MAT300 CURVES AND SURFACES

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B-splines

1 B-splines

Introduction

Given a polynomial $p \in P^n_{k,\vec{r}}[t_0, \ldots, t_n]$

$$p(t) = \begin{cases} p_1(t), \; t \in [t_0, t_1), \\ p_2(t), \; t \in [t_1, t_2), \\ \vdots \\ p_n(t), \; t \in [t_{n-1}, t_n], \end{cases}$$

with $p_i \in P_k$ for $i = 1, \ldots, n$ satisfying

$$p_j^{(m)}(t_j) = p_{j+1}^{(m)}(t_j), \quad j = 1, \ldots, n-1, \quad m = 0, \ldots, r_j \tag{1}$$

we want to write it as an affine combination of some control points c_i

$$p(t) = \sum c_i \mathcal{B}_i^k(t) \tag{2}$$

where $\mathcal{B}_i^k$ form a basis of splines called B-splines.

- How many control points and B-splines do we have for defining the curve?
- What is the interval of definition?

Recursive definition of B-splines

Definition

Given a knot sequence $\vec{t} = (\bar{t}_0, \bar{t}_1, \ldots, \bar{t}_N)$, the N B-splines of order zero are defined as

$$\mathcal{B}_i^0(t) = \begin{cases} 1 & \textit{if } t \in [\bar{t}_i, \bar{t}_{i+1}) \\ 0 & \textit{otherwise} \end{cases} \tag{3}$$

if $\bar{t}_i < \bar{t}_{i+1}$, and $\mathcal{B}_i^0(t) = 0$ if $\bar{t}_i = \bar{t}_{i+1}$ for $i = 0, \ldots, N-1$. Next we define the higher order B-splines through the following recursion for $i = 0, \ldots, N-m-1$:

$$\mathcal{B}_i^m(t) = {\color{blue}\frac{t - \bar{t}_i}{\bar{t}_{i+m} - \bar{t}_i}\mathcal{B}_i^{m-1}(t)} + {\color{red}\frac{\bar{t}_{i+m+1} - t}{\bar{t}_{i+m+1} - \bar{t}_{i+1}}\mathcal{B}_{i+1}^{m-1}(t)} \tag{4}$$

if $\bar{t}_{i+m} \neq \bar{t}_i$ and $\bar{t}_{i+m+1} \neq \bar{t}_{i+1}$. The blue term is zero if $\bar{t}_{i+m} = \bar{t}_i$, and the red term is zero if $\bar{t}_{i+m+1} = \bar{t}_{i+1}$.

Example

$\vec{t} = (-2, -1, 0, 1, 2, 3)$ knot sequence. We have 5 B-splines of order zero, they are piecewise defined functions that sum one in the interval $[-2, 3)$

- $\mathcal{B}_0^0(t) = \begin{cases} 1 & \textit{if } t \in [-2, -1) \\ 0 & \textit{otherwise} \end{cases}$
- $\mathcal{B}_1^0(t) = \begin{cases} 1 & \textit{if } t \in [-1, 0) \\ 0 & \textit{otherwise} \end{cases}$
- $\mathcal{B}_2^0(t) = \begin{cases} 1 & \textit{if } t \in [0, 1) \\ 0 & \textit{otherwise} \end{cases}$
- $\mathcal{B}_3^0(t) = \begin{cases} 1 & \textit{if } t \in [1, 2) \\ 0 & \textit{otherwise} \end{cases}$
- $\mathcal{B}_4^0(t) = \begin{cases} 1 & \textit{if } t \in [2, 3) \\ 0 & \textit{otherwise} \end{cases}$

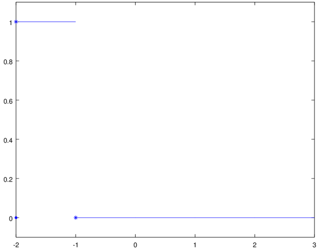


Figure: $\mathcal{B}_0^0(t)$

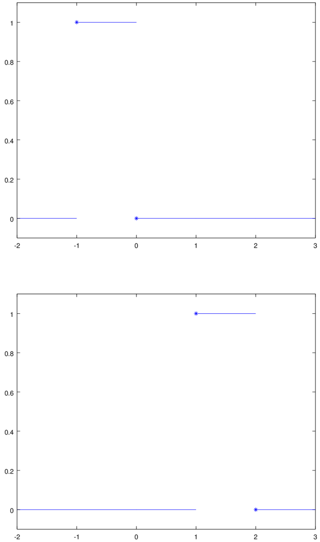


Figure: $\mathcal{B}_1^0(t)$, $\mathcal{B}_2^0(t)$, $\mathcal{B}_3^0(t)$, $\mathcal{B}_4^0(t)$

Example

We have 4 B-splines of order one, they are tent functions, non-negative and their sum is one in the interval $[-1, 2]$

- $\mathcal{B}_0^1(t) = \frac{t+2}{-1+2}\mathcal{B}_0^0(t) + \frac{0-t}{0+1}\mathcal{B}_1^0(t) = \begin{cases} 2+t & \textit{if } t \in [-2,-1) \\ -t & \textit{if } t \in [-1,0) \\ 0 & \textit{otherwise} \end{cases}$
- $\mathcal{B}_1^1(t) = \frac{t+1}{0+1}\mathcal{B}_1^0(t) + \frac{1-t}{1-0}\mathcal{B}_2^0(t) = \begin{cases} 1+t & \textit{if } t \in [-1,0) \\ 1-t & \textit{if } t \in [0,1) \\ 0 & \textit{otherwise} \end{cases}$
- $\mathcal{B}_2^1(t) = \frac{t}{1-0}\mathcal{B}_2^0(t) + \frac{2-t}{2-1}\mathcal{B}_3^0(t) = \begin{cases} t & \textit{if } t \in [0,1) \\ 2-t & \textit{if } t \in [1,2) \\ 0 & \textit{otherwise} \end{cases}$
- $\mathcal{B}_3^1(t) = \frac{t-1}{2-1}\mathcal{B}_3^0(t) + \frac{3-t}{3-2}\mathcal{B}_4^0(t) = \begin{cases} -1+t & \textit{if } t \in [1,2) \\ 3-t & \textit{if } t \in [2,3) \\ 0 & \textit{otherwise} \end{cases}$

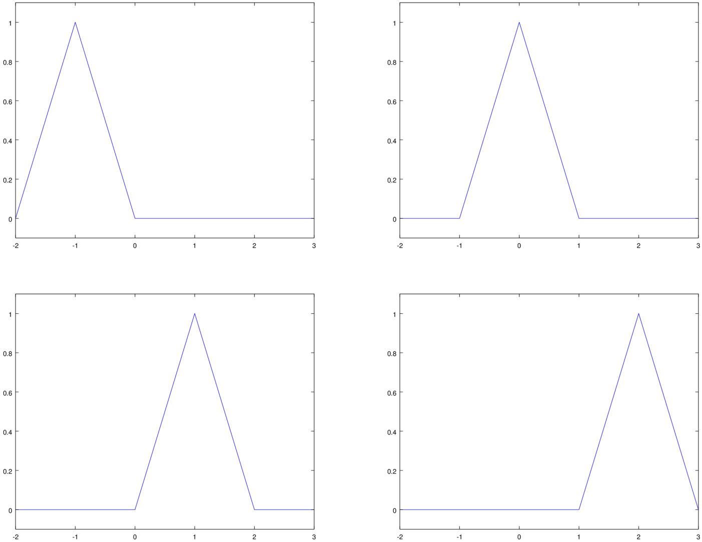


Figure: $\mathcal{B}_0^1(t)$, $\mathcal{B}_1^1(t)$, $\mathcal{B}_2^1(t)$, $\mathcal{B}_3^1(t)$

Example

We have 3 B-splines of order two, they are continuous and differentiable piecewise functions, non-negative and their sum is one in the interval $[0, 1]$

- $\mathcal{B}_0^2(t) = \frac{t+2}{0+2}\mathcal{B}_0^1(t) + \frac{1-t}{1+1}\mathcal{B}_1^1(t) = \begin{cases} 2 + 2t + \frac{t^2}{2} & \textit{if } t \in [-2, -1) \\ \frac{1}{2} - t - t^2 & \textit{if } t \in [-1, 0) \\ \frac{1}{2} - t + \frac{t^2}{2} & \textit{if } t \in [0, 1) \\ 0 & \textit{otherwise} \end{cases}$

- $\mathcal{B}_1^2(t) = \frac{t+1}{1+1}\mathcal{B}_1^1(t) + \frac{2-t}{2-0}\mathcal{B}_2^1(t) = \begin{cases} \frac{1}{2} + t + \frac{t^2}{2} & \textit{if } t \in [-1, 0) \\ \frac{1}{2} + t - t^2 & \textit{if } t \in [0, 1) \\ 2 - 2t + \frac{t^2}{2} & \textit{if } t \in [1, 2) \\ 0 & \textit{otherwise} \end{cases}$

- $\mathcal{B}_2^2(t) = \frac{t-0}{2-0}\mathcal{B}_2^1(t) + \frac{3-t}{3-1}\mathcal{B}_3^1(t) = \begin{cases} \frac{t^2}{2} & \textit{if } t \in [0, 1) \\ -\frac{3}{2} + 3t - t^2 & \textit{if } t \in [1, 2) \\ \frac{9}{2} - 3t + \frac{t^2}{2} & \textit{if } t \in [2, 3) \\ 0 & \textit{otherwise} \end{cases}$

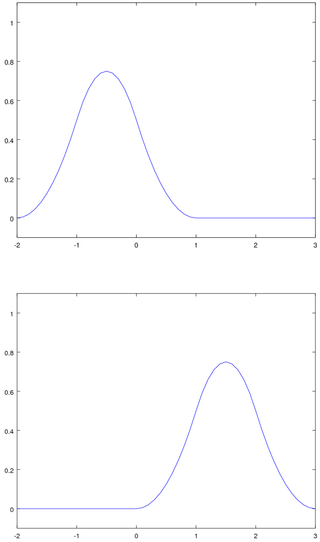


Figure: $\mathcal{B}_0^2(t)$, $\mathcal{B}_1^2(t)$, $\mathcal{B}_2^2(t)$, $\sum_{i=0}^2 \mathcal{B}_i^2(t)$

Properties of B-splines

For a knot sequence $\vec{t} = \{\bar{t}_0, \bar{t}_1, \ldots, \bar{t}_N\}$:

- $\mathcal{B}_k(\vec{t}\,) = \{\mathcal{B}_0^k(t), \mathcal{B}_1^k(t), \ldots, \mathcal{B}_{N-k-1}^k(t)\}$ is a set of $N-k$ B-splines of degree k associated to $\vec{t}$.

- Each B-spline $\mathcal{B}_i^k(t)$ is nonzero for $t \in (\bar{t}_i, \bar{t}_{i+k+1})$ and zero otherwise

- $\sum_{i=0}^{N-k-1} \mathcal{B}_i^k(t) = 1$ for $t \in [\bar{t}_k, \bar{t}_{N-k})$

- The order of continuity of a B-spline $\mathcal{B}_i^k(t)$ at each knot $\bar{t}_i, \ldots, \bar{t}_{i+k+1}$ is $k - m(\bar{t}_j)$ for $j = i, \ldots, i+k+1$ taking only into account the subsequence $(\bar{t}_i, \ldots, \bar{t}_{i+k+1})$

Example: $\vec{t} = \{0, 1, 2, 2, 3, 3, 3, 4, 5, 6\}$

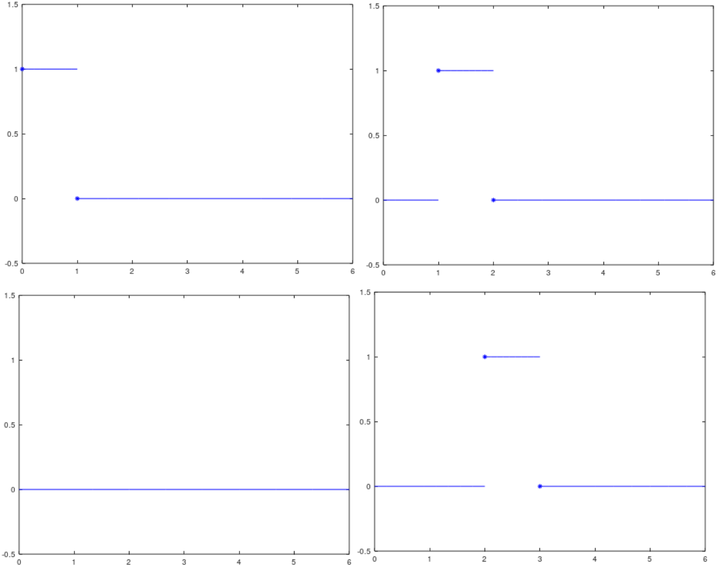
$$\mathcal{B}_0^0(t) = \begin{cases} 1 & \textit{if } t \in [0, 1) \\ 0 & \textit{otherwise} \end{cases} \qquad \mathcal{B}_1^0(t) = \begin{cases} 1 & \textit{if } t \in [1, 2) \\ 0 & \textit{otherwise} \end{cases}$$

$$\mathcal{B}_2^0(t) = 0 \qquad \mathcal{B}_3^0(t) = \begin{cases} 1 & \textit{if } t \in [2, 3) \\ 0 & \textit{otherwise} \end{cases} \qquad \mathcal{B}_4^0(t) = 0$$

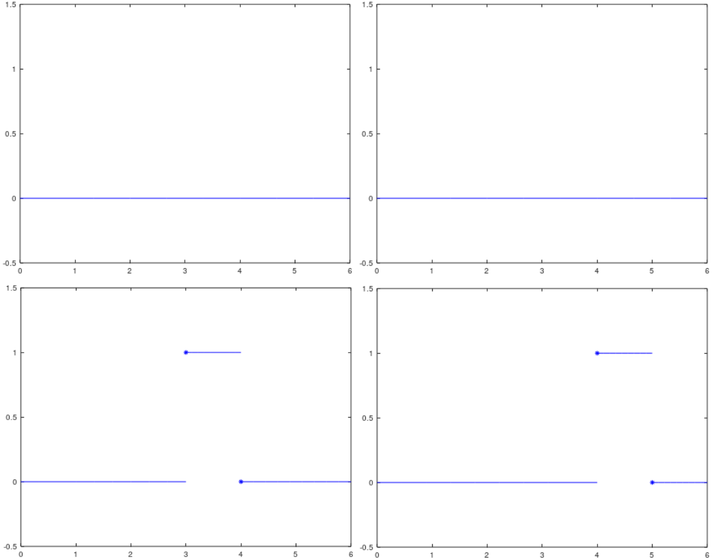
$$\mathcal{B}_5^0(t) = 0 \qquad \mathcal{B}_6^0(t) = \begin{cases} 1 & \textit{if } t \in [3, 4) \\ 0 & \textit{otherwise} \end{cases}$$

$$\mathcal{B}_7^0(t) = \begin{cases} 1 & \textit{if } t \in [4, 5) \\ 0 & \textit{otherwise} \end{cases} \qquad \mathcal{B}_8^0(t) = \begin{cases} 1 & \textit{if } t \in [5, 6) \\ 0 & \textit{otherwise} \end{cases}$$

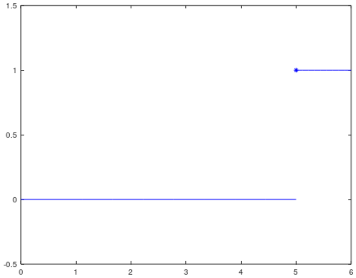
B-splines of order zero: $\mathcal{B}_0^0$, $\mathcal{B}_1^0$, $\mathcal{B}_2^0$, $\mathcal{B}_3^0$



B-splines of order zero: $\mathcal{B}_4^0$, $\mathcal{B}_5^0$, $\mathcal{B}_6^0$, $\mathcal{B}_7^0$



B-splines of order zero: $\mathcal{B}_8^0$



Example: $\vec{t} = \{0, 1, 2, 2, 3, 3, 3, 4, 5, 6\}$

$$\mathcal{B}_0^1(t) = \tfrac{t-0}{1-0}\mathcal{B}_0^0(t) + \tfrac{2-t}{2-1}\mathcal{B}_1^0(t) = \begin{cases} t & \textit{if } t \in [0,1) \\ 2-t & \textit{if } t \in [1,2) \\ 0 & \textit{otherwise} \end{cases}$$

$$\mathcal{B}_1^1(t) = \tfrac{t-1}{2-1}\mathcal{B}_1^0(t) = \begin{cases} t-1 & \textit{if } t \in [1,2) \\ 0 & \textit{otherwise} \end{cases}$$

$$\mathcal{B}_2^1(t) = \tfrac{3-t}{3-2}\mathcal{B}_3^0(t) = \begin{cases} 3-t & \textit{if } t \in [2,3) \\ 0 & \textit{otherwise} \end{cases}$$

$$\mathcal{B}_3^1(t) = \tfrac{t-2}{3-2}\mathcal{B}_3^0(t) = \begin{cases} t-2 & \textit{if } t \in [2,3) \\ 0 & \textit{otherwise} \end{cases}$$

Example: $\vec{t} = \{0, 1, 2, 2, 3, 3, 3, 4, 5, 6\}$

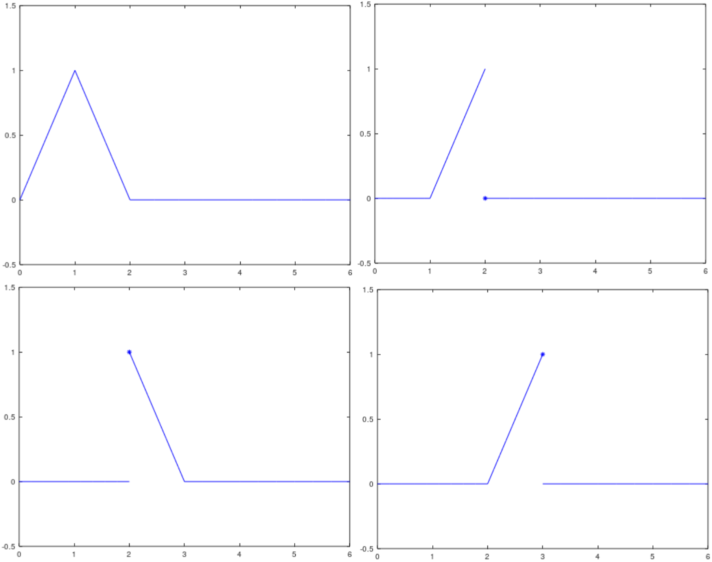
$$\mathcal{B}_4^1(t) = 0$$

$$\mathcal{B}_5^1(t) = \tfrac{4-t}{4-3}\mathcal{B}_6^0(t) = \begin{cases} 4-t & \text{if } t \in [3,4) \\ 0 & \text{otherwise} \end{cases}$$

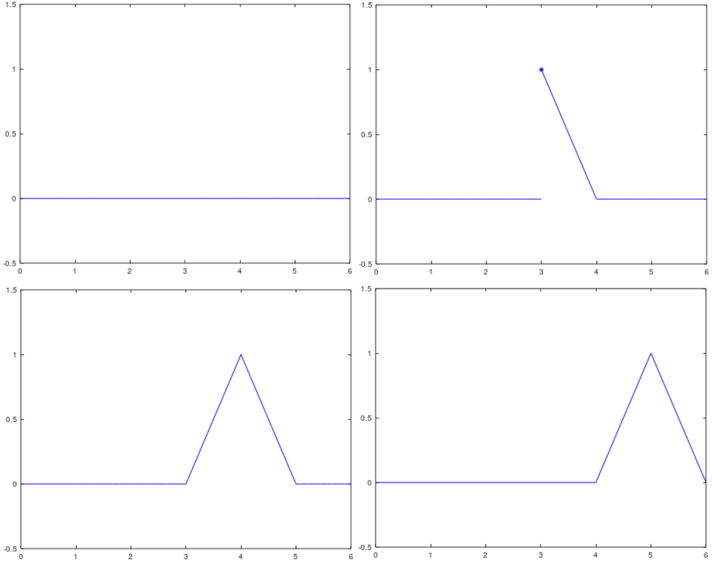
$$\mathcal{B}_6^1(t) = \tfrac{t-3}{4-3}\mathcal{B}_6^0(t) + \tfrac{5-t}{5-4}\mathcal{B}_7^0(t) = \begin{cases} t-3 & \text{if } t \in [3,4) \\ 5-t & \text{if } t \in [4,5) \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{B}_7^1(t) = \tfrac{t-4}{5-4}\mathcal{B}_7^0(t) + \tfrac{6-t}{6-5}\mathcal{B}_8^0(t) = \begin{cases} t-4 & \text{if } t \in [4,5) \\ 6-t & \text{if } t \in [5,6) \\ 0 & \text{otherwise} \end{cases}$$

B-splines of order one: $\mathcal{B}_0^1$, $\mathcal{B}_1^1$, $\mathcal{B}_2^1$, $\mathcal{B}_3^1$



B-splines of order one: $\mathcal{B}_4^1$, $\mathcal{B}_5^1$, $\mathcal{B}_6^1$, $\mathcal{B}_7^1$



Curry-Schoenberg theorem

Theorem

$P^n_{k,\vec{r}}[t_0, \dots, t_n]$ with $\vec{r} = (r_1, \dots, r_{n-1})$ and $\vec{m} = (m_1, \dots, m_{n-1})$ has a basis of B-splines associated to a knot sequence $\vec{t} = (\bar{t}_0, \dots, \bar{t}_N)$ given by

- *the first $k+1$ knots are $\bar{t}_0 \le \dots \le \bar{t}_k \le t_0$*
- *the last $k+1$ knots are $t_n \le \bar{t}_{N-k} \le \dots \le \bar{t}_N$*
- *the middle part of the sequence $(\bar{t}_{k+1}, \dots, \bar{t}_{N-k-1})$ are the break points $t_1, \dots, t_{n-1}$ taking into account multiplicities.*

Moreover we have $N + 1 = k + 1 + k + 1 + \sum_{i=1}^{n-1} m_i$ and so

$$\dim(P^n_{k,\vec{r}}[t_0, \dots, t_n]) = k + 1 + \sum_{i=1}^{n-1} m_i = N - k \tag{5}$$

Example

Find a basis of B-splines for $P_{2,\vec{r}}[0,1,2,4]$ with $\vec{r} = (0,-1)$ and $\vec{m} = (2,3)$.

We have $k = 2$ therefore

- first 3 knots ≤ 0 $\quad \vec{t} = (0,0,0,\ldots)$
- next knots 1, 2 with multiplicities $\vec{t} = (0,0,0,1,1,2,2,2,\ldots)$
- last 3 knots ≥ 4 $\quad \vec{t} = (0,0,0,1,1,2,2,2,4,4,4)$

Now we have to construct a basis of B-splines of order 2 recursively

We have 10 B-splines of order zero: $\mathcal{B}_0^0(t) = 0$, $\mathcal{B}_1^0(t) = 0$,

$$\mathcal{B}_2^0(t) = \begin{cases} 1 & \textit{if } \ t \in [0,1) \\ 0 & \textit{otherwise} \end{cases}, \quad \mathcal{B}_3^0(t) = 0, \quad \mathcal{B}_4^0(t) = \begin{cases} 1 & \textit{if } \ t \in [1,2) \\ 0 & \textit{otherwise} \end{cases},$$

$$\mathcal{B}_5^0(t) = 0, \quad \mathcal{B}_6^0(t) = 0, \quad \mathcal{B}_7^0(t) = \begin{cases} 1 & \textit{if } t \in [2,4) \\ 0 & \textit{otherwise} \end{cases}, \quad \mathcal{B}_8^0(t) = 0,$$
$$\mathcal{B}_9^0(t) = 0$$

We have 9 B-splines of order 1:

$$\mathcal{B}_0^1(t) = 0,\ \mathcal{B}_1^1(t) = \begin{cases} 1-t & \textit{if } t \in [0,1) \\ 0 & \textit{otherwise} \end{cases},\ \mathcal{B}_2^1(t) = \begin{cases} t & \textit{if } t \in [0,1) \\ 0 & \textit{otherwise} \end{cases},$$

$$\mathcal{B}_3^1(t) = \begin{cases} 2-t & \textit{if } t \in [1,2) \\ 0 & \textit{otherwise} \end{cases},\ \mathcal{B}_4^1(t) = \begin{cases} t-1 & \textit{if } t \in [1,2) \\ 0 & \textit{otherwise} \end{cases},\ \mathcal{B}_5^1(t) = 0,$$

$$\mathcal{B}_6^1(t) = \begin{cases} \frac{4-t}{2} & \textit{if } t \in [2,4) \\ 0 & \textit{otherwise} \end{cases},\ \mathcal{B}_7^1(t) = \begin{cases} \frac{t-2}{2} & \textit{if } t \in [2,4) \\ 0 & \textit{otherwise} \end{cases},\ \mathcal{B}_8^1(t) = 0$$

we have 8 B-splines of order 2

$$\mathcal{B}_0^2(t) = \begin{cases} 1 - 2t + t^2 & \textit{if } t \in [0,1) \\ 0 & \textit{otherwise} \end{cases}, \ \mathcal{B}_1^2(t) = \begin{cases} 2t - 2t^2 & \textit{if } t \in [0,1) \\ 0 & \textit{otherwise} \end{cases},$$

$$\mathcal{B}_2^2(t) = \begin{cases} t^2 & \textit{if } t \in [0,1) \\ 4 - 4t + t^2 & \textit{if } t \in [1,2) \\ 0 & \textit{otherwise} \end{cases}, \ \mathcal{B}_3^2(t) = \begin{cases} -4 + 6t - 2t^2 & \textit{if } t \in [1,2) \\ 0 & \textit{otherwise} \end{cases},$$

$$\mathcal{B}_4^2(t) = \begin{cases} 1 - 2t + t^2 & \textit{if } t \in [1,2) \\ 0 & \textit{otherwise} \end{cases}, \ \mathcal{B}_5^2(t) = \begin{cases} 4 - 2t + \frac{t^2}{4} & \textit{if } t \in [2,4) \\ 0 & \textit{otherwise} \end{cases},$$

$$\mathcal{B}_6^2(t) = \begin{cases} -4 + 3t - \frac{t^2}{2} & \textit{if } t \in [2,4) \\ 0 & \textit{otherwise} \end{cases}, \ \mathcal{B}_7^2(t) = \begin{cases} 1 - t + \frac{t^2}{4} & \textit{if } t \in [2,4) \\ 0 & \textit{otherwise} \end{cases}$$

the above B-spline satisfy the conditions for being B-splines.

Properties of B-splines

For a knot sequence $\vec{t} = \{\bar{t}_0, \bar{t}_1, \ldots, \bar{t}_N\}$:
For a knot sequence $\vec{t} = (0, 0, 0, 1, 1, 2, 2, 2, 4, 4, 4)$:

- $\mathcal{B}_k(\vec{t}\,) = \{\mathcal{B}_0^k(t), \mathcal{B}_1^k(t), \ldots, \mathcal{B}_{N-k-1}^k(t)\}$ is a set of $N - k$ B-splines of degree k associated to $\vec{t}$.

  $\mathcal{B}_2(\vec{t}\,) = \{\mathcal{B}_0^2(t), \mathcal{B}_1^2(t), \ldots, \mathcal{B}_7^2(t)\}$ is a set of $10 - 2 = 8$ B-splines of degree 2 associated to $\vec{t}$.

- Each B-spline $\mathcal{B}_i^k(t)$ is nonzero for $t \in (\bar{t}_i, \bar{t}_{i+k+1})$ and zero otherwise.

  $\mathcal{B}_0^2(t) \neq 0\ t \in (0, 1)$, $\mathcal{B}_1^2(t) \neq 0\ t \in (0, 1)$, $\mathcal{B}_2^2(t) \neq 0\ t \in (0, 2)$,
  $\mathcal{B}_3^2(t) \neq 0\ t \in (1, 2)$, $\mathcal{B}_4^2(t) \neq 0\ t \in (1, 2)$, $\mathcal{B}_5^2(t) \neq 0\ t \in (2, 4)$,
  $\mathcal{B}_6^2(t) \neq 0\ t \in (2, 4)$, $\mathcal{B}_7^2(t) \neq 0\ t \in (2, 4)$,

Properties of B-splines

- $\sum_{i=0}^{N-k-1} \mathcal{B}_i^k(t) = 1$ for $t \in [\bar{t}_k, \bar{t}_{N-k})$

  $\sum_{i=0}^{7} \mathcal{B}_i^2(t) = 1$ for $t \in [0, 4)$

- The order of continuity of a B-spline $\mathcal{B}_i^k(t)$ at each knot $\bar{t}_i, \ldots, \bar{t}_{i+k+1}$ is $k - m(\bar{t}_j)$ for $j = i, \ldots, i+k+1$ taking only into account the subsequence $(\bar{t}_i, \ldots, \bar{t}_{i+k+1})$

  $\mathcal{B}_0^2(t)$ subsequence $(0, 0, 0, 1)$ differentiable at 1
  $\mathcal{B}_1^2(t)$ subsequence $(0, 0, 1, 1)$ continuous at 1
  $\mathcal{B}_2^2(t)$ subsequence $(0, 1, 1, 2)$ continuous at 1 differentiable at 2
  $\mathcal{B}_3^2(t)$ subsequence $(1, 1, 2, 2)$ continuous at 1 and 2
  $\mathcal{B}_4^2(t)$ subsequence $(1, 2, 2, 2)$ differentiable at 1 and discontinuous at 2
  $\mathcal{B}_5^2(t)$ subsequence $(2, 2, 2, 4)$ discontinuous at 2
  $\mathcal{B}_6^2(t)$ subsequence $(2, 2, 4, 4)$ continuous at 2
  $\mathcal{B}_7^2(t)$ subsequence $(2, 4, 4, 4)$ differentiable at 2

Moreover...

- $\mathcal{B}_2(\vec{t}\,) = \{\mathcal{B}_0^2(t), \mathcal{B}_1^2(t), \ldots, \mathcal{B}_7^2(t)\}$ has 8 elements which is the dimension of $P_{2,\vec{r}}[0,1,2,4]$ with $\vec{r} = (0,-1)$
- The polynomials are of degree at most 2, piecewise defined in $[0,1,2,4]$, and are at least continuous at 1.

So they form a basis for $P_{2,\vec{r}}[0,1,2,4]$ with $\vec{r} = (0,-1)$ (but we are not proving it here)

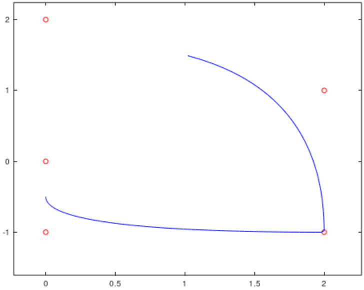
What can we do with such B-splines? construct B-spline curves.

B-spline curves

Definition

A B-spline curve $\gamma : [t_0, t_n] \to \mathbb{R}^2$ (or $\mathbb{R}^3$) is defined through a combination of B-splines and control points P_i in the following way

$$\gamma(t) = \sum_{i=0}^{N-k-1} \mathcal{B}_i^k(t) P_i \tag{6}$$



We compute such curves using the De Boor algorithm.