# Lagrange polynomials in $P_n$

**Definition**

Given $x_0, x_1, \ldots, x_n$ with $x_i \neq x_j$ for $i \neq j$, the i-th Lagrange polynomial of degree $n$ is

$$L_i^n(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}, \qquad i = 0, 1, \ldots, n \tag{2}$$

**Theorem**

*$L_i^n(x_i) = 1$ and $L_i^n(x_j) = 0$ for $j \neq i$*

**Example:** $x_0 = -2$, $x_1 = -1$, $x_2 = 1$ and $x_3 = 4$. Compute the Lagrange polynomials and verify the theorem.

# Lagrange basis

**Theorem**

*$BL = \{L_0^n,\ L_1^n,\ \ldots,\ L_n^n\}$ is a basis for $P_n$.*

We have $|BL| = n + 1 = \dim(P_n)$ so we only need to prove linearly independence or spanning.

Let $p(x) = a_0 + a_1 x + \ldots + a_n x^n \in P_n$

I take $x_0, x_1, \ldots x_n$ distinct and evaluate $p$

I obatian $p(x_0) = y_0$, $p(x_1) = y_1$, …, $p(x_n) = y_n$

So $p$ is the unique polynomial of degree at most $n$ through $(x_0, y_0)$, $(x_1, y_1)$, …, $(x_n, y_n)$.

## Neville's method

Lagrange polynomials can be generated recursively in the following way

| *Nodes* | $p_i \in P_0$ |
|---|---|
| $x_0$ | $p_0(x) = y_0$ |
| $x_1$ | $p_1(x) = y_1$ |
| $x_2$ | $p_2(x) = y_2$ |
| $x_3$ | $p_3(x) = y_3$ |
| $x_4$ | $p_4(x) = y_4$ |

In $P_0$ there is a unique Lagrange polynomial $L_0^0(x) = 1$.

So the interpolant polynomials are $p_i(x) = y_i L_0^0(x) = y_i$

We use previous information for computing interpolant polynomials in $P_1$

| *Nodes* | $p_i \in P_0$ | $p_{i,i+1} \in P_1$ |
|---|---|---|
| $x_0$ | $p_0(x) = y_0$ | |
| | | $p_{0,1}(x) = \frac{(x-x_0)p_1(x)-(x-x_1)p_0(x)}{x_1-x_0}$ |
| $x_1$ | $p_1(x) = y_1$ | |
| | | $p_{1,2}(x) = \frac{(x-x_1)p_2(x)-(x-x_2)p_1(x)}{x_2-x_1}$ |
| $x_2$ | $p_2(x) = y_2$ | |
| | | $p_{2,3}(x) = \frac{(x-x_2)p_3(x)-(x-x_3)p_2(x)}{x_3-x_2}$ |
| $x_3$ | $p_3(x) = y_3$ | |
| | | $p_{3,4}(x) = \frac{(x-x_3)p_4(x)-(x-x_4)p_3(x)}{x_4-x_3}$ |
| $x_4$ | $p_4(x) = y_4$ | |

We obtain Lagrange interpolant polynomials of degree 1 through the nodes $(x_i, y_i)$ and $(x_{i+1}, y_{i+1})$.

## Polynomials

**Definition**

Let $a_0, a_1, a_2, \ldots, a_n \in \mathbb{R}$ with $a_n \neq 0$. A polynomial $p_n$ of defree $n$ over $\mathbb{R}$ is a function $p_n : \mathbb{R} \to \mathbb{R}$ of the form

$$p_n(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \tag{1}$$

## Interpolant polynomial

**Theorem**

*Given any $n+1$ points $(x_0, y_0)$, $(x_1, y_1)$, $(x_2, y_2)$, ..., $(x_n, y_n)$ with distinct $x$-coordinate, there is a unique polynomial of degree at most $n$ passing through them. We call it the* **interpolant polynomial** *through $(x_0, y_0)$, $(x_1, y_1)$, $(x_2, y_2)$, ..., $(x_n, y_n)$.*

Substituting the points in $p_n(x) = y$ we obtain a linear system of $n+1$ equations and $n+1$ unknowns $(a_0,\ a_1,\ \dots,\ a_n)$.

$$\begin{cases} a_0 + a_1 x_0 + a_2 x_0^2 + \ldots + a_n x_0^n = y_0 \\ a_0 + a_1 x_1 + a_2 x_1^2 + \ldots + a_n x_1^n = y_1 \\ a_0 + a_1 x_2 + a_2 x_2^2 + \ldots + a_n x_2^n = y_2 \\ \vdots \\ a_0 + a_1 x_n + a_2 x_n^2 + \ldots + a_n x_n^n = y_n \end{cases} \tag{2}$$

System (2) has unique solution because ...

## Vandermonde determinant

$$\det \begin{pmatrix} 1 & x_0 & x_0^2 & \ldots & x_0^n \\ 1 & x_1 & x_1^2 & \ldots & x_1^n \\ 1 & x_2 & x_2^2 & \ldots & x_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \ldots & x_n^n \end{pmatrix} = \prod_{0 \le i < j \le n} (x_j - x_i) \neq 0 \tag{3}$$

## The space $P_n$

**Definition**

The set of polynomials of degree at most $n$, denoted with $P_n$ is defined as

$$P_n = \{p_n : \mathbb{R} \to \mathbb{R} \mid p_n(x) = a_0 + a_1 x + \ldots + a_n x^n,\ a_0, a_1, \ldots, a_n \in \mathbb{R}\}$$

**Theorem**

$(P_n, +, \cdot)$ *is a vector space.*

## Vector space

**Definition**

Let $V$ be a set of objects on which two operations $\bigodot$ and $\bigstar$ are defined.

$\bigodot$ is a binary operator that associates to each pair of objects $u$ and $v$ in $V$ an object $u \bigodot v$.

$$u,\ v \in V \to u \bigodot v$$

$\bigstar$ is a single operator that associates with each object $u$ in $V$ and each scalar $k \in \mathbb{R}$ an object $k \bigstar u$.

$$u \in V \to k \bigstar u$$

The set $V$ with the operations $\bigodot$ and $\bigstar$ denoted with $(V, \bigodot, \bigstar)$ is called a **vector space**, and its elements are called **vectors** if the following axioms are satisfied:

## $(P_n, +, \cdot)$ is a vector space

In the previous definition:

$V = P_n = \{p_n : \mathbb{R} \to \mathbb{R} \mid p_n(x) = \sum_{i=0}^n a_i x^i$ for $a_i \in \mathbb{R}\}$

$\bigodot$ is the sum $+$ so for $p_n(x) = \sum_{i=0}^n a_i x^i$ and $q_n(x) = \sum_{i=0}^n b_i x^i \in P_n$

then $(p_n + q_n)(x) = \sum_{i=0}^n a_i x^i + \sum_{i=0}^n b_i x^i = \sum_{i=0}^n (a_i + b_i) x^i$

$\bigstar$ is the scalar product $\cdot$ so for $p_n(x) = \sum_{i=0}^n a_i x^i \in P_n$ and $k \in \mathbb{R}$

$(k \cdot p_n)(x) = k \sum_{i=0}^n a_i x^i = \sum_{i=0}^n k a_i x^i$

$(P_n, +, \cdot)$ satisfies the 10 axioms

## Definition

1. if $u,\ v \in V$, then $u \bigodot v \in V$
2. $u \bigodot v = v \bigodot u$
3. $u \bigodot (v \bigodot w) = (u \bigodot v) \bigodot w$
4. $\exists 0 \in V$ such that $\forall u \in V,\ u \bigodot 0 = 0 \bigodot u = u$
5. $\forall u \in V\ \exists -u \in V$ such that $u \bigodot -u = -u \bigodot u = 0$
6. if $k \in \mathbb{R}$ and $u \in V$, then $k \bigstar u \in V$
7. for $k, m \in \mathbb{R}$ and $u \in V$, $(km) \bigstar u = k \bigstar (m \bigstar u)$
8. $1 \bigstar u = u$
9. $(k+m) \bigstar u = (k \bigstar u) \bigodot (m \bigstar u)$
10. $k \bigstar (u \bigodot v) = (k \bigstar u) \bigodot (k \bigstar v)$

- if $p,\ q \in P_n$, then $p + q \in P_n$

  take $p(x) = \sum_{i=0}^n a_i x^i,\ q(x) = \sum_{i=0}^n b_i x^i \in P_n$ then

  $(p+q)(x) = \sum_{i=0}^n (a_i + b_i) x^i$ and we have that

  $(p+q) : \mathbb{R} \to \mathbb{R}$ and that $(a_i + b_i) \in \mathbb{R}$ for $i = 0, 1, \ldots, n$

  so $(p+q) \in P_n$ ✓

- $p + q = q + p$

  take $p(x) = \sum_{i=0}^n a_i x^i,\ q(x) = \sum_{i=0}^n b_i x^i \in P_n$ then

  $(p+q)(x) = \sum_{i=0}^n (a_i + b_i) x^i = \sum_{i=0}^n (b_i + a_i) x^i = (q+p)(x)$✓

- $\forall p \in P_n\ \exists -p \in P_n$ such that $p + (-p) = -p + p = 0$

  take $p(x) = \sum_{i=0}^n a_i x^i$ and $-p(x) = \sum_{i=0}^n -a_i x^i$ then

  $(p + (-p))(x) = \sum_{i=0}^n (a_i - a_i) x^i = \sum_{i=0}^n (-a_i + a_i) x^i =$

  $\sum_{i=0}^n 0 x^i = 0(x)$✓

- if $k \in \mathbb{R}$ and $p \in P_n$, then $k \cdot p \in P_n$

  take $p(x) = \sum_{i=0}^n a_i x^i$ and $k \in \mathbb{R}$ then

  $(k \cdot p)(x) = \sum_{i=0}^n k a_i x^i$ and we have that $(k \cdot p) : \mathbb{R} \to \mathbb{R}$ and that

  $k a_i \in \mathbb{R}$ for $i = 0, 1, \ldots, n$ so $(k \cdot p) \in P_n$✓

- $(k+m) \cdot p = (k \cdot p) + (m \cdot p)$

  take $p(x) = \sum_{i=0}^n a_i x^i$ and $k,\ m \in \mathbb{R}$ then

  $(k+m) \cdot p(x) = (k+m) \sum_{i=0}^n a_i x^i = \sum_{i=0}^n (k+m) a_i x^i =$

  $\sum_{i=0}^n (k a_i x^i + m a_i x^i) = \sum_{i=0}^n k a_i x^i + \sum_{i=0}^n m a_i x^i = kp(x) + mp(x)$✓

- $k \cdot (p+q) = (k \cdot p) + (k \cdot q)$

  take $p(x) = \sum_{i=0}^n a_i x^i,\ q(x) = \sum_{i=0}^n b_i x^i$ and $k \in \mathbb{R}$ then

  $k \cdot (p(x) + q(x)) = k \cdot (\sum_{i=0}^n a_i x^i + \sum_{i=0}^n b_i x^i) =$

  $k \cdot \sum_{i=0}^n (a_i + b_i) x^i = \sum_{i=0}^n k(a_i + b_i) x^i = \sum_{i=0}^n (k a_i x^i + k b_i x^i) =$

  $\sum_{i=0}^n k a_i x^i + \sum_{i=0}^n k b_i x^i = kp(x) + kq(x)$✓

- $p + (q + r) = (p + q) + r$

  take $p(x) = \sum_{i=0}^n a_i x^i,\ q(x) = \sum_{i=0}^n b_i x^i,\ r(x) = \sum_{i=0}^n c_i x^i \in P_n$

  then $(p + (q + r))(x) = \sum_{i=0}^n a_i x^i + (\sum_{i=0}^n b_i x^i + \sum_{i=0}^n c_i x^i) =$

  $\sum_{i=0}^n a_i x^i + \sum_{i=0}^n (b_i + c_i) x^i = \sum_{i=0}^n (a_i + b_i + c_i) x^i =$

  $\sum_{i=0}^n (a_i + b_i) x^i + \sum_{i=0}^n c_i x^i =$

  $(\sum_{i=0}^n a_i x^i + \sum_{i=0}^n b_i x^i) + \sum_{i=0}^n c_i x^i = ((p+q)+r)(x)$✓

- $\exists 0 \in P_n$ such that $\forall p \in P_n,\ p + 0 = 0 + p = p$

  take $p(x) = \sum_{i=0}^n a_i x^i$ and $0(x) = \sum_{i=0}^n 0 x^i = 0 \in P_n$ then

  $(p+0)(x) = \sum_{i=0}^n (a_i + 0) x^i = \sum_{i=0}^n (0 + a_i) x^i = \sum_{i=0}^n a_i x^i = p(x)$✓

- for $k, m \in \mathbb{R}$ and $p \in P_n$, $(km) \cdot p = k \cdot (m \cdot p)$

  take $p(x) = \sum_{i=0}^n a_i x^i$ and $k,\ m \in \mathbb{R}$ then

  $(km) \cdot p(x) = (km) \sum_{i=0}^n a_i x^i = \sum_{i=0}^n k m a_i x^i = k \sum_{i=0}^n m a_i x^i =$

  $k \cdot (m \cdot p(x))$✓

- $1 \cdot p = p$

  take $p(x) = \sum_{i=0}^n a_i x^i$ then

  $1 \cdot p(x) = 1 \cdot \sum_{i=0}^n a_i x^i = \sum_{i=0}^n 1 a_i x^i = \sum_{i=0}^n a_i x^i = p(x)$✓

## Properties of vector spaces

**Theorem**

*Let $(V, \bigodot, \bigstar)$ be a vector space, $\vec{u} \in V$ and $k \in \mathbb{R}$ a scalar, then*

a) $0 \bigstar \vec{u} = \vec{0}$

b) $k \bigstar \vec{0} = \vec{0}$

c) $-1 \bigstar \vec{u} = -\vec{u}$

d) *if* $k \bigstar \vec{u} = \vec{0}$ *then* $k = 0$ *or* $\vec{u} = \vec{0}$

$0p(x) = 0(x)$

$k0(x) = 0(x)$

$-1p(x) = -p(x)$

if $kp(x) = 0$ then $k = 0$ or $p(x) = 0(x)$



## Polynomial subspaces of vector spaces

**Theorem**

*Let $m, n \in \mathbb{Z}^+$ with $m < n$, then $P_m$ is a subspace of $P_n$.*

do you remember what is a subspace of a vector space?

**Definition**

Let $(V, \bigodot, \bigstar)$ be a vector space and $W \subseteq V$ a subset of $V$ $(\vec{w} \in W \rightarrow \vec{w} \in V)$.

$W$ is a **subspace** of $V$ if $(W, \bigodot, \bigstar)$ is a vector space.

So $(P_m, +, \cdot)$ is a vector space

# Subspace main theorem

**Theorem**

*Let $(V, \bigodot, \bigstar)$ be a vector space and $W \subseteq V$ a subset of $V$.*

*$(W, \bigodot, \bigstar)$ is a subspace of $(V, \bigodot, \bigstar)$ if and only if the following hold:*

1. *if $u,\ v \in W$, then $u \bigodot v \in W$*
2. *if $k \in \mathbb{R}$ and $u \in W$, then $k \bigstar u \in W$*

- if $p,\ q \in P_m$, then $p + q \in P_m$

  take $p(x) = \sum_{i=0}^{m} a_i x^i,\ q(x) = \sum_{i=0}^{m} b_i x^i \in P_m$ then

  $(p+q)(x) = \sum_{i=0}^{m}(a_i + b_i)x^i$ and we have that

  $(p+q) : \mathbb{R} \to \mathbb{R}$ and that $(a_i + b_i) \in \mathbb{R}$ for $i = 0, 1, \ldots, m$

  so $(p+q) \in P_m \checkmark$



- if $k \in \mathbb{R}$ and $p \in P_m$, then $k \cdot p \in P_m$

  take $p(x) = \sum_{i=0}^{m} a_i x^i$ and $k \in \mathbb{R}$ then

  $(k \cdot p)(x) = \sum_{i=0}^{m} k a_i x^i$ and we have that $(k \cdot p) : \mathbb{R} \to \mathbb{R}$ and that

  $k a_i \in \mathbb{R}$ for $i = 0, 1, \ldots, m$ so $(k \cdot p) \in P_m \checkmark$

## $P_n$ and $\mathbb{R}^{n+1}$ are isomorphic

Consider the standard basis $\{1, x, x^2, \ldots, x^n\}$ in which the vector of coordinates of a polynomial

$p_n(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \in P_n$

is $(a_0, a_1, a_2, \ldots, a_n) \in \mathbb{R}^{n+1}$ and is unique.

Then we can construct a transformation $T : P_n \to \mathbb{R}^{n+1}$ given by

$$T(a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n) = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \tag{1}$$

## $T : P_n \to \mathbb{R}^{n+1}$ is linear

**Definition**

Let $(V, \odot, \bigstar)$ and $(W, \odot, \bigstar)$ be real vector spaces. A transformation $T : V \to W$ is said to be linear if for all $\vec{u}, \vec{v} \in V$ and for all $k \in \mathbb{R}$ the following hold:

- $T(\vec{u} \odot \vec{v}) = T(\vec{u}) \odot T(\vec{v})$,
- $T(k \bigstar \vec{u}) = k \bigstar T(\vec{u})$.

For $T : P_n \to \mathbb{R}^{n+1}$ with the standard operations, $T$ is linear if:

- $T(p + q) = T(p) + T(q)$,
- $T(kp) = kT(p)$.

- $T(p+q) = T(p) + T(q)$

  Take $p(x) = \sum_{i=0}^{n} a_i x^i$ and $q(x) = \sum_{i=0}^{n} b_i x^i \in P_n$.

  We have that $(p+q)(x) = \sum_{i=0}^{n} (a_i + b_i) x^i$ so

$$T(p+q) = T\left(\sum_{i=0}^{n} (a_i + b_i) x^i\right) = \begin{pmatrix} a_0 + b_0 \\ a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{pmatrix} =$$

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = T\left(\sum_{i=0}^{n} a_i x^i\right) + T\left(\sum_{i=0}^{n} b_i x^i\right) = T(p) + T(q)$$

- $T(kp) = kT(p)$

  Take $p(x) = \sum_{i=0}^{n} a_i x^i$ and $k \in \mathbb{R}$.

  We have that $(kp)(x) = \sum_{i=0}^{n} (ka_i) x^i$ so

$$T(kp) = T\left(\sum_{i=0}^{n} (ka_i) x^i\right) = \begin{pmatrix} ka_0 \\ ka_1 \\ ka_2 \\ \vdots \\ ka_n \end{pmatrix} = k \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

$$= kT\left(\sum_{i=0}^{n} a_i x^i\right) = kT(p)$$

So to assign a vector of coordinates in $\mathbb{R}^{n+1}$ to a polynomial in $P_n$ is done through a linear transformation.

# $T : P_n \to \mathbb{R}^{n+1}$ is an isomorphism

**Definition**

Let $V$ and $W$ be vector spaces. A transformation $T : V \to W$ is said to be an isomorphism if $T$ is linear and invertible.

$T$ is invertible (is a one-to-one correspondence).

The inverse of $T$ is $T^{-1} : \mathbb{R}^{n+1} \to P_n$ given by

$$T^{-1}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \tag{2}$$

As we found an isomorphism, we say the spaces are isomorphic $P_n \cong \mathbb{R}^{n+1}$. does this work for any dimension or basis?

# A basis for $P_n$

**Theorem**

*Every set $B = \{p_i(x)\ |p_i \in P_n,\ i = 0, 1, \ldots, n\}$ satisfying*

- *$B$ is linearly independent*
- *$B$ spans $P_n$*

*is a basis for $P_n$.*

*Every polynomial $q \in P_n$ can be espressed as linear combination of elemets in $B$ in a unique way*

$$q(x) = a_0 p_0(x) + a_1 p_1(x) + \ldots + a_n p_n(x) \tag{3}$$

*then $q := (a_0, a_1, \ldots, a_n)_B$ is the vector of coordinates of $q$ in the basis $B$.*

## Shifted basis

**Definition**

Let $c \in \mathbb{R}$. A shifted basis of $P_n$ is

$$B = \{1,\ x - c,\ (x - c)^2,\ \dots,\ (x - c)^n\} \tag{4}$$

**Example:** $B = \{1,\ x - 3,\ (x - 3)^2\}$ basis for $P_2$.

Find the polynomial $(3, -2, 2)_B$

$p(x) = 3(1) - 2(x - 3) + 2(x - 3)^2 = 3 - 2x + 6 + 2(x^2 - 6x + 9) =$

$27 - 14x + 2x^2$

Find the vector of coordinates of $q(x) = 2 - 6x + x^2$ in $B$.

$2 - 6x + x^2 = k_1(1) + k_2(x - 3) + k_3(x - 3)^2$

$2 - 6x + x^2 = k_1(1) + k_2(x - 3) + k_3(x^2 - 6x + 9)$

$2 - 6x + x^2 = (k_1 - 3k_2 + 9k_3) + (k_2 - 6k_3)x + (k_3)x^2$

$$\begin{cases} k_1 - 3k_2 + 9k_3 = 2 \\ k_2 - 6k_3 = -6 \\ k_3 = 1 \end{cases} \Rightarrow \begin{cases} k_1 = -7 \\ k_2 = 0 \\ k_3 = 1 \end{cases}$$

$q := (-7, 0, 1)_B$

**Theorem**

*Every set $B = \{p_i(x)\ |p_i \in P_n,\ i = 0, 1, \dots, n\}$ satisfying that $B$ is linearly independent is a basis for $P_n$.*

as we have $|B| = n + 1 = \dim(P_n)$ spanning comes for free!

**Example:** A shifted basis is a basis for $P_n$

$$B = \{1,\ x - c,\ (x - c)^2,\ \ldots,\ (x - c)^n\}$$

We have that

$$(x - c)^k = \sum_{j=0}^{k} \binom{k}{j} (-c)^{k-j} x^j$$

Applying the isomorphism (1), $T((x - c)^k) = \begin{pmatrix} \binom{k}{0} (-c)^k \\ \binom{k}{1} (-c)^{k-1} \\ \vdots \\ \binom{k}{k} (-c)^0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

We apply the isomorphism to every $(x - c)^k$ for $k = 0, 1, \ldots, n$

Introducing the vectors in the columns of a square matrix we get

$$A = \begin{pmatrix} 1 & -c & c^2 & -c^3 & \ldots & (-c)^n \\ 0 & 1 & -2c & 3c^2 & \ldots & n(-c)^{n-1} \\ 0 & 0 & 1 & -3c & \ldots & \frac{n(n-1)}{2}(-c)^{n-2} \\ 0 & 0 & 0 & 1 & \ldots & \frac{n(n-1)(n-2)}{6}(-c)^{n-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 \end{pmatrix}$$

$\det(A) = 1 \neq 0$ so the $n + 1$ vectors are linearly independent and form a basis for $\mathbb{R}^{n+1}$.

Inverting the isomorphism the $n + 1$ polynomials are linearly independent and form a basis for $Pn$.

# Vandermonde basis

**Definition**

Let $c_0, c_1, c_2 \ldots, c_n \in \mathbb{R}$ with $c_i \neq c_j$ for $i \neq j$. A Vandermonde basis of $P_n$ is

$$B = \{(x - c_0)^n,\ (x - c_1)^n,\ (x - c_2)^n,\ \ldots,\ (x - c_n)^n\} \tag{5}$$

**Example:** $B = \{x^3,\ (x-1)^3,\ (x-2)^3,\ (x-3)^3\}$ basis for $P_3$.

Find the polynomial $(1, -2, 2, 1)_B$

$p(x) = 1(x)^3 - 2(x-1)^3 + 2(x-2)^3 + (x-3)^3 =$

$x^3 - 2(x^3 - 3x^2 + 3x - 1) + 2(x^3 - 6x^2 + 12x - 8) + (x^3 - 9x^2 + 27x - 27) =$

$-41 + 45x - 15x^2 + 2x^3$

Find the vector of coordinates of $q(x) = 2 - 6x + x^2 - x^3$ in $B$.

$2 - 6x + x^2 - x^3 = k_1(x)^3 + k_2(x-1)^3 + k_3(x-2)^3 + k_4(x-3)^3 =$

$k_1x^3 + k_2(x^3 - 3x^2 + 3x - 1) + k_3(x^3 - 6x^2 + 12x - 8) + k_4(x^3 - 9x^2 + 27x - 27)$

$= (-k_2 - 8k_3 - 27k_4) + (3k_2 + 12k_3 + 27k_4)x + (-3k_2 - 6k_3 - 9k_4)x^2$

$+(k_1 + k_2 + k_3 + k_4)x^3$

$$\begin{cases} -k_2 - 8k_3 - 27k_4 = 2 \\ 3k_2 + 12k_3 + 27k_4 = -6 \\ -3k_2 - 6k_3 - 9k_4 = 1 \\ k_1 + k_2 + k_3 + k_4 = -1 \end{cases} \Rightarrow \left(\begin{array}{cccc|c} 0 & -1 & -8 & -27 & 2 \\ 0 & 3 & 12 & 27 & -6 \\ 0 & -3 & -6 & -9 & 1 \\ 1 & 1 & 1 & 1 & -1 \end{array}\right)$$

$$\Rightarrow\ RREF \Rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -37/18 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -5/2 \\ 0 & 0 & 0 & 1 & 5/9 \end{array}\right) \Rightarrow \left(-\tfrac{37}{18}, 3, -\tfrac{5}{2}, \tfrac{5}{9}\right)_B$$

# Bernstein basis

**Definition**

The Bernstein polynomials of degree n, denoted with $B_0^n$, $B_1^n$, $\ldots$, $B_n^n$ are given as

$$B_i^n(x) = \left( \begin{array}{c} n \\ i \end{array} \right) (1-x)^{n-i} x^i \tag{6}$$

The set of the Bernstein polynomials

$$\{B_0^n(x),\ B_1^n(x),\ \ldots,\ B_n^n(x)\} \tag{7}$$

form a basis for $P_n$.

**Example:** The Bernstein polynomials of degree 3 are

$$B_0^3(x) = \left( \begin{array}{c} 3 \\ 0 \end{array} \right) (1-x)^3 x^0 = (1-x)^3 = 1 - 3x + 3x^2 - x^3$$

$$B_1^3(x) = \left( \begin{array}{c} 3 \\ 1 \end{array} \right) (1-x)^2 x^1 = 3(1-x)^2 x = 3(1-2x+x^2)x = 3x-6x^2+3x^3$$

$$B_2^3(x) = \left( \begin{array}{c} 3 \\ 2 \end{array} \right) (1-x)^1 x^2 = 3(1-x)x^2 = 3x^2 - 3x^3$$

$$B_3^3(x) = \left( \begin{array}{c} 3 \\ 3 \end{array} \right) (1-x)^0 x^3 = x^3$$

$B = \{1 - 3x + 3x^2 - x^3,\ 3x - 6x^2 + 3x^3,\ 3x^2 - 3x^3,\ x^3\}$ is a basis for $P_3$

Example: find the polynomial $(3, 0, 1, 1)_B$.

Solution: $3 - 9x + 12x^2 - 5x^3$

Example: find the vector of coordinates of $p(x) = 2 - 4x + x^2 - 5x^3$ in the Bernstein basis.

Solution: $\left(2, \frac{2}{3}, -\frac{1}{3}, -6\right)_B$

# Divided differences

Let $(x_0, y_0)$, $(x_1, y_1)$, ..., $(x_n, y_n)$ be points with $x_{i-1} < x_i$ for $i = 1, \ldots, n$ and let $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(x_i) = y_i$ (interpolant polynomial, but we use the f notation to be consistent with the existing literature)

**Definition**

The $n+1$ zeroth divided differences of $f$ for the nodes $(x_0, y_0)$, $(x_1, y_1)$, ..., $(x_n, y_n)$ are

$$f[x_i] = y_i, \qquad i = 0, 1, \ldots, n \tag{1}$$

**Definition**

The $n$ first divided differences of $f$ are

$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}, \qquad i = 0, 1, \ldots, n-1 \tag{2}$$

$$\begin{array}{ccc}
\textit{Nodes} & f[x_i] & f[x_i, x_{i+1}] \\
x_0 & f[x_0] = y_0 & \\
 & & f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} \\
x_1 & f[x_1] = y_1 & \\
 & & f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1} \\
x_2 & f[x_2] = y_2 & \\
 & & f[x_2, x_3] = \frac{f[x_3] - f[x_2]}{x_3 - x_2} \\
x_3 & f[x_3] = y_3 &
\end{array}$$

The $n-1$ second divided differences of $f$ are

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}, \qquad i = 0, 1, \dots, n-2 \quad (3)$$

| *Nodes* | $f[x_i]$ | $f[x_i, x_{i+1}]$ | $f[x_i, x_{i+1}, x_{i+2}]$ |
|---|---|---|---|
| $x_0$ | $f[x_0] = y_0$ | | |
| | | $f[x_0, x_1] = \frac{f[x_1]-f[x_0]}{x_1-x_0}$ | |
| $x_1$ | $f[x_1] = y_1$ | | $f[x_0, x_1, x_2] = \frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0}$ |
| | | $f[x_1, x_2] = \frac{f[x_2]-f[x_1]}{x_2-x_1}$ | |
| $x_2$ | $f[x_2] = y_2$ | | $f[x_1, x_2, x_3] = \frac{f[x_2,x_3]-f[x_1,x_2]}{x_3-x_1}$ |
| | | $f[x_2, x_3] = \frac{f[x_3]-f[x_2]}{x_3-x_2}$ | |
| $x_3$ | $f[x_3] = y_3$ | | |

$$\begin{array}{ccccc}
\textit{Nodes} & f[x_i] & f[x_i,x_{i+1}] & f[x_i,x_{i+1},x_{i+2}] & f[x_i,x_{i+1},x_{i+2},x_{i+3}] \\
x_0 & f[x_0] & & & \\
 & & f[x_0,x_1] & & \\
x_1 & f[x_1] & & f[x_0,x_1,x_2] & \\
 & & f[x_1,x_2] & & f[x_0,x_1,x_2,x_3] \\
x_2 & f[x_2] & & f[x_1,x_2,x_3] & \\
 & & f[x_2,x_3] & & \\
x_3 & f[x_3] & & &
\end{array}$$

$f[x_0, x_1, x_2, x_3] = \frac{f[x_1,x_2,x_3]-f[x_0,x_1,x_2]}{x_3-x_0}$

## Divided differences in a general setting

Having $(x_0, y_0)$, $(x_1, y_1)$, ..., $(x_n, y_n)$ points with $x_{i-1} < x_i$ for $i = 1, \ldots, n$ and $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(x_i) = y_i$,

We compute the $n + 1$ zeroth divided differences

We compute the $n$ first divided differences

We compute the $n - 1$ second divided differences

## Newton's interpolant polynomial

**Theorem**

*Given $(x_0, y_0)$, $(x_1, y_1)$, ..., $(x_n, y_n)$ points with $x_{i-1} < x_i$ for $i = 1, \ldots, n$, the interpolant polynomial through them can be written as*

$$p(x) = f[x_0] + \sum_{k=1}^{n} \left( f[x_0, \ldots, x_k] \prod_{i=0}^{k-1} (x - x_i) \right) \tag{6}$$

In this construction we have on the one hand the first divided differences of each order

$$\begin{array}{ccccc}
\color{red}{f[x_0]} & & & & \\
 & \color{red}{f[x_0,x_1]} & & & \\
f[x_1] & & \color{red}{f[x_0,x_1,x_2]} & & \\
 & f[x_1,x_2] & & \ddots & \\
f[x_2] & & f[x_1,x_2,x_3] & \cdots & \color{red}{f[x_0,\ldots,x_n]} \\
\vdots & \vdots & \vdots & & \\
f[x_n] & & & &
\end{array} \tag{7}$$

# Newton polynomials

and on the other hand the Newton basis for $P_n$.

**Definition**

Given $x_0$, $x_1$, ..., $x_n$ with $x_{i-1} < x_i$ for $i = 1, \ldots, n$, the Newton polynomials are:

$N_0(x) = 1$

$N_1(x) = x - x_0$

$N_2(x) = (x - x_0)(x - x_1)$

$\vdots$

$N_n(x) = \prod_{i=0}^{n-1}(x - x_i)$

# The Newton basis

**Theorem**

*The set of Newton polynomials $B_N = \{N_0,\ N_1,\ \ldots,\ N_n\}$ form a basis for $P_n$.*

Construct the interpolant polynomial through $(-2, 1)$, $(-1, 3)$, $(1, 0)$ and $(4, -2)$ using divided differences and the Newton basis.

We already have the divided differences:

$$f[x_0] = 1, \qquad f[x_0, x_1] = 2, \qquad f[x_0, x_1, x_2] = -\frac{7}{6}, \qquad f[x_0, x_1, x_2, x_3] = \frac{2}{9}$$

The Newton polynomials are:

$$N_0(x) = 1, \qquad N_1(x) = x + 2, \qquad N_2(x) = (x + 2)(x + 1),$$

$$N_3(x) = (x + 2)(x + 1)(x - 1)$$

# Hermite interpolant polynomials

**Definition**

Let $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ with $x_i \neq x_j$ for $i \neq j$. Let $z_i \in \mathbb{R}$ for $i = 0, 1, \ldots, n$. A Hermite interpolant polynomial $p$ satisfies:

- $p(x_i) = y_i$ for $i = 0, 1, \ldots, n$.
- $p'(x_i) = z_i$ for some $i$.

**Example:** Find a polynomial through $(0,0)$, $(1,1)$, $(2,-1)$ satisfying $p'(0) = 0$ and $p'(2) = 0$.
Building with the conditions a system of equations we get:

$$\begin{cases} p(0) = 0 \\ p(1) = 1 \\ p(2) = -1 \\ p'(0) = 0 \\ p'(2) = 0 \end{cases} \quad \text{We have 5 equations.}$$

With the evaluation of a polynomial

$$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

and its first derivative $p'(x) = a_1 + 2a_2 x + \ldots + n a_n x^{n-1}$

we can obtain a linear system of equations.

The resulting equations will be independent, therefore for having a unique solution we need 5 unknowns. So we look for $p \in P_4$.

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

$$p'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3$$

Our system of equations will be:

$$\begin{cases} a_0 = 0 \\ a_0 + a_1 + a_2 + a_3 + a_4 = 1 \\ a_0 + 2a_1 + 4a_2 + 8a_3 + 16a_4 = -1 \\ a_1 = 0 \\ a_1 + 4a_2 + 12a_3 + 32a_4 = 0 \end{cases}$$

Solving the system

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 12 & 32 & 0 \end{array}\right) \Rightarrow RREF \Rightarrow \left(\begin{array}{c|c} & 0 \\ & 0 \\ I_5 & \frac{21}{4} \\ & -\frac{23}{4} \\ & \frac{3}{2} \end{array}\right)$$

$$p(x) = \frac{21}{4}x^2 - \frac{23}{4}x^3 + \frac{3}{2}x^4$$

We can extend Hermite interpolation to a higher order of derivatives

**Definition**

Let $(x_0, y_{00}), (x_1, y_{10}), \ldots, (x_n, y_{n0})$ with distinct x-component. Let $y_{ij} \in \mathbb{R}$ for $i = 0, 1, \ldots, n$ and $j = 0, \ldots, r_i$. An osculating interpolant polynomial $p$ satisfies:

$$p^{(j)}(x_i) = y_{ij}, \qquad i = 0, 1, \ldots, n, \qquad j = 0, \ldots, r_i \tag{1}$$

Notation: $(j)$ denotes the order of the derivative, $p^{(0)}$ is the original polynomial, $p^{(1)}$ its first derivative, $p^{(2)}$ its second derivative and so on.

Example: we want a polynomial passing through $(0,0)$, $(1,1)$ and $(2,-1)$, where $(0,0)$ and $(2,-1)$ are local maximum points.

As the polynomial passes through $(0,0)$, $(1,1)$ and $(2,-1)$ we have

$$\begin{cases} p(0) = 0 \\ p(1) = 1 \\ p(2) = -1 \end{cases}$$

$(0,0)$ and $(2,-1)$ are local maximum points, therefore $\begin{cases} p'(0) = 0 \\ p'(2) = 0 \\ p''(0) = -a \\ p''(2) = -b \end{cases}$

for $a, b > 0$. Let's take for instance $a = b = 20$.

We have 7 conditions, therefore we will construct a linear system of 7 equations with 7 unknowns.

Our resulting polynomial will be in $P_6$.

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6$$

$$p'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + 6a_6x^5$$

$$p''(x) = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4$$

Therefore the system will be

$$\begin{cases} a_0 = 0 \\ a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 1 \\ a_0 + 2a_1 + 4a_2 + 8a_3 + 16a_4 + 32a_5 + 64a_6 = -1 \\ a_1 = 0 \\ a_1 + 4a_2 + 12a_3 + 32a_4 + 80a_5 + 192a_6 = 0 \\ 2a_2 = -20 \\ 2a_2 + 12a_3 + 48a_4 + 160a_5 + 480a_6 = -20 \end{cases}$$

# Existence and uniqueness

**Theorem**

*The osculating polynomial of degree $\sum_{i=0}^{n} r_i + n$ exists and it is unique.*

We skip the proof in this course (because of really hard algebraic notation), but if you have curiosity you can look for "Confluent Vandermonde determinants" in the internet.

This way of computing osculating interpolant polynomials is quite inefficient (build a matrix, compute RREF, round off errors...) Is there any other way?

Yes, using divided differences and Newton osculating polynomials

# Divided differences & Newton osculating polynomials

The divided differences with derivatives follow the **Taylor expansion**.

**Theorem**

*Let $f : \mathbb{R} \to \mathbb{R}$ with $f \in C^n[a, b]$ and let $x_0 \in (a, b)$. We can approximate $f$ at a point $x$ near $x_0$ with a polynomial of degree $n$*

$$f(x) \simeq f(x_0) + (x - x_0)f'(x_0) + (x - x_0)^2 \frac{f''(x_0)}{2} + \ldots + (x - x_0)^n \frac{f^{(n)}(x_0)}{n!}$$
$$= \sum_{k=0}^{n} (x - x_0)^k \frac{f^{(k)}(x_0)}{k!}$$

Next we introduce the zeroth divided differences $f[x_i] = y_i$

| $x_i$ | $f[x_i]$ |
|---|---|
| 0 | 0 |
| 0 | 0 |
| 0 | 0 |
| 1 | 1 |
| 2 | $-1$ |
| 2 | $-1$ |

If one node is repeated we can not compute the first divided difference through $f[x_i, x_i] = \frac{f[x_i] - f[x_i]}{x_i - x_i}$

We compute it through a limit

$$f[x_i, x_i] = \lim_{h \to 0} f[x_i, x_i + h] = \lim_{h \to 0} \frac{f(x_i + h) - f(x_i)}{h} = f'(x_i)$$

In a general setting, if a node is repeated n times (i.e. if it appears n+1 times), the nth divided difference is

$$f[x_i, x_i, \ldots, x_i]_n = \lim_{h \to 0} \frac{f[x_i, x_i, \ldots, x_i + h]_{n-1} - f[x_i, x_i, \ldots, x_i]_{n-1}}{h}$$

$$= \frac{f^{(n)}(x_i)}{n!}$$

We finish the computation of the divided differences, now that we do not have repetitions.

We take the first divided difference of each order

| $x_i$ | $f[x_i]$ | $f[x_i, x_j]$ | | $\dots$ | | |
|---|---|---|---|---|---|---|
| 0 | 0 | | | | | |
| | | 0 | | | | |
| 0 | 0 | | 0 | | | |
| | | 0 | | 1 | | |
| 0 | 0 | | 1 | | $-\frac{9}{8}$ | |
| | | 1 | | $-\frac{5}{4}$ | | $\frac{21}{16}$ |
| 1 | 1 | | $-\frac{3}{2}$ | | $\frac{3}{2}$ | |
| | | $-2$ | | $\frac{7}{4}$ | | |
| 2 | $-1$ | | 2 | | | |
| | | 0 | | | | |
| 2 | $-1$ | | | | | |

The Newton basis in this case is

$$B_N = \{1, x, x^2, x^3, x^3(x-1), x^3(x-1)(x-2)\}$$

# Newton basis in the general case

## Definition

The Newton basis for an osculating polynomial satisfying

$$p^{(j)}(x_i) = y_{ij}, \qquad i = 0, 1, \dots, n, \qquad j = 0, \dots, r_i$$

is

$$B_N = \{1, (x-x_0), \dots, (x-x_0)^{r_0+1}, (x-x_0)^{r_0+1}(x-x_1),$$
$$\dots, (x-x_0)^{r_0+1}(x-x_1)^{r_1+1}, \dots, \left(\prod_{i=0}^{n-1}(x-x_i)^{r_i+1}\right),$$
$$\dots, \left(\prod_{i=0}^{n-1}(x-x_i)^{r_i+1}\right)(x-x_n)^{r_n}\}$$

$$|B_N| = (n+1) + \sum_{i=0}^{n} r_i$$

In the previous example $n = 2$ (3 nodes), $r_0 = 2$, $r_1 = 0$ and $r_2 = 1$.
$2 + 1 + 2 + 0 + 1 = 6 = |B_N|$

Returning to our example, if we multiply the divided differences by the Newton basis we get the osculating interpolant polynomial:

$$p(x) = x^3 - \frac{9}{8}x^3(x-1) + \frac{21}{16}x^3(x-1)(x-2)$$

which in the Newton basis has coordinates

$$(0, 0, 0, 1, -\frac{9}{8}, \frac{21}{16})_{N_B}$$

Notice that the construction of the polynomial is similar to the interpolant polynomial in the Newton basis without derivatives, i.e.

Obtain an interpolant polynomial satisfying $p(0) = 0$, $p'(0) = 0$, $p''(0) = 0$, $p(1) = 1$, $p(2) = -1$ and $p'(2) = 0$.

## Piecewise polynomials

**Definition**

A piecewise polynomial in an interval $[a, b]$ is a function

$$p : [a, b] \to \mathbb{R}$$

that is piecewise defined on $[a, x_1) \cup [x_1, x_2) \cup \ldots \cup [x_{n-1}, b]$ through polynomials $p_1 : [a, x_1) \to \mathbb{R}$, $p_2 : [x_1, x_2) \to \mathbb{R}$, ..., $p_n : [x_{n-1}, b] \to \mathbb{R}$ in the following way

$$p(x) = \begin{cases} p_1(x), & x \in [a, x_1), \\ p_2(x), & x \in [x_1, x_2), \\ \vdots \\ p_n(x), & x \in [x_{n-1}, b]. \end{cases} \tag{1}$$

## The vector space

Considering piecewise polynomials of the type

$$p(x) = \begin{cases} p_1(x), & x \in [a, x_1), \\ p_2(x), & x \in [x_1, x_2), \\ \vdots \\ p_n(x), & x \in [x_{n-1}, b], \end{cases}$$

for defining a **FINITE** dimensional vector space we **have to fix the intervals** $[a, x_1)$, $[x_1, x_2)$, ... $[x_{n-1}, b]$

Therefore denoting with $x_0 = a$ and $x_n = b$, the vector space to define depends on a mesh of nodes $x_0, x_1, \ldots, x_{n-1}, x_n$.

Next, $p_1, p_2, \ldots, p_n \in P_k$ **we have to bound the degree for the polynomials**.

**Definition**

Let $x_0 < x_1 < \ldots < x_{n-1} < x_n \in \mathbb{R}$. The set of piecewise polynomials $p : [x_0, x_n] \to \mathbb{R}$ given as

$$p(x) = \begin{cases} p_1(x), & x \in [x_0, x_1), \\ p_2(x), & x \in [x_1, x_2), \\ \vdots & \\ p_n(x), & x \in [x_{n-1}, x_n], \end{cases}$$

with $p_i \in P_k$ for $i = 1, \ldots, n$ is denoted with $P_k^n[x_0, \ldots, x_n]$.

**Theorem**

*$P_k^n[x_0, \ldots, x_n]$ is a vector space.*

The proof of the theorem consists of verifying the 10 axioms.

# The dimension of the space

**Theorem**

*The dimension of $P_k^n[x_0, \ldots, x_n]$ is $n(k+1)$.*

This is easy to check by constructing a basis of polynomials

$$p_i^j(x) = \begin{cases} x^j, & x \in [x_{i-1}, x_i) \\ 0, & x \notin [x_{i-1}, x_i) \end{cases}, \quad i = 1, \ldots, n, \quad j = 0, 1, \ldots, k \qquad (2)$$

where $i$ denotes the interval where the piecewise polynomial is not zero, and $j$ the exponent of the standard basis.

We have then $n(k+1)$ polynomials that span $P_k^n[x_0, \ldots, x_n]$, as every polynomial in $P_k^n[x_0, \ldots, x_n]$ can be written as linear combination of them. Moreover, the polynomials are linearly independent as none of them can be written as linear combination of the others. Therefore form a basis. As the basis has $n(k+1)$ elements, that is the dimension of the space.

**Example:**

$$p(x) = \begin{cases} 3 + 2x^2 + 5x^3, \ x \in [-1, 4) \\ 4 + x^2, \ x \in [4, 6] \end{cases} \qquad p \in P_3^2[-1, 4, 6]$$

$\dim(P_3^2[-1, 4, 6]) = 2(3 + 1) = 8$ and a basis for $P_3^2[-1, 4, 6]$ is

$$p_1^0(x) = \begin{cases} 1, \ x \in [-1, 4) \\ 0, \ x \in [4, 6] \end{cases} \quad p_2^0(x) = \begin{cases} 0, \ x \in [-1, 4) \\ 1, \ x \in [4, 6] \end{cases}$$

$$p_1^1(x) = \begin{cases} x, \ x \in [-1, 4) \\ 0, \ x \in [4, 6] \end{cases} \quad p_2^1(x) = \begin{cases} 0, \ x \in [-1, 4) \\ x, \ x \in [4, 6] \end{cases}$$

$$p_1^2(x) = \begin{cases} x^2, \ x \in [-1, 4) \\ 0, \ x \in [4, 6] \end{cases} \quad p_2^2(x) = \begin{cases} 0, \ x \in [-1, 4) \\ x^2, \ x \in [4, 6] \end{cases}$$

$$p_1^3(x) = \begin{cases} x^3, \ x \in [-1, 4) \\ 0, \ x \in [4, 6] \end{cases} \quad p_2^3(x) = \begin{cases} 0, \ x \in [-1, 4) \\ x^3, \ x \in [4, 6] \end{cases}$$

$B = \{p_1^0, p_1^1, p_1^2, p_1^3, p_2^0, p_2^1, p_2^2, p_2^3\}$ is this a good basis? NO!



Why not? Lets have a look to piecewise polynomials with three intervals.

$$p(x) = \begin{cases} p_1(x), & x \in [x_0, x_1) \\ p_2(x), & x \in [x_1, x_2) \\ p_3(x), & x \in [x_2, x_3] \end{cases} \qquad p \in P_2^3[x_0, x_1, x_2, x_3]$$

We have $\dim(P_2^3[x_0, x_1, x_2, x_3]) = 3(2 + 1) = 9$ and a basis can be

$B = \{p_1^0, p_1^1, p_1^2, p_2^0, p_2^1, p_2^2, p_3^0, p_3^1, p_3^2\}$ with

$$p_1^0(x) = \begin{cases} 1, & x \in [x_0, x_1) \\ 0, & x \in [x_1, x_2) \\ 0, & x \in [x_2, x_3] \end{cases} \qquad p_1^1(x) = \begin{cases} x, & x \in [x_0, x_1) \\ 0, & x \in [x_1, x_2) \\ 0, & x \in [x_2, x_3] \end{cases}$$

$$p_1^2(x) = \begin{cases} x^2, & x \in [x_0, x_1) \\ 0, & x \in [x_1, x_2) \\ 0, & x \in [x_2, x_3] \end{cases} \qquad p_2^0(x) = \begin{cases} 0, & x \in [x_0, x_1) \\ 1, & x \in [x_1, x_2) \\ 0, & x \in [x_2, x_3] \end{cases}$$

$$p_2^1(x) = \begin{cases} 0, & x \in [x_0, x_1) \\ x, & x \in [x_1, x_2) \\ 0, & x \in [x_2, x_3] \end{cases} \qquad p_2^2(x) = \begin{cases} 0, & x \in [x_0, x_1) \\ x^2, & x \in [x_1, x_2) \\ 0, & x \in [x_2, x_3] \end{cases}$$

$$p_3^0(x) = \begin{cases} 0, & x \in [x_0, x_1) \\ 0, & x \in [x_1, x_2) \\ 1, & x \in [x_2, x_3] \end{cases} \qquad p_3^1(x) = \begin{cases} 0, & x \in [x_0, x_1) \\ 0, & x \in [x_1, x_2) \\ x, & x \in [x_2, x_3] \end{cases}$$

$$p_3^2(x) = \begin{cases} 0, & x \in [x_0, x_1) \\ 0, & x \in [x_1, x_2) \\ x^2, & x \in [x_2, x_3] \end{cases}$$

The above functions have two discontinuities.

Discontinuities are a nightmare from a computational perspective!



In a space $P_k^n[x_0, x_1, \ldots, x_n]$ with $n$ intervals we have $n - 1$ discontinuities. Using a basis with

$$p_i^j(x) = \begin{cases} x^j, & x \in [x_{i-1}, x_i) \\ 0, & x \notin [x_{i-1}, x_i) \end{cases}, \quad i = 1, \ldots, n, \quad j = 0, 1, \ldots, k$$

we have $n$ intervals for each polynomial of the basis.

From an analytical perspective it is better to work with less discontinuities (the less the better) and from a computational perspective is better to work with smaller arrays.

**Moreover, if we want to construct a subspace to $P_k^n[x_0, x_1, \ldots, x_n]$ by imposing continuity and differentiability conditions at $x_1, \ldots, x_{n-1}$, it is not easy to find a basis for that subspace if we work with this type of bases.**

We will work with shifted power polynomial bases.

# Shifted power polynomial functions

**Definition**

Let $c \in \mathbb{R}$. The right continuous shifted power polynomial function of degree $n$ is

$$(x-c)_+^n = \begin{cases} 0, & x < c \\ (x-c)^n, & x \geq c \end{cases} \tag{3}$$

The left continuous shifted power polynomial function of degree $n$ is

$$(c-x)_+^n = \begin{cases} (c-x)^n, & x \leq c \\ 0, & x > c \end{cases} \tag{4}$$

**Definition**

The standard basis for $P_k^n[x_0, x_1, \dots, x_n]$ is

$B = \{1,\ x,\ \dots,\ x^k,\ (x-x_1)_+^0,\ (x-x_1)_+^1,\ \dots,\ (x-x_1)_+^k,\ \dots,$
$(x-x_{n-1})_+^0,\ (x-x_{n-1})_+^1,\ \dots,\ (x-x_{n-1})_+^k\}$



Show that $B = \{1, x, x^2, x^3, (x-4)_+^0, (x-4)_+^1, (x-4)_+^2, (x-4)_+^3\}$ is a basis for $P_3^2[-1, 4, 6]$.

We have to show:

linear independence: if

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 (x-4)_+^0 + a_5 (x-4)_+^1 + a_6 (x-4)_+^2 + a_7 (x-4)_+^3 = 0$$

then $a_0 = a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a_7 = 0$

spanning: if any $p \in P_3^2[-1, 4, 6]$ can be written as linear combination of elements in $B$.

$$p(x) = \begin{cases} 3 + 2x^2 + 5x^3, \ x \in [-1, 4) \\ 4 + x^2, \ x \in [4, 6] \end{cases} \qquad p \in P_3^2[-1, 4, 6]$$

$$B = \{1, x, x^2, x^3, (x-4)_+^0, (x-4)_+^1, (x-4)_+^2, (x-4)_+^3\}$$

Give the vector of coordinates of $p$ in the basis $B$.

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 (x-4)_+^0 + a_5 (x-4)_+^1 + a_6 (x-4)_+^2 + a_7 (x-4)_+^3$$

For $x \in [-1, 4)$

$$3 + 2x^2 + 5x^3 = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$a_0 = 3, \ a_1 = 0, \ a_2 = 2, \ a_3 = 5$$

For $x \in [4, 6]$

$$4 + x^2 = 3 + 2x^2 + 5x^3 + a_4 + a_5(x-4) + a_6(x^2 - 8x + 16) + a_7(x^3 - 12x^2 + 48x - 64)$$

$$1 - x^2 - 5x^3 = (a_4 - 4a_5 + 16a_6 - 64a_7) + (a_5 - 8a_6 + 48a_7)x + (a_6 - 12a_7)x^2 + a_7 x^3$$

Construct a linear system

$$\begin{cases} a_4 - 4a_5 + 16a_6 - 64a_7 = 1 \\ a_5 - 8a_6 + 48a_7 = 0 \\ a_6 - 12a_7 = -1 \\ a_7 = -5 \end{cases} \Rightarrow \left(\begin{array}{cccc|c} 1 & -4 & 16 & -64 & 1 \\ 0 & 1 & -8 & 48 & 0 \\ 0 & 0 & 1 & -12 & -1 \\ 0 & 0 & 0 & 1 & -5 \end{array}\right)$$

$$RREF \Rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -335 \\ 0 & 1 & 0 & 0 & -248 \\ 0 & 0 & 1 & 0 & -61 \\ 0 & 0 & 0 & 1 & -5 \end{array}\right)$$

Vector of coordinates $(3, 0, 2, 5, -335, -248, -61, -5)$