DigiPen Institute of Technology, Bilbao

MAT300 Curves and Surfaces
Spring, 2020. Midterm exam: 2/24/2020
Time Limit: 2 hours

Name: SOLUTIONS

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1. (60%) Let $x_0 = -2$, $x_1 = -1$, $x_2 = 2$ and $x_3 = 3$. Consider a polynomial $p : \mathbb{R} \to \mathbb{R}$ satisfying $p(x_0) = -57$, $p(x_1) = -15$, $p(x_2) = -9$ and $p(x_3) = -7$.
   (a) (5%) Determine the vector space in which p is the unique polynomial satisfying the above conditions, and give the standard basis for that space.
   (b) (10%) Give the Lagrange basis for the vector space using the above nodes.
   (c) (5%) Give the vector of coordinates of p in the Lagrange basis obtained in (b).
   (d) (10%) Construct a change of basis transformation from Lagrange to Standard and use it to obtain the vector of coordinates of p in the standard basis.
   (e) (5%) Give the Newton basis for the vector space using the above nodes.
   (f) (10%) Compute the divided differences to obtain the vector of coordinates of p in the Newton basis obtained in (e).
   (g) (15%) Construct a transformation for a change of basis from Lagrange to Newton and verify that the vectors of coordinates obtained in (c) and (f) correspond to the same polynomial.

2. (20%) Let $x_0 = 1$, $x_1 = 2$ and $x_2 = 4$. Consider a polynomial $p : \mathbb{R} \to \mathbb{R}$ satisfying $p(x_0) = 1$, $p'(x_0) = 3$, $p''(x_0) = 4$, $p^{(3)}(x_0) = 6$, $p(x_1) = 0$, $p'(x_1) = 1$ and $p(x_2) = -2$.
   (a) (5%) Give the vector space in which p is the unique polynomial satisfying the above conditions, and its Newton basis using the above nodes.
   (b) (15%) Compute the divided differences and give the vector of coordinates of the polynomials in the Newton basis obtained in (a).

3. (20%) Consider the polynomial

$$p(x) = \begin{cases} 1 + 2x - x^2, & x \in [0,1) \\ 2x + x^2, & x \in [1,2) \\ 1 - 3x + 2x^2, & x \in [2,4] \end{cases}$$

   (a) (5%) Determine to which vector space it belongs and give a right shifted basis for that space.
   (b) (15%) Compute the vector of coordinates of p in the basis obtained in (a).

$$p: \mathbb{R} \to \mathbb{R}$$

1. $x_0 = -2$ $\quad p(x_0) = -57$

   $x_1 = -1$ $\quad p(x_1) = -15$

   $x_2 = 2$ $\quad p(x_2) = -9$

   $x_3 = 3$ $\quad p(x_3) = -7$

a) we have 4 points with distinct x-coordinate so there is a unique polynomial in P_3 passing through them.

Vector space: P_3

Standard basis: $\quad B = \{1, x, x^2, x^3\}$

b) the Lagrange basis for P_3 with nodes x_0, x_1, x_2 and x_3 is $\quad B_L = \{L_0^3, L_1^3, L_2^3, L_3^3\}$ where $L_i^3(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}$

$$L_0^3(x) = \left(\frac{x+1}{-2+1}\right)\left(\frac{x-2}{-2-2}\right)\left(\frac{x-3}{-2-3}\right) = \frac{-x^3+4x^2-x-6}{20}$$

$$L_1^3(x) = \left(\frac{x+2}{-1+2}\right)\left(\frac{x-2}{\cancel{2}-1-2}\right)\left(\frac{x-3}{-1-3}\right) = \frac{x^3-3x^2-4x+12}{12}$$

$$L_2^3(x) = \left(\frac{x+2}{2+2}\right)\left(\frac{x+1}{2+1}\right)\left(\frac{x-3}{2-3}\right) = \frac{-x^3+7x+6}{12}$$

$$L_3^3(x) = \left(\frac{x+2}{3+2}\right)\left(\frac{x+1}{3+1}\right)\left(\frac{x-2}{3-2}\right) = \frac{x^3+x^2-4x-4}{20}$$

c) $p(x) = \sum_{k=0}^{3} y_k L_k^3(x)$ so the vector of coordinates of p in the basis B_L is $(-57, -15, -9, -7)_{B_L}$ $\quad$ 10.54

d) $T: \mathbb{R}^4 \to \mathbb{R}^4$ $\quad T(\vec{x}) = M\vec{x}$ where M is given by placing the vectors of coordinates of the Lagrange polynomials in the standard basis in columns.

$$M = \begin{pmatrix} -6/20 & 12/12 & 6/12 & -4/20 \\ -1/20 & -4/12 & 7/12 & -4/20 \\ 4/20 & -3/12 & 0 & 1/20 \\ -1/20 & 1/12 & -1/12 & 1/20 \end{pmatrix}$$

Applying T to $(-57, -15, -9, -7)$ we get

$$\begin{pmatrix} -6/20 & 12/12 & 6/12 & -4/20 \\ -1/20 & -4/12 & 7/12 & -4/20 \\ 4/20 & -3/12 & 0 & 1/20 \\ -1/20 & 1/12 & -1/12 & 1/20 \end{pmatrix} \begin{pmatrix} -57 \\ -15 \\ -9 \\ -7 \end{pmatrix}_{B_L} = \begin{pmatrix} -1 \\ 4 \\ -8 \\ 2 \end{pmatrix}_S$$

So the vector of coordinates of p in the standar basis is $(-1, 4, -8, 2)$

e) Newton basis for P_3 with nodes x_0, x_1, x_2, x_3 is

$$B_N = \{N_0, N_1, N_2, N_3\} = \{1, x+2, x^2+3x+2, x^3+x^2-4x-4\}$$

$N_0 = 1$

$N_1 = (x - x_0) = x+2$

$N_2 = (x-x_0)(x-x_1) = (x+2)(x+1) = x^2+3x+2$

$N_3 = (x-x_0)(x-x_1)(x-x_2) = (x+2)(x+1)(x-2) = x^3+x^2-4x-4$

11.05

f)

Nodes	$f[x_i]$	$f[x_i,x_j]$	$f[x_i,x_j,x_k]$	$f[x_i,x_j,x_k,x_l]$
-2	-57			
		$\frac{-15+57}{-1+2} = 42$		
-1	-15		$\frac{2-42}{2+2} = -10$	
		$\frac{-9+15}{2+1} = 2$		$\frac{0+10}{3+2} = 2$
2	-9		$\frac{2-2}{3+1} = 0$	
		$\frac{-7+9}{3-2} = 2$		
3	-7			

p in the Newton basis is given by the first divided difference of each order

$$(-57, 42, -10, 2)_{B_N}$$

g) Lagrange $\xrightarrow{T}$ Standard $\longrightarrow$ Newton

The transformation $T: \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ in (d) changes from Lagrange to Standard.

Let $S: \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be the transformation from Newton to Standard $S(\vec{x}) = N \cdot \vec{x}$ where

$$N = \begin{pmatrix} 1 & 2 & 2 & -4 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

then $(S^{-1} \circ T): \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ will be the transformation from Lagrange to Newton. $(S^{-1} \circ T)(\vec{x}) = (N^{-1} \cdot M) \cdot \vec{x}$

$$N^{-1} = \begin{pmatrix} 1 & -2 & 4 & -8 \\ 0 & 1 & -3 & 7 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ and

$$N^{-1} \cdot M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1/4 & -1/3 & 1/12 & 0 \\ -1/20 & 1/12 & -1/12 & 1/20 \end{pmatrix}$$

applying the transformation to $(-57, -15, -9, -7)$ we get

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1/4 & -1/3 & 1/12 & 0 \\ -1/20 & 1/12 & -1/12 & 1/20 \end{pmatrix} \begin{pmatrix} -57 \\ -15 \\ -9 \\ -7 \end{pmatrix}_L = \begin{pmatrix} -57 \\ 42 \\ -10 \\ 2 \end{pmatrix}_N$$

$p(1)=1 \quad p'(1)=3 \quad p''(1)=4 \quad p'''(1)=6$

$p(2)=0 \quad p'(2)=1$

$p(4)=-2$

(a) We have 7 conditions so the polynomial will be unique in P_6

$B=\{N_0, N_1, N_2, N_3, N_4, N_5, N_6\}$

where

$N_0 = 1$

$N_1 = (x-1) = x-1$

$N_2 = (x-1)^2 = x^2-2x+1$

$N_3 = (x-1)^3 = x^3-3x^2+3x-1$

$N_4 = (x-1)^4 = x^4-4x^3+6x^2-4x+1$

$N_5 = (x-1)^4(x-2) = x^5-6x^4+14x^3-16x^2+9x-2$

$N_6 = (x-1)^4(x-2)^2 = x^6-8x^5+26x^4-44x^3+41x^2-20x+4$

b) Divided differences

Nodes	0th	1st	2nd	3rd	4th
1	1	$\frac{p'(1)}{1!}=3$	$\frac{p''(1)}{2!}=2$	$\frac{p'''(1)}{3!}=1$	$\frac{-6-1}{2-1}=-7$
1	1	3	2	$\frac{-4-2}{2-1}=-6$	$\frac{6+6}{2-1}=12$
1	1	3	$\frac{-1-3}{2-1}=-4$	$\frac{2+4}{2-1}=6$	$\frac{-1-6}{4-1}=-\frac{7}{3}$
1	1	$\frac{0-1}{2-1}=-1$	$\frac{1+1}{2-1}=2$	$\frac{-1-2}{4-1}=-1$	
2	0	$\frac{p'(2)}{1!}=1$	$\frac{-1-1}{4-2}=-1$		
2	0	$\frac{-2-0}{4-2}=-1$			
4	-2				

5th 6th

$$\frac{12+7}{2-1} = 19 \qquad \frac{-\frac{43}{9} - 19}{4-1} = \frac{-214}{27}$$

$$\frac{-\frac{7}{3} - 12}{4-1} = \frac{-43}{9}$$

Vector of coordinates

$$\left(1, 3, 2, 1, -7, 19, -\frac{214}{27}\right)$$

③ $p \in P_2^3[0,1,2,4]$

a) $\lim_{p \to 1^-} p(x) = 1 + 2 - 1 = 2 \qquad \lim_{p \to 1^+} p(x) = 2 + 1 = 3$

$\to p$ is not continuous therefore $p \in P_2^3[0,1,2,4]$

basis

$$B = \{1, x, x^2, (x-1)_+^0, (x-1)_+^1, (x-1)_+^2, (x-2)_+^0, (x-2)_+^1, (x-2)_+^2\}$$

1.49

b) $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 (x-1)_+^0 + a_4 (x-1)_+^1 + a_5 (x-1)_+^2 + a_6 (x-2)_+^0 + a_7 (x-2)_+^1 + a_8 (x-2)_+^2$

For $x \in [0,1)$

$1 + 2x - x^2 = a_0 + a_1 x + a_2 x^2 \longrightarrow a_0 = 1,\ a_1 = 2$ and $a_2 = -1$

For $x \in [1,2)$

$\cancel{2x} + x^2 = 1 + \cancel{2x} - x^2 + a_3 + a_4(x-1) + a_5(x^2 - 2x + 1)$

$-1 + 2x^2 = (a_3 - a_4 + a_5) + (+a_4 - 2a_5)x + a_5 x^2$

$$\left.\begin{aligned} a_3 - a_4 + a_5 &= -1 \\ +a_4 - 2a_5 &= 0 \\ a_5 &= 2 \end{aligned}\right\} \quad \begin{aligned} a_3 &= 1 \\ a_4 &= +4 \end{aligned}$$

For $x \in [2,4]$

$$\cancel{1} - 3x + 2x^2 = \cancel{1} + 2x - x^2 + 1 + 4(x-1) + 2(x-1)^2 + a_6 + a_7(x-2) + a_8(x-2)^2$$

$$-3x + 2x^2 = 2x - x^2 + 1 + 4x - 4 + 2(x^2 - 2x + 1) + a_6 + a_7(x-2) + a_8(x^2 - 4x + 4)$$

$$-3x + \cancel{2x^2} = 6x - x^2 - 3 + \cancel{2x^2} - 4x + 2 + a_6 + a_7(x-2) + a_8(x^2 - 4x + 4)$$

$$-3x = 2x - x^2 - 1 + a_6 + a_7 x - 2a_7 + a_8 x^2 - 4a_8 x + 4a_8$$

$$-5x + x^2 + 1 = (a_6 - 2a_7 + 4a_8) + (a_7 - 4a_8)x + a_8 x^2$$

$$\left.\begin{aligned} a_6 - 2a_7 + 4a_8 &= 1 \\ a_7 - 4a_8 &= -5 \\ a_8 &= 1 \end{aligned}\right\} \quad \begin{aligned} a_6 &= -5 \\ a_7 &= -1 \end{aligned}$$

So the vector of coordinates is

$$(1, 2, -1, 1, 4, 2, -5, -1, 1)$$