MAT300 CURVES AND SURFACES

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Bezier curves

1. Polar forms of parametric curves
2. Polar forms and nested linear interpolation

Polar forms of parametric curves

$\gamma(t) = (p(t), q(t)),\ t \in [a, b]$ curve in 2D

$\gamma(t) = (p(t), q(t), r(t)),\ t \in [a, b]$ curve in 3D

The polar form of γ is given by the polar form of each component polynomial.

Idea: express the curve in point coefficient form in standard or Bernstein. Then obtain the polar form for the polynomials of the basis.

Example: $\gamma(t) = (1 - t^2, 3 + t + 2t^2) = 1(1, 3) + t(0, 1) + t^2(-1, 2)$

Polynomials belong to P_2 (smallest space) so we take polar forms for P_2

$F_0[u_1, u_2] = 1, \quad F_1[u_1, u_2] = \frac{u_1+u_2}{2}, \quad F_2[u_1, u_2] = u_1 u_2$

$F[u_1, u_2] = 1(1, 3) + \left(\frac{u_1+u_2}{2}\right)(0, 1) + u_1 u_2(-1, 2)$

Polar forms of parametric curves

Example: $\gamma(t) = B_0^2(t)(2,-1) + B_1^2(t)(3,4) + B_2^2(t)(1,3) =$

$(1-t)^2(2,-1) + 2(1-t)t(3,4) + t^2(1,3)$

Take polar forms of Bernstein polynomials in P_2

$F_0[u_1,u_2] = (1-u_1)(1-u_2), \quad F_1[u_1,u_2] = (1-u_1)u_2 + (1-u_2)u_1,$

$F_2[u_1,u_2] = u_1 u_2$

$$F[u_1,u_2] = (1-u_1)(1-u_2)(2,-1) + \Big((1-u_1)u_2 + (1-u_2)u_1\Big)(3,4) + u_1 u_2(1,3)$$

Why are polar forms interesting for us?

because through a polar form we can obtain the control points of a Bezier curve!

Polar forms and control points

$\gamma : [a, b] \to \mathbb{R}^2$ given as $\gamma(t) = (p(t), q(t))$ with $p, q \in P_n$

γ has a Bezier representation $\gamma(t) = \sum_{i=0}^{n} P_i B_i^n(t), \quad t \in [0, 1]$

Let $F[u_1, u_2, u_3, \dots u_{n-1}, u_n]$ be the polar form of γ, then:

$P_0 = F[a, a, \dots, a, a]$ evaluate with all a

$P_1 = F[a, a, \dots, a, b]$ evaluate with $n-1$ a and 1 b

$P_2 = F[a, a, \dots, b, b]$ evaluate with $n-2$ a and 2 b

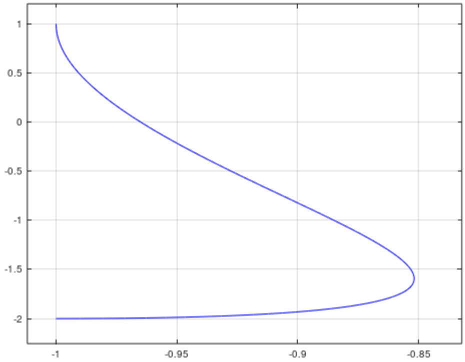
$\vdots$

$P_{n-1} = F[a, b, \dots, b, b]$ evaluate with 1 a and $n-1$ b

$P_n = F[b, b, \dots, b, b]$ evaluate with all b

A first example with $\gamma : [0, 1] \to \mathbb{R}^2$

$\gamma : [0, 1] \to \mathbb{R}^2$ with $\gamma(t) = (-1 + t - 2t^2 + t^3, -2 + 4t^2 - t^3)$



What are the control points of its Bezier representation?

$$\gamma(t) = (-1 + t - 2t^2 + t^3, -2 + 4t^2 - t^3) = 1(-1, -2) + t(1, 0) + t^2(-2, 4) + t^3(1, -1)$$

Polynomials in P_3 therefore polar form $F[u_1, u_2, u_3]$

Compute polar form for γ

$$F_0[u_1, u_2, u_3] = 1,$$

$$F_1[u_1, u_2, u_3] = \tfrac{u_1+u_2+u_3}{3}$$

$$F_2[u_1, u_2, u_3] = \tfrac{u_1 u_2+u_1 u_3+u_2 u_3}{3}$$

$$F_3[u_1, u_2, u_3] = u_1 u_2 u_3$$

$$\gamma(t) = 1(-1,-2) + t(1,0) + t^2(-2,4) + t^3(1,-1)$$

$$F[u_1, u_2, u_3] =$$

$$1(-1,-2) + \left(\tfrac{u_1+u_2+u_3}{3}\right)(1,0) + \left(\tfrac{u_1 u_2+u_1 u_3+u_2 u_3}{3}\right)(-2,4) + u_1 u_2 u_3(1,-1)$$

Once we have the polar form we obtain the control points

$\gamma : [0,1] \to \mathbb{R}^2$, $F[u_1, u_2, u_3] =$

$$1(-1,-2) + \left(\tfrac{u_1+u_2+u_3}{3}\right)(1,0) + \left(\tfrac{u_1u_2+u_1u_3+u_2u_3}{3}\right)(-2,4) + u_1u_2u_3(1,-1)$$

Control points:

$$P_0 = F[0,0,0] = 1(-1,-2) + 0(1,0) + 0(-2,4) + 0(1,-1) = (-1,-2)$$

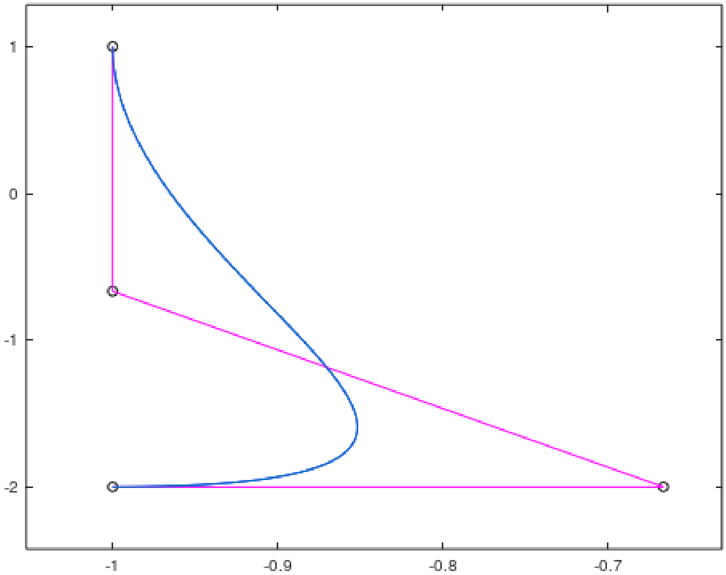
$$P_1 = F[0,0,1] = 1(-1,-2) + \tfrac{1}{3}(1,0) + 0(-2,4) + 0(1,-1) = (-\tfrac{2}{3},-2)$$

$$P_2 = F[0,1,1] = 1(-1,-2) + \tfrac{2}{3}(1,0) + \tfrac{1}{3}(-2,4) + 0(1,-1) = (-1,-\tfrac{2}{3})$$

$$P_3 = F[1,1,1] = 1(-1,-2) + 1(1,0) + 1(-2,4) + 1(1,-1) = (-1,1)$$

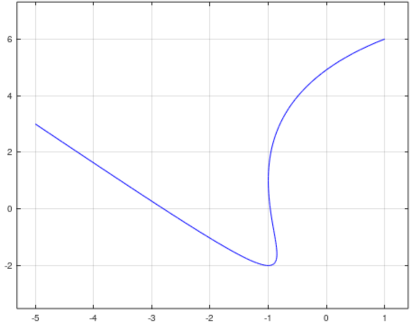
And the Bezier representation is

$$\gamma(t) = \sum_{i=0}^{3} B_i^3(t)P_i, \quad t \in [0,1]$$


1
0
-1
-2
-1
-0.9
-0.8
-0.7

A second example with $\hat{\gamma} : [a, b] \to \mathbb{R}^2$

$\hat{\gamma}(t) = (-1 + t - 2t^2 + t^3, -2 + 4t^2 - t^3) \quad t \in [-1, 2]$



What are the control points of its Bezier representation?

$F[u_1, u_2, u_3] =$

$$1(-1, -2) + \left(\frac{u_1+u_2+u_3}{3}\right)(1, 0) + \left(\frac{u_1 u_2+u_1 u_3+u_2 u_3}{3}\right)(-2, 4) + u_1 u_2 u_3(1, -1)$$

$\hat{\gamma} : [-1, 2] \to \mathbb{R}^2$, $F[u_1, u_2, u_3] =$

$$1(-1,-2) + \left(\tfrac{u_1+u_2+u_3}{3}\right)(1,0) + \left(\tfrac{u_1u_2+u_1u_3+u_2u_3}{3}\right)(-2,4) + u_1u_2u_3(1,-1)$$

Control points:

$$P_0 = F[-1,-1,-1] = (-1,-2) - (1,0) + (-2,4) - (1,-1) = (-5,3)$$

$$P_1 = F[-1,-1,2] = (-1,-2) + 0(1,0) - (-2,4) + 2(1,-1) = (3,-8)$$

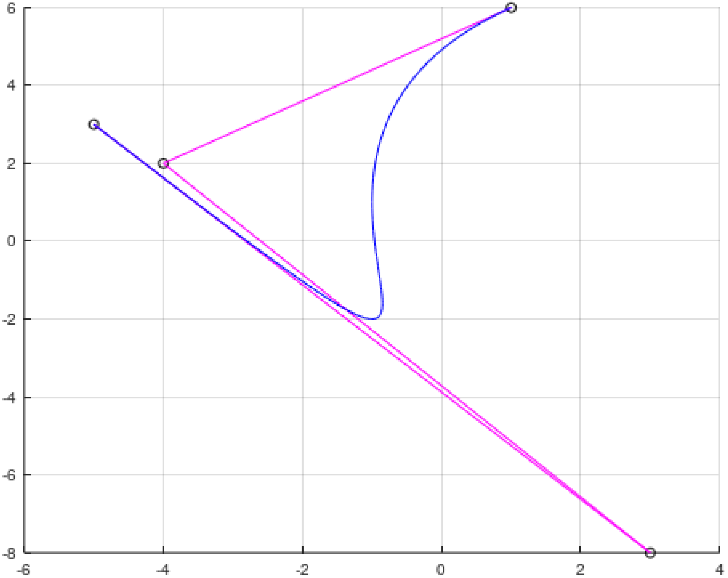
$$P_2 = F[-1,2,2] = (-1,-2) + (1,0) + 0(-2,4) - 4(1,-1) = (-4,2)$$

$$P_3 = F[2,2,2] = (-1,-2) + 2(1,0) + 4(-2,4) + 8(1,-1) = (1,6)$$

And the Bezier representation is

$$\hat{\gamma}(s) = \sum_{i=0}^{3} B_i^3(s) P_i, \quad s \in [0,1]$$

Notice the reparametrization!!!


6
4
2
0
-2
-4
-6
-8
-6
-4
-2
0
2
4

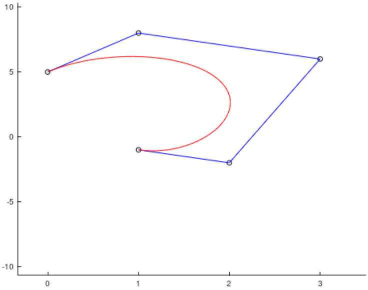
A third example: reparametrize Bezier curves

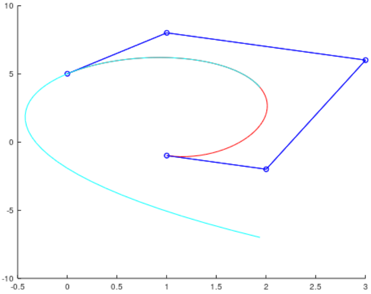
$\gamma(t) = \sum_{i=0}^{4} P_i B_i^4(t), \; t \in [0,1]$ Bezier curve,

with $P_0 = (1,-1)$, $P_1 = (2,-2)$, $P_2 = (3,6)$, $P_3 = (1,8)$ and $P_4 = (0,5)$

what is the Bezier representation $\hat{\gamma}(s) = \sum_{i=0}^{4} \hat{P}_i B_i^4(s), \; s \in [0,1]$ of $\gamma(t) = \sum_{i=0}^{4} P_i B_i^4(t), \; t \in \left[\frac{1}{2}, \frac{3}{2}\right]$?

what are the control points of such a Bezier curve?





$$F_0[u_1, u_2, u_3, u_4] = (1-u_1)(1-u_2)(1-u_3)(1-u_4)$$

$$F_1[u_1, u_2, u_3, u_4] = (1-u_1)(1-u_2)(1-u_3)u_4 + (1-u_1)(1-u_2)(1-u_4)u_3$$

$$+(1-u_1)(1-u_3)(1-u_4)u_2 + (1-u_2)(1-u_3)(1-u_4)u_1$$

$$F_2[u_1, u_2, u_3, u_4] = (1-u_1)(1-u_2)u_3u_4 + (1-u_1)(1-u_3)u_2u_4 + (1-u_1)(1-u_4)u_2u_3$$

$$+(1-u_2)(1-u_3)u_1u_4 + (1-u_2)(1-u_4)u_1u_3 + (1-u_3)(1-u_4)u_1u_2$$

$$F_3[u_1, u_2, u_3, u_4] = (1-u_1)u_2u_3u_4 + (1-u_2)u_1u_3u_4 + (1-u_3)u_1u_2u_4$$

$$+(1-u_4)u_1u_2u_3$$

$$F_4[u_1, u_2, u_3, u_4] = u_1u_2u_3u_4$$

$$F[u_1, u_2, u_3, u_4] = F_0(1, -1) + F_1(2, -2) + F_2(3, 6) + F_3(1, 8) + F_4(0, 5)$$

$$\hat{P}_0 = F[\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}] = \tfrac{1}{16}(1, -1) + \tfrac{4}{16}(2, -2) + \tfrac{6}{16}(3, 6) + \tfrac{4}{16}(1, 8) + \tfrac{1}{16}(0, 5)$$

$$= (\tfrac{31}{16}, 4)$$

$$\hat{P}_1 = F[\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}] = -\tfrac{1}{16}(1, -1) + 0(2, -2) + \tfrac{6}{16}(3, 6) + \tfrac{8}{16}(1, 8) + \tfrac{3}{16}(0, 5)$$

$$= (\tfrac{25}{16}, \tfrac{29}{4})$$

$$\hat{P}_2 = F[\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{3}{2}] = \tfrac{1}{16}(1, -1) - \tfrac{4}{16}(2, -2) - \tfrac{2}{16}(3, 6) + \tfrac{12}{16}(1, 8) + \tfrac{9}{16}(0, 5)$$

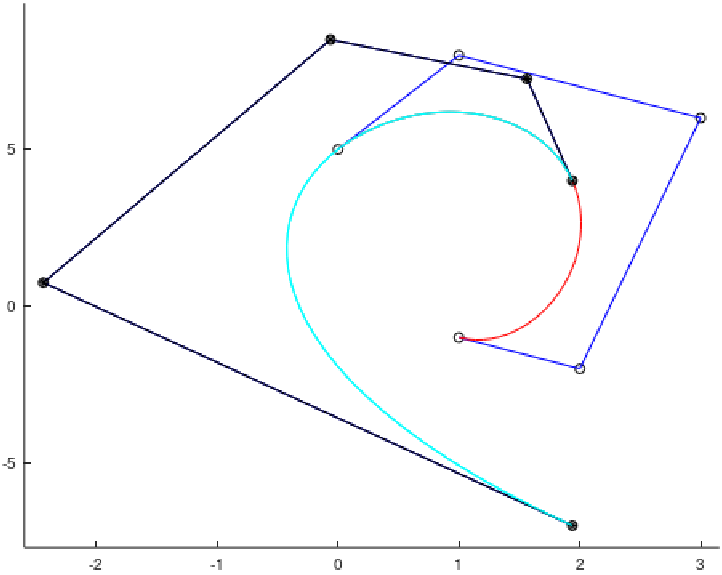
$$= (-\tfrac{1}{16}, \tfrac{17}{2})$$

$$\hat{P}_3 = F[\tfrac{1}{2}, \tfrac{3}{2}, \tfrac{3}{2}, \tfrac{3}{2}] = -\tfrac{1}{16}(1, -1) + \tfrac{8}{16}(2, -2) - \tfrac{18}{16}(3, 6) + 0(1, 8) + \tfrac{27}{16}(0, 5)$$

$$= (-\tfrac{39}{16}, \tfrac{3}{4})$$

$$\hat{P}_4 = F[\tfrac{3}{2}, \tfrac{3}{2}, \tfrac{3}{2}, \tfrac{3}{2}] = \tfrac{1}{16}(1, -1) - \tfrac{12}{16}(2, -2) + \tfrac{54}{16}(3, 6) - \tfrac{108}{16}(1, 8) + \tfrac{81}{16}(0, 5)$$

$$= (\tfrac{31}{16}, -7)$$


5
0
-5
-2
-1
0
1
2
3

Construct polar forms using recursion

Knowing the control points of a Bezier curve, its polar form can be constructed through nested linear interpolation recursively.

The algorighm is similar to the De Casteljau.

Example of polar form of a polynomial in P_2:

$$F[0,0] = P_0$$

$$F[0,1] = P_1$$

$$F[1,1] = P_2$$

We start with the control points, and apply linear interpolation with parameter u_1 for the first position leaving the second position of the array with the lowest index.

$$
\begin{array}{lll}
F[0,0] = P_0 & & \\
 & F[u_1,0] = (1-u_1)F[0,0] + u_1 F[0,1] = (1-u_1)P_0 + u_1 P_1 & \\
F[0,1] = P_1 & & \\
 & F[u_1,1] = (1-u_1)F[0,1] + u_1 F[1,1] = (1-u_1)P_1 + u_1 P_2 & \\
F[1,1] = P_2 & &
\end{array}
$$

Now we apply the recursion with parameter u_2 to the second position in the array

$$
\begin{array}{lll}
F[0,0] & & \\
 & F[u_1,0] & \\
F[0,1] & & F[u_1,u_2] = (1-u_2)F[u_1,0] + u_2 F[u_1,1] \\
 & F[u_1,1] & \\
F[1,1] & &
\end{array}
$$

where

$$F[u_1,u_2] = (1-u_2)\big((1-u_1)P_0 + u_1 P_1\big) + u_2\big((1-u_1)P_1 + u_1 P_2\big)$$

For a polynomial in P_3

$$
\begin{array}{ll}
F[0,0,0] = P_0 & \\
 & F[u_1,0,0] = (1-u_1)F[0,0,0] + u_1 F[0,0,1] \\
F[0,0,1] = P_1 & \\
 & F[u_1,0,1] = (1-u_1)F[0,0,1] + u_1 F[0,1,1] \\
F[0,1,1] = P_2 & \\
 & F[u_1,1,1] = (1-u_1)F[0,1,1] + u_1 F[1,1,1] \\
F[1,1,1] = P_3 &
\end{array}
$$

For the first level we apply nested linear interpolation with variable u_1 in the first position of the array.

$$
\begin{array}{lll}
F[0,0,0] = P_0 & & \\
 & F[u_1,0,0] & \\
F[0,0,1] = P_1 & & F[u_1,u_2,0] = (1-u_2)F[u_1,0,0] + u_2 F[u_1,0,1] \\
 & F[u_1,0,1] & \\
F[0,1,1] = P_2 & & F[u_1,u_2,1] = (1-u_2)F[u_1,0,1] + u_2 F[u_1,1,1] \\
 & F[u_1,1,1] & \\
F[1,1,1] = P_3 & &
\end{array}
$$

And finally do the same for the variable u_3 in the last position

$$
\begin{array}{ccccccc}
F[0,0,0] = P_0 & & & & & & \\
 & & F[u_1,0,0] & & & & \\
F[0,0,1] = P_1 & & & & F[u_1,u_2,0] & & \\
 & & F[u_1,0,1] & & & & F[u_1,u_2,u_3] \\
F[0,1,1] = P_2 & & & & F[u_1,u_2,1] & & \\
 & & F[u_1,1,1] & & & & \\
F[1,1,1] = P_3 & & & & & &
\end{array}
$$

where $F[u_1,u_2,u_3] = (1-u_3)F[u_1,u_2,0] + u_3 F[u_1,u_2,1]$