MAT300 CURVES AND SURFACES

Julia Sánchez Sanz

DigiPen Institiute of Technology Europe

julia.sanchez@digipen.edu

Spring 2020

Bases, vectors of coordinates and change of basis

Short review from last day

- P_n is a vector space.
- $\{1, x, x^2, \dots, x^n\}$ is the standard basis for P_n.
- $\dim(P_n) = n + 1$.
- Every polynomial $p_n(x)$ can be written in a unique way as a linear combination of the standard basis.
- The vector of coordinates of $a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$ in the standard basis is $(a_0, a_1, a_2, \dots, a_n)$ which is a vector in $\mathbb{R}^{n+1}$.

P_n and $\mathbb{R}^{n+1}$ are isomorphic

Consider the standard basis $\{1, x, x^2, \ldots, x^n\}$ in which the vector of coordinates of a polynomial

$$p_n(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \in P_n$$

is $(a_0, a_1, a_2, \ldots, a_n) \in \mathbb{R}^{n+1}$ and is unique.

Then we can construct a transformation $T : P_n \to \mathbb{R}^{n+1}$ given by

$$T(a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n) = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \tag{1}$$

$T : P_n \to \mathbb{R}^{n+1}$ is linear

Definition

Let $(V, \odot, \bigstar)$ and $(W, \odot, \bigstar)$ be real vector spaces. A transformation $T : V \to W$ is said to be linear if for all $\vec{u}, \vec{v} \in V$ and for all $k \in \mathbb{R}$ the following hold:

- $T(\vec{u} \odot \vec{v}) = T(\vec{u}) \odot T(\vec{v})$,
- $T(k \bigstar \vec{u}) = k \bigstar T(\vec{u})$.

For $T : P_n \to \mathbb{R}^{n+1}$ with the standard operations, T is linear if:

- $T(p + q) = T(p) + T(q)$,
- $T(kp) = kT(p)$.

- $T(p+q) = T(p) + T(q)$

  Take $p(x) = \sum_{i=0}^{n} a_i x^i$ and $q(x) = \sum_{i=0}^{n} b_i x^i \in P_n$.

  We have that $(p+q)(x) = \sum_{i=0}^{n}(a_i + b_i)x^i$ so

$$T(p+q) = T\left(\sum_{i=0}^{n}(a_i+b_i)x^i\right) = \begin{pmatrix} a_0+b_0 \\ a_1+b_1 \\ a_2+b_2 \\ \vdots \\ a_n+b_n \end{pmatrix} =$$

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = T\left(\sum_{i=0}^{n} a_i x^i\right) + T\left(\sum_{i=0}^{n} b_i x^i\right) = T(p) + T(q)$$

- $T(kp) = kT(p)$

  Take $p(x) = \sum_{i=0}^{n} a_i x^i$ and $k \in \mathbb{R}$.

  We have that $(kp)(x) = \sum_{i=0}^{n} (ka_i) x^i$ so

$$T(kp) = T\left(\sum_{i=0}^{n} (ka_i) x^i\right) = \begin{pmatrix} ka_0 \\ ka_1 \\ ka_2 \\ \vdots \\ ka_n \end{pmatrix} = k \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

$$= kT\left(\sum_{i=0}^{n} a_i x^i\right) = kT(p)$$

So to assign a vector of coordinates in $\mathbb{R}^{n+1}$ to a polynomial in P_n is done through a linear transformation.

$T : P_n \to \mathbb{R}^{n+1}$ is an isomorphism

Definition

Let V and W be vector spaces. A transformation $T : V \to W$ is said to be an isomorphism if T is linear and invertible.

T is invertible (is a one-to-one correspondence).

The inverse of T is $T^{-1} : \mathbb{R}^{n+1} \to P_n$ given by

$$T^{-1}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \tag{2}$$

As we found an isomorphism, we say the spaces are isomorphic $P_n \cong \mathbb{R}^{n+1}$. does this work for any dimension or basis?

Isomorphism theorem

Theorem

Let V be a finite dimensional vector space with $\dim(V) = n$ then there is an isomorphism from V to $\mathbb{R}^n$ and $V \cong \mathbb{R}^n$.

So yes, we can always find an isomorphism among P_n and $\mathbb{R}^{n+1}$ for any dimension and basis.

Moreover, we will be able to find isomorphism from other spaces in which we define curves.

Why all this? It is easier to work in Euclidean spaces: we know how to do linear independence, spanning, change of basis,... using matrices. So the idea is to switch from P_n to $\mathbb{R}^{n+1}$, work there, and return to P_n to interpret the resuts.

From an algebraic perspective is the same, as isomorphism preserves linearity and dimension!

A basis for P_n

Theorem

Every set $B = \{p_i(x) \,|p_i \in P_n,\ i = 0, 1, \ldots, n\}$ *satisfying*

- B *is linearly independent*
- B *spans* P_n

is a basis for P_n.

Every polynomial $q \in P_n$ *can be espressed as linear combination of elemets in* B *in a unique way*

$$q(x) = a_0 p_0(x) + a_1 p_1(x) + \ldots + a_n p_n(x) \tag{3}$$

then $q := (a_0, a_1, \ldots, a_n)_B$ *is the vector of coordinates of* q *in the basis* B.

Shifted basis

Definition

Let $c \in \mathbb{R}$. A shifted basis of P_n is

$$B = \{1,\ x - c,\ (x - c)^2,\ \ldots,\ (x - c)^n\} \tag{4}$$

Example: $B = \{1,\ x - 3,\ (x - 3)^2\}$ basis for P_2.

Find the polynomial $(3, -2, 2)_B$

$p(x) = 3(1) - 2(x - 3) + 2(x - 3)^2 = 3 - 2x + 6 + 2(x^2 - 6x + 9) =$

$27 - 14x + 2x^2$

Find the vector of coordinates of $q(x) = 2 - 6x + x^2$ in B.

$2 - 6x + x^2 = k_1(1) + k_2(x - 3) + k_3(x - 3)^2$

$$2 - 6x + x^2 = k_1(1) + k_2(x-3) + k_3(x^2 - 6x + 9)$$

$$2 - 6x + x^2 = (k_1 - 3k_2 + 9k_3) + (k_2 - 6k_3)x + (k_3)x^2$$

$$\begin{cases} k_1 - 3k_2 + 9k_3 = 2 \\ k_2 - 6k_3 = -6 \\ k_3 = 1 \end{cases} \Rightarrow \begin{cases} k_1 = -7 \\ k_2 = 0 \\ k_3 = 1 \end{cases}$$

$$q := (-7, 0, 1)_B$$

Theorem

Every set $B = \{p_i(x) \,|p_i \in P_n,\ i = 0, 1, \ldots, n\}$ satisfying that B is linearly independent is a basis for P_n.

as we have $|B| = n + 1 = \dim(P_n)$ spanning comes for free!

Example: A shifted basis is a basis for P_n

$$B = \{1,\ x-c,\ (x-c)^2,\ \ldots,\ (x-c)^n\}$$

We have that

$$(x-c)^k = \sum_{j=0}^{k} \binom{k}{j} (-c)^{k-j} x^j$$

Applying the isomorphism (1), $T((x-c)^k) = \begin{pmatrix} \binom{k}{0}(-c)^k \\ \binom{k}{1}(-c)^{k-1} \\ \vdots \\ \binom{k}{k}(-c)^0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

We apply the isomorphism to every $(x-c)^k$ for $k = 0, 1, \ldots, n$

Introducing the vectors in the columns of a square matrix we get

$$A = \begin{pmatrix} 1 & -c & c^2 & -c^3 & \ldots & (-c)^n \\ 0 & 1 & -2c & 3c^2 & \ldots & n(-c)^{n-1} \\ 0 & 0 & 1 & -3c & \ldots & \frac{n(n-1)}{2}(-c)^{n-2} \\ 0 & 0 & 0 & 1 & \ldots & \frac{n(n-1)(n-2)}{6}(-c)^{n-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 \end{pmatrix}$$

$\det(A) = 1 \neq 0$ so the $n+1$ vectors are linearly independent and form a basis for $\mathbb{R}^{n+1}$.

Inverting the isomorphism the $n+1$ polynomials are linearly independent and form a basis for Pn.

Vandermonde basis

Definition

Let $c_0, c_1, c_2 \ldots, c_n \in \mathbb{R}$ with $c_i \neq c_j$ for $i \neq j$. A Vandermonde basis of P_n is

$$B = \{(x-c_0)^n,\ (x-c_1)^n,\ (x-c_2)^n,\ \ldots,\ (x-c_n)^n\} \tag{5}$$

Example: $B = \{x^3,\ (x-1)^3,\ (x-2)^3,\ (x-3)^3\}$ basis for P_3.

Find the polynomial $(1, -2, 2, 1)_B$

$p(x) = 1(x)^3 - 2(x-1)^3 + 2(x-2)^3 + (x-3)^3 =$

$x^3 - 2(x^3 - 3x^2 + 3x - 1) + 2(x^3 - 6x^2 + 12x - 8) + (x^3 - 9x^2 + 27x - 27) =$

$-41 + 45x - 15x^2 + 2x^3$

Find the vector of coordinates of $q(x) = 2 - 6x + x^2 - x^3$ in B.

$$2 - 6x + x^2 - x^3 = k_1(x)^3 + k_2(x-1)^3 + k_3(x-2)^3 + k_4(x-3)^3 =$$

$$k_1x^3 + k_2(x^3 - 3x^2 + 3x - 1) + k_3(x^3 - 6x^2 + 12x - 8) + k_4(x^3 - 9x^2 + 27x - 27)$$

$$= (-k_2 - 8k_3 - 27k_4) + (3k_2 + 12k_3 + 27k_4)x + (-3k_2 - 6k_3 - 9k_4)x^2$$

$$+(k_1 + k_2 + k_3 + k_4)x^3$$

$$\begin{cases} -k_2 - 8k_3 - 27k_4 = 2 \\ 3k_2 + 12k_3 + 27k_4 = -6 \\ -3k_2 - 6k_3 - 9k_4 = 1 \\ k_1 + k_2 + k_3 + k_4 = -1 \end{cases} \Rightarrow \left(\begin{array}{cccc|c} 0 & -1 & -8 & -27 & 2 \\ 0 & 3 & 12 & 27 & -6 \\ 0 & -3 & -6 & -9 & 1 \\ 1 & 1 & 1 & 1 & -1 \end{array}\right)$$

$$\Rightarrow \ RREF \Rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -37/18 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -5/2 \\ 0 & 0 & 0 & 1 & 5/9 \end{array}\right) \Rightarrow \left(-\tfrac{37}{18}, 3, -\tfrac{5}{2}, \tfrac{5}{9}\right)_B$$

Example: A Vandermonde basis is a basis for P_n

$$B = \{(x - c_0)^n,\ (x - c_1)^n,\ (x - c_2)^n,\ \ldots,\ (x - c_n)^n\}$$

We have that

$$(x - c_i)^n = \sum_{j=0}^{n} \left(\begin{array}{c} n \\ j \end{array} \right) (-c_i)^{n-j} x^j$$

Applying the isomorphism (1), $T((x - c_i)^n) = \left(\begin{array}{c} \left(\begin{array}{c} n \\ 0 \end{array} \right) (-c_i)^n \\ \left(\begin{array}{c} n \\ 1 \end{array} \right) (-c)^{n-1} \\ \vdots \\ \left(\begin{array}{c} n \\ n \end{array} \right) (-c_i)^0 \end{array} \right)$

We apply the isomorphism to every $(x - c_i)^n$ for $i = 0, 1, \ldots, n$. Introducing the vectors in the columns of a square matrix we get

$$A = \begin{pmatrix} \binom{n}{0}(-c_0)^n & \binom{n}{0}(-c_1)^n & \ldots & \binom{n}{0}(-c_n)^n \\ \binom{n}{1}(-c_0)^{n-1} & \binom{n}{1}(-c_1)^{n-1} & \ldots & \binom{n}{1}(-c_n)^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{n}{n}(-c_0)^0 & \binom{n}{n}(-c_1)^0 & \ldots & \binom{n}{n}(-c_n)^0 \end{pmatrix}$$

$$\det(A) = \prod_{i=0}^{n} \binom{n}{i} \det \begin{pmatrix} (-c_0)^n & (-c_1)^n & \ldots & (-c_n)^n \\ (-c_0)^{n-1} & (-c_1)^{n-1} & \ldots & (-c_n)^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ (-c_0)^0 & (-c_1)^0 & \ldots & (-c_n)^0 \end{pmatrix}$$

$\det(A) \neq 0$ so the $n+1$ vectors are linearly independent and form a basis for $\mathbb{R}^{n+1}$. Inverting the isomorphism the $n+1$ polynomials are linearly independent and form a basis for Pn.

Bernstein basis

Definition

The Bernstein polynomials of degree n, denoted with $B_0^n,\ B_1^n,\ \ldots,\ B_n^n$ are given as

$$B_i^n(x) = \left(\begin{array}{c} n \\ i \end{array} \right) (1-x)^{n-i} x^i \tag{6}$$

The set of the Bernstein polynomials

$$\{B_0^n(x),\ B_1^n(x),\ \ldots,\ B_n^n(x)\} \tag{7}$$

form a basis for P_n.

Example: The Bernstein polynomials of degree 3 are

$$B_0^3(x) = \left(\begin{array}{c} 3 \\ 0 \end{array} \right) (1-x)^3 x^0 = (1-x)^3 = 1 - 3x + 3x^2 - x^3$$

$$B_1^3(x) = \left(\begin{array}{c} 3 \\ 1 \end{array} \right) (1-x)^2 x^1 = 3(1-x)^2 x = 3(1-2x+x^2)x = 3x-6x^2+3x^3$$

$$B_2^3(x) = \begin{pmatrix} 3 \\ 2 \end{pmatrix} (1-x)^1 x^2 = 3(1-x)x^2 = 3x^2 - 3x^3$$
$$B_3^3(x) = \begin{pmatrix} 3 \\ 3 \end{pmatrix} (1-x)^0 x^3 = x^3$$

$B = \{1 - 3x + 3x^2 - x^3,\ 3x - 6x^2 + 3x^3,\ 3x^2 - 3x^3,\ x^3\}$ is a basis for P_3

Example: find the polynomial $(3, 0, 1, 1)_B$.

Solution: $3 - 9x + 12x^2 - 5x^3$

Example: find the vector of coordinates of $p(x) = 2 - 4x + x^2 - 5x^3$ in the Bernstein basis.

Solution: $\left(2, \frac{2}{3}, -\frac{1}{3}, -6\right)_B$

Change of basis: a first example

Example: Let $B_1 = \{(x-0)^3,\ (x-1)^3,\ (x-2)^3,\ (x-3)^3\}$ be a Vandermonde basis.

Applying the isomorphism (1) we have $T(x^3) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$,

$$T((x-1)^3) = \begin{pmatrix} -1 \\ 3 \\ -3 \\ 1 \end{pmatrix}, \qquad T((x-2)^3) = \begin{pmatrix} -8 \\ 12 \\ -6 \\ 1 \end{pmatrix},$$

$$T((x-3)^3) = \begin{pmatrix} -27 \\ 27 \\ -9 \\ 1 \end{pmatrix}$$

$$\hat{B}_1 = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} -8 \\ 12 \\ -6 \\ 1 \end{pmatrix} \begin{pmatrix} -27 \\ 27 \\ -9 \\ 1 \end{pmatrix} \right\} \text{ is a basis for } \mathbb{R}^4.$$

Let $S_1 : \mathbb{R}^4 \to \mathbb{R}^4$ be the transformation corresponding to a change of basis from Vandermonde to standard. S_1 is given as follows:

$$S_1(\vec{x}) = \begin{pmatrix} 0 & -1 & -8 & -27 \\ 0 & 3 & 12 & 27 \\ 0 & -3 & -6 & -9 \\ 1 & 1 & 1 & 1 \end{pmatrix} \vec{x}$$

$$S_1 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}_{B_1} = \begin{pmatrix} 0 & -1 & -8 & -27 \\ 0 & 3 & 12 & 27 \\ 0 & -3 & -6 & -9 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}_{B_1} = \begin{pmatrix} -8 \\ 12 \\ -6 \\ 1 \end{pmatrix}_S$$

The change of basis from Standard to Vandermonde is given by the inverse of S_1, i.e. $S_1^{-1} : \mathbb{R}^4 \to \mathbb{R}^4$ given as

$$S_1^{-1}(\vec{x}) = \begin{pmatrix} 0 & -1 & -8 & -27 \\ 0 & 3 & 12 & 27 \\ 0 & -3 & -6 & -9 \\ 1 & 1 & 1 & 1 \end{pmatrix}^{-1} \vec{x} = \frac{1}{18} \begin{pmatrix} 3 & 6 & 11 & 18 \\ -9 & -15 & -18 & 0 \\ 9 & 12 & 9 & 0 \\ -3 & -3 & -2 & 0 \end{pmatrix} \vec{x}$$

$$S_1^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}_S = \frac{1}{18} \begin{pmatrix} 3 & 6 & 11 & 18 \\ -9 & -15 & -18 & 0 \\ 9 & 12 & 9 & 0 \\ -3 & -3 & -2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}_S = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}_{B_1}$$

Change of basis: a second example

Example: Let $B_2 = \{1 - 3x + 3x^2 - x^3,\ 3x - 6x^2 + 3x^3,\ 3x^2 - 3x^3,\ x^3\}$

be the Bernstein basis.

Applying the isomorphism (1) we have $T(B_0^3(x)) = \begin{pmatrix} 1 \\ -3 \\ 3 \\ -1 \end{pmatrix}$,

$$T(B_1^3(x)) = \begin{pmatrix} 0 \\ 3 \\ -6 \\ 3 \end{pmatrix}, \qquad T(B_2^3(x)) = \begin{pmatrix} 0 \\ 0 \\ 3 \\ -3 \end{pmatrix},$$

$$T(B_3^3(x)) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\hat{B}_2 = \left\{ \begin{pmatrix} 1 \\ -3 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ -6 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 3 \\ -3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \text{ is a basis for } \mathbb{R}^4.$$

Let $S_2 : \mathbb{R}^4 \to \mathbb{R}^4$ be the transformation corresponding to a change of basis from Bernstein to Standard. S_2 is given as follows:

$$S_2(\vec{x}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix} \vec{x}$$

$$S_2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}_{B_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}_{B_2} = \begin{pmatrix} 0 \\ 0 \\ 3 \\ -3 \end{pmatrix}_S$$

The change of basis from Standard to Bernstein is given by the inverse of S_2, i.e. $S_2^{-1} : \mathbb{R}^4 \to \mathbb{R}^4$ given as

$$S_2^{-1}(\vec{x}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix}^{-1} \vec{x} = \frac{1}{3} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 3 & 3 & 3 & 3 \end{pmatrix} \vec{x}$$

$$S_2^{-1} \begin{pmatrix} 0 \\ 0 \\ 3 \\ -3 \end{pmatrix}_S = \frac{1}{3} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 3 & 3 & 3 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 3 \\ -3 \end{pmatrix}_S = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}_{B_2}$$

Change of basis: a third example

Using the above transformations, and the composition of linear transformations in Euclidean spaces we have:

The transformation that defines a change of basis from Vandermonde to Bernstein $(S_2^{-1} \circ S_1) : \mathbb{R}^4 \to \mathbb{R}^4$ is given as

$$(S_2^{-1} \circ S_1)\vec{x} = \frac{1}{3} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 3 & 3 & 3 & 3 \end{pmatrix} \begin{pmatrix} 0 & -1 & -8 & -27 \\ 0 & 3 & 12 & 27 \\ 0 & -3 & -6 & -9 \\ 1 & 1 & 1 & 1 \end{pmatrix} \vec{x}$$

$$\begin{pmatrix} 0 & -1 & -8 & -27 \\ 0 & 0 & -4 & -18 \\ 0 & 0 & -2 & -12 \\ 1 & 0 & -1 & -8 \end{pmatrix} \vec{x}$$

Generalization change of basis

Let $B_1 = \{p_0, p_1, \ldots, p_n\}$ and $B_2 = \{q_0, q_1, \ldots, q_n\}$

bases for P_n.

We obtain the vectors of coordinates of p_i and q_i in the standard basis through the isomorphism (1).

$$T(B_1) = \{T(p_0), T(p_1), \ldots, T(p_n)\} = \{\vec{u_0}, \vec{u_1}, \ldots, \vec{u_n}\}$$

$$T(B_2) = \{T(q_0), T(q_1), \ldots, T(q_n)\} = \{\vec{v_0}, \vec{v_1}, \ldots, \vec{v_n}\}$$

bases for $\mathbb{R}^{n+1}$.

The matrix $M = [\vec{u_0}, \vec{u_1}, \ldots, \vec{u_n}]$ defines the change of basis $B_1 \rightarrow \textit{Standard}$

The matrix $N = [\vec{v_0}, \vec{v_1}, \ldots, \vec{v_n}]$ defines the change of basis $B_2 \rightarrow Standard$

Change of basis from B_1 to B_2: $B_1 \rightarrow Standard \rightarrow B_2$

$S : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$

$$S(\vec{x}) = N^{-1} M \vec{x}$$

Change of basis from B_2 to B_1: $B_2 \rightarrow Standard \rightarrow B_1$

$S^{-1} : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$

$$S^{-1}(\vec{x}) = M^{-1} N \vec{x}$$