# MAT300 CURVES AND SURFACES

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# B-splines

# Motivation

Spaces of splines with different order of continuity are the most general piecewise polynomial spaces.

We may want to construct curves $\gamma : [a, b] \to \mathbb{R}^2$

$$\gamma(t) = (p(t), q(t))$$

with $p, q \in P^n_{k,\vec{r}}[t_0, \ldots, t_n]$.

For that purpose the first thing we need to do is to code the information related to continuity conditions at the nodes $t_1, \ldots, t_{n-1}$.

This information is related to the number of shifted power functions that appear at the standard basis of $P^n_{k,\vec{r}}[t_0, \ldots, t_n]$ for a particular node $t_j$. Through this idea we now construct knot sequences.

# Knot sequences

## Definition

A knot sequence $\vec{t}$ is a nondecreasing sequence $(\bar{t}_0, \bar{t}_1, \ldots, \bar{t}_N)$, where $\bar{t}_i \in \mathbb{R}$ for $i = 0, 1, \ldots, N$.

The multiplicity of a certain knot $\bar{t}_i$ is the number of times it appears in the sequence.

A knot sequence has associated a set of shifterd power functions $sp_k(\vec{t}\,)$ with which we can construct a basis for a vector space of splines $P^n_{k,\vec{r}}[t_0, \ldots, t_n]$.

If a knot $\bar{t}_i$ has multiplicity $c$ then:

- $\bar{t}_i = \bar{t}_{i+1} = \ldots = \bar{t}_{i+c-1}$
- the shifter power functions associated to $\bar{t}_i$ are $(t - \bar{t}_i)^k_+$, $(t - \bar{t}_i)^{k-1}_+$, …, $(t - \bar{t}_i)^{k-c+1}_+$

# Example

$\vec{t} = (0, 1, 1, 2, 3, 3, 3)$ knot sequence, $k = 2$. Construct $sp_k(\vec{t})$

- knot 0 has multiplicity 1, then $(t - 0)_+^2$
- knot 1 has multiplicity 2, then $(t - 1)_+^2$, $(t - 1)_+^1$
- knot 2 has multiplicity 1, then $(t - 2)_+^2$
- knot 3 has multiplicity 3, then $(t - 3)_+^2$, $(t - 3)_+^1$, $(t - 3)_+^0$

Then

$$sp_2(\vec{t}) = \{(t-0)_+^2, (t-1)_+^2, (t-1)_+^1, (t-2)_+^2, (t-3)_+^2, (t-3)_+^1, (t-3)_+^0\}$$

# Example

$\vec{t} = (0,0,0,2,3,3)$ knot sequence, $k = 2$. Construct $sp_k(\vec{t}\,)$

- knot 0 has multiplicity 3, then $(t-0)_+^2$, $(t-0)_+^1$, $(t-0)_+^0$
- knot 2 has multiplicity 1, then $(t-2)_+^2$
- knot 3 has multiplicity 2, then $(t-3)_+^2$, $(t-3)_+^1$

Then

$sp_2(\vec{t}\,) = \{(t-0)_+^2, (t-0)_+^1, (t-0)_+^0, (t-2)_+^2, (t-3)_+^2, (t-3)_+^1\}$

$sp_2(\vec{t}\,)$ is a basis for $P^3_{2,\vec{r}}[0,2,3,a]$ with $\vec{r} = (1,0)$ and $a > 3$ a real number.

We can define bases of spaces through knot sequences and viceversa

# Example

Find a knot sequence for the space $P^4_{3,\vec{r}}[0,2,3,5,8]$ with $\vec{r} = (1,2,0)$

$B = \{(t-0)^0_+, (t-0)^1_+, (t-0)^2_+, (t-0)^3_+, \cancel{(t-2)^0_+}, \cancel{(t-2)^1_+},$

$(t-2)^2_+, (t-2)^3_+, \cancel{(t-3)^0_+}, \cancel{(t-3)^1_+}, \cancel{(t-3)^2_+}, (t-3)^3_+, \cancel{(t-5)^0_+},$

$(t-5)^1_+, (t-5)^2_+, (t-5)^3_+\}$

by reordering $B$ we obtain $sp_3(\vec{t}\,)$

$\vec{t} = (0,0,0,0,2,2,3,5,5,5)$

# Continuity and multiplicity vectors

**Definition**

Let $P^n_{k,\vec{r}}[t_0, \ldots, t_n]$ with $\vec{r} = (r_1, \ldots, r_{n-1})$.

$\vec{r}$ is the **continuity vector** of $P^n_{k,\vec{r}}[t_0, \ldots, t_n]$.

$\vec{m} = (m_1, \ldots, m_{n-1})$ is the **multiplicity vector** of $P^n_{k,\vec{r}}[t_0, \ldots, t_n]$ where

$$m_i = k - r_i \tag{1}$$

for $i = 1, \ldots, n-1$.

Remarks:

- For a knot $\bar{t}_j$ with value $t_i$ in a knot sequence, $m_i$ is its multiplicity and so the corresponding number of shifted power functions.
- If at $t_i$ the polynomials are not continuous then $r_i = -1$ and so $m_i = k + 1$.

# Basis and dimension

**Theorem**

*The dimension of* $P^n_{k,\vec{r}}[t_0, \dots, t_n]$ *with* $\vec{r} = (r_1, \dots, r_{n-1})$ *is*

$$\dim(P^n_{k,\vec{r}}[t_0, \dots, t_n]) = k + 1 + \sum_{i=1}^{n-1}(k - r_i) = k + 1 + \sum_{i=1}^{n-1} m_i \qquad (2)$$

We construct a basis for $P^n_{k,\vec{r}}[t_0, \dots, t_n]$ with $\vec{r} = (r_1, \dots, r_{n-1})$ as follows:

- Build a knot sequence $\vec{t}$ satisfying:
  - The first $k+1$ knots $\bar{t}_0 \leq \bar{t}_1 \leq \dots \bar{t}_k \leq t_0$
  - For $i = 1, \dots, n-1$ add knots with value $t_i$ and multiplicity $m_i$
- Construct $sp_k(\vec{t}\,)$ which is a basis.

# Examples

Find a basis for the space $P^3_{2,\vec{r}}[1,2,3,4]$ with $\vec{r} = (1,0)$

We have $\vec{m} = (1,2)$

Construct $\vec{t} = (0,0,0,2,3,3)$

$B = \{(t-0)^2_+, (t-0)^1_+, (t-0)^0_+, (t-2)^2_+, (t-3)^2_+, (t-3)^1_+\}$

B is not the only basis for $P^3_{2,\vec{r}}[1,2,3,4]$

Construct $\vec{t} = (1,1,1,2,3,3)$

$B = \{(t-1)^2_+, (t-1)^1_+, (t-1)^0_+, (t-2)^2_+, (t-3)^2_+, (t-3)^1_+\}$

Construct $\vec{t} = (-1,0,1,2,3,3)$

$B = \{(t+1)^2_+, (t-0)^2_+, (t-1)^2_+, (t-2)^2_+, (t-3)^2_+, (t-3)^1_+\}$