MAT300 CURVES AND SURFACES

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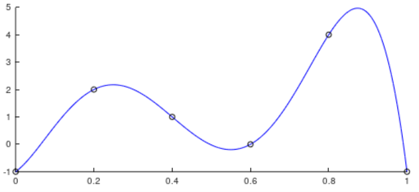
The interpolation problem

Brief introduction

Interpolation consists of determining a **continuous function** that exactly represents a **discrete collection of data**.



In this course the problem is to obtain a **polynomial curve** (or surface) passing through given points in $\mathbb{R}^2$ or $\mathbb{R}^3$.

Depending on the desired curve/surface we use a different **interpolation technique**.

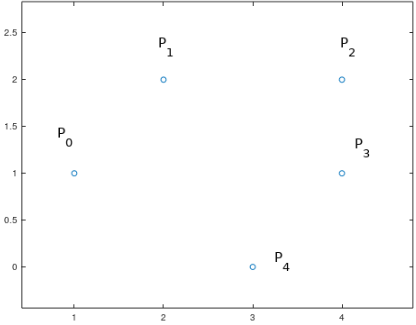
Depending on what do we want to do with the curve, we use a different **basis for the polynomial vector space**.

Example

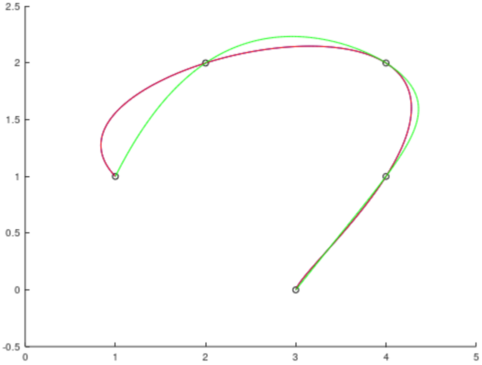
We want to obtain a curve in $\mathbb{R}^2$ through:

$P_0 = (1,1)$, $P_1 = (2,2)$, $P_2 = (4,2)$, $P_3 = (4,1)$ and $P_4 = (3,0)$

Exactly in that order!



There are infinitely many curves passing through them, for instance:



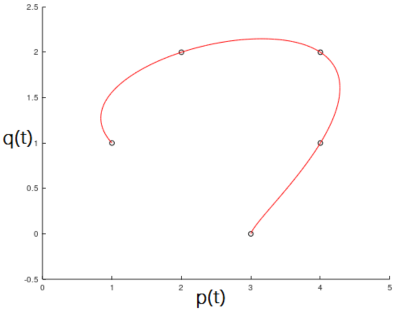
How do we obtain one of these curves?

Polynomial interpolation for curves in $\mathbb{R}^2$

Define a parametric curve $\gamma : [a, b] \to \mathbb{R}^2$ given as

$$\gamma(t) = (p(t), q(t)) \tag{1}$$

where p and q are polynomials. The graph will be

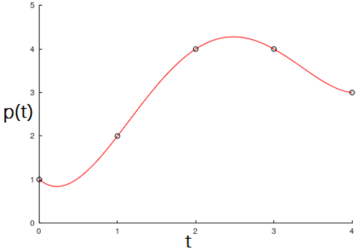


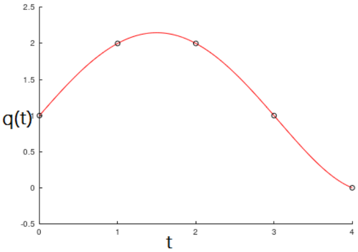
We want the curve passing through 5 points and use polynomials for each coordinate.

We take a mesh $a = t_0 < t_1 < t_2 < t_3 < t_4 = b$ (HOW?)

We divide our problem into two interpolation problems:

- finding an interpolant polynomial $p(t)$ through $(t_0, 1)$, $(t_1, 2)$, $(t_2, 4)$, $(t_3, 4)$ and $(t_4, 3)$. Unique in P_4!
- finding an interpolant polynomial $q(t)$ through $(t_0, 1)$, $(t_1, 2)$, $(t_2, 2)$, $(t_3, 1)$ and $(t_4, 0)$. Unique in P_4!

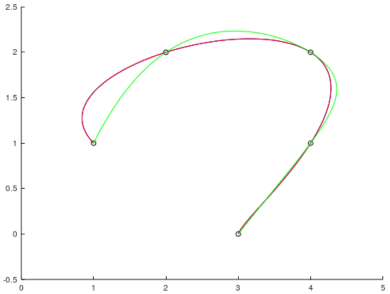




Discussion

Two important questions arise here:

- How do we select $[a, b]$ and the mesh $t_0 < t_1 < t_2 < t_3 < t_4$? this selection will determine the shape of the curve.



Red curve has a regular mesh in $[0, 4]$

Green curve has a mesh of Chebyshev extremal nodes in $[-1, 1]$

- How do we construct the curve for plotting it? this is related to speed, accuracy and number of nodes.

How do we solved the problem in MAT250

For the first interpolation problem we have:

$p(t_0) = 1,\ p(t_1) = 2,\ p(t_2) = 4,\ p(t_3) = 4$ and $p(t_4) = 3$

There is a unique interpolant $p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4$.

Construct linear system of 5 equations in 5 unknowns:

$$\begin{cases} a_0 + a_1 t_0 + a_2 t_0^2 + a_3 t_0^3 + a_4 t_0^4 = 1 \\ a_0 + a_1 t_1 + a_2 t_1^2 + a_3 t_1^3 + a_4 t_1^4 = 2 \\ a_0 + a_1 t_2 + a_2 t_2^2 + a_3 t_2^3 + a_4 t_2^4 = 4 \\ a_0 + a_1 t_3 + a_2 t_3^2 + a_3 t_3^3 + a_4 t_3^4 = 4 \\ a_0 + a_1 t_4 + a_2 t_4^2 + a_3 t_4^3 + a_4 t_4^4 = 3 \end{cases} \qquad \left(\begin{array}{ccccc|c} 1 & t_0 & t_0^2 & t_0^3 & t_0^4 & 1 \\ 1 & t_1 & t_1^2 & t_1^3 & t_1^4 & 2 \\ 1 & t_2 & t_2^2 & t_2^3 & t_2^4 & 4 \\ 1 & t_3 & t_3^2 & t_3^3 & t_3^4 & 4 \\ 1 & t_4 & t_4^2 & t_4^3 & t_4^4 & 3 \end{array}\right)$$

System has unique solution obtained by computing the RREF.

For the second interpolation problem we have:

$$q(t_0) = 1,\ q(t_1) = 2,\ q(t_2) = 2,\ q(t_3) = 1 \text{ and } q(t_4) = 0$$

There is a unique interpolant $q(t) = b_0 + b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4$.

Construct linear system of 5 equations in 5 unknowns:

$$\begin{cases} b_0 + b_1 t_0 + b_2 t_0^2 + b_3 t_0^3 + b_4 t_0^4 = 1 \\ b_0 + b_1 t_1 + b_2 t_1^2 + b_3 t_1^3 + b_4 t_1^4 = 2 \\ b_0 + b_1 t_2 + b_2 t_2^2 + b_3 t_2^3 + b_4 t_2^4 = 2 \\ b_0 + b_1 t_3 + b_2 t_3^2 + b_3 t_3^3 + b_4 t_3^4 = 1 \\ b_0 + b_1 t_4 + b_2 t_4^2 + b_3 t_4^3 + b_4 t_4^4 = 0 \end{cases} \qquad \left(\begin{array}{ccccc|c} 1 & t_0 & t_0^2 & t_0^3 & t_0^4 & 1 \\ 1 & t_1 & t_1^2 & t_1^3 & t_1^4 & 2 \\ 1 & t_2 & t_2^2 & t_2^3 & t_2^4 & 2 \\ 1 & t_3 & t_3^2 & t_3^3 & t_3^4 & 1 \\ 1 & t_4 & t_4^2 & t_4^3 & t_4^4 & 0 \end{array}\right)$$

System has unique solution obtained by computing the RREF.

The left side of the augmented matrices is the same!

We can solve both problems at the same time

$$RREF \left(\begin{array}{ccccc|c|c} 1 & t_0 & t_0^2 & t_0^3 & t_0^4 & 1 & 1 \\ 1 & t_1 & t_1^2 & t_1^3 & t_1^4 & 2 & 2 \\ 1 & t_2 & t_2^2 & t_2^3 & t_2^4 & 4 & 2 \\ 1 & t_3 & t_3^2 & t_3^3 & t_3^4 & 4 & 1 \\ 1 & t_4 & t_4^2 & t_4^3 & t_4^4 & 3 & 0 \end{array} \right) = \left(\begin{array}{ccccc|c|c} 1 & 0 & 0 & 0 & 0 & s_0 & \hat{s}_0 \\ 0 & 1 & 0 & 0 & 0 & s_1 & \hat{s}_1 \\ 0 & 0 & 1 & 0 & 0 & s_2 & \hat{s}_2 \\ 0 & 0 & 0 & 1 & 0 & s_3 & \hat{s}_3 \\ 0 & 0 & 0 & 0 & 1 & s_4 & \hat{s}_4 \end{array} \right)$$

and obtain:

$$p(t) = s_0 + s_1 t + s_2 t^2 + s_3 t^3 + s_4 t^4$$

$$q(t) = \hat{s}_0 + \hat{s}_1 t + \hat{s}_2 t^2 + \hat{s}_3 t^3 + \hat{s}_4 t^4$$

Therefore $\gamma : [t_0, t_4] \to \mathbb{R}^2$ is given as

$$\gamma(t) = (s_0 + s_1 t + s_2 t^2 + s_3 t^3 + s_4 t^4, \hat{s}_0 + \hat{s}_1 t + \hat{s}_2 t^2 + \hat{s}_3 t^3 + \hat{s}_4 t^4)$$

By evaluating the curve over a mesh of nodes we obtain the graph.

How do we proceed in $\mathbb{R}^3$?

For 3D curves the idea is the same, we just add a dimension.

Example: curve through $P_0 = (1, 1, 1)$, $P_1 = (2, 2, -1)$, $P_2 = (4, 2, -2)$, $P_3 = (4, 1, 0)$ and $P_4 = (3, 0, 0)$

Define a parametric curve $\gamma : [a, b] \to \mathbb{R}^3$ given as

$$\gamma(t) = (p(t), q(t), r(t))$$

where p, q and r are polynomials.

We take a mesh $a = t_0 < t_1 < t_2 < t_3 < t_4 = b$

We divide our problem into three interpolation problems.

We can solve the three problems at the same time

$$RREF\left(\begin{array}{ccccc|c|c|r} 1 & t_0 & t_0^2 & t_0^3 & t_0^4 & 1 & 1 & 1 \\ 1 & t_1 & t_1^2 & t_1^3 & t_1^4 & 2 & 2 & -1 \\ 1 & t_2 & t_2^2 & t_2^3 & t_2^4 & 4 & 2 & -2 \\ 1 & t_3 & t_3^2 & t_3^3 & t_3^4 & 4 & 1 & 0 \\ 1 & t_4 & t_4^2 & t_4^3 & t_4^4 & 3 & 0 & 0 \end{array}\right) = \left(\begin{array}{c|c|c|c} & s_0 & \hat{s}_0 & \bar{s}_0 \\ & s_1 & \hat{s}_1 & \bar{s}_1 \\ I_5 & s_2 & \hat{s}_2 & \bar{s}_2 \\ & s_3 & \hat{s}_3 & \bar{s}_3 \\ & s_4 & \hat{s}_4 & \bar{s}_4 \end{array}\right)$$

and obtain:

$$p(t) = s_0 + s_1 t + s_2 t^2 + s_3 t^3 + s_4 t^4$$

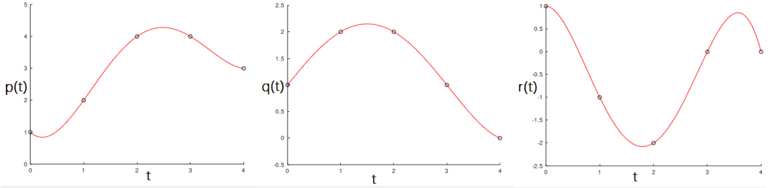
$$q(t) = \hat{s}_0 + \hat{s}_1 t + \hat{s}_2 t^2 + \hat{s}_3 t^3 + \hat{s}_4 t^4$$

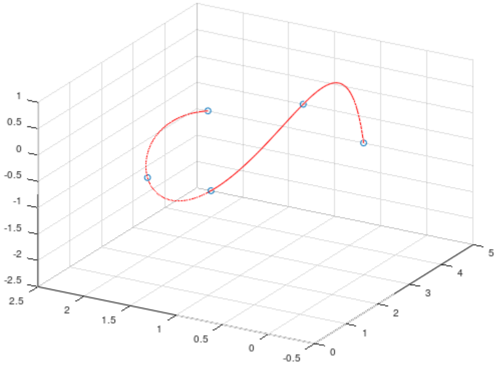
$$r(t) = \bar{s}_0 + \bar{s}_1 t + \bar{s}_2 t^2 + \bar{s}_3 t^3 + \bar{s}_4 t^4$$

Therefore $\gamma : [t_0, t_4] \to \mathbb{R}^2$ is given as

$$\gamma(t) = \left(\sum_{k=0}^{4} s_k t^k, \sum_{k=0}^{4} \hat{s}_k t^k, \sum_{k=0}^{4} \bar{s}_k t^k, \right)$$

By evaluating the curve over a mesh of nodes we obtain the graph.





Why RREF is not good for interpolation

The method used in MAT250 for interpolation is not efficient.

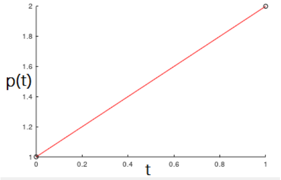
Lets do the following test:

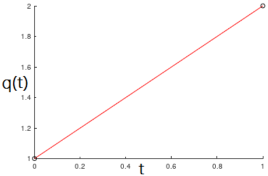
- We start with two nodes and solve an interpolation problem.
- We increase the number of nodes and see what happens.

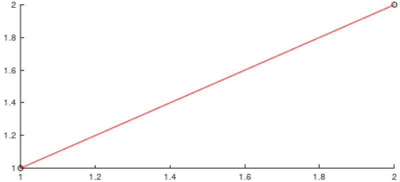
2 nodes

$t_i = i$ for $i = 0,\ 1$

$P_0 = (1, 1)$ and $P_1 = (2, 2)$



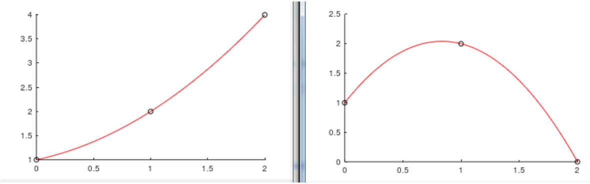


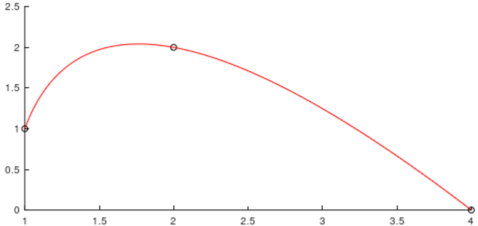


3 nodes

$t_i = i$ for $i = 0,\ 1,\ 2$

$P_0 = (1, 1)$, $P_1 = (2, 2)$ and $P_2 = (4, 0)$

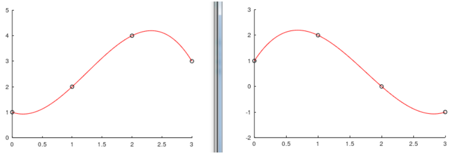


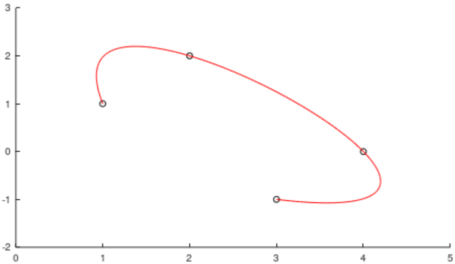


4 nodes

$t_i = i$ for $i = 0,\ 1,\ 2,\ 3,$

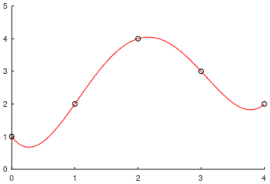
$P_0 = (1,1)$, $P_1 = (2,2)$, $P_2 = (4,0)$ and $P_3 = (3,-1)$

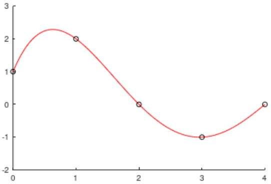


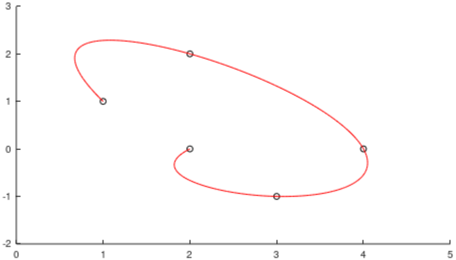


5 nodes

$t_i = i$, $P_0 = (1,1)$, $P_1 = (2,2)$, $P_2 = (4,0)$, $P_3 = (3,-1)$ and $P_4 = (2,0)$

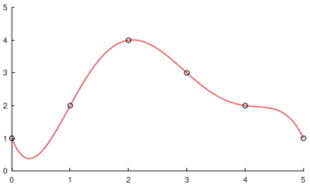


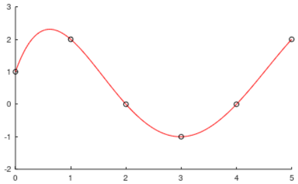


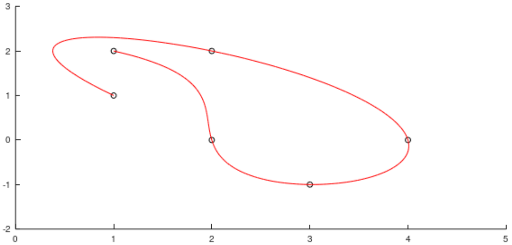


6 nodes

$t_i = i$, $P_0 = (1,1)$, $P_1 = (2,2)$, $P_2 = (4,0)$, $P_3 = (3,-1)$, $P_4 = (2,0)$ and $P_5 = (1,2)$

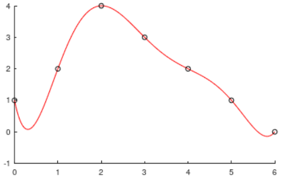


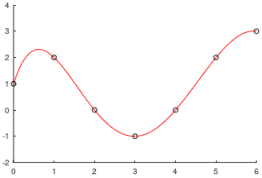


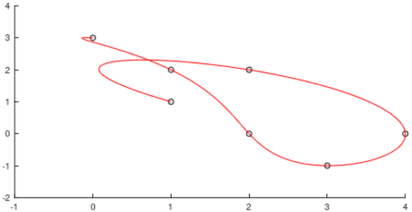


7 nodes

$t_i = i$, $P_0 = (1,1)$, $P_1 = (2,2)$, $P_2 = (4,0)$, $P_3 = (3,-1)$, $P_4 = (2,0)$, $P_5 = (1,2)$ and $P_6 = (0,3)$

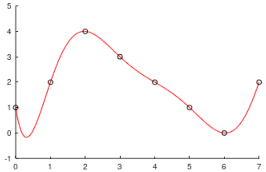


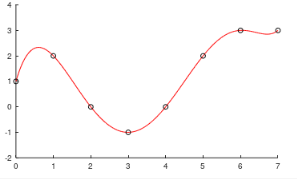


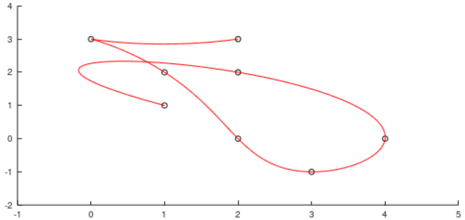


8 nodes

$t_i = i$, $P_0 = (1,1)$, $P_1 = (2,2)$, $P_2 = (4,0)$, $P_3 = (3,-1)$, $P_4 = (2,0)$, $P_5 = (1,2)$, $P_6 = (0,3)$ and $P_7 = (2,3)$

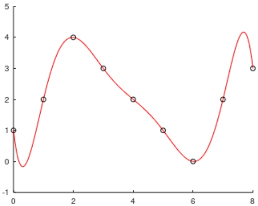


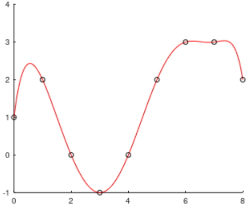


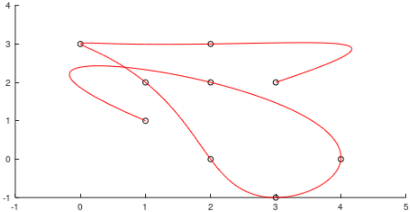


9 nodes

$t_i = i$, $P_0 = (1,1)$, $P_1 = (2,2)$, $P_2 = (4,0)$, $P_3 = (3,-1)$, $P_4 = (2,0)$, $P_5 = (1,2)$, $P_6 = (0,3)$, $P_7 = (2,3)$ and $P_8 = (3,2)$

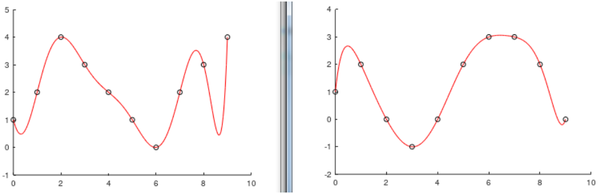


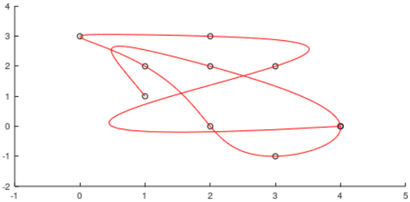




10 nodes

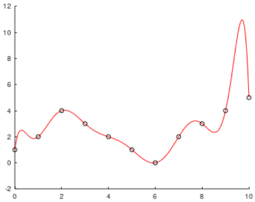
$t_i = i$, $P_0 = (1,1)$, $P_1 = (2,2)$, $P_2 = (4,0)$, $P_3 = (3,-1)$, $P_4 = (2,0)$, $P_5 = (1,2)$, $P_6 = (0,3)$, $P_7 = (2,3)$, $P_8 = (3,2)$ and $P_9 = (4,0)$

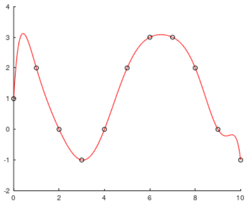


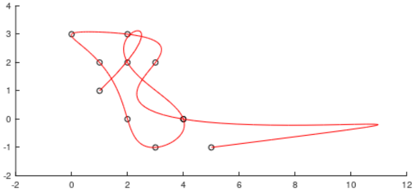


11 nodes

$t_i = i$, $P_0 = (1,1)$, $P_1 = (2,2)$, $P_2 = (4,0)$, $P_3 = (3,-1)$, $P_4 = (2,0)$, $P_5 = (1,2)$, $P_6 = (0,3)$, $P_7 = (2,3)$, $P_8 = (3,2)$, $P_9 = (4,0)$ and $P_{10} = (5,-1)$

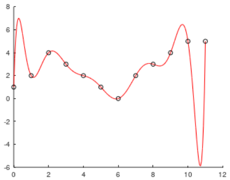


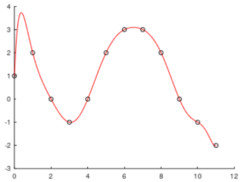


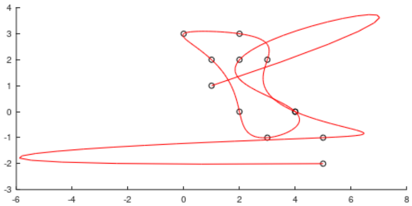


12 nodes

$t_i = i$, $P_0 = (1,1)$, $P_1 = (2,2)$, $P_2 = (4,0)$, $P_3 = (3,-1)$, $P_4 = (2,0)$, $P_5 = (1,2)$, $P_6 = (0,3)$, $P_7 = (2,3)$, $P_8 = (3,2)$, $P_9 = (4,0)$, $P_{10} = (5,-1)$ and $P_{11} = (5,-2)$

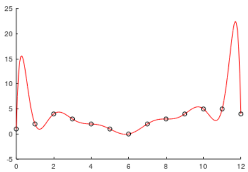


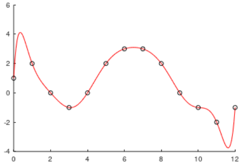


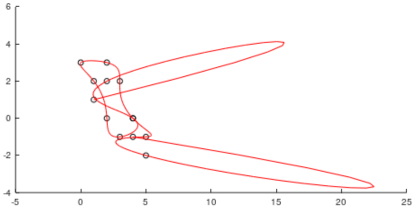


13 nodes

$t_i = i$, $P_0 = (1,1)$, $P_1 = (2,2)$, $P_2 = (4,0)$, $P_3 = (3,-1)$, $P_4 = (2,0)$, $P_5 = (1,2)$, $P_6 = (0,3)$, $P_7 = (2,3)$, $P_8 = (3,2)$, $P_9 = (4,0)$, $P_{10} = (5,-1)$, $P_{11} = (5,-2)$ and $P_{12} = (4,-1)$

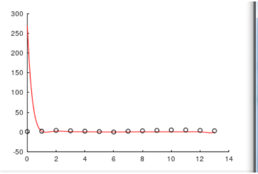


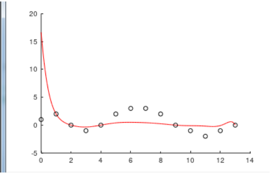


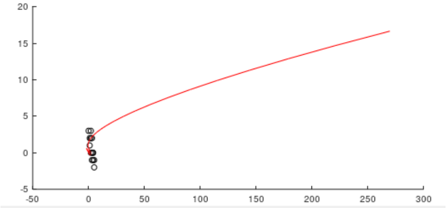


14 nodes

$t_i = i$, $P_0 = (1,1)$, $P_1 = (2,2)$, $P_2 = (4,0)$, $P_3 = (3,-1)$, $P_4 = (2,0)$, $P_5 = (1,2)$, $P_6 = (0,3)$, $P_7 = (2,3)$, $P_8 = (3,2)$, $P_9 = (4,0)$, $P_{10} = (5,-1)$, $P_{11} = (5,-2)$, $P_{12} = (4,-1)$ and $P_{13} = (3,0)$

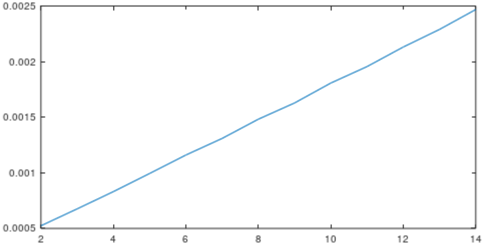






Observations

- When increasing the number of points, the curve has big variation in the extremes, due to regular mesh $t_i = i$.
- For 14 nodes the program fails due to round off errors and machine precision, RREF of Vandermonde matrix is not I_{14}.
- The increase of time of computation is given by



How to improve from observations?

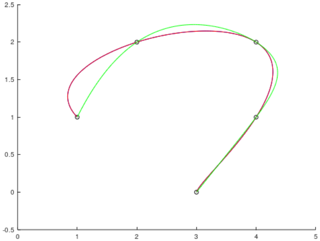
- Using other mesh of nodes
- Using other interpolation techniques

Recovering the interpolation problem for $\mathbb{R}^2$

$$RREF\left(\begin{array}{ccccc|c|c} 1 & t_0 & t_0^2 & t_0^3 & t_0^4 & \color{red}{1} & \color{blue}{1} \\ 1 & t_1 & t_1^2 & t_1^3 & t_1^4 & \color{red}{2} & \color{blue}{2} \\ 1 & t_2 & t_2^2 & t_2^3 & t_2^4 & \color{red}{4} & \color{blue}{2} \\ 1 & t_3 & t_3^2 & t_3^3 & t_3^4 & \color{red}{4} & \color{blue}{1} \\ 1 & t_4 & t_4^2 & t_4^3 & t_4^4 & \color{red}{3} & \color{blue}{0} \end{array}\right) = \left(\begin{array}{ccccc|c|c} 1 & 0 & 0 & 0 & 0 & \color{red}{s_0} & \color{blue}{\hat{s}_0} \\ 0 & 1 & 0 & 0 & 0 & \color{red}{s_1} & \color{blue}{\hat{s}_1} \\ 0 & 0 & 1 & 0 & 0 & \color{red}{s_2} & \color{blue}{\hat{s}_2} \\ 0 & 0 & 0 & 1 & 0 & \color{red}{s_3} & \color{blue}{\hat{s}_3} \\ 0 & 0 & 0 & 0 & 1 & \color{red}{s_4} & \color{blue}{\hat{s}_4} \end{array}\right)$$

The solution of the interpolation problem will depend on how do we select $t_0,\ t_1,\ t_2,\ t_3,\ t_4$ on an interval $[a, b]$.

We here consider two type of meshes: regular and Chebyshev.



Regular meshes

Definition

A regular mesh of $n+1$ nodes t_0, t_1, ..., t_n in an interval $[a, b]$ is given by

$$t_i = a + \frac{i}{n}(b-a), \qquad i = 0, 1, \dots, n \tag{2}$$

Regular meshes are uniformly distributed.

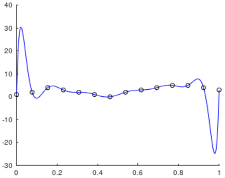
In the above examples the mesh was regular in the interval $[0, n]$ and the method breaks for $n = 13$.

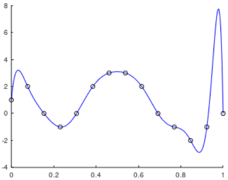
Taking a regular mesh over the interval $[0, 1]$ the method will work for more nodes!

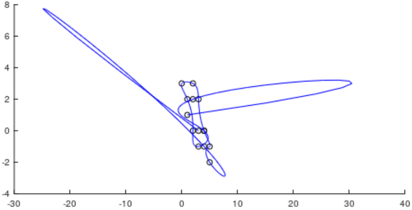
For a short number of nodes, all regular meshes give the same graph for the curve.

Regular mesh in $[0,1]$ with $n = 13$

$P_0 = (1,1)$, $P_1 = (2,2)$, $P_2 = (4,0)$, $P_3 = (3,-1)$, $P_4 = (2,0)$, $P_5 = (1,2)$, $P_6 = (0,3)$, $P_7 = (2,3)$, $P_8 = (3,2)$, $P_9 = (4,0)$, $P_{10} = (5,-1)$, $P_{11} = (5,-2)$, $P_{12} = (4,-1)$ and $P_{13} = (3,0)$

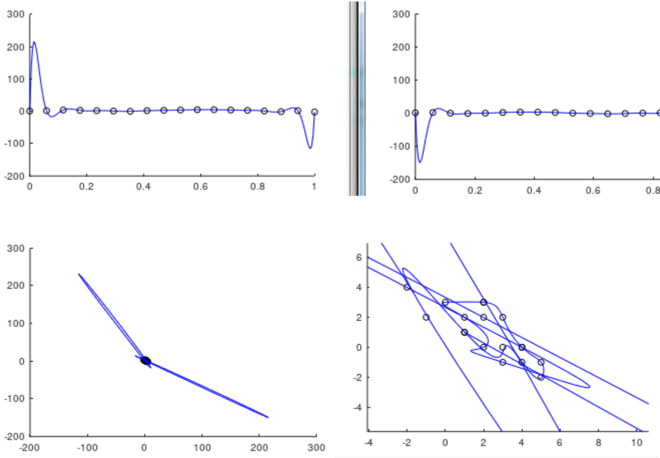






Regular mesh in $[0, 1]$ with $n = 17$

It works, but the deviation in the extremes increases!



For $n = 19$ the program breaks.

Chebyshev mesh

We saw that using regular meshes interpolant polynomials have big variation in the extremes.

We can avoid this by using Chebyshev extremal nodes.

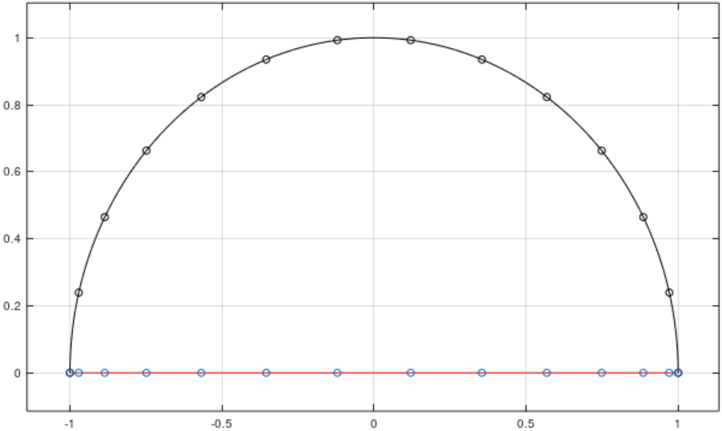
Definition

A mesh of $n+1$ Chebyshev extremal nodes t_0, t_1, …, t_n in the interval $[a, b]$ is given by

$$t_i = \frac{a+b}{2} - \left(\frac{b-a}{2}\right)\cos\left(\frac{\pi i}{n}\right), \qquad i = 0, 1, \ldots, n \tag{3}$$

Chebyshev extremal nodes are uniformly distributed in the half circle, then their projection on the interval $[a, b]$ concentrate them more on the extremes having a better aproximation there.

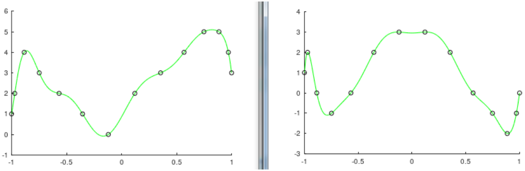
Chebyshev mesh for $n = 13$ in $[-1, 1]$

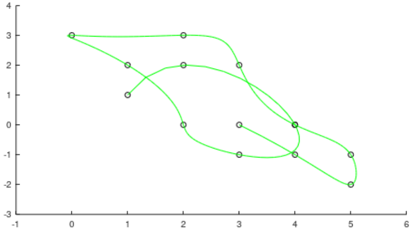


$$t_i = \frac{a+b}{2} - \left(\frac{b-a}{2}\right)\cos\left(\frac{\pi i}{n}\right) \qquad t_i = -\cos\left(\frac{\pi i}{13}\right),\ i = 0, 1, \ldots, 13$$

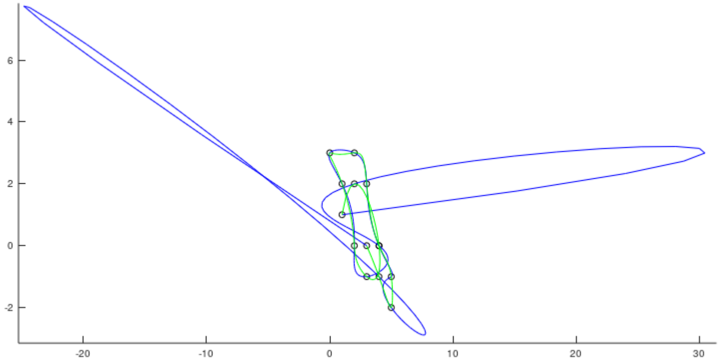
Chebyshev mesh in $[-1,1]$ with $n = 13$

$P_0 = (1,1)$, $P_1 = (2,2)$, $P_2 = (4,0)$, $P_3 = (3,-1)$, $P_4 = (2,0)$,
$P_5 = (1,2)$, $P_6 = (0,3)$, $P_7 = (2,3)$, $P_8 = (3,2)$, $P_9 = (4,0)$,
$P_{10} = (5,-1)$, $P_{11} = (5,-2)$, $P_{12} = (4,-1)$ and $P_{13} = (3,0)$

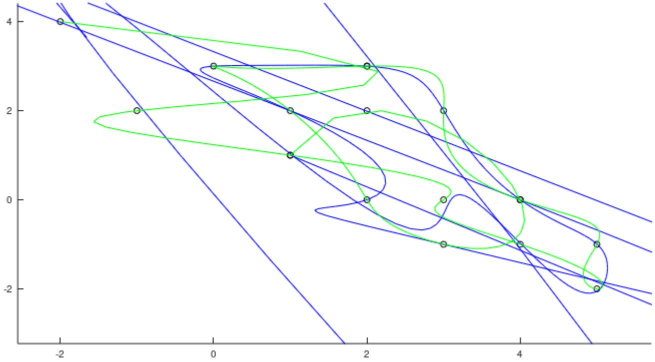




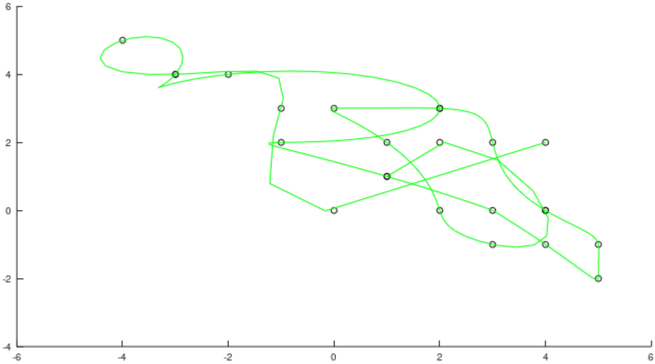
Comparison regular and Chebyshev meshes $n = 13$



Comparison regular and Chebyshev meshes $n = 17$



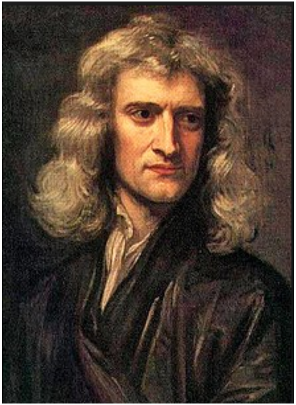
Chebyshev mesh in $[-1, 1]$ with $n = 23$

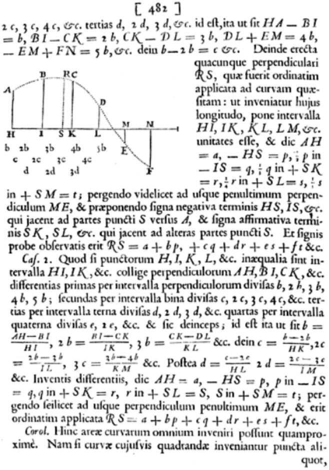


Observations

- We solved the problem of big variation in the extremes.
- At some point the Chebyshev mesh with RREF will fail as well, as the method is very unstable.
- The computational cost continues increasing.
- The behavior of the whole curve is very sensible to variations of one node.

Interpolation is an old issue





[482]

2 c, 3 c, 4 c, &c. tertias d, 2 d, 3 d, &c. id eſt, ita ut ſit $HA - BI = b$, $BI - CK = 2b$, $CK - DL = 3b$, $DL + EM = 4b$, $-EM + FN = 5b$, &c. dein $b - 2b = c$ &c. Deinde erecta quacunque perpendiculari RS, quæ fuerit ordinatim applicata ad curvam quæſitam: ut inveniatur hujus longitudo, pone intervalla HI, IK, KL, LM, &c. unitates eſſe, & dic $AH = a$, $-HS = p$, $\frac{1}{2}p$ in $-IS = q$, $\frac{1}{3}q$ in $+SK = r$, $\frac{1}{4}r$ in $+SL = s$, $\frac{1}{5}s$ in $+SM = t$; pergendo videlicet ad uſque penultimum perpendiculum ME, & præponendo ſigna negativa terminis HS, IS, &c. qui jacent ad partes puncti S verſus A, & ſigna affirmativa terminis SK, SL, &c. qui jacent ad alteras partes puncti S. Et ſignis probe obſervatis erit $RS = a + bp, + cq + dr + es + ft$ &c.

Caſ. 2. Quod ſi punctorum H, I, K, L, &c. inæqualia ſint intervalla HI, IK, &c. collige perpendiculorum AH, BI, CK, &c. differentias primas per intervalla perpendiculorum diviſas b, 2 b, 3 b, 4 b, 5 b; ſecundas per intervalla bina diviſas c, 2 c, 3 c, 4 c, &c. tertias per intervalla terna diviſas d, 2 d, 3 d, &c. quartas per intervalla quaterna diviſas e, 2 e, &c. & ſic deinceps; id eſt ita ut ſit $b = \frac{AH - BI}{HI}$, $2b = \frac{BI - CK}{IK}$, $3b = \frac{CK - DL}{KL}$ &c. dein $c = \frac{b - 2b}{HK}$, $2c = \frac{2b - 3b}{IL}$, $3c = \frac{3b - 4b}{KM}$ &c. Poſtea $d = \frac{c - 2c}{HL}$ $2d = \frac{2c - 3c}{IM}$ &c. Inventis differentiis, dic $AH = a$, $-HS = p$, p in $-IS = q$, q in $+SK = r$, r in $+SL = S$, S in $+SM = t$; pergendo ſcilicet ad uſque perpendiculum penultimum ME, & erit ordinatim applicata $RS = a + bp + cq + dr + es + ft$, &c.

Corol. Hinc areæ curvarum omnium inveniri poſſunt quamproxime. Nam ſi curvæ cujuſvis quadrandæ inveniantur puncta aliquot,

1687

How people interpolated in the past? Not with a calculator!