Homework 5

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Consider the parametrized polynomial curve $\gamma : [0,1] \to \mathbb{R}^2$ given by

$$\gamma(t) = (1 - 4t + 8t^2 - 3t^3, 2 + 4t - 5t^2 + 2t^3) \tag{1}$$

1. (20%) Compute its polar form.

   $\gamma(t) = (1,2) + t(-4,4) + t^2(8,-5) + t^3(-3,2)$

   Create polar forms for the standard basis in P_3 for $\{1, t, t^2, t^3\}$:

   $F_0[u_1, u_2, u_3] = 1$

   $F_1[u_1, u_2, u_3] = \frac{u_1+u_2+u_3}{3}$

   $F_2[u_1, u_2, u_3] = \frac{u_1u_2+u_1u_3+u_2u_3}{3}$

   $F_3[u_1, u_2, u_3] = u_1u_2u_3$

   Create polar form from γ:

   $F[u_1, u_2, u_3] = F_0(1,2) + F_1(-4,4) + F_2(8,-5) + F_3(-3,2)$

   Substituting:

   $F[u_1, u_2, u_3] = (1,2) + \frac{u_1+u_2+u_3}{3}(-4,4) + \frac{u_1u_2+u_1u_3+u_2u_3}{3}(8,-5) + u_1u_2u_3(-3,2)$

   $F[u_1, u_2, u_3] = \begin{pmatrix} 1 - \frac{4}{3}(u_1+u_2+u_3) + \frac{8}{3}(u_1u_2+u_1u_3+u_2u_3) - 3(u_1u_2u_3) \\ 2 + \frac{4}{3}(u_1+u_2+u_3) - \frac{5}{3}(u_1u_2+u_1u_3+u_2u_3) + 2(u_1u_2u_3) \end{pmatrix}$

2. (15%) Use the polar form to obtain the control points of its Bezier representation and give the Bezier representation of the curve.

   γ has a Bezier representation $\gamma(t) = \sum_{i=0}^{3} P_i B_i^3(t), \quad t \in [0,1]$

   Control points are P_0, P_1, P_2 and P_3.

   $P_0 = F[0,0,0] = \begin{pmatrix} 1 - \frac{4}{3}(0) + \frac{8}{3}(0) - 3(0) \\ 2 + \frac{4}{3}(0) - \frac{5}{3}(0) + 2(0) \end{pmatrix} = (1,2)$

$$P_1 = F[0,0,1] = \begin{pmatrix} 1 - \frac{4}{3}(1) + \frac{8}{3}(0) - 3(0) \\ 2 + \frac{4}{3}(1) - \frac{5}{3}(0) + 2(0) \end{pmatrix} = (-\frac{1}{3}, \frac{10}{3})$$

$$P_2 = F[0,1,1] = \begin{pmatrix} 1 - \frac{4}{3}(2) + \frac{8}{3}(1) - 3(1) \\ 2 + \frac{4}{3}(2) - \frac{5}{3}(1) + 2(1) \end{pmatrix} = (2,5)$$

$$P_3 = F[1,1,1] = \begin{pmatrix} 1 - \frac{4}{3}(3) + \frac{8}{3}(3) - 3(1) \\ 2 + \frac{4}{3}(3) - \frac{5}{3}(3) + 2(1) \end{pmatrix} = (2,3)$$

3. (15%) Now consider the curve $\gamma : [-2,-1] \to \mathbb{R}^2$ given with the above formula.Obtain the control points of its Bezier representation and give the Bezier representation of the curve.

   γ has a Bezier representation $\gamma(t) = \sum_{i=0}^{3} P_i B_i^3(t), \quad t \in [-2,-1]$

   Control points are P_0, P_1, P_2 and P_3.

$$P_0 = F[-2,-2,-2] = \begin{pmatrix} 1 - \frac{4}{3}(-6) + \frac{8}{3}(12) - 3(-8) \\ 2 + \frac{4}{3}(-6) - \frac{5}{3}(12) + 2(-8) \end{pmatrix} = (65,-42)$$

$$P_1 = F[-2,-2,-1] = \begin{pmatrix} 1 - \frac{4}{3}(-5) + \frac{8}{3}(10) - 3(-4) \\ 2 + \frac{4}{3}(-5) - \frac{5}{3}(10) + 2(-4) \end{pmatrix} = (\frac{139}{3}, -\frac{88}{3})$$

$$P_2 = F[-2,-1,-1] = \begin{pmatrix} 1 - \frac{4}{3}(-4) + \frac{8}{3}(5) - 3(-2) \\ 2 + \frac{4}{3}(-4) - \frac{5}{3}(5) + 2(-2) \end{pmatrix} = (\frac{77}{3}, -\frac{47}{3})$$

$$P_3 = F[-1,-1,-1] = \begin{pmatrix} 1 - \frac{4}{3}(-3) + \frac{8}{3}(3) - 3(-1) \\ 2 + \frac{4}{3}(-3) - \frac{5}{3}(3) + 2(-1) \end{pmatrix} = (\frac{48}{3}, -9)$$

4. (20%) Compute the derivative of the Bezier curve in exercise 3.

   $\gamma'(t) = (p'(t), q'(t)) \quad$ where $p', q' \in P_2$.

   $p'(t) = -4 + 16t - 9t^2$

   $q'(t) = 4 - 10t + 6t^2$

5. (10%) Compute the tangent line to the curve in exercise 3 at $t = \frac{3}{4}$.

   The expression for tangent line is $f(x,y) = \gamma(\frac{3}{4}) + \lambda\gamma'(\frac{3}{4}), \quad \lambda \in \mathbb{R}$

   Evaluating γ and γ' at $t = \frac{3}{4}$ we get:

   $f(x,y) = (\frac{79}{64}, \frac{97}{32}) + \lambda(\frac{47}{16}, -\frac{1}{8})$

6. (20%) Let $P_0 = (1,4), P_1 = (2,3)$ and $P_2 = (-1,-1)$ be the control points of a quadratic Bezier curve. Give the implicit expression $f(x,y) = 0$ of the quadratic curve on which it lies.

First check that γ is not **degenerate**:

If $v \nparallel w$, being $v = P_1 - P_0$ and $w = P_2 - P_0$

$\nparallel$ if $v \cdot w \neq |v| \cdot |w|$

$(v \cdot w = 3) \neq (\sqrt{2} \cdot \sqrt{29})$ ✓

Let consider lines

$l_0 : a_0x + b_0y + c_0 = 0$ and $L_0(x, y) = a_0x + b_0y + c_0$ through P_0 and P_1

$l_1 : a_1x + b_1y + c_1 = 0$ and $L_1(x, y) = a_1x + b_1y + c_1$ through P_0 and P_2

$l_2 : a_2x + b_2y + c_2 = 0$ and $L_2(x, y) = a_2x + b_2y + c_2$ through P_1 and P_2

Evaluating at P_0 and P_1 with $\overrightarrow{P_0P_1} = (1, -1)$ with normal vector $\overrightarrow{n} = (1, 1)x+y+c_0$ substitute P_0 $1 + 4 + c_0 = 0$ so $l_0 : x + y - 5 = 0$ we obtain $L_0(x, y) = x + y - 5$

Evaluating at P_0 and P_2 with $\overrightarrow{P_0P_2} = (-2, -5)$ with normal vector $\overrightarrow{n} = (5, -2)5x-2y+c_1$ substitute P_0 $5-8+c_1 = 0$ so $l_0 : 5x-2y+3 = 0$ we obtain $L_1(x, y) = 5x-2y+3$

Evaluating at P_1 and P_2 with $\overrightarrow{P_1P_2} = (-3, -4)$ with normal vector $\overrightarrow{n} = (4, -3)4x-3y+c_2$ substitute P_1 $8-9+c_2 = 0$ so $l_0 : 4x-3y+1 = 0$ we obtain $L_2(x, y) = 4x-3y+1$

The implicit expression for the quadratic curve is

$$f_k(x, y) = 0$$

where

$$f_k(x, y) = L_0(x, y)L_2(x, y) + kL_1(x, y)^2$$

so the equation is

$$L_0(x, y)L_2(x, y) + kL_1(x, y)^2 = 0$$

substituting

$$(x + y - 5)(4x - 3y + 1) + k(5x - 2y + 3)^2 = 0$$

expanding we get

$$(4x^2 - 3y^2 + xy - 19x + 16y - 5) + k(25x^2 + 4y^2 - 20xy + 30x - 12y + 9) = 0$$

To find k we need a point in the curve that satisfies the equation. Find $\gamma(\frac{1}{2})$ through midpoint subdivision.

$$\begin{array}{ccc} P_0 = (1, 4) & & \\ & P_0^1 = (\frac{3}{2}, \frac{7}{2}) & \\ P_1 = (2, 3) & & P_0^2 = (1, \frac{9}{4}) \\ & P_1^1 = (\frac{1}{2}, 1) & \\ P_2 = (-1, -1) & & \end{array}$$

$\gamma(\frac{1}{2}) = (1, \frac{9}{4})$ point in the curve $(x, y) = (1, \frac{9}{4})$ satisfies $f_k(x, y) = 0$, so we substitute the point in the equation

$$(4(1)^2 - 3(\frac{9}{4})^2 + (1)(\frac{9}{4}) - 19(1) + 16(\frac{9}{4}) - 5) + k(25(1)^2 + 4(\frac{9}{4})^2 - 20(1)(\frac{9}{4}) + 30(1) - 12(\frac{9}{4}) + 9) = 0$$

$$\frac{49}{16} + k(\frac{49}{4}) = 0$$

$$k = \frac{49 \cdot 4}{16 \cdot 49} = \frac{1}{4}$$

so the curve lies in the parabola

$$(4x^2 - 3y^2 + xy - 19x + 16y - 5) + \frac{1}{4}(25x^2 + 4y^2 - 20xy + 30x - 12y + 9) = 0$$

which simplifying is

$$\frac{41}{4}x^2 - 2y^2 - 4xy - \frac{23}{2}x + 13y - \frac{11}{4} = 0$$