MAT300 CURVES AND SURFACES

Julia Sánchez Sanz

DigiPen Institute of Technology Europe

julia.sanchez@digipen.edu

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Interpolation with derivatives

Motivation

In previous lectures we worked with interpolation problems, i.e.

Given n+1 points (x_0, y_0), (x_1, y_1), ..., (x_n, y_n) with $x_i \neq x_j$ for $i \neq j$ find a polynomial p satisfying $p(x_i) = y_i$ for $i = 0, 1, \ldots, n$.

In such problems we only restrict the polynomial to pass through certain points.

We may want as well to restrict the slope of the polynomial at those points,

to make it increasing or decreasing at a certain point,

or to have maximum or minimum values at those points.

For doing so **we have to impose conditions to the derivatives of** p.

Hermite interpolant polynomials

Definition

Let (x_0, y_0), (x_1, y_1), ..., (x_n, y_n) with $x_i \neq x_j$ for $i \neq j$. Let $z_i \in \mathbb{R}$ for $i = 0, 1, \ldots, n$. A Hermite interpolant polynomial p satisfies:

- $p(x_i) = y_i$ for $i = 0, 1, \ldots, n$.
- $p'(x_i) = z_i$ for some i.

Example: Find a polynomial through $(0, 0)$, $(1, 1)$, $(2, -1)$ satisfying $p'(0) = 0$ and $p'(2) = 0$.
Building with the conditions a system of equations we get:

$$\begin{cases} p(0) = 0 \\ p(1) = 1 \\ p(2) = -1 \\ p'(0) = 0 \\ p'(2) = 0 \end{cases} \qquad \text{We have 5 equations.}$$

With the evaluation of a polynomial

$$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

and its first derivative $p'(x) = a_1 + 2a_2 x + \ldots + n a_n x^{n-1}$

we can obtain a linear system of equations.

The resulting equations will be independent, therefore for having a unique solution we need 5 unknowns. So we look for $p \in P_4$.

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

$$p'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3$$

Our system of equations will be:

$$\begin{cases} a_0 = 0 \\ a_0 + a_1 + a_2 + a_3 + a_4 = 1 \\ a_0 + 2a_1 + 4a_2 + 8a_3 + 16a_4 = -1 \\ a_1 = 0 \\ a_1 + 4a_2 + 12a_3 + 32a_4 = 0 \end{cases}$$

Solving the system

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 12 & 32 & 0 \end{array}\right) \Rightarrow \; RREF \Rightarrow \left(\begin{array}{c|c} & 0 \\ & 0 \\ I_5 & \frac{21}{4} \\ & -\frac{23}{4} \\ & \frac{3}{2} \end{array}\right)$$

$p(x) = \frac{21}{4}x^2 - \frac{23}{4}x^3 + \frac{3}{2}x^4$

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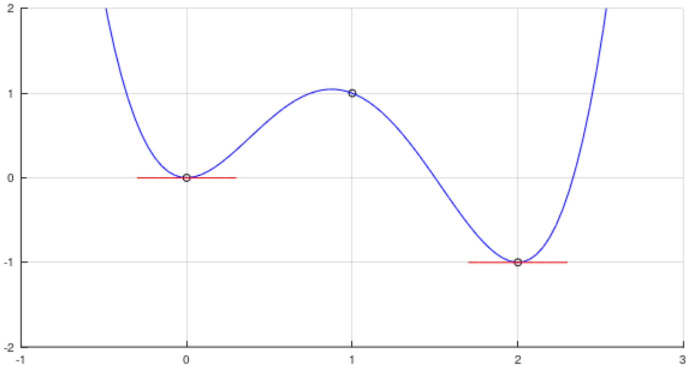

p through $(0,0)$, $(1,1)$ and $(2,-1)$ derivatives at 0 and 2 are zero

Osculating polynomials

We can extend Hermite interpolation to a higher order of derivatives

Definition

Let (x_0, y_{00}), (x_1, y_{10}), ..., (x_n, y_{n0}) with distinct x-component. Let $y_{ij} \in \mathbb{R}$ for $i = 0, 1, \ldots, n$ and $j = 0, \ldots, r_i$. An osculating interpolant polynomial p satisfies:

$$p^{(j)}(x_i) = y_{ij}, \qquad i = 0, 1, \ldots, n, \qquad j = 0, \ldots, r_i \tag{1}$$

Notation: (j) denotes the order of the derivative, $p^{(0)}$ is the original polynomial, $p^{(1)}$ its first derivative, $p^{(2)}$ its second derivative and so on.

Example: we want a polynomial passing through $(0, 0)$, $(1, 1)$ and $(2, -1)$, where $(0, 0)$ and $(2, -1)$ are local maximum points.

As the polynomial passes through $(0,0)$, $(1,1)$ and $(2,-1)$ we have

$$\begin{cases} p(0) = 0 \\ p(1) = 1 \\ p(2) = -1 \end{cases}$$

$(0,0)$ and $(2,-1)$ are local maximum points, therefore $\begin{cases} p'(0) = 0 \\ p'(2) = 0 \\ p''(0) = -a \\ p''(2) = -b \end{cases}$

for $a, b > 0$. Let's take for instance $a = b = 20$.

We have 7 conditions, therefore we will construct a linear system of 7 equations with 7 unknowns.

Our resulting polynomial will be in P_6.

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6$$

$$p'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + 6a_6x^5$$

$$p''(x) = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4$$

Therefore the system will be

$$\begin{cases} a_0 = 0 \\ a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 1 \\ a_0 + 2a_1 + 4a_2 + 8a_3 + 16a_4 + 32a_5 + 64a_6 = -1 \\ a_1 = 0 \\ a_1 + 4a_2 + 12a_3 + 32a_4 + 80a_5 + 192a_6 = 0 \\ 2a_2 = -20 \\ 2a_2 + 12a_3 + 48a_4 + 160a_5 + 480a_6 = -20 \end{cases}$$

Building the augmented coefficient matrix and solving the system

$$\left(\begin{array}{ccccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 & 32 & 64 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 12 & 32 & 80 & 192 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & -20 \\ 0 & 0 & 2 & 12 & 48 & 160 & 480 & -20 \end{array}\right) \Rightarrow RREF \Rightarrow \left(\begin{array}{c|c} & 0 \\ & 0 \\ & -10 \\ I_7 & \frac{163}{4} \\ & -\frac{793}{16} \\ & \frac{381}{16} \\ & -4 \end{array}\right)$$

$$p(x) = -10x^2 + \frac{163}{4}x^3 - \frac{793}{16}x^4 + \frac{381}{16}x^5 - 4x^6$$

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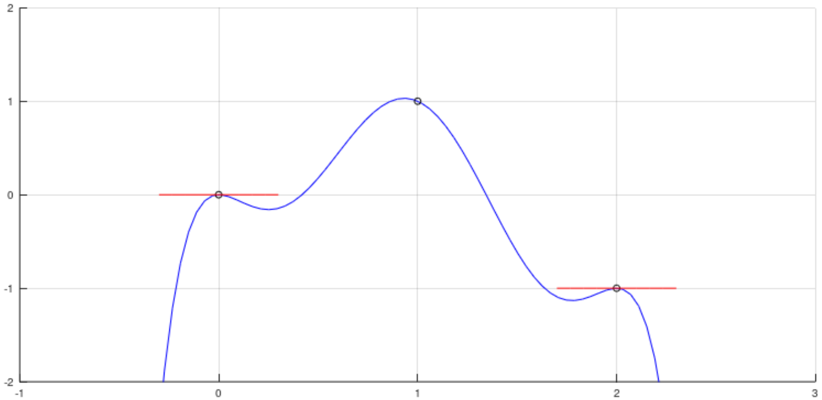


Existence and uniqueness

Theorem

The osculating polynomial of degree $\sum_{i=0}^{n} r_i + n$ exists and it is unique.

We skip the proof in this course (because of really hard algebraic notation), but if you have curiosity you can look for "Confluent Vandermonde determinants" in the internet.

This way of computing osculating interpolant polynomials is quite inefficient (build a matrix, compute RREF, round off errors...) Is there any other way?

Yes, using divided differences and Newton osculating polynomials

Divided differences & Newton osculating polynomials

The divided differences with derivatives follow the **Taylor expansion**.

Theorem

Let $f : \mathbb{R} \to \mathbb{R}$ with $f \in C^n[a, b]$ and let $x_0 \in (a, b)$. We can approximate f at a point x near x_0 with a polynomial of degree n

$$\begin{aligned} f(x) &\simeq f(x_0) + (x - x_0)f'(x_0) + (x - x_0)^2 \frac{f''(x_0)}{2} + \ldots + (x - x_0)^n \frac{f^{(n)}(x_0)}{n!} \\ &= \sum_{k=0}^{n} (x - x_0)^k \frac{f^{(k)}(x_0)}{k!} \end{aligned}$$

How is a Taylor expansion related to the divided differences? lets see an example.

Example of osculating polynomials and divided differences

Obtain the divided differences for a function f satisfying $f(0) = 0$, $f'(0) = 0$, $f''(0) = 0$, $f(1) = 1$, $f(2) = -1$ and $f'(2) = 0$.

We first introduce in the first column the nodes x_0, x_1, ... x_n with their repetitions in increasing order:

x_i
0
0
0
1
2
2

Next we introduce the zeroth divided differences $f[x_i] = y_i$

x_i	$f[x_i]$
0	0
0	0
0	0
1	1
2	-1
2	-1

If one node is repeated we can not compute the first divided difference through $f[x_i, x_i] = \frac{f[x_i]-f[x_i]}{x_i - x_i}$

We compute it through a limit

$$f[x_i, x_i] = \lim_{h\to 0} f[x_i, x_i + h] = \lim_{h\to 0} \frac{f(x_i + h) - f(x_i)}{h} = f'(x_i)$$

Next we introduce the first divided differences $f[x_i, x_i] = f'(x_i)$ and $f[x_i, x_{i+1}] = \frac{f[x_{i+1}]-f[x_i]}{x_{i+1}-x_i}$

$$
\begin{array}{ccc}
x_i & f[x_i] & f[x_i, x_j] \\
 & & \\
0 & 0 & \\
 & & 0 \\
0 & 0 & \\
 & & 0 \\
0 & 0 & \\
 & & \frac{1-0}{1-0} = 1 \\
1 & 1 & \\
 & & \frac{-1-1}{2-1} = -2 \\
2 & -1 & \\
 & & 0 \\
2 & -1 &
\end{array}
$$

If one node is repeated twice we can not compute the second divided difference through $f[x_i, x_i, x_i] = \frac{f[x_i,x_i]-f[x_i,x_i]}{x_i-x_i}$

We compute it through a limit

$$f[x_i, x_i, x_i] = \lim_{h\to 0} f[x_i, x_i, x_i + h] = \lim_{h\to 0} \frac{f[x_i, x_i + h] - f[x_i, x_i]}{x_i + h - x_i} =$$

$$\lim_{h\to 0} \frac{\frac{f(x_i+h)-f(x_i)}{h} - f'(x_i)}{h} = \lim_{h\to 0} \frac{f(x_i + h) - f(x_i) - hf'(x_i)}{h^2} =$$

Taking the Taylor expansion of order 2

$f(x_i + h) \simeq f(x_i) + ((x_i + h) - x_i)f'(x_i) + ((x_i + h) - x_i)^2 \frac{f''(x_i)}{2}$

$= f(x_i) + hf'(x_i) + h^2 \frac{f''(x_i)}{2}$

$$= \lim_{h\to 0} \frac{f(x_i) + hf'(x_i) + h^2 \frac{f''(x_i)}{2} - f(x_i) - hf'(x_i)}{h^2} = \frac{f''(x_i)}{2}$$

Next we introduce the second divided differences $f[x_i, x_i, x_i] = \frac{f''(x_i)}{2}$,

$f[x_i, x_i, x_{i+1}] = \frac{f[x_i,x_{i+1}]-f[x_i,x_i]}{x_{i+1}-x_i}$ and $f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1},x_{i+2}]-f[x_i,x_{i+1}]}{x_{i+2}-x_i}$

$$
\begin{array}{cccc}
x_i & f[x_i] & f[x_i, x_j] & f[x_i, x_j, x_k] \\
 & & & \\
0 & 0 & & \\
 & & 0 & \\
0 & 0 & & \frac{0}{2!} = 0 \\
 & & 0 & \\
0 & 0 & & \frac{1-0}{1-0} = 1 \\
 & & 1 & \\
1 & 1 & & \frac{-2-1}{2-0} = -\frac{3}{2} \\
 & & -2 & \\
2 & -1 & & \frac{0+2}{2-1} = 2 \\
 & & 0 & \\
2 & -1 & &
\end{array}
$$

In a general setting, if a node is repeated n times (i.e. if it appears n+1 times), the nth divided difference is

$$f[x_i, x_i, \ldots, x_i]_n = \lim_{h \to 0} \frac{f[x_i, x_i, \ldots, x_i + h]_{n-1} - f[x_i, x_i, \ldots, x_i]_{n-1}}{h}$$

$$= \frac{f^{(n)}(x_i)}{n!}$$

We finish the computation of the divided differences, now that we do not have repetitions.

x_i	$f[x_i]$	$f[x_i, x_j]$	$f[x_i, x_j, x_k]$	$f[x_i, x_j, x_k, x_l]$
0	0			
		0		
0	0		0	
		0		$\frac{1-0}{1-0} = 1$
0	0		1	
		1		$\frac{-\frac{3}{2}-1}{2-0} = -\frac{5}{4}$
1	1		$-\frac{3}{2}$	
		-2		$\frac{2+\frac{3}{2}}{2-0} = \frac{7}{4}$
2	-1		2	
		0		
2	-1			

x_i	$f[x_i]$	$f[x_i, x_j]$	$f[x_i, x_j, x_k]$	$f[x_i, x_j, x_k, x_l]$	$f[x_i, x_j, x_k, x_l, x_m]$
0	0				
		0			
0	0		0		
		0		1	
0	0		1		$\frac{-\frac{5}{4}-1}{2-0} = -\frac{9}{8}$
		1		$-\frac{5}{4}$	
1	1		$-\frac{3}{2}$		$\frac{\frac{7}{4}+\frac{5}{4}}{2-0} = \frac{3}{2}$
		-2		$\frac{7}{4}$	
2	-1		2		
		0			
2	-1				

$$
\begin{array}{ccccccc}
x_i & f[x_i] & f[x_i,x_j] & & \dots & & \\
 & & & & & & \\
0 & 0 & & & & & \\
 & & 0 & & & & \\
0 & 0 & & 0 & & & \\
 & & 0 & & 1 & & \\
0 & 0 & & 1 & & -\frac{9}{8} & \\
 & & 1 & & -\frac{5}{4} & & \frac{\frac{3}{2}+\frac{9}{8}}{2-0} = \frac{21}{16} \\
1 & 1 & & -\frac{3}{2} & & \frac{3}{2} & \\
 & & -2 & & \frac{7}{4} & & \\
2 & -1 & & 2 & & & \\
 & & 0 & & & & \\
2 & -1 & & & & &
\end{array}
$$

We take the first divided difference of each order

$$
\begin{array}{ccccccc}
x_i & f[x_i] & f[x_i, x_j] & & \dots & & \\
 & & & & & & \\
0 & \color{red}{0} & & & & & \\
 & & \color{red}{0} & & & & \\
0 & 0 & & \color{red}{0} & & & \\
 & & 0 & & \color{red}{1} & & \\
0 & 0 & & 1 & & \color{red}{-\frac{9}{8}} & \\
 & & 1 & & -\frac{5}{4} & & \color{red}{\frac{21}{16}} \\
1 & 1 & & -\frac{3}{2} & & \frac{3}{2} & \\
 & & -2 & & \frac{7}{4} & & \\
2 & -1 & & 2 & & & \\
 & & 0 & & & & \\
2 & -1 & & & & &
\end{array}
$$

The Newton basis in this case is

$$B_N = \{1, x, x^2, x^3, x^3(x-1), x^3(x-1)(x-2)\}$$

Newton basis in the general case

Definition

The Newton basis for an osculating polynomial satisfying

$$p^{(j)}(x_i) = y_{ij}, \qquad i = 0, 1, \ldots, n, \qquad j = 0, \ldots, r_i$$

is

$$B_N = \{1, (x - x_0), \ldots, (x - x_0)^{r_0+1}, (x - x_0)^{r_0+1}(x - x_1),$$
$$\ldots, (x - x_0)^{r_0+1}(x - x_1)^{r_1+1}, \ldots, \left(\prod_{i=0}^{n-1}(x - x_i)^{r_i+1}\right),$$
$$\ldots, \left(\prod_{i=0}^{n-1}(x - x_i)^{r_i+1}\right)(x - x_n)^{r_n}\}$$

$$|B_N| = (n + 1) + \sum_{i=0}^{n} r_i$$

In the previous example $n = 2$ (3 nodes), $r_0 = 2$, $r_1 = 0$ and $r_2 = 1$.
$2 + 1 + 2 + 0 + 1 = 6 = |B_N|$

Returning to our example, if we multiply the divided differences by the Newton basis we get the osculating interpolant polynomial:

$$p(x) = x^3 - \frac{9}{8}x^3(x-1) + \frac{21}{16}x^3(x-1)(x-2)$$

which in the Newton basis has coordinates

$$(0, 0, 0, 1, -\frac{9}{8}, \frac{21}{16})_{N_B}$$

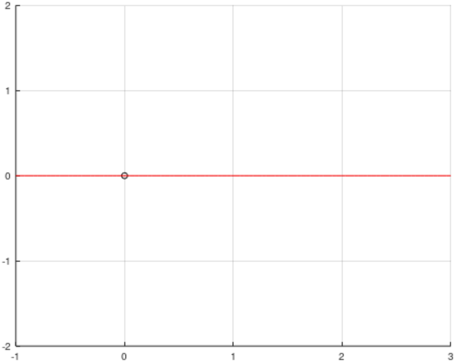
Notice that the construction of the polynomial is similar to the interpolant polynomial in the Newton basis without derivatives, i.e.

Obtain an interpolant polynomial satisfying $p(0) = 0$, $p'(0) = 0$, $p''(0) = 0$, $p(1) = 1$, $p(2) = -1$ and $p'(2) = 0$.

Adding a degree-adding a condition

Obtain an interpolant polynomial satisfying $p(0) = 0$, $p'(0) = 0$, $p''(0) = 0$, $p(1) = 1$, $p(2) = -1$ and $p'(2) = 0$.

$$p(x) = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + x^3 - \frac{9}{8}x^3(x-1) + \frac{21}{16}x^3(x-1)(x-2)$$



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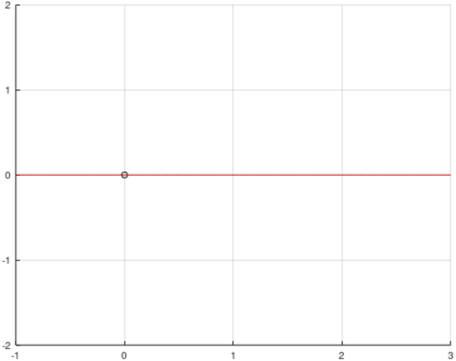

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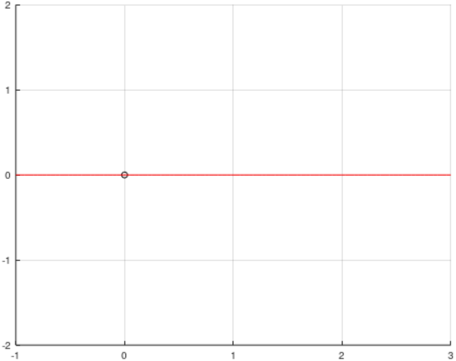

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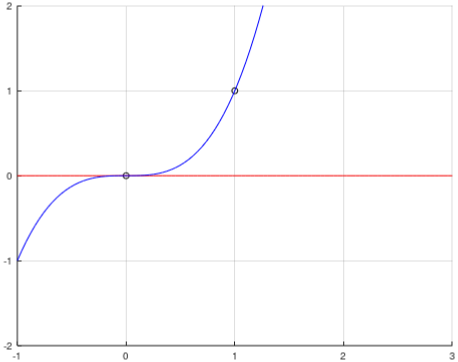

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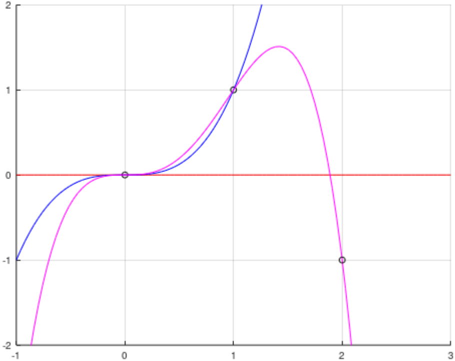

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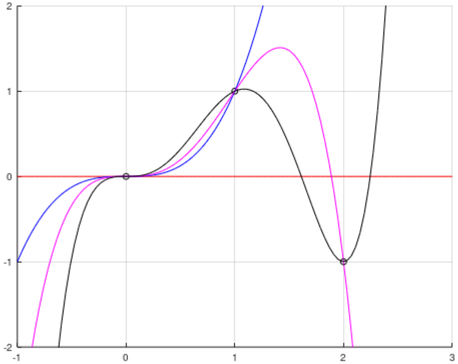

Discussion

- With osculating interpolant polynomials we can impose more restrictions to the curves.
- Each time I impose a condition I increment the degree.
- When increasing the degree of the polynomials we have more variability, in particular in the extremes of the curve.
- We may want smooth curves through many points, without much variability. This motivates the necessity of splines, i.e. piecewise osculating interpolant polynomials.

Interpolant polynomial versus cubic spline

video