MAT300 CURVES AND SURFACES

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Piecewise interpolation

Piecewise polynomials

Definition

A piecewise polynomial in an interval $[a, b]$ is a function

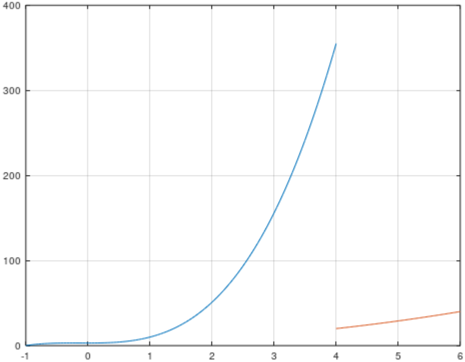
$$p : [a, b] \to \mathbb{R}$$

that is piecewise defined on $[a, x_1) \cup [x_1, x_2) \cup \ldots \cup [x_{n-1}, b]$ through polynomials $p_1 : [a, x_1) \to \mathbb{R}$, $p_2 : [x_1, x_2) \to \mathbb{R}$, …, $p_n : [x_{n-1}, b] \to \mathbb{R}$ in the following way

$$p(x) = \begin{cases} p_1(x), \ x \in [a, x_1), \\ p_2(x), \ x \in [x_1, x_2), \\ \vdots \\ p_n(x), \ x \in [x_{n-1}, b]. \end{cases} \tag{1}$$

Example:

$$p(x) = \begin{cases} 3 + 2x^2 + 5x^3, & x \in [-1, 4) \\ 4 + x^2, & x \in [4, 6] \end{cases}$$



Some questions arising

- To which **vector spaces** belong such polynomials?
- How can I construct **bases** for those vector spaces?
- How can I use these polynomials for **interpolation**?

The vector space

Considering piecewise polynomials of the type

$$p(x) = \begin{cases} p_1(x), \ x \in [a, x_1), \\ p_2(x), \ x \in [x_1, x_2), \\ \vdots \\ p_n(x), \ x \in [x_{n-1}, b], \end{cases}$$

for defining a **FINITE** dimensional vector space we **have to fix the intervals** $[a, x_1)$, $[x_1, x_2)$, $\ldots$ $[x_{n-1}, b]$

Therefore denoting with $x_0 = a$ and $x_n = b$, the vector space to define depends on a mesh of nodes $x_0, x_1, \ldots, x_{n-1}, x_n$.

Next, $p_1, p_2, \ldots, p_n \in P_k$ **we have to bound the degree for the polynomials**.

Definition

Let $x_0 < x_1 < \ldots < x_{n-1} < x_n \in \mathbb{R}$. The set of piecewise polynomials $p : [x_0, x_n] \to \mathbb{R}$ given as

$$p(x) = \begin{cases} p_1(x), & x \in [x_0, x_1), \\ p_2(x), & x \in [x_1, x_2), \\ \vdots \\ p_n(x), & x \in [x_{n-1}, x_n], \end{cases}$$

with $p_i \in P_k$ for $i = 1, \ldots, n$ is denoted with $P_k^n[x_0, \ldots, x_n]$.

Theorem

$P_k^n[x_0, \ldots, x_n]$ is a vector space.

The proof of the theorem consists of verifying the 10 axioms.

Examples:

$$p(x) = \begin{cases} 3 + 2x^2 + 5x^3, \ x \in [-1, 4) \\ \\ 4 + x^2, \ x \in [4, 6] \end{cases} \qquad p \in P_3^2[-1, 4, 6]$$

$$q(x) = \begin{cases} 1 - x, & x \in [-2, -1) \\ \\ 3 + 2x^2 + 5x^3, & x \in [-1, 4) \\ \\ 4 + x^2, & x \in [4, 6) \\ \\ x^6, & x \in [6, 10] \end{cases} \qquad q \in P_6^4[-2, -1, 4, 6, 10]$$

The dimension of the space

Theorem

The dimension of $P_k^n[x_0, \dots, x_n]$ is $n(k+1)$.

This is easy to check by constructing a basis of polynomials

$$p_i^j(x) = \begin{cases} x^j, & x \in [x_{i-1}, x_i) \\ 0, & x \notin [x_{i-1}, x_i) \end{cases}, \quad i = 1, \dots, n, \quad j = 0, 1, \dots, k \qquad (2)$$

where i denotes the interval where the piecewise polynomial is not zero, and j the exponent of the standard basis.

We have then $n(k+1)$ polynomials that span $P_k^n[x_0, \dots, x_n]$, as every polynomial in $P_k^n[x_0, \dots, x_n]$ can be written as linear combination of them. Moreover, the polynomials are linearly independent as none of them can be written as linear combination of the others. Therefore form a basis. As the basis has $n(k+1)$ elements, that is the dimension of the space.

Example:

$$p(x) = \begin{cases} 3 + 2x^2 + 5x^3, \; x \in [-1, 4) \\ 4 + x^2, \; x \in [4, 6] \end{cases} \qquad p \in P_3^2[-1, 4, 6]$$

$\dim(P_3^2[-1, 4, 6]) = 2(3 + 1) = 8$ and a basis for $P_3^2[-1, 4, 6]$ is

$$p_1^0(x) = \begin{cases} 1, \; x \in [-1, 4) \\ 0, \; x \in [4, 6] \end{cases} \qquad p_2^0(x) = \begin{cases} 0, \; x \in [-1, 4) \\ 1, \; x \in [4, 6] \end{cases}$$

$$p_1^1(x) = \begin{cases} x, \; x \in [-1, 4) \\ 0, \; x \in [4, 6] \end{cases} \qquad p_2^1(x) = \begin{cases} 0, \; x \in [-1, 4) \\ x, \; x \in [4, 6] \end{cases}$$

$$p_1^2(x) = \begin{cases} x^2, \; x \in [-1, 4) \\ 0, \; x \in [4, 6] \end{cases} \qquad p_2^2(x) = \begin{cases} 0, \; x \in [-1, 4) \\ x^2, \; x \in [4, 6] \end{cases}$$

$$p_1^3(x) = \begin{cases} x^3, \; x \in [-1, 4) \\ 0, \; x \in [4, 6] \end{cases} \qquad p_2^3(x) = \begin{cases} 0, \; x \in [-1, 4) \\ x^3, \; x \in [4, 6] \end{cases}$$

$B = \{p_1^0, p_1^1, p_1^2, p_1^3, p_2^0, p_2^1, p_2^2, p_2^3\}$ is this a good basis? NO!

Why not? Lets have a look to piecewise polynomials with three intervals.

$$p(x) = \begin{cases} p_1(x), & x \in [x_0, x_1) \\ p_2(x), & x \in [x_1, x_2) \\ p_3(x), & x \in [x_2, x_3] \end{cases} \qquad p \in P_2^3[x_0, x_1, x_2, x_3]$$

We have $\dim(P_2^3[x_0, x_1, x_2, x_3]) = 3(2+1) = 9$ and a basis can be

$B = \{p_1^0, p_1^1, p_1^2, p_2^0, p_2^1, p_2^2, p_3^0, p_3^1, p_3^2\}$ with

$$p_1^0(x) = \begin{cases} 1, & x \in [x_0, x_1) \\ 0, & x \in [x_1, x_2) \\ 0, & x \in [x_2, x_3] \end{cases} \qquad p_1^1(x) = \begin{cases} x, & x \in [x_0, x_1) \\ 0, & x \in [x_1, x_2) \\ 0, & x \in [x_2, x_3] \end{cases}$$

$$p_1^2(x) = \begin{cases} x^2, & x \in [x_0, x_1) \\ 0, & x \in [x_1, x_2) \\ 0, & x \in [x_2, x_3] \end{cases} \qquad p_2^0(x) = \begin{cases} 0, & x \in [x_0, x_1) \\ 1, & x \in [x_1, x_2) \\ 0, & x \in [x_2, x_3] \end{cases}$$

$$p_2^1(x) = \begin{cases} 0, & x \in [x_0, x_1) \\ x, & x \in [x_1, x_2) \\ 0, & x \in [x_2, x_3] \end{cases} \qquad p_2^2(x) = \begin{cases} 0, & x \in [x_0, x_1) \\ x^2, & x \in [x_1, x_2) \\ 0, & x \in [x_2, x_3] \end{cases}$$

$$p_3^0(x) = \begin{cases} 0, & x \in [x_0, x_1) \\ 0, & x \in [x_1, x_2) \\ 1, & x \in [x_2, x_3] \end{cases} \qquad p_3^1(x) = \begin{cases} 0, & x \in [x_0, x_1) \\ 0, & x \in [x_1, x_2) \\ x, & x \in [x_2, x_3] \end{cases}$$

$$p_3^2(x) = \begin{cases} 0, & x \in [x_0, x_1) \\ 0, & x \in [x_1, x_2) \\ x^2, & x \in [x_2, x_3] \end{cases}$$

The above functions have two discontinuities.

Discontinuities are a nightmare from a computational perspective!

In a space $P_k^n[x_0, x_1, \ldots, x_n]$ with n intervals we have $n-1$ discontinuities. Using a basis with

$$p_i^j(x) = \begin{cases} x^j, & x \in [x_{i-1}, x_i) \\ 0, & x \notin [x_{i-1}, x_i) \end{cases}, \quad i = 1, \ldots, n, \quad j = 0, 1, \ldots, k$$

we have n intervals for each polynomial of the basis.

From an analytical perspective it is better to work with less discontinuities (the less the better) and from a computational perspective is better to work with smaller arrays.

Moreover, if we want to construct a subspace to $P_k^n[x_0, x_1, \ldots, x_n]$ by imposing continuity and differentiability conditions at $x_1, \ldots, x_{n-1}$, it is not easy to find a basis for that subspace if we work with this type of bases.

We will work with shifted power polynomial bases.

Shifted power polynomial functions

Definition

Let $c \in \mathbb{R}$. The right continuous shifted power polynomial function of degree n is

$$(x-c)_+^n = \begin{cases} 0, & x < c \\ (x-c)^n, & x \geq c \end{cases} \tag{3}$$

The left continuous shifted power polynomial function of degree n is

$$(c-x)_+^n = \begin{cases} (c-x)^n, & x \leq c \\ 0, & x > c \end{cases} \tag{4}$$

Definition

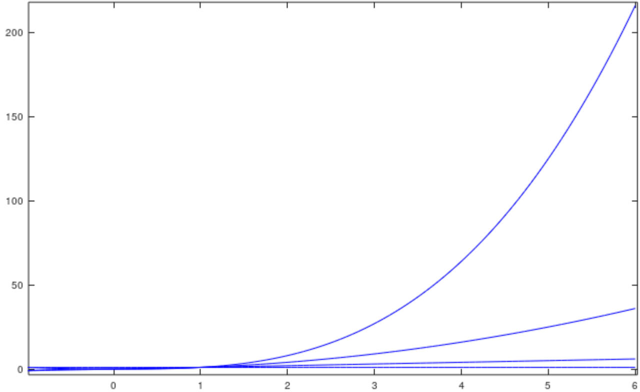
The standard basis for $P_k^n[x_0, x_1, \ldots, x_n]$ is

$$B = \{1,\ x,\ \ldots,\ x^k,\ (x-x_1)_+^0,\ (x-x_1)_+^1,\ \ldots,\ (x-x_1)_+^k,\ \ldots,\ (x-x_{n-1})_+^0,\ (x-x_{n-1})_+^1,\ \ldots,\ (x-x_{n-1})_+^k\}$$

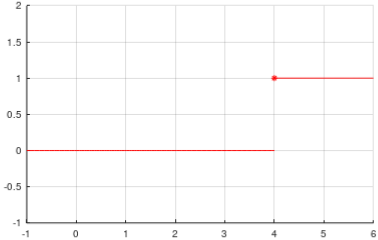
Example:

$$p(x) = \begin{cases} 3 + 2x^2 + 5x^3, \ x \in [-1, 4) \\ 4 + x^2, \ x \in [4, 6] \end{cases} \qquad p \in P_3^2[-1, 4, 6]$$

$$B = \{1, x, x^2, x^3, (x-4)_+^0, (x-4)_+^1, (x-4)_+^2, (x-4)_+^3\}$$

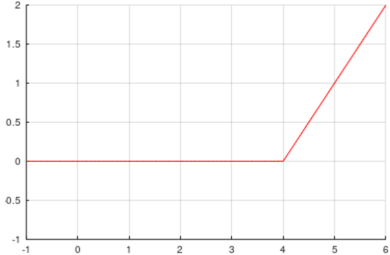


not piecewise



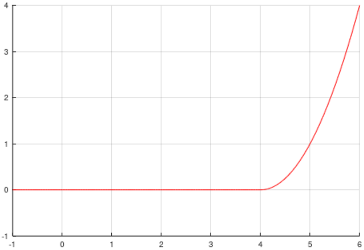
$$(x-4)_+^0 = \begin{cases} 0, & x < 4 \\ 1, & x \geq 4 \end{cases}$$

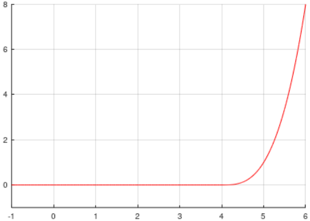
discontinuous



$$(x-4)_+^1 = \begin{cases} 0, & x < 4 \\ x-4, & x \geq 4 \end{cases}$$

$$(x-4)_+^2 = \begin{cases} 0, & x < 4 \\ x^2 - 8x + 16, & x \geq 4 \end{cases}$$





$$(x-4)_+^3 = \begin{cases} 0, & x < 4 \\ x^3 - 12x^2 + 48x - 64, & x \geq 4 \end{cases}$$

Show that $B = \{1, x, x^2, x^3, (x-4)_+^0, (x-4)_+^1, (x-4)_+^2, (x-4)_+^3\}$ is a basis for $P_3^2[-1, 4, 6]$.

We have to show:

linear independence: if

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 (x-4)_+^0 + a_5 (x-4)_+^1 + a_6 (x-4)_+^2 + a_7 (x-4)_+^3 = 0$$

then $a_0 = a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a_7 = 0$

spanning: if any $p \in P_3^2[-1, 4, 6]$ can be written as linear combination of elements in B.

WHITEBOARD

Example:

$$p(x) = \begin{cases} 3 + 2x^2 + 5x^3, \ x \in [-1, 4) \\ 4 + x^2, \ x \in [4, 6] \end{cases} \qquad p \in P_3^2[-1, 4, 6]$$

$$B = \{1, x, x^2, x^3, (x-4)_+^0, (x-4)_+^1, (x-4)_+^2, (x-4)_+^3\}$$

Give the vector of coordinates of p in the basis B.

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 (x-4)_+^0 + a_5 (x-4)_+^1 + a_6 (x-4)_+^2 + a_7 (x-4)_+^3$$

For $x \in [-1, 4)$

$$3 + 2x^2 + 5x^3 = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$a_0 = 3, \ a_1 = 0, \ a_2 = 2, \ a_3 = 5$

For $x \in [4, 6]$

$$4 + x^2 = 3 + 2x^2 + 5x^3 + a_4 + a_5(x-4) + a_6(x^2 - 8x + 16) + a_7(x^3 - 12x^2 + 48x - 64)$$

$$1-x^2-5x^3=(a_4-4a_5+16a_6-64a_7)+(a_5-8a_6+48a_7)x+(a_6-12a_7)x^2+a_7x^3$$

Construct a linear system

$$\begin{cases} a_4-4a_5+16a_6-64a_7=1 \\ a_5-8a_6+48a_7=0 \\ a_6-12a_7=-1 \\ a_7=-5 \end{cases} \Rightarrow \left(\begin{array}{rrrr|r} 1 & -4 & 16 & -64 & 1 \\ 0 & 1 & -8 & 48 & 0 \\ 0 & 0 & 1 & -12 & -1 \\ 0 & 0 & 0 & 1 & -5 \end{array}\right)$$

$$RREF \Rightarrow \left(\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & -335 \\ 0 & 1 & 0 & 0 & -248 \\ 0 & 0 & 1 & 0 & -61 \\ 0 & 0 & 0 & 1 & -5 \end{array}\right)$$

Vector of coordinates $(3, 0, 2, 5, -335, -248, -61, -5)$