

Number Theory

Martina Tscheckl

November 30, 2015

Please send feedback to martina@tscheckl.eu.

Contents

0	Basics	2
0.1	Divisibility	2
0.2	Primes	4
0.3	Congruences	5
0.4	Arithmetic functions	6
0.5	Structure of \mathbb{Z}_n^\times	8
0.5.1	Case 1: $\alpha = 1$	8
0.5.2	Case 2: $\alpha \geq 2; p \geq 3$	8
1	Diophantine Approximation	10
1.1	Dirichlet's Theorem	10
1.2	Continued fractions	11
1.3	Liouville's Theorem	19
1.4	4 Theorems of Thue- Siegel and Poth	21
1.5	5 Simultaneous Diophantine approximation and the Subset Theorem	25
1.6	6. Further generalizations and open problems	30
2	Geometry of Numbers	32
2.1	Basic notions	32
2.2	The Theorems of Blichfeldt and Minkowski	35
2.3	3 Basis reduction	37
2.4	4. Minkowski's Second Theorem	41

Organizatorial stuff

Dates (in TUGrazOnline):

Mon	14:15–15:45	C208	Exercises (starting 19.10. first exercise class)
Tue	14:15–15:45	C307	Lecture (starting 20.10. first (real) lecture)
Wed	08:15–09:45	C208	Lecture

From now until 15.12. lectures by Martin Widmer. Then C. Frei.
End: oral exams

Exercises: Find details on website of the instructor Dijana Kreso. math.tugraz.at/~kreso

0 Basics

$$\mathbb{N} = \{1, 2, \dots\} \quad (1)$$

$$\mathbb{N}_0 = \mathbb{N} \cup \{0\} \quad (2)$$

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\} \quad (3)$$

0.1 Divisibility

Definition 0.1.1. Let $a, b \in \mathbb{Z}$. a divides b (written $a \mid b$) if $\exists q \in \mathbb{Z} : b = qa$.
Some properties: Let $a, b, c \in \mathbb{Z}$. Then the following statements hold:

$$a \mid b \Rightarrow ac \mid bc \quad (4)$$

$$a \mid b \wedge b \mid c \Rightarrow a \mid c \quad (5)$$

$$a \mid b \wedge b \mid a \Leftrightarrow a = b \quad (6)$$

$$a \mid b \wedge a \mid c \Rightarrow a \mid (b + c) \quad (7)$$

Definition 0.1.2 (Remainder). Let $a \in \mathbb{Z}$, $b \in \mathbb{N}$. Then there are unique $q, r \in \mathbb{Z}$ such that

$$a = qb + r \text{ and } 0 \leq r < b.$$

Remark. 1. $b \mid a \Leftrightarrow r = 0$

2. $q = \lfloor \frac{a}{b} \rfloor$ (largest integer $\leq \frac{a}{b}$)

3. we will sometimes write: $a \bmod b := c$

Definition 0.1.3. Let $a_1, a_2, \dots, a_n, d \in \mathbb{Z}$. d is a greatest common divisor (gcd) of a_1, \dots, a_n if $d \mid a_i \forall 1 \leq i \leq n$, and for every $e \in \mathbb{Z}$ with $e \mid a_i \forall 1 \leq i \leq n$, $e \mid d$.

Remark. 1. a gcd of a_1, \dots, a_n is unique up to sign

2. we write $d = \gcd(a_1, \dots, a_n)$ if d is a gcd of a_1, \dots, a_n

3. for $a_1, \dots, a_n \in \mathbb{Z}$, a gcd exists and can be written as a linear combination of a_1, \dots, a_n , i.e., $\exists x_1, \dots, x_n \in \mathbb{Z}$ such that

$$\gcd(a_1, \dots, a_n) = x_1 a_1 + \dots + x_n a_n$$

4. $\gcd(a_1, \dots, a_n) = \gcd(\gcd(a_1, \dots, a_{n-1}), a_n)$

5. if $a \mid bc$ and $\gcd(a, b) = 1$ then $a \mid c$.

6. let $a' := \frac{a}{\gcd(a, b)}$, $b' = \frac{b}{\gcd(a, b)}$. Then $\gcd(a', b') = 1$

The algorithm is correct, since $\gcd(a, b) = \gcd(b, a \bmod b)$.
The algorithm terminates because b decreases in each step.
The algorithm is fast: ($\mathcal{O}(\log b)$)

The Euclidean algorithm also allows us to find x, y such that $\gcd(a, b) = ax + by$ by doing all computations backwards.

Hier verwendest du $:=$, sonst aber nur $=$, evtl. einheitlich machen für alle Definitionen?

sollte ausgebaut werden, 1. $\mathcal{O}(\log n)$ steps, 2. stimmt nur wenn $|r| \leq b/2$

Algorithm 1 Compute the gcd of two integers: Euclidean algorithm

Given: $a, b \in \mathbb{Z}$. $|a| \geq |b|$ **Find:** $a := \gcd(a, b)$ replace a by $|a|$, b by $|b|$ **while** $b \neq 0$ **do**write $a = qb + r$, $0 \leq r < b$ $a := b$ $b := r$ **end while****return** a

Example. $\gcd(56, 22) = ?$

$$a = 56, b = 22$$

$$56 = 2 \cdot 22 + 12$$

$$a = 22, b = 12 \neq 0$$

$$22 = 1 \cdot 12 + 10$$

$$a = 12, b = 10 \neq 0$$

$$12 = 1 \cdot 10 + 2$$

$$a = 10, b = 2 \neq 0$$

$$10 = 5 \cdot 2 + 0$$

$$a = 2, b = 0$$

$$\Rightarrow \gcd(56, 22) = 2$$

Doing the computations backwards:

$$2 = 12 - 10 = 12 - (22 - 12) = -22 + 2 \cdot 12 = -22 + 2(56 - 2 \cdot 22) = 2 \cdot 56 - 5 \cdot 22$$

$$x = 2, y = -5$$

Application (linear diophantine equations). Let $a, b, c \in \mathbb{Z}$, $a, b, c \neq 0$. Find all $(x, y) \in \mathbb{Z}^2$ which satisfy

$$ax + by = c. \quad (8)$$

Existence of solution let $d = \gcd(a, b)$.

$$(d \mid a \Rightarrow d \mid xa) \wedge (d \mid b \Rightarrow d \mid yb)$$

$$\Rightarrow d \mid xa + yb = c$$

$$\Rightarrow eq. (8)$$

can have solutions only if $d \mid c$.**Solution in case $d = 1$** Let $x_0, y_0 \in \mathbb{Z}$ such that $ax_0 + by_0 = 1$ using the Euclidean algorithm. Then from $acx_0 + bcy_0 = c$ the solution (cx_0, cy_0) of (eq. (8)) follows: for all $n \in \mathbb{Z}$: $(x, y) := (cx_0 + nb, cy_0 + na)$ is a solution.

Indeed,

$$ax + by = acx_0 + anb + bcy_0 + bna = c \quad \checkmark$$

These (x, y) are all solutions: let (x, y) be a solution. Then

$$\begin{aligned} ax + by &= c \\ acx_0 + bcy_0 &= c \\ \Rightarrow a(x - cx_0) &= b(cy_0 - y) \\ \gcd(a, b) = 1 &\Rightarrow b \mid x - cx_0 \Rightarrow x = cx_0 + nb, n \in \mathbb{Z} \\ \Rightarrow a \mid cy_0 - y &\Rightarrow y = cy_0 + ma, m \in \mathbb{Z} \\ c = ax + by &= acx_0 + anb + bcy_0 + bma \\ &= c + (n + m)ab \Rightarrow (n + m)ab = 0 \Rightarrow m = -n \end{aligned}$$

Solutions in the general case Assume $d = \gcd(a, b)$ and $d \mid c$, let

$$a' = \frac{a}{d} \quad b' = \frac{b}{d} \quad c' := \frac{c}{d}$$

Then $\gcd(a', b') = 1$ and the solution to (eq. (8)) is exactly the solution of $a'x + b'y = c'$.

0.2 Primes

Definition 0.2.1. $p \in \mathbb{N}$, $p > 1$ is a prime number if the only positive divisors of p are 1 and p , i.e., $a \in \mathbb{N}$, $a \mid p \Rightarrow a \in \{1, p\}$. $\mathbb{P} := \{\text{primes}\} \subset \mathbb{N}$, $\mathbb{P} = \{2, 3, 5, 7, 11, 13, \dots\}$. p prime and $p \mid ab \Rightarrow p \mid a$ or $p \mid b$

Theorem 0.2.2 (Fundamental theorem of arithmetic). Every $n \in \mathbb{N}$ can be written uniquely (up to reordering) as a product of primes. i.e. there are distinct primes p_1, \dots, p_l , and $\alpha_1, \dots, \alpha_l \in \mathbb{N}$ such that $n = p_1^{\alpha_1} \dots p_l^{\alpha_l}$

Sketch.

Existence let $p_0 > 1$ be the smallest divisor > 1 of n . Then p_0 is prime. $n = p_0 n_0$, induction \checkmark

Uniqueness let $p_1 \dots p_m = q_1 \dots q_l = n$, p_i, q_j primes. $p_1 \mid q_1 \dots q_l \Rightarrow \exists i : p_1 \mid q_i$, both prime $\Rightarrow p_1 = q_i$, wlog: $i = 1$. $p_1 \dots p_m = q_1 \dots q_l$, induction \checkmark

□

Theorem 0.2.3 (Euclid). There are ∞ -many primes.

Proof. Given primes $p_1, \dots, p_n \in \mathbb{P}$. We construct one more prime

$$N := p_1 \dots p_n + 1.$$

Assume P is a prime factor of N . If $P \in \{p_1, \dots, p_n\}$ then $P \mid N$ and $P \mid p_1 \dots p_n \Rightarrow P \mid 1$ \nexists

□

Remark (prime factors and gcds). Let $a_1, \dots, a_n \in \mathbb{Z}$, write

$$a_i = \prod_{p \in \mathbb{P}} p^{\alpha_{p,i}}, \quad \alpha_{p,i} \in \mathbb{N}_0,$$

almost all $a_i = 0$, then

$$\gcd(a_1, \dots, a_n) = \prod_{p \in \mathbb{P}} p^{\min_{1 \leq i \leq n} \{\alpha_{p,i}\}}$$

1. Beistriche für bessere Lesbarkeit
2. faustregel, vor und nach "i.e." gehört eigentlich beistrich

0.3 Congruences

All rings are commutative with 1.

Definition 0.3.1. Let $a, b \in \mathbb{Z}$, $n \in \mathbb{N}$. Then a is congruent to $b \pmod{n}$, $a \equiv b \pmod{n}$, if $n \mid a - b$. We write $\bar{a} = [a]_n := \{b \in \mathbb{Z} : b \equiv a \pmod{n}\}$

Remark. 1. Congruence $\text{mod } n$ is an equivalence relation

2. $\bar{0}, \bar{1}, \dots, \overline{n-1}$ is a partition of \mathbb{Z} .

3. if $a \equiv b \pmod{n}$, $c \equiv d \pmod{n}$, then $-a \equiv -b \pmod{n}$, $a \pm d \pmod{n}$.

Definition 0.3.2. $\mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n := \{[a]_n : a \in \mathbb{Z}\} = \{\bar{0}, \bar{1}, \dots, \overline{n-1}\}$ residue class ring modulo n

Remark. \mathbb{Z}_n is a ring with operation $\bar{a} \pm \bar{b} := \overline{a \pm b}$ (well defined due to item 3 of section 0.3) $\mathbb{Z}_n^\times = \{\bar{a} \in \mathbb{Z}_n : \exists \bar{b} \in \mathbb{Z}_n : \bar{a}\bar{b} = \bar{1}\}$... group of units \pmod{n}

Lemma 0.3.3. Let $a \in \mathbb{Z}$. Then $\bar{a} \in \mathbb{Z}_n^\times \Leftrightarrow \gcd(a, n) = 1$.

Proof.

“ \Rightarrow ” $\bar{a}\bar{b} = \bar{1} \Leftrightarrow ab \equiv 1 \pmod{n} \Leftrightarrow n \mid ab - 1$
 \Rightarrow no prime factor of n divides a
 $\Rightarrow \gcd(a, n) = 1$.

“ \Leftarrow ” $1 = \gcd(a, n) = ax + ny \Rightarrow \bar{1} = \bar{a}\bar{x}$

□

Remark. The inverse of \bar{a} can be computed by the Euclidean algorithm.

Example (Simultaneous congruences). Find $x \in \mathbb{Z}$ such that

$$x \equiv 2 \pmod{3} \tag{9}$$

$$x \equiv 1 \pmod{5} \tag{10}$$

$$x \equiv 0 \pmod{7} \tag{11}$$

Theorem 0.3.4 (Chinese remainder theorem (CRT)). Let

$$n_1, \dots, n_l \in \mathbb{N} \text{ subject to } \gcd(n_i, n_j) = 1 \ \forall i \neq j$$

$$x_1, \dots, x_l \in \mathbb{Z}.$$

Then

$$\exists x \in \mathbb{Z} \text{ such that } x \equiv x_i \pmod{n_i} \ \forall 1 \leq i \leq l$$

where x is unique modulo $n_1 \cdots n_l$.

Proof. How to compute x ? For $i \in \{1, \dots, l\}$, let

$$N_i := \prod_{j \neq i} n_j = n_1 \dots n_{i-1} n_{i+1} \dots n_l$$

and let

$$N := \prod_i n_i = n_1 N_1 = n_2 N_2 = \dots = n_l N_l$$

because $\gcd(n_i, N_i) = 1 \Rightarrow N_i$ is invertible mod n_i . Let

$$m_i N_i \equiv 1 \pmod{n_i}$$

and let

$$x := N_1 m_1 x_1 + \cdots + N_l m_l x_l.$$

We have $N_i m_i x_i \equiv 0 \pmod{n_j, j \neq i}$ □

Example.

$$\begin{aligned} n_1 &= 3, & n_2 &= 5, & n_3 &= 7 \\ x_1 &= 2, & x_2 &= 1, & x_3 &= 0 \\ N_1 &= 35, & N_2 &= 21, & N_3 &=? \\ \bar{m}_1 &= \overline{35}^{-1} \pmod{3} = \bar{2}^{-1} \pmod{3} = \bar{2} \pmod{3} \Rightarrow m_1 = 2 \\ \bar{m}_2 &= \overline{21}^{-1} \pmod{5} = \bar{1}^{-1} \pmod{5} = \bar{1} \pmod{5} \Rightarrow m_2 = 1 \end{aligned}$$

$$\begin{aligned} x &= 35 \cdot 2 \cdot 2 + 21 \cdot 1 \cdot 1 + 0 \\ &= 140 + 21 \\ &= 161 \\ &\equiv 56 \pmod{105} \end{aligned}$$

Example (more abstract CRT). Let $n_1, \dots, n_l \in \mathbb{N}$, with $\gcd(n_i, n_j) = 1 \ \forall i \neq j$. There is a ring isomorphism $f : \mathbb{Z}_{n_1 \dots n_l} \xrightarrow{\cong} \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_l}$ that satisfies $f([a]_{n_1 \dots n_l}) = ([a]_{n_1}, \dots, [a]_{n_l}) \ \forall a \in \mathbb{Z}$. In particular: $\mathbb{Z}_{n_1 \dots n_l}^\times \cong \mathbb{Z}_{n_1}^\times \times \cdots \times \mathbb{Z}_{n_l}^\times$ (restrict f to $\mathbb{Z}_{n_1 \dots n_l}^\times$)

0.4 Arithmetic functions

Definition 0.4.1. $f : \mathbb{N} \rightarrow \mathbb{C}$ is an arithmetic function. f is multiplicative if $\forall m, n$ it holds that $\gcd(m, n) = 1$. We have $f(mn) = f(m)f(n)$. f is completely multiplicative if $\forall m, n : f(mn) = f(m)f(n)$. Let $f : \mathbb{N} \rightarrow \mathbb{C}$. Its summatory function is $S_f(n) := \sum_{d|n} f(d)$.

Proof. If $\gcd(m, n) = 1$ and $d \mid mn$, then \exists unique d_1, d_2 such that $d = d_1 \cdot d_2$ with $d_1 \mid m, d_2 \mid n$.

$$S_f(mn) = \sum_{d|mn} f(d) = \sum_{d_1|m} \sum_{d_2|n} f(d_1 d_2) = \sum_{d_1|m} f(d_1) \sum_{d_2|n} f(d_2) = S_f(m) S_f(n)$$

□

Example.

$$\begin{aligned} \tau(n) &:= S_1(n) = \sum_{d|n} 1 && \dots \text{number of divisors of } n \\ \sigma(n) &:= S_{id}(n) = \sum_{d|n} d && \dots \text{divisor sum of } n \end{aligned}$$

Definition 0.4.2. The function $\phi(n) := |\mathbb{Z}_n^\times|$ is called Euler's ϕ -function.

Remark. 1. $\phi(n) = |\{0 \leq a < n : \gcd(a, n) = 1\}|$

2. ϕ is multiplicative (CRT: $\gcd(m, n) = 1$. $\mathbb{Z}_{nm}^\times \cong \mathbb{Z}_n^\times \times \mathbb{Z}_m^\times$)

3. $\phi(p) = p - 1$ (\mathbb{Z}_p is a field)

Lemma 0.4.3. $\phi(p^n) = p^n - p^{n-1}$

Proof.

$$\begin{aligned}\phi(p^n) &= |\{0 \leq a < p^n\}| - |\{0 \leq a < p^n : \gcd(a, p^n) \neq 1\}| \\ &= p^n - |\{0 \leq a < p^n : p|a\}| \\ &= p^n - p^{n-1}\end{aligned}$$

□

Proposition 0.4.4. If $n = p_1^{\alpha_1} \dots p_l^{\alpha_l}$ with $p_i \neq p_j$ primes, $\alpha_i \in \mathbb{N}$. Then

$$\phi(n) = \prod_{i=1}^l p_i^{\alpha_i} \left(1 - \frac{1}{p_i}\right) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$$

Theorem 0.4.5 (Euler-Fermat). Then $a^{\phi(n)} \equiv 1 \pmod{n}$. In particular: $a^{p-1} \equiv 1 \pmod{p} \forall p \nmid a$ (little Fermat).

Proof 1. Lagrange's Theorem, $G = \mathbb{Z}_n^\times$, $\bar{a} \in G \Rightarrow \bar{a}^{|G|} = \bar{1}$, $|G| = \phi(n)$. □

Proof 2. $\prod_{x \in \mathbb{Z}_n^\times} x = \prod_{x \in \mathbb{Z}_n^\times} (\bar{a}x) = \bar{a}^{\phi(n)} \prod_{x \in \mathbb{Z}_n^\times} x \Rightarrow a^{\phi(n)} \equiv 1 \pmod{n}$ □

Definition 0.4.6. The Möbius function $\mu : \mathbb{N} \rightarrow \{-1, 0, +1\}$ is defined as

$$\mu(n) = \begin{cases} (-1)^l & n = p_1 \dots p_l, p_i \neq p_j, i \neq j, p_i \text{ primes} \\ 0 & \text{otherwise i.e. if } \exists p : p^2 \mid n \end{cases}$$

Remark.

1. $\mu(1) = 1$, $\mu(2) = -1$, $\mu(3) = -1$, $\mu(4) = 0$, $\mu(5) = -1$, $\mu(6) = 1$, ...

2. μ is multiplicative

Lemma 0.4.7.

$$S_\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$$

Proof.

$$S_\mu(1) = \sum_{d|1} \mu(d) = \mu(1) = 1$$

By multiplicativity, it suffices to prove $S_\mu(p^n) = 0 \forall p, n$.

$$\begin{aligned}S_\mu(p^n) &= \sum_{d|p^n} \mu(d) \\ &= \sum_{i=0}^n \mu(p^i) \\ &= \mu(1) + \mu(p) + 0 + \dots + 0 \\ &= 0\end{aligned}$$

□

Theorem 0.4.8 (Möbius inversion formula). *Let $f : \mathbb{N} \rightarrow \mathbb{C}$. Then*

$$f(n) = \sum_{d|n} \mu(d) S_f\left(\frac{n}{d}\right).$$

Proof.

$$\begin{aligned} \sum_{d|n} \mu(d) S_f\left(\frac{n}{d}\right) &= \sum_{d|n} \mu(d) \sum_{e|\frac{n}{d}} f(e) \\ &= \sum_{e|n} f(e) \sum_{\substack{d|n \\ s.t. e|\frac{n}{d}}} \mu(d) \end{aligned}$$

$$\begin{aligned} \text{For the next step we use } d | n \wedge e | \frac{n}{d} &\Leftrightarrow ed | n \Leftrightarrow e | n \wedge d | \frac{n}{e} \\ &= \sum_{e|n} f(e) \sum_{d|\frac{n}{e}} \mu(d) \\ &= f(n) \end{aligned}$$

$$\text{since } \sum_{d|\frac{n}{e}} \mu(d) = \begin{cases} 1 & \frac{n}{e} = 1 \\ 0 & \text{otherwise} \end{cases}$$

□

0.5 Structure of \mathbb{Z}_n^\times

$n = p_1^{\alpha_1} \dots p_l^{\alpha_l}$ with $p_i \neq p_j, i \neq j, \alpha_i \in \mathbb{N}$ where p_i are primes

From the CRT it follows that $\mathbb{Z}_n^\times \cong \mathbb{Z}_{p_1^{\alpha_1}}^\times \times \dots \times \mathbb{Z}_{p_l^{\alpha_l}}^\times$. So we only consider prime powers $p^\alpha, p \in \mathbb{P}, \alpha \in \mathbb{N}$

0.5.1 Case 1: $\alpha = 1$

Theorem 0.5.1. \mathbb{Z}_p^\times is cyclic, i.e. $\mathbb{Z}_p^\times \cong \mathbb{Z}_{(p-1)}$

Proof. Use structure theorem for finite abelian groups. If G is a finite abelian group then $\exists d_1, \dots, d_l \in \mathbb{N}$ such that $1 < d_1 | d_2 | d_3 | \dots | d_l$, and $G \cong \mathbb{Z}_{d_1}^\times \times \dots \times \mathbb{Z}_{d_l}^\times$ thus, $\mathbb{Z}_p^\times \cong \mathbb{Z}_{d_1}^\times \times \dots \times \mathbb{Z}_{d_l}^\times$ (every element $x \in \mathbb{Z}_{d_1}^\times \times \dots \times \mathbb{Z}_{d_l}^\times$ satisfies $d_l x = 0 \Rightarrow$ every $x \in \mathbb{Z}_p^\times$ satisfies $x^{d_l} = 1$). $x^{d_l} - 1$ is a polynomial of degree d_l over the field $\mathbb{Z}_p \Rightarrow x^{d_l} - 1$ has $\leq d_l$ roots $\Rightarrow p-1 \leq d_l$, but $p-1 = d_1 \dots d_l \Rightarrow l = 1, p-1 = d_l$ □

Remark. The same proof shows: Let F be a field, $G \leq F^\times, |G| < \infty$. Then G is cyclic.

0.5.2 Case 2: $\alpha \geq 2; p \geq 3$

Denote $|x|$ as the order of x in $\mathbb{Z}_{p^\alpha}^\times$; i.e. $|x| = \min \{l \in \mathbb{N} : x^l \equiv 1 \pmod{p^\alpha}\}$

$|\mathbb{Z}_{p^\alpha}^\times| = \phi(p^\alpha) = p^{\alpha-1}(p-1)$, find $x, y \in \mathbb{Z}_{p^\alpha}^\times$ such that $|x| = p^{\alpha-1}, |y| = p-1$ then $|xy| = |x||y| = p^{\alpha-1}(p-1)$, since $\gcd(|x|, |y|) = 1$

Lemma 0.5.2.

$$(1+p)^{p^{n-1}} \begin{cases} \equiv 1 & \pmod{p^n} \\ \not\equiv 1 & \pmod{p^{n+1}} \end{cases}$$

Proof. Proof by induction

$n = 1$ ✓

$n \rightarrow n + 1$

$$\begin{aligned}(1+p)^{p^{n-1}} &= 1 + ap^n, p \nmid a \\ (1+p)^{p^n} &= (1+ap^n)^p \\ &= 1 + pap^n + \sum_{i=2}^{p-1} \binom{p}{i} (ap^n)^i + (ap^n)^p\end{aligned}$$

$$\begin{aligned}p^{np} \mid \bullet, \quad np \geq n+2, \quad (\text{or } p \geq 3), \quad p^{2n+1} \mid \bullet, \quad 2n+1 \geq n+2 \\ p \mid \binom{p}{i} = \frac{p!}{i!(p-i)!}, 1 \leq i < p \Rightarrow (1+p)^{p^n} \equiv 1 + ap^{n+1} \pmod{p^{n+2}}, p \nmid a\end{aligned}$$

□

2× Lemma: $x = 1 + p$ satisfies $|x| = p^{\alpha-1}$, now find y .

1. $\exists z \in \mathbb{Z} : |\bar{z}| = p - 1$ is \mathbb{Z}_p^\times
2. let $l := |E|$ is $\mathbb{Z}_{p^\alpha}^\times$
3. Then $p^\alpha \mid z^l - 1 \Rightarrow z^l \equiv 1 \pmod{p}$
4. $\Rightarrow p - 1 \mid l$.
5. Let $y := z^{\frac{l}{p-1}}$, then $|\bar{y}| = p - 1$.

We have proven: Theorem: $\mathbb{Z}_{p^\alpha}^\times$ is cyclic, i.e. $\mathbb{Z}_{p^\alpha}^\times \cong \mathbb{Z}_{p^{\alpha-1}(p-1)}$, if $p \geq 3, \alpha \geq 1$.
 $p = 2$: $\mathbb{Z}_{2^\alpha}^\times \cong \{0, \alpha = 1 \quad \mathbb{Z}_2, \alpha = 2 \quad \mathbb{Z}_2 \times \mathbb{Z}_{p^{\alpha-2}}, \alpha \geq 3\}$

Corollary 0.5.3. Let $m \in \mathbb{N}$. Then \mathbb{Z}_m^\times is cyclic iff m has one of the following forms:

- $m = 2$
- $m = 4$
- $m = p^\alpha, p \geq 3, \alpha \in \mathbb{N}$
- $m = 2p^\alpha, p \geq 3, \alpha \in \mathbb{N}$

In these cases a generator of \mathbb{Z}_m^\times is called a *primitive root modulo m* .

New Lecturer

Chapter 1:

1. Approximation to algebraic numbers; Wolfgang M. Schmidt, 1972 L'Ehseignement Mathématique
2. Lectures Notes in Mathematics 785; W.M.Schmidt, Springer
3. LNM 1467, W.M.S., Springer
4. For section 2 (continued fractions) he will strictly follow the lecture notes of MT421 of Professor James McKee

1 Diophantine Approximation

1.1 Dirichlet's Theorem

Let $\alpha \in \mathbb{R}$. As \mathbb{Q} is dense in \mathbb{R} any $\alpha \in \mathbb{R}$ can be approximated arbitrarily well, by rational numbers p/q ($p \in \mathbb{Z}, q \in \mathbb{N} = \{1, 2, 3, \dots\}$).

The question is how well can we approximate α in terms of the denominator q , e.g., is it true that for every $\alpha \in \mathbb{R}$ there exists infinitely many $p/q \in \mathbb{Q}$ ($q \in \mathbb{N}$) such that $|\alpha - \frac{p}{q}| < \frac{1}{q^2}$?

The answer is no!

Take $\alpha = r/s$ ($s \in \mathbb{N}$) a rational number. Then

$$|\alpha - \frac{p}{q}| = |\frac{r}{s} - \frac{p}{q}| = |\frac{qr - ps}{sq}| \stackrel{\geq}{\geq} \frac{p}{q} \frac{1}{sq} > \frac{1}{q^2} \text{ provided } q > s.$$

This shows that we have only finitely many solutions $p/q \in \mathbb{Q}$ for $|\alpha - \frac{p}{q}| < \frac{1}{q^2}$.

Theorem 1.1.1 (Dirichlet's Theorem). *Suppose $\alpha, Q \in \mathbb{R}$ and $Q > 1$. Then $\exists p, p \in \mathbb{Z}$ s.t. $0 < q < Q$ and $|q\alpha - p| \leq \frac{1}{Q}$.*

Proof. for $\xi \in \mathbb{R}$ put $\{\xi\} = \xi - \lfloor \xi \rfloor$. so $0 \leq \{\xi\} \leq 1$. First suppose $Q \in \mathbb{Z}$. Consider the $Q + 1$ numbers $0, 1, \{\alpha\}, \{2\alpha\}, \dots, \{(Q-1)\alpha\}$.

They all lie in $[0, 1]$. We split it up in Q subintervals:

$$[0, 1] = [0, \frac{1}{Q}] \cup [\frac{1}{Q}, \frac{2}{Q}] \cup \dots \cup [\frac{Q-1}{Q}, 1]$$

By the pigeon hole principle two of the previous numbers lie in the same subinterval. Thus $\exists r_1, r_2, s_1, s_2 \in \mathbb{Z}$ with $0 \leq r_1 < r_2 \leq Q-1$ such that $|(r_1\alpha - s_1) - (r_2\alpha - s_2)| \leq \frac{1}{Q}$. Then with $q = r_2 - r_1$ and $p = s_2 - s_1$ we get $|q\alpha - p| \leq \frac{1}{Q}$ and $0 < q < Q$. This proves the Theorem when $Q \in \mathbb{Z}$. Now suppose $Q \notin \mathbb{Z}$. We apply the previous with $Q' = \lfloor Q \rfloor + 1 > 1$. Hence, $\exists p, q \in \mathbb{Z}$ with $|q\alpha - p| \leq \frac{1}{Q'}$ and $0 < q < Q'$, and so $|q\alpha - p| \leq \frac{1}{Q}$ and $0 < q < Q$. \square

Corollary 1.1.2. *Suppose $\alpha \in \mathbb{R}/\mathbb{Q}$. Then there exist infinitely many solutions $p/q \in \mathbb{Q}$ ($q \in \mathbb{N}$) of $|\alpha - \frac{p}{q}| < \frac{1}{q^2}$.*

Proof. Take $Q_1 > 1$. By Theorem 1.1.1 we get $(p_1, q_1) \in \mathbb{Z}^2$ with $0 < q_1 < Q_1$, and $|q_1\alpha - p_1| \leq \frac{1}{Q_1}$. Thus $|\alpha - \frac{p_1}{q_1}| \leq \frac{1}{q_1 Q_1} < \frac{1}{q_1^2}$. Next take $Q_2 = |\alpha - \frac{p_1}{q_1}|^{-1} + 1$. Then Theorem 1.1.1 again yields $\frac{p_2}{q_2} \in \mathbb{Q}$ with $|\alpha - \frac{p_2}{q_2}| < \frac{1}{q_2^2}$ and $|\alpha - \frac{p_2}{q_2}| \leq \frac{1}{q_1 q_2} \leq \frac{1}{q_2} < |\alpha - \frac{p_1}{q_1}|$. So $\frac{p_2}{q_2}$ is a better approx than $\frac{p_1}{q_1}$. Repeating this process indefinitely proves the claim. \square

Theorem 1.1.3 (Pell-equation). *Suppose $m \in \mathbb{N}$ is not a square (i.e., $m \neq n^2 \forall n \in \mathbb{Z}$).*

Then

$$x^2 - my^2 = 1$$

has infinitely many solutions $(x, y) \in \mathbb{Z}^2$.

Proof. Apply Corollary 1.1.2 with $\alpha = \sqrt{m}$. So $\alpha \in \mathbb{R}/\mathbb{Q}$. We get $|\alpha - \frac{p}{q}| < \frac{1}{q^2}$ and $|\alpha + \frac{p}{q}| \leq 1 + 2\alpha$. Thus

$$|p^2 - mq^2| = q^2 \left| \alpha - \frac{p}{q} \right| \cdot \left| \alpha + \frac{p}{q} \right| < 1 + 2\sqrt{m}.$$

Hence, there exists $k \in \mathbb{Z}$ with $|k| < 1 + 2\sqrt{m}$ such that $p^2 - mq^2 = k$ for infinitely many $(p, q) \in \mathbb{Z}^2$ and p/q all distinct. As m is not a square we have $k \neq 0$.

Let S be the set of solutions $(p, q) \in \mathbb{Z}^2$ of $p^2 - mq^2 = k$. The map $S \rightarrow (\mathbb{Z}/k\mathbb{Z}) \times (\mathbb{Z}/k\mathbb{Z})$. This map is not injective ($S = \infty$) hence, $\exists (p_1, q_1) \neq (p_2, q_2)$ both in S such that $p_1 \equiv p_2, q_1 \equiv q_2 \pmod{k}$. (MOD)
Now we compute

$$k^2 = (p_1^2 - mq_1^2)(p_2^2 - mq_2^2) \quad (12)$$

$$= (p_1 + \sqrt{m}q_1)(p_2 - \sqrt{m}q_2) \quad (13)$$

$$= (r - \sqrt{m}s)(r + \sqrt{m}s) = r^2 - ms^2 \quad (14)$$

$$\text{where } r = p_1p_2 - mq_1q_2 \quad (15)$$

$$s = p_1q_2 - q_1p_2 = \frac{1}{q_1q_2} \left(\frac{p_1}{q_1} - \frac{p_2}{q_2} \right) \neq 0. \quad (16)$$

because of (MOD) $k \mid s$. Hence, $k^2 \mid s^2$. Thus $k^2 \mid r^2$. Hence $k \mid r$. Then $x = \frac{r}{k}$ and $y = \frac{s}{k}$ are both integers and

$$x^2 - my^2 = 1.$$

We have one solution but we need infinitely many! To this end we replace m by md^2 ($d \in \mathbb{N}$). The above argument yields a solution $(x', y') \in \mathbb{Z}^2$ of $x'^2 - md^2y'^2 = 1$. Thus, $(x, y) = (x', dy')$ is a new solution of $x^2 - my^2 = 1$. (Critical: $s \neq 0$) \square

1.2 Continued fractions

Let $\theta \in \mathbb{R}$. Put $a_0 = \lfloor \theta \rfloor$. If $a_0 \neq \theta$ then we find $\theta_1 > 1$ such that

$$\theta = a_0 + \frac{1}{\theta_1}$$

and we put $a_1 = \lfloor \theta_1 \rfloor$. If $a_1 \neq \theta_1$ then we can find $\theta_2 > 1$ such that

$$\theta_1 = a_1 + \frac{1}{\theta_2}$$

and we put $a_2 = \lfloor \theta_2 \rfloor$. This process can be continued indefinitely, unless $a_n = \theta_n$ for some n . Note that a_0 can be zero or negative but a_1, a_2, a_3, \dots are all positive integers.

We call this process the *continued fraction process*. The a_i are called *partial quotients* of θ .

Example.

$$\theta = \frac{19}{11}$$

Then $a_0 = \lfloor \theta \rfloor = 1$

Now $\theta = \frac{19}{11} = a_0 + \frac{1}{\theta_1} = 1 + \frac{8}{11} = 1 + \frac{1}{\frac{11}{8}}$

So $\theta_1 = \frac{11}{8}$.

Thus $a_1 = \lfloor \theta_1 \rfloor = 1$.

Now

$$\theta_1 = \frac{11}{8} = a_1 + \frac{1}{\theta_2} = 1 + \frac{3}{8} = 1 + \frac{1}{\frac{8}{3}}$$

Thus $\theta_2 = \frac{8}{3}$ and $a_2 = \lfloor \theta_2 \rfloor = 2$

and so on...

If the continued fraction process terminates then we have

$$\theta = a_0 + \frac{1}{\theta_1} \tag{17}$$

$$= a_0 + \frac{1}{a_2 + \frac{1}{\theta_2}} \tag{18}$$

$$= a_0 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{\theta_3}}} \quad \dots = a_0 + \frac{1}{a_1 + \frac{1}{\dots}} \tag{19}$$

In this case we write $\theta = [a_0, \dots, a_n]$.

We use the same notation when the a_i are any real numbers, not necessarily integers.

In particular

$$\theta = [a_0, \dots, a_i, \theta_{i+1}]$$

where $a \leq i < n$.

If the continued fraction process does not terminate then we write $\theta = [a_0, a_1, a_2, \dots]$.

Note that in this case, for every $n \geq 0$, we have

$$\theta = [a_0, \dots, a_n, \theta_{n+1}]$$

where a_0, \dots, a_n are integers but θ_{n+1} is not! For $n \geq 0$ we set

$$\frac{p_n}{q_n} = [a_0, \dots, a_n]$$

where $\gcd(p_n, q_n) = 1$. We shall say that $\frac{p_n}{q_n}$ is the n -th convergent of θ . We will prove that $\frac{p_n}{q_n} \rightarrow \theta$ as $n \rightarrow \infty$. Next we shall see that $p_n, q_n > 0$ both satisfy the same simple recurrence relation $x_n = a_n x_{n-1} + x_{n-2}$ with different starting values.

Lemma 1.2.1. *Let a_0, a_1, a_2, \dots be a sequence of integers with $a_i > 0$ ($i > 0$).*

Define p_n, q_n :

$$p_0 = a_0 \quad (20)$$

$$q_0 = 1 \quad (21)$$

$$p_1 = a_0 a_1 + 1 \quad (22)$$

$$q_1 = a_1 \quad (23)$$

$$p_n = a_n p_{n-1} + p_{n-2} \text{ for } n \geq 2 \quad (24)$$

$$q_n = a_n q_{n-1} + q_{n-2} \text{ for } n \geq 2. \quad (25)$$

Then:

$$1. \ p_n q_{n+1} - p_{n+1} q_n = (-1)^{n+1}$$

$$2. \ \gcd(p_n, q_n) = 1$$

$$3. \ p_n/q_n = [a_0, \dots, a_n]$$

4. If the a_i are produced by the continued fraction process for θ , then, for every $n \geq 1$, $\frac{p_n}{q_n}$ is the n -th convergent of θ and

$$\theta = \frac{p_n \theta_{n+1} + p_{n-1}}{q_n \theta_{n+1} + q_{n-1}}$$

Proof. 1. We use induction on n . For $n = 0$ we note that

$$p_0 q_1 - p_1 q_0 = a_0 a_1 - a_0 a_1 - 1 = -1.$$

So the result holds for $n = 0$.

Now suppose result holds for $n = m - 1$.

consider case $n = m$. Using the recurrence relation, we set

$$p_m q_{m+1} - p_{m+1} q_m = p_m (a_m q_m + q_{m-1}) - q_m (a_m p_m + p_{m-1}) \quad (26)$$

$$= p_m q_{m-1} - p_{m-1} q_m = -(-1)^m = (-1)^{m+1}. \quad (27)$$

This proves claim for $n = m$.

2. Immediate from (a)

3. (c) + (d):

Remark about $\frac{p_n}{q_n}$ in (d) follows directly from (c). We prove the rest of (d), along with (c), using induction on n . Remember that (c) a priori does not require that the a_i are produced by the continued fraction process. Consider base case $n = 1$. For (c) note that $\frac{p_1}{q_1} = a_0 + \frac{1}{a_1} = [a_0, a_1]$. For (d) we note that

$$\frac{p_1 \theta_2 + p_0}{q_1 \theta_2 + q_0} = \frac{(a_0 a_1 + 1) \theta_2 + a_0}{a_1 \theta_2 + 1} = a_0 + \frac{\theta_2}{a_1 \theta_2 + 1} = a_0 + \frac{1}{a_1 + \frac{1}{\theta_2}} = \theta$$

Next suppose (c) and (d) both hold for $n = m - 1$, and consider $n = m$.

Using (d) with $n = m - 1$ we get

$$[a_0, \dots, a_m] = \frac{p_{m-1} a_m + p_{m-2}}{q_{m-1} a_m + q_{m-2}} = \frac{p_m}{q_m} \text{ by recurrence relation.}$$

This proves (c) for $n = m$.

To prove (d) with $n = m$ we observe that

$$\theta = [a_0, \dots, a_m, \theta m + 1] \quad (28)$$

$$= [a_0, \dots, a_m + \frac{1}{\theta_{m+1}}] \quad (29)$$

$$\stackrel{(d)}{for} n = m - 1 \frac{p_{m-1}(a_m + \frac{1}{\theta_{m+1}}) + p_{m-2}}{q_{m-1}(a_m \frac{1}{\theta_{m-1}}) + q_{m-2}} \quad (30)$$

$$\stackrel{rec.rel}{=} \frac{p_m + p_{m-1}(\frac{1}{\theta_{m+1}})}{q_m + q_{m-1}(\frac{1}{\theta_{m+1}})} \quad (31)$$

$$= \frac{p_m \theta_{m+1} + p_{m-1}}{q_m \theta_{m+1} + q_{m-1}} \quad (32)$$

which is (d) for $n = m$. □

Next we deduce some properties of continued fraction convergents.

Theorem 1.2.2. *Let $\theta = [a_0, a_1, a_2, \dots]$ with convergents $\frac{p_n}{q_n}$. For (a) - (d) we assume that the continued fraction process does not terminate*

1. For all $n \in \mathbb{N}_0$, θ lies between $\frac{p_n}{q_n}$ and $\frac{p_{n+1}}{q_{n+1}}$.
2. For all $n \in \mathbb{N}_0 : |\theta - \frac{p_n}{q_n}| \leq \frac{1}{q_n q_{n+1}}$
3. For $n \geq 1$ we have $q_{n+2} \geq 2 \cdot q_n$
4. $\frac{p_n}{q_n} \rightarrow \theta$ as $n \rightarrow \infty$
5. The continued fraction process terminates if and only if θ is rational.

Proof. 1. Note $\theta = [a_0, \dots, a_n, \theta_{n+1}] = [a_0, \dots, a_n + \frac{1}{\theta_{n+1}}]$ where $0 < \frac{1}{\theta_{n+1}} < \frac{1}{a_{n+1}}$. So that θ lies between $[a_0, \dots, a_n]$ and $[a_0, \dots, a_n + \frac{1}{a_{n+1}}]$. But $[a_0, \dots, a_n + \frac{1}{a_{n+1}}] = [a_0, \dots, a_{n+1}]$. This shows (a).

2. By (a) we have $|\theta - \frac{p_n}{q_n}| \leq |\frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}}| = |\frac{p_n q_{n+1} - p_{n+1} q_n}{q_n q_{n+1}}| \stackrel{Lemma 1.2.1(a)}{=} \frac{1}{q_n q_{n+1}}$

3. Follows from the fact that $a_i > 0 (i > 0)$ using Lemma 1.2.1.

4. Follows from (b) and (c)

5. Only if part is obvious.

Conversely suppose $\theta = \frac{a}{b} \in \mathbb{Q}$ but the process does *not* terminate. Taking n such that $q_n > b$ yields

$$|\theta - \frac{p_n}{q_n}| \geq \frac{a}{b} - \frac{p_n}{q_n} \geq \frac{p_n}{q_n} \text{ as } q_n > b \text{ and } \gcd(p_n, q_n) = 1 \geq \frac{1}{b q_n} > \frac{1}{q_n q_{n+1}}$$

contradicting (b). □

Example. Take $\theta = \frac{16}{9}$. We have $a_0 = 1$. Then $\theta = 1 + \frac{7}{9}$ so $\theta_1 = \frac{9}{7}$ and $a_1 = 1$. From $\theta_1 = \frac{9}{7} = 1 + \frac{2}{7}$ we get $\theta_2 = \frac{7}{2}$ and $a_2 = 3$. From $\theta_2 = \frac{7}{2} = 3 + \frac{1}{2}$ we get $\theta_3 = 2$ and $a_3 = 2$. Thus $\theta = \frac{16}{9} = [1, 1, 3, 2]$ and the convergents are $\frac{p_0}{q_0} = \frac{1}{1}$, $\frac{p_1}{q_1} = 1 + \frac{1}{1} = \frac{2}{1}$, $\frac{p_2}{q_2} = 1 + \frac{1}{1+\frac{1}{3}} = 1 + \frac{1}{\frac{4}{3}} = \frac{7}{4}$ and $\frac{p_3}{q_3} = \frac{16}{9}$.

Let's check some of the properties claimed.

$$p_1q_2 + p_2q_1 = 2 \cdot 4 - 7 \cdot 1 = 1\checkmark, p_2q_3 - p_3q_2 = 7 \cdot 9 - 16 \cdot 4 = -1\checkmark, \frac{p_2\theta_3 + p_1}{q_2\theta_3 + q_1} = \frac{7 \cdot 2 + 2}{4 \cdot 2 + 1} = \frac{16}{9} = \theta\checkmark$$

We now show that convergents give best-possible rational approximations.

Theorem 1.2.3. Let θ be an irrational real number, and let $\frac{p_n}{q_n}$ be the convergents ($n \geq 0$) with partial quotients a_n ($n \geq 0$).

Then

1. $|\theta - \frac{p_n}{q_n}|$ strictly decreases as n increases.
2. the convergents give successively closer approximations to θ .
3. $\frac{1}{(a_{n+1}+2)q_n^2} < |\theta - \frac{p_n}{q_n}| < \frac{1}{a_{n+1}q_n^2} \leq \frac{1}{q_n^2}$
4. If $p, q \in \mathbb{Z}$ with $0 < q < q_{n+1}$ then

$$|q\theta - p| \geq |q_n\theta - p_n|$$

Moreover, "=" only if $(p, q) = (p_n, q_n)$.

(In this sense convergents are best-possible approximations.)

5. If $(p, q) \in \mathbb{Z} \times \mathbb{N}$ and $|\theta - \frac{p}{q}| < \frac{1}{2 \cdot q^2}$ then $\frac{p}{q}$ is a convergent to θ .

Proof. 1. From Lemma 1.2.1(d) we have $\theta = \frac{p_n\theta_{n+1} + p_{n-1}}{q_n\theta_{n+1} + q_{n-1}}$. Using Lemma 1.2.1(a) we get

$$|q_n\theta - p_n| = \left| \frac{q_n p_n \theta_{n+1} + q_n p_{n-1} - p_n q_n \theta_{n+1} - p_n q_{n-1}}{q_n \theta_{n+1} + q_{n-1}} \right| \quad (33)$$

$$= \frac{1}{q_n \theta_{n+1} + q_{n-1}} \quad (34)$$

$$< \frac{1}{q_n + q_{n-1}} \quad (35)$$

$$= \frac{1}{(a_n + 1)q_{n-1} + q_{n-2}} \quad (36)$$

$$< \frac{1}{\theta_n q_{n-1} + q_{n-2}} \quad (37)$$

$$= |q_{n-1}\theta - p_{n-1}| \quad (38)$$

This shows (a) and (b) because the q_n are increasing.

- c We use $a_{n+1}q_n^2 < \theta_{n+1}q_n^2 + q_n q_{n-1} < (a_{n+1} + 2)q_n^2$ and combine it with the equation (proof part (a)),

$$|\theta - \frac{p}{q}| = \frac{1}{q_n^2 \theta_{n+1} + q_n q_{n-1}}$$

d) By Lemma 1.2.1(a) we can find $\begin{pmatrix} u \\ v \end{pmatrix} \in n\mathbb{Z}^2$ such that

$$\begin{pmatrix} p_n & p_{n+1} \\ q_n & q_{n+1} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}.$$

As $0 < q < q_{n+1}$ we have $u \neq 0$. If $v = 0$ then $(p, q) = u \cdot (p_n, q_n)$ and the claim is trivial. ($u = 1 \Rightarrow$ equality, $u > 1 \Rightarrow$ strictly \downarrow)

So let's assume $v \neq 0$. Then u and v cannot both be negative (as $q > 0$) nor both be positive (as $q < q_{n+1}$). So they have opposite signs.

By Theorem 1.2.2(a) also $q_n\theta - p_n$ and $q_{n+1}\theta - p_{n+1}$ have opposite signs. Hence, $|q\theta - p| = |u(q_n\theta - p_n) + v(q_{n+1}\theta - p_{n+1})| > |q_n\theta - p_n|$.

e) Take n with $q_n \leq q < q_{n+1}$. Then

$$\begin{aligned} \left| \frac{p}{q} - \frac{p_n}{q_n} \right| &\leq \left| \theta - \frac{p}{q} \right| + \left| \theta - \frac{p_n}{q_n} \right| \\ &= \frac{|q\theta - p|}{q} + \frac{|q_n\theta - p_n|}{q_n} \\ &\stackrel{(d)}{\leq} \left(\frac{1}{q} + \frac{1}{q_n} \right) |q\theta - p| \\ &\leq \frac{2}{q_n} \frac{1}{2q} \\ &= \frac{1}{qq_n} \end{aligned}$$

Hence, $\frac{p}{q} = \frac{p_n}{q_n}$.

□

Remark. • (d) implies that if $p, q \in \mathbb{Z}$, $0 < q \leq p_n$ then

$$\begin{aligned} \left| \theta - \frac{p}{q} \right| &\geq \left| \theta - \frac{p_n}{q_n} \right| \cdot \frac{p_n}{q} \\ &\geq \left| \theta - \frac{p_n}{q_n} \right| \end{aligned}$$

with "=" only if $\frac{p}{q} = \frac{p_n}{q_n}$.

• We say $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is badly approximable if

$$\exists c > 0 \text{ such that } \left| \alpha - \frac{p}{q} \right| > \frac{c}{q^2} \forall (p, q) \in \mathbb{Z} \times \mathbb{N}$$

• By (c) and (d) we see that $\theta = [a_0, a_1, a_2, \dots]$ is badly approximable if and only if the partial quotients a_i are uniformly bounded, i.e., $\exists M > 0$ such that $a_i < M \forall i$.

• (c) suggests that the "worst-approximable" number is $\theta = [1, 1, 1, \dots]$. That's indeed the case c.f Exercise sheet 2 # 5,6 (using that $\theta = 1 + \frac{1}{1+\dots} = 1 + \frac{1}{\theta}$. So $\theta^2 - \theta - 1 = 0$. So $\theta = \frac{1 \pm \sqrt{5}}{2}$ but $a_0 = 1$ so $\theta = \frac{1 + \sqrt{5}}{2}$).

Counting Diophantine approximations 1:

Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and let $\phi : [1, \infty) \rightarrow (0, \infty)$ be decreasing. Consider the number of " ϕ -good" approximations:

$$N_\alpha(\phi, Q) = \#\left\{\frac{p}{q} \in \mathbb{Q}; \left|\alpha - \frac{p}{q}\right| < \phi(q), 1 \leq q \leq Q\right\}$$

We put $S_\alpha(\phi, Q) = \{(x, y) \in \mathbb{R}^2 : \left|\alpha - \frac{x}{y}\right| < \phi(y), 1 \leq y \leq Q\}$. Then

$$N_\alpha(\phi, Q) = \#\{(p, q) \in \mathbb{Z} \times \mathbb{N} : \gcd(p, q) = 1\} \cap S_\alpha(\phi, Q)$$

Note that by Corollary 1.1.2 we have $N_\alpha(\phi, Q) \rightarrow \infty$ as $Q \rightarrow \infty$ provided $\phi(y) \geq \frac{1}{y^2}$, and by Exercise sheet 2, even when $\phi(y) \geq \frac{1}{\sqrt{5}y^2}$. If ϕ decays slowly enough then one can easily show that

$$N_\alpha(\phi, Q) = 2 \cdot \underbrace{\int_1^Q y\phi(y)dy}_{\text{Vol } S_\alpha(\phi, Q)} \underbrace{(\text{It } o(1) \text{ tends to 0 as } Q \rightarrow \infty)}_{\text{as } Q \rightarrow \infty}$$

More specifically, using tools we develop in Chapter 3, one can easily show that

$$\#\mathbb{Z}^2 \cap S_\alpha(\phi, Q) = 2 \cdot \int_1^Q y\phi(y)dy + \mathcal{O}(Q),$$

using Möbius-inversion, one can show that

$$N_\alpha(\phi, Q) = \frac{2}{S(2)} \cdot \int_1^Q y\phi(y)dy + \mathcal{O}(Q \log Q).$$

So we get an asymptotic formula

$$N_\alpha(\phi, Q) \sim \frac{2}{S(2)} \text{Vol } S_\alpha(\phi, Q)$$

provided

$$\frac{Q \log Q}{\int_1^Q y\phi(y)dy} \rightarrow 0 \text{ as } Q \rightarrow \infty.$$

So, e.g., if $\phi(y) \geq \frac{(\log y)^2}{y}$.

However, the case when $\phi(y)$ decays much quicker is more interesting. Serge Lang in 1967 proved that if α is real quadratic then

$$N_\alpha\left(\frac{1}{x^2}, Q\right) = c_\alpha \cdot \log(Q) + \mathcal{O}(1). \quad (c_\alpha > 0).$$

He mentioned that it would seem quite difficult to prove an asymptotic result for algebraic α , let alone transcended.

Adams showed

$$N_e\left(\frac{1}{x^2}, Q\right) = c_e \cdot \frac{\log Q}{\log \log Q} + \mathcal{O}(1) \quad (c_e > 0)$$

where $e = 2.7122 \dots$

Lang and Adams both used continuous fractions expansion. How can one prove asymptotics for $N_\alpha(\phi, Q)$? Here is an example.

Example. Suppose $\phi(x) = \frac{1}{2x^2}$. Consider the continuous fraction expansion $\alpha = [a_0, a_1, a_2, \dots]$. By Theorem 1.2.3 we know $|\alpha - \frac{p}{q}| < \phi(q) \Rightarrow \frac{p}{q}$ is a convergent. Moreover, if $\frac{p}{q} = \frac{p_n}{q_n}$ is the n -th convergent then $|\alpha - \frac{p}{q}| < \frac{1}{a_{n+1}q^2}$. So if all $a_i > 1$ then $|\alpha - \frac{p}{q}| < \phi(q) \forall$ convergent $\frac{p}{q}$. Hence, $N_\alpha(\phi, Q) = \#\{n : q_n \leq Q\}$. So, we need to compute the number of convergents $\frac{p_n}{q_n}$ with $q_n \leq Q$. We shall soon see that this is rather simple if $\alpha = [b, a, b, a, b, a, \dots]$ with $a \mid b$. We will get back to this after Theorem 1.2.5.

A continued fraction $[a_0, a_1, a_2, \dots]$ is called *periodic* if

$$\exists k \in \mathbb{N} \text{ and } L \in \mathbb{N}_0 \text{ such that } a_{k+l} a_l \forall l \geq L.$$

In this case we write $[a_0, a_1, a_2, \dots] = [a_0, \dots, a_L, a_{L+1}, \bar{}, a_{L+k-1}]$.

Theorem 1.2.4. $\theta = [a_0, a_1, a_2, \dots]$ is periodic $\iff \theta$ is real quadratic (θ is real quadratic means $\exists D \in \mathbb{Z}[x] \setminus 0$ with $D(\theta) = 0$, but $\theta \notin \mathbb{Q}$ and $\theta \in \mathbb{R}$)

See Ex Sheet 2 #3 for a special instance.

A proof can be found, e.g., in Hardy & Wright "The Theory of numbers", Oxford University press

Let's go back to the problem of computing p_n, q_n of the n -th convergent. The general recursion formula is unhandy. But in certain cases there is a simple explicit formula. Consider $\theta = [b, a, b, a, \dots] = [b, \bar{a}]$ and suppose $b = a \cdot c$ for some $c \in \mathbb{N}$. Now $\theta = b + \frac{1}{a + \frac{1}{\bar{\theta}}} = b + \frac{1}{a + \frac{1}{\bar{\theta}}}$. Thus $\underbrace{a\theta^2 - ab\theta - b\theta^2 - b\theta - c}_{=0} = 0$, so $\theta = \frac{b + \sqrt{b^2 + 4c}}{2}$ and we put $\bar{\theta} = \frac{b - \sqrt{b^2 + 4c}}{2}$.

Theorem 1.2.5. The p_n and q_n of the n -th convergent $\frac{p_n}{q_n}$ of $\theta = [b, \bar{a}]$ ($b = ac$) are give by

$$p_n = c^{-\lfloor \frac{n+1}{2} \rfloor} \cdot U_{n+2}, q_n = c^{-\lfloor \frac{n+q}{2} \rfloor} \cdot u_{n+1}$$

where

$$u_n = \frac{\theta^n - \bar{\theta}^n}{\theta - \bar{\theta}}.$$

(Recall: $\theta = \frac{b + \sqrt{b^2 + 4c}}{2}, \bar{\theta} = \frac{b - \sqrt{b^2 + 4c}}{2}$, so $\theta - b\theta - c = 0, \bar{\theta}^2 - b\bar{\theta} - c = 0$)

Proof. For $n = 0, 1$ we note that

$$q_0 = q = u_1 \tag{39}$$

$$q_1 = a = \frac{b}{c} = \frac{u_2}{c} \tag{40}$$

$$p_0 = b = \theta + \bar{\theta} = u_2 \tag{41}$$

$$p_1 = ab + 1 = \frac{b^2 + c}{c} = \frac{(\theta + \bar{\theta})^2 - \theta\bar{\theta}}{c} = \frac{u_3}{c} \tag{42}$$

Put $\omega_{n+2} = c^{-\lfloor \frac{n+1}{2} \rfloor} u_{n+2}$.

So we need to show that $p_n = \omega_{n+2}$.

Using that $\theta^{n+2} = b\theta^{n+1} + c\theta^n$ and $\bar{\theta}^{n+2} = b\bar{\theta}^{n+1} + c\bar{\theta}^n$ and hence $u_{n+2} = \frac{\theta^{n+2} - \bar{\theta}^{n+2}}{\theta - \bar{\theta}} = bu_{n+1} + cu_n$.

Moreover, $u_{2m+2} = c^m \omega_{2m} + 2$, $u_{2m+1} = c^m \omega_{2m+1}$. Inserting this into the above, distinguishing n even or odd yields:

$$\omega_{2m+2} = b\omega_{2m+1} + \omega_{2m} \quad (43)$$

$$\omega_{2m+1} = a\omega_{2m} + \omega_{2m-1} \quad (44)$$

Hence, p_n and ω_{n+2} satisfy the same recurrence relation. and here the same two starting values, so $p_n = \omega_{n+2}$.

Similar for q_n . □

Counting Diophantine Approximation 2:

We can use Theorem 1.2.5 to show that if $\theta = [\bar{b}, a]$ with $b = ac, a > 1$ then

$$N_\theta\left(\frac{1}{2x^2}, Q\right) = \frac{\log Q}{\log\left(\frac{Q}{\sqrt{c}}\right)} + \mathcal{O}(1)$$

Indeed, we have already seen, that

$$N_\theta\left(\frac{1}{2x^2}, Q\right) = \#\{n : q_n \leq Q\}$$

By Theorem 1.2.5 we know

$$q_n \leq Q \iff c^{-\lfloor \frac{n+1}{2} \rfloor} \frac{\theta^n - \bar{\theta}^n}{\theta - \bar{\theta}} = \left(\frac{\theta}{\sqrt{c}}\right)^n \left(1 - \left(\frac{\bar{\theta}}{\theta}\right)^n\right) \epsilon \leq Q$$

$$\text{where } \epsilon = \begin{cases} \frac{1}{\theta - \bar{\theta}} & 2 \mid n \\ \frac{1}{\sqrt{c}(\theta - \bar{\theta})} & 2 \nmid n \end{cases}$$

$$\iff n \log\left(\frac{\theta}{\sqrt{c}}\right) + \log\left(1 - \left(\frac{\bar{\theta}}{\theta}\right)^n\right) + \log \epsilon \leq \log Q$$

Using Taylor series expansion we see that

$$\left|\log\left(1 - \left(\frac{\bar{\theta}}{\theta}\right)^n\right)\right| \leq \left|\frac{\bar{\theta}}{\theta - \bar{\theta}}\right|$$

This proves the claim.

1.3 Liouville's Theorem

Let $\alpha \in \mathbb{C}$. If $\exists D(x) \in \mathbb{Z}[x]$, $D \neq 0$ and $D(\alpha) = 0$ then we say α is *algebraic*. In this case $\exists D(x) = a_0 x^d + \dots + a_d \in \mathbb{Z}[x]$ with

- $D(\alpha) = 0$
- $a_0 > 0$
- $\gcd(a_0, \dots, a_d) = 1$
- $\deg D(x)$ minimal

Imposing all these condition renders D unique; We write $D_\alpha(x)$ and call this the *minimal polynomial* of α . If α is algebraic then we say $\deg D_\alpha$ is the *degree* of α .

Example. • $\alpha = 0, D_\alpha(x) = x$

• $\alpha = \sqrt{2} + 1, D_\alpha(x) = (x - 1)^2 - 2 = x^2 - 2x - 1$

• $\alpha = \frac{1}{\sqrt{2}}, D_\alpha(x) = 2x^2 - 1$

Theorem 1.3.1 (1.3.1 Liouville's Theorem). *Suppose α is a real, algebraic number of degree d . Then $\exists c(\alpha) > 0$ such that*

$$|\alpha - \frac{p}{q}| > \frac{c(\alpha)}{q^d}$$

for every $(p, q) \in \mathbb{Z} \times \mathbb{N}$ with $\alpha \neq \frac{p}{q}$.

Proof. Suppose $|\alpha - \frac{p}{q}| > 1$ then the claim holds for every $c(\alpha) > 1$. Now suppose $|\alpha - \frac{p}{q}| \leq 1$. Taylor series expansion at D_α about α gives:

$$D_\alpha(x) = \sum_{i=1}^d (x - \alpha)^i \frac{1}{i!} D_\alpha^{(i)}(\alpha)$$

Hence,

$$|D_\alpha\left(\frac{p}{q}\right)| = \left| \sum_{i=1}^d \left(\frac{p}{q} - \alpha\right)^i \frac{1}{i!} D_\alpha^{(i)}(\alpha) \right| \leq |D_\alpha^{(d)}(\alpha)| \left| \frac{p}{q} - \alpha \right| \frac{1}{c(\alpha)}$$

where

$$c(\alpha) = \left(1 + \sum_{i=1}^d \frac{1}{i!} |D_\alpha^{(i)}(\alpha)| \right)^{-1}$$

Now if D_α has a rational root then it must have degree one, so have only *one* root. Thus $D_\alpha\left(\frac{p}{q}\right) \neq 0$ unless $\alpha = \frac{p}{q}$. Hence, if $\alpha \neq \frac{p}{q}$ we get

$$|D_\alpha\left(\frac{p}{q}\right)| = \left| \frac{\text{non-zero integer}}{q^d} \right| \geq \frac{1}{q^d}.$$

Combing this with (D)label yields

$$|\alpha - \frac{p}{q}| > \frac{c(\alpha)}{q^d}.$$

□

We say a real number α is a *Liouville number* if for every $n \in \mathbb{N}$

$$0 < |\alpha - \frac{p}{q}| < \frac{1}{q^n}$$

has a solution. $p, q \in \mathbb{Z}$ with $q > 1$.

Example. $\alpha = \sum_{k=1}^{\infty} 10^{-k^k}$ is a *Liouville number*. Let $n \in \mathbb{N}$ and put $p = \sum_{k=1}^n 10^{n-k^k}$ and $q = 10^{n^n}$. Then $0 < |\alpha - \frac{p}{q}| = \sum_{k>n} 10^{-k} \leq 2 \cdot 10^{-(n+1)^{(n+1)}} < 10^{-n^{(n+1)}} = q^{-n}$

Corollary 1.3.2 (1.3.2). *Every Liouville number is transcendental (i.e., not algebraic).*

Proof. Immediate from Theorem 1.3.1 (Liouville's Theorem). \square

Algebraic numbers are enumerable and thus have Lebesgue measure zero. It's not difficult to show that the set of Liouville numbers, while *not* enumerable, also has measure zero. In fact "most" real numbers are "not very far" from badly approximable as the following theorem shows.

Theorem 1.3.3 (Khinchine). *Suppose $\psi : \mathbb{N} \rightarrow (0, \infty)$ is monotone decreasing (not necessarily strictly). The set*

$$A_\psi = \left\{ \alpha \in \mathbb{R} : \left| \alpha - \frac{p}{q} \right| < \frac{\psi(q)}{q} \text{ has } \infty\text{-many solutions } (p, q) \in \mathbb{Z} \times \mathbb{N} \right\}$$

has a Lebesgue measure zero of $\sum_{q=1}^{\infty} \psi(q)$ converges and has full Lebesgue measure (i.e. the complement has measure zero) if $\sum_{q=1}^{\infty} \psi(q)$ diverges.

We will not prove this Theorem. (For a proof see e.g. Glyn Harman "Metric number theory".)

Example. • Take $\psi(q) = \frac{1}{q}$. We already know that $A_\psi = \mathbb{R} \setminus \mathbb{Q}$. And indeed $\sum \psi(q)$ diverges...

- $\psi(q) = \frac{1}{q \log(q-1)}$. Then $\sum \psi(q)$ diverges and thus A_ψ has full measure.
- $\psi(q) = \frac{1}{q(\log(q+1))^{1+\epsilon}}$ ($\epsilon > 0$) then $\sum \psi(q)$ converges, so A_ψ has measure zero.

1.4 4 Theorems of Thue- Siegel and Poth

In Section 1 we have seen that ∞ -many solutions $\frac{p}{q}$ to $\left| \sqrt{2} - \frac{p}{q} \right| < \frac{1}{q^2}$ leads to ∞ -many solutions $(x, y) \in \mathbb{Z}^2$ of $x^2 - 2y^2 = 1$. What about $x^3 - 2y^3 = 1$? Starting as for $x^2 - 2y^2$ we get

$$y^3 \left| \frac{x}{y} - 2^{1/3} \right| \underbrace{\left| \frac{x}{y} - 2^{1/3} \omega \right|}_{\geq \text{Im } \omega} \underbrace{\left| \frac{x}{y} - 2^{1/3} \omega^2 \right|}_{\geq (\text{Im } \omega)^2}$$

where $\omega = e^{\frac{2\pi i}{3}}$.

So to get boundedness of $x^3 - 2y^3$ for ∞ -many (x, y) we need $\exists c > 0$ such that

$$\left| \frac{x}{y} - 2^{1/3} \right| < \frac{c}{y^3}$$

has ∞ -many solutions $(x, y) \in \mathbb{Z} \times \mathbb{N}$.

Theorem 1.3.3 tells us that we would be extremely lucky if that were the case. And even if so, we still would lack the group structure for $\mathbb{Z} + \sqrt{2}\mathbb{Z}$ (closed under multiplication but $\mathbb{Z} + 2^{1/3}\mathbb{Z}$ is not). On the other hand, suppose we could show that

$$\left| \frac{x}{y} - 2^{1/3} \right| < 1/y^\lambda$$

has only finitely many solutions $(x, y) \in \mathbb{Z} \times \mathbb{N}$ for some fixed $\lambda < 3$. As $x^3 - 2y^3 = 1$, and $y \neq 0$ yields:

$$\left| \frac{x}{y} - 2^{1/3} \right| < \frac{1}{2^{1/3}(\text{Im } \omega)^2 y^3}$$

We would conclude that $x^3 - 2y^3 = 1$ has only finitely many solutions $(x, y) \in \mathbb{Z}^2$. Note that "deg" $2^{1/3} = 3(D(x) = x^3 - 2)$ and so Liouville's Theorem yields only $\lambda = 3$ not $\lambda < 3$. So the big challenge is to improve Liouville's Theorem. After Liouville it has taken 65 years until the first breakthrough was obtained by Axel Thue in 1909.

Theorem 1.4.1 (1.4.1 Thue). *Let α be a real algebraic number of degree $d \geq 2$, and let $\lambda > \frac{d}{2} + 1$. Then $\exists c = c(\alpha, \lambda) > 0$ such that*

$$\left| \alpha - \frac{p}{q} \right| > \frac{c}{q^\lambda}, \quad \forall (p, q) \in \mathbb{Z} \times \mathbb{N}.$$

- Note that for $d = 2$ Liouville is stronger
- Given α and λ there is no method to determine a feasible value for c . This is in stark contrast to Liouville's Theorem.

Just as for $x^3 - 2y^2 = 1$ one can now very easily show that if $f(X, Y) = a_0(X - \alpha_1 Y) \cdots (X - \alpha_d Y) \in \mathbb{Q}[X, Y]$ with $a_0 \neq 0, d \geq 3$, and $\alpha_1, \dots, \alpha_d$ pairwise distinct, and $b \in \mathbb{Q} \setminus \{0\}$, then

$$f(x, y) = b$$

has only finitely many solutions $(x, y) \in \mathbb{Z}^2$.

Wrong if $d = 2$:

$$X^2 - 2Y^2 = 1$$

or $b = 0$:

$$X^3 - Y^3 = 0$$

or $\alpha_1, \dots, \alpha_d$ not pairwise distinct:

$$(X - Y)^5 = 1$$

We will show that Theorem 1.4.1 implies even the following stronger result.

Theorem 1.4.2 (1.4.2 Generalized Thue equations). *Let $f(X, Y) = a_0(X - \alpha_1 Y) \cdots (X - \alpha_d Y) \in \mathbb{Q}[X, Y]$ with $a_0 \neq 0, d \geq 3$ and $\alpha_1, \dots, \alpha_d$ pairwise distinct. Let $g(X, Y) \in \mathbb{Q}[X, Y]$ of total degree $< \frac{d}{2} - 1$. Then there are only finitely many $(X, Y) \in \mathbb{Z}^2$ with*

$$f(x, y) = g(x, y)$$

and $g(x, y) \neq 0$.

Example.

$$x^5 - 2y^5 = x - y$$

has only finitely many solutions $(x, y) \in \mathbb{Z}^2$. Indeed if $x - y = 0$ then $x^5 - 2y^5 = 0$ thus $x = y = 0$. Note Theorem can go wrong if $\alpha_1 = \alpha_2$:

$$(X^2 - 2Y^2)^2 = 1.$$

(assuming Theorem 1.4.1). If $y = 0$ then we have at most d possibilities for x . So we can assume $y \neq 0$. We claim that

$$|x| \leq c_1 |y|$$

for some $c_1 = c_1(f, g)$. Clearly true when $|x| \leq |y|$, so let's assume $|x| > |y|$. Then we write

$$f(x, y) = \sum_{i=0}^d a_i x^{d-i} y^i = \sum_{j+k \leq d-1} b_{jk} x^j y^k = g(x, y)$$

Dividing by x^{d-i} yields

$$a_0 x = - \sum_{i=0}^d a_i \frac{y^i}{x^{i-1}} + \sum_{j+k \leq d-1} b_{jk} x^{j-d+1} y^k$$

We have

$$\left| \frac{y^i}{x^{i-1}} \right| \leq |y|$$

and

$$\left| \frac{y^k}{x^{d-1-j}} \right| \leq |y|^{j+k-(d-1)} \leq 1$$

Therefore $|x| \leq c_1 |y|$, e.g. with $c_1 = \frac{1}{|a_0|} (\sum |a_i| + \sum |b_{jk}|) + 1$. From

$$f(x, y) = g(x, y), (*)$$

we get

$$|\alpha_0| \prod_{j=1}^d \left| \frac{x}{y} - \alpha_j \right| \leq c_2 |y|^{e-d}$$

where $c_2 = c_2(c_1, g)$ and $e < \frac{d}{2} - 1$. So assume $(*)$ has ∞ -many solutions $(x, y) \in \mathbb{Z}^2$. Then $\exists i$, say $i = 1$, such that $\left| \frac{x}{y} - \alpha_1 \right| \leq \mu := \frac{1}{2} \min_{j \neq i} \{|\alpha_j - \alpha_1|\} > 0$ for ∞ -many (x, y) of these solutions of $(*)$. Now

$$\left| \frac{x}{y} - \alpha_j \right| \geq \left| |\alpha_j - \alpha_i| - \left| \frac{x}{y} - \alpha_1 \right| \right| \geq 2\mu - \mu = \mu > 0$$

Hence, we conclude

$$\left| \frac{x}{y} - \alpha_1 \right| \leq \frac{c_2}{|a_0|} \mu^{1-d} |y|^{e-d}, (**)$$

for these solutions (x, y) . Here we can assume $y > 0$ (just replace x by $-x$). Now let d_1 be the degree of α_1 . As $f(x, 1) \in \mathbb{Q}[x]$, $f(x, 1) \neq 0$ and $f(\alpha_1, 1) = 0$. Thus $d_1 \leq d$. Moreover, $d - e > \frac{d}{2} + 1$ and this $\exists \lambda$ such that

$$d - e > \lambda > \frac{d_1}{2} + 1.$$

If $d_1 \geq 2$ then Theorem 1.4.1 implies that $(*)$ has only finitely many solutions $(x, y) \in \mathbb{Z}^2$. Finally suppose $d_1 = 1$. Then $\alpha_1 = \frac{p}{q}$, and $(**)$ yields:

$$\left| x - \frac{p}{q} y \right| \leq c_3 y^{e-d+1} \leq c_3 y^{-\frac{d}{2}}.$$

Thus $x = \frac{p}{q} y = \alpha_1 y$ for y large enough. But then $0 = f(x, y) = g(x, y)$ a contradiction. \square

After Thue came Siegel (1921) who improved the exponent $\frac{d}{2} + 1$ to $2\sqrt{d}$. This was slightly improved by Dyson and Gelfand (1947) to $\sqrt{2d}$. Finally in 1955 came Roth:

Theorem 1.4.3 (1.4.3 (Roth)). *Let α be a real, algebraic irrational number, and $\lambda > 2$. Then $\exists c = c(\alpha, \lambda) > 0$ such that*

$$\left| \alpha - \frac{p}{q} \right| \geq \frac{c}{q^\lambda}, \quad \forall (p, q) \in \mathbb{Z} \times \mathbb{N}.$$

By Corollary 1.1.2 $\lambda > 2$ is best-possible. But if we allow more general functions $\phi(q)$, not only powers of q , then an improvement might be possible. However, since 1955 nobody was able to replace $q^{-\lambda}$ by a function $\phi(q)$ that decays more slowly, e.g. $\phi(q) = q^{-2}(\log q)^{-1}$. However, back to the case where $\phi(q)$ is a power of q . From Theorem 1.3.3. we know that for a generic real α

$$\left| \frac{p}{q} - \alpha \right| < q^{-\lambda}$$

has only finitely many solutions $p, q \in \mathbb{Z} \times \mathbb{N}$ provided $\lambda > 2$. Any by Corollary 1.1.2 every irrational real number has ∞ -many solutions when $\lambda = 2$. And so from Roth's Theorem we see an algebraic irrational behaves "essentially" like a generic number.

Roth's Theorem has various new applications to, e.g., Diophantine equations and transcendence. Let's consider just one now transcendence result: Take $\alpha = \sum_{k=1}^{\infty} 2^{-3^k}$; put $q_n = 2^{3^n}$ and $p_n = q_n \sum_{k=1}^n 2^{-3^k}$. Then $0 < \left| \alpha - \frac{p_n}{q_n} \right| = \sum_{k=n+1}^{\infty} 2^{-3^k} < 2 \cdot 2^{-3^{n+1}} = 2 \cdot 2 \cdot q_n^{-1}$ so by Roth's Theorem α is transcendental.

How does one prove results like Roth's Theorem of the kind

$$\left| \alpha - \frac{p}{q} \right| \geq \phi(q)?$$

The idea is to find good rational approximations.

$$\left| \alpha - \frac{p_n}{q_n} \right| \leq \delta_n$$

with δ_n "pretty small". Then

$$\left| \alpha - \frac{p}{q} \right| \geq \left| \frac{p_n}{q_n} - \frac{p}{q} \right| - \left| \alpha - \frac{p_n}{q_n} \right|$$

If

$$\frac{p_n}{q_n} \neq \frac{p}{q} \tag{45}$$

then

$$\left| \alpha - \frac{p}{q} \right| \geq \frac{1}{qq_n} - \delta_n.$$

If we are lucky then $\delta_n < \frac{1}{qq_n}$ and we get a positive lower bound. How do we find these $\frac{p_n}{q_n}$?

Usually this is a difficult task, but sometimes one can easily see these approximations $\frac{p_n}{q_n}$. Here is an example.

Take again $\alpha = \sum_{k=1}^{\infty} 2^{-3^k}$. Then we can take again $q_n = 2^{3^n}$, $p_n = q_n \sum_{k=1}^n 2^{-3^k}$; so $\left| \alpha - \frac{p_n}{q_n} \right| < 2 \cdot q_n^{-3}$. Hence, if

$$\frac{p_n}{q_n} \neq \frac{p}{q}$$

then

$$\left| \alpha - \frac{p}{q} \right| \geq \frac{1}{qq_n} - \frac{2}{q_n^3}$$

If $q_n^2 > 4 \cdot q$ then

$$\frac{1}{qq_n} - \frac{2}{q_n^3} \geq \frac{q}{2 \cdot qq_n}$$

As $\frac{p_n}{q_n}$ tends strictly monotonously to α , we have $\frac{p_n}{q_n} \neq \frac{p}{q}$ or $\frac{p_{n+1}}{q_{n+1}} \neq \frac{p}{q}$. Let m be minimal with $q_m > 4 \cdot q$. Hence

$$q_m^{\frac{2}{3}} = q_{m-1}^2 \leq 4 \cdot q < q_m^2$$

If $\frac{p_m}{q_m} \neq \frac{p}{q}$ we take $n = m$ and $n = m + 1$ else. We conclude

$$\left| \alpha - \frac{p}{q} \right| \geq \frac{1}{2qq_n} \geq \frac{1}{2qq_{m+1}} \geq \frac{1}{2q} \frac{1}{q_m^3} \geq \frac{1}{2q} \frac{1}{(4q)^{\frac{9}{2}}} = 2^{-10} q^{-\frac{11}{2}}$$

In this example everything works out nicely, e.g., (ref*) could easily be guaranteed by using $\frac{p_n}{q_n}$ tending strictly monotonously to α . However, in Roth's Theorem (ref*) becomes the major-problem.

1.5 5 Simultaneous Diophantine approximation and the Subset Theorem

Suppose $\alpha_1, \dots, \alpha_n$ are real numbers. Theorem 1.1.1 can be generated to yield a solution $(x_1, \dots, x_n, y) \in \mathbb{Z}^n \times \mathbb{N}$ at the system

$$\left| \frac{x_i}{y} - \alpha_i \right| \leq \frac{1}{y \cdot Q} (1 \leq i \leq n), 0 < y < Q.$$

(c.f. Exercise sheet 4). This in turn yields ∞ -many solutions $(x_1, \dots, x_n, y) \in \mathbb{Z}^n \times \mathbb{N}$ of the system

$$\left| \frac{x_i}{y} - \alpha_i \right| < \frac{1}{y^{1+\frac{1}{n}}} (1 \leq i \leq n).$$

provided at least one of the α_i 's is irrational. So Corollary 1.1.2 extends to simultaneous approximation. A much deeper fact is that Roth's Theorem also extends to simultaneous approximation.

For $\underline{x} \in \mathbb{R}^n$ we write $\|\underline{x}\| = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$ for the Euclidean length.

Theorem 1.5.1 (Subspace Theorem, Schmidt). *Suppose $L_i(\underline{x}) = \sum_{j=1}^n a_{ij}x_j$ ($1 \leq i \leq n$) are linearly independent linear forms with algebraic coefficients a_{ij} . Let $\delta > 0$. Then the solutions $\underline{x} \in \mathbb{Z}^n \setminus \underline{0}$ of*

$$|L_1(\underline{x}) \dots L_n(\underline{x})| < \|\underline{x}\|^{-\delta}$$

lie in finitely many proper subspaces of \mathbb{Q}^n .

Remark. *linearly independent linear forms means the coefficient vectors (a_{i1}, \dots, a_{in}) are linearly independent over \mathbb{C} .*

Corollary 1.5.2 (1.5.2). *Let $\delta > 0$, suppose $\alpha_1, \dots, \alpha_n$ are algebraic and $1, \alpha_1, \dots, \alpha_n$ are linearly independent over \mathbb{Q} . Then there are only finitely many $(x_1, \dots, x_n, y) \in \mathbb{Z}^n \times \mathbb{N}$ with*

$$(5.1) \left| \frac{x_i}{y} - \alpha_i \right| < \frac{1}{y^{1+\frac{1}{n}+\delta}} \quad (1 \leq i \leq n) \quad (46)$$

Proof. (assuming Theorem 1.5.1) Put $\underline{X} = (X_1, \dots, X_n, Y)$, $L_i(\underline{X}) = \alpha_i Y - X_i$ ($1 \leq i \leq n$), $L_n(\underline{X}) = Y$. These $n+1$ linear forms in $n+1$ unknowns are linearly independent. With $\underline{x} = (x_1, \dots, x_n, y)$ the solutions of (5.1) yield

$$|L_1(\underline{x}) \dots L_{n+1}(\underline{x})| < \frac{1}{y^\delta} < \frac{1}{\|\underline{x}\|^{\frac{\delta}{2}}}$$

if y is large enough. so by Theorem 1.5.1 (in $n+1$ dimensions), we set that the solutions lie in finitely many proper sub spaces at \mathbb{Q}^{n+1} . Pick one of these (of co-dimension I say). It is given by an equation $c_1 x_1 + \dots + c_n x_n + c_{n+1} y = 0$ where $c_i \in \mathbb{Q}$ not all zero. On this subspace we have

$$(c_1 \alpha_1 + \dots + c_n \alpha_n + c_{n+1})y = c_1(\alpha_1 y - x_1) + \dots + c_n(\alpha_n y - x_n).$$

Put $\gamma = c_1 \alpha_1 + \dots + c_n \alpha_n + c_{n+1}$. By \mathbb{Q} -linearly independence of $1, \alpha_1, \dots, \alpha_n$ we have $\gamma \neq 0$. Hence,

$$|\gamma||y| \leq |c_1||\alpha_1 y - x_1| + \dots + |c_n||\alpha_n y - x_n| \leq (|c_1| + \dots + |c_n|) \frac{1}{y^{1+\frac{1}{n}+\delta}} \leq |c_1| + \dots + |c_n|$$

So $|y|$ is bounded and we are done. \square

In applications one sometimes needs a "p-adic" version of the subspace Theorem in which one approximates with respect to also the so called p-adic absolute values.

Definition (Absolute values). *An absolute value on a field K is a map $|\bullet| : K \rightarrow [0, \infty)$ such that*

- $|x| = 0 \iff x = 0$
- $|x \cdot y| = |x| \cdot |y|$
- $|x + y| \leq |x| + |y|$

Example. • K arbitrary. $|x| = \begin{cases} 0 & x = 0 \\ 1 & x \neq 0 \end{cases}$ the trivial absolute value.

- $K = \mathbb{Q}$, $|\bullet|$ = standard absolute value on \mathbb{Q} . To distinguish it from other absolute values let's write it as $|\bullet| = |\bullet|_\infty$.
- $K = \mathbb{Q}$ and let $p \in \mathbb{N}$ be a prime number. If $x \in \mathbb{Q}, x \neq 0, \pm 1$, then \exists a unique prime factorisation $x = \pm p_1^{a_1} \dots p_s^{a_s}$ where p_1, \dots, p_s primes and $a_i \in \mathbb{Z} \setminus \{0\}$.

For any prime $p \in \mathbb{N}$ write $\text{ord}_p(x)$ for the exponent of p in the prim-factorisation of x (e.g. $\text{ord}_{p_i} x = a_i$). For $x = \pm 1$ we put $\text{ord}_p x = 0 \forall p_i$. The p -adic absolute value $1 \cdot 1_p$ on \mathbb{Q} is defined by

$$|x|_p = \begin{cases} 0 & : x = 0 \\ p^{-\text{ord}_p(x)} & : x \neq 0 \end{cases}$$

The multiplicativity is clear. Note that $\text{ord}_p(x_1 + x_2) \geq \min\{\text{ord}_p(x_1), \text{ord}_p(x_2)\}$.

Hence, $|x_1 + x_2|_p = p^{-\text{ord}_p(x_1 + x_2)} \leq p^{-\min\{\text{ord}_p(x_1), \text{ord}_p(x_2)\}} = \underbrace{\max\{|x_1|_p, |x_2|_p\}}_{\text{strong triangle inequality}}$

$|x_1|_p + |x_2|_p$ An absolute value that satisfies the strong triangle inequality is called non-Archimedean.

Definition 1.5.3. We set $M_{\mathbb{Q}} = \{\text{primes in } \mathbb{N}\} \cup \{\infty\}$. Then for each $v \in M_{\mathbb{Q}}$ we get an absolute value $|\cdot|_v$. Note that if $v \in M_{\mathbb{Q}}$ and p a prime, $a \in \mathbb{Z}$, then

$$|\pm p^a|_v = \begin{cases} p & : v = p \\ p^a & : v = \infty \\ 1 & : v \neq p, v \neq \infty \end{cases}$$

Hence

$$\prod_{v \in M_{\mathbb{Q}}} |1 \pm p^a|_v = 1$$

and so by multiplicativity we conclude

$$\prod_{v \in M_{\mathbb{Q}}} |x|_v = 1$$

for all $x \in \mathbb{Q}$, $x \neq 0$. (PF) This is the so-called product formula (PF) on \mathbb{Q} .

Next, we want to introduce a notion of "arithmetic complexity" on elements in \mathbb{Q}^{n+1} , the so-called projective height:

$$H_{\mathbb{P}^n} : \mathbb{Q}^{n+1} \rightarrow [1, \infty)$$

defined by

$$H_{\mathbb{P}^n}(\underline{x}) = \prod_{v \in M_{\mathbb{Q}}} |\underline{x}|_v$$

where $|\underline{x}|_v = \max\{|x_0|_v, \dots, |x_n|_v\}$.

Example. If $\underline{x} = (x_0, \dots, x_n)$ where $x_0, \dots, x_n \in \mathbb{Z}$ and $\gcd(x_0, \dots, x_n) = 1$. Then $|x|_p = 1$ for all primes p . Hence,

$$H_{\mathbb{P}^n}(\underline{x}) = \max\{|x_0|_{\infty}, \dots, |x_n|_{\infty}\}.$$

Note that $H_{\mathbb{P}^n}(\lambda \cdot \underline{x}) = H_{\mathbb{P}^n}(\underline{x}) \forall \lambda \in \mathbb{Q} \setminus 0$.

Theorem 1.5.4 (1.5.3 p-adic Subspace Theorem, Schlickewei and Schnidt). Let $\delta > 0$ and let $S \subset M_{\mathbb{Q}}$ be finite and with $\infty \in S$. For $v \in S$ let L_{v_1}, \dots, L_{v_n} be n linearly independent linear forms in n variables with coefficients in \mathbb{Q} . Then the set of solutions $\underline{x} \in \mathbb{Q}^{n+1} \setminus 0$ of

$$\prod_{v \in S} \prod_{i=1}^n \frac{|L_{v_i}(\underline{x})|_v}{|\underline{x}|_v} < H_{\mathbb{P}^{n-1}}(\underline{x})^{-n-\delta}$$

lie in finitely many proper subspace of \mathbb{Q}^n .

An interesting consequence is a finiteness result for S -unit equations.
 S -integers and S -units:

Let v be a non-Archimedean absolute value on a field K . Then

$$O_v = \{x \in K : |x|_v \leq 1\}$$

is called the valuation ring of v . It is indeed a ring, e.g., $|x|_v, |y|_v \leq 1$ then

$$|x + y|_v \leq \max\{|x|_v, |y|_v\} \leq 1.$$

In particular, if $K = \mathbb{Q}$ and $v = p$ then O_v is a sub-ring of \mathbb{Q} .

Now let $S \subset M_{\mathbb{Q}}$ be finite and $\infty \in S$. We define the set of S -integers O_S to be

$$O_S = \cap_{v \notin S} O_v.$$

As $\infty \in S$, this is an intersection of rings, hence a ring.

Example. If $S = \{\infty\}$, then $O_S = \mathbb{Z}$. If $S = \{\infty, p_1, \dots, p_s\}$ then

$$O_S = \left\{ \frac{m}{p_1^{a_1} \dots p_s^{a_s}} : m \in \mathbb{Z}, a_1, \dots, a_s \in \mathbb{N}_0 \right\}$$

We say $x \in \mathbb{Q}$ is an S -unit if $x \neq 0$ and x, x^{-1} are both in O_S . So if $S = \{\infty\}$, then ± 1 are the only S -units. If $S = \{\infty, p_1, \dots, p_s\}$ then x is an S -unit $\iff x = \pm \prod_{p \in S \setminus \infty} p_p^a$ (and $ap \in \mathbb{Z}$).

Theorem 1.5.5 (1.5.4 S -unit equation). *Let $S \subset M_{\mathbb{Q}}$ be finite, and $\infty \in S$. Let $\alpha_0, \dots, \alpha_n$ be non-zero and in \mathbb{Q} . Then*

$$\alpha_0 x_0 + \dots + \alpha_n x_n = 0$$

has only finitely many solutions $\underline{x} = (x_0, \dots, x_n)$ if:

- x_0, \dots, x_n are S -units
- we identify proportional solutions (i.e., $\underline{x} = \lambda \underline{x}$ for $\lambda \in \mathbb{Q} \setminus 0$).
- no proper sub-sum vanishes, i.e., $\sum_I \alpha_i x_i \neq 0$ for all $\emptyset \subsetneq I \subsetneq \{0, 1, \dots, n\}$.

Remark. $S = \{\infty, p\}$ $x_0 + x_1 + x_2 + x_3 = 0$ then $x_0 = -x_1 = 1$, $x_2 = -x_3 = p^a$ ($a \in \mathbb{Z}$) are solutions in S -units. So non-vanishing condition is needed!

Example. The exponential Diophantine equation

$$3^x + 5^y - 7^z = 1$$

has solutions, e.g., $(x, y, z) = (0, 0, 0)$ or $(x, y, z) = (1, 1, 1)$. However, with $S = \{\infty, 3, 5, 7\}$ each solution (x, y, z) yields a solution $u_0 = 3x$, $u_1 = 5y$, $u_2 = -7z$, $u_3 = -1$ of the S -unit equation $u_0 + u_1 + u_2 + u_3 = 0$. These solutions are all non-proportional. Moreover, no sub-sum vanishes unless $xyz = 0$ but then we easily see that $x = y = z = 0$. Hence, Theorem 1.5.4 yields finiteness.

assuming Theorem 1.5.3. Induction on n . If $n = 1$ then $\alpha_0 x_0 + \alpha_1 x_1 = 0$, so all solutions are proportional to $(1, -\frac{\alpha_0}{\alpha_1})$. Now suppose the claim holds for all S -unit equations in $\leq n$ variables. As x_i are S -units we have

$$|x_i|_v = 1 \forall v \notin S.$$

By the product formula (PF)

$$1 = \prod_{v \in M_{\mathbb{Q}}} |x_i|_v = \prod_{v \in S} |x_i|_v,$$

and thus

$$\prod_{v \in S} \prod_{i=0}^n |x_i|_v = 1.$$

Let $\tilde{x} = (x_0, \dots, x_{n-1})$. For each $v \in S$ pick $i(v)$ with $0 \leq i(v) \leq n-1$. So we get $n^{\#S}$ such tuples $(i(v))_{v \in S}$. Choose one of those tuples and consider all solutions of $\alpha_0 x_0 + \dots + \alpha_n x_n = 0$ with

$$|\tilde{x}|_v = |x_{i(v)}|_v$$

Choose the set of linear forms

$$\{L_{v_j} : 1 \leq j \leq n\} = \{X_0, X_1, \dots, X_{n-1}, \frac{\alpha_0}{\alpha_n} X_0 + \dots + \frac{\alpha_{n-1}}{\alpha_n} X_{n-1}\} \setminus \{X_{i(v)}\}$$

Then

$$\prod_{v \in S} \prod_{j=1}^n \frac{|L_{v_j}(\tilde{x})|_v}{|\tilde{x}|_v} = \frac{1}{H_{\mathbb{P}^{n+1}}(\tilde{x})n+1}$$

By Theorem 1.5.3 the solutions \tilde{x} lie in finitely many proper subspaces. Take one of these then all elements in this subspace satisfy an equation

$$c_0 x_0 + \dots + c_{n-1} x_{n-1} = 0 (c_i \in \mathbb{Q}, \text{ not all } = 0!)$$

Let J_0 be the set of i with $c_i \neq 0$. Then

$$\sum_{i \in J_0} c_i x_i = 0 \text{ marker}(S) \tag{47}$$

is an S -unit equation in $\leq n$ unknowns. For every solution of "marker(S)" there is a set $J \subset J_0$, $J \neq \emptyset$, such that

$$\sum_{i \in J} c_i x_i = 0$$

and no sub-sum vanishes. By the induction hypotheses, up to proportionality, we get only finitely many solutions. Moreover, the number of possible choices J is finite. Therefore it suffices to consider solutions $\{x_i\}_{i \in J}$ that are proportional to a fixed $\{u_i\}_{i \in J}$, i.e., $x_i = \xi u_i (i \in J)$. Returning to our initial equation $\sum_{i=0}^n \alpha_i x_i = 0$ we get

$$\xi \left(\sum_{i \in J} \alpha_i u_i \right) + \sum_{i \notin J} \alpha_i x_i = 0$$

If $\sum_{i \in J} \alpha_i x_i \neq 0$ then the above is an S -unit equation in $1 + (n+1) - \#J \leq n$ unknowns, namely $\xi, x_i (i \notin J)$. By the induction hypothesis we get only finitely many non-proportional solutions $\{x_i\}_{i=0}^n$ for which no sub-sum vanishes. Finally, if $\sum_{i \in J} \alpha_i x_i = 0$ then $\sum_{i \notin J} \alpha_i x_i = 0$ and we ignore these solutions by assumption of the Theorem. \square

1.6 6. Further generalizations and open problems

Let $\alpha, \beta \in \mathbb{R} \setminus \mathbb{Q}$ and consider the linearly independent linear forms $L_1(\underline{x}) = x_0\alpha - x_1$, $L_2(\underline{x}) = x_0\beta - x_2$, $L_3(\underline{x}) = x_0$. If α, β are algebraic then Theorem 1.5.1 implies that the solutions $\underline{x} \in \mathbb{Z}^3 \setminus \underline{0}$ of

$$|L_1(\underline{x}) \cdot L_2(\underline{x}) \cdot L_3(\underline{x})| < \|\underline{x}\|^{-\delta} \quad (\delta > 0)$$

lie in finitely many proper subspaces of \mathbb{Q}^3 .

However, in the following is a long-standing conjecture.

Conjecture 1.6.1 (Littlewood-conjecture, around 1920). *Let $\alpha, \beta \in \mathbb{R} \setminus \mathbb{Q}$ and $\varepsilon > 0$. Then $\exists \underline{x} \in \mathbb{Z}^3$ such that*

$$0 < |L_1(\underline{x})L_2(\underline{x})L_3(\underline{x})| < \varepsilon$$

Remark. *Note that the conjecture is obviously true if $\alpha = \beta$ (by Corollary 1.1.2) or if α or β are not badly approximable.*

Let's consider again approximations to one real α . So far our approximations were $\frac{p}{q} \in \mathbb{Q}$. If we replace \mathbb{Q} by a smaller or larger set then we get now interesting problems.

Open Problem 1.6.2 (1.6.2). *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $\lambda < 2$. Does $\left| \alpha - \frac{p}{q} \right| < q^{-\lambda}$ have ∞ -many solutions $(p, q) \in \mathbb{Z} \times \mathbb{N}$ with:*

- *p and q are both square-free.*
Best-result (Hoath-Brown 1984): Yes, if $\lambda < \frac{5}{3}$.
- *q is prime?*
Best-result (Matomaki, 2009): Yes, if $\lambda < \frac{4}{3}$.

Let's now consider problems in which \mathbb{Q} is replaced by a certain subset A of

$$\overline{\mathbb{Q}} = \{\alpha \in \mathbb{C} : \alpha \text{ algebraic}\}.$$

If we assume that $A \subset \mathbb{R}$ then we still can use the (usual) absolute value on \mathbb{R} to measure

$$|\alpha - x| \quad (x \in A).$$

But usually we have no "natural denominators" for $x \in A$. But there is a natural way to interpret the original setting that easily generalizes from $A = \mathbb{Q}$ to $A = \overline{\mathbb{Q}} \cap \mathbb{R}$. To this end we introduce the so-called multiplicative absolute Weil height:

$$H : \mathbb{Q} \rightarrow [1, \infty)$$

defined by

$$H(\alpha) = M(D_\alpha(x))$$

where $D_\alpha(x) = a_0(x - \alpha_1) \dots (x - \alpha_d) \in \mathbb{Z}[x]$ is the minimal polynomial of α and

$$M(D_\alpha(x)) = |a_0| \cdots \prod_{i=1}^n \max\{1, |\alpha_i|\}$$

M is called the Mahler-measure.

Example. • $\alpha = \frac{p}{q} \in \mathbb{Q}$ ($q > 0$, $\gcd(p, q) = 1$).

$\deg(\alpha) = 1$, $D_\alpha = qx - p$. So $H(\alpha) = M(D_\alpha) = q \max\{1, \left|\frac{p}{q}\right|\} = \max\{q, |p|\} = H_{\mathbb{P}^1}((1, \alpha))$.

• $\alpha = 2^{\frac{1}{d}}$, $D_\alpha = x^d - 2$ (2-Eisenstein), $\deg \alpha = d$, $H(\alpha) = M(D_\alpha)^{\frac{1}{d}} = \prod_{i=1}^d \max\{1, \left|\xi_d^{i-1} 2^{\frac{1}{d}}\right|\} = 2^{\frac{1}{d}}$

One can easily show (c.f. sheet 4) that

$$\#\{\alpha \in \overline{\mathbb{Q}} : \deg d \leq d, H(\alpha) \leq X\} < \infty \quad \forall d \in \mathbb{N} \quad X \geq 1. \quad (48)$$

Back to Diophantine approximation with $A = \mathbb{Q}$. As

$$a + m - \frac{p}{q} = a - \left(\frac{p - mq}{q}\right)$$

we can assume $\alpha \in (0, 1)$ So all good enough approximations $\frac{p}{q}$ lie also in $(0, 1)$. Now if $\frac{p}{q} \in (0, 1)$ then

$$H\left(\frac{p}{q}\right) = q.$$

So

$$\left|\alpha - \frac{p}{q}\right| < \phi(q) \iff \left|\alpha - \frac{p}{q}\right| < \phi\left(H\left(\frac{p}{q}\right)\right).$$

So now the denominator plays no role any more and we can write more easily:

$$|\alpha - x| < \phi(H(x)) \quad (49)$$

(49) makes sense as long as $x - \alpha \in \mathbb{R}$, so $x \in \mathbb{R}$, and $x \in \overline{\mathbb{Q}}$. So let's assume $A \subset \overline{\mathbb{Q}} \cap \mathbb{R}$. However, if x_1, x_2, x_3, \dots is a sequence of pairwise distinct solutions of (49) then we want to conclude that $x_i \rightarrow \alpha$ (with respect to $|\bullet|$). Now as $\phi(t) \rightarrow 0$ as $t \rightarrow \infty$ but we don't know a priori that $H(x_i) \rightarrow \infty$. So cannot conclude from (49) that $x_i \rightarrow \alpha$. But if $A \subset \mathbb{Q}_{(d)} = \{\alpha \in \overline{\mathbb{Q}} : \deg \alpha \leq d\}$ then (48) tells us that $H(x_i) \rightarrow \infty$ and so $x_i \rightarrow \alpha$.

More generally this is true if

$$\#\{\alpha \in A : H(\alpha) \leq X\} < \infty \quad \forall X \geq 1.$$

In this case we say A has property N.

Theorem 1.6.3 (1.6.3 Wirsing 1961). Let $d \in \mathbb{N}$, $d > 1$ and $\alpha \in \mathbb{R} \setminus \mathbb{Q}_{(d)}$. Then $\exists \infty$ -many $x \in \mathbb{Q}_{(d)}$ with

$$|\alpha - x| < H(x)^{-\left(\frac{d+3}{2}\right)}$$

Conjecture 1.6.4 (1.6.4 Wirsing's Conjecture, around 1961). Suppose $\alpha \in \mathbb{R} \setminus \mathbb{Q}_{(d)}$, ($d \in \mathbb{N}$) and $\lambda < d + 1$ then $\exists \infty$ -many $x \in \mathbb{Q}_{(d)}$ with

$$|\alpha - x| < H(x)^{-\lambda}.$$

Theorem 1.6.5 (1.6.5 Davenport and Schmidt). Wirsing's conjecture (1.6.4) holds for $d \leq 2$.

Instead of taking $A = \mathbb{Q}_{(d)} \cap \mathbb{R}$ let's replace $\mathbb{Q}_{(d)}$ with the smallest field that contains $\mathbb{Q}_{(d)}$; let's call this field $\mathbb{Q}^{(d)}$. Unfortunately, nobody knows whether $\mathbb{Q}^{(d)}$ has property (N), except when $d \leq 2$.

Theorem 1.6.6 (1.6.6 Bombien-Zannier, 2001). $\mathbb{Q}^{(2)}$ has Property (N).

Open Problem 1.6.7 (1.6.7). Find an analogue of Corollary 1.1.2 for $A = \mathbb{Q}(2) \cap \mathbb{R}$. How quickly can $\phi : [1, \infty) \rightarrow (0, \infty)$ decay if for every $\alpha \in \mathbb{R} \setminus A$

$$|\alpha - x| < \phi(H(x))$$

has ∞ -many solutions $x \in A$? It is not difficult to show an inequality in the other direction provided α is algebraic, e.g., if $\alpha \in \overline{\mathbb{Q}} \setminus \mathbb{Q}^{(2)}$ then

$$|\alpha - x| > (2 \cdot H(\alpha)H(x))^{-\deg \alpha \cdot 2^{(2H(x))}}$$

How much can this be improved?

2 Geometry of Numbers

References:

- J.W.S. Cassels "An Introduction to the Geometry of Numbers"
- W.M. Schmidt Lecture Notes M. 785 and 1467

2.1 Basic notions

Let R be an integral domain (with 1), and $n \in \mathbb{N}$. We write $\text{Mat}_n(R) = \{n \times n \text{ matrices with entries in } R\}$ and $\text{GL}_n(R) = \{A \in \text{Mat}_n(R) : A \text{ is invertible and } A^{-1} \in \text{Mat}_n(R)\}$. Then $(\text{GL}_n(R), \cdot)$ is a group. We say $u \in R$ is a unit (in R) if $\exists u' \in R$ such that $u' \cdot u = 1$.

If $A \in \text{GL}_n(R)$ then $(\det A)^{-1} = \det A^{-1} \in R$. So $\det A$ is a unit in R . On the other hand if $A \in \text{Mat}_n(R)$ and $\det A$ is a unit in R then $A^{-1} = (\det A)^{-1} \text{adj}(A) \in \text{Mat}_n(R)$ as the adjugate matrix $\text{adj}(A)$ of A clearly is in $\text{Mat}_n(R)$. Thus we have

$$\text{GL}_n(R) = \{A \in \text{Mat}_n(R) : \det A \text{ is a unit in } R\}.$$

In particular, $\text{GL}_n(\mathbb{Z}) = \{A \in \text{Mat}_n(\mathbb{Z}) : \det A = \pm 1\}$.

Let $n \in \mathbb{N}$. A lattice Λ in \mathbb{R}^n is a set of the form

$$\Lambda = A\mathbb{Z}^n = \{Ax : x \in \mathbb{Z}^n\}$$

where $A \in \text{GL}_n(\mathbb{R})$. The column vectors of A are called a basis of Λ .

Lemma 2.1.1 (2.1.1). Let $A, B \in \text{GL}_n(\mathbb{R})$. Then

$$A\mathbb{Z}^n = B\mathbb{Z}^n \iff \exists T \in \text{GL}_n(\mathbb{Z}) \text{ such that } B = AT$$

Proof. " \Leftarrow " If $B = AT$ with $T \in \text{GL}_n(\mathbb{Z})$ then $T\mathbb{Z}^n = \mathbb{Z}^n$. Hence, $B\mathbb{Z}^n = A\mathbb{Z}^n$.
" \Rightarrow " If $A\mathbb{Z}^n = B\mathbb{Z}^n$ then each column vector of B lies in $A\mathbb{Z}^n$, thus $\exists T \in \text{Mat}_n(\mathbb{Z})$ such that $B = AT$. Similarly $\exists T' \in \text{Mat}_n(\mathbb{Z})$ such that $A = BT'$. Hence, $A = ATT'$. Thus $T' = T^{-1}$. So $T \in \text{GL}_n(\mathbb{Z})$. \square

By Lemma 2.1.1 we see that if $\Lambda = A\mathbb{Z}^n$ then $|\det A|$ is uniquely determined by Λ . We call it the determinant of Λ

$$\det \Lambda = |\det A|$$

Let b_1, \dots, b_n be a basis of Λ and $v \in \Lambda$. We set

$$F_v = [0, 1) \cdot b_1 + \dots + [0, 1) \cdot b_n + v$$

and call it a fundamental cell of Λ .

Note that

- $\det \Lambda = \text{Vol } F$
- $\mathbb{R}^n = \bigcup_{v \in \Lambda} \text{disjointwith} F_v$ is a partition of \mathbb{R}^n (cf sheet 5).

Recall that $C \subset \mathbb{R}^n$ is called convex if:

$$x, y \in C \implies tx + (1-t)y \in C \forall t \in [0, 1]$$

We say C is symmetric if:

$$x \in C \implies -x \in C$$

Recall that every convex set is measurable.

Let C be a convex, compact and symmetric set in \mathbb{R}^n with the origin in the interior of C . Let Λ be a lattice in \mathbb{R}^n . Then we define the successive minima $\lambda_1, \dots, \lambda_n$ of Λ with respect to C by

$$\lambda_i = \inf\{\lambda : \lambda C \cap \Lambda \text{ contains } i \text{ linearly independent vectors}\}.$$

Note that $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n < \infty$.

Example. • $\Lambda = \mathbb{Z}^n$, $C = [-1, 1]^n$. Then $\lambda_1 = \dots = \lambda_n = 1$.

$$\bullet \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \mathbb{Z}^2, C = [-1, 1]^2. \text{ Then } \lambda_1 = 1, \lambda_2 = 2.$$

$$\bullet \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \mathbb{Z}^2, C = [-1, 1] \times [-2, 2]. \text{ Then } \lambda_1 = \lambda_2 = 1.$$

Theorem 2.1.2. Let Λ be in \mathbb{R}^n . Then Λ is a lattice in \mathbb{R}^n if and only if:

- i) $(\Lambda, +)$ is a group
- ii) Λ contains n linearly independent vectors
- iii) Λ is discrete, $\#S \cap \Lambda < \infty \forall$ compact $S \subset \mathbb{R}^n$.

Proof. First suppose Λ is a lattice. Then i) and ii) are clear. And iii) is clear, at least if $\Lambda = \mathbb{Z}^n$. But if $\Lambda = A\mathbb{Z}^n$ then $\#\Lambda \cap S = \#\mathbb{Z}^n \cap A^{-1}S$ as $x \mapsto A^{-1}x$ is continuous we have S compact $\implies A^{-1}S$ compact. So this proves the first direction.

Now let's suppose $\Lambda \subset \mathbb{R}^n$ such that i), ii), iii) hold. We use induction on n . Let $n = 1$. Then Λ contains a non-zero vector b that is closest to the origin (using ii) and iii)). By i) we easily see that $\Lambda = b\mathbb{Z}$. So Λ is a lattice in \mathbb{R}^1 .

Now suppose the claim holds in \mathbb{R}^m if $m < n$. Let u_1, \dots, u_n be n linearly independent vectors in Λ . Consider the subspace $U = \langle u_1, \dots, u_{n-1} \rangle_{\mathbb{R}}$; thus $\dim U = n - 1$. Let $\tilde{e}_1, \dots, \tilde{e}_{n-1}$ be an orthonormal basis of U . Let

$$O \in O_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) : A^T A = I_n\} \text{ (the orthogonal group)}$$

with

$$O(\tilde{e}_i) = e_i \text{ (} 1 \leq i \leq n - 1 \text{)}$$

where e_1, \dots, e_n is the canonical basis of \mathbb{R}^n . Hence, $O(U) = \mathbb{R}^{n-1} \times \{0\}$, and

$$O(\Lambda \cap U) \subset \mathbb{R}^{n-1} \times \{0\}$$

is a discrete additive group that contains the $n - 1$ linearly independent vectors $O(u_1), \dots, O(u_{n-1})$. Let $\Pi : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$, $\Pi(x) = (x_1, \dots, x_{n-1})$. Then $\Pi \circ O(\Lambda \cap U)$ is also a discrete additive group that contains $n - 1$ linearly independent vectors and it is also in \mathbb{R}^{n-1} . Hence, by induction hypothesis $\Pi \circ O(\Lambda \cap U)$ is a lattice in \mathbb{R}^{n-1} . So $\exists A_{n-1} \subset GL_{n-1}(\mathbb{R})$ such that $\Pi \circ O(\Lambda \cap U) = A_{n-1} \mathbb{Z}^{n-1}$. So

$$O(\Lambda \cap U) = \left(\underbrace{\begin{pmatrix} A_{n-1} & 0 \\ 0 & 0 \end{pmatrix}}_{\tilde{A}} \right) \mathbb{Z}^n$$

Now let $\mu = \inf\{|w_n| : w = (w_1, \dots, w_n) \in O(\Lambda \setminus U)\}$. Suppose $v_1, v_2, v_3, \dots \in O(\Lambda \setminus U)$ with

$$|v_{in}| \rightarrow \mu \text{ (as } i \rightarrow \infty \text{)}$$

where v_{in} is the last coordinate.

Adding elements from $O(U)$ does not change the last coordinate. Hence, we can assume that the vectors

$$(v_{i1}, \dots, v_{in-1}) \in [0, 1)a_1 + \dots + [0, 1)a_{n-1}$$

where a_i = column vector of A_{n-1} . In particular, the first $n - 1$ coordinates of v_i are bounded in absolute value. But the absolute value of the last coordinate also tends to μ , so is also bounded. As Λ is discrete by iii) also $O(\Lambda)$ is discrete. Thus the sequence v_i contains only finitely many vectors, in particular

$$\exists v \in O(\Lambda \setminus U) \text{ such that } v_n \xrightarrow{\text{after } v \rightarrow -v} \mu \text{ and } \mu > 0.$$

Let $u \in O(\Lambda)$. Then also $u' = u - \left\lfloor \frac{u_n}{\mu} \right\rfloor \cdot v$ is in $O(\Lambda)$ ($O(\Lambda)$ is a group as Λ is). So $0 \leq u'_n < \mu$, so by minimality of μ , $u'_n = 0$. Hence, $u' \in O(\Lambda) \cap \mathbb{R}^{n-1} \times \{0\} = O(\Lambda) \cap O(U) = O(\Lambda \cap U)$. Now

$$u = u' + \left\lfloor \frac{u_n}{\mu} \right\rfloor \cdot v \in O(\Lambda \cap U) + \mathbb{Z} \cdot v$$

and thus

$$u \in \underbrace{\tilde{A} \mathbb{Z}^n + v \cdot \mathbb{Z}}_{= AZ^n} = AZ^n$$

where $\tilde{a}_i = i$ -th column vector of \tilde{A} . Thus $O(\Lambda) \subset AZ^n$. Clearly (as $O(\Lambda)$ is a group) also $AZ^n \subset O(\Lambda)$. Now the rows of A are linearly independent. Thus $A \in GL_n(\mathbb{R})$. So $O(\Lambda)$ is a lattice and thus Λ is a lattice. \square

Corollary 2.1.3 (2.1.3). Let $n \in \mathbb{N}$, $m_1, \dots, m_n \in \mathbb{N}$ and $a_{ij} \in \mathbb{Z}$ ($1 \leq i, j \leq n$). Then

$$\Lambda = \{x \in \mathbb{Z}^n : \sum_{j=1}^n a_{ij}x_j \equiv 0 \pmod{m_i} (1 \leq i \leq n)\}$$

is a lattice in \mathbb{R}^n .

Proof. As $\Lambda \subset \mathbb{Z}^n$, it is discrete. Clearly $0 \in \Lambda$, and if $x_1, x_2 \in \Lambda$ then $x_1 + x_2 \in \Lambda$. So Λ is a discrete additive subgroup of \mathbb{Z}^n . Moreover, the n linearly independent vectors $(m, 0, \dots, 0), \dots, (0, \dots, 0, m)$ where $m = m_1 \cdots m_n$ are all in Λ . Hence, by Theorem 2.1.2 we conclude that Λ is a lattice in \mathbb{R}^n . \square

2.2 The Theorems of Blichfeldt and Minkowski

Minkowski's First and Second Theorem are possibly some of the most useful theorems in number theory. We will deduce Minkowski's First Theorem via Blichfeldt's Theorem which is of interest for its own sake.

Theorem 2.2.1 (Blichfeldt). Let Λ be a lattice in \mathbb{R}^n , and let $S \subset \mathbb{R}^n$ be measurable such that $\text{Vol } S > \det \Lambda$ ($\text{Vol } S = \infty$ is allowed). Then

$$\exists x_1, x_2 \in S, x_1 \neq x_2 \text{ and } x_1 - x_2 \in \Lambda.$$

Proof. Let b_1, \dots, b_n be a basis of Λ and let $F = [0, 1)b_1 + \dots + [0, 1)b_n$ be a fundamental cell. Thus $\text{Vol } F = \det \Lambda$, and if $x \in \mathbb{R}^n$ then \exists unique $v \in \Lambda$ and $\theta \in F$ such that

$$x = v + \theta.$$

Now for each $v \in \Lambda$ consider

$$\mathcal{R}(v) = \{\theta \in F : v + \theta \in S\}$$

Hence,

$$\sum_{v \in \Lambda} \text{Vol}(\mathcal{R}(v)) = \text{Vol } S$$

Now if $\text{Vol } S > \det \Lambda$ then (2.1) implies $\sum_{v \in \Lambda} \text{Vol}(\mathcal{R}(v)) > \det \Lambda = \text{Vol } F$. But $\cup_{v \in \Lambda} \mathcal{R}(v) \subset F$; so the union cannot be disjoint. Thus $\exists v_1, v_2 \in \Lambda, v_1 \neq v_2$ such that $\theta_0 \in \mathcal{R}(v_1) \cap \mathcal{R}(v_2)$. Hence, the points $x_1 = v_1 + \theta_0, x_2 = v_2 + \theta_0$ are both in S and $x_1 - x_2 = v_1 - v_2 \in \Lambda \setminus 0$. \square

Theorem 2.2.2 (Minkowski's First Theorem). Let Λ be a lattice in \mathbb{R}^n , and let $S \subset \mathbb{R}^n$ be convex and symmetric. Suppose that either

- $\text{Vol } S > 2^n \det \Lambda$ ($\text{Vol } S = \infty$ allowed)
- or
- $\text{Vol } S \geq 2^n \det \Lambda$ and S is compact

Then S contains a non-zero lattice point.

Remark. 2^n is sharp, take $\Lambda = \mathbb{Z}^n$ and $S = (-1, 1)^n$, then $\text{Vol } S = 2^n$, $\det \Lambda = 1$, S is symmetric and convex, but $S \cap \Lambda = \{0\}$.

Proof. First suppose $\text{Vol } S > 2^n \cdot \det \Lambda$. Now $\text{Vol}(\frac{1}{2}S) = 2^{-n} \text{Vol } S > \det \Lambda$. By Theorem 2.2.1 (with the set $\frac{1}{2}S$) we see that $\exists x_1, x_2 \in S$, $x_1 \neq x_2$ such that $\frac{1}{2}x_1 - \frac{1}{2}x_2 \in \Lambda \setminus 0$. But S is symmetric, thus $-x_2 \in S$. As S is convex we conclude that $\frac{1}{2}x_1 + \frac{1}{2}(-x_2) \in S$. This proves the first part. Now suppose S is compact and $\text{Vol } S = 2^n \det \Lambda$. If $v \in \Lambda \setminus S$ then $\exists \varepsilon_v > 0$ such that $B_{\varepsilon_v}(v) \cap S = \emptyset$ (S^c is open!) where $B_r(y) = \{x \in \mathbb{R}^n : |x - y| < r\}$. As S is compact $\exists R > 0$ such that $\lambda S \subset B_R(0)$ for all λ with $0 < \lambda < 2$. So $(\Lambda \setminus S) \cap B_R(0)$ is finite by Theorem 2.1.2 and hence $\exists \varepsilon > 0$ such that

$$B_\varepsilon(v) \cap S = \emptyset \quad \forall v \in (\Lambda \setminus S) \cap B_R(0).$$

Hence, $\exists \lambda > 1$ such that

$$\lambda S \cap \Lambda = S \cap \Lambda.$$

By the first part we know that λS contains a non-zero lattice point, and this completes the proof. \square

Corollary 2.2.3. *Let Λ be a lattice in \mathbb{R}^n and let $a_{ij} \in \mathbb{R}$ ($1 \leq i, j \leq n$). Suppose $c_1, \dots, c_n > 0$ and $c_1 \dots c_n \geq |\det A| \det \Lambda$. Then $\exists u \in \Lambda \setminus 0$ such that (2.2)*

$$\begin{cases} \left| \sum_{j=1}^n a_{1j} u_j \right| \leq c_1 \\ \left| \sum_{j=1}^n a_{ij} u_j \right| < c_i \quad (2 \leq i \leq n). \end{cases}$$

Proof. First suppose $\det A \neq 0$. Then $\mathcal{L} = A - \Lambda$ is a lattice in \mathbb{R}^n with $\det \mathcal{L} = |\det A| \det \Lambda$. Then (2.2) means we are looking for a non-zero lattice point $x \in \mathcal{L}$ such that

$$\begin{aligned} |x_1| &\leq c_1 \\ |x_i| &< c_i \quad (2 \leq i \leq n) \end{aligned}$$

These inequalities define a symmetric, convex set of points $x \in \mathbb{R}^n$ with volume $= 2^n c_1 \dots c_n$. So if $c_1 \dots c_n > |\det A| \det \Lambda$ then we can apply Theorem 2.2.2 and the claim follows at once. Next let $0 < \varepsilon < 1$. Then the set

$$\begin{aligned} S_\varepsilon : |x_1| &\leq c_1 + \varepsilon < c_1 + 1 \\ |x_i| &< c_i \quad (2 \leq i \leq n) \end{aligned}$$

still has a non-zero lattice point in \mathcal{L} . But these sets S_ε all lie in S_1 which lies in compact set and hence has only finitely many lattice points. Hence, there must be a non-zero lattice point $o \in \mathcal{L}$ in S_0 . This proves the Corollary if $\det A \neq 0$. Now if $\det A = 0$ then (2.2) defines a set of points $u \in \mathbb{R}^n$ of infinite volume and so Theorem 2.2.2 applies again and yields the claim. \square

Corollary 2.2.4 (Lagrange). *Every positive integer is the sum of four squares.*

Proof. First we observe that

$$(x_1^2 + x_2^2 + x_3^2 + x_4^2) \cdot (y_1^2 + y_2^2 + y_3^2 + y_4^2) = (x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4)^2 + (-x_1 y_2 + x_2 y_1 - x_3 y_4 + x_4 y_3)^2 + (-x_1 y_3 + x_2 y_4 + x_3 y_1 - x_4 y_2)^2 + (-x_1 y_4 + x_2 y_3 + x_3 y_2 - x_4 y_1)^2$$

Now $1 = 1^2 + 0^2 + 0^2 + 0^2$. So it suffices to prove the claim for primes p . Now $2 = 1^2 + 1^2 + 0^2 + 0^2$; so we can assume $p \neq 2$. Now a^2 and $-(b^2 + 1)$ run through exactly $\frac{p+1}{2}$ distinct residue classes modulo p as $a \bmod p$ run through an entire

system of residue classes. $(0^2, 1^2, \dots, (\frac{p-1}{2})^2)$ are all distinct in \mathcal{F}_p . Hence, they have a common residue class; thus

$$\exists a, b \in \mathbb{Z} \text{ such that } a^2 + b^2 + 1 \equiv 0 \pmod{p}.$$

With this choice of $a \pmod{b}$ consider

$$\Lambda = A \cdot \mathbb{Z}^4 \text{ where } A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a & b & p & 0 \\ b & -a & 0 & p \end{pmatrix}.$$

So Λ is a lattice in \mathbb{R}^4 with $\det \Lambda = p^2$. Next consider the convex, symmetric set

$$S = \{(x_1, \dots, x_4) \in \mathbb{R}^4 : x_1^2 + \dots + x_4^2 < 2p\}$$

Then $\text{Vol } S = \frac{\pi^2}{2}(2p)^4 = (2\pi)^2 p^4 > 16 \cdot p^2 = 2^4 \det \Lambda$. By Theorem 2.2.2 $\exists x \in \Lambda \cap S$, $x \neq 0$. Now

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = A \cdot z = \begin{pmatrix} z_1 \\ z_2 \\ az_1 + bz_2 + pz_3 \\ bz_1 - az_2 + pz_4 \end{pmatrix} \quad (z \in \mathbb{Z}^4 \setminus 0)$$

Hence,

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 \equiv z_1^2 + z_2^2 + (az_1 + bz_2)^2 + (bz_1 - az_2)^2 = \underbrace{(1 + a^2 + b^2)}_{\equiv 0 \pmod{p}} (z_1^2 + z_2^2) \equiv 0 \pmod{p}$$

Since $x \in S$, and $x \neq 0$ we conclude that $x_1^2 + \dots + x_4^2 = p$. This proves the corollary. \square

2.3 3 Basis reduction

Let M be a lattice in \mathbb{R}^n and let Λ be a sublattice of M , i.e., $\Lambda \subset M$ and Λ is a lattice in \mathbb{R}^n . Hence, there exists a matrix $C \in \text{Mat}_n(\mathbb{Z})$, $\det C \neq 0$ such that $\Lambda = C \cdot M$. We define the index of Λ in M by

$$I = |\det C| = \frac{\det \Lambda}{\det M}.$$

Note that $I \cdot C^{-1} = \text{adj}(C) \in \text{Mat}_n(\mathbb{Z})$, and thus $I \cdot M = IC^{-1}\Lambda \subset \Lambda$, so

$$IMC \subset \Lambda \subset M \tag{50}$$

Theorem 2.3.1 (.). *Let Λ be a sublattice of the lattice $M \subset \mathbb{R}^n$, and let b_1, \dots, b_n be a basis of M . Then there exists a basis a_1, \dots, a_n of Λ with*

$$a_1 = v_{11}b_1 \tag{51}$$

$$a_2 = v_{21}b_1 + v_{22}b_2 \tag{52}$$

$$\vdots \tag{53}$$

$$a_n = v_{n1}b_1 + \dots + v_{nn}b_n \tag{54}$$

with $v_{ij} \in \mathbb{Z}$, $v_{ii} \neq 0$ ($1 \leq j \leq i \leq n$). Conversely, if a_1, \dots, a_n is a basis of Λ then there exists a basis b_1, \dots, b_n of M such that (51) holds.

Proof. By (50) we know that $\exists v_{ij} \in \mathbb{Z}$ with $v_{ii} \neq 0$ and $|v_{ii}|$ minimal such that

$$a_i = v_{i1}b_1 + \cdots + v_{ii}b_i \in \Lambda$$

We will show that a_1, \dots, a_n is a basis for Λ . Let $c \in \Lambda$ and suppose c is not a \mathbb{Z} -linear combination of the a_i 's. As $c \in M \exists t_i \in \mathbb{Z}$ such that

$$c = t_1b_1 + \cdots + t_kb_k \quad (1 \leq k \leq n, t_k \neq 0)$$

If there exist several such c 's then we choose one where k is minimal. Next we note that $v_{kk} \neq 0$. Hence, there exists a $s \in \mathbb{Z}$ such that

$$|t_k - sv_{kk}| < |v_{kk}| \quad (55)$$

Thus

$$c - sa_k = (t_1 - sv_{k1})b_1 + \cdots + (t_k - sv_{kk})b_k$$

lies in Λ (as c and a_k do) and is not a \mathbb{Z} -linear combination of the a_i 's as C is not. Thus, by minimality of k we must have $t_k - sv_{kk} \neq 0$. But then (55) contradicts the minimality of $|v_{kk}|$. Hence, c must be a \mathbb{Z} -linear combination of the a_i 's and thus a_1, \dots, a_n is a basis. This proves the first part.

For the second part suppose a_1, \dots, a_n is a basis of Λ . By the first part and (50) there exists a basis b_1, \dots, b_n of $I \cdot M \subset \Lambda$ with

$$Ib_1 = w_{11}a_1 \quad (56)$$

$$Ib_2 = w_{21}a_1 + w_{22}a_2 \quad (57)$$

$$\vdots \quad (58)$$

$$Ib_n = w_{n1}a_1 + \cdots + w_{nn}a_n \quad (59)$$

with $w_{ij} \in \mathbb{Z}$, $w_{ij} \neq 0$. Successively solving the above system for a_i we get a system as in (51) but a priori with $v_{ij} \in \mathbb{Q}$. But b_1, \dots, b_n is a basis of M and the $a_i \in M$. As the representation

$$a = t_1b_1 + \cdots + t_nb_n \quad (t_i \in \mathbb{R})$$

is unique we conclude that $v_{ij} \in \mathbb{Z}$, and this proves the second part. \square

Lemma 2.3.2 (2.3.2 Hadamard's inequality). *Let $a_1, \dots, a_n \in \mathbb{R}^n$. Then*

$$|\det(a_1, \dots, a_n)| \leq |a_1| \cdots |a_n|$$

Proof. This is geometrically obvious as volume of a parallelepiped is no larger than product of the lengths of the spanning vectors. However, here is a formal proof. If a_1, \dots, a_n are linearly dependent then trivial. Now assume a_1, \dots, a_n are linearly independent. Put $c_i = a_i$.

$$c_i = a_i - \sum_{j < i} a_j c_j |c_j|^{-2} \cdot c_j$$

Then

$$c_i \cdot c_j = 0 \quad (i \neq j) \quad (60)$$

and

$$a_i = t_{i1}c_1 + \cdots + t_{ii-1}c_{i-1} + c_i$$

By (60) and (2.3) we get

$$|a_i|^2 = a_i a_i = \left(\sum_{j=1}^{i-1} t_{ij}^2 |c_j|^2 \right) + |c_i|^2 \geq |c_j|^2$$

and $\det(a_1, \dots, a_n) = \det(c_1, \dots, c_n)$ (by linearity in columns). Moreover,

$$(\det(c_1, \dots, c_n))^2 = \det \left(\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix} \right) = \prod_{i=1}^n |c_i|^2 = \prod_{i=1}^n |a_i|^2. \quad (61)$$

□

A distance function f on \mathbb{R}^n is a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

- $f(x) \geq 0 \ \forall x \in \mathbb{R}^n$
- $f(tx) = |t|f(x) \ \forall x \in \mathbb{R}^n, \ \forall t \in \mathbb{R}$
- f is continuous.

We say C is a star body in \mathbb{R}^n if

- $C \subset \mathbb{R}^n$ and compact
- origin lies in the interior of C
- $x \in C \implies t \cdot x \in C \ (0 \leq t \leq 1)$

Remark. • For a star body C we have $t \cdot C \subset C$ for $0 \leq t \leq 1$.

- Every compact, convex $C \subset \mathbb{R}^n$ with the origin in its interior is a star body in \mathbb{R}^n .
- To every symmetric star body C in \mathbb{R}^n we can associate a distance function f_C defined by

$$f_C(x) = \inf\{\lambda : x \in \lambda \cdot C\}$$

- Note that $f_C(x) = 0 \implies x = 0$. Why? If $x \neq 0$, then there exists a $\lambda > 0$ such that $\lambda x \notin C$. Hence, $f_C(x) \geq \frac{1}{\lambda}$.
- If C is symmetric and convex then f_C is actually a norm on \mathbb{R}^n (cf sheet 5) In particular, f_C satisfies the triangle-inequality.

Lemma 2.3.3 (2.3.3). Let C be a convex, symmetric star body in \mathbb{R}^n , and let Λ be a lattice in \mathbb{R}^n with successive minima $\lambda_1, \dots, \lambda_n$ with respect to C . Then there exist linearly independent $a_1, \dots, a_n \in \Lambda$ with $f_C(a_i) = \lambda_i$. Moreover, if $a \in \Lambda$ and $f_C(a) < \lambda_j$ then a_1, \dots, a_{j-1}, a are linearly dependent.

Proof. The set $(\lambda_n + 1) \cdot C$ is compact and by definition of λ_n contains n linearly independent lattice points. By the definition of the λ_i 's it suffices to consider these points. But by Theorem ?? there are only finitely many of these, and so the claim easily follows. □

Corollary 2.3.4 (2.3.4). *Let C be a convex, symmetric star body in \mathbb{R}^n , let Λ be a lattice in \mathbb{R}^n with successive minima $\lambda_1, \dots, \lambda_n$ with respect to C . Then there exists a basis b_1, \dots, b_n of Λ such that for $j = 1, \dots, n$: $x \in \Lambda$ and $f_C(x) < \lambda_j \implies x = u_1 b_1 + \dots + u_{j-1} b_{j-1}$ for some $u_1, \dots, u_{j-1} \in \mathbb{Z}$.*

Proof. Let $a_1, \dots, a_n \in \Lambda$ be as in Lemma 2.3.3. Let $\Lambda' = (a_1, \dots, a_n)\mathbb{Z}^n$ be the sublattice of Λ with basis a_1, \dots, a_n . By Theorem 2.3.1 there exists a basis b_1, \dots, b_n of Λ with (51); so a_j is dependent only on b_1, \dots, b_j . By Lemma 2.3.3 if $f_C(x) < \lambda_j$, then $x = s_1 a_1 + \dots + s_{j-1} a_{j-1} = u_1 b_1 + \dots + u_{j-1} b_{j-1}$ with $a_i \in \mathbb{Q}$. As $x \in \Lambda$ and b_1, \dots, b_{j-1} are linearly independent we conclude that $u_1, \dots, u_{j-1} \in \mathbb{Z}$. \square

Example (Exercise). $C = B_0(1)$ and $\Lambda = \begin{pmatrix} 2 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \mathbb{Z}^5$ There exists no

basis b_1, \dots, b_5 such that $|b_i| = \lambda_i$ ($1 \leq i \leq 5$) ($\lambda_i = \lambda_i(\Lambda, C)$)

Lemma 2.3.5 (2.3.5). *Let C be a convex, symmetric star body in \mathbb{R}^n , let Λ be a lattice in \mathbb{R}^n , and let a_1, \dots, a_n be linearly independent vectors in Λ . Then there exists a basis b_1, \dots, b_n of Λ such that*

$$f_C(b_j) \leq \max\{f_C(a_j), \frac{1}{2}(f_C(a_1) + \dots + f_C(a_j))\}$$

Proof. Consider the sublattice $\Lambda' = (a_1 \dots a_n)\mathbb{Z}^n \subset \Lambda$. By Theorem 2.3.1 there exists a basis c_1, \dots, c_n of Λ such that

$$a_1 = v_{11}c_1 \tag{62}$$

$$a_2 = v_{21}c_1 + v_{22}c_2 \tag{63}$$

$$\vdots \tag{64}$$

$$a_n = v_{n1}c_1 + \dots + v_{nn}c_n \tag{65}$$

with $v_{ij} \in \mathbb{Z}$ and $v_{ii} \neq 0$. Consider

$$b_j = c_j + t_{jj-1}a_{j-1} + \dots + t_{j1}a_1$$

where $t_{ji} \in \mathbb{R}$. If b_1, \dots, b_n are in Λ then by (62) they form a basis of Λ . How do we choose t_{ji} ? If $v_{jj} = \pm 1$ then we put $b_j = \pm a_j$, which clearly is in the required form () and obviously

$$f_C(b_j) = f_C(a_j).$$

Now suppose $|v_{jj}| \geq 2$. Now solving (62) for c_j yields

$$c_j = v_{jj}^{-1}a_j + k_{jj-1}a_{j-1} + \dots + k_{j1}a_1$$

with $k_{ji} \in \mathbb{Q}$. We choose $t_{ji} \in \mathbb{Z}$ such that

$$|k_{ji} + t_{ji}| \leq \frac{1}{2}$$

Then

$$b_j \in \Lambda \text{ and } b_j = l_{jj}a_j + l_{jj-1}a_{j-1} + \dots + l_{j1}a_1$$

with

$$|l_{jj}| = |v_{jj}| \leq \frac{1}{2} \text{ and}$$

$$|l_{ji}| = |t_{ji} + k_{ji}| \leq \frac{1}{2} \text{ (} i < j \text{)}$$

Using that C is a convex, symmetric star body we have the triangle-inequality. Hence,

$$f_C(b_j) \leq f_C(l_{jj}a_j) + \dots + f_C(l_{j1}a_1) = |l_{jj}|f_C(a_j) + \dots + |l_{j1}|f_C(a_1) \leq \frac{1}{2}(f_C(a_j) + \dots + f_C(a_1)).$$

□

Corollary 2.3.6. *Let C be a convex, symmetric star body, and let Λ be a lattice in \mathbb{R}^n with successive minima $\lambda_1, \dots, \lambda_n$ with respect to C . Then there exists a basis b_1, \dots, b_n of Λ with*

$$f_C(b_j) \leq \max\{\lambda_j, \frac{1}{2}(\lambda_1 + \dots + \lambda_n)\}$$

Proof. Immediate from Lemmas 2.3.3 and 2.3.5. □

2.4 4. Minkowski's Second Theorem

Minkowski's Second Theorem is a refinement of his First Theorem and a central result in Geometry of Numbers. Let's start by rephrasing Minkowski 1.

First note if $C \subset \mathbb{R}^n$ is convex, symmetric and of positive volume then there exist $\varepsilon > 0$, $x \in C$ such that

$$B_\varepsilon(x) \subset C.$$

But then there exists $\varepsilon' > 0$ such that $B_{\varepsilon'}(0) \subset C$. So the origin lies in the interior of C . Let Λ be a lattice in \mathbb{R}^n . So we can consider the successive minima $\lambda_1, \dots, \lambda_n$ of Λ with respect to C .

Note that by definition of λ_1 :

$$\forall \varepsilon > 0: (\lambda_1 - \varepsilon)C \text{ contains } \underline{\text{no}} \text{ non-zero lattice point.}$$

Minkowski's First Theorem yields:

$$\lambda_1^n \cdot \text{Vol}(C) = \text{Vol}(\lambda_1 C) \leq 2^n \cdot \det \Lambda \quad (66)$$

On the other hand (66) and $\text{Vol } C > 2^n \det \Lambda$ implies $\lambda_1 < 1$ and hence C contains a non-zero lattice point. The following theorem is much more precise than (66)!

Theorem 2.4.1 (2.4.1 Minkowski's Second Theorem). *Let C be a convex, symmetric star body in \mathbb{R}^n , and let Λ be a lattice in \mathbb{R}^n with successive minima $\lambda_1, \dots, \lambda_n$ with respect to C . Then*

$$\frac{2^n}{n!} \det \Lambda \leq \lambda_1 \dots \lambda_n \cdot \text{Vol } C \leq 2^n \det \Lambda$$

Remark. • Both bounds are sharp. For the upper bound take $\Lambda = \mathbb{Z}^n$ and $C = [-1, 1]^n$; so $\lambda_1 = \dots = \lambda_n = 1 = \det \Lambda$, and $\text{Vol } C = 2^n$. For the lower bound take $\Lambda = \mathbb{Z}^n$ and C defined by $|x_1| + \dots + |x_n| \leq 1$. Then $\text{Vol } C = \frac{2^n}{n!}$ and $\lambda_1 = \dots = \lambda_n = 1 = \det \Lambda$.

- The upper bound is much harder to prove. We will prove Theorem 2.4.1 only for the ball $C = B_1(0)$.

Proof. (Special case $C = B_1(0)$)

Put

$$\delta(C) = \sup_M \frac{\lambda_1^n(M, C)}{\det M}$$

where the supremum runs over all lattices M in \mathbb{R}^n . By Minkowski 1 (cf (66)). We have

$$\delta(C) \leq \frac{2^n}{\text{Vol } C}.$$

We will show that if $C = B_1(0)$ then

$$\det \Lambda \leq \lambda_1 \dots \lambda_n \leq \delta(C) \det \Lambda. \quad (67)$$

In particular, as $\text{Vol}(C) \geq \frac{2^n}{n!}$, we get

$$\frac{2^n}{n!} \lambda_1 \dots \lambda_n \leq \text{Vol } C \lambda_1 \dots \lambda_n \leq 2^n \det \Lambda.$$

For the lower bound in (67) take linearly independent $a_1, \dots, a_n \in \Lambda$ with $|a_i| = \lambda_i$ using Lemma 2.3.3. For the sublattice $\Lambda' = (a_1 \dots a_n) \mathbb{Z}^n \subset \Lambda$ we have $\det \Lambda' = I \det \Lambda$, where I is the index of Λ' in Λ . By Hadamard's inequality:

$$|a_1| \dots |a_n| \geq \det \Lambda' \geq \det \Lambda.$$

Thus

$$\det \Lambda \leq \lambda_1 \dots \lambda_n.$$

Now we prove the upper bound in (67). Let b_1, \dots, b_n be a basis of Λ as in Corollary 2.3.4. As in the prove of Lemma ?? we can find mutually orthogonal vectors c_1, \dots, c_n such that

$$b_j = t_{j1}c_1 + \dots + t_{jj}c_j \quad (t_{ji} \in \mathbb{R})$$

By scaling we can assume $|c_j|^2 = 1$ ($1 \leq j \leq n$). Now

$$\sum_{j=1}^n u_j b_j = \sum_{i=1}^n \left(\sum_{j \geq i} u_j t_{ji} \right) c_i$$

thus

$$\left| \sum_{j=1}^n u_j b_j \right|^2 = \sum_{i=1}^n \left(\sum_{j \geq i} u_j t_{ji} \right)^2 \quad (68)$$

Next we show that

$$\sum_{i=1}^n \lambda_i^{-2} \left(\sum_{j \geq i} u_j t_{ji} \right)^2 \geq 1 \quad (69)$$

where $u = (u_1, \dots, u_n) \in \mathbb{Z}^n \setminus 0$. Let $u \in \mathbb{Z}^n \setminus 0$,

$$u_J \neq 0 \text{ and } u_j = 0 \text{ for } j > J. \quad (70)$$

Then $u_1 b_1 + \dots + u_n b_n, b_1, \dots, b_{J-1}$ are linearly independent and by Corollary 2.3.4 we have

$$\left| \sum_{j=1}^n u_j b_j \right|^2 \geq \lambda_J^2 \quad (71)$$

Moreover, (70) implies that summands with $j > J$ in (68) and (69) are zero. Thus, the left hand-side in (69) is equal to

$$\sum_{i \leq J} \lambda_i^{-2} \left(\sum_{i \geq j} u_j t_{ji} \right)^2 \geq \sum_{i \leq J} \lambda_J^{-2} \left(\sum_{i \geq j} u_j t_{ji} \right)^2 = \lambda_J^{-2} \left| \sum_{j=1}^n u_j b_j \right|^2 \underset{\text{by (71) and (68)}}{\geq} 1 \quad (72)$$

So if Λ' is the lattice with basis

$$b'_j = t_{j1} \lambda_1^{-1} c_1 + \dots + t_{jj} \lambda_j^{-1} c_j \quad (1 \leq j \leq n) \quad (73)$$

Then

$$\left| \sum_{j=1}^n u_j b'_j \right| \geq 1$$

for every point $\sum u_j b'_j \in \Lambda' \setminus 0$. Hence,

$$\lambda_1(\Lambda', C) \geq 1 \quad (74)$$

But

$$\det \Lambda' = \lambda_1 \det \Lambda = -1 \dots \lambda_n \det \Lambda = -1 \det \Lambda \quad (75)$$

$$\lambda_i = \lambda_i(\Lambda, C)$$

Moreover, by definition

$$\frac{\lambda_1^n(\Lambda', C)}{\det \Lambda'} \leq \sup_M \frac{\lambda_1^n(M, C)}{\det M} = \delta(C) \quad (76)$$

Combining (74), (75) and (76) we conclude

$$\lambda_1 \dots \lambda_n = \det \Lambda \frac{1}{\det \Lambda'} \leq \det \Lambda \frac{\lambda_1^n(\Lambda', C)}{\det \Lambda'} \leq \det \Lambda \cdot \delta(C) \quad (77)$$

□