MAT300 CURVES AND SURFACES

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Bezier curves

Motivation: review

We saw how to obtain two Bezier curves by subdivision.

$$\gamma(t) = \sum_{i=0}^{n} P_i B_i^n(t), \quad t \in [0,1] \tag{1}$$

- select a point $\bar{t} \in (0,1)$ for the subdivision
- construct two curves

$$\gamma_1(t) = \sum_{i=0}^{n} P_i B_i^n(t), \;\; t \in [0, \bar{t}\,] \qquad \gamma_2(t) = \sum_{i=0}^{n} P_i B_i^n(t), \;\; t \in [\bar{t}, 1]$$

- reparametrize the curves for obtaining their Bezier representation

$$\gamma_1(s) = \sum_{i=0}^{n} \bar{P}_i B_i^n(s), \;\; s \in [0,1] \qquad \gamma_2(s) = \sum_{i=0}^{n} \hat{P}_i B_i^n(s), \;\; s \in [0,1]$$

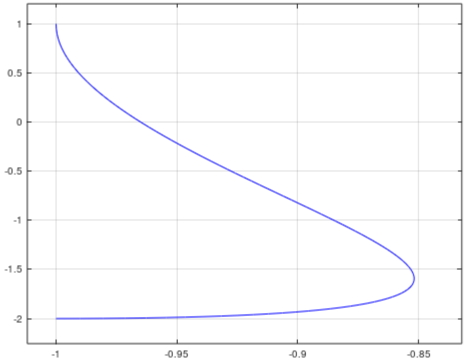
the reparametrization is based on selecting new control points.

Motivation: new problems

Given a polynomial curve $\gamma : [0,1] \to \mathbb{R}^2$ defined as

$$\gamma(t) = (p(t), q(t)) \tag{2}$$

where $p, q \in P_n$. What are the control points of its Bezier representation?



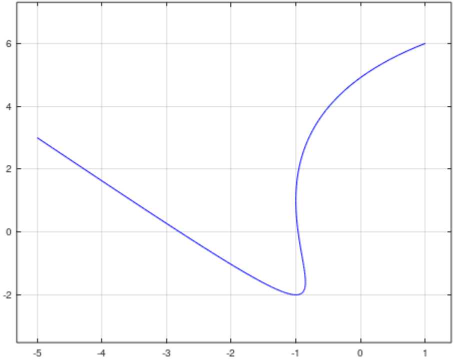
$\gamma(t) = (-1 + t - 2t^2 + t^3, -2 + 4t^2 - t^3)$

Motivation: new problems

Given a polynomial curve $\gamma : [a, b] \to \mathbb{R}^2$ defined as

$$\gamma(t) = (p(t), q(t)) \tag{3}$$

where $p, q \in P_n$. What are the control points of its Bezier representation?



$\gamma(t) = (-1 + t - 2t^2 + t^3, -2 + 4t^2 - t^3) \quad t \in [-1, 2]$

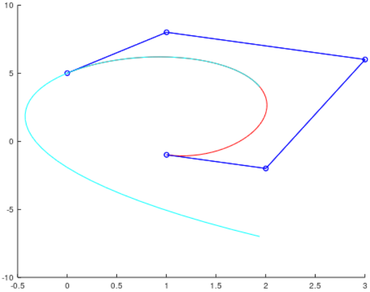
Motivation: new problems

Given a Bezier curve $\gamma(t) = \sum_{i=0}^{n} P_i B_i^n(t),\ t \in [0,1]$,

what is the Bezier representation of $\hat{\gamma}(t) = \sum_{i=0}^{n} P_i B_i^n(t),\ t \in [a,b]$?

what are the control points of such a Bezier curve?

$$\hat{\gamma}(s) = \sum_{i=0}^{n} \hat{P}_i B_i^n(s),\ s \in [0,1] \tag{4}$$



Affine functions

Definition

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be affine if $f(x) = \alpha x + \beta$.

Affine functions satisfy

$$f((1-t)a + tb) = (1-t)f(a) + tf(b) \tag{5}$$

Remark: the above notation denotes a reparametrization $x = (1-t)a + tb$

Proof:

$f((1-t)a+tb) = \alpha((1-t)a+tb)+\beta = (1-t)\alpha a+t\alpha b+\beta = (1-t)\alpha a+ t\alpha b + \beta(1-t+t) = (1-t)(\alpha a+\beta) + t(\alpha b+\beta) = (1-t)f(a) + tf(b)$

Affine functions preserve barycentric coordinates.

Polar forms

Definition

A polar form $F[u_1, u_2, \ldots, u_n]$ for a polynomial $p \in P_n$ is a multivariable multivalued function satisfying:

- Symmetry

$$F[u_{\sigma_1}, u_{\sigma_2}, \ldots, u_{\sigma_n}] = F[u_1, u_2, \ldots, u_n] \tag{6}$$

  for $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ any permutation of the set $\{1, 2, \ldots, n\}$

- Construction by substitution

$$F[t, t, \ldots, t] = p(t) \tag{7}$$

- Affinity: the function

$$F[u_1, u_2, \ldots, x, \ldots, u_n] \tag{8}$$

  is affine for x and for every position, being the other u_i constants.

Examples of polar forms

We select the space: P_3

- $p(t) = 1$ has polar form $F[u_1, u_2, u_3] = 1$
- $p(t) = t$ has polar form $F[u_1, u_2, u_3] = \frac{u_1+u_2+u_3}{3}$
- $p(t) = t^2$ has polar form $F[u_1, u_2, u_3] = \frac{u_1 u_2+u_1 u_3+u_2 u_3}{3}$
- $p(t) = t^3$ has polar form $F[u_1, u_2, u_3] = u_1 u_2 u_3$

Show that the above polar forms satisfy the properties in the definition

whiteboard

More examples of polar forms

We select the space: P_4

- $p(t) = 1$ has polar form $F[u_1, u_2, u_3, u_4] = 1$
- $p(t) = t$ has polar form $F[u_1, u_2, u_3, u_4] = \frac{u_1+u_2+u_3+u_4}{4}$
- $p(t) = t^2$ has $F[u_1, u_2, u_3, u_4] = \frac{u_1 u_2+u_1 u_3+u_1 u_4+u_2 u_3+u_2 u_4+u_3 u_4}{6}$
- $p(t) = t^3$ has $F[u_1, u_2, u_3, u_4] = \frac{u_1 u_2 u_3+u_1 u_2 u_4+u_1 u_3 u_4+u_2 u_3 u_4}{4}$
- $p(t) = t^4$ has polar form $F[u_1, u_2, u_3, u_4] = u_1 u_2 u_3 u_4$

The above polar forms satisfy the properties in the definition

Polar forms of the standard basis for P_n

$p(t) = t^j$ for $j = 0, 1, \ldots, n$ therefore $F[u_1, u_2, \ldots, u_n]$

- For $p(t) = 1$ then $F[u_1, u_2, \ldots, u_n] = 1$, otherwise:
- Take the set $\{u_1, u_2, \ldots, u_n\}$ and from this set do the $\left(\begin{array}{c} n \\ j \end{array} \right)$ combinations of j elements.
- Multiply the elements of each combination and sum the result.
- Divide the result by the number of j-combinations $\left(\begin{array}{c} n \\ j \end{array} \right)$

Example: obtain the polar forms of the standard basis for P_5.

Polar form of an arbitrary polynomial

Consider the space P_n and its standard basis $\{1, t, t^2, \ldots, t^n\}$

We denote with $F_j := F_j[u_1, u_2, \ldots, u_n]$ the polar form for $p(t) = t^j$

The polar form of a polynomial $p(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n$ is given by

$$F[u_1, u_2, \ldots, u_n] = a_0 F_0 + a_1 F_1 + a_2 F_2 + \ldots + a_n F_n \tag{9}$$

Example: obtain the polar form of $p(t) = 2 - 3t + 4t^2$ in P_2

$F_0 = 1,\ F_1 = \frac{u_1+u_2}{2},\ F_2 = u_1 u_2$ then

$F = 2F_0 - 3F_1 + 4F_2 = 2 - 3\left(\frac{u_1+u_2}{2}\right) + 4u_1 u_2$

Example: obtain the polar form of $p(t) = 2 - 3t + 4t^2$ in P_3 whiteboard

Polar forms for Bernstein polynomials

We select the space: P_2

- $B_0^2(t) = \begin{pmatrix} 2 \\ 0 \end{pmatrix} (1-t)^2 t^0$ has polar form

  $F[u_1, u_2] = (1-u_1)(1-u_2)$

- $B_1^2(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} (1-t)^1 t^1$ has polar form

  $F[u_1, u_2] = (1-u_1)u_2 + (1-u_2)u_1$

- $B_2^2(t) = \begin{pmatrix} 2 \\ 2 \end{pmatrix} (1-t)^0 t^2$ has polar form $F[u_1, u_2] = u_1 u_2$

Show that the above polar forms satisfy the properties in the definition

whiteboard

Polar forms of the Bernstein basis for P_n

$p(t) = \binom{n}{j}(1-t)^{n-j}t^j$ for $j = 0, 1, \ldots, n$ therefore $F[u_1, u_2, \ldots, u_n]$

- Take the set $U = \{u_1, u_2, \ldots, u_n\}$ and from this set do the

  $\binom{n}{n-j}$ combinations of $n-j$ elements.

- For each combination $\{u_{\sigma_1}, \ldots, u_{\sigma_{n-j}}\}$ construct a product

  $\prod_{k=1}^{n-j}(1-u_{\sigma_k})$ and multiply it by the elements that are in

  $U \setminus \{u_{\sigma_1}, \ldots, u_{\sigma_{n-j}}\}$.

- Sum the result of each combination.

Example: obtain the polar forms of the Bernstein basis for P_3.

Polar form of a polynomial in Bernstein basis

Consider the space P_n and its Bernstein basis $\{B_0^n, B_1^n, B_2^n, \ldots, B_n^n\}$

We denote with $F_j := F_j[u_1, u_2, \ldots, u_n]$ the polar form for $B_j^n(t)$

The polar form of a polynomial $p(t) = a_0 B_0^n + a_1 B_1^n + a_2 B_2^n + \ldots + a_n B_n^n$ is given by

$$F[u_1, u_2, \ldots, u_n] = a_0 F_0 + a_1 F_1 + a_2 F_2 + \ldots + a_n F_n \tag{10}$$

Theorem

The polar form of a polynomial $p \in P_n$ exists and is unique.