MAT300 CURVES AND SURFACES

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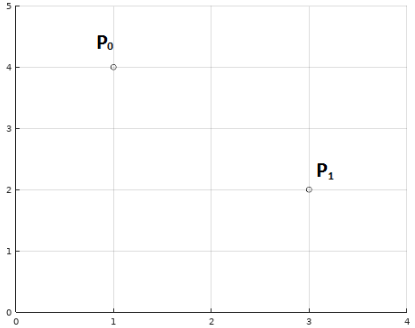
Spring 2020

Lagrange interpolation

1. Lagrange interpolation
   - The line and P_1
   - Generalization to P_n
   - Recursive generation of Lagrange polynomials
   - Interpolation of surfaces

The simplest example: a line

We want a line through $P_0 = (x_0, y_0)$ and $P_1 = (x_1, y_1)$ having $x_0 \neq x_1$

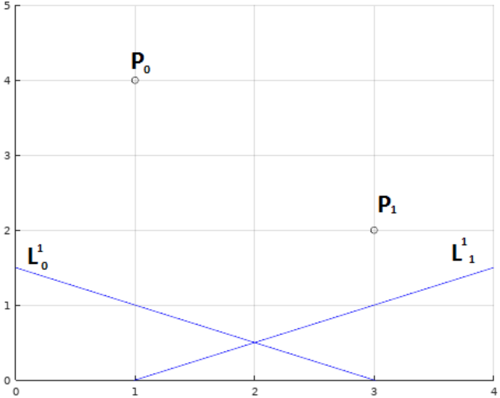


How can we construct it? We can define two polynomials of degree 1

$$L_0^1(x) = \frac{x - x_1}{x_0 - x_1}, \qquad L_1^1(x) = \frac{x - x_0}{x_1 - x_0}$$

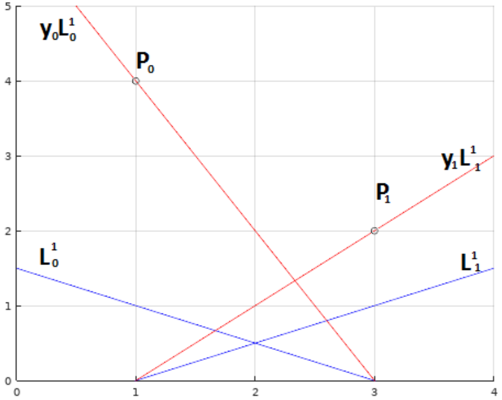
How are these polynomials?

- $L_0^1(x_0) = \frac{x_0 - x_1}{x_0 - x_1} = 1$ and $L_0^1(x_1) = \frac{x_1 - x_1}{x_0 - x_1} = 0$
- $L_1^1(x_0) = \frac{x_0 - x_0}{x_1 - x_0} = 0$ and $L_1^1(x_1) = \frac{x_1 - x_0}{x_1 - x_0} = 1$



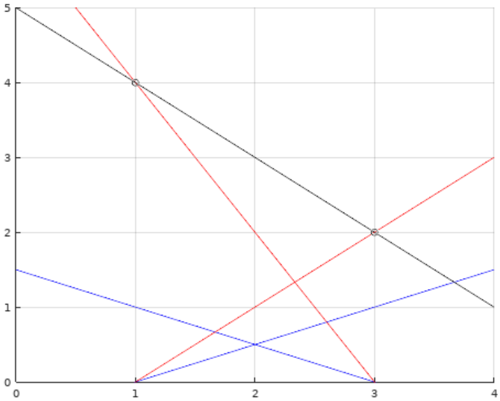
if I multiply L_0^1 by y_0 and L_1^1 by y_1 I obtain

- $y_0 L_0^1(x_0) = y_0 \frac{x_0 - x_1}{x_0 - x_1} = y_0$ and $y_0 L_0^1(x_1) = y_0 \frac{x_1 - x_1}{x_0 - x_1} = 0$
- $y_1 L_1^1(x_0) = y_1 \frac{x_0 - x_0}{x_1 - x_0} = 0$ and $y_1 L_1^1(x_1) = y_1 \frac{x_1 - x_0}{x_1 - x_0} = y_1$



finally I sum them to obtain the interpolant polynomial

$$p(x) = y_0 L_0^1(x) + y_1 L_1^1(x) = y_0 \frac{x - x_1}{x_0 - x_1} + y_1 \frac{x - x_0}{x_1 - x_0} \tag{1}$$



The Lagrange basis for P_1

The polynomials L_0^1 and L_1^1 are the Lagrange polynomials of degree 1 for the nodes x_0 and x_1 with $x_0 \neq x_1$.

$\{L_0^1, L_1^1\}$ is a basis for P_1

$$L_0^1(x) = \frac{x-x_1}{x_0-x_1} \text{ therefore } T(L_0^1(x)) = \begin{pmatrix} -\frac{x_1}{x_0-x_1} \\ \\ \frac{1}{x_0-x_1} \end{pmatrix}$$

$$L_1^1(x) = \frac{x-x_0}{x_1-x_0} \text{ therefore } T(L_1^1(x)) = \begin{pmatrix} -\frac{x_0}{x_1-x_0} \\ \\ \frac{1}{x_1-x_0} \end{pmatrix}$$

$$\det \begin{pmatrix} -\frac{x_1}{x_0-x_1} & -\frac{x_0}{x_1-x_0} \\ \\ \frac{1}{x_0-x_1} & \frac{1}{x_1-x_0} \end{pmatrix} = \frac{1}{(x_1-x_0)^2} \det \begin{pmatrix} x_1 & -x_0 \\ \\ -1 & 1 \end{pmatrix} = \frac{1}{x_1-x_0} \neq 0$$

as the vectors are linearly independent, inverting the isomorphism L_0^1 and L_1^1 are also linearly independent.

$|\{L_0^1, L_1^1\}| = 2 = \dim(P_1)$

So it is proved that it is a basis.

Every polynomial in P_1 can be expressed as $p(x) = y_0 L_0^1(x) + y_1 L_1^1(x)$

The vector of coordinates of $p(x)$ in the Lagrange basis is $(y_0, y_1)_{LB}$

To recover the original polynomial (in the standard basis) we do

$$p(x) = \begin{pmatrix} 1 & x \end{pmatrix} \begin{pmatrix} -\frac{x_1}{x_0 - x_1} & -\frac{x_0}{x_1 - x_0} \\ \frac{1}{x_0 - x_1} & \frac{1}{x_1 - x_0} \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}$$

Lagrange versus RREF

When solving an interpolation problem by constructing a linear system of equations, the expression for the polynomial is not explicitely given. We have to apply Gauss-Jordan to obtain the RREF to get the explicit expression for ploting the solution curve (increase time of computation and decreaces accuracy).

When we use Lagrange polynomials $p(x) = y_0 L_0^1(x) + y_1 L_1^1(x)$ is already a explicit expression, so we only need to substitute values of x to obtain the solution curve.

Lagrange polynomials in P_n

Definition

Given x_0, x_1, $\ldots$, x_n with $x_i \neq x_j$ for $i \neq j$, the i-th Lagrange polynomial of degree n is

$$L_i^n(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}, \qquad i = 0, 1, \ldots, n \tag{2}$$

Theorem

$L_i^n(x_i) = 1$ and $L_i^n(x_j) = 0$ for $j \neq i$

Example: $x_0 = -2$, $x_1 = -1$, $x_2 = 1$ and $x_3 = 4$. Compute the Lagrange polynomials and verify the theorem.

Lagrange basis

Theorem

$BL = \{L_0^n,\ L_1^n,\ \dots,\ L_n^n\}$ is a basis for P_n.

We have $|BL| = n + 1 = \dim(P_n)$ so we only need to prove linearly independence or spanning.

Let $p(x) = a_0 + a_1 x + \ldots + a_n x^n \in P_n$

I take x_0, x_1, … x_n distinct and evaluate p

I obatian $p(x_0) = y_0$, $p(x_1) = y_1$, …, $p(x_n) = y_n$

So p is the unique polynomial of degree at most n through (x_0, y_0), (x_1, y_1), …, (x_n, y_n).

I construct $q(x) = y_0 L_0^n(x) + y_1 L_1^n(x) + \ldots + y_n L_n^n(x)$

through linear combination of the Lagrange polynomials.

We know that $L_i^n(x_i) = 1$ and $L_i^n(x_j) = 0$ for $j \neq i$ therefore

$$q(x_0) = y_0 L_0^n(x_0) + y_1 L_1^n(x_0) + \ldots + y_n L_n^n(x_0) = y_0 \text{ through } (x_0, y_0)$$

$$q(x_1) = y_0 L_0^n(x_1) + y_1 L_1^n(x_1) + \ldots + y_n L_n^n(x_1) = y_1 \text{ through } (x_1, y_1)$$

$$\vdots$$

$$q(x_n) = y_0 L_0^n(x_n) + y_1 L_1^n(x_n) + \ldots + y_n L_n^n(x_n) = y_n \text{ through } (x_n, y_n)$$

q has also degree at most n and passes through (x_0, y_0), (x_1, y_1), $\ldots$, (x_n, y_n), so $q = p$. As we can express every $p \in P_n$ as linear combination of BL we have $P_n = span(BL)$ so BL is a basis for P_n.

Interpolant polynomial in Lagrange basis

Definition

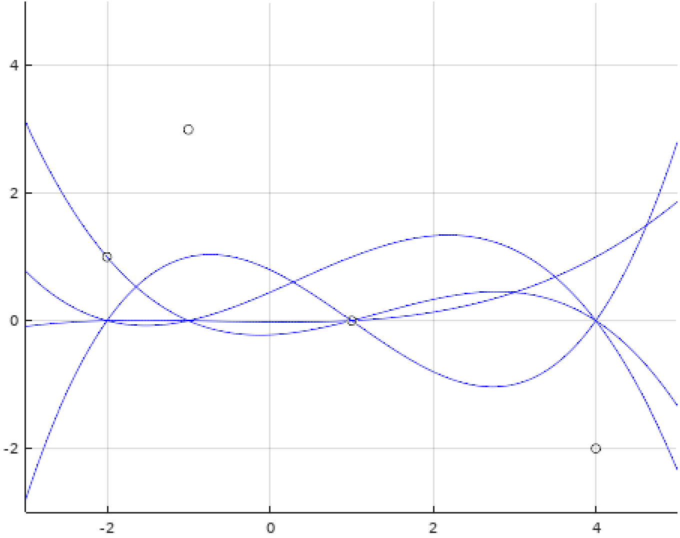
Given $n+1$ points (x_i, y_i) for $i = 0, 1, \ldots, n$ with $x_i \neq x_j$ for $i \neq j$, the Lagrange interpolnat polynomial of degree at most n is given as

$$p(x) = \sum_{i=0}^{n} y_i L_i^n(x) \tag{3}$$

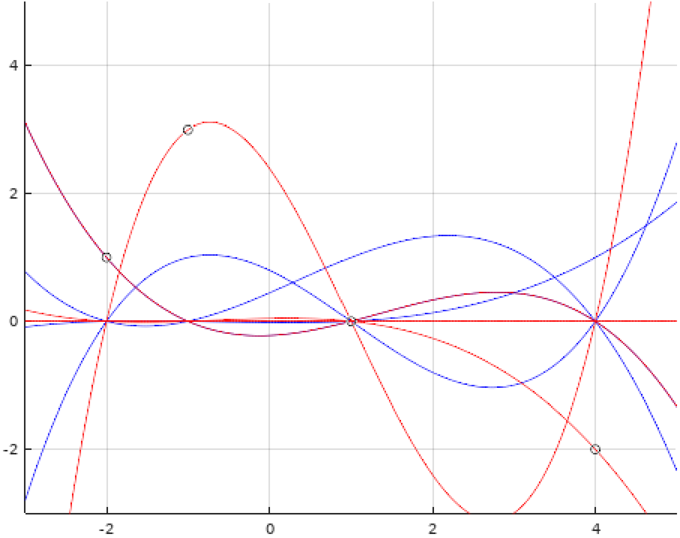
The vector of coordinates of p in the Lagrange basis is $(y_0, y_1, \ldots, y_n)_{BL}$.

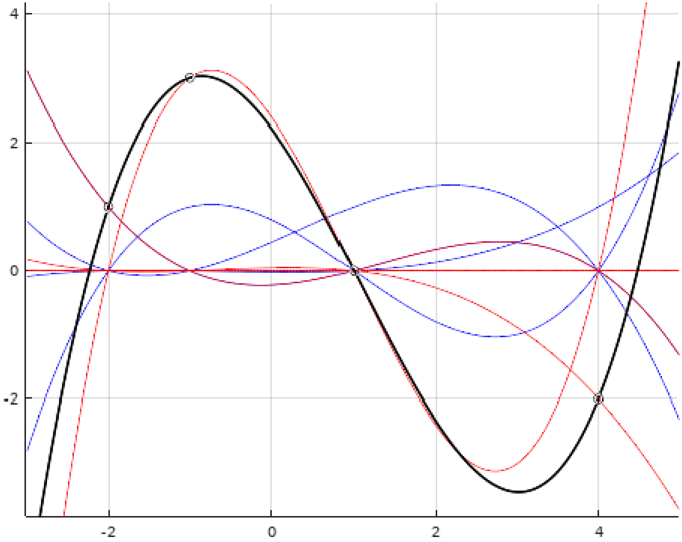
Example: Use the Lagrange basis to obtain the interpolant polynomial through $(-2, 1)$, $(-1, 3)$, $(1, 0)$ and $(4, -2)$. Give the vector of coordinates of the polynomial in the Lagrange and in the standard basis.

The geometry of the previous example

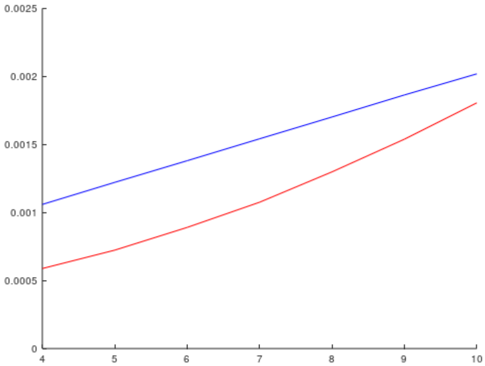


The geometry of the previous example




4
2
0
-2
-2
0
2
4

Lagrange interpolation is faster than RREF



Motivation

Let (x_0, y_0), (x_1, y_1), …, (x_n, y_n) be points with $x_{i-1} < x_i$ for $i = 1, \dots, n$

and let $p(x) = \sum_{i=0}^{n} y_i L_i^n(x)$ its Lagrange interpolant polynomial.

Now we introduce another point (x_{n+1}, y_{n+1}) with $x_n < x_{n+1}$

How do we construct the new interpolant polynomial? we want to use the previous work and save computational cost, therefore a good idea is to construct the polynomials through recursion.

Notation: we denote with $p_{m_0,m_1,\dots,m_k}$ the Lagrange interpolant polynomial of degree at most k through the nodes (x_{m_0}, y_{m_0}), (x_{m_1}, y_{m_1}), …, (x_{m_k}, y_{m_k})

Example: $p_{0,2,3,5}$ is the Lagrange interpolant polynomial through (x_0, y_0), (x_2, y_2), (x_3, y_3) and (x_5, y_5).

Neville's method

Lagrange polynomials can be generated recursively in the following way

Nodes	$p_i \in P_0$
x_0	$p_0(x) = y_0$
x_1	$p_1(x) = y_1$
x_2	$p_2(x) = y_2$
x_3	$p_3(x) = y_3$
x_4	$p_4(x) = y_4$

In P_0 there is a unique Lagrange polynomial $L_0^0(x) = 1$.

So the interpolant polynomials are $p_i(x) = y_i L_0^0(x) = y_i$

We use previous information for computing interpolant polynomials in P_1

Nodes	$p_i \in P_0$	$p_{i,i+1} \in P_1$
x_0	$p_0(x) = y_0$	
		$p_{0,1}(x) = \frac{(x-x_0)p_1(x)-(x-x_1)p_0(x)}{x_1-x_0}$
x_1	$p_1(x) = y_1$	
		$p_{1,2}(x) = \frac{(x-x_1)p_2(x)-(x-x_2)p_1(x)}{x_2-x_1}$
x_2	$p_2(x) = y_2$	
		$p_{2,3}(x) = \frac{(x-x_2)p_3(x)-(x-x_3)p_2(x)}{x_3-x_2}$
x_3	$p_3(x) = y_3$	
		$p_{3,4}(x) = \frac{(x-x_3)p_4(x)-(x-x_4)p_3(x)}{x_4-x_3}$
x_4	$p_4(x) = y_4$	

We obtain Lagrange interpolant polynomials of degree 1 through the nodes (x_i, y_i) and (x_{i+1}, y_{i+1}).

We use previous information for computing interpolant polynomials in P_2

Nodes	$p_i \in P_0$	$p_{i,i+1} \in P_1$	$p_{i,i+1,i+2} \in P_2$
x_0	p_0		
		$p_{0,1}$	
x_1	p_1		$p_{0,1,2}(x) = \frac{(x-x_0)p_{1,2}(x)-(x-x_2)p_{0,1}(x)}{x_2-x_0}$
		$p_{1,2}$	
x_2	p_2		$p_{1,2,3}(x) = \frac{(x-x_1)p_{2,3}(x)-(x-x_3)p_{1,2}(x)}{x_3-x_1}$
		$p_{2,3}$	
x_3	p_3		$p_{2,3,4}(x) = \frac{(x-x_2)p_{3,4}(x)-(x-x_4)p_{2,3}(x)}{x_4-x_2}$
		$p_{3,4}$	
x_4	p_4		

We obtain Lagrange interpolant polynomials of degree 2 through the nodes (x_i, y_i), (x_{i+1}, y_{i+1}) and (x_{i+2}, y_{i+2}).

We use previous information for computing interpolant polynomials in P_3

Nodes	P_0	P_1	P_2	$p_{i,i+1,i+2,i+3} \in P_3$
x_0	p_0			
		$p_{0,1}$		
x_1	p_1		$p_{0,1,2}$	
		$p_{1,2}$		$p_{0,1,2,3}(x) = \frac{(x-x_0)p_{1,2,3}(x)-(x-x_3)p_{0,1,2}(x)}{x_3-x_0}$
x_2	p_2		$p_{1,2,3}$	
		$p_{2,3}$		$p_{1,2,3,4}(x) = \frac{(x-x_1)p_{2,3,4}(x)-(x-x_4)p_{1,2,3}(x)}{x_4-x_1}$
x_3	p_3		$p_{2,3,4}$	
		$p_{3,4}$		
x_4	p_4			

We obtain Lagrange interpolant polynomials of degree 3 through the nodes (x_i, y_i), (x_{i+1}, y_{i+1}), (x_{i+2}, y_{i+2}) and (x_{i+3}, y_{i+3}).

Finally we obtain the interpolant polynomial in P_4

$$
\begin{array}{cccccc}
Nodes & P_0 & P_1 & P_2 & P_3 & p_{i,i+1,i+2,i+3,i+4} \in P_4 \\
 & & & & & \\
x_0 & p_0 & & & & \\
 & & p_{0,1} & & & \\
x_1 & p_1 & & p_{0,1,2} & & \\
 & & p_{1,2} & & p_{0,1,2,3} & \\
x_2 & p_2 & & p_{1,2,3} & & p_{0,1,2,3,4} \\
 & & p_{2,3} & & p_{1,2,3,4} & \\
x_3 & p_3 & & p_{2,3,4} & & \\
 & & p_{3,4} & & & \\
x_4 & p_4 & & & &
\end{array}
$$

where

$$p_{0,1,2,3,4}(x) = \frac{(x - x_0)p_{1,2,3,4}(x) - (x - x_4)p_{0,1,2,3}(x)}{x_4 - x_0}$$

What happens if we add an extra node?

We can use the previous information and continue the process

Nodes	P_0	P_1	P_2	P_3	P_4	P_5
x_0	p_0					
		$p_{0,1}$				
x_1	p_1		$p_{0,1,2}$			
		$p_{1,2}$		$p_{0,1,2,3}$		
x_2	p_2		$p_{1,2,3}$		$p_{0,1,2,3,4}$	
		$p_{2,3}$		$p_{1,2,3,4}$		$p_{0,1,2,3,4,5}$
x_3	p_3		$p_{2,3,4}$		$p_{1,2,3,4,5}$	
		$p_{3,4}$		$p_{2,3,4,5}$		
x_4	p_4		$p_{3,4,5}$			
		$p_{4,5}$				
x_5	p_5					

Observations

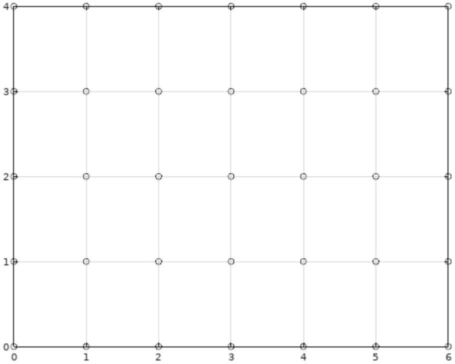
- Neville's method is good if we use a regular mesh in $[0, n]$, but not for regular mesh in $[0, 1]$ or Chebyshev due to displacement of nodes.
- We have to repeat the method for every output node.
- For computing polynomial curves, the idea is the same as the explained in the last lecture, but using Lagrange basis.

Example: Use Neville's method to obtain the Lagrange interpolant polynomial through $(-2, 1)$, $(-1, 3)$, $(1, 0)$ and $(4, -2)$

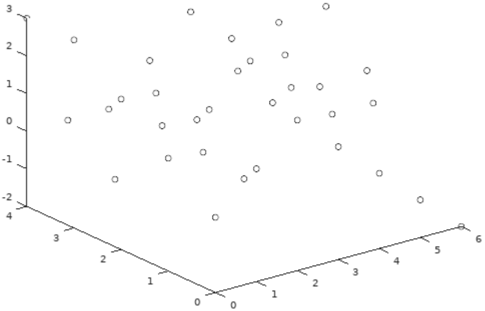
Interpolation of Surfaces using Lagrange polynomials

Consider a rectangular region $[a, b] \times [c, d] \in \mathbb{R}^2$

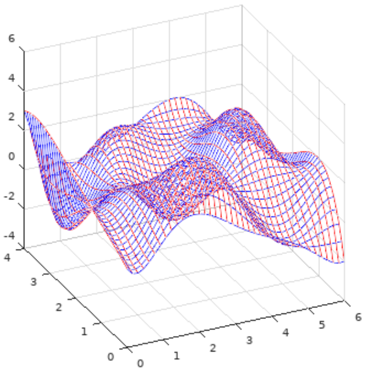
Consider a grid of $(m+1) \times (n+1)$ nodes in that region

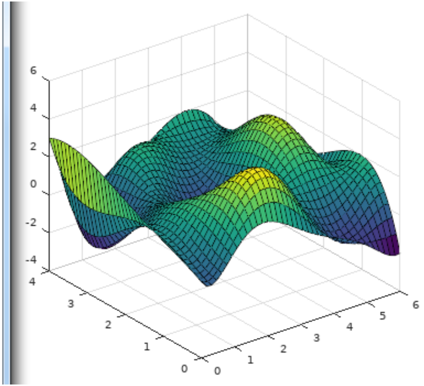


We can use Lagrange interpolation to construct a surface through (x_i, y_j, z_{ij}) for $i = 0, 1, \ldots, m$ and $j = 0, 1, \ldots, n$.



The surface will be of the form $\sigma : [a, b] \times [c, d] \in \mathbb{R}^2 \to \mathbb{R}$ with $\sigma(x, y)$ an interpolant polynomial in two variables.



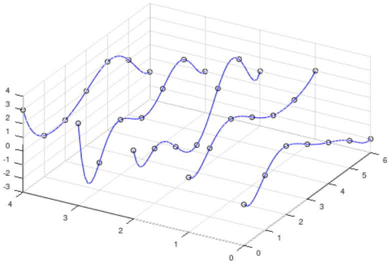


How do we compute it?

Fixing y-coordinate

If we fix the y-component to a certain y_j for $j = 0, 1, \ldots n$, then σ restricted to y_j will be like $\hat{\sigma}_{y_j} : [a, b] \in \mathbb{R} \to \mathbb{R}$ with $\hat{\sigma}_{y_j}(x)$ an interpolant polynomial through the nodes (x_i, y_j, z_{ij}). The result will be a curve for each y_j and can be computed with Lagrange interpolant polynomials as follows

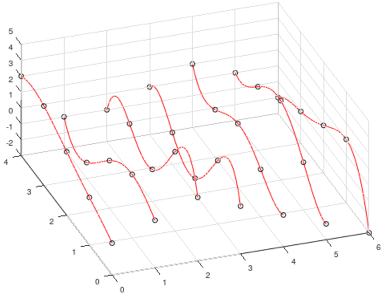
$$\hat{\sigma}_{y_j}(x) = \sum_{i=0}^{m} z_{ij} L_i^m(x)$$



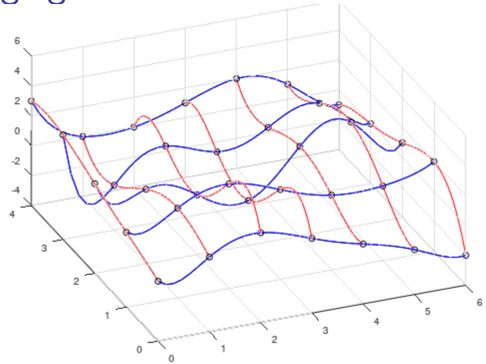
Fixing x-coordinate

If we fix the x-component to a certain x_i for $i = 0, 1, \ldots m$, then σ restricted to x_i will be like $\hat{\sigma}_{x_i} : [c, d] \in \mathbb{R} \to \mathbb{R}$ with $\hat{\sigma}_{x_i}(y)$ an interpolant polynomial through the nodes $({\color{red}x_i}, y_j, z_{{\color{red}i}j})$. The result will be a curve for each x_i and can be computed with Lagrange interpolant polynomials as follows

$$\hat{\sigma}_{x_i}(y) = \sum_{j=0}^{n} z_{ij} L_j^n(y)$$



Merging the curves into a surface



the functions $\hat{\sigma}_{x_i}$ determine how the nodes move along the y-axis for the function $\hat{\sigma}_{y_j}$ and viceversa, i.e.

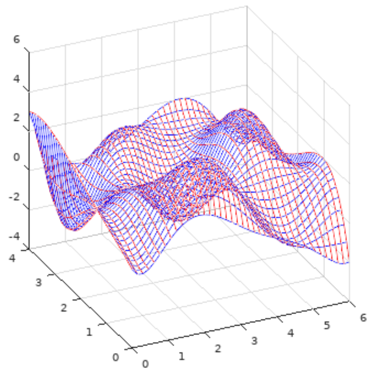
$$\sigma(x,y) = \sum_{i=0}^{m} \hat{\sigma}_{x_i}(y) L_i^m(x) = \sum_{i=0}^{m} \sum_{j=0}^{n} z_{ij} L_j^n(y) L_i^m(x)$$

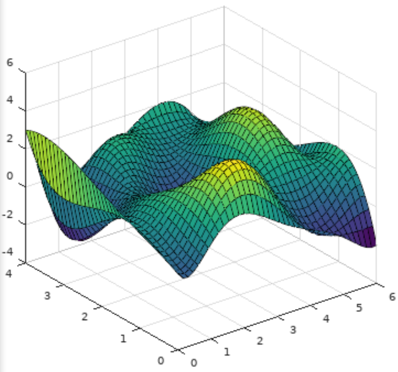
$\hat{\sigma}_{x_i}(y_j) = z_{ij}$

Evalutating

$$\sigma(x,y) = \sum_{i=0}^{m} \sum_{j=0}^{n} z_{ij} L_j^n(y) L_i^m(x)$$

at a bigger mesh of nodes we obtain





Tensor product spaces

To which polynomial vector space belongs the interpolant surface?

Definition

Let V and W be vector spaces with $\dim(V) = m + 1$ and $\dim(W) = n + 1$.

The tensor product $V \otimes W$ is the vector space with basis

$$B = \{v_i \times w_j,\ for\ i = 0, 1, \ldots, m,\ j = 0, 1, \ldots n\}$$

being $B_1 = \{v_i,\ i = 0, 1, \ldots, m\}$ and $B_2 = \{w_j,\ j = 0, 1, \ldots, n\}$ respective bases for V and W.

$$\dim(V \otimes W) = \dim(V) \times \dim(W) = (m + 1)(n + 1)$$

So $\sigma \in P_m \times P_n$ which basis is $\{1,\ x,\ y,\ x^2,\ y^2,\ xy, \ldots, x^m y^n\}$