MAT300 CURVES AND SURFACES

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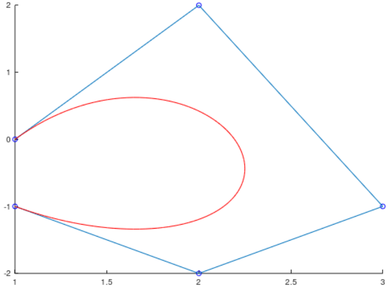
Bezier curves

1. The Bernstein polynomials
2. Bezier Curves

Introduction

A technique used in industry, in particular in automotive design, is the **computation of smooth curves using control points and Bernstein polynomials**. Such curves are known as **Bezier curves**.



The technique was developed in the 50s of the 20th century by Bezier in Renault and by de Casteljau in Citroën. The curves are also used in digital typography, architecture, and graphics design.

Moving control points does not affect much the curve.

The Bernstein polynomials and P_n

Definition

The Bernstein polynomials of degree n are given as

$$B_i^n(x) = \left(\begin{array}{c} n \\ i \end{array} \right) (1-x)^{n-i} x^i, \quad i = 0, 1, \ldots, n. \tag{1}$$

The set of Bernstein polynomials of degree n

$$B_B = \{B_0^n,\ B_1^n,\ \ldots,\ B_n^n\} \tag{2}$$

form a basis for P_n.

Any polynomial $p \in P_n$ can be represented in a unique way as a linear combination of Bernstein polynomials. Such a linear combination is called the **Bezier representation of** p.

Example in P_3

$$B_0^3(x) = \begin{pmatrix} 3 \\ 0 \end{pmatrix} (1-x)^3 x^0 = 1 - 3x + 3x^2 - x^3$$

$$B_1^3(x) = \begin{pmatrix} 3 \\ 1 \end{pmatrix} (1-x)^2 x^1 = 3x - 6x^2 + 3x^3$$

$$B_2^3(x) = \begin{pmatrix} 3 \\ 2 \end{pmatrix} (1-x)^1 x^2 = 3x^2 - 3x^3$$

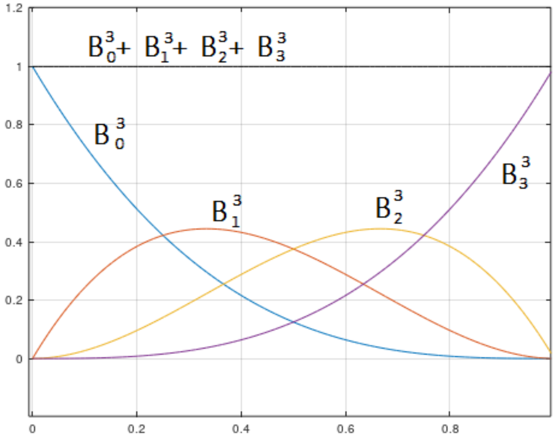
$$B_3^3(x) = \begin{pmatrix} 3 \\ 3 \end{pmatrix} (1-x)^0 x^3 = x^3$$

$p(x) = 2 - 9x + 24x^2 - 19x^3 = 2B_0^3(x) - B_1^3(x) + 4B_2^3(x) - 2B_3^3(x)$

WHITEBOARD!

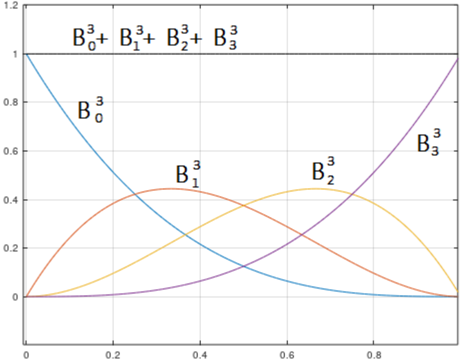
Properties of Bernstein polynomials: the sum is 1

- $\sum_{i=0}^{n} B_i^n(x) = \sum_{i=0}^{n} \binom{n}{i} (1-x)^{n-i} x^i = ((1-x)+x)^n = 1^n = 1$



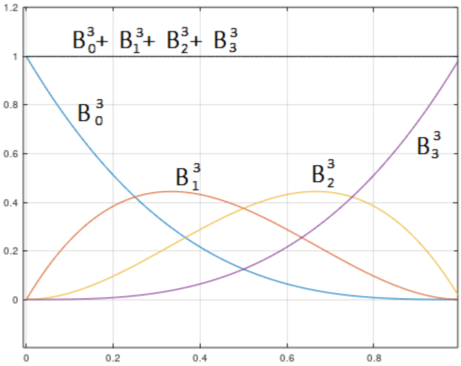
Properties: symmetry w.r.t. $x = 0.5$

- $B_i^n(x) = \binom{n}{i}(1-x)^{n-i}x^i = \binom{n}{n-i}x^i(1-x)^{n-i} = \binom{n}{n-i}(1-(1-x))^{n-(n-i)}(1-x)^{n-i} = B_{n-i}^n(1-x)$



Properties: evaluation at $x \in [0,1]$

- For $x \in (0,1)$ $B_i^n(x) = \left(\begin{array}{c} n \\ i \end{array} \right) (1-x)^{n-i} x^i > 0$
- $B_0^n(0) = 1$ and $B_i^n(0) = 0$ for $i = 1, \ldots, n$
- $B_n^n(1) = 1$ and $B_i^n(1) = 0$ for $i = 0, \ldots, n-1$



Properties: definition through recursion

Bernstein polynomials of degree n can be generated through recursion having the Bernestein polynomials of degree $n-1$ as follows

$$B_i^n(x) = xB_{i-1}^{n-1}(x) + (1-x)B_i^{n-1}(x) \tag{3}$$

where $B_{-1}^{n-1}(x) := 0$ and $B_n^{n-1}(x) := 0$.

For $i = 0$:

$$B_0^n(x) = \begin{pmatrix} n \\ 0 \end{pmatrix} (1-x)^n x^0 = (1-x)^n = (1-x)(1-x)^{n-1}$$

$$= (1-x) \begin{pmatrix} n-1 \\ 0 \end{pmatrix} (1-x)^{n-1} x^0 = (1-x)B_0^{n-1}(x)$$

$$= \cancel{xB_{-1}^{n-1}(x)} + (1-x)B_0^{n-1}(x) \quad \checkmark$$

For $i = n$:

$$B_n^n(x) = \left(\begin{array}{c} n \\ n \end{array} \right) (1-x)^0 x^n = x^n = x \cdot x^{n-1}$$

$$= x \left(\begin{array}{c} n-1 \\ n-1 \end{array} \right) (1-x)^{(n-1)-(n-1)} x^{n-1} = x B_{n-1}^{n-1}(x)$$

$$= x B_{n-1}^{n-1}(x) + \cancel{(1-x) B_n^{n-1}(x)} \quad \checkmark$$

For $i = 1, \ldots, n-1$:

$$B_i^n(x) = \left(\begin{array}{c} n \\ i \end{array} \right) (1-x)^{n-i} x^i = \left[\left(\begin{array}{c} n-1 \\ i-1 \end{array} \right) + \left(\begin{array}{c} n-1 \\ i \end{array} \right) \right] (1-x)^{n-i} x^i$$

$$= \left(\begin{array}{c} n-1 \\ i-1 \end{array} \right) (1-x)^{n-i} x^i + \left(\begin{array}{c} n-1 \\ i \end{array} \right) (1-x)^{n-i} x^i$$

$$= \binom{n-1}{i-1}(1-x)^{(n-1)-(i-1)}x^i + \binom{n-1}{i}(1-x)^{(n-1)-i+1}x^i =$$

$$x\binom{n-1}{i-1}(1-x)^{(n-1)-(i-1)}x^{i-1} + (1-x)\binom{n-1}{i}(1-x)^{(n-1)-i}x^i$$

$$= xB_{i-1}^{n-1}(x) + (1-x)B_i^{n-1}(x) \quad \checkmark$$

Properties: derivative of Bernstein polynomial

The derivative of a Bernstein polynomial can be defined through recursion as well

$$\frac{d}{dx}B_i^n(x) = n\big(B_{i-1}^{n-1}(x) - B_i^{n-1}(x)\big) \tag{4}$$

We have $B_i^n(x) = xB_{i-1}^{n-1}(x) + (1-x)B_i^{n-1}(x)$ for $i = 0, 1, \ldots, n$ where $B_{-1}^{n-1}(x) := 0$ and $B_n^{n-1}(x) := 0$

Using this result and recursion we will prove (4)

For n=1 we have:

$\frac{d}{dx}B_i^1(x) = \frac{d}{dx}(xB_{i-1}^0(x) + (1-x)B_i^0(x)) =$

$B_{i-1}^0(x) + \cancel{x\frac{d}{dx}B_{i-1}^0(x)} - B_i^0(x) + \cancel{(1-x)\frac{d}{dx}B_i^0(x)} = B_{i-1}^0(x) - B_i^0(x)$ ✓

For n=2 we have:

$$\frac{d}{dx}B_i^2(x) = \frac{d}{dx}(xB_{i-1}^1(x) + (1-x)B_i^1(x)) =$$

$$B_{i-1}^1(x) + x\frac{d}{dx}B_{i-1}^1(x) - B_i^1(x) + (1-x)\frac{d}{dx}B_i^1(x) =$$

$$B_{i-1}^1(x) + x(B_{i-2}^0(x) - B_{i-1}^0(x)) - B_i^1(x) + (1-x)(B_{i-1}^0(x) - B_i^0(x)) =$$

$$B_{i-1}^1(x) - B_i^1(x) + [xB_{i-2}^0(x) + (1-x)B_{i-1}^0(x)] - [xB_{i-1}^0(x) + (1-x)B_i^0(x)] =$$

$$2B_{i-1}^1(x) - 2B_i^1(x) = 2(B_{i-1}^1(x) - B_i^1(x)) \quad \checkmark$$

Now assuming true for $n-1$, for n we have:

$$\frac{d}{dx}B_i^n(x) = \frac{d}{dx}(xB_{i-1}^{n-1}(x) + (1-x)B_i^{n-1}(x)) =$$

$$B_{i-1}^{n-1}(x) + x\frac{d}{dx}B_{i-1}^{n-1}(x) - B_i^{n-1}(x) + (1-x)\frac{d}{dx}B_i^{n-1}(x) =$$

$$B_{i-1}^{n-1}(x) + x(n-1)(B_{i-2}^{n-2}(x) - B_{i-1}^{n-2}(x)) - B_i^{n-1}(x)$$

$$+(1-x)(n-1)(B_{i-1}^{n-2}(x) - B_i^{n-2}(x)) =$$

$$B_{i-1}^{n-1}(x) - B_i^{n-1}(x) + (n-1)[xB_{i-2}^{n-2}(x) + (1-x)B_{i-1}^{n-2}(x)]$$

$$-(n-1)[xB_{i-1}^{n-2}(x) + (1-x)B_i^{n-2}(x)] =$$

$$B_{i-1}^{n-1}(x) - B_i^{n-1}(x) + (n-1)B_{i-1}^{n-1}(x) - (n-1)B_i^{n-1}(x) =$$

$$= n(B_{i-1}^{n-1}(x) - B_i^{n-1}(x)) \quad \checkmark$$

Bezier curves

Definition

A Bezier curve is a polynomial curve given as a linear combination of Bernstein polynomials as follows:

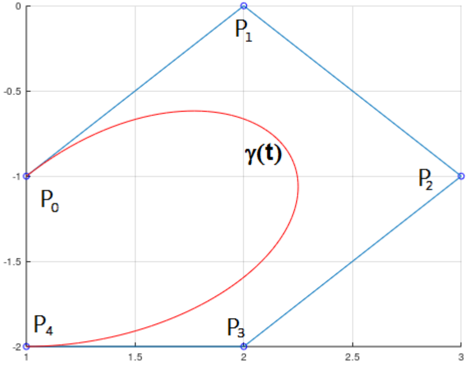
$$\gamma(t) = \sum_{i=0}^{n} P_i B_i^n(t), \qquad t \in [0,1]. \tag{5}$$

The points P_i for $i = 0, 1, \ldots, n$ are the control points of the curve.

If $P_i \in \mathbb{R}^2$ the curve $\gamma : [0,1] \to \mathbb{R}^2$ is planar.

If $P_i \in \mathbb{R}^3$ the curve $\gamma : [0,1] \to \mathbb{R}^3$ is a curve in 3D.

Example



$P_0 = (1, -1)$, $P_1 = (2, 0)$, $P_2 = (3, -1)$, $P_3 = (2, -2)$ and $P_4 = (1, -2)$

$\gamma(t) = \sum_{i=0}^{4} P_i B_i^4(t)$, $\quad t \in [0, 1]$ obtain curve whiteboard

$\gamma(t) = (1 + 4t - 8t^3 + 4t^4, -1 + 4t - 12t^2 + 8t^3 - t^4), \quad t \in [0, 1]$

Properties: sum of Bernstein polynomials is 1

$\sum_{i=0}^{n} B_i^n(t) = 1 \; \forall t$ and their evaluation in $[0, 1]$ is nonnegative, then every point $\gamma(t)$ has barycentric coordinates with respect to the control points P_0, P_1, ..., P_n.

Good choice for affine transformations!

Remember that: barycentric coordinates are invariant under affine transformations.

This means that if we apply an affine transformation to $\gamma(t)$, instead of applying the transformation to every single point in the curve, we just have to apply the transformation to the control points and then compute the curve with the new control points.

Affine transformation

$$T : \mathbb{R}^2 \to \mathbb{R}^2 \qquad T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$