then all elements in this subspace satisfy an equation

$$c_0 x_0 + \dots + c_{n-1} x_{n-1} = 0 (c_i \in \mathbb{Q}, \text{ not all } = 0!)$$

Let J_0 be the set of i with $c_i \neq 0$. Then

$$\sum_{i \in J_0} c_i x_i = 0 \text{ marker}(S) \tag{47}$$

is an S -unit equation in $\leq n$ unknowns. For every solution of "marker(S)" there is a set $J \subset J_0$, $J \neq \emptyset$, such that

$$\sum_{i \in J} c_i x_i = 0$$

and no subsum vanishes. By the induction hypotheses, up to proportionality, we get only finitely many solutions. Moreover, the number of possible choices J is finite. Therefore it suffices to consider solutions $\{x_i\}_{i \in J}$ that are proportional to a fixed $\{u_i\}_{i \in J}$, i.e., $x_i = \xi u_i (i \in J)$. Returning to our initial equation $\sum_{i=0}^n \alpha_i x_i = 0$ we get

$$\xi \left(\sum_{i \in J} \alpha_i u_i \right) + \sum_{i \notin J} \alpha_i x_i = 0$$

If $\sum_{i \in J} \alpha_i x_i \neq 0$ then the above is an S -unit equation in $1 + (n+1) - \#J \leq n$ unknowns, namely $\xi, x_i (i \notin J)$. By the induction hypothesis we get only finitely many non-proportional solutions $\{x_i\}_{i=0}^n$ for which no subsum vanishes. Finally, if $\sum_{i \in J} \alpha_i x_i = 0$ then $\sum_{i \notin J} \alpha_i x_i = 0$ and we ignore these solutions by assumption of the Theorem. \square

1.6 6. Further generalizations and open problems

Let $\alpha, \beta \in \mathbb{R} \setminus \mathbb{Q}$ and consider the linearly independent linear forms $L_1 \underline{x} = x_0 \alpha - x_1$, $L_2(\underline{x}) = x_0 \beta - x_2$, $L_3(\underline{x}) = x_0$. If α, β are algebraic then Theorem 1.5.1 implies that the solutions $\underline{x} \in \mathbb{Z}^3 \setminus \underline{0}$ of

$$|L_1(\underline{x}) \cdot L_2(\underline{x}) \cdot L_3(\underline{x})| < \|\underline{x}\|^{-\delta} \quad (\delta > 0)$$

lie in finitely many proper subspaces of \mathbb{Q}^3 .

However, in the following is a long-standing conjecture.

Conjecture 1.6.1 (Littlewood-conjecture, around 1920). *Let $\alpha, \beta \in \mathbb{R} \setminus \mathbb{Q}$ and $\varepsilon > 0$. Then $\exists \underline{x} \in \mathbb{Z}^3$ such that*

$$0 < |L_1(\underline{x})L_2(\underline{x})L_3(\underline{x})| < \varepsilon$$

Remark. *Note that the conjecture is obviously true if $\alpha = \beta$ (by Corollary 1.1.2) or if α or β are not badly approximable.*

Let's consider again approximations to one real α . So far our approximations were $\frac{p}{q} \in \mathbb{Q}$. If we replace \mathbb{Q} by a smaller or larger set then we get now interesting problems.

Open Problem 1.6.2 (1.6.2). *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $\lambda < 2$. Does $\left| \alpha - \frac{p}{q} \right| < q^{-\lambda}$ have ∞ -many solutions $(p, q) \in \mathbb{Z} \times \mathbb{N}$ with:*

- *p and q are both square-free.*
Best-result (Hoath-Brown 1984): Yes, if $\lambda < \frac{5}{3}$.
- *q is prime?*
Best-result (Matomaki, 2009): Yes, if $\lambda < \frac{4}{3}$.

Let's now consider problems in which \mathbb{Q} is replaced by a certain subset A of

$$\overline{\mathbb{Q}} = \{\alpha \in \mathbb{C} : \alpha \text{ algebraic}\}.$$

If we assume that $A \subset \mathbb{R}$ then we still can use the (usual) absolute value on \mathbb{R} to measure

$$|\alpha - x| \quad (x \in A).$$

But usually we have no "natural denominators" for $x \in A$. But there is a natural way to interpret the original setting that easily generalizes from $A = \mathbb{Q}$ to $A = \overline{\mathbb{Q}} \cap \mathbb{R}$. To this end we introduce the so-called multiplicative absolute Weil height:

$$H : \mathbb{Q} \rightarrow [1, \infty)$$

defined by

$$H(\alpha) = M(D_\alpha(x))$$

where $D_\alpha(x) = a_0(x - \alpha_1) \dots (x - \alpha_d) \in \mathbb{Z}[x]$ is the minimal polynomial of α and

$$M(D_\alpha(x)) = |a_0| \cdots \prod_{i=1}^n \max\{1, |\alpha_i|\}$$

M is called the Mahler-measure.

Example. • $\alpha = \frac{p}{q} \in \mathbb{Q}$ ($q > 0$, $\gcd(p, q) = 1$).

$\deg(\alpha) = 1$, $D_\alpha = qx - p$. So $H(\alpha) = M(D_\alpha) = q \max\{1, \left|\frac{p}{q}\right|\} = \max\{q, |p|\} = H_{\mathbb{P}^1}((1, \alpha))$.

• $\alpha = 2^{\frac{1}{d}}$, $D_\alpha = x^d - 2$ (2-Eisenstein), $\deg \alpha = d$, $H(\alpha) = M(D_\alpha)^{\frac{1}{d}} = \prod_{i=1}^d \max\{1, \left|\xi_d^{i-1} 2^{\frac{1}{d}}\right|\} = 2^{\frac{1}{d}}$

One can easily show (cf. sheet 4) that

$$\#\{\alpha \in \overline{\mathbb{Q}} : \deg d \leq d, H(\alpha) \leq X\} < \infty \quad \forall d \in \mathbb{N} \quad X \geq 1. \quad (48)$$

Back to Diophantine approximation with $A = \mathbb{Q}$. As

$$a + m - \frac{p}{q} = a - \left(\frac{p - mq}{q}\right)$$

we can assume $\alpha \in (0, 1)$ So all good enough approximations $\frac{p}{q}$ lie also in $(0, 1)$. Now if $\frac{p}{q} \in (0, 1)$ then

$$H\left(\frac{p}{q}\right) = q.$$

So

$$\left|\alpha - \frac{p}{q}\right| < \phi(q) \iff \left|\alpha - \frac{p}{q}\right| < \phi\left(H\left(\frac{p}{q}\right)\right).$$

So now the denominator plays no role anymore and we can write more easily:

$$|\alpha - x| < \phi(H(x)) \quad (49)$$

(49) makes sense as long as $x - \alpha \in \mathbb{R}$, so $x \in \mathbb{R}$, and $x \in \overline{\mathbb{Q}}$. So let's assume $A \subset \overline{\mathbb{Q}} \cap \mathbb{R}$. However, if x_1, x_2, x_3, \dots is a sequence of pairwise distinct solutions of (49) then we want to conclude that $x_i \rightarrow \alpha$ (with respect to $|\bullet|$). Now as $\phi(t) \rightarrow 0$ as $t \rightarrow \infty$ but we don't know a priori that $H(x_i) \rightarrow \infty$. So cannot conclude from (49) that $x_i \rightarrow \alpha$. But if $A \subset \mathbb{Q}_{(d)} = \{\alpha \in \overline{\mathbb{Q}} : \deg \alpha \leq d\}$ then (48) tells us that $H(x_i) \rightarrow \infty$ and so $x_i \rightarrow \alpha$.

More generally this is true if

$$\#\{\alpha \in A : H(\alpha) \leq X\} < \infty \quad \forall X \geq 1.$$

In this case we say A has Property N.

Theorem 1.6.3 (1.6.3 Wirsing 1961). Let $d \in \mathbb{N}$, $d > 1$ and $\alpha \in \mathbb{R} \setminus \mathbb{Q}_{(d)}$. Then $\exists \infty$ -many $x \in \mathbb{Q}_{(d)}$ with

$$|\alpha - x| < H(x)^{-\left(\frac{d+3}{2}\right)}$$

Conjecture 1.6.4 (1.6.4 Wirsing's Conjecture, around 1961). Suppose $\alpha \in \mathbb{R} \setminus \mathbb{Q}_{(d)}$, ($d \in \mathbb{N}$) and $\lambda < d + 1$ then $\exists \infty$ -many $x \in \mathbb{Q}_{(d)}$ with

$$|\alpha - x| < H(x)^{-\lambda}.$$

Theorem 1.6.5 (1.6.5 Davenport and Schmidt). Wirsing's conjecture (2.2.4) holds for $d \leq 2$.

Instead of taking $A = \mathbb{Q}_{(d)} \cap \mathbb{R}$ let's replace $\mathbb{Q}_{(d)}$ with the smallest field that contains $\mathbb{Q}_{(d)}$; let's call this field $\mathbb{Q}^{(d)}$. Unfortunately, nobody knows whether $\mathbb{Q}^{(d)}$ has Property (N), except when $d \leq 2$.

Theorem 1.6.6 (1.6.6 Bombien-Zannier, 2001). $\mathbb{Q}^{(2)}$ has Property (N).

Open Problem 1.6.7 (1.6.7). Find an analog of Corollary 1.1.2 for $A = \mathbb{Q}(2) \cap \mathbb{R}$. How quickly can $\phi : [1, \infty) \rightarrow (0, \infty)$ decay if for every $\alpha \in \mathbb{R} \setminus A$

$$|\alpha - x| < \phi(H(x))$$

has ∞ -many solutions $x \in A$? It is not difficult to show an inequality in the other direction provided α is algebraic, e.g., if $\alpha \in \overline{\mathbb{Q}} \setminus \mathbb{Q}^{(2)}$ then

$$|\alpha - x| > (2 \cdot H(\alpha)H(x))^{-\deg \alpha \cdot 2^{(2H(x))}}$$

How much can this be improved?