DigiPen Institute of Technology, Bilbao

MAT300 Curves & Surfaces

Spring 2020. Homework 1: Deadline: 1-27-2020

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1. (10%) Consider $x_0 = -3$, $x_1 = -1$, $x_2 = 0$, $x_3 = 2$ and $x_4 = 3$. Construct a polynomial $p : \mathbb{R} \to \mathbb{R}$ of minimum degree satisfying $p(x_0) = 100$, $p(x_1) = 14$, $p(x_2) = 1$, $p(x_3) = 65$ and $p(x_4) = 154$.

2. (10%) Is p (the result in exercise 1) the unique polynomial of degree at most 4 satisfying that conditions? justify your answer with algebraic statements (equivalent statements from MAT250). If p is not unique give a polynomial $q \neq p$ satisfying the conditions in exercise 1.

3. (10%) Is p (the result in exercise 1) the unique polynomial of degree at most 5 satisfying that conditions? justify your answer with algebraic statements. If p is not unique give a polynomial $q \neq p$ satisfying the conditions in exercise 1.

4. (5%) Construct the Vandermonde basis with constants x_0, x_1, x_2, x_3 and x_4 (those given in exercise 1). What is the space for which it is a basis?

5. (5%) Construct a shifted basis for the space in exercise 4 taking x_0 of exercise 1 as a constant.

6. (15%) Show that the Bernstein polynomials of degree n form a basis for P_n (for n arbitrary).

7. (5%) Construct the Bernstein basis for the space in exercise 4.

8. (10%) Construct a transformation corresponding to a change of basis from Bernstein to Vandermonde (using bases from the previous exercises).

9. (10%) Construct a transformation corresponding to a change of basis from Shifted to Vandermonde (using bases from the previous exercises).

10. (10%) Construct a transformation corresponding to a change of basis from the Bernstein to Shifted (using bases from the previous exercises).

11. (10%) Give the vector of coordinates of p (obtained in exercise 1) in the standard, the Vandermonde, the Shifted, and the Bernstein basis (using the above transformations).

MAT300 Spring 2020 SOLUTIONS HW 1

(1) $x_0=-3$, $x_1=-1$, $x_2=0$, $x_3=2$, $x_4=3$

$p:\mathbb{R}\to\mathbb{R}$

$p(x)=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4$

$p(x_0)=100$, $p(x_1)=14$, $p(x_2)=1$, $p(x_3)=65$

$p(x_4)=154$

We build a linear system of 5 equations and 5 unknowns

$$
\begin{array}{ll}
p(x_0)=100 & \\
p(x_1)=14 & \\
p(x_2)=1 & \\
p(x_3)=65 & \\
p(x_4)=154 &
\end{array}
\qquad
\left.\begin{array}{l}
a_0-3a_1+9a_2-27a_3+81a_4=100\\
a_0-a_1+a_2-a_3+a_4=14\\
a_0=1\\
a_0+2a_1+4a_2+8a_3+16a_4=65\\
a_0+3a_1+9a_2+27a_3+81a_4=154
\end{array}\right\}
$$

We get the augmented coefficient matrix and its RREF

$$
\left(\begin{array}{ccccc|c}
1 & -3 & 9 & -27 & 81 & 100\\
1 & -1 & 1 & -1 & 1 & 14\\
1 & 0 & 0 & 0 & 0 & 1\\
1 & 2 & 4 & 8 & 16 & 65\\
1 & 3 & 9 & 27 & 81 & 154
\end{array}\right)
\xrightarrow{RREF}
\left(\begin{array}{c|c}
 & 1\\
 & 0\\
I & 14\\
 & 1\\
 & 0
\end{array}\right)
\longrightarrow
\begin{array}{l}
a_0=1\\
a_1=0\\
a_2=14\\
a_3=1\\
a_4=0
\end{array}
$$

so $\boxed{p(x)=1+14x^2+x^3}$

(2) Yes, it is. As the nodes x_0, x_1, x_2, x_3 and x_4 are all different, then the system of equations $A\vec{x}=\vec{b}$ has matrix

$$
A=\begin{pmatrix}
1 & x_0 & x_0^2 & x_0^3 & x_0^4\\
1 & x_1 & x_1^2 & x_1^3 & x_1^4\\
1 & x_2 & x_2^2 & x_2^3 & x_2^4\\
1 & x_3 & x_3^2 & x_3^3 & x_3^4\\
1 & x_4 & x_4^2 & x_4^3 & x_4^4
\end{pmatrix}
$$

The matrix A is a Vandermonde matrix satisfying $\det(A)\neq 0$, and so the system of equations has a unique solution.

Therefore the polynomial is unique for degree at most 4.

③ $q : \mathbb{R} \longrightarrow \mathbb{R}$ $\quad q(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$

We build a system of equations by substituting the conditions in exercise 1 into $q(x)$

$$\left.\begin{array}{ll} q(-3)=100 & a_0 - 3a_1 + 9a_2 - 27a_3 + 81a_4 - 243a_5 = 100 \\ q(-1) = 14 & a_0 - a_1 + a_2 - a_3 + a_4 - a_5 = 14 \\ q(0) = 1 & a_0 = 1 \\ q(2) = 65 & a_0 + 2a_1 + 4a_2 + 8a_3 + 16a_4 + 32a_5 = 65 \\ q(3) = 154 & a_0 + 3a_1 + 9a_2 + 27a_3 + 81a_4 + 243a_5 = 154 \end{array}\right\}$$

The system has 6 unknowns and 5 equations. If we take the coefficient matrix A of $A\vec{x} = \vec{b}$

$$A = \begin{pmatrix} 1 & -3 & 9 & -27 & 81 & -243 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 4 & 8 & 16 & 32 \\ 1 & 3 & 9 & 27 & 81 & 243 \end{pmatrix}$$

we have that

$\operatorname{rank}(A) + \operatorname{null}(A) = 6$

~~and~~ as $\operatorname{rank}(A) = 5$ we have $\operatorname{null}(A) = 1$, so the dimension of the nullspace of A is 1 and so the system $A\vec{x} = \vec{b}$ can not have unique solution, so p is not unique of degree at most 5. Solving the system we get

$$\operatorname{RREF}\left(\begin{array}{cccccc|c} 1 & -3 & 9 & -27 & 81 & -243 & 100 \\ 1 & -1 & 1 & -1 & 1 & -1 & 14 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 2 & 4 & 8 & 16 & 32 & 65 \\ 1 & 3 & 9 & 27 & 81 & 243 & 154 \end{array}\right) = \left(\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & -18 & 0 \\ 0 & 0 & 1 & 0 & 0 & -9 & 14 \\ 0 & 0 & 0 & 1 & 0 & 11 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right)$$

$$\longrightarrow \left\{\begin{array}{l} a_0 = 1 \\ a_1 - 18a_5 = 0 \\ a_2 - 9a_5 = 14 \\ a_3 + 11a_5 = 1 \\ a_4 + a_5 = 0 \end{array}\right.$$

Giving $a_5 = \lambda$ the parametric solution is

$$\begin{cases} a_0 = 1 \\ a_1 = 18\lambda \\ a_2 = 14 + 9\lambda \\ a_3 = 1 - 11\lambda \\ a_4 = -\lambda \\ a_5 = \lambda \end{cases}$$

$\lambda \in \mathbb{R}$ So polynomials of degree at most 5 are

$$\boxed{q(x) = 1 + 18\lambda x + (14+9\lambda)x^2 + (1-11\lambda)x^3 - \lambda x^4 + \lambda x^5}$$

in particular for $\lambda = 1$

$$\boxed{q_1(x) = 1 + 18x + 23x^2 - 10x^3 - x^4 + x^5}$$

4 $\boxed{B_v = \{(x+3)^4, (x+1)^4, (x-0)^4, (x-2)^4, (x-3)^4\}}$

where:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

$(x+3)^4 = x^4 + 4x^3 3 + 6x^2 3^2 + 4x 3^3 + 3^4$

$= x^4 + 12x^3 + 54x^2 + 108x + 81$

$(x+1)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1$

$(x-0)^4 = x^4$

$(x-2)^4 = x^4 + 4x^3(-2) + 6x^2(-2)^2 + 4x(-2)^3 + (-2)^4 = x^4 - 8x^3 + 24x^2 - 32x + 16$

$(x-3)^4 = x^4 - 12x^3 + 54x^2 - 108x + 81$

$\boxed{B_v \text{ is a basis for } P_4}$

5 $B_S = \{1, x+3, (x+3)^2, (x+3)^3, (x+3)^4\}$

$x_0 = -3$ $\quad (x+3)^2 = x^2 + 6x + 9$

$(x+3)^3 = x^3 + 3x^2 3 + 3x 3^2 + 3^3 = x^3 + 9x^2 + 27x + 27$

$$\boxed{B_S = \{1, x+3, x^2+6x+9, x^3+9x^2+27x+27, x^4+12x^3+54x^2+108x+81\}}$$

6 $B_B = \{B_0^4(x), B_1^n(x), \dots B_n^n(x)\}$ where

$B_i^n(x) = \binom{n}{i}(1-x)^{n-i} x^i$

$\dim(P_n) = n+1$ and $|B_B| = n+1$, so if the polynomials are linearly independent they form a basis.

Applying an isomorphism $T: P_n \longrightarrow \mathbb{R}^{n+1}$ given as

$$T\left(\sum_{i=0}^{n} a_i x^i\right) = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix}$$

to the Bernstein polynomials we have

$$T(B_0^n(x)) = T\left(\binom{n}{0}(1-x)^n x^0\right) = \binom{n}{0} T((1-x)^n) = \binom{n}{0} \begin{pmatrix} \binom{n}{0} \\ -\binom{n}{1} \\ +\binom{n}{2} \\ \vdots \\ (-1)^n \binom{n}{n} \end{pmatrix}$$

$$T(B_1^n(x)) = T\left(\binom{n}{1}(1-x)^{n-1} x\right) = \binom{n}{1} T((1-x)^{n-1} x) = \binom{n}{1} \begin{pmatrix} 0 \\ \binom{n-1}{0} \\ -\binom{n-1}{1} \\ \vdots \\ (-1)^{n-1} \binom{n-1}{n-1} \end{pmatrix}$$

$$T(B_2^n(x)) = T\left(\binom{n}{2}(1-x)^{n-2} x^2\right) = \binom{n}{2} T((1-x)^{n-2} x^2) = \binom{n}{2} \begin{pmatrix} 0 \\ 0 \\ \binom{n-2}{0} \\ \vdots \\ (-1)^{n-2} \binom{n-2}{n-2} \end{pmatrix}$$

we repeat the process for all of them

$$T(B_n^n(x)) = T\left(\binom{n}{n}(1-x)^{n-n} x^n\right) = \binom{n}{n} T(x^n) = \binom{n}{n} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

Introducing the vectors in $\mathbb{R}^{n+1}$ in a matrix M we get

$$\det(M) = \prod_{i=0}^{n} \binom{n}{i} \underbrace{\det \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -\binom{n}{1} & 1 & \cdots & 0 \\ \binom{n}{2} & -\binom{n-1}{1} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ (-1)^n \binom{n}{n} & (-1)^{n-1} \binom{n-1}{n-1} & \cdots & 1 \end{pmatrix}}_{\overset{\shortparallel}{1}} = \prod_{i=0}^{n} \binom{n}{i} \neq 0$$

so the vectors in $\mathbb{R}^{n+1}$ are linearly independent. As T is invertible and preserves linearities, the polynomials are also linearly independent and so they form a basis for P_n.

⑦ $B_B = \{ B_0^4(x), B_1^4(x), B_2^4(x), B_3^4(x), B_4^4(x) \}$

where $B_0^4(x) = \binom{4}{0}(1-x)^4 x^0 = (1-x)^4 = 1 - 4x + 6x^2 - 4x^3 + x^4$

$B_1^4(x) = \binom{4}{1}(1-x)^3 x = 4(1-3x+3x^2-x^3)x = 4x - 12x^2 + 12x^3 - 4x^4$

$B_2^4(x) = \binom{4}{2}(1-x)^2 x^2 = 6(1-2x+x^2)x^2 = 6x^2 - 12x^3 + 6x^4$

$B_3^4(x) = \binom{4}{3}(1-x)x^3 = 4(1-x)x^3 = 4x^3 - 4x^4$

$B_4^4(x) = \binom{4}{4}(1-x)^0 x^4 = x^4$

$$\boxed{B_B = \{1-4x+6x^2-4x^3+x^4,\ 4x-12x^2+12x^3-4x^4,\ 6x^2-12x^3+6x^4,\ 4x^3-4x^4,\ x^4\}}$$

⑧ Bernstein to Vandermonde

$T_{B\to V} : \mathbb{R}^5 \longrightarrow \mathbb{R}^5$ $\quad T_{B\to V}(x) = M\vec{x}$ where M is given as follows:

- Bernstein to standard: $T_{B\to St} : \mathbb{R}^5 \longrightarrow \mathbb{R}^5$

$T_{B\to St}(x) = M_1 \cdot x$ where

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -4 & 4 & 0 & 0 & 0 \\ 6 & -12 & 6 & 0 & 0 \\ -4 & 12 & -12 & 4 & 0 \\ 1 & -4 & 6 & -4 & 1 \end{pmatrix}$$

- Standard to Vandermonde $T_{St\to V} : \mathbb{R}^5 \longrightarrow \mathbb{R}^5$

$T_{St\to V}(x) = M_2\, x$ where

$$M_2 = \begin{pmatrix} 81 & 1 & 0 & 16 & 81 \\ 108 & 4 & 0 & -32 & -108 \\ 54 & 6 & 0 & 24 & 54 \\ 12 & 4 & 0 & -8 & -12 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}^{-1} = \frac{1}{2160} \begin{pmatrix} 12 & 12 & 2 & -18 & 0 \\ -90 & -45 & 135 & 405 & 0 \\ 120 & 30 & -220 & -270 & 2160 \\ -72 & 18 & 108 & -162 & 0 \\ 30 & -15 & -25 & 45 & 0 \end{pmatrix}$$

- Bernstein to Vandermonde as defined above with $M = M_2^{-1} M_1$

$$T_{B\to V}: \mathbb{R}^5 \longrightarrow \mathbb{R}^5$$

$$T(\vec{x}) = \frac{1}{360}\begin{pmatrix} 8 & -32 & 38 & -12 & 0 \\ -120 & 510 & -675 & 270 & 0 \\ 320 & -1520 & 2480 & -1620 & 360 \\ 192 & -528 & 432 & -108 & 0 \\ -40 & 130 & -115 & 30 & 0 \end{pmatrix} \vec{x}$$

9 Shifted to Vandermonde

$T_{S\to V}: \mathbb{R}^5 \longrightarrow \mathbb{R}^5$ $\quad T_{S\to V}(\vec{x}) = M\vec{x}$ where M is given as follows:

- Shifted to standard: $T_{S\to St}: \mathbb{R}^5 \longrightarrow \mathbb{R}^5$

  $T_{S\to St}(\vec{x}) = M_3 \vec{x}$ where

$$M_3 = \begin{pmatrix} 1 & 3 & 9 & 27 & 81 \\ 0 & 1 & 6 & 27 & 108 \\ 0 & 0 & 1 & 9 & 54 \\ 0 & 0 & 0 & 1 & 12 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- Standard to Vandermonde see exercise 8.

Shifted to Vandermonde as defined above with $M = M_2^{-1} M_3$

$$T_{S\to V}: \mathbb{R}^5 \longrightarrow \mathbb{R}^5$$

$$T(\vec{x}) = \frac{1}{2160}\begin{pmatrix} 12 & 48 & 182 & 648 & 2160 \\ -90 & -315 & -945 & -2025 & 0 \\ 120 & 390 & 1040 & 1800 & 0 \\ -72 & -198 & -432 & -648 & 0 \\ 30 & 75 & 155 & 225 & 0 \end{pmatrix} \vec{x}$$

(10) Bernstein to Shifted

$$\boxed{T_{B\to S}: \mathbb{R}^5 \to \mathbb{R}^5 \quad T_{B\to S}(\vec{x}) = M\vec{x} \quad \text{where } M = M_3^{-1}\cdot M_1}$$

$$\boxed{T(\vec{x}) = \begin{pmatrix} 256 & -768 & 864 & -432 & 81 \\ -256 & 832 & -1008 & 540 & -108 \\ 96 & -336 & 438 & -252 & 54 \\ -16 & 60 & -84 & 52 & -12 \\ 1 & -4 & 6 & -4 & 1 \end{pmatrix} \vec{x}}$$

(11)

$$P_{St} = \begin{pmatrix} 1 & 0 & 14 & 1 & 0 \end{pmatrix}_{St} \qquad \text{standard}$$

$$P_V = M_2^{-1}\cdot P_{St}^T = \frac{1}{2160}\begin{pmatrix} 22 \\ 2205 \\ -3230 \\ 1278 \\ -275 \end{pmatrix}_V \qquad \text{Vandermonde}$$

$$P_S = M_3^{-1} P_{St}^T = \begin{pmatrix} 100 \\ -57 \\ 5 \\ 1 \\ 0 \end{pmatrix}_S \qquad \text{Shifted}$$

$$P_B = M_1^{-1} P_{St}^T = \begin{pmatrix} 1 \\ 1 \\ 10/3 \\ 33/4 \\ 16 \end{pmatrix}_B \qquad \text{Bernstein}$$