DigiPen Institute of Technology, Bilbao

MAT300 Curves & Surfaces

Spring 2020. Class exercises: 30-3-2020

Consider the piecewise polynomial

$$p(x) = \begin{cases} 1 + 2x^2 & x \in [0,1) \\ 2x + x^2 & x \in [1,2) \\ 10 - 9x + 4x^2 & x \in [2,3) \\ 27 - 20x + 6x^2 & x \in [3,4) \\ 27 - 20x + 6x^2 & x \in [4,5] \end{cases}$$

1. Determine the space to which $p(x)$ belongs.

   All the polynomials are of degree 2 so $p \in P_2^5[0,1,2,3,4,5]$.

   We compute the derivatives of p

   $$p'(x) = \begin{cases} 4x & x \in [0,1) \\ 2 + 2x & x \in [1,2) \\ -9 + 8x & x \in [2,3) \\ -20 + 12x & x \in [3,4) \\ -20 + 12x & x \in [4,5] \end{cases} \quad p''(x) = \begin{cases} 4 & x \in [0,1) \\ 2 & x \in [1,2) \\ 8 & x \in [2,3) \\ 12 & x \in [3,4) \\ 12 & x \in [4,5] \end{cases}$$

   We check orders of differentiability at the intermediate points.

   (a) At $x_1 = 1$ order of differentiability r_1:

   $$p: \lim_{x \to 1^-} 1 + 2x^2 = 1 + 2 = 3 \qquad \lim_{x \to 1^+} 2x + x^2 = 2 + 1 = 3 \qquad \Rightarrow r_1 \geq 0$$

   $$p': \lim_{x \to 1^-} 4x = 4 \qquad \lim_{x \to 1^+} 2 + 2x = 2 + 2 = 4 \qquad \Rightarrow r_1 \geq 1$$

   $$p'': \lim_{x \to 1^-} 4 = 4 \qquad \lim_{x \to 1^+} 2 = 2 \qquad \Rightarrow r_1 = 1$$

   (b) At $x_2 = 2$ order of differentiability r_2:

   $$p: \lim_{x \to 2^-} 2x + x^2 = 4 + 4 = 8 \qquad \lim_{x \to 2^+} 10 - 9x + 4x^2 = 10 - 18 + 16 = 8 \Rightarrow r_2 \geq 0$$

   $$p': \lim_{x \to 2^-} 2 + 2x = 2 + 4 = 6 \qquad \lim_{x \to 2^+} -9 + 8x = -9 + 16 = 7 \qquad \Rightarrow r_2 = 0$$

(c) At $x_3 = 3$ order of differentiability r_3:

$$p: \lim_{x\to 3^-} 10 - 9x + 4x^2 = 10 - 27 + 36 = 19$$
$$\lim_{x\to 3^+} 27 - 20x + 6x^2 = 27 - 60 + 36 = 3 \ \Rightarrow r_3 = -1$$

(d) At $x_4 = 4$ order of differentiability r_4:

$$p_4(x) = 27 - 20x + 6x^2 = p_5(x) \qquad \Rightarrow r_4 = 2$$

We conclude that $p \in P^5_{2,\vec{r}}[0,1,2,3,4,5]$ with $\vec{r} = (1, 0, -1, 2)$.

2. Construct the standard basis for that space.

We start with the standard basis for $P^5_2[0,1,2,3,4,5]$ which is

$$B = \{1, x, x^2, (x-1)^0_+, (x-1)^1_+, (x-1)^2_+, (x-2)^0_+, (x-2)^1_+, (x-2)^2_+, (x-3)^0_+, (x-3)^1_+, (x-3)^2_+, (x-4)^0_+, (x-4)^1_+, (x-4)^2_+, \}$$

(a) At $x = 1$ order of differentiability is 1:

$$B = \{1, x, x^2, \cancel{(x-1)^0_+}, \cancel{(x-1)^1_+}, (x-1)^2_+, (x-2)^0_+, (x-2)^1_+, (x-2)^2_+, (x-3)^0_+, (x-3)^1_+, (x-3)^2_+, (x-4)^0_+, (x-4)^1_+, (x-4)^2_+, \}$$

(b) At $x = 2$ order of differentiability is 0:

$$B = \{1, x, x^2, \cancel{(x-1)^0_+}, \cancel{(x-1)^1_+}, (x-1)^2_+, \cancel{(x-2)^0_+}, (x-2)^1_+, (x-2)^2_+, (x-3)^0_+, (x-3)^1_+, (x-3)^2_+, (x-4)^0_+, (x-4)^1_+, (x-4)^2_+, \}$$

(c) At $x = 3$ order of differentiability is -1:

$$B = \{1, x, x^2, \cancel{(x-1)^0_+}, \cancel{(x-1)^1_+}, (x-1)^2_+, \cancel{(x-2)^0_+}, (x-2)^1_+, (x-2)^2_+, (x-3)^0_+, (x-3)^1_+, (x-3)^2_+, (x-4)^0_+, (x-4)^1_+, (x-4)^2_+, \}$$

(d) At $x = 4$ order of differentiability is 2:

$$B = \{1, x, x^2, \cancel{(x-1)^0_+}, \cancel{(x-1)^1_+}, (x-1)^2_+, \cancel{(x-2)^0_+}, (x-2)^1_+, (x-2)^2_+, (x-3)^0_+, (x-3)^1_+, (x-3)^2_+, \cancel{(x-4)^0_+}, \cancel{(x-4)^1_+}, \cancel{(x-4)^2_+}, \}$$

After deleting the elements that break continuity the standard basis is

$$B = \{1, x, x^2, (x-1)^2_+, (x-2)^1_+, (x-2)^2_+, (x-3)^0_+, (x-3)^1_+, (x-3)^2_+, \}$$

3. Give the vector of coordinates of p in the standard basis.

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3(x-1)^2_+ + a_4(x-2)^1_+ + a_5(x-2)^2_+ + a_6(x-3)^0_+ + a_7(x-3)^1_+ + a_8(x-3)^2_+$$

We do the construction piecewise.

(a) for $x \in [0,1)$, $p(x) = 1 + 2x^2$:

$$1 + 2x^2 = a_0 + a_1 x + a_2 x^2 + a_3 \cdot 0 + a_4 \cdot 0 + a_5 \cdot 0 + a_6 \cdot 0 + a_7 \cdot 0 + a_8 \cdot 0$$

so $a_0 = 1$, $a_1 = 0$ and $a_2 = 2$.

(b) for $x \in [1,2)$, $p(x) = 2x + x^2$:

$$2x + x^2 = 1 + 2x^2 + a_3(x-1)^2 \quad \rightarrow -1 + 2x - x^2 = a_3(x^2 - 2x + 1)$$

so $a_3 = -1$.

(c) for $x \in [2,3)$, $p(x) = 10 - 9x + 4x^2$:

$$10 - 9x + 4x^2 = 1 + 2x^2 - (x-1)^2 + a_4(x-2) + a_5(x-2)^2$$

$$\begin{aligned} 10 - 9x + 4x^2 &= 1 + 2x^2 - x^2 + 2x - 1 + a_4(x-2) + a_5(x^2 - 4x + 4) \\ 10 - 11x + 3x^2 &= a_4(x-2) + a_5(x^2 - 4x + 4) \\ 10 - 11x + 3x^2 &= (-2a_4 + 4a_5) + (a_4 - 4a_5)x + a_5 x^2 \end{aligned}$$

so $a_4 = 1$ and $a_5 = 3$.

(d) for $x \in [3,4)$, $p(x) = 27 - 20x + 6x^2$:

$$27 - 20x + 6x^2 = 1 + 2x^2 - (x-1)^2 + (x-2) + 3(x-2)^2 + a_6 + a_7(x-3) + a_8(x-3)^2$$

$$27 - 20x + 6x^2 = 1 + 2x^2 - x^2 + 2x - 1 + x - 2 + 3x^2 - 12x + 12 + a_6 + a_7(x-3) + a_8(x-3)^2$$

$$\begin{aligned} 27 - 20x + 6x^2 &= 4x^2 - 9x + 10 + a_6 + a_7(x-3) + a_8(x^2 - 6x + 9) \\ 17 - 11x + 2x^2 &= (a_6 - 3a_7 + 9a_8) + (a_7 - 6a_8)x + a_8 x^2 \end{aligned}$$

so $a_6 = 2$, $a_7 = 1$ and $a_8 = 2$.

Then the vector of coordinates is

$$(1, 0, 2, -1, 1, 3, 2, 1, 2).$$