# MAT300 CURVES AND SURFACES

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# Splines

# Continuous piecewise polynomials

## Definition

Let $x_0 < x_1 < \ldots < x_{n-1} < x_n \in \mathbb{R}$. The set of continuous piecewise polynomials $p : [x_0, x_n] \to \mathbb{R}$ given as

$$p(x) = \begin{cases} p_1(x), & x \in [x_0, x_1), \\ p_2(x), & x \in [x_1, x_2), \\ \vdots \\ p_n(x), & x \in [x_{n-1}, x_n], \end{cases}$$

with $p_i \in P_k$ for $i = 1, \ldots, n$ satisfying

$$p_j(x_j) = p_{j+1}(x_j)$$

for $j = 1, \ldots, n-1$ is denoted with $P^n_{k,0}[x_0, \ldots, x_n]$.

**Theorem**

$P^n_{k,0}[x_0,\ldots,x_n]$ *is a subspace of* $P^n_k[x_0,\ldots,x_n]$.

$p, q \in P^n_{k,0}[x_0,\ldots,x_n] \to p+q \in P^n_{k,0}[x_0,\ldots,x_n]$

$p \in P^n_{k,0}[x_0,\ldots,x_n],\ \lambda \in \mathbb{R} \to \lambda p \in P^n_{k,0}[x_0,\ldots,x_n]$

To find a basis for $P^n_{k,0}[x_0,\ldots,x_n]$ we start with the standard basis for $P^n_k[x_0,\ldots,x_n]$, i.e.

$$B = \{1,\ x,\ \ldots,\ x^k,\ (x-x_1)^0_+,\ (x-x_1)^1_+,\ \ldots,\ (x-x_1)^k_+,\ \ldots,\ (x-x_{n-1})^0_+,\ (x-x_{n-1})^1_+,\ \ldots,\ (x-x_{n-1})^k_+\}$$

and delete the elements that break the continuity ($n-1$ elements)

$$B = \{1,\ x,\ \ldots,\ x^k,\ \cancel{(x-x_1)^0_+},\ (x-x_1)^1_+,\ \ldots,\ (x-x_1)^k_+,\ \ldots,\ \cancel{(x-x_{n-1})^0_+},\ (x-x_{n-1})^1_+,\ \ldots,\ (x-x_{n-1})^k_+\}$$

## Theorem

$\dim(P^n_{k,0}[x_0, \ldots, x_n]) = nk + 1$

Proof: $\dim(P^n_{k,0}[x_0, \ldots, x_n]) = \dim(P^n_k[x_0, \ldots, x_n]) - (n-1) = n(k+1) - (n-1) = nk + n - n + 1 = nk + 1$

Example: $p(x) = \begin{cases} 3 + 2x - x^2, & x \in [-1, 4) \\ -5 - 4x - 3x^2 + x^3, & x \in [4, 6] \end{cases}$

Show that $p \in P^2_{3,0}[-1, 4, 6]$.

$p \in P^2_3[-1, 4, 6]$ and $p_1(4) = -5 = p_2(4)$ so $p \in P^2_{3,0}[-1, 4, 6]$

$B = \{1, x, x^2, x^3, (x-4)^1_+, (x-4)^2_+, (x-4)^3_+\}$ basis

Find the vector of coordinates of $p$ in that basis. WHITEBOARD

# Differentiable piecewise polynomials

## Definition

Let $x_0 < x_1 < \ldots < x_{n-1} < x_n \in \mathbb{R}$. The set of differentiable piecewise polynomials $p : [x_0, x_n] \to \mathbb{R}$ given as

$$p(x) = \begin{cases} p_1(x), & x \in [x_0, x_1), \\ p_2(x), & x \in [x_1, x_2), \\ \vdots \\ p_n(x), & x \in [x_{n-1}, x_n], \end{cases}$$

with $p_i \in P_k$ for $i = 1, \ldots, n$ satisfying

$$p_j(x_j) = p_{j+1}(x_j), \qquad p_j'(x_j) = p_{j+1}'(x_j)$$

for $j = 1, \ldots, n-1$ is denoted with $P_{k,1}^n[x_0, \ldots, x_n]$.

**Theorem**

*$P^n_{k,1}[x_0, \ldots, x_n]$ is a subspace of $P^n_{k,0}[x_0, \ldots, x_n]$.*

To find a basis for $P^n_{k,1}[x_0, \ldots, x_n]$ we start with the standard basis for $P^n_k[x_0, \ldots, x_n]$, i.e.

$$B = \{1,\ x,\ \ldots,\ x^k,\ (x-x_1)^0_+,\ (x-x_1)^1_+,\ (x-x_1)^2_+,\ \ldots,\ (x-x_1)^k_+,$$

$$\ldots,\ (x-x_{n-1})^0_+,\ (x-x_{n-1})^1_+,\ (x-x_{n-1})^2_+,\ \ldots,\ (x-x_{n-1})^k_+\}$$

and delete the elements that break the continuity ($n-1$ elements) and differentiability ($n-1$ elements)

$$B = \{1,\ x,\ \ldots,\ x^k,\ \cancel{(x-x_1)^0_+},\ \cancel{(x-x_1)^1_+},\ (x-x_1)^2_+,\ \ldots,\ (x-x_1)^k_+,$$

$$\ldots,\ \cancel{(x-x_{n-1})^0_+},\ \cancel{(x-x_{n-1})^1_+},\ (x-x_{n-1})^2_+,\ \ldots,\ (x-x_{n-1})^k_+\}$$

**Theorem**

$\dim(P^n_{k,1}[x_0,\dots,x_n]) = (k-1)n+2$

Proof: $\dim(P^n_{k,1}[x_0,\dots,x_n]) = \dim(P^n_k[x_0,\dots,x_n]) - 2(n-1) =$

$n(k+1) - 2(n-1) = n(k+1-2)+2 = (k-1)n+2$

What is going on for higher order differentiability?

We can construct subspaces of piecewise polynomials twice differentiable

three times differentiable

Up till $k$ times differentiable (in that case our function will be in $C^\infty$)

# r-differentiable piecewise polynomials

**Definition**

Let $x_0 < x_1 < \ldots < x_{n-1} < x_n \in \mathbb{R}$. The set of r-differentiable piecewise polynomials $p : [x_0, x_n] \to \mathbb{R}$ given as

$$p(x) = \begin{cases} p_1(x), \ x \in [x_0, x_1), \\ p_2(x), \ x \in [x_1, x_2), \\ \vdots \\ p_n(x), \ x \in [x_{n-1}, x_n], \end{cases}$$

with $p_i \in P_k$ for $i = 1, \ldots, n$ satisfying

$$p_j^{(m)}(x_j) = p_{j+1}^{(m)}(x_j), \quad m = 0, 1, \ldots, r$$

for $j = 1, \ldots, n-1$ is denoted with $P_{k,r}^n[x_0, \ldots, x_n]$.

**Theorem**

*$P_k = P^n_{k,k}[x_0, \ldots, x_n]$ is a subspace of $P^n_{k,k-1}[x_0, \ldots, x_n]$,*

*which is a subspace of $P^n_{k,k-2}[x_0, \ldots, x_n]$,*

$\vdots$

*which is a subspace of $P^n_{k,0}[x_0, \ldots, x_n]$,*

*which is a subspace of $P^n_{k}[x_0, \ldots, x_n]$.*

To find a basis for $P^n_{k,r}[x_0, \ldots, x_n]$ we start with the standard basis for $P^n_k[x_0, \ldots, x_n]$, and delete the elements that break the differentiability till order r, so we delete $(r+1)(n-1)$ elements.

## Theorem

$$\dim(P^n_{k,r}[x_0,\ldots,x_n]) = n(k+1) - (r+1)(n-1)$$

In particular:

$$\dim(P^n_{k,k}[x_0,\ldots,x_n]) = n(k+1) - (k+1)(n-1) = k+1,$$

$$\dim(P^n_{k,k-1}[x_0,\ldots,x_n]) = n(k+1) - (k)(n-1) = n+k,$$

$$\dim(P^n_{k,k-2}[x_0,\ldots,x_n]) = n(k+1) - (k-1)(n-1) = 2n+k-1.$$

Example: determine to which vector space (the smallest one) the following polynomial belongs. Find a right shifted basis and obtain vector of coordinates of $p$ in that basis

$$p(x) = \begin{cases} x^3 - 2x^2 + x + 5, & x \in [0,2) \\ 2x^3 - 8x^2 + 13x - 3, & x \in [2,4] \end{cases}$$

$$p(x) = \begin{cases} x^3 - 2x^2 + x + 5, & x \in [0,2) \\ 2x^3 - 8x^2 + 13x - 3, & x \in [2,4] \end{cases}$$

$p \in P_3^2[0,2,4]$

Check continuity at $x = 2$

$p_1(2) = 2^3 - 2 \cdot 2^2 + 2 + 5 = 7 \qquad p_2(2) = 2 \cdot 2^3 - 8 \cdot 2^2 + 13 \cdot 2 - 3 = 7$

$p_1(2) = p_2(2)$ so $p$ is continuous, $p \in P_{3,0}^2[0,2,4]$

Check differentiability at $x = 2$

$$p'(x) = \begin{cases} 3x^2 - 4x + 1, & x \in [0,2) \\ 6x^2 - 16x + 13, & x \in [2,4] \end{cases}$$

$p_1'(2) = 3 \cdot 2^2 - 4 \cdot 2 + 1 = 5 \qquad p_2'(2) = 6 \cdot 2^2 - 16 \cdot 2 + 13 = 5$

$p_1'(2) = p_2'(2)$ so $p$ is differentiable, $p \in P^2_{3,1}[0,2,4]$

Check twice differentiability at $x = 2$

$$p''(x) = \begin{cases} 6x - 4, & x \in [0,2) \\ 12x - 16, & x \in [2,4] \end{cases}$$

$p_1''(2) = 6 \cdot 2 - 4 = 8 \qquad p_2''(2) = 12 \cdot 2 - 16 = 8$

$p_1''(2) = p_2''(2)$ so $p$ is twice differentiable, $p \in P^2_{3,2}[0,2,4]$

Check three times differentiability at $x = 2$

$$p^{(3)}(x) = \begin{cases} 6, & x \in [0,2) \\ 12, & x \in [2,4] \end{cases}$$

$p_1^{(3)}(2) = 6 \neq 12 = p_2^{(3)}(2)$ therefore $p \notin P^2_{3,3}[0,2,4]$

Find a basis for $p \in P^2_{3,2}[0,2,4]$

We start with the standard basis for $P^2_3[0,2,4]$, i.e.

$$B = \{1, x, x^2, x^3, (x-2)^0_+, (x-2)^1_+, (x-2)^2_+, (x-2)^3_+\}$$

We delete elements that break continuity and obtain basis for $P^2_{3,0}[0,2,4]$

$$B_0 = \{1, x, x^2, x^3, \cancel{(x-2)^0_+}, (x-2)^1_+, (x-2)^2_+, (x-2)^3_+\}$$

$$= \{1, x, x^2, x^3, (x-2)^1_+, (x-2)^2_+, (x-2)^3_+\}$$

We delete elements that break differentiability and obtain basis for $P^2_{3,1}[0,2,4]$

$$B_1 = \{1, x, x^2, x^3, \cancel{(x-2)^1_+}, (x-2)^2_+, (x-2)^3_+\}$$

$$= \{1, x, x^2, x^3, (x-2)^2_+, (x-2)^3_+\}$$

We delete elements that break twice differentiability and obtain basis for $P^2_{3,2}[0,2,4]$

$B_2 = \{1, x, x^2, x^3, \cancel{(x-2)^2_+}, (x-2)^3_+\} = \{1, x, x^2, x^3, (x-2)^3_+\}$

Find vector of coordinates of p in basis $B_2$

$$p(x) = \begin{cases} x^3 - 2x^2 + x + 5, & x \in [0,2) \\ 2x^3 - 8x^2 + 13x - 3, & x \in [2,4] \end{cases}$$

$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 (x-2)^3_+$

For $x \in [0,2)$ we have

$x^3 - 2x^2 + x + 5 = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

so $a_0 = 5$, $a_1 = 1$, $a_2 = -2$ and $a_3 = 1$

For $x \in [2,4]$ we have

$$2x^3 - 8x^2 + 13x - 3 = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 (x-2)^3$$

substituting $a_0 = 5$, $a_1 = 1$, $a_2 = -2$ and $a_3 = 1$ we have

$$2x^3 - 8x^2 + 13x - 3 = 5 + x - 2x^2 + x^3 + a_4(x^3 - 6x^2 + 12x - 8)$$

$x^3 - 6x^2 + 12x - 8 = a_4(x^3 - 6x^2 + 12x - 8)$ and so $a_4 = 1$

$$(5, 1, -2, 1, 1)_{B_2}$$

Polynomials of degree at most 3 are easy to compute. Moreover, piecewise polynomials twice differentiable look like smooth functions. These are reasons for considering the polynomials in $P^n_{3,2}[x_0, \ldots, x_n]$ good for interpolation purposes. Such polynomials are called cubic splines

# The cubic spline interpolation problem

## Definition

Given $n+1$ points $(x_0, y_0)$, ..., $(x_n, y_n)$ with $x_i < x_{i+1}$ for $i = 0, \ldots, n-1$, cubic spline interpolation consists of finding an interpolant polynomial $p \in P^n_{3,2}[x_0, \ldots, x_n]$ through the given points.

As we have $n+1$ constrains (equations) and $\dim(P^n_{3,2}[x_0, \ldots, x_n]) = 4n - 3(n-1) = n+3$ (unknowns) we need to impose two more conditions (equations) to have a unique solution:

$p''(x_0) = 0$ and $p''(x_n) = 0$.

**Example:** Find a cubic spline through $(0,1)$, $(1,3)$, $(2,-1)$ and $(4,0)$.

$$p(x) = \begin{cases} p_1(x), & x \in [0,1) \\ p_2(x), & x \in [1,2) \\ p_3(x), & x \in [2,4] \end{cases}$$

with $p \in P^3_{3,2}[0,1,2,4]$.

$B = \{1, x, x^2, x^3, (x-1)^3_+, (x-2)^3_+, \}$ and $p = (a_0, a_1, a_2, a_3, a_4, a_5)_B$

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 (x-1)^3_+ + a_5 (x-2)^3_+$$

$$p'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 3a_4 (x-1)^2_+ + 3a_5 (x-2)^2_+$$

$$p''(x) = 2a_2 + 6a_3 x + 6a_4 (x-1)_+ + 6a_5 (x-2)_+$$

$$\begin{cases} p(0) = 1 \\ p''(0) = 0 \\ p(1) = 3 \\ p(2) = -1 \\ p(4) = 0 \\ p''(4) = 0 \end{cases} \Rightarrow \begin{cases} a_0 = 1 \\ 2a_2 = 0 \\ a_0 + a_1 + a_2 + a_3 = 3 \\ a_0 + 2a_1 + 4a_2 + 8a_3 + a_4 = -1 \\ a_0 + 4a_1 + 16a_2 + 64a_3 + 27a_4 + 8a_5 = 0 \\ 2a_2 + 24a_3 + 18a_4 + 12a_5 = 0 \end{cases}$$

$$\left(\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 3 \\ 1 & 2 & 4 & 8 & 1 & 0 & -1 \\ 1 & 4 & 16 & 64 & 27 & 8 & 0 \\ 0 & 0 & 2 & 24 & 18 & 12 & 0 \end{array}\right) RREF \to \left(\begin{array}{c|c} & 1 \\ & \frac{173}{46} \\ I_6 & 0 \\ & -\frac{81}{46} \\ & \frac{105}{23} \\ & -\frac{153}{46} \end{array}\right)$$

$p = \left(1, \frac{173}{46}, 0, -\frac{81}{46}, \frac{105}{23}, -\frac{153}{46}\right)_B$ therefore

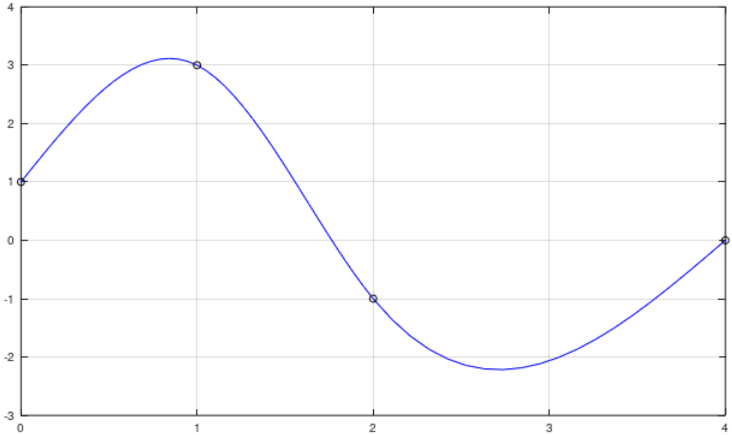
$p_1(x) = 1 + \frac{173}{46}x - \frac{81}{46}x^3$

$p_2(x) = 1 + \frac{173}{46}x - \frac{81}{46}x^3 + \frac{105}{23}(x-1)^3 = -\frac{82}{23} + \frac{803}{46}x - \frac{315}{23}x^2 + \frac{129}{46}x^3$

$p_3(x) = 1 + \frac{173}{46}x - \frac{81}{46}x^3 + \frac{105}{23}(x-1)^3 - \frac{153}{46}(x-2)^3 =$

$\frac{530}{23} - \frac{1033}{46}x + \frac{144}{23}x^2 - \frac{12}{23}x^3$

$$p(x) = \begin{cases} 1 + \frac{173}{46}x - \frac{81}{46}x^3, & x \in [0,1) \\ -\frac{82}{23} + \frac{803}{46}x - \frac{315}{23}x^2 + \frac{129}{46}x^3, & x \in [1,2) \\ \frac{530}{23} - \frac{1033}{46}x + \frac{144}{23}x^2 - \frac{12}{23}x^3, & x \in [2,4] \end{cases}$$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