MAT300 CURVES AND SURFACES

Julia Sánchez Sanz

DigiPen Institute of Technology Europe

julia.sanchez@digipen.edu

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Bezier curves

Bezier curves

Definition

A Bezier curve is a polynomial curve given as a linear combination of Bernstein polynomials as follows:

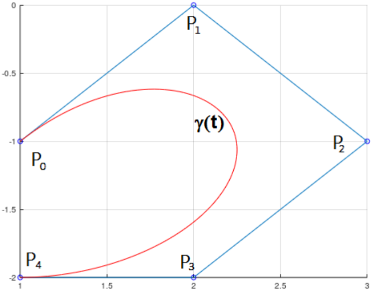
$$\gamma(t) = \sum_{i=0}^{n} P_i B_i^n(t), \qquad t \in [0,1]. \tag{1}$$

The points P_i for $i = 0, 1, \ldots, n$ are the control points of the curve.

If $P_i \in \mathbb{R}^2$ the curve $\gamma : [0,1] \to \mathbb{R}^2$ is planar.

If $P_i \in \mathbb{R}^3$ the curve $\gamma : [0,1] \to \mathbb{R}^3$ is a curve in 3D.

Example



$P_0 = (1, -1),\ P_1 = (2, 0),\ P_2 = (3, -1),\ P_3 = (2, -2)$ and $P_4 = (1, -2)$

$\gamma(t) = \sum_{i=0}^{4} P_i B_i^4(t) =$

$(1 + 4t - 8t^3 + 4t^4, -1 + 4t - 12t^2 + 8t^3 - t^4), \qquad t \in [0, 1]$

Properties: sum of Bernstein polynomials is 1

$\sum_{i=0}^{n} B_i^n(t) = 1 \; \forall t$ then every point $\gamma(t)$ has barycentric coordinates with respect to the control points P_0, P_1, $\ldots$, P_n.

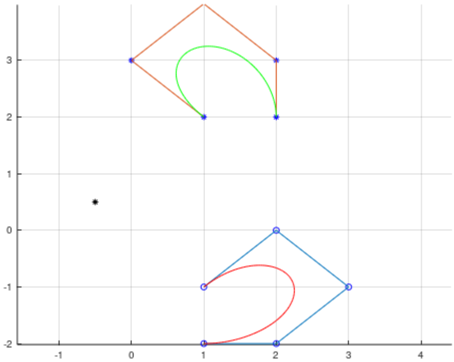
Good choice for affine transformations!

Remember that: barycentric coordinates are invariant under affine transformations.

This means that if we apply an affine transformation to $\gamma(t)$, instead of applying the transformation to every single point in the curve, we just have to apply the transformation to the control points and then compute the curve with the new control points.

Affine transformation

$$T : \mathbb{R}^2 \to \mathbb{R}^2 \qquad T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

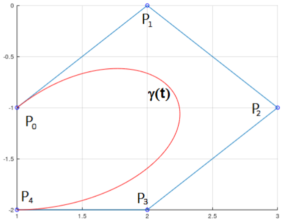


Properties: evaluation at $[0, 1]$

- $B_0^n(0) = 1$ and $B_i^n(0) = 0$ for $i = 1, \ldots, n \Rightarrow \gamma(0) = P_0$

- $B_n^n(1) = 1$ and $B_i^n(1) = 0$ for $i = 0, \ldots, n-1 \Rightarrow \gamma(1) = P_n$

- For $t \in (0,1)$ $B_i^n(t) = \begin{pmatrix} n \\ i \end{pmatrix} (1-t)^{n-i} t^i > 0$ and as for each t Bernstein polynomials define barycentric coordinates, $\gamma(t)$ is contained in the hull formed by the polygonal line through the control points $P_0 - P_1 - \ldots - P_n - P_0$.



Properties: symmetry of Bernstein polynomials

- $B_i^n(t) = B_{n-i}^n(1-t)$ therefore we can parametrize the curve in both directions.

  Taking $s = 1 - t$

  $$\gamma(t) = \sum_{i=0}^{n} P_i B_i^n(t) = \sum_{i=0}^{n} P_i \left(\begin{array}{c} n \\ i \end{array} \right) (1-t)^{n-i} t^i$$

  $$= \sum_{i=0}^{n} P_i \left(\begin{array}{c} n \\ i \end{array} \right) s^{n-i}(1-s)^i = \sum_{i=0}^{n} P_i \left(\begin{array}{c} n \\ n-i \end{array} \right) (1-s)^{n-(n-i)} s^{n-i}$$

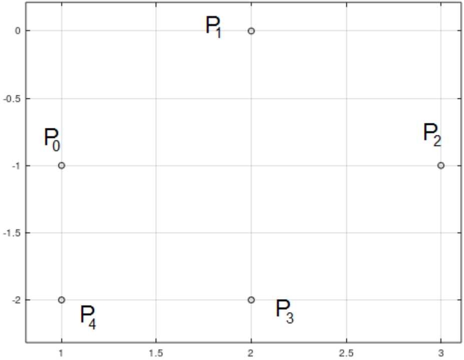
  $$= \sum_{i=0}^{n} P_i B_{n-i}^n(s)$$

Construction through recursion

The Bezier curve

$$\gamma(t) = \sum_{i=0}^{n} P_i B_i^n(t) \qquad t \in [0,1]$$

can be constructed through recursion using linear interpolation.



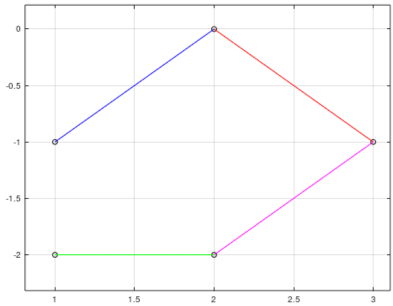
$P_0 = (1,-1)$

$P_1 = (2,0)$

$P_2 = (3,-1)$

$P_3 = (2,-2)$

$P_4 = (1,-2)$



P_0

$$P_0^1(t) = tP_1 + (1-t)P_0$$

P_1

$$P_1^1(t) = tP_2 + (1-t)P_1$$

P_2

$$P_2^1(t) = tP_3 + (1-t)P_2$$

P_3

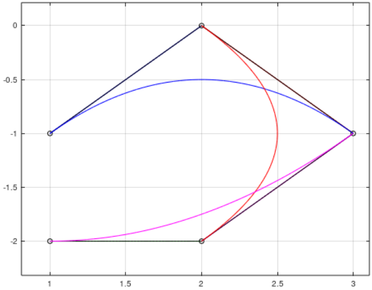
$$P_3^1(t) = tP_4 + (1-t)P_3$$

P_4

where

$$P_0^1(t) = (1+t, -1+t), \qquad P_1^1(t) = (2+t, -t),$$

$$P_2^1(t) = (3-t, -1-t), \qquad P_3^1(t) = (2-t, -2),$$



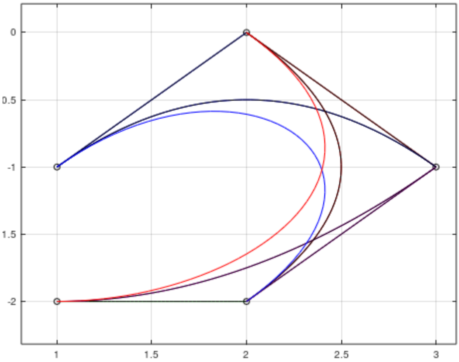
$$
\begin{array}{lll}
P_0 & & \\
 & P_0^1(t) & \\
P_1 & & P_0^2(t) = tP_1^1(t) + (1-t)P_0^1(t) \\
 & P_1^1(t) & \\
P_2 & & P_1^2(t) = tP_2^1(t) + (1-t)P_1^1(t) \\
 & P_2^1(t) & \\
P_3 & & P_2^2(t) = tP_3^1(t) + (1-t)P_2^1(t) \\
 & P_3^1(t) & \\
P_4 & &
\end{array}
$$

where

$$P_0^2(t) = t(2+t, -t) + (1-t)(1+t, -1+t) = (1+2t, -1+2t-2t^2)$$

$$P_1^2(t) = t(3-t, -1-t) + (1-t)(2+t, -t) = (2+2t-2t^2, -2t)$$

$$P_2^2(t) = t(2-t, -2) + (1-t)(3-t, -1-t) = (3-2t, -1-2t+t^2)$$

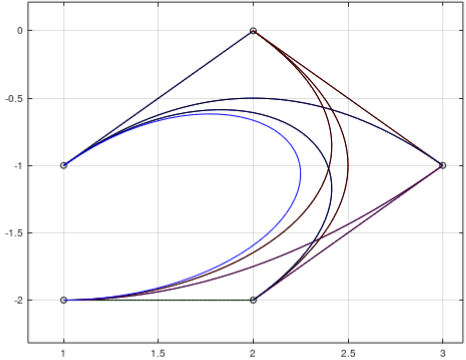


$$
\begin{array}{cccc}
P_0 & & & \\
 & P_0^1(t) & & \\
P_1 & & P_0^2(t) & \\
 & P_1^1(t) & & \color{blue}{P_0^3(t)} \\
P_2 & & P_1^2(t) & \\
 & P_2^1(t) & & \color{red}{P_1^3(t)} \\
P_3 & & P_2^2(t) & \\
 & P_3^1(t) & & \\
P_4 & & &
\end{array}
$$

where

$$\color{blue}{P_0^3(t) = tP_1^2(t) + (1-t)P_0^2(t) = (1 + 3t - 2t^3, -1 + 3t - 6t^2 + 2t^3)}$$

$$\color{red}{P_1^3(t) = tP_2^2(t) + (1-t)P_1^2(t) = (2 + 3t - 6t^2 + 2t^3, -3t + t^3)}$$



$$
\begin{array}{ccccc}
P_0 & & & & \\
 & P_0^1(t) & & & \\
P_1 & & P_0^2(t) & & \\
 & P_1^1(t) & & P_0^3(t) & \\
P_2 & & P_1^2(t) & & \gamma(t) \\
 & P_2^1(t) & & P_1^3(t) & \\
P_3 & & P_2^2(t) & & \\
 & P_3^1(t) & & & \\
P_4 & & & &
\end{array}
$$

where

$$\gamma(t) = tP_1^3(t) + (1-t)P_0^3(t) =$$

$$t(2+3t-6t^2+2t^3, -3t+t^3)+(1-t)(1+3t-2t^3, -1+3t-6t^2+2t^3) =$$

$$(1+4t-8t^3+4t^4, -1+4t-12t^2+8t^3-t^4)$$

More properties

- Linear interpolation implies convexity.

- The curvature is influenced by the control points.

The De Casteljau Algorithm

The implementation of the above technique is known as the De Casteljau algorithm.

Given $P_0, \ P_1, \ \ldots, \ P_n$ points in $\mathbb{R}^2$ or $\mathbb{R}^3$ the Bezier curve

$\gamma(t) = \sum_{i=0}^{n} P_i B_i^n(t), \ t \in [0,1]$ can be computed as follows:

- Construct a mesh of $m+1$ nodes in $[0,1]$, for instance $t_j = \frac{j}{m}$ for $j = 0, 1, \ldots, m$.
- For each node apply recursively linear interpolation to obtain $\gamma(t_j)$.
- Store the values of γ in an array for displaying the plot.

The De Casteljau Algorithm: Example

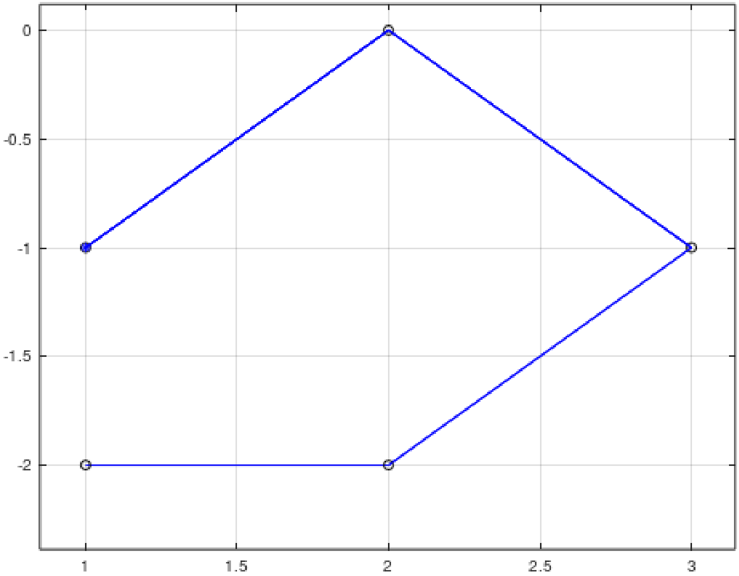
$P_0 = (1, -1)$, $P_1 = (2, 0)$, $P_2 = (3, -1)$, $P_3 = (2, -2)$ and $P_4 = (1, -2)$

6 points in $[0, 1]$ where $t_j = \frac{j}{5}$ for $j = 0, 1, \ldots, 5$.

For $t_0 = 0$ having $P_i^k(t_j) = t_j P_{i+1}^{k-1} + (1 - t_j) P_i^{k-1}$

$$
\begin{array}{lllll}
P_0 = (1, -1) & & & & \\
 & P_0^1 = (1, -1) & & & \\
P_1 = (2, 0) & & P_0^2 = (1, -1) & & \\
 & P_1^1 = (2, 0) & & P_0^3 = (1, -1) & \\
P_2 = (3, -1) & & P_1^2 = (2, 0) & & \gamma(0) = (1, -1) \\
 & P_2^1 = (3, -1) & & P_1^3 = (2, 0) & \\
P_3 = (2, -2) & & P_2^2 = (3, -1) & & \\
 & P_3^1 = (2, -2) & & & \\
P_4 = (1, -2) & & & &
\end{array}
$$

Doing a polygonal line for each level we can see better the linearity (this can be appreciated graphically for next values of t)

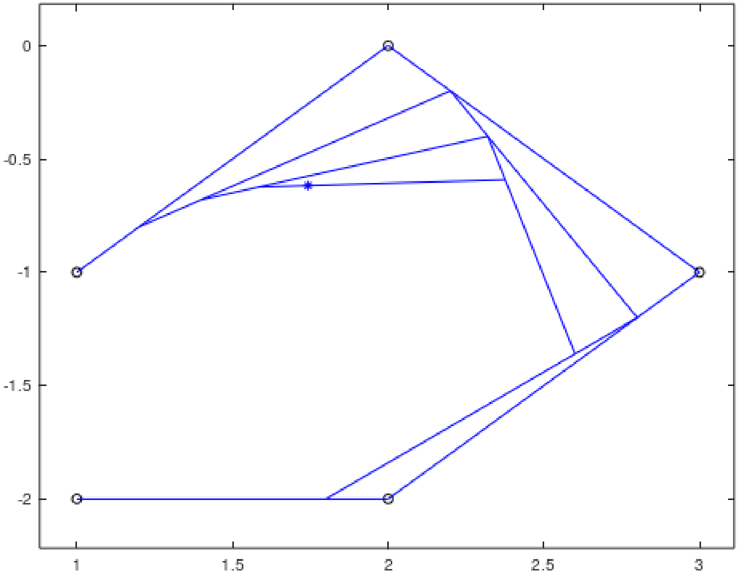

0
-0.5
-1
-1.5
-2
1
1.5
2
2.5
3

For $t_1 = \frac{1}{5}$ having $P_i^k(t_j) = t_j P_{i+1}^{k-1} + (1 - t_j) P_i^{k-1}$

$$
\begin{array}{lllll}
P_0 = (1,-1) & & & & \\
 & P_0^1 = (\frac{6}{5}, -\frac{4}{5}) & & & \\
P_1 = (2,0) & & P_0^2 = (\frac{7}{5}, -\frac{17}{25}) & & \\
 & P_1^1 = (\frac{11}{5}, -\frac{1}{5}) & & P_0^3 = (\frac{198}{125}, -\frac{78}{125}) & \\
P_2 = (3,-1) & & P_1^2 = (\frac{58}{25}, -\frac{2}{5}) & & \gamma(\frac{1}{5}) \\
 & P_2^1 = (\frac{14}{5}, -\frac{6}{5}) & & P_1^3 = (\frac{297}{125}, -\frac{74}{125}) & \\
P_3 = (2,-2) & & P_2^2 = (\frac{13}{5}, -\frac{34}{25}) & & \\
 & P_3^1 = (\frac{9}{5}, -2) & & & \\
P_4 = (1,-2) & & & &
\end{array}
$$

with $\gamma\left(\frac{1}{5}\right) = (\frac{1089}{625}, -\frac{386}{625})$

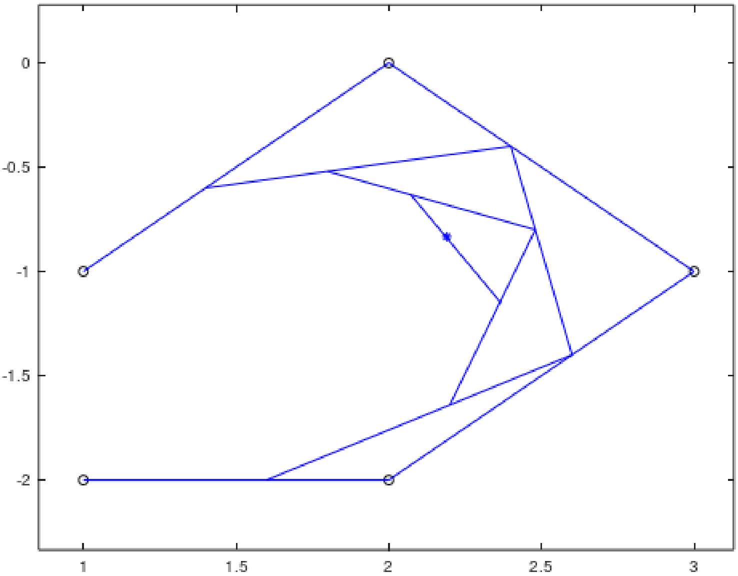
The intermediate resulting points are known as *intermediate Bezier points*. We can use these to obtain curves by subdivision.


0
-0.5
-1
-1.5
-2
1
1.5
2
2.5
3

For $t_2 = \frac{2}{5}$ having $P_i^k(t_j) = t_j P_{i+1}^{k-1} + (1-t_j) P_i^{k-1}$

$$\begin{array}{lllll}
P_0 = (1,-1) & & & & \\
 & P_0^1 = (\frac{7}{5}, -\frac{3}{5}) & & & \\
P_1 = (2,0) & & P_0^2 = (\frac{9}{5}, -\frac{13}{25}) & & \\
 & P_1^1 = (\frac{12}{5}, -\frac{2}{5}) & & P_0^3 = (\frac{259}{125}, -\frac{79}{125}) & \\
P_2 = (3,-1) & & P_1^2 = (\frac{62}{25}, -\frac{4}{5}) & & \gamma(\frac{2}{5}) \\
 & P_2^1 = (\frac{13}{5}, -\frac{7}{5}) & & P_1^3 = (\frac{296}{125}, -\frac{144}{125}) & \\
P_3 = (2,-2) & & P_2^2 = (\frac{11}{5}, -\frac{41}{25}) & & \\
 & P_3^1 = (\frac{8}{5}, -2) & & & \\
P_4 = (1,-2) & & & &
\end{array}$$

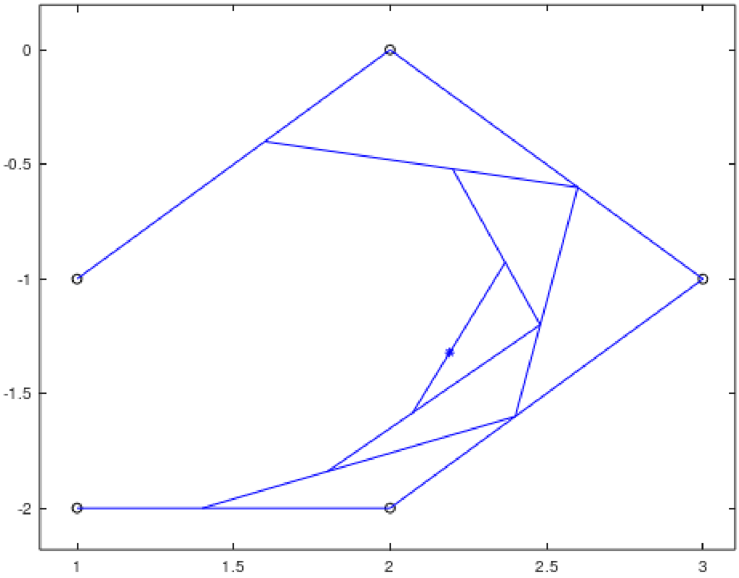
with $\gamma\left(\frac{2}{5}\right) = \left(\frac{1369}{625}, -\frac{521}{625}\right)$


0
-0.5
-1
-1.5
-2
1
1.5
2
2.5
3

For $t_3 = \frac{3}{5}$ having $P_i^k(t_j) = t_j P_{i+1}^{k-1} + (1 - t_j) P_i^{k-1}$

$$
\begin{array}{ccccc}
P_0 = (1,-1) & & & & \\
 & P_0^1 = (\frac{8}{5}, -\frac{2}{5}) & & & \\
P_1 = (2,0) & & P_0^2 = (\frac{11}{5}, -\frac{13}{25}) & & \\
 & P_1^1 = (\frac{13}{5}, -\frac{3}{5}) & & P_0^3 = (\frac{296}{125}, -\frac{116}{125}) & \\
P_2 = (3,-1) & & P_1^2 = (\frac{62}{25}, -\frac{6}{5}) & & \gamma(\frac{3}{5}) \\
 & P_2^1 = (\frac{12}{5}, -\frac{8}{5}) & & P_1^3 = (\frac{259}{125}, -\frac{198}{125}) & \\
P_3 = (2,-2) & & P_2^2 = (\frac{9}{5}, -\frac{46}{25}) & & \\
 & P_3^1 = (\frac{7}{5}, -2) & & & \\
P_4 = (1,-2) & & & &
\end{array}
$$

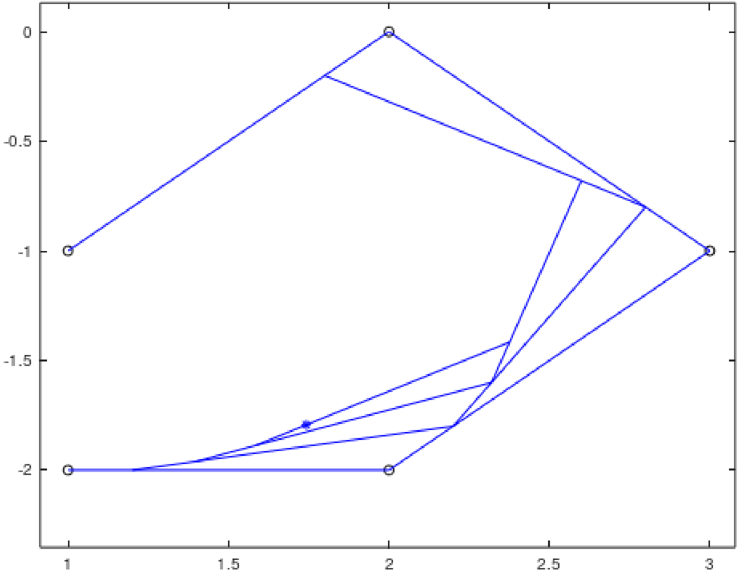
with $\gamma\left(\frac{3}{5}\right) = \left(\frac{1369}{625}, -\frac{826}{625}\right)$


0
-0.5
-1
-1.5
-2
1
1.5
2
2.5
3

For $t_4 = \frac{4}{5}$ having $P_i^k(t_j) = t_j P_{i+1}^{k-1} + (1-t_j) P_i^{k-1}$

$$
\begin{array}{lllll}
P_0 = (1,-1) & & & & \\
 & P_0^1 = (\frac{9}{5}, -\frac{1}{5}) & & & \\
P_1 = (2,0) & & P_0^2 = (\frac{13}{5}, -\frac{17}{25}) & & \\
 & P_1^1 = (\frac{14}{5}, -\frac{4}{5}) & & P_0^3 = (\frac{297}{125}, -\frac{177}{125}) & \\
P_2 = (3,-1) & & P_1^2 = (\frac{58}{25}, -\frac{8}{5}) & & \gamma(\frac{4}{5}) \\
 & P_2^1 = (\frac{11}{5}, -\frac{9}{5}) & & P_1^3 = (\frac{198}{125}, -\frac{236}{125}) & \\
P_3 = (2,-2) & & P_2^2 = (\frac{7}{5}, -\frac{49}{25}) & & \\
 & P_3^1 = (\frac{6}{5}, -2) & & & \\
P_4 = (1,-2) & & & &
\end{array}
$$

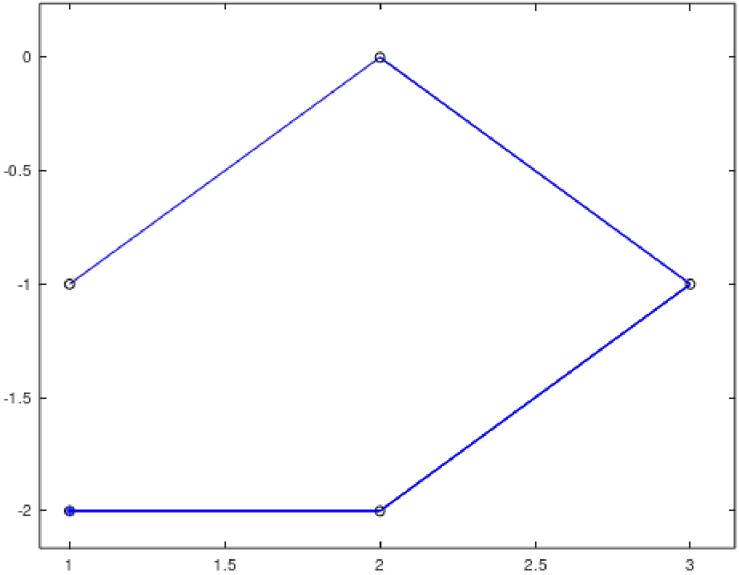
with $\gamma\left(\frac{4}{5}\right) = (\frac{1089}{625}, -\frac{1121}{625})$


0
-0.5
-1
-1.5
-2
1
1.5
2
2.5
3

For $t_5 = 1$ having $P_i^k(t_j) = t_j P_{i+1}^{k-1} + (1 - t_j) P_i^{k-1}$

$$
\begin{array}{ccccc}
P_0 = (1,-1) & & & & \\
 & P_0^1 = (2,0) & & & \\
P_1 = (2,0) & & P_0^2 = (3,-1) & & \\
 & P_1^1 = (3,-1) & & P_0^3 = (2,-2) & \\
P_2 = (3,-1) & & P_1^2 = (2,-2) & & \gamma(1) \\
 & P_2^1 = (2,-2) & & P_1^3 = (1,-2) & \\
P_3 = (2,-2) & & P_2^2 = (1,-2) & & \\
 & P_3^1 = (1,-2) & & & \\
P_4 = (1,-2) & & & &
\end{array}
$$

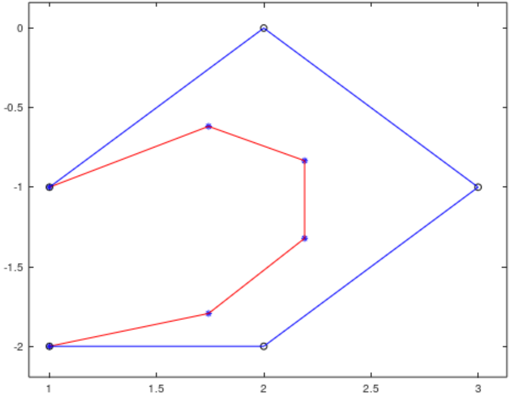
with $\gamma(1) = (1,-2)$


0
-0.5
-1
-1.5
-2
1
1.5
2
2.5
3

Now we take the obtained values $\gamma(0) = (1, -1)$, $\gamma\left(\frac{1}{5}\right) = \left(\frac{1089}{625}, -\frac{386}{625}\right)$,

$\gamma\left(\frac{2}{5}\right) = \left(\frac{1369}{625}, -\frac{521}{625}\right)$, $\gamma\left(\frac{3}{5}\right) = \left(\frac{1369}{625}, -\frac{826}{625}\right)$, $\gamma\left(\frac{4}{5}\right) = \left(\frac{1089}{625}, -\frac{1121}{625}\right)$ and

$\gamma(1) = (1, -2)$ and plot them using a polygonal line.



Direct evaluation using Bernstein polynomials

Bezier curves can be computed by direct evaluation of the Bernstein polynomials.

Given $P_0,\ P_1,\ \dots,\ P_n$ points in $\mathbb{R}^2$ or $\mathbb{R}^3$ the Bezier curve

$\gamma(t) = \sum_{i=0}^{n} P_i B_i^n(t),\ t \in [0,1]$ can be computed as follows:

- Construct a mesh of $m+1$ nodes in $[0,1]$, for instance $t_j = \frac{j}{m}$ for $j = 0, 1, \dots, m$.
- For each node t_j evaluate the Bernstein polynomials $B_i^n(t_j)$ for $i = 0, 1, \dots, n$ and store the values.
- For each node compute the sum $\gamma(t_j) = \sum_{i=0}^{n} P_i B_i^n(t_j)$ and store the values.
- Plot the resulting curve.

Example

$P_0 = (1, -1)$, $P_1 = (2, 0)$, $P_2 = (3, -1)$, $P_3 = (2, -2)$ and $P_4 = (1, -2)$

6 points in $[0, 1]$ where $t_j = \frac{j}{5}$ for $j = 0, 1, \ldots, 5$.

The Bernstein polynomials are:

$$B_0^4(t) = \left(\begin{array}{c} 4 \\ 0 \end{array} \right) (1-t)^4 t^0 = 1 - 4t + 6t^2 - 4t^3 + t^4$$

$$B_1^4(t) = \left(\begin{array}{c} 4 \\ 1 \end{array} \right) (1-t)^3 t^1 = 4t - 12t^2 + 12t^3 - 4t^4$$

$$B_2^4(t) = \left(\begin{array}{c} 4 \\ 2 \end{array} \right) (1-t)^2 t^2 = 6t^2 - 12t^3 + 6t^4$$

$$B_3^4(t) = \left(\begin{array}{c} 4 \\ 3 \end{array} \right) (1-t)^1 t^3 = 4t^3 - 4t^4$$

$$B_4^4(t) = \left(\begin{array}{c} 4 \\ 4 \end{array} \right) (1-t)^0 t^4 = t^4$$

Evaluation of the Bernstein polynomials at the nodes

```
t =

   0.00000   0.20000   0.40000   0.60000   0.80000   1.00000

>> B04=1-4*t+6*t.^2-4*t.^3+t.^4
B04 =

   1.00000   0.40960   0.12960   0.02560   0.00160   0.00000

>> B14=4*t-12*t.^2+12*t.^3-4*t.^4
B14 =

   0.00000   0.40960   0.34560   0.15360   0.02560   0.00000

>> B24=6*t.^2-12*t.^3+6*t.^4
B24 =

   0.00000   0.15360   0.34560   0.34560   0.15360   0.00000

>> B34=4*t.^3-4*t.^4
B34 =

   0.00000   0.02560   0.15360   0.34560   0.40960   0.00000

>> B44=t.^4
B44 =

   0.00000   0.00160   0.02560   0.12960   0.40960   1.00000
```

For each node compute the sum $\gamma(t_j) = \sum_{i=0}^{n} P_i B_i^n(t_j)$, store the values

```
>> gamma=P0.*B04+P1.*B14+P2.*B24+P3.*B34+P4.*B44
gamma =

   1.00000   1.74240   2.19040   2.19040   1.74240   1.00000
  -1.00000  -0.61760  -0.83360  -1.32160  -1.79360  -2.00000
```

and plot

```
>> plot(gamma(1,:),gamma(2,:),'r')
```