MAT300 CURVES AND SURFACES

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Bezier curves

1. Derivatives of Bezier Curves
2. Implicit form of quadratic Bezier curves

Differentiable curves

Consider a polynomial curve $\gamma : [a, b] \to \mathbb{R}^2$ given as

$$\gamma(t) = (p(t), q(t)) \tag{1}$$

where $p, q \in P_n$.

If we differentiate both polynomials we can obtain a function $\gamma' : [a, b] \to \mathbb{R}^2$ given as

$$\gamma'(t) = (p'(t), q'(t)) \tag{2}$$

where $p', q' \in P_{n-1}$.

Under which conditions is γ differentiable?

Definition

γ is differentiable at $\bar{t} \in (a, b)$ if $\gamma'(\bar{t})$ exists.

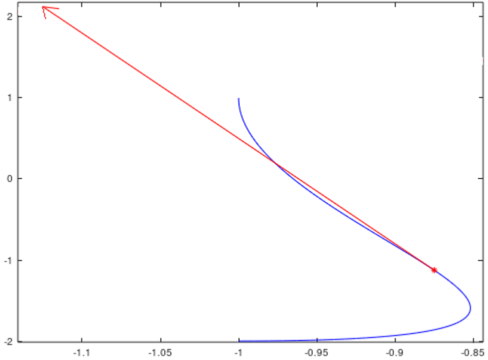
As $p, q \in P_n$, p' and q' are defined in $[a, b]$ therefore γ is differentiable.

Smooth curves

$\gamma'(t)$ can be interpreted as the velocity vector in the direction of increasing t along $\gamma(t)$.

$\gamma : [0,1] \to \mathbb{R}^2$ with $\gamma(t) = (-1 + t - 2t^2 + t^3, -2 + 4t^2 - t^3)$

$\gamma' : [0,1] \to \mathbb{R}^2$ with $\gamma'(t) = (1 - 4t + 3t^2, 8t - 3t^2)$



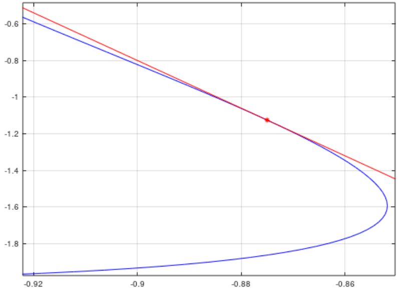
If $\gamma'(\bar{t}) = \vec{0}$ we stop abruptly the path along γ at $\bar{t}$.

Definition

The curve γ is smooth at $\bar{t} \in (a, b)$ if $\gamma'(\bar{t})$ exists and $\gamma'(\bar{t}) \neq \vec{0}$

A tangent line to γ at $\gamma(\bar{t})$ has vector director $\gamma'(\bar{t})$

$$l: \ (x, y) = \gamma(\bar{t}) + \lambda\gamma'(\bar{t}), \quad \lambda \in \mathbb{R} \tag{3}$$



Derivatives of Bezier curves

How do we compute directions and tangents along Bezier curves?

$\gamma(t) = \sum_{i=0}^{n} P_i B_i^n(t), \quad t \in [0,1]$ therefore

$\gamma'(t) = \sum_{i=0}^{n} P_i \frac{d}{dt} B_i^n(t), \quad t \in [0,1]$

Remember that the derivatives of the Bernstein polynomials can be obtained through

$$\frac{d}{dt} B_i^n(t) = n(B_{i-1}^{n-1}(t) - B_i^{n-1}(t))$$

with $B_{-1}^{n-1}(t) = 0$ and $B_n^{n-1}(t) = 0$

so we have

$$\gamma'(t) = \sum_{i=0}^{n} P_i n(B_{i-1}^{n-1}(t) - B_i^{n-1}(t)), \quad t \in [0,1] \tag{4}$$

Cumulative Bernstein polynomials

(4) doesn't have a geometrical interpretation.

How the control points of a Bezier curve make influence in the tangent to the curve at a certain point?

We will rewrithe (4) using cumulative Bernstein polynomials.

Definition

The ith cumulative Bernstein polynomial of degree n is given as

$$C_i^n(t) = \sum_{j=i}^{n} B_j^n(t) \tag{5}$$

$C_i^n(t) = \sum_{j=i}^n B_j^n(t) = B_i^n(t) + \sum_{j=i+1}^n B_j^n(t) = B_i^n(t) + C_{i+1}^n(t)$

Therefore $B_i^n(t) = C_i^n(t) - C_{i+1}^n(t)$

Bezier curve in cumulative form

The Bezier curve in cumulative form is given as follows:

$$\gamma(t) = \sum_{i=0}^{n} P_i B_i^n(t) = \sum_{i=0}^{n} P_i (C_i^n(t) - C_{i+1}^n(t)) = \sum_{i=0}^{n} P_i C_i^n(t) - \sum_{i=0}^{n} P_i C_{i+1}^n(t) =$$

(with $C_{n+1}^n(t) = 0$)

$$= \sum_{i=0}^{n} P_i C_i^n(t) - \sum_{i=0}^{n-1} P_i C_{i+1}^n(t) = P_0 C_0^n(t) + \sum_{i=1}^{n} P_i C_i^n(t) - \sum_{i=1}^{n} P_{i-1} C_i^n(t) =$$

(with $C_0^n(t) = \sum_{j=0}^{n} B_j^n(t) = 1$)

$$= P_0 + \sum_{i=1}^{n} (P_i - P_{i-1}) C_i^n(t)$$

we denote with $\vec{v_i} = P_i - P_{i-1}$ then

$$\gamma(t) = P_0 + \sum_{i=1}^{n} \vec{v_i} C_i^n(t), \quad t \in [0,1] \tag{6}$$

Derivative of Cumulative Bernstein polynomiasls

$$\frac{d}{dt}C_i^n(t) = \frac{d}{dt}\sum_{j=i}^{n} B_j^n(t) = \sum_{j=i}^{n}\frac{d}{dt}B_j^n(t) = \sum_{j=i}^{n} n(B_{j-1}^{n-1}(t) - B_j^{n-1}(t))$$

$$n\left(\sum_{j=i}^{n} B_{j-1}^{n-1}(t) - \sum_{j=i}^{n} B_j^{n-1}(t)\right) = n\left(\sum_{j=i-1}^{n-1} B_j^{n-1}(t) - \sum_{j=i}^{n} B_j^{n-1}(t)\right)$$

$$= n\left(B_{i-1}^{n-1}(t) + \sum_{j=i}^{n-1}(B_j^{n-1}(t) - B_j^{n-1}(t)) - B_n^{n-1}(t)\right) = nB_{i-1}^{n-1}(t)$$

So the derivative of a Bezier curve is

$$\gamma'(t) = \frac{d}{dt}\left(P_0 + \sum_{i=1}^{n}\vec{v_i}C_i^n(t)\right) = \sum_{i=1}^{n}\vec{v_i}\frac{d}{dt}C_i^n(t) = n\sum_{i=1}^{n}\vec{v_i}B_{i-1}^{n-1}(t)$$

$$= n\sum_{i=0}^{n-1}\vec{v_{i+1}}B_i^{n-1}(t)$$

Example

The tangent line to a Bezier curve with control points $P_0 = (0,0)$,

$P_1 = (2,3)$ and $P_2 = (5,7)$ at $t = 1$ is

$$(x, y) = \gamma(1) + \lambda\gamma'(1), \quad \lambda \in \mathbb{R} \tag{7}$$

$\gamma(1) = P_2 = (5,7)$

$\gamma'(1) = 2(\vec{v_1} B_0^1(1) + \vec{v_2} B_1^1(1)) = 2((P_1 - P_0)(1-1) + (P_2 - P_1)1)$

$= 2(3,4) = (6,8)$

$$(x, y) = (5,7) + \lambda(6,8), \quad \lambda \in \mathbb{R}$$

Implicit form for quadratic Bezier curves

Given P_0, P_1, P_2 points in $\mathbb{R}^2$. A Bezier curve $\gamma : [0,1] \to \mathbb{R}^2$ is given as $\gamma(t) = \sum_{i=0}^{2} P_i B_i^2(t)$

If we forget about the t dependence, we can pass from parametric to implicit form

$$f(x,y) = 0 \tag{8}$$

so our curve will be an arc of the above implicit equation.

If the points P_0, P_1, P_2 are aligned, then the Bezier curve will be a segment of a straight line, and we say γ is **degenerate.**

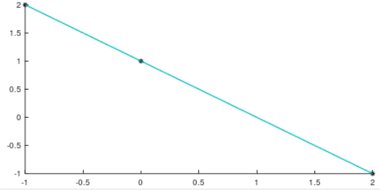
$\gamma(t) = P_0 + \sum_{i=1}^{2} \vec{v_i} C_i^2(t), \quad t \in [0,1]$

$\vec{v_1}$ and $\vec{v_2}$ are parallel.

$l : (x,y) = P_0 + \lambda \vec{v_1}, \quad \lambda \in \mathbb{R}$

Example of degenerate quadratic curve

$P_0 = (2, -1),\ P_1 = (0, 1),\ P_2 = (-1, 2)$



For obtaining the implicit expression we start computing the curve in the standard basis

$\gamma(t) = (1-t)^2(2,-1) + 2(1-t)t(0,1) + t^2(-1,2)$

$= (2 - 4t + t^2, -1 + 4t - t^2)$ then $x = 2 - 4t + t^2$ and so

$y = -1 + 4t - t^2 = 1 - 2 + 4t - t^2 = 1 - (2 - 4t + t^2) = 1 - x$

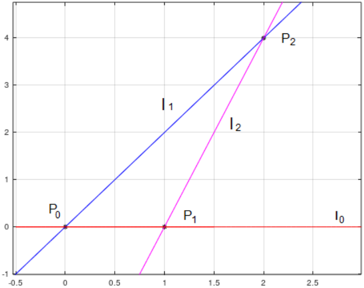
So the implicit form is $x + y - 1 = 0$

Non degenerate quadratic Bezier curves

If the points P_0, P_1, P_2 are not aligned, then the Bezier curve will be an arc of a parabola, and we say γ is **non degenerate.**

How do we find the implicit expression? Consider

$$l_0 : a_0 x + b_0 y + c_0 = 0 \text{ and } L_0(x, y) = a_0 x + b_0 y + c_0$$
$$l_1 : a_1 x + b_1 y + c_1 = 0 \text{ and } L_1(x, y) = a_1 x + b_1 y + c_1$$
$$l_2 : a_2 x + b_2 y + c_2 = 0 \text{ and } L_2(x, y) = a_2 x + b_2 y + c_2$$



The quadratic Bezier curve with control points P_0, P_1, P_2 is given by

$$f_k(x, y) = 0 \tag{9}$$

where

$$f_k(x, y) = L_0(x, y)L_2(x, y) + kL_1(x, y)^2 \tag{10}$$

for a certain constant k.

Example: $P_0 = (0, 0)$, $P_1 = (1, 0)$, $P_2 = (2, 4)$

$l_0 : a_0x + b_0y + c_0 = 0$ evaluating at P_0 and P_1 $\begin{cases} c_0 = 0 \\ a_0 + c_0 = 0 \end{cases}$ so the curve $l_0 : y = 0$ and $L_0(x, y) = y$

$l_1 : a_1x + b_1y + c_1 = 0$ evaluating at P_0 and P_2 $\begin{cases} c_1 = 0 \\ 2a_1 + 4b_1 + c_1 = 0 \end{cases}$ so the curve $l_1 : 2x - y = 0$ and $L_1(x, y) = 2x - y$

$l_2 : a_2x + b_2y + c_2 = 0$ evaluating at P_1 and P_2 $\begin{cases} a_2 + c_2 = 0 \\ 2a_2 + 4b_2 + c_2 = 0 \end{cases}$
so the curve $l_2 : 4x - y - 4 = 0$ and $L_2(x, y) = 4x - y - 4$

$$f_k(x, y) = L_0(x, y)L_2(x, y) + kL_1(x, y)^2 = y(4x - y - 4) + k(2x - y)^2$$

k is determined by evaluating one point inside the curve (not the extremes).

For instance $\gamma(\frac{1}{2}) = (1, 1)$

$f_k(1, 1) = 0$ and obtain $k = 1$

Then the implicit equation is $y(4x - y - 4) + (2x - y)^2 = 0$

$$4xy - y^2 - 4y + 4x^2 - 4xy + y^2 = 0$$

$$x^2 - y = 0$$