MAT300 CURVES AND SURFACES

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1. De Boor Algorithm

Introduction of the problem

We want to compute a B-spline curve $\gamma : [t_0, t_n] \to \mathbb{R}^2$ of order k given as

$$\gamma(t) = \sum_{i=0}^{N-k-1} \mathcal{B}_i^k(t) P_i \tag{1}$$

where:

- P_i for $i = 0, \ldots, N-k-1$ are control points in $\mathbb{R}^2$.
- The B-splines $\mathcal{B}_i^k(t)$ for $i = 0, \ldots, N-k-1$ are associated to

  a knot sequence $\vec{t} = (\bar{t}_0, \ldots, \bar{t}_N)$.

The curve that we are computing is of the form $\gamma(t) = (p(t), q(t))$ where

$p, q \in P^n_{k,\vec{r}}[t_0, \ldots, t_n]$.

Main difficulty

The main difficulty for approaching the problem is the piecewise definition of the B-splines.

B-splines of order k are piecewise defined in at most $k + 2$ pieces. The pieces change for each B-spline.

B-splines are defined recursively.

It is not a good idea to construct a explicit expression for each B-spline.

It is not a good idea to check for each t in the output mesh which piece to evaluate for each B-spline

The trick

We know that $\mathcal{B}_i^k(t) \neq 0$ for $t \in (\bar{t}_i, \bar{t}_{i+k+1}) \subset [\bar{t}_i, \bar{t}_{i+k+1})$

Therefore the idea is to fix the inteval $[\bar{t}_j, \bar{t}_{j+1})$

where $\mathcal{B}_{j-k}^k(t) \neq 0, \ldots, \mathcal{B}_j^k(t) \neq 0$ and evaluate piecewise

$$\gamma(t) = \sum_{i=j-k}^{j} P_i \mathcal{B}_i^k(t), \ t \in [\bar{t}_j, \bar{t}_{j+1}) \tag{2}$$

The evaluation of the $\mathcal{B}_i^k(t)$ is done through nested linear interpolation (a technique similar to De Casteljau) for each t. As we evaluate directly and we already set the interval we do not have to care about piecewise definition.

We do care about multiplicity of nodes.

De Boor algorithm

Input data: k, $\vec{t} = (\bar{t}_0, \ldots, \bar{t}_N)$, P_i for $i = 0, \ldots, N-k-1$.

- Construct output mesh in $[\bar{t}_k, \bar{t}_{N-k}]$ (regular)
- **for** each t in the mesh:
  - Determine index j such that $t \in [\bar{t}_j, \bar{t}_{j+1})$
  - Apply nested linear interpolation:

$$\begin{array}{llllll}
\color{red}{P_{j-k}} & P^0_{j-k} & & & & \\
\color{red}{P_{j-k+1}} & P^0_{j-k+1} & P^1_{j-k+1} & & & \\
\color{red}{\vdots} & \vdots & \vdots & \ddots & & \\
\color{red}{P_{j-1}} & P^0_{j-1} & P^1_{j-1} & P^2_{j-1} & \ddots & \\
\color{red}{P_j} & P^0_j & P^1_j & P^2_j & \ldots & P^k_j
\end{array}$$

where for $n = 1, \ldots, k$ and $i = j-k+n, \ldots, j$

$$P^n_i = \frac{\bar{t}_{i+k-n+1} - t}{\bar{t}_{i+k-n+1} - \bar{t}_i} P^{n-1}_{i-1} + \frac{t - \bar{t}_i}{\bar{t}_{i+k-n+1} - \bar{t}_i} P^{n-1}_i$$

  - $\gamma(t) = P^k_j$

Relation of the algorithm and the recursive construction

for $t \in [\bar{t}_j, \bar{t}_{j+1})$ we have

$$\gamma(t) = \sum_{i=j-k}^{j} P_i \mathcal{B}_i^k(t) = \sum_{i=j-k}^{j} P_i \left(\frac{t - \bar{t}_i}{\bar{t}_{i+k} - \bar{t}_i} \mathcal{B}_i^{k-1}(t) + \frac{\bar{t}_{i+k+1} - t}{\bar{t}_{i+k+1} - \bar{t}_{i+1}} \mathcal{B}_{i+1}^{k-1}(t) \right)$$

with $\mathcal{B}_i^{k-1}(t)$ defined on $[\bar{t}_i, \bar{t}_{i+k})$ and $\mathcal{B}_{i+1}^{k-1}(t)$ defined on $[\bar{t}_{i+1}, \bar{t}_{i+k+1})$

so $\mathcal{B}_{j-k}^{k-1}(t) = 0$ and $\mathcal{B}_{j+1}^{k-1}(t) = 0$

$$\gamma(t) = \sum_{i=j-k+1}^{j} P_i \frac{t - \bar{t}_i}{\bar{t}_{i+k} - \bar{t}_i} \mathcal{B}_i^{k-1}(t) + \sum_{i=j-k}^{j-1} P_i \frac{\bar{t}_{i+k+1} - t}{\bar{t}_{i+k+1} - \bar{t}_{i+1}} \mathcal{B}_{i+1}^{k-1}(t)$$

$$= \sum_{i=j-k+1}^{j} P_i \frac{t - \bar{t}_i}{\bar{t}_{i+k} - \bar{t}_i} \mathcal{B}_i^{k-1}(t) + \sum_{i=j-k+1}^{j} P_{i-1} \frac{\bar{t}_{i+k} - t}{\bar{t}_{i+k} - \bar{t}_i} \mathcal{B}_i^{k-1}(t)$$

$$= \sum_{i=j-k+1}^{j} \left(P_i \frac{t - \bar{t}_i}{\bar{t}_{i+k} - \bar{t}_i} + P_{i-1} \frac{\bar{t}_{i+k} - t}{\bar{t}_{i+k} - \bar{t}_i} \right) \mathcal{B}_i^{k-1}(t)$$

$$= \sum_{i=j-k+1}^{j} \left(\frac{\bar{t}_{i+k-1+1} - t}{\bar{t}_{i+k-1+1} - \bar{t}_i} P_{i-1}^{1-0} + \frac{t - \bar{t}_i}{\bar{t}_{i+k-1+1} - \bar{t}_i} P_i^{1-0} \right) \mathcal{B}_i^{k-1}(t)$$

$$= \sum_{i=j-k+1}^{j} P_i^1 \mathcal{B}_i^{k-1}(t)$$

Therefore

$$\gamma(t) = \sum_{i=j-k}^{j} P_i \mathcal{B}_i^k(t) = \sum_{i=j-k+1}^{j} P_i^1 \mathcal{B}_i^{k-1}(t) = \sum_{i=j-k+2}^{j} P_i^2 \mathcal{B}_i^{k-2}(t)$$

$$\dots = \sum_{i=j-k+k}^{j} P_i^k \mathcal{B}_i^{k-k}(t) = P_j^k \mathcal{B}_j^0(t) = P_j^k$$

Example

We want to compute a B-spline curve $\gamma : [0,3] \to \mathbb{R}^2$ $\gamma(t) = (p(t), q(t))$ where $p, q \in P_{3,\vec{r}}[0,1,2,3]$ with $\vec{r} = (0,2)$.

We have $k = 3$ and $\vec{m} = (3,1)$.

$\vec{t} = (0,0,0,0,1,1,1,2,3,3,3,3)$ is a valid knot sequence.

$N = 11$ so we need 8 control points.

$P_0 = (0,0)$, $P_1 = (0,-1)$, $P_2 = (1,-1)$, $P_3 = (2,0)$, $P_4 = (2,1)$, $P_5 = (1,2)$, $P_6 = (0,2)$, $P_7 = (-1,1)$

We do a mesh for t $(0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3)$

For each t in the mesh we apply De Boor algorithm

$\vec{t} = (0,0,0,0,1,1,1,2,3,3,3,3)$

$$P_i^n = \frac{\bar{t}_{i+k-n+1} - t}{\bar{t}_{i+k-n+1} - \bar{t}_i} P_{i-1}^{n-1} + \frac{t - \bar{t}_i}{\bar{t}_{i+k-n+1} - \bar{t}_i} P_i^{n-1}$$

For $t = 0$ $t \in [0,1) = [\bar{t}_3, \bar{t}_4)$ $j = 3$

$$P_0 = (0,0)$$

$$P_1 = (0,-1) \quad P_1^1 = (0,0)$$

$$P_2 = (1,-1) \quad P_2^1 = (0,-1) \quad P_2^2 = (0,0)$$

$$P_3 = (2,0) \quad P_3^1 = (1,-1) \quad P_3^2 = (0,-1) \quad P_3^3 = (0,0)$$

$\gamma(0) = (0,0)$

$\vec{t} = (0, 0, 0, 0, 1, 1, 1, 2, 3, 3, 3, 3)$

$$P_i^n = \frac{\bar{t}_{i+k-n+1} - t}{\bar{t}_{i+k-n+1} - \bar{t}_i} P_{i-1}^{n-1} + \frac{t - \bar{t}_i}{\bar{t}_{i+k-n+1} - \bar{t}_i} P_i^{n-1}$$

For $t = \frac{1}{2}$ $t \in [0, 1) = [\bar{t}_3, \bar{t}_4)$ $j = 3$

$P_0 = (0, 0)$

$P_1 = (0, -1) \quad P_1^1 = (0, -\frac{1}{2})$

$P_2 = (1, -1) \quad P_2^1 = (\frac{1}{2}, -1) \quad P_2^2 = (\frac{1}{4}, -\frac{3}{4})$

$P_3 = (2, 0) \quad P_3^1 = (\frac{3}{2}, -\frac{1}{2}) \quad P_3^2 = (1, -\frac{3}{4}) \quad P_3^3 = (\frac{5}{8}, -\frac{3}{4})$

$\gamma(\frac{1}{2}) = (\frac{5}{8}, -\frac{3}{4})$

$\vec{t} = (0, 0, 0, 0, 1, 1, 1, 2, 3, 3, 3, 3)$

$$P_i^n = \frac{\bar{t}_{i+k-n+1} - t}{\bar{t}_{i+k-n+1} - \bar{t}_i} P_{i-1}^{n-1} + \frac{t - \bar{t}_i}{\bar{t}_{i+k-n+1} - \bar{t}_i} P_i^{n-1}$$

For $t = 1$ $t \in [1, 2) = [\bar{t}_6, \bar{t}_7)$ $j = 6$

$$P_3 = (2, 0)$$

$$P_4 = (2, 1) \quad P_4^1 = (2, 0)$$

$$P_5 = (1, 2) \quad P_5^1 = (2, 1) \quad P_5^2 = (2, 0)$$

$$P_6 = (0, 2) \quad P_6^1 = (1, 2) \quad P_6^2 = (2, 1) \quad P_6^3 = (2, 0)$$

$\gamma(1) = (2, 0)$

$\vec{t} = (0,0,0,0,1,1,1,2,3,3,3,3)$

$$P_i^n = \frac{\bar{t}_{i+k-n+1} - t}{\bar{t}_{i+k-n+1} - \bar{t}_i} P_{i-1}^{n-1} + \frac{t - \bar{t}_i}{\bar{t}_{i+k-n+1} - \bar{t}_i} P_i^{n-1}$$

For $t = \frac{3}{2}$ $t \in [1,2) = [\bar{t}_6, \bar{t}_7)$ $j = 6$

$P_3 = (2,0)$

$P_4 = (2,1) \quad P_4^1 = (2, \frac{1}{2})$

$P_5 = (1,2) \quad P_5^1 = (\frac{7}{4}, \frac{5}{4}) \quad P_5^2 = (\frac{15}{8}, \frac{7}{8})$

$P_6 = (0,2) \quad P_6^1 = (\frac{3}{4}, 2) \quad P_6^2 = (\frac{3}{2}, \frac{23}{16}) \quad P_6^3 = (\frac{27}{16}, \frac{37}{32})$

$\gamma(\frac{3}{2}) = (\frac{27}{16}, \frac{37}{32})$

$\vec{t} = (0,0,0,0,1,1,1,2,3,3,3,3)$

$$P_i^n = \frac{\bar{t}_{i+k-n+1} - t}{\bar{t}_{i+k-n+1} - \bar{t}_i} P_{i-1}^{n-1} + \frac{t - \bar{t}_i}{\bar{t}_{i+k-n+1} - \bar{t}_i} P_i^{n-1}$$

For $t = 2$ $t \in [2,3) = [\bar{t}_7, \bar{t}_8)$ $j = 7$

$$\begin{array}{llll} P_4 = (2,1) & & & \\ P_5 = (1,2) & P_5^1 = (\frac{3}{2}, \frac{3}{2}) & & \\ P_6 = (0,2) & P_6^1 = (\frac{1}{2}, 2) & P_6^2 = (1, \frac{7}{4}) & \\ P_7 = (-1,1) & P_7^1 = (0,2) & P_7^2 = (\frac{1}{2}, 2) & P_7^3 = (1, \frac{7}{4}) \end{array}$$

$\gamma(2) = (1, \frac{7}{4})$

$\vec{t} = (0, 0, 0, 0, 1, 1, 1, 2, 3, 3, 3, 3)$

$$P_i^n = \frac{\bar{t}_{i+k-n+1} - t}{\bar{t}_{i+k-n+1} - \bar{t}_i} P_{i-1}^{n-1} + \frac{t - \bar{t}_i}{\bar{t}_{i+k-n+1} - \bar{t}_i} P_i^{n-1}$$

For $t = \frac{5}{2}$ $t \in [2, 3) = [\bar{t}_7, \bar{t}_8)$ $j = 7$

$P_4 = (2, 1)$

$P_5 = (1, 2) \qquad P_5^1 = (\frac{5}{4}, \frac{7}{4})$

$P_6 = (0, 2) \qquad P_6^1 = (\frac{1}{4}, 2) \qquad P_6^2 = (\frac{1}{2}, \frac{31}{16})$

$P_7 = (-1, 1) \quad P_7^1 = (-\frac{1}{2}, \frac{3}{2}) \quad P_7^2 = (-\frac{1}{8}, \frac{7}{4}) \quad P_7^3 = (\frac{3}{16}, \frac{59}{32})$

$\gamma(\frac{5}{2}) = (\frac{3}{16}, \frac{59}{32})$

$\vec{t} = (0, 0, 0, 0, 1, 1, 1, 2, 3, 3, 3, 3)$

For $t = 3$ we can not apply the algorithm. The last point is out of the domain!