MAT300 CURVES AND SURFACES

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Newton interpolation

Motivation

Last day we saw Neville's method to obtain Lagrange interpolant polynomials recursively.

The recursion has to be done for every node of the output mesh.

It would be nice to do such a recursion only once.

Such a method is the **divided differences** and the interpolant polynomial is constructed in the **Newton basis**.

Divided differences

Let (x_0, y_0), (x_1, y_1), ..., (x_n, y_n) be points with $x_{i-1} < x_i$ for $i = 1, \dots, n$ and let $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(x_i) = y_i$ (interpolant polynomial, but we use the f notation to be consistent with the existing literature)

Definition

The $n+1$ zeroth divided differences of f for the nodes (x_0, y_0), (x_1, y_1), ..., (x_n, y_n) are

$$f[x_i] = y_i, \qquad i = 0, 1, \dots, n \tag{1}$$

Nodes	$f[x_i]$
x_0	$f[x_0] = y_0$
x_1	$f[x_1] = y_1$
x_2	$f[x_2] = y_2$
x_3	$f[x_3] = y_3$

The rest of the divided differences are defined recursively

Definition

The n first divided differences of f are

$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}, \qquad i = 0, 1, \ldots, n-1 \tag{2}$$

$$\begin{array}{ccc}
\textit{Nodes} & f[x_i] & f[x_i, x_{i+1}] \\
 & & \\
x_0 & f[x_0] = y_0 & \\
 & & f[x_0, x_1] = \frac{f[x_1]-f[x_0]}{x_1-x_0} \\
x_1 & f[x_1] = y_1 & \\
 & & f[x_1, x_2] = \frac{f[x_2]-f[x_1]}{x_2-x_1} \\
x_2 & f[x_2] = y_2 & \\
 & & f[x_2, x_3] = \frac{f[x_3]-f[x_2]}{x_3-x_2} \\
x_3 & f[x_3] = y_3 &
\end{array}$$

Definition

The $n-1$ second divided differences of f are

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}, \qquad i = 0, 1, \ldots, n-2 \quad (3)$$

Nodes	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$
x_0	$f[x_0] = y_0$		
		$f[x_0, x_1] = \frac{f[x_1]-f[x_0]}{x_1-x_0}$	
x_1	$f[x_1] = y_1$		$f[x_0, x_1, x_2] = \frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0}$
		$f[x_1, x_2] = \frac{f[x_2]-f[x_1]}{x_2-x_1}$	
x_2	$f[x_2] = y_2$		$f[x_1, x_2, x_3] = \frac{f[x_2,x_3]-f[x_1,x_2]}{x_3-x_1}$
		$f[x_2, x_3] = \frac{f[x_3]-f[x_2]}{x_3-x_2}$	
x_3	$f[x_3] = y_3$		

Definition

The $n-2$ third divided differences of f for $i = 0, 1, \ldots, n-3$ are

$$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}] = \frac{f[x_{i+1}, x_{i+2}, x_{i+3}] - f[x_i, x_{i+1}, x_{i+2}]}{x_{i+3} - x_i}. \qquad (4)$$

Nodes	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$
x_0	$f[x_0]$			
		$f[x_0, x_1]$		
x_1	$f[x_1]$		$f[x_0, x_1, x_2]$	
		$f[x_1, x_2]$		$f[x_0, x_1, x_2, x_3]$
x_2	$f[x_2]$		$f[x_1, x_2, x_3]$	
		$f[x_2, x_3]$		
x_3	$f[x_3]$			

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}$$

Divided differences in a general setting

Having (x_0, y_0), (x_1, y_1), ..., (x_n, y_n) points with $x_{i-1} < x_i$ for $i = 1, \ldots, n$ and $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(x_i) = y_i$,

We compute the $n+1$ zeroth divided differences

We compute the n first divided differences

We compute the $n-1$ second divided differences

$\vdots$

We compute the unique nth divided difference

The general formula for the divided difference is

$$f[x_i, x_{i+1}, \ldots, x_{i+k}] = \frac{f[x_{i+1}, \ldots, x_{i+k}] - f[x_i, \ldots, x_{i+k-1}]}{x_{i+k} - x_i} \tag{5}$$

Example

Compute the divided differences for a polynomial through $(-2, 1)$, $(-1, 3)$, $(1, 0)$ and $(4, -2)$.

Nodes	$f[x_i]$
$x_0 = -2$	$f[x_0] = 1$
$x_1 = -1$	$f[x_1] = 3$
$x_2 = 1$	$f[x_2] = 0$
$x_3 = 4$	$f[x_3] = -2$

Compute the divided differences for a polynomial through $(-2, 1)$, $(-1, 3)$, $(1, 0)$ and $(4, -2)$.

Nodes	$f[x_i]$	$f[x_i, x_{i+1}]$
$x_0 = -2$	$f[x_0] = 1$	
		$f[x_0, x_1] = \frac{f[x_1]-f[x_0]}{x_1-x_0} = \frac{3-1}{-1+2} = 2$
$x_1 = -1$	$f[x_1] = 3$	
		$f[x_1, x_2] = \frac{f[x_2]-f[x_1]}{x_2-x_1} = \frac{0-3}{1+1} = -\frac{3}{2}$
$x_2 = 1$	$f[x_2] = 0$	
		$f[x_2, x_3] = \frac{f[x_3]-f[x_2]}{x_3-x_2} = \frac{-2-0}{4-1} = -\frac{2}{3}$
$x_3 = 4$	$f[x_3] = -2$	

Compute the divided differences for a polynomial through $(-2, 1)$, $(-1, 3)$, $(1, 0)$ and $(4, -2)$.

Nodes	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$
$x_0 = -2$	1		
		$f[x_0, x_1] = 2$	
$x_1 = -1$	3		$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{-\frac{3}{2} - 2}{1+2} = -\frac{7}{6}$
		$f[x_1, x_2] = -\frac{3}{2}$	
$x_2 = 1$	0		$f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1} = \frac{-\frac{2}{3} + \frac{3}{2}}{4+1} = \frac{1}{6}$
		$f[x_2, x_3] = -\frac{2}{3}$	
$x_3 = 4$	-2		

Compute the divided differences for a polynomial through $(-2,1)$, $(-1,3)$, $(1,0)$ and $(4,-2)$.

Nodes	$f[x_i]$	$f[x_i,x_{i+1}]$	$f[x_i,x_{i+1},x_{i+2}]$	$f[x_i,x_{i+1},x_{i+2},x_{i+3}]$
$x_0=-2$	1			
		2		
$x_1=-1$	3		$f[x_0,x_1,x_2]=-\frac{7}{6}$	
		$-\frac{3}{2}$		$f[x_0,x_1,x_2,x_3]=(*)$
$x_2=1$	0		$f[x_1,x_2,x_3]=\frac{1}{6}$	
		$-\frac{2}{3}$		
$x_3=4$	-2			

$$f[x_0,x_1,x_2,x_3]=\frac{f[x_1,x_2,x_3]-f[x_0,x_1,x_2]}{x_3-x_0}=\frac{\frac{1}{6}+\frac{7}{6}}{4+2}=\frac{2}{9}$$

Compute the divided differences for a polynomial through $(-2, 1)$, $(-1, 3)$, $(1, 0)$ and $(4, -2)$.

Nodes	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$
$x_0 = -2$	1			
		2		
$x_1 = -1$	3		$-\frac{7}{6}$	
		$-\frac{3}{2}$		$\frac{2}{9}$
$x_2 = 1$	0		$\frac{1}{6}$	
		$-\frac{2}{3}$		
$x_3 = 4$	-2			

What can we do with these numbers?

Newton's interpolant polynomial

Theorem

Given (x_0, y_0), (x_1, y_1), ..., (x_n, y_n) points with $x_{i-1} < x_i$ for $i = 1, \ldots, n$, the interpolant polynomial through them can be written as

$$p(x) = f[x_0] + \sum_{k=1}^{n} \left(f[x_0, \ldots, x_k] \prod_{i=0}^{k-1} (x - x_i) \right) \tag{6}$$

In this construction we have on the one hand the first divided differences of each order

$$\begin{array}{ccccc}
f[x_0] & & & & \\
 & f[x_0, x_1] & & & \\
f[x_1] & & f[x_0, x_1, x_2] & & \\
 & & & \ddots & \\
 & f[x_1, x_2] & & & \\
f[x_2] & & f[x_1, x_2, x_3] & \ldots & f[x_0, \ldots, x_n] \\
\vdots & \vdots & \vdots & & \\
f[x_n] & & & &
\end{array} \tag{7}$$

Newton polynomials

and on the other hand the Newton basis for P_n.

Definition

Given x_0, x_1, ..., x_n with $x_{i-1} < x_i$ for $i = 1, \ldots, n$, the Newton polynomials are:

$$N_0(x) = 1$$

$$N_1(x) = x - x_0$$

$$N_2(x) = (x - x_0)(x - x_1)$$

$$\vdots$$

$$N_n(x) = \prod_{i=0}^{n-1}(x - x_i)$$

The Newton basis

Theorem

The set of Newton polynomials $B_N = \{N_0,\ N_1,\ \ldots,\ N_n\}$ form a basis for P_n.

Construct the interpolant polynomial through $(-2, 1)$, $(-1, 3)$, $(1, 0)$ and $(4, -2)$ using divided differences and the Newton basis.

We already have the divided differences:

$$f[x_0] = 1, \qquad f[x_0, x_1] = 2, \qquad f[x_0, x_1, x_2] = -\frac{7}{6}, \qquad f[x_0, x_1, x_2, x_3] = \frac{2}{9}$$

The Newton polynomials are:

$$N_0(x) = 1, \qquad N_1(x) = x + 2, \qquad N_2(x) = (x + 2)(x + 1),$$

$$N_3(x) = (x + 2)(x + 1)(x - 1)$$

The interpolant polynomial is:

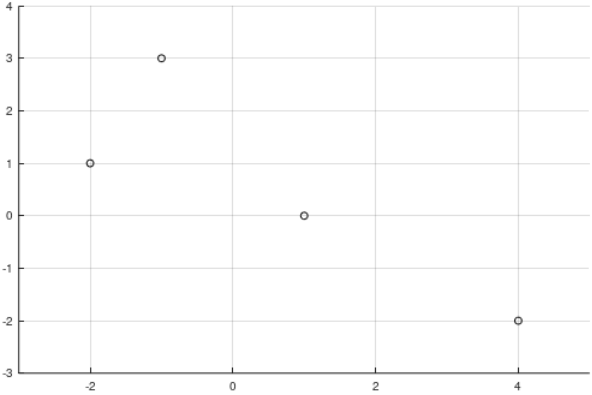
$$p(x) = f[x_0]N_0(x) + f[x_0, x_1]N_1(x) + f[x_0, x_1, x_2]N_2(x) + f[x_0, x_1, x_2, x_3]N_3(x)$$

$$1 + 2(x + 2) - \frac{7}{6}(x + 2)(x + 1) + \frac{2}{9}(x + 2)(x + 1)(x - 1)$$

HOW DOES THIS WORK?

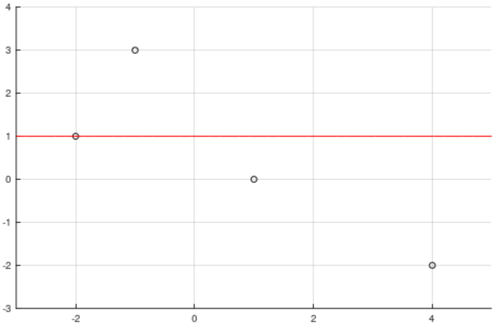
$$p(x) = 1 + 2(x+2) - \frac{7}{6}(x+2)(x+1) + \frac{2}{9}(x+2)(x+1)(x-1)$$

We start with the points that we want to interpolate



We take the first point $(-2, 1)$ and compute the interpolant polynomial of degree zero passing through it

$$p(x) = 1 + 2(x+2) - \frac{7}{6}(x+2)(x+1) + \frac{2}{9}(x+2)(x+1)(x-1)$$



$p(x) = 1$

We now take the second point $(-1, 3)$ and compute the interpolant polynomial of degree one passing through it and through $(-2, 1)$.

We have to increase the degree of the polynomial by one, so the increment will be of the type $a(x + b)$.

$p(x) = 1 + a(x + b)$

The increment has to vanish at $x = -2$ because we want the polynomial to pass through $(-2, 1)$

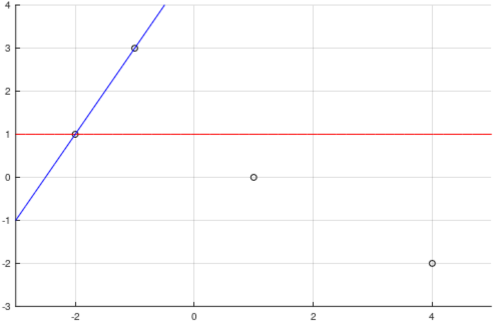
$p(x) = 1 + a(x + 2)$

The resulting polynomial has to pass through $(-1, 3)$ therefore

$a = \frac{3-1}{-1+2} = \frac{y_1 - y_0}{x_1 - x_0} = f[x_0, x_1] = 2$

$p(x) = 1 + 2(x + 2)$

$$p(x) = 1 + 2(x+2) - \frac{7}{6}(x+2)(x+1) + \frac{2}{9}(x+2)(x+1)(x-1)$$



We now take the third point $(1, 0)$ and compute the interpolant polynomial of degree two passing through it and through $(-2, 1)$ and $(-1, 3)$.

We have to increase the degree of the polynomial by one, so the increment will be of the type $a(x + b)(x + c)$.

$$p(x) = 1 + 2(x + 2) + a(x + b)(x + c)$$

The increment has to vanish at $x = -2$ and at $x = -1$ because we want the polynomial to pass through $(-2, 1)$ and $(-1, 3)$

$$p(x) = 1 + 2(x + 2) + a(x + 2)(x + 1)$$

The resulting polynomial has to pass through $(1, 0)$ therefore

$$y_2 = f[x_0] + f[x_0, x_1](x_2 - x_0) + a(x_2 - x_0)(x_2 - x_1)$$

restructuring the equation we shall obtain that $a = f[x_0, x_1, x_2]$

$$y_2 = f[x_0] + f[x_0, x_1](x_2 - x_0) + a(x_2 - x_0)(x_2 - x_1)$$

$$y_2 = y_0 + f[x_0, x_1](x_2 - x_0) + a(x_2 - x_0)(x_2 - x_1)$$

$$y_2 - y_0 = f[x_0, x_1](x_2 - x_0) + a(x_2 - x_0)(x_2 - x_1)$$

$$(y_2 - y_1) + (y_1 - y_0) = f[x_0, x_1](x_2 - x_0) + a(x_2 - x_0)(x_2 - x_1)$$

$$\frac{(y_2 - y_1)}{(x_2 - x_1)} + \frac{(y_1 - y_0)}{(x_2 - x_1)} = f[x_0, x_1]\frac{(x_2 - x_0)}{(x_2 - x_1)} + a(x_2 - x_0)$$

$$f[x_1, x_2] + \frac{(y_1 - y_0)(x_1 - x_0)}{(x_2 - x_1)(x_1 - x_0)} = f[x_0, x_1]\frac{(x_2 - x_0)}{(x_2 - x_1)} + a(x_2 - x_0)$$

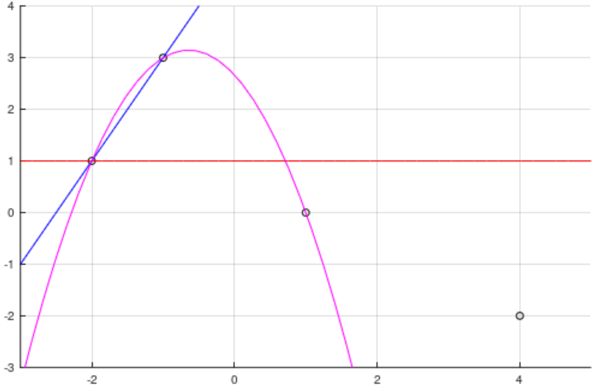
$$f[x_1, x_2] + f[x_0, x_1]\frac{(x_1 - x_0)}{(x_2 - x_1)} = f[x_0, x_1]\frac{(x_2 - x_0)}{(x_2 - x_1)} + a(x_2 - x_0)$$

$$f[x_1, x_2] + f[x_0, x_1]\left(\frac{(x_1 - x_0)}{(x_2 - x_1)} - \frac{(x_2 - x_0)}{(x_2 - x_1)}\right) = a(x_2 - x_0)$$

$$f[x_1, x_2] - f[x_0, x_1] = a(x_2 - x_0)$$

$$a = \frac{f[x_1, x_2] - f[x_0, x_1]}{(x_2 - x_0)} = f[x_0, x_1, x_2]$$

$$p(x) = 1 + 2(x+2) - \frac{7}{6}(x+2)(x+1) + \frac{2}{9}(x+2)(x+1)(x-1)$$



Repeating the process we obtain the interpolant polynomial.

The advantage of Newton interpolation is that we can add new nodes without loosing work

It is faster than the previous techniques.