DigiPen Institute of Technology, Bilbao

MAT300 Curves & Surfaces

Spring 2020. Homework 2: Deadline: 5-2-2020

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1. (20%) Consider $x_0 = -1$, $x_1 = 0$, $x_2 = 1$, $x_3 = 2$ and $x_4 = 4$. Construct the Lagrange polynomials associated to that nodes and give their vectors of coordinates in the standard basis.

2. (15%) Show that the set of Lagrange polynomials obtained in exercise 1 is a basis for a certain polynomial vector space. Which space?

3. (10%) Consider a polynomial $p : \mathbb{R} \to \mathbb{R}$ satisfying $p(x_0) = 0$, $p(x_1) = 2$, $p(x_2) = 1$, $p(x_3) = 3$ and $p(x_4) = 3$. Give the vector of coordinates of p in the Lagrange basis.

4. (20%) Construct the divided differences for the nodes $x_0 = -1$, $x_1 = 0$, $x_2 = 1$, $x_3 = 2$ and $x_4 = 4$ being $f = p$ in exercise 3.

5. (15%) Construct the Newton basis associated to the nodes $x_0 = -1$, $x_1 = 0$, $x_2 = 1$, $x_3 = 2$ and $x_4 = 4$.

6. (10%) Show that the Newton basis in exercise 5 is indeed a basis of a certain polynomial vector space. Which space?

7. (10%) Give the vector of coordinates of p (the one in exercise 3) in the Newton basis (the one in exercise 5). **Hint:** you have computed the divided differences for that polynomial in exercise 4.

(1)
$$\left.\begin{aligned} x_0 &= -1 \\ x_1 &= 0 \\ x_2 &= 1 \\ x_3 &= 2 \\ x_4 &= 4 \end{aligned}\right\} \quad 5 \text{ nodes} \Rightarrow \text{polynomials in } P_4$$

$$L_i^4(x) = \prod_{j\neq i} \frac{(x-x_j)}{(x_i-x_j)} \qquad i,j = 0,1,2,3,4$$

$$L_0^4(x) = \left(\frac{x-0}{-1-0}\right)\left(\frac{x-1}{-1-1}\right)\left(\frac{x-2}{-1-2}\right)\left(\frac{x-4}{-1-4}\right) = \frac{x}{-1}\cdot\frac{x-1}{-2}\cdot\frac{x-2}{-3}\cdot\frac{x-4}{-5}$$

$$= \frac{x^4}{30} - \frac{7x^3}{30} + \frac{7x^2}{15} - \frac{4x}{15} \qquad \left(0, \frac{-4}{15}, \frac{7}{15}, \frac{-7}{30}, \frac{1}{30}\right)_{St}$$

$$L_1^4(x) = \left(\frac{x+1}{0+1}\right)\left(\frac{x-1}{0-1}\right)\left(\frac{x-2}{0-2}\right)\left(\frac{x-4}{0-4}\right) = \frac{x+1}{1}\cdot\frac{x-1}{-1}\cdot\frac{x-2}{-2}\cdot\frac{x-4}{-4}$$

$$= -\frac{x^4}{8} + \frac{3x^3}{4} - \frac{7x^2}{8} - \frac{3x}{4} + 1 \qquad \left(1, \frac{-3}{4}, \frac{-7}{8}, \frac{3}{4}, \frac{-1}{8}\right)_{St}$$

$$L_2^4(x) = \left(\frac{x+1}{1+1}\right)\left(\frac{x-0}{1-0}\right)\left(\frac{x-2}{1-2}\right)\left(\frac{x-4}{1-4}\right) = \frac{x+1}{2}\cdot\frac{x}{1}\cdot\frac{x-2}{-1}\cdot\frac{x-4}{-3}$$

$$= \frac{x^4}{6} - \frac{5x^3}{6} + \frac{x^2}{3} + \frac{4x}{3} \qquad \left(0, \frac{4}{3}, \frac{1}{3}, -\frac{5}{6}, \frac{1}{6}\right)_{St}$$

$$L_3^4(x) = \left(\frac{x+1}{2+1}\right)\left(\frac{x-0}{2-0}\right)\left(\frac{x-1}{2-1}\right)\left(\frac{x-4}{2-4}\right) = \frac{x+1}{3}\cdot\frac{x}{2}\cdot\frac{x-1}{1}\cdot\frac{x-4}{-2}$$

$$= -\frac{x^4}{12} + \frac{x^3}{3} + \frac{x^2}{12} - \frac{x}{3} \qquad \left(0, -\frac{1}{3}, \frac{1}{12}, \frac{1}{3}, -\frac{1}{12}\right)_{St}$$

$$L_4^4(x) = \left(\frac{x+1}{4+1}\right)\left(\frac{x-0}{4-0}\right)\left(\frac{x-1}{4-1}\right)\left(\frac{x-2}{4-2}\right) = \frac{x+1}{5}\cdot\frac{x}{4}\cdot\frac{x-1}{3}\cdot\frac{x-2}{2}$$

$$= \frac{x^4}{120} - \frac{x^3}{60} - \frac{x^2}{120} + \frac{x}{60} \qquad \left(0, \frac{1}{60}, -\frac{1}{120}, -\frac{1}{60}, \frac{1}{120}\right)_{St}$$

(2) $LB = \{L_0^4, L_1^4, L_2^4, L_3^4, L_4^4\}$ is a basis for P_4.

- $\dim(P_4) = 5 = |LB|$ so we only need to verify linear independence.
- For linear independence $P_4 \simeq \mathbb{R}^5$ so to prove that the vectors of coordinates in $\mathbb{R}^5$ are linearly independent is the same as proving that the polynomials in P_4 are linearly independent. Introducing the vectors of coordinates in a matrix and computing its determinant we get

$$\det\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -4/15 & -3/4 & 4/3 & -1/3 & 1/60 \\ 7/15 & -7/8 & 1/3 & 1/12 & -1/120 \\ -7/30 & 3/4 & -5/6 & 1/3 & -1/60 \\ 1/30 & -1/8 & 1/6 & -1/12 & 1/120 \end{pmatrix} \simeq 6.94 \cdot 10^4 \neq 0$$

So the vectors are linearly independent in $\mathbb{R}^5$ and so the polynomials are linearly independent in P_4. We can conclude they form a basis for P_4.

(3) $p: \mathbb{R} \to \mathbb{R}$ $\quad p(x_0) = 0$, $p(x_1) = 2$, $p(x_2) = 1$, $p(x_3) = 3$

$p(x_4) = 3$

$$p(x) = \sum_{i=0}^{4} y_i L_i^4(x) \quad \text{therefore} \quad p = (0, 2, 1, 3, 3)_{LB}$$

(4) Divided differences:

Nodes	$f[x_i]=y_i$	$f[x_i,x_{i+1}]$	$f[x_i,x_{i+1},x_{i+2}]$	$f[x_i,x_{i+1},x_{i+2},x_{i+3}]$	$f[x_i,x_{i+1},x_{i+2},x_{i+3},x_{i+4}]$
$x_0=-1$	$f[x_0]=0$				
		$f[x_0,x_1]=\frac{2-0}{0+1}=2$			
$x_1=0$	$f[x_1]=2$		$f[x_0,x_1,x_2]=\frac{-1-2}{1+1}=-\frac{3}{2}$		
		$f[x_1,x_2]=\frac{1-2}{1-0}=-1$		$f[x_0,x_1,x_2,x_3]=\frac{\frac{3}{2}+\frac{3}{2}}{2+1}=1$	
$x_2=1$	$f[x_2]=1$		$f[x_1,x_2,x_3]=\frac{2+1}{2-0}=\frac{3}{2}$		$f[x_0,x_1,x_2,x_3,x_4]=(*)$
		$f[x_2,x_3]=\frac{3-1}{2-1}=2$		$f[x_1,x_2,x_3,x_4]=\frac{-\frac{2}{3}-\frac{3}{2}}{4-0}=-\frac{13}{24}$	
$x_3=2$	$f[x_3]=3$		$f[x_2,x_3,x_4]=\frac{0-2}{4-1}=-\frac{2}{3}$		
		$f[x_3,x_4]=\frac{3-3}{4-2}=0$			
$x_4=4$	$f[x_4]=3$				

$$(*)=\frac{-\frac{13}{24}-1}{4+1}=\frac{-37}{120}$$

(5) Newton basis

5 nodes $\rightarrow$ basis for $\mathbb{P}_4$

$BN=\{N_0, N_1, N_2, N_3, N_4\}$ where

$N_0(x)=1$

$N_1(x)=x+1$

$N_2(x)=(x+1)(x-0)=x^2+x$

$N_3(x)=(x+1)(x-0)(x-1)=x^3-x$

$N_4(x)=(x+1)(x-0)(x-1)(x-2)=x^4-2x^3-x^2+2x$

(6) $BN = \{N_0, N_1, N_2, N_3, N_4\}$ is a basis for P_4

- $\dim(P_4) = 5 = |BN|$ so we only need to prove linear independence.

Obtaining the vectors of coordinates of the Newton polynomials in the standard basis we have

$$\left.\begin{aligned} N_0 &= (1,0,0,0,0)_{St} \\ N_1 &= (1,1,0,0,0)_{St} \\ N_2 &= (0,1,1,0,0)_{St} \\ N_3 &= (0,-1,0,1,0)_{St} \\ N_4 &= (0,2,-1,-2,1)_{St} \end{aligned}\right\}$$

Vectors in $\mathbb{R}^5$.
To verify that they are linearly independent we introduce the vectors in a matrix and compute its determinant.

$$\det\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = 1 \neq 0 \rightarrow$$ Vectors are linearly independent in $\mathbb{R}^5$ $\longrightarrow$

Polynomials are linearly independent in P_4, so they form a basis.

(7) Taking the first divided differences in exercise 4, the coordinates of p are

$$\left(0, 2, -\frac{3}{2}, 1, \frac{-37}{120}\right)_{NB}$$