MAT300 CURVES AND SURFACES

Julia Sánchez Sanz

DigiPen Institiute of Technology Europe

julia.sanchez@digipen.edu

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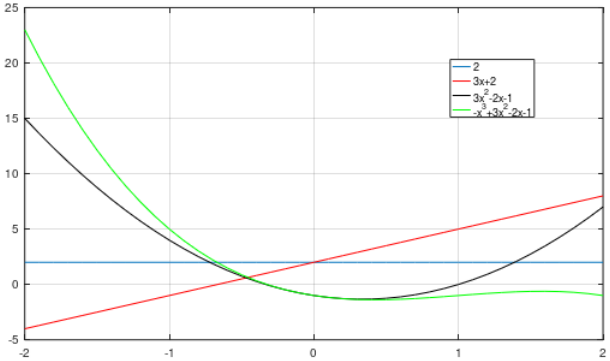
Polynomial vector spaces

Polynomials

Definition

Let $a_0, a_1, a_2, \ldots, a_n \in \mathbb{R}$ with $a_n \neq 0$. A polynomial p_n of defree n over $\mathbb{R}$ is a function $p_n : \mathbb{R} \to \mathbb{R}$ of the form

$$p_n(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \tag{1}$$



Interpolant polynomial

Theorem

Given any $n+1$ points (x_0, y_0), (x_1, y_1), (x_2, y_2), …, (x_n, y_n) with distinct x-coordinate, there is a unique polynomial of degree at most n passing through them. We call it the **interpolant polynomial** *through (x_0, y_0), (x_1, y_1), (x_2, y_2), …, (x_n, y_n).*

Substituting the points in $p_n(x) = y$ we obtain a linear system of $n+1$ equations and $n+1$ unknowns (a_0, a_1, …, a_n).

$$\begin{cases} a_0 + a_1 x_0 + a_2 x_0^2 + \ldots + a_n x_0^n = y_0 \\ a_0 + a_1 x_1 + a_2 x_1^2 + \ldots + a_n x_1^n = y_1 \\ a_0 + a_1 x_2 + a_2 x_2^2 + \ldots + a_n x_2^n = y_2 \\ \vdots \\ a_0 + a_1 x_n + a_2 x_n^2 + \ldots + a_n x_n^n = y_n \end{cases} \tag{2}$$

System (2) has unique solution because ...

Vandermonde determinant

$$\det \begin{pmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ 1 & x_2 & x_2^2 & \dots & x_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^n \end{pmatrix} = \prod_{0 \le i < j \le n} (x_j - x_i) \neq 0 \tag{3}$$

$$\begin{vmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ 1 & x_2 & x_2^2 & \dots & x_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^n \end{vmatrix} = \text{subtract row 1 to rest of rows}$$

$$\begin{vmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 0 & x_1 - x_0 & x_1^2 - x_0^2 & \dots & x_1^n - x_0^n \\ 0 & x_2 - x_0 & x_2^2 - x_0^2 & \dots & x_2^n - x_0^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & x_n - x_0 & x_n^2 - x_0^2 & \dots & x_n^n - x_0^n \end{vmatrix} = \text{column k - } x_0 \text{ times column } k-1$$

$$\begin{vmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & x_1 - x_0 & x_1(x_1 - x_0) & \dots & x_1^{n-1}(x_1 - x_0) \\ 0 & x_2 - x_0 & x_2(x_2 - x_0) & \dots & x_2^{n-1}(x_2 - x_0) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & x_n - x_0 & x_n(x_n - x_0) & \dots & x_n^{n-1}(x_n - x_0) \end{vmatrix} = \text{cofactor expansion}$$

$$\begin{vmatrix} x_1 - x_0 & x_1(x_1 - x_0) & \dots & x_1^{n-1}(x_1 - x_0) \\ x_2 - x_0 & x_2(x_2 - x_0) & \dots & x_2^{n-1}(x_2 - x_0) \\ \vdots & \vdots & \ddots & \vdots \\ x_n - x_0 & x_n(x_n - x_0) & \dots & x_n^{n-1}(x_n - x_0) \end{vmatrix} = \text{multiply row by constant}$$

$$(x_1 - x_0)\begin{vmatrix} 1 & x_1 & \dots & x_1^{n-1} \\ x_2 - x_0 & x_2(x_2 - x_0) & \dots & x_2^{n-1}(x_2 - x_0) \\ \vdots & \vdots & \ddots & \vdots \\ x_n - x_0 & x_n(x_n - x_0) & \dots & x_n^{n-1}(x_n - x_0) \end{vmatrix} = \text{same other rows}$$

$$= \prod_{0<i\leq n}(x_i - x_0) \begin{vmatrix} 1 & x_1 & \dots & x_1^{n-1} \\ 1 & x_2 & \dots & x_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \dots & x_n^{n-1} \end{vmatrix} \text{ repeat process and obtain}$$

$$= \prod_{0<i\leq n}(x_i - x_0) \prod_{1<i\leq n}(x_i - x_1) \begin{vmatrix} 1 & x_2 & \dots & x_2^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \dots & x_n^{n-2} \end{vmatrix} \text{ repeat process ...}$$

$$= \prod_{0\leq i<j\leq n}(x_j - x_i) \neq 0$$

Solution of interpolant polynomial

Computing the augmented coefficient matrix of (2) and solving the system

$$\left(\begin{array}{ccccc|c} 1 & x_0 & x_0^2 & \dots & x_0^n & y_0 \\ 1 & x_1 & x_1^2 & \dots & x_1^n & y_1 \\ 1 & x_2 & x_2^2 & \dots & x_2^n & y_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^n & y_n \end{array}\right) \Rightarrow RREF$$

$$\Rightarrow \left(\begin{array}{ccccc|c} 1 & 0 & 0 & \dots & 0 & s_0 \\ 0 & 1 & 0 & \dots & 0 & s_1 \\ 0 & 0 & 1 & \dots & 0 & s_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & s_n \end{array}\right)$$

Therefore $p_n(x) = s_0 + s_1 x + s_2 x^2 + \ldots + s_n x^n$

Example

Obtain a polynomial through $P_0 = (0, -2)$, $P_1 = (1, -2)$, $P_2 = (2, -2)$ and $P_3 = (3, 4)$.

The polynomial will be of degree at most 3, and will have the form $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$.

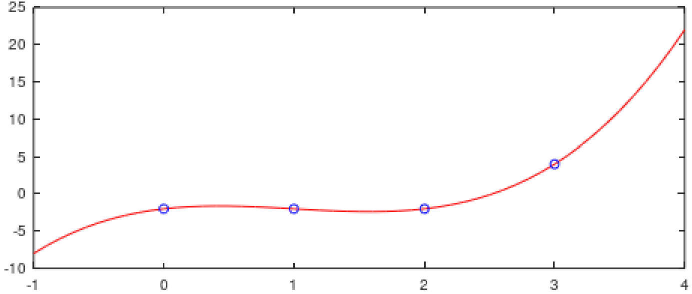
Substituting P_0, P_1, P_2 and P_3 in $a_0 + a_1 x + a_2 x^2 + a_3 x^3 = y$ we create a linear system of 4 equations and 4 unknowns with unique solution.

$$\begin{cases} a_0 = -2 \\ a_0 + a_1 + a_2 + a_3 = -2 \\ a_0 + 2a_1 + 4a_2 + 8a_3 = -2 \\ a_0 + 3a_1 + 9a_2 + 27a_3 = 4 \end{cases}$$

We create the augmented coefficient matrix and apply Gauss-Jordan to solve the system

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -2 \\ 1 & 1 & 1 & 1 & -2 \\ 1 & 2 & 4 & 8 & -2 \\ 1 & 3 & 9 & 27 & 4 \end{array}\right) \Rightarrow RREF \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \Rightarrow \begin{cases} a_0 = -2 \\ a_1 = 2 \\ a_2 = -3 \\ a_4 = 1 \end{cases}$$

The interpolant polynomial is $p(x) = -2 + 2x - 3x^2 + x^3$



The space P_n

Definition

The set of polynomials of degree at most n, denoted with P_n is defined as

$$P_n = \{p_n : \mathbb{R} \to \mathbb{R} \mid p_n(x) = a_0 + a_1 x + \ldots + a_n x^n,\ a_0, a_1, \ldots, a_n \in \mathbb{R}\}$$

Theorem

$(P_n, +, \cdot)$ *is a vector space.*

do you remember what is a vector space?

Vector space

Definition

Let V be a set of objects on which two operations $\odot$ and $\bigstar$ are defined.

$\odot$ is a binary operator that associates to each pair of objects u and v in V an object $u \odot v$.

$$u,\ v \in V \to u \bigodot v$$

$\bigstar$ is a single operator that associates with each object u in V and each scalar $k \in \mathbb{R}$ an object $k \bigstar u$.

$$u \in V \to k \bigstar u$$

The set V with the operations $\odot$ and $\bigstar$ denoted with $(V, \odot, \bigstar)$ is called a **vector space**, and its elements are called **vectors** if the following axioms are satisfied:

Definition

1. if $u,\ v \in V$, then $u \bigodot v \in V$
2. $u \bigodot v = v \bigodot u$
3. $u \bigodot (v \bigodot w) = (u \bigodot v) \bigodot w$
4. $\exists 0 \in V$ such that $\forall u \in V,\ u \bigodot 0 = 0 \bigodot u = u$
5. $\forall u \in V\ \exists - u \in V$ such that $u \bigodot -u = -u \bigodot u = 0$
6. if $k \in \mathbb{R}$ and $u \in V$, then $k \bigstar u \in V$
7. for $k, m \in \mathbb{R}$ and $u \in V$, $(km) \bigstar u = k \bigstar (m \bigstar u)$
8. $1 \bigstar u = u$
9. $(k+m) \bigstar u = (k \bigstar u) \bigodot (m \bigstar u)$
10. $k \bigstar (u \bigodot v) = (k \bigstar u) \bigodot (k \bigstar v)$

$(P_n, +, \cdot)$ is a vector space

In the previous definition:

$V = P_n = \{p_n : \mathbb{R} \to \mathbb{R} \mid p_n(x) = \sum_{i=0}^{n} a_i x^i \text{ for } a_i \in \mathbb{R}\}$

$\odot$ is the sum $+$ so for $p_n(x) = \sum_{i=0}^{n} a_i x^i$ and $q_n(x) = \sum_{i=0}^{n} b_i x^i \in P_n$

then $(p_n + q_n)(x) = \sum_{i=0}^{n} a_i x^i + \sum_{i=0}^{n} b_i x^i = \sum_{i=0}^{n} (a_i + b_i) x^i$

$\bigstar$ is the scalar product $\cdot$ so for $p_n(x) = \sum_{i=0}^{n} a_i x^i \in P_n$ and $k \in \mathbb{R}$

$(k \cdot p_n)(x) = k \sum_{i=0}^{n} a_i x^i = \sum_{i=0}^{n} k a_i x^i$

$(P_n, +, \cdot)$ satisfies the 10 axioms

- if $p,\ q \in P_n$, then $p + q \in P_n$

  take $p(x) = \sum_{i=0}^{n} a_i x^i,\ q(x) = \sum_{i=0}^{n} b_i x^i \in P_n$ then

  $(p+q)(x) = \sum_{i=0}^{n}(a_i + b_i)x^i$ and we have that

  $(p+q) : \mathbb{R} \to \mathbb{R}$ and that $(a_i + b_i) \in \mathbb{R}$ for $i = 0, 1, \dots, n$

  so $(p+q) \in P_n$ ✓

- $p + q = q + p$

  take $p(x) = \sum_{i=0}^{n} a_i x^i,\ q(x) = \sum_{i=0}^{n} b_i x^i \in P_n$ then

  $(p+q)(x) = \sum_{i=0}^{n}(a_i + b_i)x^i = \sum_{i=0}^{n}(b_i + a_i)x^i = (q+p)(x)$✓

- $p+(q+r)=(p+q)+r$

  take $p(x)=\sum_{i=0}^{n} a_i x^i,\ q(x)=\sum_{i=0}^{n} b_i x^i,\ r(x)=\sum_{i=0}^{n} c_i x^i \in P_n$

  then $(p+(q+r))(x)=\sum_{i=0}^{n} a_i x^i+\left(\sum_{i=0}^{n} b_i x^i+\sum_{i=0}^{n} c_i x^i\right)=$

  $\sum_{i=0}^{n} a_i x^i+\sum_{i=0}^{n}(b_i+c_i)x^i=\sum_{i=0}^{n}(a_i+b_i+c_i)x^i=$

  $\sum_{i=0}^{n}(a_i+b_i)x^i+\sum_{i=0}^{n} c_i x^i=$

  $\left(\sum_{i=0}^{n} a_i x^i+\sum_{i=0}^{n} b_i x^i\right)+\sum_{i=0}^{n} c_i x^i=((p+q)+r)(x)\checkmark$

- $\exists 0\in P_n$ such that $\forall p\in P_n,\ p+0=0+p=p$

  take $p(x)=\sum_{i=0}^{n} a_i x^i$ and $0(x)=\sum_{i=0}^{n} 0x^i=0\in P_n$ then

  $(p+0)(x)=\sum_{i=0}^{n}(a_i+0)x^i=\sum_{i=0}^{n}(0+a_i)x^i=\sum_{i=0}^{n} a_i x^i=p(x)\checkmark$

- $\forall p \in P_n \; \exists -p \in P_n$ such that $p + (-p) = -p + p = 0$

  take $p(x) = \sum_{i=0}^{n} a_i x^i$ and $-p(x) = \sum_{i=0}^{n} -a_i x^i$ then

  $(p + (-p))(x) = \sum_{i=0}^{n}(a_i - a_i)x^i = \sum_{i=0}^{n}(-a_i + a_i)x^i =$

  $\sum_{i=0}^{n} 0x^i = 0(x)$✓

- if $k \in \mathbb{R}$ and $p \in P_n$, then $k \cdot p \in P_n$

  take $p(x) = \sum_{i=0}^{n} a_i x^i$ and $k \in \mathbb{R}$ then

  $(k \cdot p)(x) = \sum_{i=0}^{n} k a_i x^i$ and we have that $(k \cdot p) : \mathbb{R} \to \mathbb{R}$ and that

  $k a_i \in \mathbb{R}$ for $i = 0, 1, \ldots, n$ so $(k \cdot p) \in P_n$✓

- for $k, m \in \mathbb{R}$ and $p \in P_n$, $(km) \cdot p = k \cdot (m \cdot p)$

  take $p(x) = \sum_{i=0}^{n} a_i x^i$ and $k,\ m \in \mathbb{R}$ then

  $(km) \cdot p(x) = (km) \sum_{i=0}^{n} a_i x^i = \sum_{i=0}^{n} k m a_i x^i = k \sum_{i=0}^{n} m a_i x^i =$

  $k \cdot (m \cdot p(x)) \checkmark$

- $1 \cdot p = p$

  take $p(x) = \sum_{i=0}^{n} a_i x^i$ then

  $1 \cdot p(x) = 1 \cdot \sum_{i=0}^{n} a_i x^i = \sum_{i=0}^{n} 1 a_i x^i = \sum_{i=0}^{n} a_i x^i = p(x) \checkmark$

- $(k+m)\cdot p = (k\cdot p) + (m\cdot p)$

  take $p(x) = \sum_{i=0}^{n} a_i x^i$ and $k,\ m \in \mathbb{R}$ then

  $(k+m)\cdot p(x) = (k+m)\sum_{i=0}^{n} a_i x^i = \sum_{i=0}^{n}(k+m)a_i x^i =$

  $\sum_{i=0}^{n}(ka_i x^i + ma_i x^i) = \sum_{i=0}^{n} ka_i x^i + \sum_{i=0}^{n} ma_i x^i = kp(x) + mp(x) \checkmark$

- $k\cdot(p+q) = (k\cdot p) + (k\cdot q)$

  take $p(x) = \sum_{i=0}^{n} a_i x^i$, $q(x) = \sum_{i=0}^{n} b_i x^i$ and $k \in \mathbb{R}$ then

  $k\cdot(p(x)+q(x)) = k\cdot\left(\sum_{i=0}^{n} a_i x^i + \sum_{i=0}^{n} b_i x^i\right) =$

  $k\cdot\sum_{i=0}^{n}(a_i + b_i)x^i = \sum_{i=0}^{n} k(a_i+b_i)x^i = \sum_{i=0}^{n}(ka_i x^i + kb_i x^i) =$

  $\sum_{i=0}^{n} ka_i x^i + \sum_{i=0}^{n} kb_i x^i = kp(x) + kq(x) \checkmark$

Properties of vector spaces

Theorem

Let $(V, \bigodot, \bigstar)$ be a vector space, $\vec{u} \in V$ and $k \in \mathbb{R}$ a scalar, then

a) $0 \bigstar \vec{u} = \vec{0}$

b) $k \bigstar \vec{0} = \vec{0}$

c) $-1 \bigstar \vec{u} = -\vec{u}$

d) *if* $k \bigstar \vec{u} = \vec{0}$ *then* $k = 0$ *or* $\vec{u} = \vec{0}$

$0p(x) = 0(x)$

$k0(x) = 0(x)$

$-1p(x) = -p(x)$

if $kp(x) = 0$ then $k = 0$ or $p(x) = 0(x)$

Polynomial subspaces of vector spaces

Theorem

Let $m, n \in \mathbb{Z}^+$ with $m < n$, then P_m is a subspace of P_n.

do you remember what is a subspace of a vector space?

Definition

Let $(V, \bigodot, \bigstar)$ be a vector space and $W \subseteq V$ a subset of V $(\vec{w} \in W \rightarrow \vec{w} \in V)$.

W is a **subspace** of V if $(W, \bigodot, \bigstar)$ is a vector space.

So $(P_m, +, \cdot)$ is a vector space

Subspace main theorem

Theorem

Let $(V, \odot, \bigstar)$ be a vector space and $W \subseteq V$ a subset of V.

$(W, \odot, \bigstar)$ is a subspace of $(V, \odot, \bigstar)$ if and only if the following hold:

1. *if $u,\ v \in W$, then $u \odot v \in W$*
2. *if $k \in \mathbb{R}$ and $u \in W$, then $k \bigstar u \in W$*

- if $p,\ q \in P_m$, then $p + q \in P_m$

  take $p(x) = \sum_{i=0}^{m} a_i x^i,\ q(x) = \sum_{i=0}^{m} b_i x^i \in P_m$ then

  $(p+q)(x) = \sum_{i=0}^{m} (a_i + b_i) x^i$ and we have that

  $(p+q) : \mathbb{R} \to \mathbb{R}$ and that $(a_i + b_i) \in \mathbb{R}$ for $i = 0, 1, \ldots, m$

  so $(p+q) \in P_m \checkmark$

- if $k \in \mathbb{R}$ and $p \in P_m$, then $k \cdot p \in P_m$

  take $p(x) = \sum_{i=0}^{m} a_i x^i$ and $k \in \mathbb{R}$ then

  $(k \cdot p)(x) = \sum_{i=0}^{m} k a_i x^i$ and we have that $(k \cdot p) : \mathbb{R} \to \mathbb{R}$ and that

  $k a_i \in \mathbb{R}$ for $i = 0, 1, \ldots, m$ so $(k \cdot p) \in P_m \checkmark$

$\{0 : \mathbb{R} \to \mathbb{R} \mid 0(x) = 0\}$ subspace of

$P_0 = \{p_0 : \mathbb{R} \to \mathbb{R} \mid p_0(x) = a_0$ for $a_0 \in \mathbb{R}\}$ subspace of

$P_1 = \{p_1 : \mathbb{R} \to \mathbb{R} \mid p_1(x) = a_0 + a_1 x$ for $a_0, a_1 \in \mathbb{R}\}$ subspace of

$P_2 = \{p_2 : \mathbb{R} \to \mathbb{R} \mid p_2(x) = a_0 + a_1 x + a_2 x^2$ for $a_0, a_1, a_2 \in \mathbb{R}\}$

and so on

The standard basis

Theorem

$\{1,\ x,\ x^2,\ \ldots,\ x^n\}$ *is a basis for* P_n *called the standard basis.*

The dimension of P_n *is* $n+1$ $\dim(P_n) = n+1$

do you remember what are basis and dimension of a vector space?

Definition

If $(V, \odot, \bigstar)$ is any vector space and $S = \{v_1, v_2, \ldots, v_n\}$ is a set of vectors in V, then S is called a **basis** for V if the following two conditions hold:

1. S is linearly independent.
2. S spans V.

We need $\{1,\ x,\ x^2,\ \ldots,\ x^n\}$ to be linearly independent and span P_n.

Do you remember what is linearly independence and spanning?

Definition

Let $(V, \bigodot, \bigstar)$ be a vector space and $S = \{\vec{v_1},\ \vec{v_2},\ \ldots,\ \vec{v_n}\}$ a set of vectors in V. We say that the vectors in S are **linearly independent** and that S is a **linearly independent set** if the equation

$$(k_1 \bigstar \vec{v_1}) \bigodot (k_2 \bigstar \vec{v_2}) \bigodot \ldots \bigodot (k_n \bigstar \vec{v_n}) = \vec{0} \tag{4}$$

has as unique solution $k_1 = 0,\ k_2 = 0,\ \ldots,\ k_n = 0$. The set is **linearly dependent** if it is not linearly independent.

So $\{1,\ x,\ x^2,\ \ldots,\ x^n\}$ is linearly independent if

$$k_0 + k_1 x + k_2 x^2 + \ldots + k_n x^n = 0 \tag{5}$$

has unique solution $k_0 = k_1 = k_2 = \ldots = k_n = 0$, and it has!
(whiteboard with Vandermonde matrix)

Definition

If $S = \{v_1,\ v_2,\ \dots,\ v_n\}$ is a set of vectors in a vector space $(V, \odot, \bigstar)$, then the subspace $(W, \odot, \bigstar)$ consisting of all the linear combinations of the vectors in S is called the **space spanned by** $v_1,\ v_2,\ \dots,\ v_n$, and we say that $v_1,\ v_2,\ \dots,\ v_n$ **span** W.

$$W = span(S) \qquad W = span\{v_1,\ v_2,\ \dots,\ v_n\}$$

So $P_n = span\{1,\ x,\ x^2,\ \dots,\ x^n\}$ if every polynomial of degree at most n can be written as linear combination of $1,\ x,\ x^2,\ \dots,\ x^n$ and it does! (definition of P_n)

Definition

The dimension of a finite-dimensional vector space $(V, \odot, \bigstar)$, denoted with $dim(V)$, is the number of vectors in one of its basis.

The dimension of $(\{\vec{0}\}, \odot, \bigstar)$ is zero.

So as $\{1,\ x,\ x^2,\ \ldots,\ x^n\}$ is a basis for P_n containing $n+1$ elements, then $\dim(P_n) = n+1$

$\{0 : \mathbb{R} \to \mathbb{R} \mid 0(x) = 0\}$ has dimension 0.

Theorem

If $S = \{v_1, v_2, \ldots, v_n\}$ is a basis for a vector space $(V, \bigodot, \bigstar)$, then every vector in V can be expressed in the form $v = (c_1 \bigstar v_1) \bigodot (c_2 \bigstar v_2) \bigodot \ldots \bigodot (c_n \bigstar v_n)$ in exactly one way.

Definition

$S = \{v_1, v_2, \ldots, v_n\}$ basis for $(V, \bigodot, \bigstar)$ and $v = (c_1 \bigstar v_1) \bigodot (c_2 \bigstar v_2) \bigodot \ldots \bigodot (c_n \bigstar v_n)$, then $v_S = (c_1, c_2, \ldots, c_n)$ is the vector of coordinates.

$a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ has vector of coordinates $(a_0, a_1, a_2, \ldots, a_n)$ in the standard basis. VECTORS OF COORDINATES ARE IN $\mathbb{R}^{n+1}$