MAT300 CURVES AND SURFACES

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B-splines

Review: Piecewise polynomials

Definition

Let $x_0 < x_1 < \ldots < x_n \in \mathbb{R}$. The set of piecewise polynomials $p : [x_0, x_n] \to \mathbb{R}$ given as

$$p(x) = \begin{cases} p_1(x), \ x \in [x_0, x_1), \\ p_2(x), \ x \in [x_1, x_2), \\ \vdots \\ p_n(x), \ x \in [x_{n-1}, x_n], \end{cases}$$

with $p_i \in P_k$ for $i = 1, \ldots, n$ is denoted with $P_k^n[x_0, \ldots, x_n]$.

Theorem

$P_k^n[x_0, \ldots, x_n]$ is a vector space.

Review: Piecewise polynomials

Definition

The standard basis for $P_k^n[x_0, x_1, \ldots, x_n]$ is

$$B = \{1,\ x,\ \ldots,\ x^k,\ (x - x_1)_+^0,\ (x - x_1)_+^1,\ \ldots,\ (x - x_1)_+^k,\ \ldots,$$

$$(x - x_{n-1})_+^0,\ (x - x_{n-1})_+^1,\ \ldots,\ (x - x_{n-1})_+^k\}$$

We constructed subspaces of $P_k^n[x_0, x_1, \ldots, x_n]$ by adding orders of continuity:

- continuity $\leftrightarrow$ $p_j(x_j) = p_{j+1}(x_j)$ for $j = 1, \ldots, n-1$.
  $p \in P_{k,0}^n[x_0, \ldots, x_n]$.
- r-differentiability $\leftrightarrow$ $p_j^{(m)}(x_j) = p_{j+1}^{(m)}(x_j)$ for $j = 1, \ldots, n-1$ and $m = 0, 1, \ldots, r$.
  $p \in P_{k,r}^n[x_0, \ldots, x_n]$.

Review: Piecewise polynomials

Theorem

$P_k = P^n_{k,k}[x_0, \dots, x_n]$ is a subspace of $P^n_{k,k-1}[x_0, \dots, x_n]$,

which is a subspace of $P^n_{k,k-2}[x_0, \dots, x_n]$,

$\vdots$

which is a subspace of $P^n_{k,0}[x_0, \dots, x_n]$,

which is a subspace of $P^n_k[x_0, \dots, x_n]$.

To find a basis for $P^n_{k,r}[x_0, \dots, x_n]$ we start with the standard basis for $P^n_k[x_0, \dots, x_n]$, and delete the elements that break the differentiability till order r, so we delete $(r+1)(n-1)$ elements.

Motivation

When considering piecewise polynomials in previous sections we assumed the same order of continuity for the polynomial in $[x_0, x_n]$.

Which means that the polynomial has the same order of continuity at each intermediate point x_1, x_2, ..., x_{n-1}

What if we assume different orders of continuity for each intermediate point x_1, x_2, ..., x_{n-1}?

Splines with different order of continuity

Consider the space $P_k^n[x_0,\dots,x_n]$ of piecewise polynomials of the form

$$p(x) = \begin{cases} p_1(x),\ x \in [x_0, x_1), \\ p_2(x),\ x \in [x_1, x_2), \\ \vdots \\ p_n(x),\ x \in [x_{n-1}, x_n], \end{cases}$$

with $p_i \in P_k$ for $i = 1, \dots, n$.

We want to find a subspace of $P_k^n[x_0,\dots,x_n]$ by setting different orders of continuity at x_1, x_2, ..., x_{n-1}. For such a purpose we define a vector

$$\vec{r} = (r_1, r_2, \dots, r_{n-1}) \tag{1}$$

of orders of continuity for each node.

r_i denotes the order of continuity at x_i being -1 discontinuous, 0 continuous, 1 differentiable, and so on.

The subspace $P^n_{k,\vec{r}}[x_0, \ldots, x_n]$ is the one containing the polynomials

$$p(x) = \begin{cases} p_1(x), \; x \in [x_0, x_1), \\ p_2(x), \; x \in [x_1, x_2), \\ \vdots \\ p_n(x), \; x \in [x_{n-1}, x_n], \end{cases}$$

with $p_i \in P_k$ for $i = 1, \ldots, n$ satisfying

$$p_j^{(m)}(x_j) = p_{j+1}^{(m)}(x_j), \quad j = 1, \ldots, n-1, \quad m = 0, \ldots, r_j \tag{2}$$

Notice that when $r_j = -1$ we do not have continuity condition

To find a basis for $P^n_{k,\vec{r}}[x_0, \ldots, x_n]$ we start with the standard basis for $P^n_k[x_0, \ldots, x_n]$, and delete the elements that break the differentiability till order r_j at the node x_j.

Example

$P^4_{3,\vec{r}}[0,1,2,3,4]$ with $\vec{r} = (1,2,0)$ is the space of polynomials of the form

$$p(x) = \begin{cases} a_0 + a_1 x + a_2 x^2 + a_3 x^3, & x \in [0,1) \\ b_0 + b_1 x + b_2 x^2 + b_3 x^3, & x \in [1,2) \\ c_0 + c_1 x + c_2 x^2 + c_3 x^3, & x \in [2,3) \\ d_0 + d_1 x + d_2 x^2 + d_3 x^3, & x \in [3,4] \end{cases} \tag{3}$$

that satisfy:

- at $x = 1$ p is continuous and differentiable

$$\lim_{x \to 1^-} p(x) = \lim_{x \to 1^+} p(x) \qquad a_0 + a_1 + a_2 + a_3 = b_0 + b_1 + b_2 + b_3$$

$$\lim_{x \to 1^-} p'(x) = \lim_{x \to 1^+} p(x)' \qquad a_1 + 2a_2 + 3a_3 = b_1 + 2b_2 + 3b_3$$

- at $x = 2$ p is continuous, differentiable and twice differentiable

$$\lim_{x\to 2^-} p(x) = \lim_{x\to 2^+} p(x) \qquad b_0 + 2b_1 + 4b_2 + 8b_3 = c_0 + 2c_1 + 4c_2 + 8c_3$$

$$\lim_{x\to 2^-} p'(x) = \lim_{x\to 2^+} p(x)' \qquad b_1 + 4b_2 + 12b_3 = c_1 + 4c_2 + 12c_3$$

$$\lim_{x\to 2^-} p''(x) = \lim_{x\to 2^+} p(x)'' \qquad 2b_2 + 12b_3 = 2c_2 + 12c_3$$

- at $x = 3$ p is continuous

$$\lim_{x\to 3^-} p(x) = \lim_{x\to 3^+} p(x) \quad c_0 + 3c_1 + 9c_2 + 27c_3 = d_0 + 3d_1 + 9d_2 + 27d_3$$

What is the standard basis for $P^4_{3,\vec{r}}[0, 1, 2, 3, 4]$ with $\vec{r} = (1, 2, 0)$?

We start with the standard basis for $P_3^4[0,1,2,3,4]$

$$B = \{1, x, x^2, x^3, (x-1)_+^0, (x-1)_+^1, (x-1)_+^2, (x-1)_+^3, (x-2)_+^0,$$

$$(x-2)_+^1, (x-2)_+^2, (x-2)_+^3, (x-3)_+^0, (x-3)_+^1, (x-3)_+^2, (x-3)_+^3\}$$

- At $x = 1$ p is continuous and differentiable, so we delete the elements that break continuity and differentiability at that point:

  $$B = \{1, x, x^2, x^3, \xcancel{(x-1)_+^0}, \xcancel{(x-1)_+^1}, (x-1)_+^2, (x-1)_+^3, (x-2)_+^0,$$

  $$(x-2)_+^1, (x-2)_+^2, (x-2)_+^3, (x-3)_+^0, (x-3)_+^1, (x-3)_+^2, (x-3)_+^3\}$$

- At $x = 2$ p is continuous and twice differentiable, so we delete the elements that break continuity, first and second differentiability:

  $$B = \{1, x, x^2, x^3, (x-1)_+^2, (x-1)_+^3, \xcancel{(x-2)_+^0}, \xcancel{(x-2)_+^1}, \xcancel{(x-2)_+^2},$$

  $$(x-2)_+^3, (x-3)_+^0, (x-3)_+^1, (x-3)_+^2, (x-3)_+^3\}$$

- At $x = 3$ p is continuous, so we delete the element that break continuity at that point

  $B = \{1, x, x^2, x^3, (x-1)_+^2, (x-1)_+^3, (x-2)_+^3, \cancel{(x-3)_+^0}, (x-3)_+^1,$

  $(x-3)_+^2, (x-3)_+^3\}$

The basis for $P_{3,\vec{r}}^4[0,1,2,3,4]$ with $\vec{r} = (1,2,0)$ is

$B = \{1, x, x^2, x^3, (x-1)_+^2, (x-1)_+^3, (x-2)_+^3, (x-3)_+^1, (x-3)_+^2, (x-3)_+^3\}$

Every piecewise polynomial in $P_{3,\vec{r}}^4[0,1,2,3,4]$ can be expressed as linear combination of the elements in B.

Example in the whiteboard

Determine to which space

$$p(x) = \begin{cases} 1 + 2x^2, & x \in [0, 1) \\ 2x + x^2, & x \in [1, 2) \\ 10 - 9x + 4x^2, & x \in [2, 3) \\ 27 - 20x + 6x^2, & x \in [3, 4) \\ 27 - 20x + 6x^2, & x \in [4, 5] \end{cases} \tag{4}$$

belongs.

Construct the standard basis for that space.

Give the vector of coordinates of p in the standard basis.

Solution example in whiteboard

$p \in P^5_{2,\vec{r}}[0, 1, 2, 3, 4, 5]$ with $\vec{r} = (1, 0, -1, 2)$

$B = \{1, x, x^2, (x-1)^2_+, (x-2)^1_+, (x-2)^2_+, (x-3)^0_+, (x-3)^1_+, (x-3)^2_+\}$

$p := (1, 0, 2, -1, 1, 3, 2, 1, 2)$