DigiPen Institute of Technology, Bilbao

MAT300 Curves & Surfaces

Spring 2020. Homework 4: Deadline: 23-3-2020

In this assignment the algorithms are applied by hand calculations step by step. If you present any sort of screenshoot of a matrix created by a computer, or any code you will get a zero. Notice that the purpose of this assignment is to have something to test with your code.

1. (80%) Let $P_0 = (1, -1)$, $P_1 = (2, 3)$ and $P_2 = (0, 4)$ be the control points of a Bezier curve γ.
   - (20%) For $t_0 = 0$, $t_1 = \frac{1}{4}$, $t_2 = \frac{1}{2}$, $t_3 = \frac{3}{4}$ and $t_4 = 1$ apply the De Casteljau algorithm (numerically) to obtain the evaluation of the curve γ at the nodes.
   - (20%) Use the previous numerical results to draw the evaluation of the curve at every node doing the recursively linear interpolation (graphically). You have to plot the shells for each node in a separate graph where you identify numerically each intermediate control point, giving the corresponding label, and the evaluation of the curve in the plots. Then, you have to combine all the plots in the same graph using a different color for each node. You can do the plots by hand, use octave or any other program for that purpose. Revise the grading policy for graphs in the syllabus.
   - (20%) Apply three iterations of the midpoint subdivision algorithm (numerically), showing how many curves and points of γ do you get after each iteration, and what are the control points for the curves in the following iteration.
   - (20%) Use the previous numerical results to draw the evaluation of the curve after each iteration. Construct a plot for each iteration showing the control points of each curve (using different colors) and the obtained curve γ. You can do the plots by hand, use octave or any other program for that purpose. Revise the grading policy for graphs in the syllabus.

2. (20%) Find the quadratic Bezier curve that starts at $P_0 = (0, 0)$, goes through $Q = (1, 1)$ at $t = \frac{3}{4}$ and ends at $P_2 = (2, 4)$. Give the expression of the Bezier curve as linear combination of Bernstein polynomials, providing the corresponding interval.

MAT300 HW 4 SOLUTIONS

(1) $P_0=(1,-1)$ $P_1=(2,3)$ $P_2=(0,4)$ control points

a) We apply the recursion $P_i^k(t_j)=t_j P_{i+1}^{k-1}(t_j)+(1-t_j)P_i^{k-1}(t_j)$ for $k=1,2$, $i=0,\dots,2-k$ and $j=0,1,2,3,4$.

For $j=0$ $t_0=0$:

$P_0=(1,-1)$ $P_0^1=0(2,3)+1(1,-1)=(1,-1)$ $P_0^2=0(2,3)+1(1,-1)=(1,-1)$

$P_1=(2,3)$ $P_1^1=0(0,4)+1(2,3)=(2,3)$

$P_2=(0,4)$

$\boxed{\gamma(0)=(1,-1)}$

For $j=1$ $t_1=1/4$:

$P_0=(1,-1)$ $P_0^1=\frac{1}{4}(2,3)+\frac{3}{4}(1,-1)=\left(\frac{5}{4},0\right)$ $P_0^2=\frac{1}{4}\left(\frac{3}{2},\frac{13}{4}\right)+\frac{3}{4}\left(\frac{5}{4},0\right)=\left(\frac{21}{16},\frac{13}{16}\right)$

$P_1=(2,3)$ $P_1^1=\frac{1}{4}(0,4)+\frac{3}{4}(2,3)=\left(\frac{3}{2},\frac{13}{4}\right)$

$P_2=(0,4)$

$\boxed{\gamma\left(\frac{1}{4}\right)=\left(\frac{21}{16},\frac{13}{16}\right)}$

For $j=2$ $t_2=1/2$:

$P_0=(1,-1)$ $P_0^1=\frac{1}{2}(2,3)+\frac{1}{2}(1,-1)=\left(\frac{3}{2},1\right)$ $P_0^2=\frac{1}{2}\left(1,\frac{7}{2}\right)+\frac{1}{2}\left(\frac{3}{2},1\right)=\left(\frac{5}{4},\frac{9}{4}\right)$

$P_1=(2,3)$ $P_1^1=\frac{1}{2}(0,4)+\frac{1}{2}(2,3)=\left(1,\frac{7}{2}\right)$

$P_2=(0,4)$

$\boxed{\gamma\left(\frac{1}{2}\right)=\left(\frac{5}{4},\frac{9}{4}\right)}$

For $j=3$ $t_3=3/4$:

$P_0=(1,-1)$ $P_0^1=\frac{3}{4}(2,3)+\frac{1}{4}(1,-1)=\left(\frac{7}{4},2\right)$ $P_0^2=\frac{3}{4}\left(\frac{1}{2},\frac{15}{4}\right)+\frac{1}{4}\left(\frac{7}{4},2\right)=\left(\frac{13}{16},\frac{53}{16}\right)$

$P_1=(2,3)$ $P_1^1=\frac{3}{4}(0,4)+\frac{1}{4}(2,3)=\left(\frac{1}{2},\frac{15}{4}\right)$

$P_2=(0,4)$

$\boxed{\gamma\left(\frac{3}{4}\right)=\left(\frac{13}{16},\frac{53}{16}\right)}$

For $j=4$ $t_4=1$:

$P_0=(1,-1)$ $P_0^1=1(2,3)+0(1,-1)=(2,3)$ $P_0^2=1(0,4)+0(2,3)=(0,4)$

$P_1=(2,3)$ $P_1^1=1(0,4)+0(2,3)=(0,4)$

$P_2=(0,4)$

$\boxed{\gamma(1)=(0,4)}$

c) We apply the recursion $P_i^k = \frac{1}{2} P_{i+1}^{k-1} + \frac{1}{2} P_i^{k-1}$

for $k = 1, 2$ and $i = 0, \dots, 2-k$.

1st iteration:

$P_0 = (1,-1)$ $\quad P_0^1 = \frac{1}{2}(2,3) + \frac{1}{2}(1,-1) = \left(\frac{3}{2}, 1\right)$ $\quad P_0^2 = \frac{1}{2}\left(1,\frac{7}{2}\right) + \frac{1}{2}\left(\frac{3}{2},1\right) = \left(\frac{5}{4}, \frac{9}{4}\right)$

$P_1 = (2,3)$ $\quad P_1^1 = \frac{1}{2}(0,4) + \frac{1}{2}(2,3) = \left(1, \frac{7}{2}\right)$

$P_2 = (0,4)$

$$\boxed{\begin{array}{l} \text{3 Points of } \gamma: \quad (1,-1), \left(\frac{5}{4}, \frac{9}{4}\right), (0,4) \\ \text{Divide } \gamma \text{ in two curves:} \\ \quad \gamma_{11} \text{ with control points } (1,-1), \left(\frac{3}{2}, 1\right), \left(\frac{5}{4}, \frac{9}{4}\right) \\ \quad \gamma_{12} \text{ with control points } \left(\frac{5}{4}, \frac{9}{4}\right), \left(1, \frac{7}{2}\right), (0,4) \end{array}}$$

2nd iteration:

γ_{11}:

$P_0 = (1,-1)$ $\quad P_0^1 = \left(\frac{1+3/2}{2}, \frac{-1+1}{2}\right) = \left(\frac{5}{4}, 0\right)$ $\quad P_0^2 = \left(\frac{\frac{5}{4}+\frac{11}{8}}{2}, \frac{0+13/8}{2}\right) = \left(\frac{21}{16}, \frac{13}{16}\right)$

$P_1 = \left(\frac{3}{2}, 1\right)$ $\quad P_1^1 = \left(\frac{\frac{3}{2}+\frac{5}{4}}{2}, \frac{1+\frac{9}{4}}{2}\right) = \left(\frac{11}{8}, \frac{13}{8}\right)$

$P_2 = \left(\frac{5}{4}, \frac{9}{4}\right)$

γ_{12}

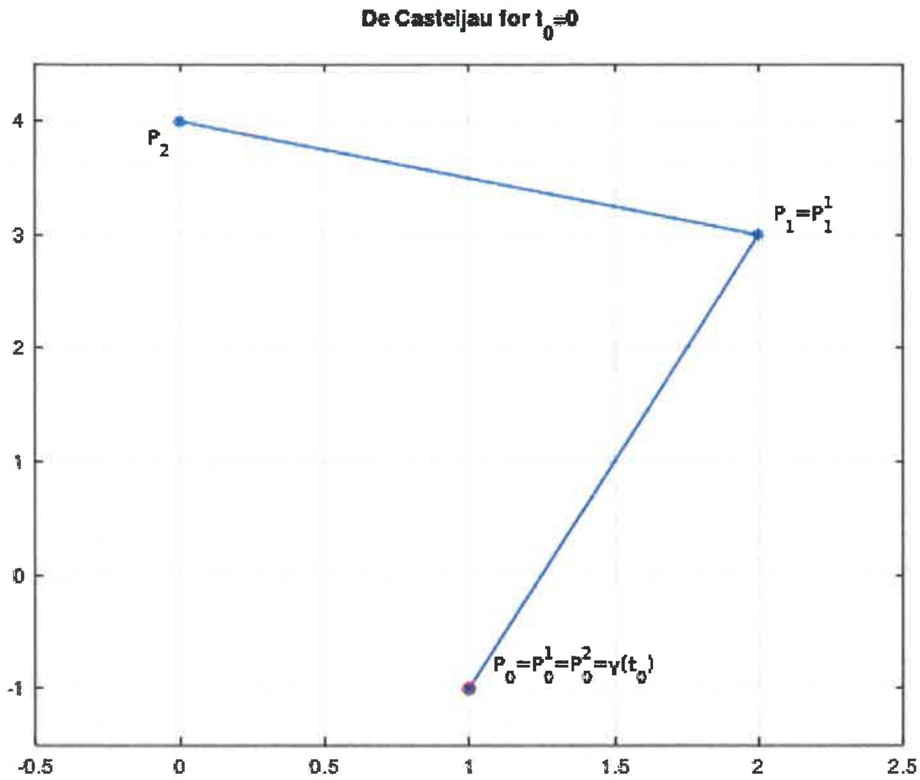
$P_0 = \left(\frac{5}{4}, \frac{9}{4}\right)$ $\quad P_0^1 = \left(\frac{\frac{5}{4}+1}{2}, \frac{\frac{9}{4}+\frac{7}{2}}{2}\right) = \left(\frac{9}{8}, \frac{23}{8}\right)$ $\quad P_0^2 = \left(\frac{\frac{9}{8}+\frac{1}{2}}{2}, \frac{\frac{23}{8}+\frac{15}{4}}{2}\right) = \left(\frac{13}{16}, \frac{53}{16}\right)$

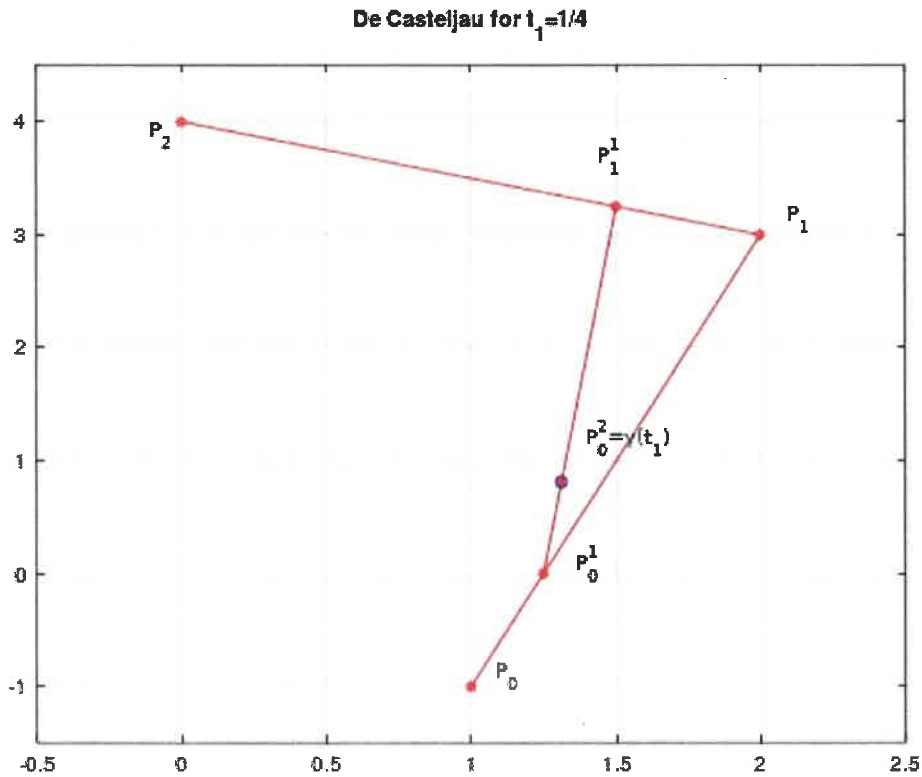
$P_1 = \left(1, \frac{7}{2}\right)$ $\quad P_1^1 = \left(\frac{1+0}{2}, \frac{\frac{7}{2}+4}{2}\right) = \left(\frac{1}{2}, \frac{15}{4}\right)$

$P_2 = (0,4)$

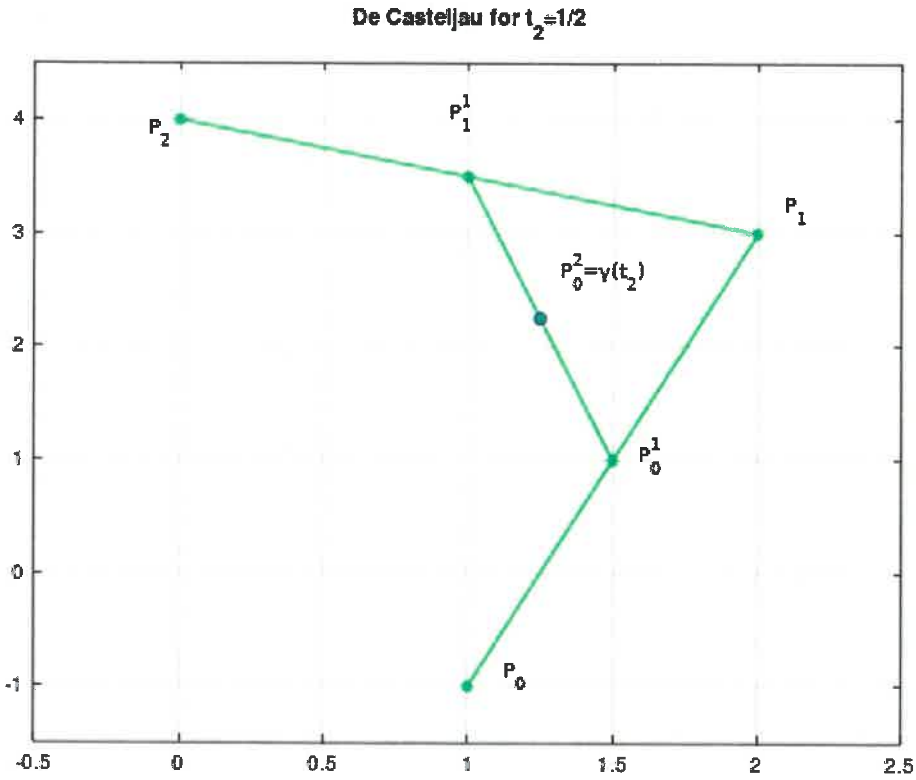
$$\boxed{\begin{array}{l} \text{5 points of } \gamma: (1,-1), \left(\frac{21}{16}, \frac{13}{16}\right), \left(\frac{5}{4}, \frac{9}{4}\right), \left(\frac{13}{16}, \frac{53}{16}\right), (0,4) \\ \text{Divide } \gamma_{11} \text{ and } \gamma_{12} \text{ into 2 curves each, so 4 curves:} \\ \quad \gamma_{21} \text{ with control points } (1,-1), \left(\frac{5}{4}, 0\right), \left(\frac{21}{16}, \frac{13}{16}\right) \\ \quad \gamma_{22} \text{ with control points } \left(\frac{21}{16}, \frac{13}{16}\right), \left(\frac{11}{8}, \frac{13}{8}\right), \left(\frac{5}{4}, \frac{9}{4}\right) \\ \quad \gamma_{23} \text{ with control points } \left(\frac{5}{4}, \frac{9}{4}\right), \left(\frac{9}{8}, \frac{23}{8}\right), \left(\frac{13}{16}, \frac{53}{16}\right) \\ \quad \gamma_{24} \text{ with control points } \left(\frac{13}{16}, \frac{53}{16}\right), \left(\frac{1}{2}, \frac{15}{4}\right), (0,4) \end{array}}$$

b)

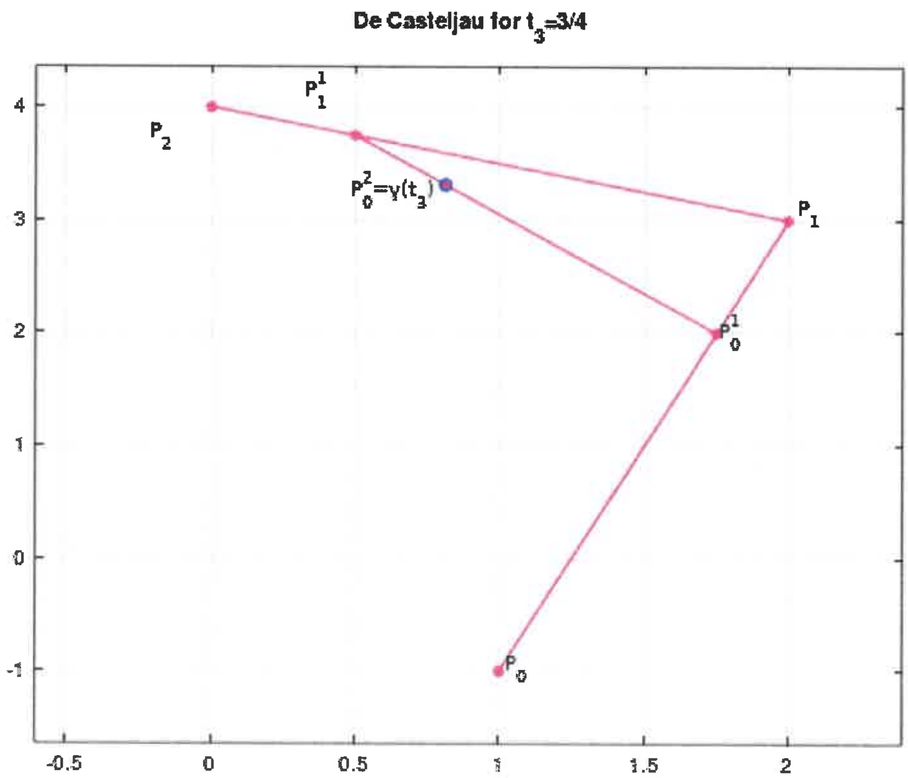




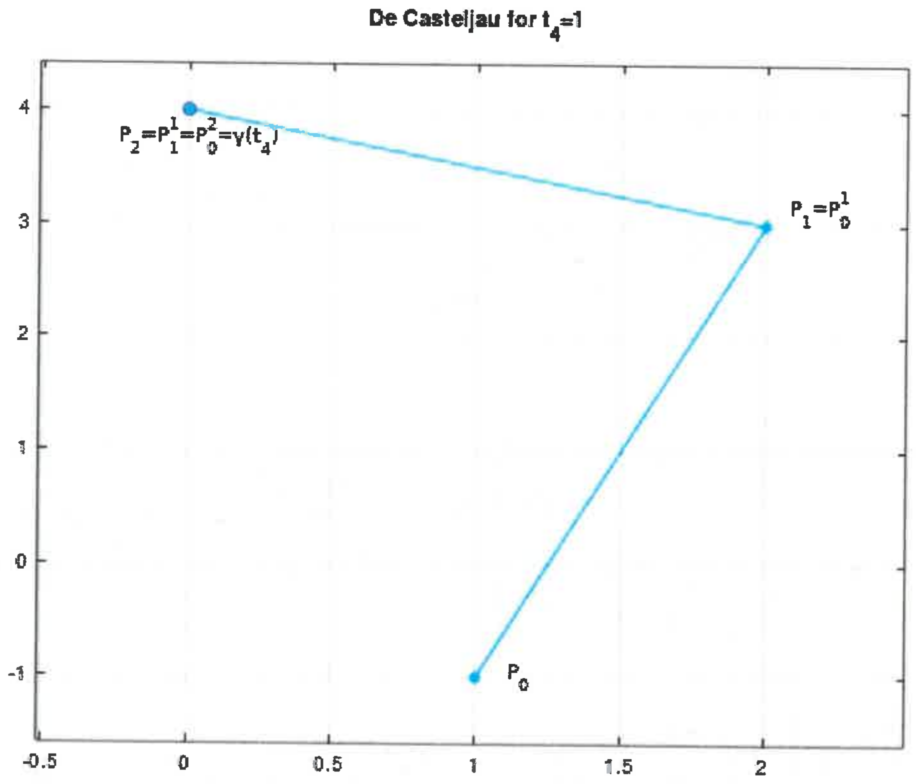
The numerical values of $P_0, P_1, P_2, P_0^1, P_1^1$ and P_0^2 for each t_j are those in a)



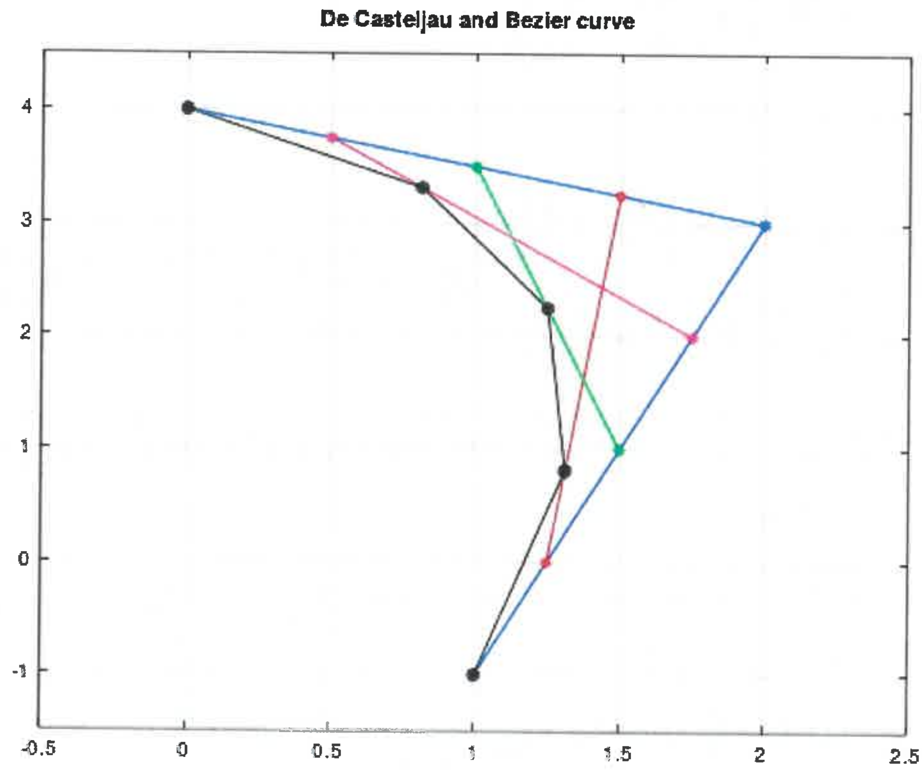
De Casteljau for t_2=1/2
P_2
P_1^1
P_1
$P_0^2=\gamma(t_2)$
P_0^1
P_0
4
3
2
1
0
-1
-0.5
0
0.5
1
1.5
2
2.5



De Casteljau for t_3=3/4
P_1^1
P_2
$P_0^2=\gamma(t_3)$
P_1
P_0^1
P_0
4
3
2
1
0
-1
-0.5
0
0.5
1
1.5
2



De Casteljau for t_4=1
$P_2=P_1^1=P_0^2=\gamma(t_4)$
$P_1=P_0^1$
P_0
4
3
2
1
0
-1
-0.5
0
0.5
1
1.5
2



De Casteljau and Bezier curve
4
3
2
1
0
-1
-0.5
0
0.5
1
1.5
2
2.5

c) (continuation)

3rd iteration:

γ_{21}:

$P_0=(1,-1)$ $\quad P_0^1=\left(\frac{1+5/4}{2}, \frac{-1+0}{2}\right)=\left(\frac{9}{8}, \frac{-1}{2}\right)$

$P_1=(\frac{5}{4},0)$ $\quad P_0^2=\left(\frac{\frac{9}{8}+\frac{41}{32}}{2}, \frac{-\frac{1}{2}+\frac{13}{32}}{2}\right)=\left(\frac{77}{64}, \frac{-3}{64}\right)$

$P_2=(\frac{21}{16},\frac{13}{16})$ $\quad P_1^1=\left(\frac{\frac{5}{4}+\frac{21}{16}}{2}, \frac{0+13/16}{2}\right)=\left(\frac{41}{32}, \frac{13}{32}\right)$

γ_{22}:

$P_0=(\frac{21}{16},\frac{13}{16})$ $\quad P_0^1=\left(\frac{\frac{21}{16}+\frac{11}{8}}{2}, \frac{\frac{13}{16}+\frac{13}{8}}{2}\right)=\left(\frac{43}{32}, \frac{39}{32}\right)$

$P_1=(\frac{11}{8},\frac{13}{8})$ $\quad P_0^2=\left(\frac{\frac{43}{32}+\frac{21}{16}}{2}, \frac{\frac{39}{32}+\frac{31}{16}}{2}\right)=\left(\frac{85}{64}, \frac{101}{64}\right)$

$P_2=(\frac{5}{4},\frac{9}{4})$ $\quad P_1^1=\left(\frac{\frac{11}{8}+\frac{5}{4}}{2}, \frac{\frac{13}{8}+\frac{9}{4}}{2}\right)=\left(\frac{21}{16}, \frac{31}{16}\right)$

γ_{23}:

$P_0=(\frac{5}{4},\frac{9}{4})$ $\quad P_0^1=\left(\frac{\frac{5}{4}+\frac{9}{8}}{2}, \frac{\frac{9}{4}+\frac{23}{8}}{2}\right)=\left(\frac{19}{16}, \frac{41}{16}\right)$

$P_1=(\frac{9}{8},\frac{23}{8})$ $\quad P_0^2=\left(\frac{\frac{19}{16}+\frac{31}{32}}{2}, \frac{\frac{41}{16}+\frac{99}{32}}{2}\right)=\left(\frac{69}{64}, \frac{181}{64}\right)$

$P_2=(\frac{13}{16},\frac{53}{16})$ $\quad P_1^1=\left(\frac{\frac{9}{8}+\frac{13}{16}}{2}, \frac{\frac{23}{8}+\frac{53}{16}}{2}\right)=\left(\frac{31}{32}, \frac{99}{32}\right)$

γ_{24}:

$P_0=(\frac{13}{16},\frac{53}{16})$ $\quad P_0^1=\left(\frac{\frac{13}{16}+\frac{1}{2}}{2}, \frac{\frac{53}{16}+\frac{15}{4}}{2}\right)=\left(\frac{21}{32}, \frac{113}{32}\right)$

$P_1=(\frac{1}{2},\frac{15}{4})$ $\quad P_0^2=\left(\frac{\frac{21}{32}+\frac{1}{4}}{2}, \frac{\frac{113}{32}+\frac{31}{8}}{2}\right)=\left(\frac{29}{64}, \frac{237}{64}\right)$

$P_2=(0,4)$ $\quad P_1^1=\left(\frac{1/2+0}{2}, \frac{\frac{15}{4}+4}{2}\right)=\left(\frac{1}{4}, \frac{31}{8}\right)$

9 points of γ: $(1,-1), (\frac{77}{64},\frac{-3}{64}), (\frac{21}{16},\frac{13}{16}), (\frac{85}{64},\frac{101}{64}), (\frac{5}{4},\frac{9}{4}), (\frac{69}{64},\frac{181}{64})$, $(\frac{13}{16},\frac{53}{16}), (\frac{29}{64},\frac{237}{64}), (0,4)$

Divide $\gamma_{21}, \gamma_{22}, \gamma_{33}$ and γ_{24} into two curves each, so 8 curves!

γ_{31} control points: $(1,-1), (\frac{9}{8},-\frac{1}{2}), (\frac{77}{64},\frac{-3}{64})$

γ_{32} control points: $(\frac{77}{64},\frac{-3}{64}), (\frac{41}{32},\frac{13}{32}), (\frac{21}{16},\frac{13}{16})$

γ_{33} control points: $(\frac{21}{16},\frac{13}{16}), (\frac{43}{32},\frac{39}{32}), (\frac{85}{64},\frac{101}{64})$

γ_{34} control points: $(\frac{85}{64},\frac{101}{64}), (\frac{21}{16},\frac{31}{16}), (\frac{5}{4},\frac{9}{4})$

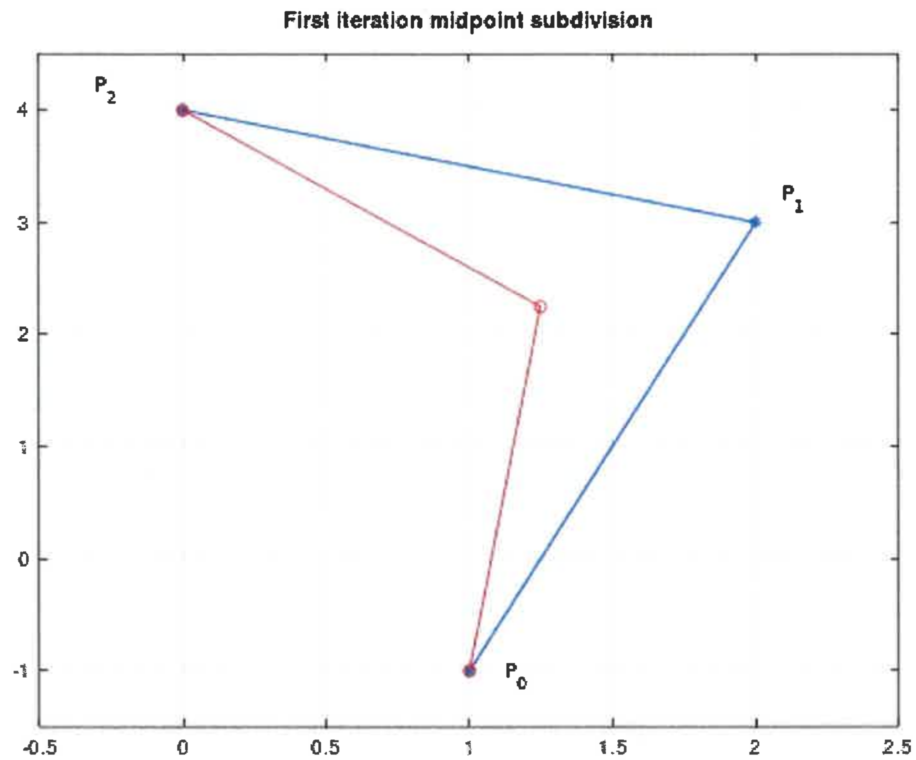
γ_{35} control points: $(\frac{5}{4},\frac{9}{4}), (\frac{19}{16},\frac{41}{16}), (\frac{69}{64},\frac{181}{64})$

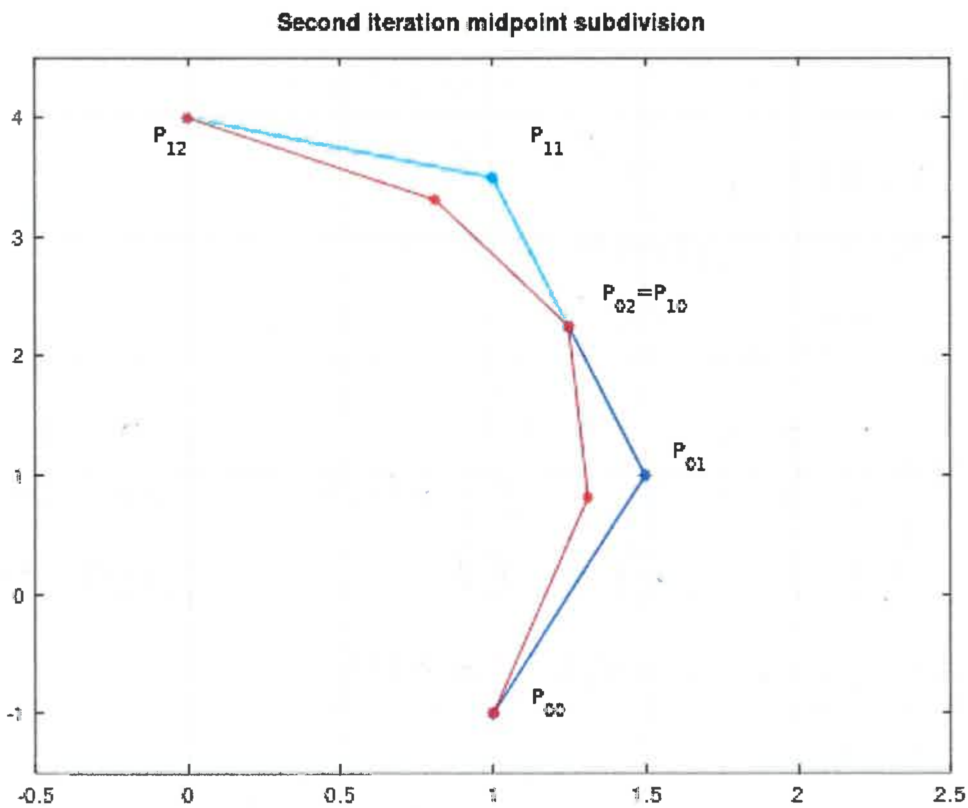
γ_{36} control points: $(\frac{69}{64},\frac{181}{64}), (\frac{31}{32},\frac{99}{32}), (\frac{13}{16},\frac{53}{16})$

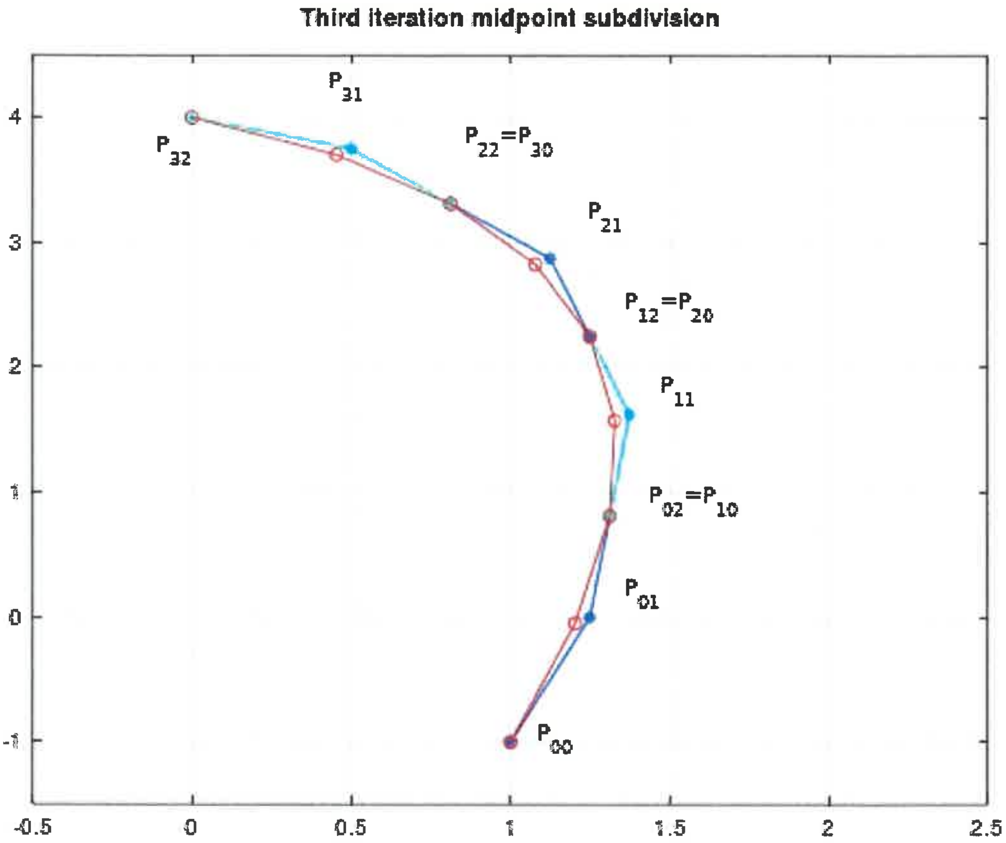
γ_{37} control points: $(\frac{13}{16},\frac{53}{16}), (\frac{21}{32},\frac{113}{32}), (\frac{29}{64},\frac{237}{64})$

γ_{38} control points: $(\frac{29}{64},\frac{237}{64}), (\frac{1}{4},\frac{31}{8}), (0,4)$

d)


First iteration midpoint subdivision
P2
P1
P0
4
3
2
1
0
-1
-0.5
0
0.5
1
1.5
2
2.5


Second iteration midpoint subdivision
P12
P11
P02=P10
P01
P00
4
3
2
1
0
-1
-0.5
0
0.5
1
1.5
2
2.5



② $\gamma(t) = \sum_{i=0}^{2} B_i^2(t) P_i \quad t \in [0,1] \qquad \gamma : [0,1] \longrightarrow \mathbb{R}^2$

$P_0 = (0,0) \qquad B_0^2(t) = \binom{2}{0}(1-t)^2 t^0 = 1 - 2t + t^2$

$P_1 = (x,y) \qquad B_1^2(t) = \binom{2}{1}(1-t)^1 t^1 = 2t - 2t^2$

$P_2 = (2,4) \qquad B_2^2(t) = \binom{2}{2}(1-t)^0 t^2 = t^2$

$\gamma(\frac{3}{4}) = (1,1)$

$$\gamma(t) = (1, t, t^2) \begin{pmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ x & y \\ 2 & 4 \end{pmatrix} = (1, t, t^2) \begin{pmatrix} 0 & 0 \\ 2x & 2y \\ -2x+2 & -2y+4 \end{pmatrix}$$

$= (2xt + (-2x+2)t^2, \ 2yt + (-2y+4)t^2) = \gamma(t)$

Evaluate $\gamma(\frac{3}{4}) = (1,1)$

$\rightarrow (1,1) = \left(2x\frac{3}{4} + (-2x+2)\frac{9}{16}, \ 2y\frac{3}{4} + (-2y+4)\frac{9}{16}\right)$

$$\begin{cases} \frac{3}{2}x - \frac{9x}{8} + \frac{9}{8} = 1 \\ \frac{3}{2}y - \frac{9}{8}y + \frac{9}{4} = 1 \end{cases} \rightarrow \begin{cases} 12x - 9x + 9 = 8 \\ 12y - 9y + 18 = 8 \end{cases} \rightarrow \begin{cases} 3x = -1 \\ 3y = -10 \end{cases}$$

$\rightarrow \quad x = -\frac{1}{3} \quad y = -\frac{10}{3}$ then $P_1 = \left(-\frac{1}{3}, -\frac{10}{3}\right)$

So $\gamma : [0,1] \longrightarrow \mathbb{R}^2$ is given by $\gamma(t) = \sum_{i=0}^{2} B_i^2(t) P_i$ for $B_i^2(t)$ the Bernstein polynomials already given and $P_0 = (0,0)$, $P_1 = \left(-\frac{1}{3}, -\frac{10}{3}\right)$, $P_2 = (2,4)$.