Advanced Algorithms March 21, 2019

Lecture 10: Simplex Method

Notes by Ola Svensson[1]

In this lecture we consider one of the most important algorithms: the simplex method for solving linear programs. Instead of introducing a lot of notation, we explain the method by considering an example. This will clarify the description of the algorithm. We then discuss some technical issues that have to be overcome to do an accurate and fast implementation.

These lecture notes are from [1]. For an excellent description of the simplex method and further information see Chapter 5 in the book [2].

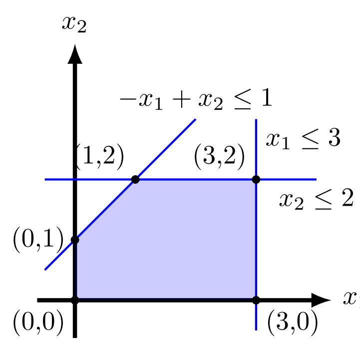
1 The Simplex Method

In this section we will talk about the simplex method which is used to solve LPs. It was invented in 1947 by George Dantzig[2] and remains one of the most important algorithm of the 20th century. Despite the fact that its running time can be proved to be exponential in the worst case, it runs very fast in practice and is widely used in industry even though we now know other algorithms that do run in polynomial time but that are slower for practical purposes.

1.1 Explanation of Simplex via an Example

We will introduce this method using a concrete example. This allows us to avoid heavy notation and to focus on the core idea of the algorithm. Let us now focus on the following LP with its associated polytope:

$$
\begin{array}{rl}
\textbf{maximize} & x_1 + x_2 \\
\textbf{subject to} & -x_1 + x_2 \le 1 \\
 & x_1 \le 3 \\
 & x_2 \le 2 \\
 & x_1,\ x_2 \ge 0
\end{array}
$$



It will be convenient to not work with inequalities and so we change the inequalities to equalities by introducing slack variables $s_1, .., s_m$ that *compensate* for the equalities. We also add a new line that will represent our current objective function which is expressed as $z =$ *objective function.* At the beginning the objective function is the one given in the original problem. Also we will remember that all variables must be non-negative including the slack variables. This gives us the following modified LP:

$$-x_1 + x_2 + s_1 = 1 \tag{1}$$

$$x_1 + s_2 = 3 \tag{2}$$

$$x_2 + s_3 = 2 \tag{3}$$

$$z = x_1 + x_2$$

[1]**Disclaimer:** These notes were written as notes for the lecturer. They have not been peer-reviewed and may contain inconsistent notation, typos, and omit citations of relevant works.

[2]`http://www.ams.org/notices/200703/fea-cottle.pdf`

In this initial configuration we set all $x_i := 0$ which implies the following assignation for the slack variables $s_1 := 1$, $s_2 := 3$, $s_3 = 2$ according to the constraints. Now the solving procedure can really start. We will maintain a *simplex tableau* which consists of all the constraints and the objective function, and we will always write the constraints such that the non-zero variables are on the left-hand side. These are called *basic variables* (and the variables on the right-hand-side are called nonbasic). This gives us:

$$s_1 = 1 + x_1 - x_2 \tag{1}$$
$$s_2 = 3 - x_1 \tag{2}$$
$$s_3 = 2 - x_2 \tag{3}$$

$$z = x_1 + x_2$$

$$x_1 := 0 \quad x_2 := 0 \quad s_1 := 1 \quad s_2 := 3 \quad s_3 := 2$$

Now the goal is to maximize the objective function $x_1 + x_2$. We will iteratively increase as much as possible one variable appearing in the objective function that has a positive coefficient (since we want to maximize). In our case let us consider x_2.[3] Clearly since there is a positive coefficient in front of it (namely 1), an increase of x_2 will get us closer to the optimal solution (remember that currently $x_1 := 0$, $x_2 := 0 \Rightarrow z = 0$) but in the process we will necessarily have to decrease other variables to preserve the equalites. We now need to check by how much we can increase x_2 without breaking non-negativity for all variables. (1) tells us that $x_2 \leq 1$, (2) says nothing about x_2 and (3) implies $x_2 \leq 2$ therefore we increase x_2 by 1.

We will compensate this increase by modifying all the variables appearing on the left-hand sides of the constraints. Note that for the constraint that imposes the value for the variable ((1) in our case) the basic variable will become zero. Therefore we decrease s_1 and s_3 by 1.

We then rewrite the constraint that dictates the value for the chosen variable (constraint (1)) by swapping the left-hand side and the variable ($s_1 \leftrightarrow x_2$). This operation is called *pivoting*. We then substitute the new right-hand side of x_2 in the other constraints and in the objective function:

$$\begin{aligned} x_2 &= 1 + x_1 - s_1 && (1) \\ s_2 &= 3 - x_1 && (2) \\ s_3 &= 2 - \underbrace{(1 + x_1 - s_1)}_{=x_2} && (3) \\ \hline z &= x_1 + \underbrace{(1 + x_1 - s_1)}_{=x_2} \end{aligned} \implies \begin{aligned} x_2 &= 1 + x_1 - s_1 && (1) \\ s_2 &= 3 - x_1 && (2) \\ s_3 &= 1 - x_1 + s_1 && (3) \\ \hline z &= 1 + 2x_1 - s_1 \end{aligned}$$

$$x_1 := 0 \quad x_2 := 1 \quad s_1 := 0 \quad s_2 := 3 \quad s_3 := 1$$

Now we see that only one variable has a positive coefficient in the objective function, namely x_1 and we will therefore try to increase it. Constraint (1) does not give any upperbound on x_1, (2) imposes $x_1 \leq 3$ and (3) enforces $x_1 \leq 1$ thus we increase x_1 by 1 which leads s_2 and s_3 to decrease by 1 and x_2 to increase by 1. We now pivote x_1 in constraint (3) and substitute its right-hand side in the rest :

[3] Note that we could have chosen to increase x_1 since it has also a positive coefficient. Actually there is a whole *industry* behind the choice of the variable to increase in each step. A simple idea could be to choose the one with the largest coefficient but there are many strategies.

$$
\begin{aligned}
x_2 &= 1 + \underbrace{(1 - s_3 + s_1)}_{=x_1} - s_1 && (1) \\
s_2 &= 3 - \underbrace{(1 - s_3 + s_1)}_{=x_1} && (2) \\
x_1 &= 1 - s_3 + s_1 && (3) \\
\hline
z &= 1 + 2\underbrace{(1 - s_3 + s_1)}_{=x_1} - s_1
\end{aligned}
\implies
\begin{aligned}
x_2 &= 2 - s_3 && (1) \\
s_2 &= 2 + s_3 - s_1 && (2) \\
x_1 &= 1 - s_3 + s_1 && (3) \\
\hline
z &= 3 - 2s_3 + s_1
\end{aligned}
$$

$$x_1 := 1 \quad x_2 := 2 \quad s_1 := 0 \quad s_2 := 2 \quad s_3 := 0$$

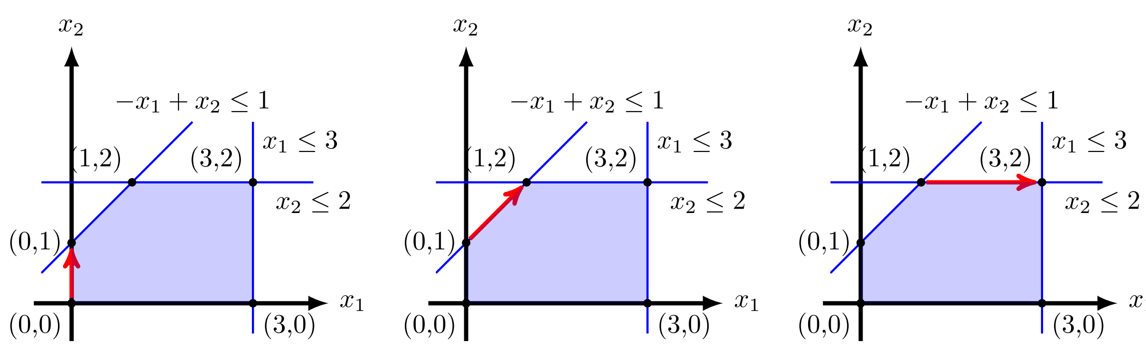
Again the only choice is to increase s_1 since it is the only variable that has a positive coefficient. (1) gives nothing about s_1, (2) dictates $s_1 \leq 2$ and (3) does not upperbound s_1. We therefore increase s_1 by 2, this implies to decrease s_2 by 2 and increase x_1 by 2, pivote s_1 and s_2 in (2) and substitute the new right-hand side of s_1.

$$
\begin{aligned}
x_2 &= 2 - s_3 && (1) \\
s_1 &= 2 + s_3 - s_2 && (2) \\
x_1 &= 1 - s_3 + \underbrace{(2 + s_3 - s_2)}_{=s_1} && (3) \\
\hline
z &= 3 - 2s_3 + \underbrace{(2 + s_3 - s_2)}_{=s_1}
\end{aligned}
\implies
\begin{aligned}
x_2 &= 2 - s_3 && (1) \\
s_1 &= 2 + s_3 - s_2 && (2) \\
x_1 &= 3 - s_2 && (3) \\
\hline
z &= 5 - s_3 - s_2
\end{aligned}
$$

$$x_1 := 3 \quad x_2 := 2 \quad s_1 := 2 \quad s_2 := 0 \quad s_3 := 0$$

Now we cannot increase any variables since they all have negative coefficients which would deteriorate the objective function. Therefore we stop the procedure and we can read the optimal solution $z = 5 - 0 - 0 = 5$ achieved with $x_1 = 3$, $x_2 = 2$. (We ignore the values of the slack variables).

If we look carefully at the values taken by x_1 and x_2 throughout the procedure we can see that at each step we actually follow an edge of the polytope that brings us closer to the optimal solution:



How can we be sure that this is the optimal solution? Let us consider any feasible solution $\{\bar{x_1}, \bar{x_2}, \bar{s_1}, \bar{s_2}, \bar{s_3}\}$. We know that any such feasible solution has to satisfy the equalities in the tableau and since all variables must be non-negative it is clear that the objective function is upperbounded by 5: $z = 5 - \bar{s_3} - \bar{s_2} \leq 5$.

Therefore we cannot hope to achieve better than 5 in the objective function and the values we got for x_1 and x_2 lead exactly to 5.

1.2 Technicalities

We present here some problems that can arise when trying to apply the simplex method. We will not go into the details on how to get around these issues but it is good to be aware of them.

Unboundness : It might happen that the chosen variable to increase is not bounded by any constraint. This is the case when the polytope defined by the original constraints (the ones with inequalities) is not bounded and the optimal solution is infinite.

Degeneracy : During the execution of the simplex algorithm it might happen that we cannot increase any variable but still need to pivot two of them in order to proceed. The swapped variable will become a basic variable (on the left-hand side) even though its current value will be zero. This does not prevent the algorithm to find the final solution but we have to be careful not to cycle if we keep pivoting the same variables without changing their values. This can be avoid using different strategies such as a lexicographic ordering of the variables.

Initial vertex : In our exemple it was quite clear that $x_1 := 0$, $x_2 := 0$ was a feasible starting solution. In the general case the initial assignment is not trival and demands to solve another linear program to get the starting values. This is called the Phase I of the simplex method followed by the Phase II which is what we showed above.

Infeasibility : If the constraints define an empty polytope then there is no feasible solution. This can be detected easily by Phase I of the simplex algorithm.

References

[1] Vincent Eggerling and Simon Rodriguez. Lecture 11 in topics in tcs 2015. `http://theory.epfl.ch/courses/topicstcs/Lecture112015.pdf`.

[2] Jiri Matousek and Bernd Gärtner. *Understanding and using linear programming.* Springer Science & Business Media, 2007.