Homework 4

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1.

(80%) Let $P_0 = (1, -1), P_1 = (2, 3)$ and $P_2 = (0, 4)$ be the control points of a Bezier curve γ.

- (20%) For $t_0 = 0, t_1 = \frac{1}{4}, t_2 = \frac{1}{2}, t_3 = \frac{3}{4}, t_4 = 1$ apply the De Casteljau algorithm (numerically) to obtain the evaluation of the curve γ at the nodes.

  Having $P_i^k(t_j) = t_j P_{i+1}^{k-1} + (1 - t_j) P_i^{k-1}$

  Let's calculate P_i^k polynomials:

$$\begin{array}{ccc} Node & k=1 & k=2 \\ P_0 & & \\ & P_0^1 & \\ P_1 & & P_0^2 = \gamma(t) \\ & P_1^1 & \\ P_2 & & \end{array} \tag{1}$$

$$P_0^1(t) = t(2,3) + (1-t)(1,-1) = (2t, 3t) + (1-t, t-1) = (1+t, 4t-1)$$

$$P_1^1(t) = t(0,4) + (1-t)(2,3) = (0,4t) + (2-2t, 3-3t) = (2-2t, 3+t)$$

$$P_0^2(t) = \gamma(t) = tP_1^1(t) + (1-t)P_0^1(t) = (-3t^2 + 2t + 1, -3t^2 + 8t - 1)$$

Now let's evaluate at each t_j value, being $j = 0, \ldots, 4$:

- For $t_0 = 0$

  $P_0^1(0) = (1, -1) \quad P_1^1(0) = (2, 3) \quad \gamma(0) = P_0^2(0) = (1, -1)$

- For $t_1 = \frac{1}{4}$

  $P_0^1(\frac{1}{4}) = (\frac{5}{4}, 0) \quad P_1^1(\frac{1}{4}) = (\frac{3}{2}, \frac{13}{4}) \quad \gamma(\frac{1}{4}) = P_0^2(\frac{1}{4}) = (\frac{21}{16}, \frac{13}{16})$

- For $t_2 = \frac{1}{2}$

$$P_0^1(\tfrac{1}{2}) = (\tfrac{3}{2}, 1) \quad P_1^1(\tfrac{1}{2}) = (1, \tfrac{7}{2}) \quad \gamma(\tfrac{1}{2}) = P_0^2(\tfrac{1}{2}) = (\tfrac{5}{4}, \tfrac{9}{4})$$

– For $t_3 = \frac{3}{4}$

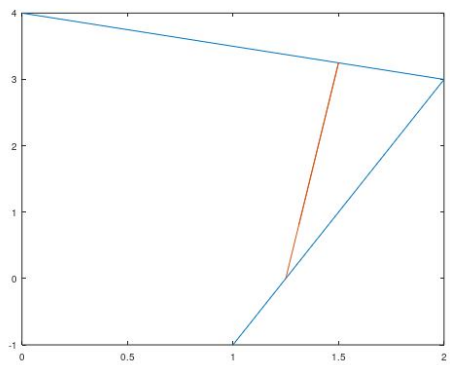
$$P_0^1(\tfrac{3}{4}) = (\tfrac{7}{4}, 2) \quad P_1^1(\tfrac{3}{4}) = (\tfrac{1}{2}, \tfrac{15}{4}) \quad \gamma(\tfrac{3}{4}) = P_0^2(\tfrac{3}{4}) = (\tfrac{13}{16}, \tfrac{53}{16})$$

– For $t_4 = 1$

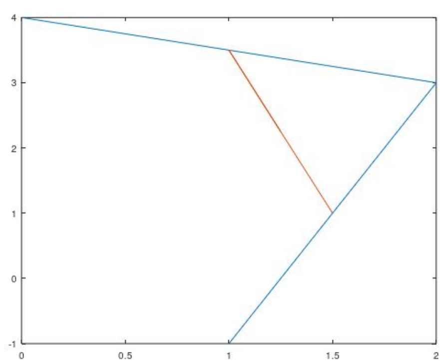
$$P_0^1(1) = (2, 3) \quad P_1^1(1) = (0, 4) \quad \gamma(1) = P_0^2(1) = (0, 4)$$

- (20%) Use the previous numerical results to draw the evaluation of the curve at every node doing the recursively linear interpolation (graphically). You have to plot the shells for each node in a separate graph where you identify numerically each intermediate control point, giving the corresponding label, and the evaluation of the curve in the plots. Then, you have to combine all the plots in the same graph using a different color for each node.

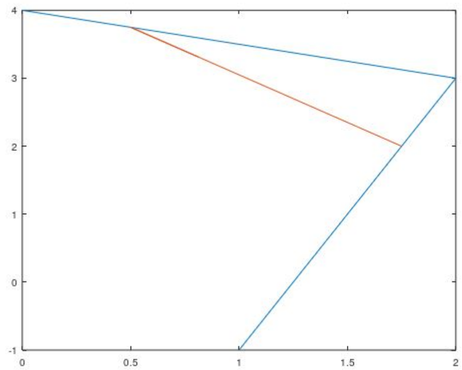
We only show intermediate shells, since $\gamma(t_0) = P_0$ and $\gamma(t_4) = P_2$ already.



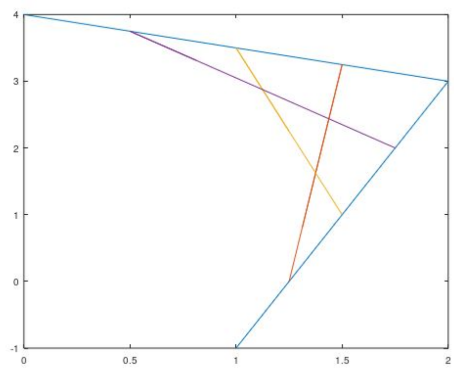
(i) Shell for $t_1 = \frac{1}{4}$



(i) Shell for $t_2 = \frac{1}{2}$



(i) Shell for $t_3 = \frac{3}{4}$



(i) All shells

- (20%) Apply three iterations of the midpoint subdivision algorithm (numerically), showing how many curves and points of γ do you get after each iteration, and what are the control points for the curves in the following iteration.

  Iterations $k = 3$,
  Total number of curves $m = \sum_{i=0}^{k-1} 2^i = 1 + 2 + 4 = 7$,
  Total number of points $n = \sum_{j=0}^{m-1}(2 + \sum_{i=0}^{k-1} 2^i) = \sum_{j=0}^{m-1}(2^k + 1) = 3 + 5 + 9 = 17$

  - 1st iteration, curve (γ_0, $t = \frac{1}{2}$)
    For γ_0

$$\begin{array}{ccc} P_0 = (1,-1) & & \\ & P_0^1 = (\frac{3}{2}, 1) & \\ P_1 = (2,3) & & \gamma_0(\frac{1}{2}) = (\frac{5}{4}, \frac{9}{4}) \\ & P_1^1 = (1, \frac{7}{2}) & \\ P_2 = (0,4) & & \end{array} \tag{2}$$

- 2nd iteration, curves (γ_1, $t = \frac{1}{4}$) and (γ_2, $t = \frac{3}{4}$):
  For γ_1

$$\begin{matrix} P_0 = (1, -1) & & \\ & P_0^1 = (\frac{9}{8}, \frac{-1}{2}) & \\ P_1 = (\frac{3}{2}, 1) & & \gamma_1(\frac{1}{4}) = (\frac{77}{64}, \frac{-3}{64}) \\ & P_1^1 = (\frac{23}{16}, \frac{21}{16}) & \\ P_2 = (\frac{5}{4}, \frac{9}{4}) & & \end{matrix} \tag{3}$$

  For γ_2

$$\begin{matrix} P_0 = (\frac{5}{4}, \frac{9}{4}) & & \\ & P_0^1 = (\frac{17}{16}, \frac{51}{16}) & \\ P_1 = (1, \frac{7}{2}) & & \gamma_2(\frac{3}{4}) = (\frac{29}{64}, \frac{237}{64}) \\ & P_1^1 = (\frac{1}{4}, \frac{31}{8}) & \\ P_2 = (0, 4) & & \end{matrix} \tag{4}$$

- 3rd iteration, curves $(\gamma_3, t = \frac{1}{8}), (\gamma_4, t = \frac{3}{8}), (\gamma_5, t = \frac{5}{8})$ and $(\gamma_6, t = \frac{7}{8})$
  For γ_3

$$\begin{matrix} P_0 = (1, -1) & & \\ & P_0^1 = (\frac{65}{64}, \frac{-15}{16}) & \\ P_1 = (\frac{9}{8}, \frac{-1}{2}) & & \gamma_2(\frac{1}{8}) = (\frac{4221}{4096}, \frac{-3587}{4096}) \\ & P_1^1 = (\frac{581}{512}, \frac{-227}{512}) & \\ P_2 = (\frac{77}{64}, \frac{-3}{64}) & & \end{matrix} \tag{5}$$

  For γ_4

$$\begin{matrix} P_0 = (\frac{77}{64}, \frac{-3}{64}) & & \\ & P_0^1 = (\frac{661}{512}, \frac{237}{512}) & \\ P_1 = (\frac{23}{16}, \frac{21}{16}) & & \gamma_2(\frac{3}{8}) = (\frac{5405}{4096}, \frac{3741}{4096}) \\ & P_1^1 = (\frac{175}{128}, \frac{213}{128}) & \\ P_2 = (\frac{5}{4}, \frac{9}{4}) & & \end{matrix} \tag{6}$$

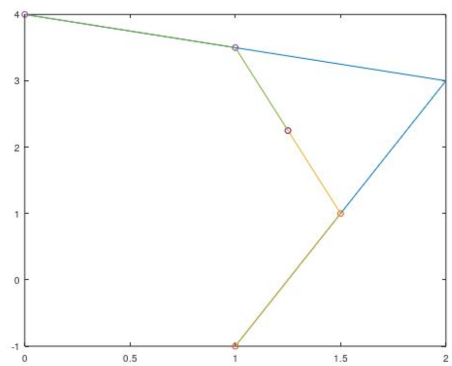
  For γ_5

$$\begin{matrix} P_0 = (\frac{5}{4}, \frac{9}{4}) & & \\ & P_0^1 = (\frac{145}{128}, \frac{333}{128}) & \\ P_1 = (\frac{17}{16}, \frac{51}{16}) & & \gamma_2(\frac{5}{8}) = (\frac{3485}{4096}, \frac{12981}{4096}) \\ & P_1^1 = (\frac{349}{512}, \frac{1797}{512}) & \\ P_2 = (\frac{29}{64}, \frac{237}{64}) & & \end{matrix} \tag{7}$$

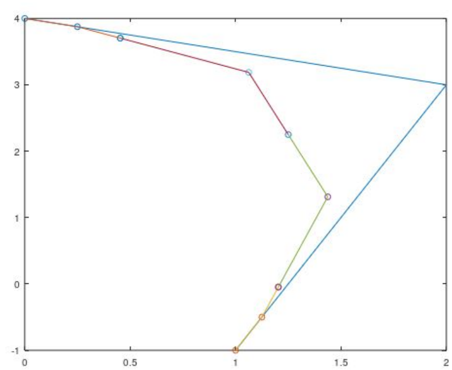
  For γ_6

$$\begin{matrix} P_0 = (\frac{29}{64}, \frac{237}{64}) & & \\ & P_0^1 = (\frac{141}{512}, \frac{1973}{512}) & \\ P_1 = (\frac{1}{4}, \frac{31}{8}) & & \gamma_2(\frac{7}{8}) = (\frac{253}{4096}, \frac{16253}{4096}) \\ & P_1^1 = (\frac{1}{32}, \frac{255}{64}) & \\ P_2 = (0, 4) & & \end{matrix} \tag{8}$$

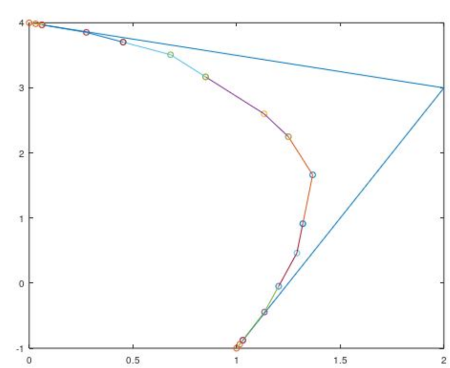
- (20%) Use the previous numerical results to draw the evaluation of the curve after each iteration. Construct a plot for each iteration showing the control points of each curve (using different colors) and the obtained γ.



(i) 1st iteration



(i) 2nd iteration



(i) 3rd iteration

2.

(20%) Find the quadratic Bezier curve that starts at $P_0 = (0,0)$, goes through $Q = (1,1)$ at $t = \frac{3}{4}$ and ends at $P_2 = (2,4)$. Give the expression of the Bezier curve as linear combination of Bernstein polynomials, providing the corresponding interval.

We start splitting the Bezier curve into two curves γ_1 and γ_2

- Curve γ_1 starting at P_0 and ending at $\gamma(\frac{3}{4}) = Q$.
- Curve γ_2 starting at $\gamma(\frac{3}{4}) = Q$ and ending at P_2.

Evaluating $\gamma(\frac{3}{4})$ through recursion we get the control points for the curve:

$$\begin{array}{ccc} P_0 = (0,0) & & \\ & P_0^1 & \\ P_1 & & P_0^2 = \gamma(\frac{3}{4}) = Q = (1,1) \\ & P_1^1 & \\ P_2 = (2,4) & & \end{array} \tag{9}$$

Through algebra we get that

$P_1 = \frac{P_0^2 - t^2 P_2 + P_0(-t^2+2t-1)}{-2t^2+2t}$

then $P_1 = (\frac{-1}{3}, \frac{-10}{3}), P_0^1 = (\frac{-1}{4}, \frac{-5}{2}), P_1^1 = (\frac{17}{12}, \frac{13}{6})$

γ_1 lies in the hull of P_0, P_0^1 and P_0^2.

γ_2 lies in the hull of P_0^2, P_1^1 and P_2.

γ lies in the hull of P_0, P_1 and P_2.

We can define each curve as such:

$\gamma_1(t) = \sum_{i=0}^{2} P_0^i B_i^2(t)$, $t \in [0,1]$

$\gamma_2(t) = \sum_{i=0}^{2} P_i^{2-i} B_i^2(t)$, $t \in [0,1]$

$\gamma(t) = \sum_{i=0}^{2} P_i B_i^2(t)$, $t \in [0,1]$

Let's calculate the Bernstein polynomials that form the basis of P_2 (the polynomial space, not the point).

$B_B = \{B_0^2, B_1^2, B_2^2\}$

$B_0^2(t) = \binom{2}{0} (1-t)^2 t^0 = 1 - 2t + t^2$

$B_1^2(t) = \binom{2}{1} (1-t)^1 t^1 = 2t - 2t^2$

$B_2^2(t) = \binom{2}{2} (1-t)^0 t^2 = t^2$

Finally we get the curves as linear combination:

$\gamma_1(t) = (0,0)(1-2t+t^2) + (\frac{-1}{4}, \frac{-5}{2})(2t-2t^2) + (1,1)t^2, \quad t \in [0,1]$

$$\gamma_2(t) = (1,1)(1-2t+t^2) + (\tfrac{17}{12}, \tfrac{13}{6})(2t-2t^2) + (2,4)t^2, \quad t \in [0,1]$$
$$\gamma(t) = (0,0)(1-2t+t^2) + (\tfrac{-1}{3}, \tfrac{-10}{3})(2t-2t^2) + (2,4)t^2, \quad t \in [0,1]$$