MAT 300 Supplementary Notes Spring 2013

(Note: These notes are meant to supplement the lectures. Full lecture notes can only be obtained by attending class lectures and taking notes.)

Lecture 1, T Jan.8, 2013

Main Points:

- **Syllabus - grading, homework and quiz policies.**
- **Introduction to Project Part I - Bezier Curves, Nested Linear Interpolation**
- **Review of Linear Algebra**
- **Introduction to Vector Spaces of Polynomials and bases**

Refer to project specification for full details. In class we discuss the basic steps and requirements without full justification, which will come later. The idea is to get going quickly on the project.

The projects are all about different methods for rendering curves in 2D. The curves that we deal with in this course are either parametric polynomial or parametric piecewise polynomial. We also discuss some implicit curves briefly.

- A 2D parametric polynomial curve is:
$$\gamma(t) = (p_1(t), p_2(t))$$
where $p_1(t)$ and $p_2(t)$ are polynomials.
- A 3D parametric polynomial curve is:
$$\gamma(t) = (p_1(t), p_2(t), p_3(t))$$
where $p_1(t)$, $p_2(t)$ and $p_3(t)$ are polynomials.
- A 2D parametric piecewise polynomial curve is:
$$\gamma(t) = (f_1(t), f_2(t))$$
where $f_1(t)$, and $f_2(t)$ are piecewise polynomials.
- An 2D implicit curve is given by an equation $f(x, y) = 0$, such as the unit circle: $x^2 + y^2 - 1 = 0$.

Algorithms for rendering curves can use explicit polynomial formulas like $\gamma(t) = (p_1(t), p_2(t))$, or they can use other methods, as we see below, such as Nested Linear Interpolation.

In the project, you will design an interface (or use one of the frameworks) which brings up a window and allows the user to click on points to select *input points*, which are either control points (which influence the shape of the curve) or interpolation points (which the curve must pass through). The points are typically labelled P_0, $P_1, \ldots, P_d$, in the order in which they are clicked by the user. In the first part of the project, for each of 3 subparts, you use the control points to generate a Bezier curve. The method is different for each subpart, but the curve should be the same if you work with high enough resolution.

Nested Linear Interpolation (Refer to Project Part I, subpart 1)

- t-values should be chosen and one point generated on the curve for each t. It is up to you to choose enough t-values to the make the curve look smooth.
- the basic interval for the t-values is $[0, 1]$ (optionally you can extend the curve to fill the screen with $t < 0$ and $t > 1$.)
- basic linear interpolation between points: $(1-t)P_0 + tP_1$. (This is an affine sum since the coefficients $1-t$ and t add up to 1.)
- nested linear interpolation for three points:

$$\gamma_{[P_0,P_1,P_2]}(t) = (1-t)\left[(1-t)P_0 + tP_1\right] + t\left[(1-t)P_1 + tP_2\right]$$

- recursive form for Nested Linear Interpolation:

$$\gamma(t) = \gamma_{[P_0,P_1\ldots P_d]}(t) = (1-t)\gamma_{[P_0,P_1\ldots P_{d-1}]}(t) + t\gamma_{[P_1\ldots P_d]}(t).$$

- Bezier point recursion for stages $k = 1, \ldots, d$, with base case $P_i^0 = P_i$ (control points)

$$P_i^k = (1-t)P_i^{k-1} + tP_{i+1}^{k-1}.$$

- Array of Bezier Points for a degree three Bezier curve with 4 control points P_0, P_1, P_2 and P_3: (and some chosen value of t)

$$\begin{array}{cccc}
P_0^0 & & & \\
 & P_0^1 & & \\
P_1^0 & & P_0^2 & \\
 & P_1^1 & & P_0^3 = \gamma(t) \\
P_2^0 & & P_1^2 & \\
 & P_2^1 & & \\
P_3^0 & & &
\end{array}$$

- Note: for any triangle of 3 points in the Bezier point diagram:

$$\begin{array}{cc}
P & \\
 & R \\
Q &
\end{array}$$

  R is obtained as: $R = (1-t)P + tQ$.
- Note: when computing $\gamma(t)$ it is important to use arrays based on Bezier points instead of recursive functions (even though the recursive form is nice to write down.)
- the **polyline** is the collection of line segments joining the control points together in order from P_0 to P_d.
- the **shells** consist of the polyline together with the line segments joining the Bezier points for each stage (superscript) in order from P_0^k to P_{d-k}^k.
- Note: This is the ONLY subpart of the Project Part I which uses the shells. The other parts do not use shells.

BB-(Bernestein-Bezier) form (Refer to Project Part I, subpart 2)

- BB-form:

$$\gamma(t) = \gamma_{[P_0, P_1 \dots P_d]}(t) = \sum_{i=0}^{d} B_i^d(t) P_i.$$

- Bernstein polynomials:

$$B_i^d(t) = \binom{d}{i} (1-t)^{d-i} t^i, \quad i = 0, \dots, d,$$

where the binomial coefficient $\binom{d}{i}$ is defined as:

$$\binom{d}{i} = \frac{d!}{(d-i)!i!}.$$

- Note that the BB-form is an affine sum, since the coefficients $B_i^d(t)$ all add up to 1 (to be proved later)
- Binomial coefficients should be computed with Pascal's Identity: (not factorials)

$$\binom{d}{i} = \binom{d-1}{i-1} + \binom{d-1}{i},$$

with base cases: $\binom{d}{0} = \binom{d}{d} = 1$.

Important Points to Review from Linear Algebra:

- linear independence (for a finite set of vectors)
  - a set of n vectors $\{v_1, \dots, v_n\}$ in a vector space V is linearly *independent* if the vector equation:

  $$c_1 v_1 + \cdots + c_n v_n = 0_V$$

  is only true when all the coefficients are zero. (The coefficients are real number scalars, and 0_V represents the zero vector for the vector space V.)
  - a set of n column vectors in R^n is linearly independent if and only if the $n \times n$ matrix of column vectors is nonsingular, which means it has nonzero determinant.
- linear dependence (for a finite set of vectors)
  - a set of n vectors $\{v_1, \dots, v_n\}$ in a vector space V is linearly *dependent* if it is not linearly independent.
  - equivalently, a set of n vectors $\{v_1, \dots, v_n\}$ in a vector space V is linearly *dependent* if the vector equation

  $$c_1 v_1 + \cdots + c_n v_n = 0_V$$

  has a nontrivial solution, which means that some coefficients can be found which are not all zero and which make the equation true.
  - equivalently, a set of n vectors $\{v_1, \dots, v_n\}$ in a vector space V is linearly *dependent* if at least one of the vectors can be written as a linear combination of the others.
- spanning property (for a finite set of vectors), and the $Span(S)$.
  - a set of n vectors $\{v_1, \dots, v_n\}$ in a vector space V is said to *span* V if any vector in V can be written as a linear combination of $v_1, \dots, v_n$.
  - The set $Span(S)$ for $S = \{v_1, \dots, v_n\}$ in a vector space V means the set of linear combinations of vectors in S.
  - The set $Span(S)$ is always a vector sybspace of V.
- basis and dimension

- a set of n vectors $\{v_1, \ldots, v_n\}$ in a vector space V is a basis of V if it is both linearly independent and spans V. (Note: Any basis must have the same number of vectors.)
- The dimension of a vector space V is the number of vectors in a basis.

- coordinate vectors with respect to a basis
  - If $B = \{v_1, \ldots, v_n\}$ is a basis of a vector space V and v is any vector in V, then we can write

$$v = c_1 v_1 + \cdots + c_n v_n$$

for some real scalar coefficients. The column vector

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

is then called the *coordinate vector* for v with respect to the basis B.

- change of basis matrix
  - If B_1 and B_2 are bases of a vector space V, and a vector v in V has coordinate vectors $\mathbf{x}_1$ and $\mathbf{x}_2$, then these column vectors are related by the equation

$$A\mathbf{x}_1 = \mathbf{x}_2$$

where A is an invertible matrix called the *change of basis matrix* from B_1 to B_2. Also, A^{-1} is then the change of basis matrix from B_2 to B_1.

- determinant of a square matrix (with cofactors)
  - the 2×2 determinant formula:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

  - the 3×3 determinant formula:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

- determinant of a triangular matrix
  - A matrix is called *upper triangular* if the entries below the main diagonal are zero, and *lower triangular* if the entries above the main diagonal are zero. A matrix which is upper or lower triangular is called a *triangular* matrix. A matrix which is both upper and lower triangular is called a *diagonal* matrix.
  - The determinant of a triangular matrix is equal the product of the diagonal entries.
  - a 3×3 upper triangular determinant:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & 0 & c_3 \end{vmatrix} = a_1 b_2 c_3.$$

- Inverse of a square matrix
  - A square matrix A is invertible if and only if the determinant of A is nonzero:

$$A^{-1} \text{ exists } \iff det(A) \neq 0.$$

  - 2×2 matrix inversion formula:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

  - Gaussian elimination method for 3×3 (or higher) matrix inversion. The arrow means apply Gaussian elimination to entire matrix, reducing A to the identity I, and reducing I to A^{-1}.

$$(A|I) \longrightarrow \left(I|A^{-1}\right).$$

- subspace of a vector space:

  A subset U of a vector space V is called a vector subspace of V if it satisfies the axioms of a vector space with the same addition and scalar multiplication inherited from V.

- criterion for a subset to be a subspace:

  A subset U of a vector space V is a subspace of V if and only if U is closed under vector addition and scalar multiplication. (Equivalently, if u and v are in U, then $u + v$ must also be in U, and for any scalar c, $c \cdot u$ must also be in U.)

Important Theorem from Linear Algebra:

If V is a vector space of dimension n, and $S = \{v_1, \ldots, v_n\}$ is a set of n distinct vectors in V, then S is linearly independent if and only if S is a spanning set. We use this in most proofs of new bases for P_d or for vector spaces of splines.

Lecture 2, Th Jan.10, 2013

Main Points:

- Finish Overview of Project Part I - Bezier Curves, BB-form, Midpoint Subdivision
- Vector Spaces of Polynomials: Shifted, Vandermonde, and Top-down bases.
- Bernstein Polynomials and Bernstein basis.

Midpoint Subdivision (Refer to Project Part I subpart 3)

The third sub-part of Project I, Midpoint Subdivision, differs from the other two in an important way: the points which are generated are not necessarily on the curve. However, they are always either on the curve or on a tangent line to the curve. Since the number of points increases, this guarantees that the line segments converge to the actual curve. Another way in which it differs is that it produces successive approximations to the curve recursively, rather than producing a list of points to be connected.

Polynomial vector spaces

P_d is the vector space of polynomials of degree at most d. (In this course we will typically use the variable t for polynomials.)

Examples:

- The vector space P_3 consists of polynomials of degree at most 3 such as $2 - 4t + 8t^2 - 6t^3$, $4t - t^3$, $1 + 6t$, or even constant polynomials (degree zero) such as 3, -2, or 0, etc.

Addition and scalar multiplication in P_d are done in the usual way for polynomials, which is the usual addition of polynomials and multiplication by real numbers that is familiar from algebra.

The **standard basis** of P_d is: $\{1, t, t^2, \ldots, t^d\}$. The **dimension** of P_d is $d + 1$.

The **coordinate vector** of a polynomial with respect to the standard basis is a column vector of coefficients. The default order for the coefficients is with increasing degree corresponding to order in the column vector from top to bottom.

Examples:

- The polynomial $2 - 3t + 4t^2$ in P_2 has coordinate vector with respect to the standard basis given by: $\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$.

Shifted bases are formed with one constant c as: $\{1, t - c, (t - c)^2, \ldots, (t - c)^d\}$.

Examples:

- A shifted basis of P_2 is: $\{1, t-3, (t-3)^2\}$. The polynomial $3-2(t-3)+2(t-3)^2$ has coorinate vector with respect to this basis: $\begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$. Since $3-2(t-3)+2(t-3)^2$ is equal (after multiplying out) to: $27-14t+2t^2$, we also see that this polynomial has coordinate vector with respect to the standard basis: $\begin{pmatrix} 27 \\ -14 \\ 2 \end{pmatrix}$. It is important to understand that these two coordinate vectors represent exactly the same polynomial, but the coordinates are different because they represent the coefficients with respect to different bases.

Vandermonde bases are formed with $d+1$ constants $t_0, t_1, \ldots, t_d$ as: $\{(t-t_0)^d, (t-t_1)^d, \ldots, (t-t_d)^d\}$.

Examples:

- $\{(t-1)^2, (t-3)^2, (t-9)^2\}$ is a Vandermond basis of P_2, with $t_0 = 1$, $t_1 = 3$, $t_2 = 9$.
- $\{t^3, (t+1)^3, (t-4)^3, (t-5)^3\}$ is a Vandermond basis of P_3, with $t_0 = 0$, $t_1 = -1$, $t_2 = 4$, and $t_3 = 5$.

Top-down bases (intuitive description)

A **top-down basis** of P_d is a set of $d+1$ polynomials which can be either i) Vandermonde, or ii) Shifted, or iii) in between. The third case can be thought of as follows: Take several shifted bases and list them in columns from highest degree on top, to lowest degree on bottom. From these columns, choose any number of polynomials from the top downward, without skipping any, so that the total number of polynomials adds up to $d+1$. This is a top-down basis. If you only use the top elements of each column, you get a Vandermonde basis. If you go all the way down one column, you get a shifted basis. The in between cases give you pieces of shifted bases collected together, with the requirement that the pieces start with highest degree and work down (top-down).

Examples:

- $\{(t-1)^2, t-2, (t-3)^2\}$ is a basis which is not a top-down basis of P_2.
- Exactly 10 top-down bases can be formed from following the grid of polynomials:

$$\begin{matrix} (t-1)^2 & (t-2)^2 & (t-3)^2 \\ t-1 & t-2 & t-3 \\ 1 & 1 & 1 \end{matrix}$$

  One of them is also Vandermonde: $\{(t-1)^2, (t-2)^2, (t-3)^2\}$. Each column makes up one shifted basis. The other cases are of the type: $\{(t-1)^2, t-1, (t-2)^2\}$, which take two from the top of one stack and one from another. (Check that there are ten in all.)

Top-down bases (detailed technical description)

The **top-down bases** are the sets obtained from a grid of polynomials in the following way: Choose distinct real numbers $t_0 < t_1 < \cdots < t_r$. The grid $G(t_0, t_1, \ldots, t_r)$ consists of rows $(t-t_0)^{d-i}, (t-t_1)^{d-i}, \ldots, (t-t_r)^{d-i}$ for $i = 0, \ldots, d$. (Another way to define this grid is as a $(d+1)\times(r+1)$ matrix with (i,j) entry given as: $(t-t_{j-1})^{d-i+1}$.) A **top-down set** S is a set of at most $d+1$ polynomials taken from this grid by choosing from the top of some subset of the $r+1$ columns and working down. Any number of polynomials can be taken from each column, up to a maximum of $d+1$. Thus, if $(t-t_j)^{d-i}$ is a member of S, then $(t-t_j)^d, (t-t_j)^{d-1}, \ldots, (t-t_j)^{d-i+1}$ must also be members of S. If a set of $d+1$ polynomials is chosen in this way, it is called a top-down basis of P_d. We can also describe these sets without the visual aid of the grid as follows: Choose distinct real numbers $t_0 < t_1 < \cdots < t_r$, and indices $m_0, m_1, \ldots, m_r$ with $0 \le m_i \le d+1$, and the sum $\sum_{i=0}^r m_i = d+1$. Then the top-down set associated to this data is the union of the sets

$$\{(t-t_i)^d, (t-t_i)^{d-1}, \ldots, (t-t_i)^{d-m_i+1}\}, \quad i = 0, \ldots, r$$

where if $m_i = 0$ then the set is empty.

Change of basis

To change between one basis and another, we need a change of basis matrix. The simplest change of basis matrix is one which can be used to convert from some basis to the standard basis. This is obtained simply by writing down the coordinate vectors of the basis polynomials and putting them into a matrix. The process can also be reversed by using the inverse matrix.

Examples:

- Find the change of basis matrix which converts from the Vandermonde basis $\{(t-1)^2, (t-2)^2, (t-3)^2\}$ to the standard basis of P_2. Since $(t-1)^2 = 1 - 2t + t^2$, it has coordinate vector with respect to the standard basis: $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$. Similarly, $(t-2)^2$ and $(t-3)^2$ have coordinate vectors: $\begin{pmatrix} 4 \\ -4 \\ 1 \end{pmatrix}$, and $\begin{pmatrix} 9 \\ -6 \\ 1 \end{pmatrix}$. The change of basis matrix is thus: $\begin{pmatrix} 1 & 4 & 9 \\ -2 & -4 & -6 \\ 1 & 1 & 1 \end{pmatrix}$. This can be used to convert the polynomial $3(t-1)^2 - 2(t-2)^2 + 4(t-3)^2$ to the standard basis as follows: Since the coordinate vector of this polynomial is $\begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}$ we multiply to get:

$$\begin{pmatrix} 1 & 4 & 9 \\ -2 & -4 & -6 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 31 \\ -22 \\ 5 \end{pmatrix}$$

  which says that $3(t-1)^2 - 2(t-2)^2 + 4(t-3)^2 = 31 - 22t + 5t^2$. But this is no surprise, since we can also work this out by simply multiplying these binomials and adding terms. What is a little more subtle, is the fact that we can also reverse this process with an inverse matrix.

- In the previous example we apply the inverse matrix to the matrix equation and obtain:

$$\begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{5}{4} & 3 \\ -1 & -2 & -3 \\ \frac{1}{2} & \frac{3}{4} & 1 \end{pmatrix} \begin{pmatrix} 31 \\ -22 \\ 5 \end{pmatrix}$$

  This gives the same information about the polynomials in reverse. It can also be used to find the coefficients for any polynomial in standard basis, converted to the Vandermonde basis. For example, suppose we want to convert the polynomial $2 - 3t + t^2$ into the Vandermonde basis from the previous example. This means we want to find coefficients so that:

$$2 - 3t + t^2 = a_0(t-1)^2 + a_1(t-2)^2 + a_2(t-3)^2.$$

  So we simply apply the inverse matrix to the stanard basis coordinate vector, to obtain:

$$\begin{pmatrix} \frac{1}{2} & \frac{5}{4} & 3 \\ -1 & -2 & -3 \\ \frac{1}{2} & \frac{3}{4} & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ 1 \\ -\frac{1}{4} \end{pmatrix}$$

  This is saying that $2 - 3t + t^2 = \frac{1}{4}(t-1)^2 + (t-2)^2 - \frac{1}{4}(t-3)^2$ which can also be checked by multiplying out the right side.

Bernstein polynomials and Bernstein basis

The Bernstein polynomials of degree d are labelled as:

$$B_0^d(t), B_1^d(t), B_2^d(t), \ldots, B_d^d(t).$$

Each one is defined as:

$$B_i^d(t) = \binom{d}{i}(1-t)^{d-i}t^i,$$

where the binomial coefficient $\binom{d}{i}$ is defined as:

$$\binom{d}{i} = \frac{d!}{(d-i)!i!}.$$

The binomial coefficients are better computed with Pascal's Identity:

$$\binom{d}{i} = \binom{d-1}{i-1} + \binom{d-1}{i}$$

and the fact that $\binom{d}{0} = \binom{d}{d} = 1$. This identity is the basis for Pascal's Triangle, in which row d consists of $\binom{d}{0}\ \binom{d}{1} \cdots \binom{d}{d}$.

$$\begin{array}{c} 1 \\ 1\ \ 1 \\ 1\ \ 2\ \ 1 \\ 1\ \ 3\ \ 3\ \ 1 \\ 1\ \ 4\ \ 6\ \ 4\ \ 1 \\ 1\ \ 5\ \ 10\ \ 10\ \ 5\ \ 1 \\ \text{etc.} \end{array}$$

The binomial coefficients also count the number of subsets of size i in a set of size d.

Examples:

- The number of subsets of size two in the set $\{1,2,3,4\}$ is six, and sets are clearly $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{2,3\}$, $\{2,4\}$, $\{3,4\}$. This is the binomial coefficient $\binom{4}{2}$ which equals 6. It is also the middle element of the row

  $$1\ \ 4\ \ 6\ \ 4\ \ 1$$

  in Pascal's Triangle, and Pascal's Identity is:

  $$\binom{4}{2} = \binom{3}{1} + \binom{3}{2} = 3 + 3 = 6.$$

Bernstein basis

The set of Bernstein polynomials $\{B_0^d(t), B_1^d(t), B_2^d(t), \ldots, B_d^d(t)\}$ is a basis of P_d, called the **Bernstein basis** of P_d.

Examples:

- The Bernstein basis of P_1 is $\{B_0^1(t), B_1^1(t)\} = \{1-t, t\}$.
- The Bernstein basis of P_2 is $\{B_0^2(t), B_1^2(t), B_2^2(t)\} = \{(1-t)^2, 2(1-t)t, t^2\}$.
- The Bernstein basis of P_3 is $\{B_0^3(t), B_1^3(t), B_2^3(t), B_3^3(t)\} = \{(1-t)^3, 3(1-t)^2t, 3(1-t)t^2, t^3\}$.

More Change of Basis

Find the change of basis matrix from $B(2) = \{B_0^2(t), B_1^2(t), B_2^2(t)\}$ to the top-down basis $B_1 = \{(t-1)^2, t-1, (t-2)^2\}$.

The simplest method is to first change from $B(2)$ to standard basis S_2 and then from S_2 to B_1. The change of basis from $B(2)$ to S_2 is obtained as above, by first expanding the polynomials:

$B_0^2(t) = (1-t)^2 = 1 - 2t + t^2$, with coordinate vector $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$,

$B_1^2(t) = 2(1-t)t = 2t - 2t^2$, with coordinate vector $\begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}$, and

$B_2^2(t) = t^2$, with coordinate vector $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Then the change of basis matrix which converts from $B(2)$ to S_2 is:

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{pmatrix}.$$

Next we need the change of basis matrix from S_2 to B_1. Again, we first convert the other way, from B_1 to S_2:

$(t-1)^2 = 1 - 2t + t^2$, has coordinate vector $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$,

$t - 1$, has coordinate vector $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, and

$(t-2)^2 = 4 - 4t + t^2$, has coordinate vector $\begin{pmatrix} 4 \\ -4 \\ 1 \end{pmatrix}$.

So the change of basis matrix which converts from B_1 to S_2 is:

$$A_2 = \begin{pmatrix} 1 & -1 & 4 \\ -2 & 1 & -4 \\ 1 & 0 & 1 \end{pmatrix}.$$

We need the inverse, which changes from S_2 to B_1:

$$A_2^{-1} = \begin{pmatrix} -1 & -1 & 0 \\ 2 & 3 & 4 \\ 1 & 1 & 1 \end{pmatrix}.$$

The final matrix which changes from $B(2)$ to B_1 is then the product:

$$A_2^{-1}A_1 = \begin{pmatrix} -1 & -1 & 0 \\ 2 & 3 & 4 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & -2 & 4 \\ 0 & 0 & 1 \end{pmatrix}.$$

We can test this on a particular polynomial, say $2(1-t)^2 + 3(2(1-t)t) + 4t^2$. This has coordinate vector $\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ with respect to the basis $B(2)$. Using the matrix above, we can convert it to B_1:

$$\begin{pmatrix} 1 & -2 & 0 \\ 0 & -2 & 4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -4 \\ 10 \\ 4 \end{pmatrix},$$

which is saying that $2(1-t)^2 + 3(2(1-t)t) + 4t^2 = -4(t-1)^2 + 10(t-1) + 4(t-2)^2$.

This is confirmed by multiplying both polynomials out to the standard basis:

$$2(1-t)^2 + 3(2(1-t)t) + 4t^2 = 2t + 2,$$

$$-4(t-1)^2 + 10(t-1) + 4(t-2)^2 = 2t + 2.$$

In the exercises, we construct polynomials that pass through certain points, or have required derivatives at certain points. This is a special case of polynomial interpolation. Here we do a few examples with the method of linear systems.

Examples:

- Find the polynomial $p(t) = a_0 + a_1 t + a_2 t^2$ which satisfies: $p(0) = 2$, $p(1) = -3$, and $p(2) = 0$. To solve for the coefficients, we set up three linear equations:

  $$\begin{array}{ccccccc} a_0 & + & a_1 \cdot 0 & + & a_2 \cdot 0^2 & = & 2 \\ a_0 & + & a_1 \cdot 1 & + & a_2 \cdot 1^2 & = & -3 \\ a_0 & + & a_1 \cdot 2 & + & a_2 \cdot 2^2 & = & 0 \end{array}$$

  This is equivalent to the augmented matrix:

  $$\begin{pmatrix} 1 & 0 & 0 & | & 2 \\ 1 & 1 & 1 & | & -3 \\ 1 & 2 & 4 & | & 0 \end{pmatrix}$$

  The solution can be obainted by Gaussian Elimination:

  $$\begin{pmatrix} 1 & 0 & 0 & | & 2 \\ 1 & 1 & 1 & | & -3 \\ 1 & 2 & 4 & | & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & | & 2 \\ 0 & 1 & 1 & | & -5 \\ 0 & 2 & 4 & | & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & | & 2 \\ 0 & 1 & 1 & | & -5 \\ 0 & 0 & 2 & | & 8 \end{pmatrix}$$

  $$\longrightarrow \begin{pmatrix} 1 & 0 & 0 & | & 2 \\ 0 & 1 & 1 & | & -5 \\ 0 & 0 & 1 & | & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & | & 2 \\ 0 & 1 & 0 & | & -9 \\ 0 & 0 & 1 & | & 4 \end{pmatrix}$$

  which means that $a_0 = 2$, $a_1 = -9$, and $a_2 = 4$, and the polynomial is $p(t) = 2 - 9t + 4t^2$. Checking, we see that indeed $p(0) = 2$, $p(1) = -3$, and $p(2) = 0$.

- Find the polynomial $p(t) = a_0 + a_1 t + a_2 t^2$ which satisfies: $p(0) = 2$, $p'(0) = 1$, and $p(1) = 3$. To solve for the coefficients, we set up three linear equations, this time using both $p(t)$ and $p'(t) = a_1 + 2a_2 t$:

  $$\begin{array}{ccccccc} a_0 & + & a_1 \cdot 0 & + & a_2 \cdot 0^2 & = & 2 \\ & & a_1 & + & 2 \cdot a_2 \cdot 0 & = & 1 \\ a_0 & + & a_1 \cdot 1 & + & a_2 \cdot 1^2 & = & 3 \end{array}$$

  This is equivalent to the augmented matrix:

$$\begin{pmatrix} 1 & 0 & 0 & | & 2 \\ 0 & 1 & 0 & | & 1 \\ 1 & 1 & 1 & | & 3 \end{pmatrix}$$

The solution can be obainted by Gaussian Elimination:

$$\begin{pmatrix} 1 & 0 & 0 & | & 2 \\ 0 & 1 & 0 & | & 1 \\ 1 & 1 & 1 & | & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & | & 2 \\ 0 & 1 & 0 & | & 1 \\ 0 & 1 & 1 & | & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & | & 2 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 1 & | & 0 \end{pmatrix}$$

which means that $a_0 = 2$, $a_1 = 1$, and $a_2 = 0$, and the polynomial is $p(t) = 2 + t + 0 \cdot t^2 = 2 + t$. Checking, we see that indeed $p(0) = 2$, $p'(0) = 1$, and $p(1) = 3$. Note: this could have been predicted by the conditions, since heading out from $(0, 2)$ with a slope of 1 will take you directly to $(1, 3)$. We will see later, in the section on interpolation, that the solution to this type of problem is unique within the vector space P_d (in this case P_2) so in this case we know that there is no parabola satisfying the conditions.

Lecture 3, T Jan.15, 2013

Main Points:

- Quiz 1 - Polynomial vector spaces and bases
- Proofs of basis properties of polynomials

Linear independence for polynomials

To prove that a set of polynomials is linearly independent, we refer to the coordinate vectors with respect to a basis, usually the standard basis. If the set of coordinate vectors is linearly independent, then the corresponding polynomials are also independent. (This follows from the important theorem from linear algebra from lecture 1.) In order to check for linear independence of coordinate vectors, we use the determinant criterion:

Determinant Criterion for linear independence of column vectors:

A set of n column vectors in R^n is linearly independent if and only if the determinant of the matrix of column vectors is nonzero. (Note: the columns can be in any order in the matrix.)

Examples:

- The **shifted basis** $\{1, t-c, (t-c)^2\}$ can be seen to be linearly independent by writing the matrix of column vectors:
$$\begin{pmatrix} 1 & -c & c^2 \\ 0 & 1 & -2c \\ 0 & 0 & 1 \end{pmatrix}$$
which has determinant 1 (equal to the product of diagonal entries since the matrix is upper triangular.)
- The **Bernstein basis** $\{(1-t)^2, 2t(1-t), t^2\}$ can be seen to be linearly independent by writing the matrix of column vectors:
$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{pmatrix}$$
which has determinant 2 (equal to the product of diagonal entries since the matrix is lower triangular.)
- The **Vandermonde basis** $\{(t-1)^2, (t-2)^2, (t-3)^2\}$ can be seen to be linearly independent by writing the matrix of column vectors:
$$\begin{pmatrix} 1 & 4 & 9 \\ -2 & -4 & -6 \\ 1 & 1 & 1 \end{pmatrix}$$
which has determinant 2 (which can be computed with the method of cofactor expansion.)
- The **top-down basis** $\{(t-1)^2, t-1, (t-2)^2\}$ can be seen to be linearly independent by writing the matrix of column vectors:
$$\begin{pmatrix} 1 & -1 & 4 \\ -2 & 1 & -4 \\ 1 & 0 & 1 \end{pmatrix}$$
which has determinant -1 (which can be computed with the method of cofactor expansion.)

In order to do higher degree cases of shifted or Bernstein bases, the matrices are still triangular and so the determinants follow the same pattern.

It is useful to recall the binomial expansion:

$$(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-2}a^2 b^{n-2} + \binom{n}{n-1}a^1 b^{n-1} + \binom{n}{n}a^0 b^n.$$

Examples:

- The general shifted basis $\{1, t-c, (t-c)^2, \ldots, (t-c)^d\}$ has upper triangular change of basis matrix (to the standard basis), which has determinant 1:

$$\begin{vmatrix} 1 & -c & c^2 & \cdots & (-c)^d \\ 0 & 1 & -2c & \cdots & \binom{d}{d-1}(-c)^{d-1} \\ 0 & 0 & 1 & \cdots & \binom{d}{d-2}(-c)^{d-2} \\ 0 & 0 & 0 & \cdots & \binom{d}{d-3}(-c)^{d-3} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix} = 1.$$

- The general Bernstein basis

$$\{B_0^d(t), B_1^d(t), \ldots, B_d^d(t)\} = \{(1-t)^d, d(1-t)^{d-1}t, \binom{d}{2}(1-t)^{d-2}t^2, \ldots, t^d\}$$

  has lower triangular change of basis matrix (to the standard basis), which has nonzero determinant (since all the binomial coefficients are nonzero):

$$\begin{vmatrix} \binom{d}{0} & 0 & 0 & \cdots & 0 \\ * & \binom{d}{1} & 0 & \cdots & 0 \\ * & * & \binom{d}{2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ * & * & * & \cdots & \binom{d}{d} \end{vmatrix} = \binom{d}{0}\binom{d}{1}\binom{d}{2}\cdots\binom{d}{d} \neq 0.$$

For Vandermonde and top-down bases, however, we need different formulas. We start with the Vandermonde determinant.

Vandermonde Determinant Formula:

$$\begin{vmatrix} 1 & t_0 & t_0^2 & \cdots & t_0^d \\ 1 & t_1 & t_1^2 & \cdots & t_1^d \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & t_d & t_d^2 & \cdots & t_d^d \end{vmatrix} = \prod_{0 \le i < j \le d} (t_j - t_i)$$

Examples:

- A Vandermonde determinant with $t_0 = 2$, $t_1 = 3$, $t_2 = 4$:

$$\begin{vmatrix} 1 & 2 & 2^2 \\ 1 & 3 & 3^2 \\ 1 & 4 & 4^2 \end{vmatrix} = (3-2)(4-2)(4-3) = 2$$

- A Vandermonde determinant with $t_0 = -2$, $t_1 = 5$, $t_2 = 3$:

$$\begin{vmatrix} 1 & -2 & 2^2 \\ 1 & 5 & 3^2 \\ 1 & 3 & 4^2 \end{vmatrix} = (5-(-2))(3-(-2))(3-5) = -70$$

Proof of Vandermonde Determinant Formula:

Let

$$D = \begin{vmatrix} 1 & t_0 & t_0^2 & \cdots & t_0^d \\ 1 & t_1 & t_1^2 & \cdots & t_1^d \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & t_d & t_d^2 & \cdots & t_d^d \end{vmatrix}.$$

This is an $n \times n$ determinant, with $n = d+1$. To prove this formula, we will use an induction argument. The induction hypothsis is that the formula is true for $n \leq d$. The base case is $d = 0$. We can check this case immediately, since the formula then simply says that $1 = 1$. The case $d = 1$ can also be checked directly, which is:

$$D = \begin{vmatrix} 1 & t_0 \\ 1 & t_1 \end{vmatrix} = t_1 - t_0$$

which we know to be true from the 2×2 determinant formula:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

So we want to show that $D = \prod_{0 \leq i < j \leq d} (t_j - t_i)$. In order to show this, we introduce a polynomial which is defined as a determinant:

$$P(x) = \begin{vmatrix} 1 & x & x^2 & \cdots & x^d \\ 1 & t_1 & t_1^2 & \cdots & t_1^d \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & t_d & t_d^2 & \cdots & t_d^d \end{vmatrix}.$$

Because we can expand by cofactors along the top row of the determinant, it is clear that $P(x)$ is a polynomial of degree at most d, with coefficients given by determinants using the values $t_1, \ldots, t_d$. From the definitions, we also have $D = P(t_0)$. Next, we will factor the polynomial $P(x)$ in order to arrive at the determinant formula.

To factor $P(x)$ we simply notice that it must have zeros given by $t_1, \ldots, t_d$. This follows from the property of a determinant which says that if two rows or two columns are identical, then the determinant equals zero. Each of these zeros corresponds to a factor and so we have d factors:

$$P(x) = C \cdot (x - t_1)(x - t_2) \cdots (x - t_d),$$

where C is some constant. Since this polynomial has degree d, there can be no other factors. We can also determine the constant C by looking at the original definition of $P(x)$ using the determinant. This tells us that the coefficient of x^d must be a cofactor determinant given by:

$$\begin{vmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^{d-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & t_d & t_d^2 & \cdots & t_d^{d-1} \end{vmatrix}.$$

This has the same form as the original determinant, so we can use the induction step: We assume that the original formula works for all smaller Vandermonde determinants, such as this one. This says that the coefficient C must be:

$$C = \prod_{1 \leq i < j \leq d} (t_j - t_i)$$

But then

$$
\begin{aligned}
D = P(t_0) &= C \cdot (t_0 - t_1)(t_0 - t_2)\cdots(t_0 - t_d) && (1)\\
&= \left(\prod_{1\le i<j\le d}(t_j - t_i)\right)\cdot(t_0 - t_1)(t_0 - t_2)\cdots(t_0 - t_d) && (2)\\
&= \prod_{0\le i<j\le d}(t_j - t_i) && (3)
\end{aligned}
$$

which completes the proof.

Examples:

- To see how this proof applies to a Vandermonde basis of degree 2 we can start with the basis:

  $$\{(t-t_0)^2, (t-t_1)^2, (t-t_2)^2\}.$$

  Now we would like to show that these polynomials are linearly independent, so we first write down the matrix of column vectors:

  $$\begin{pmatrix} t_0^2 & t_1^2 & t_2^2 \\ -2t_0 & -2t_1 & -2t_2 \\ 1 & 1 & 1 \end{pmatrix}$$

  We can apply a few properties of determinants to see that this is a constant times a Vandermonde determinant. Call this matrix A. Then by swapping the first and last rows, then factoring out a -2 from the second row, and taking a transpose, and applying the Vandermonde formula yields:

  $$det(A) = -\begin{vmatrix} 1 & 1 & 1 \\ -2t_0 & -2t_1 & -2t_2 \\ t_0^2 & t_1^2 & t_2^2 \end{vmatrix} = -(-2)\begin{vmatrix} 1 & 1 & 1 \\ t_0 & t_1 & t_2 \\ t_0^2 & t_1^2 & t_2^2 \end{vmatrix} = 2\cdot\begin{vmatrix} 1 & t_0 & t_0^2 \\ 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \end{vmatrix} = 2\cdot(t_1-t_0)(t_2-t_0)(t_2-t_1)$$

  This uses the properties of determinants:

  - swapping two rows changes a determinant by a factor of -1
  - a nonzero constant can be factored out of a single row or column
  - transpose does not change a determinant

Linearity of Determinant Operator:

The determinant of an $n \times n$ matrix produces a single number, as a function of all n^2 entries of the matrix. However, the determinant can also be considered as a function of a single row or column, with constants in all the other entries. For instance, in the 3×3 case we can write:

$$\begin{vmatrix} x & y & z \\ a & b & c \\ d & e & f \end{vmatrix} = \begin{vmatrix} b & c \\ e & f \end{vmatrix} x - \begin{vmatrix} a & c \\ d & f \end{vmatrix} y + \begin{vmatrix} a & b \\ d & e \end{vmatrix} z = (bf-ce)x - (af-dc)y + (ae-bd)z,$$

which is a linear function of the first row. Further, if the first row is a sum $[x, y, z] = [x_1, y_1, z_1] + [x_2, y_2, z_2]$, then we have

$$\begin{vmatrix} x & y & z \\ a & b & c \\ d & e & f \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & z_1 \\ a & b & c \\ d & e & f \end{vmatrix} + \begin{vmatrix} x_1 & y_1 & z_1 \\ a & b & c \\ d & e & f \end{vmatrix},$$

and also:

$$\begin{vmatrix} cx & cy & cz \\ a & b & c \\ d & e & f \end{vmatrix} = c\begin{vmatrix} x & y & z \\ a & b & c \\ d & e & f \end{vmatrix}.$$

Another way to write this is to let $v = [x, y, z]$, $v_1 = [x_1, y_1, z_1]$, and $v_2 = [x_2, y_2, z_2]$, and define:

$$L(v) = \begin{vmatrix} x & y & z \\ a & b & c \\ d & e & f \end{vmatrix}.$$

Then by the properties above, we have:

$$L(v_1 + v_2) = L(v_1) + L(v_2) \text{ and } L(cv) = cL(v),$$

and thus L is a linear operator.

Linear independence of Top-down bases

This depends on the Confluent Vandermonde determinant formula. A confluent Vandermonde determinant is a generalization of the regular Vandermonde determinant, where rows may be followed by derivatives. A few examples will give the idea:

Examples:

- A confluent Vandermonde with a regular row followed by a derivative row, followed by another regular row. The first two rows use the parameter a, and the third row uses the parameter b. For the purpose of taking derivatives, a is considered as a variable.

$$D(aab) = D(a^2b) = \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & 2a \\ 1 & b & b^2 \end{vmatrix} = (b-a)^2$$

- Now there are two derivative rows:

$$D(aaab) = D(a^3b) = \begin{vmatrix} 1 & a & a^2 & a^3 \\ 0 & 1 & 2a & 3a^2 \\ 0 & 0 & 2 & 6a \\ 1 & b & b^2 & b^3 \end{vmatrix} = 2(b-a)^3$$

- Following the same pattern:

$$D(aaaabb) = D(a^4b^2) = \begin{vmatrix} 1 & a & a^2 & a^3 & a^4 & a^5 \\ 0 & 1 & 2a & 3a^2 & 4a^3 & 5a^4 \\ 0 & 0 & 2 & 6a & 12a^2 & 20a^3 \\ 0 & 0 & 0 & 6 & 24a & 60a^2 \\ 1 & b & b^2 & b^3 & b^4 & b^5 \\ 0 & 1 & 2b & 3b^2 & 4b^3 & 5b^4 \end{vmatrix} = 12(b-a)^8$$

In all of the above examples we can see the factor $(b - a)$ occurring to various powers. The powers are given by the product of the exponents on a and b. The constants 2 and 12 come from the diagonal entries in the a-rows.

We can also see that the case $D(abc)$ is then just the regular Vandermonde formula, which gives:

$$D(abc) = (b-a)(c-a)(c-b).$$

To adapt the regular Vandermonde formula to the confluent case, we can take all backward differences and delete those that are zero (like $(a - a)$) and then put in the diagonal factors. This yields the general formula:

$$D(a_1^{e_1} a_2^{e_2} \cdots a_n^{e_n}) = \prod_{1 \le i < j \le n} (a_j - a_i)^{e_i e_j} \prod_{k=1}^{n} (e_k - 1)!!$$

where the double factorial means:

$$N!! = N!(N-1)!(N-2)!\cdots 2!1!.$$

The proof of this formula is given in a separate paper on the website.

Lecture 4, Th Jan.17, 2013

Main Points:

- Properties of Bernstein Polynomials
- Cumulative Bernstein Polynomials

Properties of Bernstein Polynomials

1. Partition of Unity:
$$\sum_{i=0}^{d} B_i^d(t) = 1$$
2. Symmetry:
$$B_i^d(t) = B_{d-i}^d(1-t)$$
3. Positivity:
$$B_i^d(t) > 0, \quad \text{for } 0 < t < 1$$
4. Recursion:
$$B_i^d(t) = tB_{i-1}^{d-1}(t) + (1-t)B_i^{d-1}(t)$$
5. Derivative:
$$\frac{d}{dt}B_i^d(t) = d\left(B_{i-1}^{d-1}(t) - B_i^{d-1}(t)\right)$$

Proofs of Bernstein Properties:

1. Partition of Unity:
$$\sum_{i=0}^{d} B_i^d(t) = 1$$
This follows from the binomial theorem:
$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$$
with the substitution: $a = 1-t$, $b = t$, and $n = d$.
2. Symmetry:
$$B_i^d(t) = B_{d-i}^d(1-t)$$
This follows from the fact that for any binomial coefficient
$$\binom{d}{i} = \binom{d}{d-i}$$
which in turn follows from Pascal's Triangle, or also from the counting property of binomial coefficients: $\binom{d}{i}$ counts the number of subsets of size i in a set of size d, so also $\binom{d}{d-i}$ counts the number of complementary subsets of size $d-i$ in a set of size d. The subsets of size d are in $1-1$ correspondence with the complementary subsets of size $d-i$.

3. Positivity:
$$B_i^d(t) > 0, \quad \text{for } 0 < t < 1$$
This follows directly from the definition.

4. Recursion:
$$B_i^d(t) = tB_{i-1}^{d-1}(t) + (1-t)B_i^{d-1}(t)$$
This follows from Pascal's Identity:
$$\binom{d}{i} = \binom{d-1}{i-1} + \binom{d-1}{i},$$
and the definition of $B_i^d(t)$:
$$\begin{aligned} B_i^d(t) &= \binom{d}{i}(1-t)^{d-i}t^i \\ &= \left(\binom{d-1}{i-1} + \binom{d-1}{i}\right)(1-t)^{d-i}t^i \\ &= \binom{d-1}{i-1}(1-t)^{d-i}t^i + \binom{d-1}{i}(1-t)^{d-i}t^i \\ &= t\binom{d-1}{i-1}(1-t)^{d-i}t^{i-1} + (1-t)\binom{d-1}{i}(1-t)^{d-i-1}t^i \\ &= tB_{i-1}^{d-1}(t) + (1-t)B_i^{d-1}(t). \end{aligned} \tag{1}$$

5. Derivative:
$$\frac{d}{dt}B_i^d(t) = d\left(B_{i-1}^{d-1}(t) - B_i^{d-1}(t)\right)$$

Here we use a combination of the recursion property for Bernstein polynomials, the product rule for derivatives, and an induction argument (using the derivative formula for case $d-1$).

$$\begin{aligned} \frac{d}{dt}B_i^d(t) &= \frac{d}{dt}\left[tB_{i-1}^{d-1}(t) + (1-t)B_i^{d-1}(t)\right] \\ &= 1 \cdot B_{i-1}^{d-1}(t) + t \cdot \frac{d}{dt}B_{i-1}^{d-1}(t) + (-1) \cdot B_i^{d-1}(t) + (1-t) \cdot \frac{d}{dt}B_i^{d-1}(t) \\ &= B_{i-1}^{d-1}(t) + t \cdot (d-1)\left(B_{i-2}^{d-2}(t) - B_{i-1}^{d-2}(t)\right) - \cdot B_i^{d-1}(t) + (1-t) \cdot (d-1)\left(B_{i-1}^{d-2}(t) - B_i^{d-2}(t))\right) \\ &= B_{i-1}^{d-1}(t) - B_i^{d-1}(t) + (d-1)\left(tB_{i-2}^{d-2}(t) + (1-t)B_{i-1}^{d-2}(t)\right) - (d-1)\left(tB_{i-1}^{d-2}(t) + (1-t)B_i^{d-2}(t))\right) \\ &= B_{i-1}^{d-1}(t) - B_i^{d-1}(t) + (d-1)\left(B_{i-1}^{d-1}(t) - B_i^{d-1}(t)\right) \\ &= d\left(B_{i-1}^{d-1}(t) - B_i^{d-1}(t)\right). \end{aligned}$$

Cumulative Bernstein polynomials and basis

Define the Cumulative Bernstein polynomials as:
$$C_i^d(t) = \sum_{j=i}^{d} B_j^d(t).$$

The Cumulative Bernstein basis is defined as:

$$\{C_0^d(t), C_1^d(t), \dots C_d^d(t)\}.$$

Note: $C_0^d(t) = 1$ by the Partition of Unity Property for Bernstein polynomials.

Examples:

- The Cumulative Bernstein basis for $d = 1$:
$$\{1, t\}.$$
- The Cumulative Bernstein basis for $d = 2$:
$$\{1, 2t(1-t) + t^2, t^2\} = \{1, 2t - t^2, t^2\}.$$

Bezier Curves

A 2D Bezier curve is simply a piecewise polynomial curve written with the polynomials in the Bernstein basis:

$$\begin{aligned}
\gamma(t) &= (p_1(t), p_2(t)) \\
&= (a_0(1-t)^2 + a_1 2(1-t)t + a_2 t^2, b_0(1-t)^2 + b_1 2(1-t)t + b_2 t^2) \\
&= (a_0, b_0)(1-t)^2 + (a_1, b_1)2(1-t)t + (a_2, b_2)t^2 \\
&= (1-t)^2(a_0, b_0) + 2(1-t)t(a_1, b_1) + t^2(a_2, b_2) \\
&= B_0^2(t)P_0 + B_1^2(t)P_1 + B_2^2(t)P_2.
\end{aligned}$$

Notes:

- The last line is the BB-form. It is a sum of points with coefficients which are Berstein polynomials evaluated at some value t. Since the Bernstein polynomials add up to 1 (the Partition of Unity Property), this is an affine sum of points.
- The third line could be called "Point-Coefficient Form". What does it mean? Can we multiply points times polynomials, as if they were scalars? The answer is no, and all we mean by this line is that it is another notation for the second line. It is simply an easy and slightly more compact way of writing the second line. It can be done for any basis of polynomials, but the Bernstein basis leads to the BB-form, which has special meaning as an affine sum.

Lecture 5, T Jan.22, 2013

Main Points:

- Quiz 2
- Ordered pairs of polynomials
- Piecewise polynomial vector spaces

The vector space P_d^2:

P_d^2 is simply the vector space of ordered pairs of polynomials of degree $\leq d$.

Addition and scalar multiplication in P_d^2 are defined componentwise:

$$(p_1(t), p_2(t)) + (q_1(t), q_2(t)) = (p_1(t) + q_1(t), p_2(t) + q_2(t)),$$

$$c(p_1(t), p_2(t)) = (cp_1(t), cp_2(t)).$$

Examples:

- some elements of P_2^2 are: $(2 - t + 3t^2, 1 - t)$ and $(3 + t^2, 5 - 2t - 7t^2)$.
- some elements of P_3^2 are: $(1 - 5t + 3t^3, 1 - 4t^2)$ and $(3 + 2t^2, 5 - 2t - 7t^2)$. Note: the latter ordered pair has no degree 3 terms, which is fine since the degrees must simply be at most 3.

A basis for P_2^2:

A basis for P_2^2 can be given by:

$$\{(1, 0), (t, 0), (t^2, 0), (0, 1), (0, t), (0, t^2)\}.$$

We can check that this set spans P_2^2, and is also linearly independent. The spanning property is easily seen since:

$$\begin{aligned} (p_1(t), p_2(t)) &= (a_0 + a_1 t + a_2 t^2, b_0 + b_1 t + b_2 t^2) \\ &= a_0(1, 0) + a_1(t, 0) + a_2(t^2, 0) + b_0(0, 1) + b_1(0, t) + b_2(0, t^2). \end{aligned}$$

The linear independence can be seen by appealing to the definition of linear independence. We need to show that any linear combination of the ordered pairs can only be zero if all the coefficients are zero. Suppose we have such a linear combination set equal to the zero vector:

$$a_0(1, 0) + a_1(t, 0) + a_2(t^2, 0) + b_0(0, 1) + b_1(0, t) + b_2(0, t^2) = (0, 0).$$

This is equivalent to having:

$$(a_0 + a_1 t + a_2 t^2, b_0 + b_1 t + b_2 t^2) = (0, 0),$$

which in turn implies that

$$a_0 + a_1 t + a_2 t^2 = 0, \quad \text{and} \quad b_0 + b_1 t + b_2 t^2 = 0.$$

Since these are both linear combinations in P_2 of the standard basis polynomials, we know that all of the coefficients must be zero, which follows from the linear independence of 1, t, and t^2 in P_2.

Other bases for P_2^2:

We can mimic the previous basis which used the standard basis polynomials of P_2, and switch to the Bernstein basis of P_2, to get the basis of P_2^2:

$$\{(B_0^2(t),0),(B_1^2(t),0),(B_2^2(t),0),(0,B_0^2(t)),(0,B_1^2(t)),(0,B_2^2(t))\}.$$
$$=\{((1-t)^2,0),(2(1-t)t,0),(t^2,0),(0,(1-t)^2),(0,2(1-t)t),(0,t^2)\}.$$

Similarly, this can be done with any other basis of polynomials, such as $\{b_0(t),b_1(t),b_2(t)\}$, to form the basis of P_2^2:

$$\{(b_0(t),0),(b_1(t),0),(b_2(t),0),(0,b_0(t)),(0,b_1(t)),(0,b_2(t))\}.$$

For example, we could take $\{b_0(t),b_1(t),b_2(t)\}$ equal to a Vandermonde basis of P_2 such as $\{(t-1)^2,(t-2)^2,(t-3)^2\}$ to get:

$$\{((t-1)^2,0),((t-2)^2,0),((t-3)^2,0),(0,(t-1)^2),(0,(t-2)^2),(0,(t-3)^2)\}.$$

Basis for P_d^k:

The set of k-tuples
$$(t^i,0,\ldots,0),(0,t^i,0,\ldots,0),\ldots,(0,\ldots,0,t^i)$$
for $i=0\ldots,d$, is a basis of P_d^k. There are k entries in the list for each i, so $(d+1)k$ in all, which says that the dimension of P_d^k is $(d+1)k$.

Examples:

- P_1^3 has dimension 6 with basis:
$$\{(1,0,0),(0,1,0),(0,0,1),$$
$$(t,0,0),(0,t,0),(0,0,t)\}$$
- P_3^4 has dimension 16 with basis:
$$\{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1),$$
$$(t,0,0,0),(0,t,0,0),(0,0,t,0),(0,0,0,t),$$
$$(t^2,0,0,0),(0,t^2,0,0),(0,0,t^2,0),(0,0,0,t^2),$$
$$(t^3,0,0,0),(0,t^3,0,0),(0,0,t^3,0),(0,0,0,t^3)\}$$
- P_2^5 has dimension 15 with basis:
$$\{(1,0,0,0,0),(0,1,0,0,0),(0,0,1,0,0),(0,0,0,1,0),(0,0,0,0,1),$$
$$(t,0,0,0,0),(0,t,0,0,0),(0,0,t,0,0),(0,0,0,t,0),(0,0,0,0,t),$$
$$(t^2,0,0,0,0),(0,t^2,0,0,0),(0,0,t^2,0,0),(0,0,0,t^2,0),(0,0,0,0,t^2)\}$$

Next, we develop a different basis of P_d^k. We start by including all the elements:
$$(1,1,\ldots,1),(t,t,\ldots,t),\ldots,(t^d,t^d,\ldots,t^d)$$
These are $d+1$ elements, corresponding to the standard basis of P_d. By choosing coefficients for these elements we can obtain any polynomial of degree d or less, repeated on each component. In order to form a basis we need to obtain any element of P_d^k of the form:
$$(p_1,p_2,\ldots,p_k).$$
We can do this with the following additional elements:
$$(0,1,\ldots,1),(0,t,\ldots,t),\ldots,(0,t^d,\ldots,t^d),$$

$$
\begin{gathered}
(0,0,1,\dots,1),(0,0,t,\dots,t),\dots,(0,0,t^d,\dots,t^d),\\
\vdots\\
(0,\dots,0,1),(0,\dots,0,t),\dots,(0,\dots,0,t^d).
\end{gathered}
$$

In order to see that the above elements span P_d^k, we first write:

$$
\begin{aligned}
(p_1,p_2,\dots,p_k) &= (p_1,p_1,\dots,p_1)\\
&= +(0,p_2-p_1,\dots,p_2-p_1)\\
&= +(0,0,p_3-p_2\dots,p_3-p_2)\\
&= +(0,0,0,p_4-p_3\dots,p_4-p_3)\\
&= \vdots\\
&= +(0,0,0,\dots,0,p_k-p_{k-1})
\end{aligned}
$$

Note: In the above sum each line makes a correction to the previous line so that one more component has the desired polynomial. Also, each line can be spanned by exactly the corresponding line in the following basis:

$$
\begin{gathered}
\{(1,1,\dots,1),(t,t,\dots,t),\dots,(t^d,t^d,\dots,t^d),\\
(0,1,\dots,1),(0,t,\dots,t),\dots,(0,t^d,\dots,t^d),\\
(0,0,1,\dots,1),(0,0,t,\dots,t),\dots,(0,0,t^d,\dots,t^d),\\
\vdots\\
(0,\dots,0,1),(0,\dots,0,t),\dots,(0,\dots,0,t^d)\}
\end{gathered}
$$

Examples:

- P_1^3 has dimension 6 with basis:

$$\{(1,1,1),(t,t,t),(0,1,1),(0,t,t),(0,0,1),(0,0,t)\}$$

- Write the element $(2t-1,3,5t)$ in P_1^3 in terms of the two bases given so far. Call these two bases:

$$B_1=\{(1,0,0),(t,0,0),(0,1,0),(0,t,0),(0,0,1),(0,0,t)\}$$

and

$$B_2=\{(1,1,1),(t,t,t),(0,1,1),(0,t,t),(0,0,1),(0,0,t)\}.$$

In the first case:

$$(2t-1,3,5t)=-(1,0,0)+2(t,0,0)+3(0,1,0)+0(0,t,0)+0(0,0,1)+5(0,0,t).$$

For the second case, we need to write

$$p_1(t)=2t-1,\ \ p_2(t)=3,\ \text{ and } p_3(t)=5t.$$

Then we also have

$$p_2(t)-p_1(t)=4-2t,\ \text{ and }\ p_3(t)-p_2(t)=5t-3.$$

Now we can follow the method above for

$$(p_1,p_2,p_3)=(p_1,p_1,p_1)+(0,p_2-p_1,p_2-p_1)+(0,0,p_3-p_2)$$

to obtain:

$$(2t-1,3,5t)=-(1,1,1)+2(t,t,t)+4(0,1,1)-2(0,t,t)-3(0,0,1)+5(0,0,t).$$

- Find the coordinate vectors of the element $(2t-1, 3, 5t)$ in P_1^3 with respect to the bases B_1 and B_2. Based on the coefficients from the previous example we have the coordinate vectors:

$$B_1 : \begin{pmatrix} -1 \\ 2 \\ 3 \\ 0 \\ 0 \\ 5 \end{pmatrix} \text{ and } B_2 : \begin{pmatrix} -1 \\ 2 \\ 4 \\ -2 \\ -3 \\ 5 \end{pmatrix}.$$

- P_3^4 has dimension 16 with basis:

$$\begin{gathered}\{(1,1,1,1),(t,t,t,t),(t^2,t^2,t^2,t^2),(t^3,t^3,t^3,t^3),\\ (0,1,1,1),(0,t,t,t),(0,t^2,t^2,t^2),(0,t^3,t^3,t^3),\\ (0,0,1,1),(0,0,t,t),(0,0,t^2,t^2),(0,0,t^3,t^3),\\ (0,0,0,1),(0,0,0,t),(0,0,0,t^2),(0,0,0,t^3)\}\end{gathered}$$

- P_2^5 has dimension 15 with basis:

$$\begin{gathered}\{(1,1,1,1,1),(t,t,t,t,t),(t^2,t^2,t^2,t^2,t^2),\\ (0,1,1,1,1),(0,t,t,t,t),(0,t^2,t^2,t^2,t^2),\\ (0,0,1,1,1),(0,0,t,t,t),(0,0,t^2,t^2,t^2),\\ (0,0,0,1,1),(0,0,0,t,t),(0,0,0,t^2,t^2),\\ (0,0,0,0,1),(0,0,0,0,t),(0,0,0,0,t^2)\}\end{gathered}$$

Piecewise polynomial functions:

A piecewise polynomial function (or ppf) is a function defined on a sequence of intervals by various polynomials. If the sequence of intervals is $[u_0, u_1], [u_1, u_2], \ldots, [u_{k-1}, u_k]$, then we denote this sequence with the notation:

$$[u_0, u_1, \ldots, u_k].$$

The endpoints of this sequence of intervals are u_0 and u_k, and all the other values u_i for $i = 1 \ldots, k-1$ are called *breakpoints*. A ppf defined on this sequence of intervals has the following general form, where $p_1(t), \ldots, p_k(t)$ are polynomials. Note that the intervals which are used by each polynomial are closed on the left and open on the right for p_1 through p_{k-1} and then closed on both ends for p_k. This is done so that the function f is defined exactly once for all points in the closed interval $[u_0, u_k]$.

$$f(t) = \begin{cases} p_1(t), & u_0 \le t < u_1 \\ p_2(t), & u_1 \le t < u_2 \\ \vdots & \vdots \\ p_k(t), & u_{k-1} \le t \le u_k \end{cases}$$

Examples:

- A ppf defined by cubic polynomials on the sequence of intervals $[0, 2, 4]$:

$$f(t) = \begin{cases} p_1(t) = t^3 - 2t^2 + t - 5, & 0 \leq t < 2 \\ p_2(t) = 2t^3 - 8t^2 + 13t - 3, & 2 \leq t \leq 4 \end{cases}$$

- A ppf defined by linear polynomials on the sequence of intervals $[0, 1, 2, 3, 4]$:

$$f(t) = \begin{cases} p_1(t) = 2t - 5, & 0 \leq t < 1 \\ p_2(t) = 4, & 1 \leq t < 2 \\ p_3(t) = t - 1, & 2 \leq t < 3 \\ p_4(t) = 3t + 7, & 3 \leq t \leq 4 \end{cases}$$

The piecewise polynomial vector space $P_d^k[u_0, u_1, \ldots, u_k]$.

The set of all piecewise polynomial functions on the sequence of intervals $[u_0, u_1, \ldots, u_k]$ forms the vector space which we call $P_d^k[u_0, u_1, \ldots, u_k]$. Addition and scalar multiplication in this vector space are given simply by addition of functions and real number multipication of functions. So this vector space is a subspace of the vector space $\mathcal{F}[u_0, u_k]$ of all functions on the interval $[u_0, u_k]$.

Correspondence between P_d^k and $P_d^k[u_0, u_1, \ldots, u_k]$:

Any ordered k-tuple of polynomials $(p_1(t), p_2(t), \ldots, p_k(t))$ in P_d^k can be converted into a ppf in $P_d^k[u_0, u_1, \ldots, u_k]$, given by:

$$f(t) = \begin{cases} p_1(t), & u_0 \leq t < u_1 \\ p_2(t), & u_1 \leq t < u_2 \\ \vdots & \vdots \\ p_k(t), & u_{k-1} \leq t \leq u_k \end{cases}$$

This process can also be reversed by taking this ppf and stripping away all of the information about intervals and forming the ordered k-tuple of the polynomials.

Examples:

- The element $(t, 1, 2 - t)$ in P_1^3 corresponds to the ppf

$$f(t) = \begin{cases} t, & 0 \leq t < 1 \\ 1, & 1 \leq t < 2 \\ 2 - t, & 2 \leq t \leq 3 \end{cases}$$

  in $P_1^3[0, 1, 2, 3]$.

- The element $(1 - 3t + t^2, 4, 2 - t^2, t + 5)$ in P_2^4 corresponds to the ppf

$$f(t) = \begin{cases} 1 - 3t_t^2, & 0 \leq t < 3 \\ 4, & 3 \leq t < 5 \\ 2 - t^2, & 5 \leq t < 6 \\ t + 5, & 6 \leq t \leq 8 \end{cases}$$

  in $P_1^3[0, 3, 5, 6, 8]$.

Isomorphism between P_d^k and $P_d^k[u_0, u_1, \ldots, u_k]$:

The above correspondence gives a dictionary between the ordered k-tuples of polynomials, and ppf's on a given sequence of intervals. It is important to note that this correspondence preserves linear algebra constructs. In particular, if

$$(p_1(t), p_2(t), \ldots, p_k(t)) \text{ and } (q_1(t), q_2(t), \ldots, q_k(t))$$

have corresponding ppf's $f(t)$ and $g(t)$, and

$$(p_1(t), p_2(t), \ldots, p_k(t)) + (q_1(t), q_2(t), \ldots, q_k(t)) = (r_1(t), r_2(t), \ldots, r_k(t))$$

and

$$f(t) + g(t) = h(t),$$

then it must the case that the ordered k-tuple

$$(r_1(t), r_2(t), \ldots, r_k(t))$$

corresponds to the ppf $h(t)$. Further, all linear algebra calculations can be transferred back and forth between these two vector spaces under this correspondence.

Modified bases for P_d^k:

Since we have many bases for P_d, it is also possible to use these to find other bases for P_d^k. In general, if $b_0(t), \ldots, b_d(t)$ is any basis of P_d, then we can construct the set of k-tuples

$$(b_i(t), 0, \ldots, 0), (0, b_i(t), 0, \ldots, 0), \ldots, (0, \ldots, 0, b_i(t))$$

for $i = 0 \ldots, d$, which is a basis of P_d^k. There are k entries in the list for each i, so $(d+1)k$ in all, which again exhibits the fact that the dimension of P_d^k is $(d+1)k$.

Lecture 6, Th Jan.24, 2013

Main Points:

- Shifted (or truncated) power functions
- More bases for piecewise polynomial vector spaces

Shifted (or truncated) Power Functions:

The (right-continuous) shifted power function, with constant c, is defined as:

$$(t-c)_+^k = \begin{cases} 0, & t < c \\ (t-c)^k, & t \geq c \end{cases}, \quad \text{for } k \geq 1, \text{ and}$$

$$(t-c)_+^0 = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$$

The (left-continuous) shifted power function, with constant c, is defined as:

$$(c-t)_+^k = \begin{cases} 0, & c < t \\ (c-t)^k, & c \geq t \end{cases} = \begin{cases} (c-t)^k, & t \leq c \\ 0, & t > c \end{cases}, \quad \text{for } k \geq 1, \text{ and}$$

$$(c-t)_+^0 = \begin{cases} 0, & c < t \\ 1, & c \geq t \end{cases} = \begin{cases} 1, & t \leq c \\ 0, & t > c \end{cases}$$

Examples:

- The continuous function $(t-1)_+^1$ is zero to the left of $t=1$ and is equal to the linear function $t-1$ to the right of $t=1$.
- The continuous function $(1-t)_+^1$ is zero to the right of $t=1$ and is equal to the linear function $1-t$ to the left of $t=1$.
- The discontinuous (but also right-continuous) function $(t-1)_+^0$ is zero to the left of $t=1$ and is equal to the constant function 1 at $t=1$ and to the right of $t=1$.
- The discontinuous (but also left-continuous) function $(1-t)_+^0$ is zero to the right of $t=1$ and is equal to the constant function 1 at $t=1$ and to the left of $t=1$.

Bases with shifted power functions:

The standard basis of $P_d^k[u_0, \ldots, u_k]$ is:

$$\{1, t, t^2, \ldots, t^d, (t-u_1)_+^0, (t-u_1)_+^1, \ldots, (t-u_1)_+^d, \ldots, (t-u_{k-1})_+^0, (t-u_{k-1})_+^1, \ldots, (t-u_{k-1})_+^d\}.$$

Note: This basis consists of a basis of P_d (the standard basis) and also $d+1$ shifted power functions at each of the $k-1$ break-points.

Examples:

- A basis for $P_1^4[0,1,2,3,4]$ is:

  $$\{1, t, (t-1)_+^0, (t-1)_+^1, (t-2)_+^0, (t-2)_+^1, (t-3)_+^0, (t-3)_+^1\}.$$

  The dimension of $P_1^4[0,1,2,3,4]$ is 8.

- A basis for $P_3^3[0,1,2,3]$ is:

$$\{1, t, t^2, t^3, (t-1)_+^0, (t-1)_+^1, (t-1)_+^2, (t-1)_+^3, (t-2)_+^0, (t-2)_+^1, (t-2)_+^2, (t-2)_+^3, \}.$$

  The dimension of $P_3^3[0,1,2,3]$ is 12.

To see how these bases of shifted power functions are indeed bases, we relate them back to the the ordered k-tuples:

Correspondence between P_d^k and $P_d^k[u_0,\ldots,u_k]$:

$$(p_1, p_2, \ldots, p_k) \longleftrightarrow f(t) = \begin{cases} p_1(t), & u_0 \le t < u_1 \\ p_2(t), & u_1 \le t < u_2 \\ & \vdots \\ p_k(t), & u_{k-1} \le t < u_k \end{cases}$$

Proving that shifted power bases are indeed linearly independent:

We can use the isomorphism between P_d^k and $P_d^k[u_0,\ldots,u_k]$ and the correspondence between elements to verify that these shifted power bases are indeed linearly independent.

Examples:

- To show that $\{1, t, (t-1)_+^0, (t-1)_+^1, (t-2)_+^0, (t-2)_+^1, (t-3)_+^0, (t-3)_+^1\}$ is a basis of $P_1^4[0,1,2,3,4]$, we note that it corresponds with the basis of 4-tuples for P_1^4:

$$\{(1,1,1,1), (t,t,t,t), (0,1,1,1), (0,t-1,t-1,t-1), (0,0,1,1), (0,0,t-2,t-2), (0,0,0,1), (0,0,0,t-3)\}.$$

  With the interpretation of these 4-tuples as piecewise polynomial functions on the sequence of intervals $[0,1,2,3,4]$, we see that they span the vector space $\{1, t, (t-1)_+^0, (t-1)_+^1, (t-2)_+^0, (t-2)_+^1, (t-3)_+^0, (t-3)_+^1\}$ and hence are also linearly independent.

Modified shifted power bases:

A modified basis of $P_d^k[u_0,\ldots,u_k]$ can be given with any other basis of P_d, such as $\{b_0(t),\ldots,b_d(t)\}$:

$$\{b_0(t), \ldots, b_d(t), (t-u_1)_+^0, (t-u_1)_+^1, \ldots, (t-u_1)_+^d, \ldots, (t-u_{k-1})_+^0, (t-u_{k-1})_+^1, \ldots, (t-u_{k-1})_+^d\}.$$

Examples:

- A basis of $P_2^3[1,3,5,7]$ can be given by:

$$\{1, t-1, (t-1)^2, (t-3)_+^0, (t-3)_+^1, (t-3)_+^2, (t-5)_+^0, (t-5)_+^1, (t-5)_+^2\}$$

  where we have simply replaced the standard basis $\{1, t, t^2\}$ with the shifted basis $\{1, t-1, (t-1)^2\}$.

- Another way to state this basis of $P_2^3[1,3,5,7]$ is also:

$$\{(t-1)_+^0, (t-1)_+^1, (t-1)_+^2, (t-3)_+^0, (t-3)_+^1, (t-3)_+^2, (t-5)_+^0, (t-5)_+^1, (t-5)_+^2\}$$

  where we have simply replaced the shifted basis $\{1, t-1, (t-1)^2\}$ of polynomials, by the set of shifted power functions $\{(t-1)_+^0, (t-1)_+^1, (t-1)_+^2\}$. It is important to understand that these sets of functions are identical on the interval $[1,7]$, which is the declared domain of the functions in this vector space. (A polynomial has implied domain all real numbers, but when it is a member of a vector space such as $P_2^3[1,3,5,7]$ This example illustrates that we can write bases made of functions which are all of the same type.

Writing a ppf in terms of a shifted power basis:

To write a given function in terms of a specific basis means to find the coordinate vector of coefficients which expresses that function as a linear combination of the basis elements. The simplest method is to start with the polynomial that defines the function on the first interval on the left, and use this to solve for some coefficients of basis functions which are nonzero on this interval. Then proceed to the next interval and work right.

Examples:

- Write the function

$$f(t) = \begin{cases} t, & 0 \le t < 1 \\ 1, & 1 \le t < 2 \\ 3-t, & 2 \le t \le 3 \end{cases},$$

in terms of the basis:

$$\{1, t, (t-1)_+^1, (t-1)_+^0, (t-2)_+^1, (t-2)_+^0\}.$$

We seek coefficients $a_0, \ldots, a_5$ such that:

$$f(t) = a_0 \cdot 1 + a_1 \cdot t + a_2 \cdot (t-1)_+^1 + a_3 \cdot (t-1)_+^0 + a_4 \cdot (t-2)_+^1 + a_5 \cdot (t-2)_+^0.$$

Solution: We start with the interval $[0,1)$, and with $f(t) = t$:

$$\text{interval:} \quad [0,1), \quad f(t) = t$$

This gives us the equation:

$$t = a_0 \cdot 1 + a_1 \cdot t + \text{ zero terms.}$$

We don't need to include the other terms since they are zero on the interval $[0,1)$. The above equation easily gives us $a_0 = 0$ and $a_1 = 1$. Our function now reads:

$$f(t) = 0 \cdot 1 + 1 \cdot t + a_2 \cdot (t-1)_+^1 + a_3 \cdot (t-1)_+^0 + a_4 \cdot (t-2)_+^1 + a_5 \cdot (t-2)_+^0.$$

Now we move to the next interval:

$$\text{interval:} \quad [1,2), \quad f(t) = 1$$

This gives us the equation:

$$1 = t + a_2 \cdot (t-1)_+^1 + a_3 \cdot (t-1)_+^0 + \text{ zero terms.}$$

Now when we restrict the shifted power functions to the interval $[1,2)$ we can write them more simply as:

$$1 = t + a_2 \cdot (t-1) + a_3 \cdot 1,$$

or

$$1 - t = a_2 \cdot (t-1) + a_3 \cdot 1,$$

which has solution $a_2 = -1$ and $a_3 = 0$. Our function now reads:

$$f(t) = 0 \cdot 1 + 1 \cdot t - 1 \cdot (t-1)_+^1 + 0 \cdot (t-1)_+^0 + a_4 \cdot (t-2)_+^1 + a_5 \cdot (t-2)_+^0.$$

Now we move to the last interval:

$$\text{interval:} \quad [2,3), \quad f(t) = 3 - t$$

This gives us the equation:

$$3 - t = 1 + a_4 \cdot (t-2)_+^1 + a_5 \cdot (t-2)_+^0,$$

which simplifies to:

$$2 - t = a_4 \cdot (t-2) + a_5 \cdot 1,$$

which has solution $a_4 = -1$ and $a_5 = 0$. This gives us the complete solution:

$$f(t) = 0 \cdot 1 + 1 \cdot t - 1 \cdot (t-1)_+^1 + 0 \cdot (t-1)_+^0 - 1 \cdot (t-2)_+^1 + 0 \cdot (t-2)_+^0.$$

So the coordinate vector of f with respect to this basis is:

$$\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ -1 \\ 0 \end{pmatrix}.$$

Lecture 7, T Jan.29, 2013

Main Points:

- Quiz 3
- The General Interpolation Problem
- Polynomial Interpolation
- Existence and Uniqueness of interpolating polynomial in P_d

The General Interpolation Problem:

The general interpolation problem assumes that we are working with a vector space of functions V, with finite dimension n. Then we suppose that we are given n "data points": $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$, which are interpreted as the desired input and output for some function in V. In other words, the problem is to find a function f in V such that:

$$f(x_1) = y_1, f(x_2) = y_2, \dots, f(x_n) = y_n.$$

If it exists, such a function is often called an *interpolant* to the data that we specified. If we also have a basis B for V, say:

$$B = \{b_1(t), \dots, b_n(t)\}$$

then we can phrase the interpolation problem: to find coefficients $a_1, \dots, a_n$ so that the function

$$f(t) = a_1 b_1(t) + \cdots + a_n b_n(t)$$

satisfies the conditions:

$$f(x_1) = y_1, f(x_2) = y_2, \dots, f(x_n) = y_n.$$

This problem can in turn be converted to a linear system $Ax = b$ where the coefficients $a_1, \dots, a_n$ are the unkowns. There will be a solution, for the coefficients and hence for f, if and only if the matrix A is invertible. Such a function may or may not exist depending on the vector space V and the distribution of the data points.

The Polynomial Interpolation Problem:

For polynomial interpolation, we use $V = P_d$ and the variable t. We also may specify the data points by choosing distinct input values $t_0, \dots, t_d$ and then giving a data function $g(t)$ for the outputs. This is often done because it is natural to use a polynomial $p(t)$ to approximate some other function $g(t)$. The polynomial interpolation problem then takes the form:

Given (distinct) values $t_0, t_1, \dots, t_d$, and a data function $g(t)$, find $p(t) \in P_d$ satisfying: $p(t_i) = g(t_i)$ for $i = 0, \dots, d$.

Existence and Uniqueness of the Interpolating Polynomial:

It is a fact that given any (distinct) values $t_0, t_1, \dots, t_d$, and a data function $g(t)$, there always exists exactly one polynomial $p(t) \in P_d$ satisfying: $p(t_i) = g(t_i)$ for $i = 0, \dots, d$.

Examples:

- Let $t_0 = 1$ and $t_1 = 3$. Also let $g(t) = 3/t$, so that $g(1) = 3$ and $g(3) = 1$. Then the interpolation problem asks for a polynomial $p(t)$ of degree $d = 1$ that satisfies $p(1) = g(1)$ and $p(3) = g(3)$, or in other words the line through the two points $(1, 3)$ and $(3, 1)$, which is

  $$p(t) = -t + 4.$$

  Since this clearly is the only line through the two points, this answer is unique.

Proofs of Existence and Uniqueness of the Interpolating Polynomial:

We will prove the existence and uniqueness in two ways:

1. with the standard basis, and a linear system with Vandermonde determinant,
2. with the Lagrange basis.

1. Standard Basis Proof

When we first introduced the Vandermonde determinant, we did not say where it most naturally arises. The answer is that it comes up naturally in the context of solving a linear system in order to find an interpolating polynomial with respect to the standard basis.

Given (distinct) values $t_0, t_1, \ldots, t_d$, and a data function $g(t)$, we seek a polynomial

$$p(t) = a_0 + a_1 t + \cdots + a_d t^d$$

which has the property that $p(t_i) = g(t_i)$ for $i = 0, \ldots, d$. In order to solve for the coefficients of such a $p(t)$, we set up a linear system with one row for each i of the form:

$$a_0 + a_1 t_i + \cdots + a_d t_i^d = g(t_i).$$

Since the a_j's are the variables, and t_i is a constant, we can also write this as

$$1 \cdot a_0 + t_i \cdot a_1 + \cdots + t_i^d \cdot a_d = g(t_i).$$

Extracting the constants as coefficients of the a_j, we can convert this to a row of an augmented matrix:

$$[1 \;\; t_i \;\; t_i^2 \;\; \cdots \;\; t_i^d \;\; | \;\; g(t_i)].$$

The entire augmented matrix looks like this:

$$\begin{pmatrix} 1 & t_0 & t_0^2 & \cdots & t_0^d & | & g(t_0) \\ 1 & t_1 & t_1^2 & \cdots & t_1^d & | & g(t_1) \\ 1 & t_2 & t_2^2 & \cdots & t_2^d & | & g(t_2) \\ \vdots & \vdots & \vdots & \vdots & \vdots & | & \vdots \\ 1 & t_d & t_d^2 & \cdots & t_d^d & | & g(t_d) \end{pmatrix}.$$

As a matrix equation, it is equivalent to:

$$\begin{pmatrix} 1 & t_0 & t_0^2 & \cdots & t_0^d \\ 1 & t_1 & t_1^2 & \cdots & t_1^d \\ 1 & t_2 & t_2^2 & \cdots & t_2^d \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & t_d & t_d^2 & \cdots & t_d^d \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_d \end{pmatrix} = \begin{pmatrix} g(t_0) \\ g(t_1) \\ g(t_2) \\ \vdots \\ g(t_d) \end{pmatrix}.$$

We know that this matrix has Vandermonde determinant equal to the product

$$\prod_{0 \le i < j \le d} (t_j - t_i),$$

which is clearly not equal to zero as long as the t_i are all distinct, which is required since we ask for $d+1$ distinct inputs in the interpolation problem.

Since the determinant is nonzero, the matrix is invertible and the linear system has exactly one solution. So there is exactly one solution for the coefficients of the polynomial $p(t)$. This completes the first proof of existence and uniqueness for the interpolating polynomial.

2. Lagrange Basis Proof

First we define the Lagrange basis of P_d. Given a sequence of distinct numbers $t_0, t_1, \ldots, t_d$ let

$$L_i(t) = \frac{(t-t_0)(t-t_1)\cdots(t-t_{i-1})(t-t_{i+1})\cdots(t-t_d)}{(t_i-t_0)(t_i-t_1)\cdots(t_i-t_{i-1})(t_i-t_{i+1})\cdots(t_i-t_d)}, \quad i = 0, \ldots, d.$$

For example, if $d = 2$ then

$$L_0(t) = \frac{(t-t_1)(t-t_2)}{(t_0-t_1)(t_0-t_2)}, \quad L_1(t) = \frac{(t-t_0)(t-t_2)}{(t_1-t_0)(t_1-t_2)}, \quad L_2(t) = \frac{(t-t_0)(t-t_1)}{(t_2-t_0)(t_2-t_1)}.$$

An immediate property of the Lagrange polynomials is:

$$L_i(t_i) = 1, \ i = 0, \ldots, d \quad \text{and} \quad L_i(t_j) = 0, \ i \neq j.$$

It is a fact that the set of Lagrange polynomials is a basis of P_d:

$$\{L_0(t), L_1(t), \ldots, L_d(t)\}.$$

We can prove this fact by showing that they are linearly independent. We appeal to the definition of linear independence. So, suppose that there is a linear combination of these polynomials equal to the zero polynomial (we write $0(t)$ for the zero polynomial to emphasize that it is not simply the number zero):

$$c_0 L_0(t) + c_1 L_1(t) + \cdots + c_d L_d(t) = 0(t).$$

To see that all the coefficients must be zero (which is required in the definition of linear independence) we simply plug in the numbers $t_0, t_1, \ldots, t_d$ to the equation, and use the above property, giving the equations: $c_0 = 0, c_1 = 0, \ldots, c_d = 0$ respectively.

Next, we write the interpolating polynomial to the data values $t_0, \ldots, t_d$ with data function $g(t)$ as:

$$p(t) = g(t_0)L_0(t) + g(t_1)L_1(t) + \cdots + g(t_p)L_p(t).$$

Again, we can simply use the above property, and plug in the data values, to see that indeed

$$p(t_i) = g(t_i), \ i = 0, \ldots, d.$$

This shows the existence of the interpolating polynomial in P_d. In order to show the uniqueness of $p(t)$, we suppose that there is another polynomial $q(t)$ in P_d satisfying:

$$q(t_i) = g(t_i), \ i = 0, \ldots, d.$$

Then we can write such a polynomial $q(t)$ in terms of the Lagrange basis as:

$$q(t) = b_0 L_0(t) + b_1 L_1(t) + \cdots + b_p L_p(t).$$

But then we also plug in the data values to this equation and find that

$$q(t_i) = b_i, \ i = 0, \ldots, d,$$

which implies in turn that

$$b_i = q(t_i) = g(t_i), \ i = 0, \ldots, d,$$

which means that

$$q(t) = g(t_0)L_0(t) + g(t_1)L_1(t) + \cdots + g(t_p)L_p(t) = p(t).$$

This shows that $p(t)$, the interpolating polynomial, is unique in P_d, and this completes the proof of the existence and uniqueness of the interpolating polynomial using the Lagrange basis.

Examples:

- Find the interpolating polynomial $p(t)$ in P_2 which matches the data function $g(t) = 2^t$ for data values $t_0 = 0$, $t_1 = 1$ and $t_2 = 2$. With the standard basis we assume the form

  $$p(t) = a_0 + a_1 t + a_2 t^2$$

  and we then get the linear system:

  $$\begin{aligned} 1 \cdot a_0 + 0 \cdot a_1 + 0^2 \cdot a_2 &= g(0) \\ 1 \cdot a_0 + 1 \cdot a_1 + 1^2 \cdot a_2 &= g(1) \\ 1 \cdot a_0 + 2 \cdot a_1 + 2^2 \cdot a_2 &= g(2) \end{aligned}$$

  which also simplifies to the matrix equation:

  $$\begin{pmatrix} 1 & 0 & 0^2 \\ 1 & 1 & 1^2 \\ 1 & 2 & 2^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}.$$

  The augmented matrix then reduces with Gaussian elimination to:

  $$\begin{pmatrix} 1 & 0 & 0 & | & 1 \\ 1 & 1 & 1 & | & 2 \\ 1 & 2 & 4 & | & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 1 & | & 1 \\ 0 & 2 & 4 & | & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 1 & | & 1 \\ 0 & 0 & 2 & | & 1 \end{pmatrix}$$

  $$\longrightarrow \begin{pmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 1 & | & 1/2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 1 & | & 1/2 \\ 0 & 0 & 1 & | & 1/2 \end{pmatrix}$$

  which tells us that the interpolating polynomial is:

  $$p(t) = 1 + \frac{1}{2}t + \frac{1}{2}t^2.$$

- Now we find $p(t)$ with the same data using the Lagrange polynomials:

  $$\begin{aligned} p(t) &= g(0)L_0(t) + g(1)L_1(t) + g(2)L_2(t) \\ &= 1 \cdot \frac{(t-1)(t-2)}{(0-1)(0-2)} + 2 \cdot \frac{(t-0)(t-2)}{(1-0)(1-2)} + 4 \cdot \frac{(t-0)(t-1)}{(2-0)(2-1)} \\ &= \frac{1}{2}(t-1)(t-2) - 2t(t-2) + 2t(t-1) \\ &= \frac{1}{2}(t^2 - 3t + 2) - 2t^2 + 4t + 2t^2 - 2t \\ &= 1 + \frac{1}{2}t + \frac{1}{2}t^2. \end{aligned} \tag{1}$$

Lecture 8, Th Jan.31, 2013

Main Points:

- Recursive form and third proof of existence and uniqueness
- Divided differences
- Newton form and basis
- Leibniz formula

Advantages of the Newton Form

The two previous forms of the interpolating polynomial, in standard basis and Lagrange basis, have some disadvantages which can be removed by considering the Newton form. In particular, adding a new interpolation point in the previous two cases requires setting up a new linear system or a new set of Lagrange polynomials. In the case of the Newton form, we can recycle all of our work and simply add one more term to the interpolating polynomial in order to match one more point. Before we define the Newton form, we will first give another method to prove the existence and uniqueness of the interpolating polynomial, which will then give rise to the Newton form.

Third proof of Existence and Uniqueness of Interpolating Polynomial.

This proof is based on a recursive formula and an induction argument. So we will need a base case, with $d = 0$. The polynomial interpolation problem for one data point is trivial. In particular, to find the polynomial of degree zero that matches a data function $g(t)$ at t_0, we simply write the constant function

$$p(t) = g(t_0).$$

This is the base case. Now, for the induction step we make the assumption that the interpolating polynomial exists for degrees $\leq d$. This is the induction hypothesis. Then assuming that we have data values $t_0, \ldots, t_d$, and data function $g(t)$, we can define, based on the induction hypothesis, two polynomials of degree $\leq d-1$:

$$p_0(t), \quad \text{with data values } \; t_0, t_1, \ldots, t_{d-1},$$

and

$$p_1(t), \quad \text{with data values } \; t_1, t_2, \ldots, t_d.$$

This means that

$$p_0(t_i) = g(t_i), \; i = 0, \ldots, d-1, \quad \text{and} \quad p_1(t_i) = g(t_i), \; i = 1, \ldots, d.$$

Assuming the existence of these two polynomials, we then define:

$$p(t) = \frac{t - t_0}{t_d - t_0} p_1(t) + \frac{t_d - t}{t_d - t_0} p_0(t).$$

One easily checks now that

$$p(t_0) = p_0(t_0) = g(t_0) \quad \text{and} \quad p(t_d) = p_1(t_d) = g(t_d).$$

Then we can also check the middle values t_i with $1 \leq i \leq d-1$, using the fact that for such i we have $p_0(t_i) = g(t_i) = p_1(t_i)$:

$$\begin{aligned} p(t_i) &= \frac{t_i - t_0}{t_d - t_0} p_1(t_i) + \frac{t_d - t_i}{t_d - t_0} p_0(t_i) \\ &= \frac{t_i - t_0}{t_d - t_0} g(t_i) + \frac{t_d - t_i}{t_d - t_0} g(t_i) \end{aligned}$$

$$
\begin{aligned}
&= \left[\frac{t_i - t_0}{t_d - t_0} + \frac{t_d - t_i}{t_d - t_0}\right] g(t_i) \\
&= \left[\frac{t_d - t_0}{t_d - t_0}\right] g(t_i) \\
&= g(t_i)
\end{aligned}
$$

Note that since $p_0(t)$ and $p_1(t)$ each have degree $\leq d-1$ it must be that $p(t)$ is a sum of polynomials of degree $\leq d$ and hence $p(t)$ is in P_d. This shows that the interpolating polynomial $p(t)$ exists in P_d.

To prove again the uniqueness of the interpolating polynomial, we suppose that two such polynomials $p(t)$ and $q(t)$ exist in P_d, and define:

$$f(t) = p(t) - q(t).$$

Then $f(t)$ is also in P_d and since $p(t)$ and $q(t)$ satisfy the interpolation conditions, we have:

$$f(t_i) = p(t_i) - q(t_i) = g(t_i) - g(t_i) = 0, \quad i = 0, \ldots, d.$$

Thus $f(t)$ is a polynomial in P_d with $d+1$ distinct zeros $t_0, \ldots, t_d$. But a nonzero polynomial of degree at most d can have at most d zeros, since each zero corresponds to a factor of f. Only the zero polynomial in P_d can have more than d zeros, so in fact it must be that $f(t) = 0(t)$ which means that $p(t) = q(t)$. This shows that the interpolating polynomial is unique in P_d.

Definition of the Newton basis:

Given d real numbers $t_0, \ldots, t_{d-1}$, which may or may not be distinct, we define the Newton basis of P_d as

$$\{1, t - t_0, (t - t_0)(t - t_1), (t - t_0)(t - t_1)(t - t_2), \ldots, (t - t_0)(t - t_1)(t - t_2) \cdots (t - t_{d-1})\}.$$

This can easily be seen to be linearly independent since the elements are of increasing degree, giving a triangular matrix of coordinate vectors with respect to the standard basis, with determinant 1.

Examples:

- The Newton basis of P_2 for the values $t_0 = 1$ and $t_1 = 3$ is:

$$\{1, t - 1, (t - 1)(t - 3)\}.$$

- The Newton basis of P_2 for the values $t_0 = 3$ and $t_1 = 3$ is:

$$\{1, t - 3, (t - 3)^2\},$$

  which is also a shifted basis.

In order to define the Newton form we need to define divided differences. The divided difference is a number which is obtained from the data specified for a polynomial interpolation problem. In other words, the input is the data $t_0, t_1, \ldots, t_d$ and g, and the output is the number $[t_0, t_1, \ldots, t_d]g$. As we see below, this number is defined using the interpolating polynomial $p(t)$.

Definition of divided differences:

The divided difference $[t_0, t_1, \ldots, t_d]g$ is defined to be the coefficient of t^d in the interpolating polynomial $p(t)$ in P_d with data function g and data values $t_0, t_1, \cdots, t_d$. In other words, if $p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_d t^d$ in standard basis form, then

$$[t_0, t_1, \ldots, t_d]g = a_d.$$

Note: because the interpolating polynomial depends only on the input data and data function $g(t)$, it is not dependent on the order of the data. This also means that the order of the data values in the divided difference can be changed without affecting the outcome.

Examples:

- Find the divided difference $[0,1,2]g$ with $g(t) = 2^t$. Since we already worked out the interpolating polynomial $p(t)$ above to be $1 + \frac{1}{2}t + \frac{1}{2}t^2$, we see from the definition that $[0,1,2]$ is the coefficient of t^2 which is $\frac{1}{2}$, so

$$[0,1,2]g = \frac{1}{2}.$$

  Note: From the comments above, since we are only taking the coefficient of t^2, the input data can be in any order, so we have:

$$[0,1,2]g = [0,2,1]g = [1,0,2]g = [1,2,0]g = [2,0,1]g = [2,1,0]g = \frac{1}{2}.$$

- Find the divided difference $[0,1]$ with $g(t) = 2^t$. For this data, the interpolating polynomial is just the line through the points $(0, g(0))$ and $(1, g(1))$, or $(0,1)$ and $(1,2)$. This line is $p(t) = 1 + t$. So

$$[0,1]g = 1.$$

Since the sequence $t_0, t_1, \ldots, t_d$ has many possible subsequences, each of which may be used in the recursion below, we may write $t_i, t_{i+1}, \ldots, t_{i+k}$ to represent the most general such sequence.

Recursion Property for divided differences:

The divided differences, defined above, also satisfy the recursion:

$$[t_0, t_1, \ldots, t_d]g = \frac{[t_1, t_2, \ldots, t_d]g - [t_0, t_1, \ldots, t_{d-1}]g}{t_d - t_0}.$$

Similarly:

$$[t_i, t_{i+1}, \ldots, t_{i+k}]g = \frac{[t_{i+1}, t_{i+2}, \ldots, t_{i+k}]g - [t_i, t_{i+1}, \ldots, t_{i+k-1}]g}{t_{i+k} - t_i}.$$

Examples:

- The divided differences can be computed in the following table format, here for $d = 2$.

$$\begin{array}{llll}
t_0 & [t_0]g = g(t_0) & & \\
 & & [t_0,t_1]g = \frac{[t_1]g-[t_0]g}{t_1-t_0} & \\
t_1 & [t_1]g = g(t_1) & & [t_0,t_1,t_2]g = \frac{[t_1,t_2]g-[t_0,t_1]g}{t_2-t_0} \\
 & & [t_1,t_2]g = \frac{[t_2]g-[t_1]g}{t_2-t_1} & \\
t_2 & [t_2]g = g(t_2) & &
\end{array}$$

- Here is an example with specific inputs $t_0 = 0$, $t_1 = 1$, $t_2 = 2$, and $g(0) = 3$, $g(1) = -2$, and $g(2) = 1$.

$$\begin{array}{llll}
0 & 3 & & \\
 & & \frac{-2-3}{1-0} = -5 & \\
1 & -2 & & \frac{3-(-5)}{2-0} = 4 \\
 & & \frac{1-(-2)}{2-1} = 3 & \\
2 & 1 & &
\end{array}$$

Proof of the Recursive Property

To prove the recursive property for divided differences we use the recursive form of the interpolating polynomial:

$$p(t) = \frac{t - t_0}{t_d - t_0} p_1(t) + \frac{t_d - t}{t_d - t_0} p_0(t),$$

where the polynomials $p_0(t)$ and $p_1(t)$ in P_{d-1} are also interpolating polynomials:

$$p_0(t), \quad \text{with data values } \ t_0, t_1, \ldots, t_{d-1},$$

and

$$p_1(t), \quad \text{with data values } \ t_1, t_2, \ldots, t_d.$$

Now we recall the definition of the operator $[t_0, \ldots, t_d]g$ as the coefficient of t^d in $p(t)$. To extract the coefficients of t^d we suppose that

$$p_0(t) = a_0 + a_1 t + \cdots + a_{d-1} t^{d-1},$$

and

$$p_1(t) = b_0 + b_1 t + \cdots + b_{d-1} t^{d-1}.$$

Then:

$$p(t) = \frac{t - t_0}{t_d - t_0} \left(a_0 + a_1 t + \cdots + a_{d-1} t^{d-1} \right) + \frac{t_d - t}{t_d - t_0} \left(b_0 + b_1 t + \cdots + b_{d-1} t^{d-1} \right).$$

By the induction hypothesis, we also can say that

$$a_{d-1} = [t_0, t_1, \ldots, t_{d-1}]g,$$

and

$$b_{d-1} = [t_1, t_2, \ldots, t_d]g.$$

So, the coefficient of t^d in $p(t)$ is

$$\begin{aligned}
[t_0, t_1, \ldots, t_d]g &= \frac{1}{t_d - t_0} a_{d-1} + \frac{-1}{t_d - t_0} b_{d-1} \\
&= \frac{1}{t_d - t_0} [t_0, t_1, \ldots, t_{d-1}]g + \frac{-1}{t_d - t_0} [t_1, t_2, \ldots, t_d]g \\
&= \frac{[t_1, t_2, \ldots, t_d]g - [t_0, t_1, \ldots, t_{d-1}]g}{t_d - t_0}.
\end{aligned}$$

Newton Form of the interpolating polynomial

The expansion of the interpolating polynomial $p(t)$ with data $t_0, t_1, \cdots, t_d$ and g, in terms of the Newton basis is called the Newton Form, and can be written as:

$$p(t) = \sum_{i=0}^{d} [t_0, \ldots, t_i] g N_i(t),$$

where $N_0(t) = 1$, $N_1(t) = t - t_0$, $N_2(t) = (t - t_0)(t - t_1)$, $\ldots$, and $N_d(t) = (t - t_0)(t - t_1) \cdots (t - t_{d-1})$.

Examples:

- Find the Newton form of the interpolating polynomial $p(t)$ for the specific inputs $t_0 = 2$, $t_1 = 4$, and $g(2) = 6$, $g(4) = -2$. The divided difference table is:

$$\begin{array}{ccc} t_0 & [t_0]g & \\ & & [t_0, t_1]g \\ t_1 & [t_1]g & \end{array} \qquad \text{or} \qquad \begin{array}{ccc} 2 & 6 & \\ & & \frac{-2-6}{4-2} = -4 \\ 4 & -2 & \end{array}$$

The numbers along the top diagonal are the coefficients in the Newton form:

$$\begin{aligned} p(t) &= [t_0]g + [t_0,t_1]g\cdot(t-t_0) \\ &= 6 + (-4)\cdot(t-2) \\ &= 14 - 4t, \end{aligned}$$

which is easily seen to satisfy the interpolation conditions: $p(2) = 6$, $p(4) = -2$.

- Find the Newton form of the interpolating polynomial $p(t)$ for the specific inputs $t_0 = 0$, $t_1 = 1$, $t_2 = 2$, and $g(0) = 3$, $g(1) = -2$, and $g(2) = 1$. The divided difference table which we computed in a previous example is:

$$\begin{array}{cccc} t_0 & [t_0]g & & \\ & & [t_0,t_1]g & \\ t_1 & [t_1]g & & [t_0,t_1,t_2]g \\ & & [t_1,t_2]g & \\ t_2 & [t_2]g & & \end{array} \quad \text{or} \quad \begin{array}{cccc} 0 & 3 & & \\ & & \frac{-2-3}{1-0} = -5 & \\ 1 & -2 & & \frac{3-(-5)}{2-0} = 4 \\ & & \frac{1-(-2)}{2-1} = 3 & \\ 2 & 1 & & \end{array}$$

The numbers along the top diagonal are the coefficients in the Newton form:

$$\begin{aligned} p(t) &= [t_0]g + [t_0,t_1]g\cdot(t-t_0) + [t_0,t_1,t_2]g\cdot(t-t_0)(t-t_1) \\ &= 3 + (-5)\cdot(t-0) + 4\cdot(t-0)(t-1) \\ &= 3 - 5t + 4t(t-1), \end{aligned}$$

which is easily seen to satisfy the interpolation conditions: $p(0) = 3$, $p(1) = -2$, and $p(2) = 1$.

- Find the Newton form of the interpolating polynomial $p(t)$ for the specific inputs $t_0 = 0$, $t_1 = 1$, $t_2 = 2$, $t_3 = 3$, and $g(0) = 2$, $g(1) = 3$, $g(2) = 4$, and $g(3) = -1$. The divided difference table is:

$$\begin{array}{ccccc} t_0 & [t_0]g & & & \\ & & [t_0,t_1]g & & \\ t_1 & [t_1]g & & [t_0,t_1,t_2]g & \\ & & [t_1,t_2]g & & [t_0,t_1,t_2,t_3]g \\ t_2 & [t_2]g & & [t_1,t_2,t_3]g & \\ & & [t_2,t_3]g & & \\ t_3 & [t_3]g & & & \end{array} \quad \text{or} \quad \begin{array}{ccccc} 0 & 2 & & & \\ & & \frac{3-2}{1-0} = 1 & & \\ 1 & 3 & & \frac{1-1}{2-0} = 0 & \\ & & \frac{4-3}{2-1} = 1 & & \frac{-3-0}{3-0} = -1 \\ 2 & 4 & & \frac{-5-1}{3-1} = -3 & \\ & & \frac{-1-4}{3-2} = -5 & & \\ 3 & -1 & & & \end{array}$$

The numbers along the top diagonal are the coefficients in the Newton form:

$$\begin{aligned} p(t) &= [t_0]g + [t_0,t_1]g\cdot(t-t_0) + [t_0,t_1,t_2]g\cdot(t-t_0)(t-t_1)[t_0,t_1,t_2,t_3]g\cdot(t-t_0)(t-t_1)(t-t_2) \\ &= 2 + 1\cdot(t-0) + 0\cdot(t-0)(t-1) - 1\cdot(t-0)(t-1)(t-2) \\ &= 2 + t - t(t-1)(t-2), \end{aligned}$$

which is easily seen to satisfy the interpolation conditions: $p(0) = 2$, $p(1) = 3$, $p(2) = 4$ and $p(3) = -1$.

- Find the Newton form of the interpolating polynomial $p(t)$ for the specific inputs $t_0 = 0$, $t_1 = 1$, $t_2 = 2$, and $g(0) = 2$, $g(1) = 3$, and $g(2) = 4$. Note: This is just the first three data points from the previous example. Also note: the three points are collinear, so we already know that there is a linear polynomial passing through them. The divided difference table is:

$$\begin{array}{cccc}
0 & 2 & & \\
 & & \frac{3-2}{1-0}=1 & \\
1 & 3 & & \frac{1-1}{2-0}=0 \\
 & & \frac{4-3}{2-1}=1 & \\
2 & 4 & &
\end{array}$$

The numbers along the top diagonal are the coefficients in the Newton form:

$$\begin{aligned}
p(t) &= [t_0]g + [t_0,t_1]g\cdot(t-t_0) + [t_0,t_1,t_2]g\cdot(t-t_0)(t-t_1) \\
&= 2 + 1\cdot(t-0) + 0\cdot(t-0)(t-1) \\
&= 2+t,
\end{aligned}$$

which is easily seen to satisfy the interpolation conditions: $p(0)=2$, $p(1)=3$, $p(2)=4$.

Leibniz' Rule for Divided Differences

Let $f(t)=g(t)h(t)$. Then

$$[t_i,t_{i+1},\ldots,t_{i+k}]f = \sum_{r=i}^{i+k}([t_i,\ldots,t_r]g)([t_r,\ldots,t_{i+k}]h).$$

Note: Draw the divided difference triangles for g and h, and label them T_1 and T_2. Then make two lists, first along the top of T_1 from left to right, then along the bottom of T_2, from right to left. Each of these lists has length $k+1$ and can be viewed as a vector. The dot product of these two vectors is the same as the sum above.

Examples:

- For degree $d=2$ the Leibniz formula looks like this:

  $$[t_0,t_1,t_2]f = [t_0]g[t_0,t_1,t_2]h + [t_0,t_1]g[t_1,t_2]h + [t_0,t_1,t_2]g[t_2]h.$$

  The coefficients come from the two divided difference tables:

  $$\begin{array}{cccc}
t_0 & [t_0]g & & \\
 & & [t_0,t_1]g & \\
t_1 & [t_1]g & & [t_0,t_1,t_2]g \\
 & & [t_1,t_2]g & \\
t_2 & [t_2]g & &
\end{array} \qquad \text{and} \qquad \begin{array}{cccc}
t_0 & [t_0]h & & \\
 & & [t_0,t_1]h & \\
t_1 & [t_1]h & & [t_0,t_1,t_2]h \\
 & & [t_1,t_2]h & \\
t_2 & [t_2]h & &
\end{array}$$

- Let $f(t)=|t|(t-2)^2$, with $g(t)=|t|$, and $h(t)=(t-2)^2$, and take $t_0=-1$, $t_1=0$, and $t_2=1$. We then have the tables:

  $$\begin{array}{cccc}
-1 & 1 & & \\
 & & -1 & \\
0 & 0 & & 1 \\
 & & 1 & \\
1 & 1 & &
\end{array} \qquad \text{and} \qquad \begin{array}{cccc}
-1 & 9 & & \\
 & & -5 & \\
0 & 4 & & 1 \\
 & & -3 & \\
1 & 1 & &
\end{array}$$

  Then we have the corresponding Leibniz formula for $[-1,0,1]f$:

$$\begin{aligned}[-1,0,1]f &= [-1]g[-1,0,1]h + [-1,0]g[0,1]h + [-1,0,1]g[1]h \\ &= (1)(1) + (-1)(-3) + (1)(1) \\ &= 5\end{aligned}$$

We can confirm this by constructing the table for f alone:

$$\begin{array}{cccc} -1 & 9 & & \\ & & -9 & \\ 0 & 0 & & 5 \\ & & 1 & \\ 1 & 1 & & \end{array}$$

This confirms directly that $[-1,0,1]f = 5$.

- Let $f(t) = (t-2)_+^2(t-2)$, with $g(t) = (t-2)_+^2$, and $h(t) = t-2$, and take $t_0 = 1$, $t_1 = 2$, and $t_2 = 3$. We then have the tables:

$$\begin{array}{cccc} 1 & 0 & & \\ & & 0 & \\ 2 & 0 & & \frac{1}{2} \\ & & 1 & \\ 3 & 1 & & \end{array} \qquad \text{and} \qquad \begin{array}{cccc} 1 & -1 & & \\ & & 1 & \\ 2 & 0 & & 0 \\ & & 1 & \\ 3 & 1 & & \end{array}$$

Then we have the corresponding Leibniz formula for $[1,2,3]f$:

$$\begin{aligned}[1,2,3]f &= [1]g[1,2,3]h + [1,2]g[2,3]h + [1,2,3]g[3]h \\ &= (0)(0) + (0)(1) + (\frac{1}{2})(1) \\ &= \frac{1}{2}\end{aligned}$$

We can confirm this by constructing the table for f alone:

$$\begin{array}{cccc} 1 & 0 & & \\ & & 0 & \\ 2 & 0 & & \frac{1}{2} \\ & & 1 & \\ 3 & 1 & & \end{array}$$

Lecture 9, T Feb.5, 2013

Main Points:

- Quiz 4
- Proof of Newton form
- Proof of Leibniz' Rule
- Osculating polynomials

Proof of the Newton form:

To establish the validity of the Newton form, we suppose that the interpolating polynomial $p(t)$ with (distinct) data values $t_0, \ldots, t_d$ and data function $g(t)$ exists and is unique in P_d, and also that the interpolating polynomial $p_0(t)$ with the subset of data values $t_0, t_1, \ldots, t_{d-1}$ exists and is unique in P_{d-1}. For convenience, we will rename the polynomial $p_0(t)$ as $q_{d-1}(t)$, to indicate that it has degree $d-1$. Then we can consider the polynomial

$$f(t) = p(t) - q_{d-1}(t),$$

which is also in P_d, and has the property

$$f(t_i) = p(t_i) - q_{d-1}(t_i) = g(t_i) - g(t_i) = 0, \quad i = 0, \ldots, d-1.$$

This says that we can factor the polynomial $f(t)$ to obtain:

$$f(t) = C \cdot (t - t_0)(t - t_1) \cdots (t - t_{d-1}),$$

where C is the coefficient of t^d in this polynomial. But since $q_{d-1}(t)$ has degree at most $d-1$, we see that C is the coefficient of t^d in $p(t)$, which is by definition the divided difference:

$$C = [t_0, t_1, \ldots, t_d]g.$$

Thus we obtain:

$$p(t) - q_{d-1}(t) = [t_0, t_1, \ldots, t_d]g \cdot (t - t_0)(t - t_1) \cdots (t - t_{d-1}),$$

or:

$$p(t) = q_{d-1}(t) + [t_0, t_1, \ldots, t_d]g \cdot (t - t_0)(t - t_1) \cdots (t - t_{d-1}).$$

Now, the argument we just applied to $p(t)$ can also be carried out for $q_{d-1}(t)$, supposing that $q_{d-2}(t)$ is the interpolating polynomial with the data values: $t_0, t_1, \ldots, t_{d-2}$, and we obtain:

$$q_{d-1}(t) = q_{d-2}(t) + [t_0, t_1, \ldots, t_{d-1}]g \cdot (t - t_0)(t - t_1) \cdots (t - t_{d-2}),$$

Continuing in this we way, we will eventually arrive at the statement

$$q_1(t) = q_0(t) + [t_0, t_1]g \cdot (t - t_0),$$

where $q_0(t)$ is the interpolating polynomial of degree 0 with the data value t_0 and data function $g(t)$, in other words $q_0(t) = g(t_0) = [t_0]g$ is constant. Piecing all of this back together, we arrive at the Newton form:

$$p(t) = [t_0]g + [t_0, t_1]g \cdot (t - t_0) + [t_0, t_1, t_2]g \cdot (t - t_0)(t - t_1) + \cdots + [t_0, t_1, \ldots, t_d]g \cdot (t - t_0)(t - t_1) \cdots (t - t_{d-1}).$$

Proof of Leibniz' Rule for Divided Differences

Let $f(t) = g(t)h(t)$. Then the general form of Leibniz' Rule looks like:

$$[t_i, t_{i+1}, \ldots, t_{i+k}]f = \sum_{r=i}^{i+k}([t_i, \ldots, t_r]g)([t_r, \ldots, t_{i+k}]h).$$

For degree $d = 2$ the Leibniz formula looks like this:

$$[t_0, t_1, t_2]f = [t_0]g[t_0, t_1, t_2]h + [t_0, t_1]g[t_1, t_2]h + [t_0, t_1, t_2]g[t_2]h.$$

First let's check the base case, $d = 0$. In this case there is only one entry in each of the tables for f, g, and h. In particular, we have

$$[t_0]f = f(t_0) = g(t_0)h(t_0) = [t_0]g[t_0]h.$$

We can also do the linear case $d = 1$ which says that:

$$[t_0, t_1]f = [t_0]g \cdot [t_0, t_1]h + [t_0, t_1]g \cdot [t_1]h.$$

We can check this easily since the divided differences in this case are just the slope of a line through two points. In other words:

$$[t_0, t_1]f = \frac{f(t_1) - f(t_0)}{t_1 - t_0}, \qquad [t_0, t_1]g = \frac{g(t_1) - g(t_0)}{t_1 - t_0}, \qquad \text{and} \qquad [t_0, t_1]h = \frac{h(t_1) - h(t_0)}{t_1 - t_0}.$$

Then the right hand side of Leibniz' rule becomes:

$$\begin{aligned}
[t_0]g \cdot [t_0, t_1]h + [t_0, t_1]g \cdot [t_1]h &= g(t_0) \cdot \frac{h(t_1) - h(t_0)}{t_1 - t_0} + \frac{g(t_1) - g(t_0)}{t_1 - t_0} h(t_1) \\
&= \\
&= \frac{g(t_0)h(t_1) - g(t_0)h(t_0) + g(t_1)h(t_1) - g(t_0)h(t_1)}{t_1 - t_0} \\
&= \\
&= \frac{g(t_1)h(t_1) - g(t_0)h(t_0)}{t_1 - t_0} \\
&= \\
&= \frac{f(t_1) - f(t_0)}{t_1 - t_0} \\
&= \\
&= [t_0, t_1]f
\end{aligned}$$

For the degree $d = 2$ case, we give a proof which also extends to the higher degree cases, using the Newton forms with data values t_0, t_1, and t_2, and each of the data functions f, g, and h. Call these Newton forms $p(t)$, $q(t)$ and $r(t)$, respectively. We will write the Newton forms slightly differently, taking advantage of the fact that the order of the data values does not matter. In particular, we will reverse the order of the data values for the Newton form $r(t)$, with data function h. The divided difference tables for $q(t)$ and $r(t)$ then look like this:

$$\begin{array}{cccc}
t_0 & [t_0]g & & \\
 & & [t_0, t_1]g & \\
t_1 & [t_1]g & & [t_0, t_1, t_2]g \\
 & & [t_1, t_2]g & \\
t_2 & [t_2]g & &
\end{array}
\qquad \text{and} \qquad
\begin{array}{cccc}
t_2 & [t_2]h & & \\
 & & [t_1, t_2]h & \\
t_1 & [t_1]h & & [t_0, t_1, t_2]h \\
 & & [t_0, t_1]h & \\
t_0 & [t_0]h & &
\end{array}$$

It is tempting to think that the product of the Newton forms q and r is equal to the Newton form p. This is because of the property:

$$q(t_i) \cdot r(t_i) = g(t_i) \cdot h(t_i) = f(t_i), \quad i = 0, 1, 2.$$

This says that the product $q(t)r(t)$ satisfies the interpolation conditions. However, this product is only guaranteed to have degree ≤ 4, but we know that $p(t)$ exists in P_2. But even though the product does not give the Newton form p, we can still obtain p by subtracting off part of the product.

From the above tables we obtain the two Newton forms:

$$q(t) = [t_0]g + [t_0, t_1]g \cdot (t - t_0) + [t_0, t_1, t_2]g \cdot (t - t_0)(t - t_1),$$

and

$$r(t) = [t_2]h + [t_1, t_2]h \cdot (t - t_2) + [t_0, t_1, t_2]h \cdot (t - t_1)(t - t_2).$$

Then the product of these is:

$$\begin{aligned} q(t) \cdot r(t) &= ([t_0]g + [t_0, t_1]g \cdot (t - t_0) + [t_0, t_1, t_2]g \cdot (t - t_0)(t - t_1)) \\ &\quad \cdot ([t_2]h + [t_1, t_2]h \cdot (t - t_2) + [t_0, t_1, t_2]h \cdot (t - t_2)(t - t_1)) \\ &= F(t) + G(t) \end{aligned}$$

where

$$\begin{aligned} F(t) &= [t_0]g \cdot [t_2]h \quad \text{(degree 0 term)} \\ &+ [t_0]g \cdot [t_1, t_2]h \cdot (t - t_2) + [t_2]h \cdot [t_0, t_1]g \cdot (t - t_0) \quad \text{(degree 1 terms)} \\ &+ [t_0]g \cdot [t_0, t_1, t_2]h \cdot (t - t_2)(t - t_1) \\ &+ [t_0, t_1]g \cdot [t_1, t_2]h \cdot (t - t_0)(t - t_2) \quad \text{(degree 2 terms)} \\ &+ [t_0, t_1, t_2]g \cdot [t_2]h \cdot (t - t_0)(t - t_1) \end{aligned}$$

and

$$\begin{aligned} G(t) &= [t_0, t_1]g \cdot [t_0, t_1, t_2]h \cdot (t - t_0)(t - t_1)(t - t_2) \\ &+ [t_1, t_2]h \cdot [t_0, t_1, t_2]g \cdot (t - t_0)(t - t_1)(t - t_2) \quad \text{(degree 3 terms)} \\ &+ [t_0, t_1, t_2]g \cdot [t_0, t_1, t_2]h \cdot (t - t_0)(t - t_1)^2(t - t_2). \quad \text{(degree 4 term)} \end{aligned}$$

Next, we note that $G(t_0) = G(t_1) = G(t_2) = 0$. But then we have

$$\begin{aligned} q(t_i) \cdot r(t_i) &= g(t_i) \cdot h(t_i) = f(t_i) \\ &= F(t_i) + G(t_i) \\ &= F(t_i) \end{aligned}$$

for $i = 0, 1, 2$. In summary,

$$F(t_i) = f(t_i), \quad i = 0, 1, 2,$$

and $F(t)$ is in P_2. By the uniqueness of the interpolating polynomial we must have

$$F(t) = p(t).$$

Finally, to get the divided difference $[t_0, t_1, t_2]f$ we simply extract the coefficient of t^2 in $p(t)$, which is the same as $F(t)$. From the above we can see that the t^2 terms for $F(t)$ have exactly the coefficients predicted by Leibniz' rule:

$$[t_0, t_1, t_2]f = [t_0]g \cdot [t_0, t_1, t_2]h + [t_0, t_1]g \cdot [t_1, t_2]h + [t_0, t_1, t_2]g \cdot [t_2]h.$$

This completes the proof of Leibniz' rule for $d = 2$.

For the general case, we can use the same proof as for $d = 2$, obtaining the sum $F(t) + G(t)$, where $F(t)$ has degree $\leq d$ and $G(t_i) = 0$ for $i = 0, \ldots, d$. We can then extract Leibniz' rule from the sum of terms which contain t^d.

Examples:

- Let $f(t) = |t|(t-2)^2$, with $g(t) = |t|$, and $h(t) = (t-2)^2$, and take $t_0 = -1$, $t_1 = 0$, and $t_2 = 1$. We then have the tables:

$$\begin{array}{cccc} -1 & 1 & & \\ & & -1 & \\ 0 & 0 & & 1 \\ & & 1 & \\ 1 & 1 & & \end{array} \qquad \text{and} \qquad \begin{array}{cccc} -1 & 9 & & \\ & & -5 & \\ 0 & 4 & & 1 \\ & & -3 & \\ 1 & 1 & & \end{array}$$

  Then we have the corresponding Leibniz formula for $[-1,0,1]f$:

$$\begin{aligned} [-1,0,1]f &= [-1]g[-1,0,1]h + [-1,0]g[0,1]h + [-1,0,1]g[1]h \\ &= (1)(1) + (-1)(-3) + (1)(1) \\ &= 5 \end{aligned}$$

  We can confirm this by constructing the table for f alone:

$$\begin{array}{cccc} -1 & 9 & & \\ & & -9 & \\ 0 & 0 & & 5 \\ & & 1 & \\ 1 & 1 & & \end{array}$$

  This confirms directly that $[-1,0,1]f = 5$.

- Let $f(t) = (t-2)_+^2(t-2)$, with $g(t) = (t-2)_+^2$, and $h(t) = t-2$, and take $t_0 = 1$, $t_1 = 2$, and $t_2 = 3$. We then have the tables:

$$\begin{array}{cccc} 1 & 0 & & \\ & & 0 & \\ 2 & 0 & & \frac{1}{2} \\ & & 1 & \\ 3 & 1 & & \end{array} \qquad \text{and} \qquad \begin{array}{cccc} 1 & -1 & & \\ & & 1 & \\ 2 & 0 & & 0 \\ & & 1 & \\ 3 & 1 & & \end{array}$$

  Then we have the corresponding Leibniz formula for $[1,2,3]f$:

$$\begin{aligned} [1,2,3]f &= [1]g[1,2,3]h + [1,2]g[2,3]h + [1,2,3]g[3]h \\ &= (0)(0) + (0)(1) + (\frac{1}{2})(1) \\ &= \frac{1}{2} \end{aligned}$$

  We can confirm this by constructing the table for f alone:

$$
\begin{array}{cccc}
1 & 0 & & \\
 & & 0 & \\
2 & 0 & & \frac{1}{2} \\
 & & 1 & \\
3 & 1 & &
\end{array}
$$

Definition of the osculating polynomial

Instead of matching only values of a data function, we might want to also match derivative values. In the following definition, we take repeated data values to mean that we are requiring consecutive matching of derivatives. It turns out to be best to require derivatives in sequence, without any gaps, which is also referred to as *Hermite* interpolation.

Given any *nondecreasing* sequence of real numbers $t_0 \le t_1 \le \cdots \le t_d$ and a function $g(t)$ with values $g(t_i)$ at these numbers, suppose further that g is differentiable to order r_i at each t_i, where r_i is determined by $r_i = 0$ if $t_i < t_{i+1}$, and $r_i = k$ if $t_i = t_{i+1} = \cdots = t_{i+k}$ and $t_{i+k} < t_{i+k+1}$. Then define an *osculating polynomial* $p(t)$ with the data sequence $t_0, t_1, \cdots, t_d$ and data function $g(t)$ as a polynomial which satisfies:

$$p^{(j)}(t_i) = g^{(j)}(t_i) \;\; for \;\; i = 0, \dots, d, \;\; and \;\; j = 0, \dots, r_i.$$

Note: If we change the order of the sequence in such a way that equal values are still consecutive, the definition of the osculating polynomial is not affected. So we can allow changes in the order of the data as long as whenever $t_i = t_j$, with $i < j$, then also $t_i = t_k$ for all k satisfying $i < k < j$.

Examples:

- Find a polynomial which matches the data function $g(t) = \dfrac{1}{t-2}$ for the data sequence $t_0 = 0$, $t_1 = 0$, and $t_2 = 1$. This means that we want $p(t)$ in P_2 satisfying: $p(0) = g(0)$, $p'(0) = g'(0)$ and $p(1) = g(1)$. Since $g'(t) = \dfrac{-1}{(t-2)^2}$, we need to find $p(t)$ satisfying:

  $$p(0) = -\frac{1}{2}, \;\; p'(0) = -\frac{1}{4} \qquad \text{and} \qquad p(1) = -1.$$

  We can find such a $p(t)$ with the standard basis and a linear system: We solve for the coefficients a_0, a_1, and a_2 with

  $$p(t) = a_0 + a_1 t + a_2 t^2, \qquad \text{and} \qquad p'(t) = a_1 + 2a_2 t.$$

  The linear system is then:

  $$\begin{pmatrix} 1 & 0 & 0 & | & -\frac{1}{2} \\ 0 & 1 & 0 & | & -\frac{1}{4} \\ 1 & 1 & 1 & | & -1 \end{pmatrix}$$

  which has reduced form:

  $$\begin{pmatrix} 1 & 0 & 0 & | & -\frac{1}{2} \\ 0 & 1 & 0 & | & -\frac{1}{4} \\ 0 & 0 & 1 & | & -\frac{1}{4} \end{pmatrix}$$

  and solution $a_2 = \frac{1}{4}$, $a_1 + 2a_2 = -\frac{1}{4}$, or $a_1 = -\frac{3}{4}$, and $a_0 = -\frac{1}{2}$. So the osculating polynomial $p(t)$ is

  $$p(t) = -\frac{1}{2} - \frac{1}{4}t - \frac{1}{4}t^2,$$

  with

  $$p'(t) = -\frac{1}{4} - \frac{1}{2}t.$$

Existence and Uniqueness of the osculating polynomial

It is a fact that for any nondecreasing sequence $t_0 \le t_1 \le \cdots \le t_d$ of real numbers, and function g, the osculating polynomial $p(t)$ with data function g and data values $t_0, t_1, \cdots, t_d$ exists as an element of P_d, and is unique.

Lecture 10, Th Feb.7, 2013

Main Points:

- Proof of Existence and Uniqeness of Osculating polynomials
- Divided differences with derivatives
- Newton form for osculating polynomial

Definition of the osculating polynomial

Instead of matching only values of a data function, we might want to also match derivative values. In the following definition, we take repeated data values to mean that we are requiring consecutive matching of derivatives. It turns out to be best to require derivatives in sequence, without any gaps, which is also referred to as *Hermite* interpolation.

Given any *nondecreasing* sequence of real numbers $t_0 \leq t_1 \leq \cdots \leq t_d$ and a function $g(t)$ with values $g(t_i)$ at these numbers, suppose further that g is differentiable to order r_i at each t_i, where r_i is determined by $r_i = 0$ if $t_i < t_{i+1}$, and $r_i = k$ if $t_i = t_{i+1} = \cdots = t_{i+k}$ and $t_{i+k} < t_{i+k+1}$. Then define an *osculating polynomial* $p(t)$ with the data sequence $t_0, t_1, \cdots, t_d$ and data function $g(t)$ as a polynomial which satisfies:

$$p^{(j)}(t_i) = g^{(j)}(t_i) \quad \textit{for} \quad i = 0, \ldots, d, \quad \textit{and} \quad j = 0, \ldots, r_i.$$

Note: If we change the order of the sequence in such a way that equal values are still consecutive, the definition of the osculating polynomial is not affected. So we can allow changes in the order of the data as long as whenever $t_i = t_j$, with $i < j$, then also $t_i = t_k$ for all k satisfying $i < k < j$.

Existence and Uniqueness of the osculating polynomial

It is a fact that for any nondecreasing sequence $t_0 \leq t_1 \leq \cdots \leq t_d$ of real numbers, and function g, the osculating polynomial $p(t)$ with data function g and data values $t_0, t_1, \cdots, t_d$ exists as an element of P_d, and is unique.

Proof of Existence and Uniqueness of the osculating polynomial

Just as with the interpolating polynomial, we can prove the existence and uniqesness of the osculating polynomial using the standard basis and a linear system. In this case we need to specify both point values and derivative values according to the data sequence. We will see that the coefficient matrix for this linear system has Confluent Vandermonde determinant which is nonzero, again showing that the linear system has a unique solution.

The linear system may have some rows which come from equating values of a polynomial and its derivatives with values of the data function $g(t)$. Let's suppose that $t_0 = t_1 = t_2 = u_0$ and $t_3 = t_4 = u_1$.

$$\begin{array}{ccccccccccc}
a_0 & + & a_1 u_0 & + & a_2 u_0^2 & + & a_3 u_0^3 & + & a_4 u_0^4 & = & g(u_0) \\
 & & a_1 & + & 2a_2 u_0 & + & 3a_3 u_0^2 & + & 4a_4 u_0^3 & = & g'(u_0) \\
 & & & & 2a_2 & + & 6a_3 u_0 & + & 12 a_4 u_0^2 & = & g''(u_0) \\
a_0 & + & a_1 u_1 & + & a_2 u_1^2 & + & a_3 u_1^3 & + & a_4 u_1^4 & = & g(u_1) \\
 & & a_1 & + & 2a_2 u_1 & + & 3a_3 u_1^2 & + & 4a_4 u_1^3 & = & g'(u_1)
\end{array}$$

The corresponding augmented matrix of the linear system would then look like:

$$\begin{array}{cccccccc}
1 & u_1 & u_0^2 & u_0^3 & u_0^4 & | & g(u_0) \\
0 & 1 & 2u_0 & 3u_0^2 & 4u_0^3 & | & g'(u_0) \\
0 & 0 & 2 & 6u_0 & 12u_0^2 & | & g''(u_0) \\
1 & u_1 & u_1^2 & u_1^3 & u_1^4 & | & g(u_1) \\
0 & 1 & 2u_1 & 3u_1^2 & 4u_1^3 & | & g'(u_1)
\end{array}$$

As we saw earlier, Confluent Vandermonde determinants were constructed in this way, and the determinants are nonzero as long as the sequence of values are distinct. In order to keep them separate, since the t_i values are allowed to be equal in order to signify derivative matching, we can assign values $u_0, u_1, \ldots, u_k$ to mean the distinct values appearing in the the list of data values $t_0, \ldots, t_d$. If each such value u_i appears with multiplicity m_i then we understand that if $m_i = 1$ then there is only one regular Vandermonde row for u_i, and if $m_i > 1$ then there are also some consecutive derivative rows with the value u_i. The determinant of the coefficient matrix is then nonzero by the Confluent Vandermonde product formula:

$$D(u_0^{m_1} u_2^{m_2} \cdots u_k^{m_k}) = \prod_{1 \le i < j \le n} (u_j - u_i)^{m_i m_j} \prod_{i=1}^{k} (m_i - 1)!! \neq 0,$$

where the double factorial means:

$$N!! = N!(N-1)!(N-2)! \cdots 2!1!.$$

This completes the proof of existence and uniqueness of the osculating polynomial.

Divided Differences for the osculating polynomial

For a nondecreasing sequence $t_0 \le t_1 \le \cdots \le t_d$, and data function g, the divided differences are defined in the same way as they were for distinct values. In particular, we define the divided difference $[t_0, \ldots, t_d]g$ to be the coefficient of t^d in the osculating polynomial $p(t)$ with data values $t_0, \ldots, t_d$, and data function $g(t)$.

Recursive formula for the Divided Differences

For a nondecreasing sequence $t_0 \le t_1 \le \cdots \le t_d$, and function g, the divided differences satisfy the same recursion as the interpolating polynomial. For $t_0 < t_d$, we have:

$$[t_0, t_1, \ldots, t_d]g = \frac{[t_1, t_2, \ldots, t_d]g - [t_0, t_1, \ldots, t_{d-1}]g}{t_d - t_0},$$

and when $t_0 = t_d$ we have:

$$[t_0, t_1, \ldots, t_d]g = \frac{g^{(d)}(t_0)}{d!}.$$

Newton form for the osculating polynomial

The Newton form for the osculating polynomial is identical to the one used for the interpolating polynomial, but with the new interpretation of the divided differences, as indicated above.

$$p(t) = [t_0]g + [t_0, t_1]g \cdot (t - t_0) + [t_0, t_1, t_2]g \cdot (t - t_0)(t - t_1) + \cdots + [t_0, t_1, \ldots, t_d]g \cdot (t - t_0)(t - t_1) \cdots (t - t_{d-1}).$$

Special cases of the osculating polynomial

When the t_i are all distinct, the osculating polynomial reduces to the interpolating polynomial. At the other extreme, when $t_0 = t_1 = \cdots = t_d$, we have the Taylor polynomial:

$$p(t) = g(t_0) + g'(t_0)(t - t_0) + \frac{g''(t_0)}{2!}(t - t_0)^2 + \cdots + \frac{g^{(d)}(t_0)}{d!}(t - t_0)^d$$

written in terms of the Taylor basis with parameter t_0.

Examples:

- Let $t_0 = 0$, $t_1 = 0$, $t_2 = 0$, $t_3 = 1$, and $t_4 = 1$. Find the osculating polynomial which agrees with the data function $g(t)$ for the above data values if $g(0) = 4$, $g'(0) = 3$ and $g''(0) = -2$, and $g(1) = 5$, and $g'(1) = -1$.

  We form the divided difference table (noting that $\dfrac{g''(0)}{2!} = -1$):

$$
\begin{array}{cccccc}
0 & 4 & & & & \\
 & & 3 & & & \\
0 & 4 & & -1 & & \\
 & & 3 & & -1 & \\
0 & 4 & & -2 & & 1 \\
 & & 1 & & 0 & \\
1 & 5 & & -2 & & \\
 & & -1 & & & \\
1 & 5 & & & &
\end{array}
$$

  The Newton form is then

$$
\begin{aligned}
p(t) &= 4 + 3(t-0) - (t-0)^2 - (t-0)^3 + (t-0)^3(t-1) \\
&= 4 + 3t - t^2 - t^3 + t^3(t-1) \\
&= 4 + 3t - t^2 - 2t^3 + t^4
\end{aligned}
$$

  with derivatives:

$$
p'(t) = 3 - 2t - 6t^2 + 4t^3, \qquad \text{and} \qquad p''(t) = -2 - 12t + 12t^2,
$$

  which can be seen to satisfy the original conditions.

- Note that the first three conditions $g(0) = 4$, $g'(0) = 2$ and $g''(0) = -1$, in the previous example amount to the construction of the quadratic Taylor polynomial:

$$
p(t) = 4 + 3t - t^2.
$$

  This illustrates the cumulative nature of the Newton form, since the requirement of more data simply adds more terms to the Newton form.

Second proof of existence of osculating polynomial

In the second proof we again appeal to the recursive form and use induction. This time we need to verify the requirement about derivatives. The base case is the same as for interpolation: $d = 0$, with one data value t_0, so $p(t) = [t_0]g = g(t_0)$ is constant. For $d > 0$ the proof breaks into two cases:

a) $t_0 = t_d$ (and thus for all i: $t_0 = t_i = t_d$.)
b) $t_0 < t_d$.

In the first case we simply use the Taylor polynomial from Calculus. This coincides exactly with our definition for the osculating polynomial. This shows existence of the osculating polynomial in the first case. So now we assume $t_0 < t_d$.

For the induction step we assume that $p_0(t)$ and $p_1(t)$ are osculating polynomials with sequences $[t_0, \ldots, t_{d-1}]$ and $[t_1, \ldots, t_d]$ respectively. Then we form the polynomial $p(t)$:

$$
p(t) = \frac{t - t_0}{t_d - t_0} p_1(t) + \frac{t_d - t}{t_d - t_0} p_0(t).
$$

According to the definition of the osculating polynomial, we now need to verify:

$$
p^{(j)}(t_i) = g^{(j)}(t_i), \quad j = 0, \ldots, r
$$

whenever $t_i = t_{i+1} = \cdots = t_{i+r}$. In order to check this, we take some derivatives of $p(t)$, to get:

$$p^{(j)}(t) = \frac{t-t_0}{t_d-t_0} p_1^{(j)}(t) + \frac{t_d-t}{t_d-t_0} p_0^{(j)}(t) + j \cdot \frac{p_1^{(j-1)}(t) - p_0^{(j-1)}(t)}{t_d-t_0}.$$

Now to check that $p(t)$ works, assume that we have $t_i = t_{i+1} = \ldots = t_{i+r}$ for some i and r. Case b) from above now breaks into three cases:

i) $t_i = t_0$, ii) $t_i = t_d$ and iii) $t_0 < t_i < t_d$

For case i) we just plug $t_i = t_0$ into the derivative formula $p^{(j)}(t)$ and show that this equals $g^{(j)}(t_0)$, for $j = 0, \ldots, r$. The first two terms give us the correct value, since $p_0^{(j)}(t_0) = g^{(j)}(t_0)$ for $j = 0, \ldots, r$ since the sequence $t_i = t_{i+1} = \ldots = t_{i+r}$ of equal values is part of the sequence for $p_0(t)$. The only slightly tricky part is to show that the last term is zero. This follows from the fact that the sequence $t_1 = t_2 = \ldots = t_r$ has length $r-1$ and is inside the sequence for $p_1(t)$, so $p_1^{(j-1)}(t_0) = g^{(j-1)}(t_0)$ for $j = 1, \ldots, r$.

The second case is symmetric to the first, and the third case is easier since the the sequences for $p_0(t)$ and $p_1(t)$ both contain the equal values. This completes the existence part of the proof.

Before we proceed to the uniqueness proof, we establish an important fact about multiplicity of zeros of polynomials.

Multiplicity of zeros of polynomials

The usual notion of multiplicity of zero for a polynomial is given algebraically by the corresponding multiplicity of a factor. For example, the polynomial

$$p(t) = 5(t-1)^2(t-3)^5$$

has a zero at $t = 1$ of multiplicity 2, and a zero at $t = 3$ of multiplicity 5. We can also capture this information by using derivatives instead. In particular:

Definition: A function $f(t)$ has a zero of multiplicity r at $t = c$ if $f(c) = 0$, $f'(c) = 0, \ldots, f^{(r-1)}(c) = 0$. In other words:

$$f^{(j)}(c) = 0, \quad j = 0, \ldots, r-1,$$

where $f^{(0)}(t) = f(t)$.

Examples:

- Let $f(t) = 1 - \cos t$. We will show that $f(t)$ has a zero of multiplicity 2 at $t = 0$. We have $f'(t) = \sin t$, and therefore $f'(0) = 0$, as well as $f(0) = 0$. By the definition above, $f(t)$ has a zero of multiplicity 2 at $t = 0$. We can further verify this by seeing that the Taylor series for $\cos t$ gives:

$$\begin{aligned} f(t) &= 1 - \cos t \\ &= 1 - (1 - \frac{1}{2}t^2 + \frac{1}{4!}t^4 + \cdots) \\ &= \frac{1}{2}t^2 + \frac{1}{4!}t^4 + \cdots \\ &= t^2(\frac{1}{2} + \frac{1}{4!}t^2 + \cdots) \end{aligned}$$

  which is expected for a zero of multiplicity 2 at $t = 0$.

Important fact: A polynomial $p(t)$ has a zero of multiplicity r at $t = c$ if and only if $(t-c)^r$ is a factor of $p(t)$.

We can prove this fact by writing any polynomial $p(t)$ in terms of the Taylor basis $\{1, t-c, \frac{(t-c)^2}{2!}, \ldots, \frac{(t-c)^d}{d!}\}$:

$$p(t) = a_0 + a_1(t-c) + a_2\frac{(t-c)^2}{2!} + \cdots + a_d\frac{(t-c)^d}{d!}.$$

Then we can also write the derivatives of $p(t)$ as:

$$p^{(j)}(t) = a_j + a_{j+1}(t-c) + a_{j+2}\frac{(t-c)^2}{2!} + \cdots + a_d\frac{(t-c)^{d-j}}{(d-j)!}.$$

From this form it is easy to see that:

$$p^{(j)}(c) = 0 \quad \text{if and only if} \quad a_j = 0.$$

But it is also clear that

$$a_0 = a_1 = \cdots = a_{r-1} = 0 \quad \text{if and only if} \quad (t-c)^r \;\text{ is a factor of } p(t).$$

So we conclude that

$$p^{(j)}(c) = 0,\; j = 0,\ldots,r-1 \quad \text{if and only if} \quad (t-c)^r \;\text{ is a factor of } p(t),$$

which says $p(t)$ has a zero of multiplicity r at $t = c$ if and only if $(t-c)^r$ is a factor of $p(t)$.

Second proof of uniquess of osculating polynomial

For the uniqueness proof we suppose that there are two osculating polynomials $p(t)$ and $q(t)$, and we consider the difference

$$f(t) = p(t) - q(t).$$

Then $f(t)$ is in P_d and we also have:

$$f^{(j)}(t_i) = p^{(j)}(t_i) - q^{(j)}(t_i) = g^{(j)}(t_i) - g^{(j)}(t_i) = 0, \qquad j = 0,\ldots,r$$

whenever $t_i = t_{i+1} = \cdots = t_{i+r}$. By the above fact on multiplicities of zeros for polynomials, this says that whenever there are $r+1$ consecutive equal values $t_i = t_{i+1} = \cdots = t_{i+r}$, then $f(t)$ has a zero of multiplicity $r+1$ at $t = t_i$ and hence also has a factor of the form $(t-t_i)^{r+1}$.

Since there are $d+1$ values in the sequence $t_0,\ldots,t_d$ we can group them into subsequences of equal values. Each such subsequence then corresponds to a factor with exponent equal to the number of terms in the subsequence. The total degree of the product of all such factors is then $d+1$. But $f(t)$ is in P_d and can only have degree at most d or must be zero. So we conclude that $f(t)$ is zero, and hence $p(t) = q(t)$. This completes the uniqueness proof.

Proof of the Newton form for the osculating polynomial

In order to establish the Newton form, we proceed in the same way as for the interpolating polynomial. We assume that $p_0(t)$ is the osculating polynomial with data values $t_0,\ldots,t_{d-1}$ and that $p(t)$ is the osculating polynomial with data values $t_0,\ldots,t_d$. Then we consider the polynomial $f(t) = p(t) - p_0(t)$. This polynomial has the property that

$$f(t_i) = p(t_i) - p_0(t_i) = 0, \qquad i = 0,\ldots,d-1,$$

but many of these values may be repeated. But we also have for the repeated values:

$$f^{(j)}(t_i) = p^{(j)}(t_i) - p_0^{(j)}(t_i) = 0, \qquad j = 0,\ldots,r,$$

whenever $t_i = t_{i+1} = \cdots = t_r$. So, by the fact about multiplicity of zeros, we can write

$$f(t) = p(t) - p_0(t) = C\cdot(t-t_0)(t-t_1)\cdots(t-t_{d-1}),$$

where many of the factors may be repeated according to the correct multiplicities implied by the above. We can also identify the constant C by equating coefficients of t^d on both sides to obtain

$$C = [t_0,\ldots,t_d]g.$$

Just as we saw with the interpolating polynomial, we can then write

$$p(t) = q_{d-1}(t) + [t_0, \ldots, t_d]g \cdot (t - t_0)(t - t_1) \cdots (t - t_{d-1}),$$

where $q_{d-1}(t) = p_0(t)$. Repeating this process for $q_{d-1}(t)$ we obtain

$$q_{d-1}(t) = q_{d-2}(t) + [t_0, \ldots, t_{d-1}]g \cdot (t - t_0)(t - t_1) \cdots (t - t_{d-2})$$

and continuing in the same way we will arrive at the Newton form:

$$p(t) = [t_0]g + [t_0, t_1]g \cdot (t - t_0) + [t_0, t_1, t_2]g \cdot (t - t_0)(t - t_1) + \cdots + [t_0, t_1, \ldots, t_d]g \cdot (t - t_0)(t - t_1) \cdots (t - t_{d-1}).$$

Leibniz' Rule with repeated values in the divided differences.

Let $f(t) = g(t)h(t)$. Then the divided differences with repeated values also satisfy the Leibniz Rule:

$$[t_i, t_{i+1}, \ldots, t_{i+k}]f = \sum_{r=i}^{i+k} ([t_i, \ldots, t_r]g)([t_r, \ldots, t_{i+k}]h).$$

For degree $d = 2$ it is:

$$[t_0, t_1, t_2]f = [t_0]g[t_0, t_1, t_2]h + [t_0, t_1]g[t_1, t_2]h + [t_0, t_1, t_2]g[t_2]h.$$

We proved this in the case where all the data values t_i were distinct, which used the Newton forms for *interpolating* polynomials. Now we consider the case of repeated data values $t_i = t_{i+1} = \cdots = t_{i+r}$ which is used for Newton forms of *osculating* polynomials.

The base case is still $d = 0$, and since there is only one data value we cannot have any repetition, so it is identical to the previous case. For the rest of the proof, we can proceed exactly as before, using a product of Newton forms for osculating polynomials, and using the uniqueness of the osculating polynomial in P_d.

Lecture 11, T Feb.12, 2013

Main Points:

- Quiz 5
- Osculating polynomial, second proof of existence and uniqueness
- Proof of Newton form for osculating polynomial
- Leibniz' Rule with derivatives

Second proof of existence of osculating polynomial

In the second proof we again appeal to the recursive form and use induction. This time we need to verify the requirement about derivatives. The base case is the same as for interpolation: $d = 0$, with one data value t_0, so $p(t) = [t_0]g = g(t_0)$ is constant. For $d > 0$ the proof breaks into two cases:

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$$p(t) = \frac{t - t_0}{t_d - t_0} p_1(t) + \frac{t_d - t}{t_d - t_0} p_0(t).$$

According to the definition of the osculating polynomial, we now need to verify:

$$p^{(j)}(t_i) = g^{(j)}(t_i), \quad j = 0, \ldots, r$$

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Now to check that $p(t)$ works, assume that we have $t_i = t_{i+1} = \ldots = t_{i+r}$ for some i and r. Case b) from above now breaks into three cases:

i) $t_i = t_0$, ii) $t_i = t_d$ and iii) $t_0 < t_i < t_d$

For case i) we just plug $t_i = t_0$ into the derivative formula $p^{(j)}(t)$ and show that this equals $g^{(j)}(t_0)$, for $j = 0, \ldots, r$. The first two terms give us the correct value, since $p_0^{(j)}(t_0) = g^{(j)}(t_0)$ for $j = 0, \ldots, r$ since the sequence $t_i = t_{i+1} = \ldots = t_{i+r}$ of equal values is part of the sequence for $p_0(t)$. The only slightly tricky part is to show that the last term is zero. This follows from the fact that the sequence $t_1 = t_2 = \ldots = t_r$ has length $r - 1$ and is inside the sequence for $p_1(t)$, so $p_1^{(j-1)}(t_0) = g^{(j-1)}(t_0)$ for $j = 1, \ldots, r$.

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$$f(t) = p(t) - q(t).$$

Then $f(t)$ is in P_d and we also have:

$$f^{(j)}(t_i) = p^{(j)}(t_i) - q^{(j)}(t_i) = g^{(j)}(t_i) - g^{(j)}(t_i) = 0, \qquad j = 0, \dots, r$$

whenever $t_i = t_{i+1} = \cdots = t_{i+r}$. By the above fact on multiplicities of zeros for polynomials, this says that whenever there are $r+1$ consecutive equal values $t_i = t_{i+1} = \cdots = t_{i+r}$, then $f(t)$ has a zero of multiplicity $r+1$ at $t = t_i$ and hence also has a factor of the form $(t - t_i)^{r+1}$.

Since there are $d+1$ values in the sequence $t_0, \dots, t_d$ we can group them into subsequences of equal values. Each such subsequence then corresponds to a factor with exponent equal to the number of terms in the subsequence. The total degree of the product of all such factors is then $d+1$. But $f(t)$ is in P_d and can only have degree at most d or must be zero. So we conclude that $f(t)$ is zero, and hence $p(t) = q(t)$. This completes the uniqueness proof.

Proof of the Newton form for the osculating polynomial

In order to establish the Newton form, we proceed in the same way as for the interpolating polynomial. We assume that $p_0(t)$ is the osculating polynomial with data values $t_0, \dots, t_{d-1}$ and that $p(t)$ is the osculating polynomial with data values $t_0, \dots, t_d$. Then we consider the polynomial $f(t) = p(t) - p_0(t)$. This polynomial has the property that

$$f(t_i) = p(t_i) - p_0(t_i) = 0, \qquad i = 0, \dots, d-1,$$

but many of these values may be repeated. But we also have for the repeated values:

$$f^{(j)}(t_i) = p^{(j)}(t_i) - p_0^{(j)}(t_i) = 0, \qquad j = 0, \dots, r,$$

whenever $t_i = t_{i+1} = \cdots = t_r$. So, by the fact about multiplicity of zeros, we can write

$$f(t) = p(t) - p_0(t) = C \cdot (t - t_0)(t - t_1) \cdots (t - t_{d-1}),$$

where many of the factors may be repeated according to the correct multiplicities implied by the above. We can also identify the constant C by equating coefficients of t^d on both sides to obtain

$$C = [t_0, \dots, t_d]g.$$

Just as we saw with the interpolating polynomial, we can then write

$$p(t) = q_{d-1}(t) + [t_0, \dots, t_d]g \cdot (t - t_0)(t - t_1) \cdots (t - t_{d-1}),$$

where $q_{d-1}(t) = p_0(t)$. Repeating this process for $q_{d-1}(t)$ we obtain

$$q_{d-1}(t) = q_{d-2}(t) + [t_0, \dots, t_{d-1}]g \cdot (t - t_0)(t - t_1) \cdots (t - t_{d-2})$$

and continuing in the same way we will arrive at the Newton form:

$$p(t) = [t_0]g + [t_0, t_1]g \cdot (t - t_0) + [t_0, t_1, t_2]g \cdot (t - t_0)(t - t_1) + \cdots + [t_0, t_1, \dots, t_d]g \cdot (t - t_0)(t - t_1) \cdots (t - t_{d-1}).$$

Leibniz' Rule with repeated values in the divided differences.

Let $f(t) = g(t)h(t)$. Then the divided differences with repeated values also satisfy the Leibniz Rule:

$$[t_i, t_{i+1}, \dots, t_{i+k}]f = \sum_{r=i}^{i+k} ([t_i, \dots, t_r]g)([t_r, \dots, t_{i+k}]h).$$

For degree $d = 2$ it is:

$$[t_0, t_1, t_2]f = [t_0]g[t_0, t_1, t_2]h + [t_0, t_1]g[t_1, t_2]h + [t_0, t_1, t_2]g[t_2]h.$$

We proved this in the case where all the data values t_i were distinct, which used the Newton forms for *interpolating* polynomials. Now we consider the case of repeated data values $t_i = t_{i+1} = \cdots = t_{i+r}$ which is used for Newton forms of *osculating* polynomials.

The base case is still $d = 0$, and since there is only one data value we cannot have any repetition, so it is identical to the previous case. For the rest of the proof, we can proceed exactly as before, using a product of Newton forms for osculating polynomials, and using the uniqueness of the osculating polynomial in P_d.

Examples:

- Define the functions:

$$f(t) = (t-2)_+^2 \cdot \frac{1}{t}, \quad g(t) = (t-2)_+^2, \quad \text{and} \quad h(t) = \frac{1}{t}.$$

We will find the osculating polynomials with the data $[1, 1, 3, 3]$ and data functions $f(t)$, $g(t)$, and $h(t)$. Call these osculating polynomials $p(t)$, $q(t)$, and $r(t)$ respectively. We will verify the Leibniz Rule for $[1, 1, 3, 3]f$. First, we have the following divided difference tables:

$$f(t): \begin{array}{ccccc} 1 & 0 & & & \\ & & 0 & & \\ 1 & 0 & & \frac{1}{2} & \\ & & \frac{1}{6} & & \frac{1}{18} \\ 3 & \frac{1}{3} & & \frac{7}{36} & \\ & & \frac{5}{9} & & \\ 3 & \frac{1}{3} & & & \end{array} \qquad g(t): \begin{array}{ccccc} 1 & 0 & & & \\ & & 0 & & \\ 1 & 0 & & \frac{1}{4} & \\ & & \frac{1}{2} & & \frac{1}{4} \\ 3 & 1 & & \frac{3}{4} & \\ & & 2 & & \\ 3 & 1 & & & \end{array} \quad \text{and} \quad h(t): \begin{array}{ccccc} 1 & 1 & & & \\ & & -1 & & \\ 1 & 1 & & \frac{1}{3} & \\ & & -\frac{1}{3} & & -\frac{1}{9} \\ 3 & \frac{1}{3} & & \frac{1}{9} & \\ & & -\frac{1}{9} & & \\ 3 & \frac{1}{3} & & & \end{array}$$

Then we have:

$$[1, 1, 3, 3]f = \frac{1}{18}$$

and

$$\begin{aligned} &[1]g[1, 1, 3, 3]h + [1, 1]g[1, 3, 3]h + [1, 1, 3]g[3, 3]h + [1, 1, 3, 3]g[3]h \\ &= 0 \cdot (-\frac{1}{9}) + 0 \cdot \frac{1}{9} + \frac{1}{4} \cdot (-\frac{1}{9}) + \frac{1}{4} \cdot (\frac{1}{3}) = \frac{1}{18}. \end{aligned}$$

We can also verify that the osculating polynomial $p(t)$ can be found as the appropriate terms taken from the product $q(t)r(t)$. Those polynomials can be obtained with Newton forms. For $p(t)$ and $q(t)$ we use the tables above to get:

$$p(t) = 0 + 0 \cdot (t-1) + \frac{1}{12}(t-1)^2 + \frac{1}{18}(t-1)^2(t-3),$$
$$q(t) = 0 + 0 \cdot (t-1) + \frac{1}{4}(t-1)^2 + \frac{1}{4}(t-1)^2(t-3).$$

For $r(t)$ we do the divided difference table and Newton form in the reverse order, as we did in the proof.

$$\begin{array}{ccccc} 3 & \frac{1}{3} & & & \\ & & -\frac{1}{9} & & \\ 3 & \frac{1}{3} & & \frac{1}{9} & \\ & & -\frac{1}{3} & & -\frac{1}{9} \\ 1 & 1 & & \frac{1}{3} & \\ & & -1 & & \\ 1 & 1 & & & \end{array}$$

So we can write the Newton form:

$$r(t) = \frac{1}{3} - \frac{1}{9}(t-3) + \frac{1}{9}(t-3)^2 - \frac{1}{9}(t-3)^2(t-1).$$

Then we write the terms from $q(t)r(t)$ which consist of products which do not have a zero of multiplicity two at both of the values $t=1$ and $t=3$. So, we are avoiding the terms which contain factors $(t-1)^2(t-3)^2$. In the proof we called those terms that we are avoiding $G(t)$ and the remaining terms $F(t)$. Thus we have:

$$F(t) = \frac{1}{3}\cdot\frac{1}{4}(t-1)^2 + \frac{1}{3}\cdot\frac{1}{4}(t-1)^2(t-3) - \frac{1}{9}(t-3)\frac{1}{4}(t-1)^2.$$

At this point, we recall that

$$f(t) = (t-2)_+^2 \cdot \frac{1}{t},$$

and thus

$$f'(t) = 2(t-2)_+^1 \cdot \frac{1}{t} + (t-2)_+^2 \cdot (-\frac{1}{t^2}).$$

We would like to verify that indeed $F(t) = p(t)$. This can be done by checking that $F(t)$ satisfies the osculation conditions for the data $[1,1,3,3]$ with data function $f(t)$. We can check that

$$F(1) = 0 = f(1), \quad \text{and} \quad F(3) = \frac{1}{3} = f(3).$$

Taking a derivative we have:

$$F'(t) = \frac{1}{6}(t-1) + \frac{1}{12}\left[2(t-1)(t-3) + (t-1)^2\right] - \frac{1}{36}\left[(t-1)^2 + 2(t-3)(t-1)\right],$$

and we can also verify that:

$$F'(1) = 0 = f'(1), \quad \text{and} \quad F'(3) = \frac{5}{9} = f'(3).$$

Thus, since $F(t)$ is in P_3 and it satisfies the conditions above, and the osculating polnomial is unique in P_3, we must have $F(t) = p(t)$. Of course, it is also simple to check that the expression for $F(t)$ above can be simplified to obtain:

$$F(t) = \frac{1}{12}(t-1)^2 + \frac{1}{18}(t-1)^2(t-3) = p(t).$$

Lecture 12, Th Feb.14, 2013

Main Points:

- Orders of continuity for splines (piecewise polynomials)
- Standard basis for splines
- Exact orders of continuity for splines
- C^2 cubic splines

Orders of continuity for piecewise polynomials or spline functions

The vector space of piecewise polynomial functions $P_d^k[u_0, \dots, u_k]$ has many important subspaces of functions which have specific continuity or derivative conditions at the breakpoints $u_1, u_2, \dots, u_{k-1}$. The simplest of these is the subspace of continuous functions:

$$P_{d,0}^k[u_0, \dots, u_k] = \{f \in P_d^k[u_0, \dots, u_k] \ : \ f \text{ is continuous } \}.$$

Note: The subscript zero after the d indicates that the functions must have "zeroth order continuity" which simply means continuity. It is equivalent to specify that at each breakpoint the two polynomials on either side must agree in value, or in other words:

$$p_{i-1}(u_i) = p_i(u_i), \quad i = 1, \dots, k-1.$$

The next simplest such subspace is the subspace of differentiable functions:

$$P_{d,1}^k[u_0, \dots, u_k] = \{f \in P_d^k[u_0, \dots, u_k] \ : \ f \text{ is differentiable } \}.$$

Note: Now the subscript one after the d indicates that the functions must have "first order continuity" which means that they are differentiable and hence also continuous. It is equivalent to specify that at each breakpoint the two polynomials on either side must agree both in value and derivative, or in other words:

$$p_{i-1}(u_i) = p_i(u_i), \quad i = 1, \dots, k-1,$$

and

$$p'_{i-1}(u_i) = p'_i(u_i), \quad i = 1, \dots, k-1.$$

The general such subspace is the subspace of functions differentiable to order r:

$$P_{d,r}^k[u_0, \dots, u_k] = \{f \in P_d^k[u_0, \dots, u_k] \ : \ f \text{ has } r \text{ continuous derivatives } \}.$$

It is equivalent to specify that at each breakpoint the two polynomials on either side must agree both in value and derivatives up to the r^{th} derivative, or in other words:

$$p_{i-1}^{(j)}(u_i) = p_i^{(j)}(u_i), \quad i = 1, \dots, k-1, \quad \text{and} \quad j = 0, \dots, r.$$

Examples:

- Determine the appropriate vector spaces to which the function f belongs:

$$f(t) = \begin{cases} p_1(t) = t^3 - 2t^2 + t + 5, & 0 \le t < 2 \\ p_2(t) = 2t^3 - 8t^2 + 13t - 3, & 2 \le t \le 4 \end{cases}$$

  First, we can say that since f consists of cubic polynomials on the sequence of intervals $[0, 2, 4]$ that f is in the vector space $P_3^2[0, 2, 4]$. Beyond that, we need to know if f is continuous or differentiable to some order. To check continuity we find that

$$p_1(2) = 7 = p_2(2)$$

and thus $f(t)$ is continuous at $t = 2$, so we know also that f is in $P^2_{3,0}[0,2,4]$. Next, we find the following derivatives:

$$p_1'(t) = 3t^2 - 4t + 1,\ p_1'(2) = 5, \quad p_2'(t) = 6t^2 - 16t + 13,\ p_2'(2) = 5$$
$$p_1''(t) = 6t - 4,\ p_1''(2) = 8, \quad p_2''(t) = 12t - 16,\ p_2''(2) = 8$$
$$p_1'''(t) = 6,\ p_1'''(2) = 6, \quad p_2'''(t) = 12,\ p_2'''(2) = 12.$$

We can see that the derivatives match at $t = 2$ up to order 2, so the function f has two continuous derivatives and is in the vector space $P^2_{3,2}[0,2,4]$. Note: this last vector space is the most specific, since it requires two continuous derivatives. This is a subspace of $P^2_{3,1}[0,2,4]$, which also contains f. In fact, all of these vector space can be put into the sequence of subspaces:

$$P^2_{3,2}[0,2,4] \subset P^2_{3,1}[0,2,4] \subset P^2_{3,0}[0,2,4] \subset P^2_3[0,2,4].$$

Standard basis and dimension of $P^k_{d,r}[u_0,\dots,u_k]$:

The standard basis of $P^k_{d,r}[u_0,\dots,u_k]$ is constructed from the standard basis of P_d together with shifted power functions of degree d down to degree $r+1$:

$$\{1, t, t^2, \dots, t^d, (t-u_1)^d_+, \dots, (t-u_1)^{r+1}_+, \dots, (t-u_{k-1})^d_+, \dots, (t-u_{k-1})^{r+1}_+\}.$$

The dimension of $P^k_{d,r}[u_0,\dots,u_k]$ is thus:

$$\dim(P^k_{d,r}[u_0,\dots,u_k]) = d + 1 + (d-r)(k-1).$$

Examples:

- Let $V = P^4_{3,2}[0,1,2,3,4]$. Then V has basis $\{1, t, t^2, t^3, (t-1)^3_+, (t-2)^3_+, (t-3)^3_+\}$ and dimension 7.
- Let $V = P^4_{2,0}[0,1,2,3,4]$. Then V has basis $\{1, t, t^2, (t-1)^2_+, (t-1)^1_+, (t-2)^2_+, (t-2)^1_+, (t-3)^2_+, (t-3)^1_+\}$ and dimension 9.
- Let $V = P^3_{4,1}[0,1,2,3]$. Then V has basis $\{1, t, t^2, t^3, t^4, (t-1)^4_+, (t-1)^3_+, (t-1)^2_+, (t-2)^4_+, (t-2)^3_+, (t-2)^2_+\}$ and dimension 11.
- The function f from a previous example

$$f(t) = \begin{cases} p_1(t) = t^3 - 2t^2 + t + 5, & 0 \le t < 2 \\ p_2(t) = 2t^3 - 8t^2 + 13t - 3, & 2 \le t \le 4 \end{cases}$$

  was shown to be in the vector space $P^2_{3,2}[0,2,4]$. We can express f as a linear combination of the basis:

$$\{1, t, t^2, t^3, (t-2)^3_+\},$$

  we need to find coefficients so that

$$f(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 (t-2)^3_+.$$

  Clearly, the first 4 coefficients determine the polynomial $p_1(t)$ on the first interval $[0,2)$. This means we have:

$$f(t) = 5 + t - 2t^2 + t^3 + a_4 (t-2)^3_+.$$

  To obtain the correct values for f on the second subinterval, we need to have

$$a_4 (t-2)^3_+ = p_2(t) - p_1(t) = t^3 - 6t^2 + 12t - 8 = (t-2)^3,$$

  and so we see that $a_4 = 1$ and we have:

$$f(t) = 5 + t - 2t^2 + t^3 + (t-2)^3_+.$$

Exact order of continuity of shifted power functions

We say a function $f(t)$ has order of continuity r at $t = c$ if f is continuous at $t = c$ and each of the derivative functions $f', f'', \ldots, f^{(r)}$ are continuous at $t = c$. If, in addition, the function $f^{(r+1)}$ is *not* continuous at $t = c$, then we say that f has *exact order of continuity* r at $t = c$. If f and all of its derivatives are continuous at $t = c$ then we say f has infinite order of continuity, or simply f is continuous to all orders at $t = c$.

The shifted power function $(t-c)_+^k$ is continuous to all orders at all points not equal to c, and can be seen to have exact order of continuity $k-1$ at $t = c$.

If we let $f(t) = (t-c)_+^k$ then the derivatives of f are:

$$f'(t) = k(t-c)_+^{k-1}, f''(t) = k(k-1)(t-c)_+^{k-2}, \ldots, f^{(k-1)}(t) = k!(t-c)_+^1.$$

Note: The function $f(t) = (t-c)_+^1$ is not differentiable at $t = c$, although it is continuous there. The function $f(t) = (t-c)_+^0$ is neither continuous nor differentiable at $t = c$.

Note: The standard basis of $P_{d,r}^k[u_0, \ldots, u_k]$ can be obtained from the standard basis of $P_d^k[u_0, \ldots, u_k]$ by simply removing the functions which have the incorrect orders of continuity. In particular, the functions $(t-c)_+^r$ have order of continuity $r-1$ which is one lower than required. Hence all of those functions and lower powers do not belong. This is not a general rule, but rather, it is a special property of this particular collection of bases. Other collections of bases can be found at the opposite extreme, for instance the first basis of ordered k-tuples that we gave for P_d^k corresponds only to discontinuous functions. Thus it is impossible to remove any functions from this basis and end up with a basis of $P_{d,r}^k[u_0, \ldots, u_k]$.

Examples:

- The function $f(t) = (t-2)_+^2$ has exact order of continuity 1 at $t = 2$. This means that the first derivative $f'(t) = 2(t-2)_+^1$ exists and is continuous at $t = 2$ but that the second derivative fails to exist at $t = 2$. Note that it is tempting to say that since the function can be differentiated almost everywhere (except at $t = 2$) and can be given by the formula $2(t-2)_+^0$ away from $t = 2$, that this is the second derivative of f. But the function $2(t-2)_+^0$ is defined and equal to 2 at $t = 2$, but $f'(t)$ does not have a value there, so the two functions do not agree, and f' fails to be differentiable.
- The standard basis of $P_2^3[0, 2, 4, 6]$ is

  $$\{1, t, t^2, (t-2)_+^2, (t-2)_+^1, (t-2)_+^0, (t-4)_+^2, (t-4)_+^1, (t-4)_+^0\}.$$

  In order to obtain a basis for $P_{2,0}^3[0, 2, 4, 6]$ we can simply throw out the discontinuous functions from the previous basis to obtain:

  $$\{1, t, t^2, (t-2)_+^2, (t-2)_+^1, (t-4)_+^2, (t-4)_+^1\}.$$

  In order to obtain a basis for $P_{2,1}^3[0, 2, 4, 6]$ we can simply throw out the nondifferentiable functions from the previous basis to obtain:

  $$\{1, t, t^2, (t-2)_+^2, (t-4)_+^2\}.$$

- A different basis of $P_2^3[0, 2, 4, 6]$ is the one that corresponds to the set of ordered triples:

  $$\{(1, 0, 0), (t, 0, 0), (t^2, 0, 0), (0, 1, 0), (0, t, 0), (0, t^2, 0), (0, 0, 1), (0, 0, t), (0, 0, t^2)\}$$

  Note that the functions corresponding to these triples, defined on the sequence of intervals $[0, 2, 4, 6]$, are *all discontinuous* and hence no subset of this basis will be a basis of the continuous or differentiable subspaces above.

Proof of standard basis for $V = P_{d,r}^k[u_0, \ldots, u_k]$.

We will verify that the standard basis spans V and is linearly independent. For the spanning property we need to show that any function in V can be written as a linear combination of the basis functions. Let $f \in V$. Then f is a piecewise polynomial which corresponds to a k-tuple of polynomial functions $(p_1, p_2, \ldots, p_k)$ with the property:

$$p_{i-1}^{(j)}(u_i) = p_i^{(j)}(u_i), \quad i = 1, \ldots, k-1, \quad \text{and} \quad j = 0, \ldots, r.$$

Recall that we can write such a k-tuple as:

$$\begin{aligned}
(p_1, p_2, \ldots, p_k) \quad = \quad & (p_1, p_1, \ldots, p_1) \\
& + (0, p_2 - p_1, \ldots, p_2 - p_1) \\
& + (0, 0, p_3 - p_2 \ldots, p_3 - p_2) \\
& \vdots \\
& + (0, 0, 0 \ldots, 0, p_k - p_{k-1}).
\end{aligned}$$

We will show that each line in the above sum can be generated by a particular set of k-tuples, which in turn corresponds to a particular set of piecewise polynomial functions.

The first k-tuple $(p_1, \ldots, p_1)$ is easily generated as a sum of k-tuples of the form:

$$(1, 1, \ldots, 1), (t, t, \ldots, t), \ldots, (t^d, t^d, \ldots, t^d).$$

As functions, these correspond to the polynomials $1, t, t^2, \ldots, t^d$.

Next, we note that any difference $q_i = p_i - p_{i-1}$ above is a polynomial which satisfies:

$$q_i^{(j)}(u_i) = 0, \; j = 0, \ldots, r.$$

But this means that the function $q_i(t)$ has a zero of multiplicity r at $t = u_i$, which means that $(t - u_i)^r$ is a factor of q_i. Thus we can write:

$$q_i(t) = a_1(t - u_i)^{r+1} + a_2(t - u_i)^{r+2} + \cdots + a_{d-r}(t - u_i)^d,$$

which shows that $q_i(t)$ is a linear combination of a subset of a shifted basis of P_d. This is a linearly independent set (since it is a subset of a basis) and gives us the recipe for generating the ordered k-tuples of the form $(0, 0, \ldots, 0, q_i(t), q_i(t), \ldots, q_i(t))$. In fact, all such elements are obtained as linear combinations of:

$$(0, 0, \ldots, 0, (t-u_i)^{r+1}, \ldots, (t-u_i)^{r+1}), (0, 0, \ldots, 0, (t-u_i)^{r+2}, \ldots, (t-u_i)^{r+2}), \ldots, (0, 0, \ldots, 0, (t-u_i)^d, \ldots, (t-u_i)^d).$$

But these k-tuples correspond exactly to the shifted power functions

$$(t - u_i)_+^{r+1}, (t - u_i)_+^{r+2}, \ldots, (t - u_i)_+^d.$$

Thus any function in V corresponding to a k-tuple $(p_1, \ldots, p_k)$ can be represented as a sum of the functions in the suggested basis. So we know that this set spans V.

To check linear independence, we suppose that we have a linear combination of the basis elements equal to the zero function:

$$\begin{aligned}
a_0 + a_1 t + a_2 t^2 + \cdots + a_d t^d \quad & + \quad b_{1,r+1}(t - u_1)_+^{r+1} + \cdots + b_{1,d}(t - u_1)_+^d \\
& + \quad b_{2,r+1}(t - u_2)_+^{r+1} + \cdots + b_{2,d}(t - u_2)_+^d \\
& \quad \vdots \\
& + \quad b_{k-1,r+1}(t - u_{k-1})_+^{r+1} + \cdots + b_{k-1,d}(t - u_{k-1})_+^d = 0.
\end{aligned}$$

Now it is straight-forward to check that all the coefficients must be zero, by simply focusing on each subinterval from left to right. Since any shifted power function $(t - c)_+^j$ is zero for $t < c$, we can see that on the first subinterval $[u_0, u_1)$ the sum consists only of the polynomial $a_0 + a_1 t + a_2 t^2 + \cdots + a_d t^d$. Since the equation above must true on each subinterval, we can restrict the t-values to the subinterval $[u_0, u_1)$ to obtain:

$$a_0 + a_1 t + a_2 t^2 + \cdots + a_d t^d = 0, \quad \text{in } P_d.$$

But this is a sum of the standard basis, which is certainly independent in P_d, so all the coefficients a_i must be equal to zero. The above sum now reduces to:

$$\begin{aligned} b_{1,r+1}(t-u_1)_+^{r+1} + \cdots + b_{1,d}(t-u_1)_+^d \quad &+ \quad b_{2,r+1}(t-u_2)_+^{r+1} + \cdots + b_{2,d}(t-u_2)_+^d \\ &\quad \vdots \\ &+ \quad b_{k-1,r+1}(t-u_{k-1})_+^{r+1} + \cdots + b_{k-1,d}(t-u_{k-1})_+^d = 0. \end{aligned}$$

Next, we consider the subinterval $[u_1, u_2)$. The only functions remaining in the above sum which are nonzero on this interval are those of the type $(t-u_1)_+^j$. Further, on the interval $[u_1, u_2)$ these functions are simply polynomials $(t-u_1)^j$. So we obtain:

$$b_{1,r+1}(t-u_1)^{r+1} + \cdots + b_{1,d}(t-u_1)^d = 0, \quad \text{in } P_d.$$

But these polynomials are a subset of a shifted basis of P_d, so are certainly linearly independent in P_d, hence all of these coefficients $b_{i,r+1}$ must also be zero.

Proceeding this way, on each subinterval $[u_i, u_{i+1})$ we conclude that all of the coefficients must be zero and the suggested basis is indeed linearly independent, and hence a basis of V.

Lecture 13, T Feb.19, 2013

Main Points:

- Review for Midterm Exam

Points to be able to work out for midterm exam:

- Determine if a set of polynomials is linearly independent or not using a determinant of the matrix of coordinate vectors
- Determine if a set of polynomials is a top-down basis or not
- Write the coordinate vector of a polynomial expressed as a linear combination of basis polynomials.
- Find the change of basis matrix to go from one polynomial basis to another
- Find the coefficients of an interpolating polynomial in the standard basis using a linear system
- Compute a Vandermonde or Confluent Vandermonde determinant
- Apply the properties of Bernstein polynomials
- Convert a Bezier curve from standard to BB-form using change of basis for coordinate polynomials
- Write bases for P_d^k with ordered k-tuples
- Write bases for $P_d^k[u_0, \ldots, u_k]$ with shifted power functions
- Convert between $P_d^k[u_0, \ldots, u_k]$ and P_d^k using the correspondence between shifted power functions and ordered k-tuples
- Write a piecewise polynomial function in terms of a basis
- Find an interpolating polynomial using Lagrange polynomials
- Find an interpolating polynomial using Newton form
- Find an osculating polynomial using Newton form
- Compute a divided difference given information about an interpolating or osculating polynomial $p(t)$ by using the definition of divided difference
- Compute a divided difference using a table and the recursive formula
- Find an interpolating or osculating polynomial given some data by using the existence and uniqueness theorem
- Compute a divided difference with the Leibniz Rule and verify directly
- Switch the sequence of values for divided differences in the interpolating or osculating cases without changing the resulting polynomial
- Verify the induction step for interpolating or osculating polynomial of degree d based on polynomials of degree $d-1$
- Determine the exact order of continuity of a shifted power function
- Determine the exact order of continuity of a spline function in piecewise polynomial form
- Write a basis for a vector of continuous splines by removing basis functions from a previous basis

Examples of continuity conditions for splines:

- Let $V = P^4_{3,1}[0,1,2,3,4]$. Then V has basis $\{1, t, t^2, t^3, (t-1)^3_+, (t-1)^2_+, (t-2)^3_+, (t-2)^2_+, (t-3)^3_+ (t-3)^2_+\}$ and dimension 10.

- For the previous V, determine if the function

$$f(t) = \begin{cases} p_1(t) = t^3 - 2t, & 0 \le t < 2 \\ p_2(t) = t^3 + 3t^2 - 14t + 12, & 2 \le t \le 4 \end{cases}$$

  is in V, or not. We check and find that $p_1(2) = p_2(2) = 4$ and also that $p_1'(2) = p_2'(2) = 10$, but that $p_1''(2) = 12 \ne p_2''(2) = 18$. This confirms that f is indeed in V.

- Write the previous function f in terms of the standard basis of V. We need to find:

$$f(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4(t-1)^3_+ + a_5(t-1)^2_+ + a_6(t-2)^3_+ + a_7(t-2)^2_+ + a_8(t-3)^3_+ + a_9(t-3)^2_+.$$

  Clearly, the first four coefficients determine $p_1(t)$, so we have:

$$f(t) = -2t + t^3 + a_4(t-1)^3_+ + a_5(t-1)^2_+ + a_6(t-2)^3_+ + a_7(t-2)^2_+ + a_8(t-3)^3_+ + a_9(t-3)^2_+.$$

  Next, we note that there is no need to change this function on the second subinterval $[1,2)$, so we do not need the functions $a_4(t-1)^3_+$ or $a_5(t-1)^2_+$, and so

$$f(t) = -2t + t^3 + +a_6(t-2)^3_+ + a_7(t-2)^2_+ + a_8(t-3)^3_+ + a_9(t-3)^2_+.$$

  Now we need to use the next functions to obtain $p_2(t) - p_1(t)$ on the subinterval $[2,3)$. So we compute:

$$p_2(t) - p_1(t) = (t^3 + 3t^2 - 14t + 12) - (t^3 - 2t) = 3t^2 - 12t + 12 = 3(t^2 - 4t + 4) = 3(t-2)^2.$$

  So when we restrict the previous version of f to the interval $[2,3)$ we obtain

$$p_2(t) = p_1(t) + a_6(t-2)^3 + a_7(t-2)^2$$

  or

$$p_2(t) - p_1(t) = a_6(t-2)^3 + a_7(t-2)^2$$

  or

$$3(t-2)^2 = a_6(t-2)^3 + a_7(t-2)^2.$$

  But this has the obvious solution: $a_6 = 0$ and $a_7 = 3$. Next, we notice that the function f does not change from the interval $[2,3)$ to the interval $[3,4]$, so we do not need the functions $a_8(t-3)^3_+$ or $a_9(t-3)^2_+$ and we can take $a_8 = a_9 = 0$. Finally, we have:

$$f(t) = -2t + t^3 + 3(t-2)^2_+.$$

- The function $f(t) = (t-1)^3_+$ has exact order of continuity 2 at $t = 1$. This means that the first derivative $f'(t) = 3(t-2)^2_+$ and second derivative $f''(t) = 6(t-2)^1_+$ both exist and are continuous at $t = 1$ but that the third derivative fails to exist at $t = 1$. The derivative of $f''(t) = 6(t-2)^1_+$ exists almost everywhere and is equal to $6(t-2)^0_+$ except at $t = 1$, but $f''(t)$ fails to have a derivative at $t = 1$, so f is continuous to exact order 2, since f'' fails to be differentiable at $t = 1$.

- The standard basis of $P^3_1[0,2,4,6]$ is

$$\{1, t, (t-2)^1_+, (t-2)^0_+, (t-4)^1_+, (t-4)^0_+\}.$$

  In order to obtain a basis for $P^3_{1,0}[0,2,4,6]$, the *continuous* piecewise linear functions, we can simply throw out the discontinuous functions from the previous basis to obtain:

$$\{1, t, (t-2)^1_+, (t-4)^1_+\}.$$

- A different basis of $P^3_1[0,2,4,6]$ is the one that corresponds to the set of ordered triples:

$$\{(1,0,0), (t,0,0), (0,1,0), (0,t,0), (0,0,1), (0,0,t)\}$$

  Note that the functions corresponding to these triples, defined on the sequence of intervals $[0,2,4,6]$, are *all discontinuous* and hence no subset of this basis will be a basis of the continuous subspace above.

Lecture 14, T Feb.26, 2013

Main Points:

- Review of bases for splines with continuity
- Affine functions
- Polar forms for polynomials
- Polar forms for Bezier curves

Spline vector spaces with continuity

Recall that the vector space $P^k_{d,r}[u_0, \ldots, u_k]$ of piecewise polynomial functions, also called spline functions, has standard basis given by the union of the sets: i) standard basis of P_d, ii) the shifted power functions $(t - u_i)^j_+$, $j = r+1, \ldots, d$, $i = 1, \ldots, k-1$. For example, $P^4_{3,1}[0, 1, 2, 3, 4]$ has basis:

$$\{1, t, t^2, t^3, (t-1)^2_+, (t-1)^3_+, (t-2)^2_+, (t-2)^3_+, (t-3)^2_+, (t-3)^3_+\}.$$

Affine functions

An affine function $f(x)$ can be defined as:

$$f(x) = \alpha x + \beta$$

where α and β are constants.

An important property of affine functions is that they respect affine sums. In particular:

$$f((1-t)a + tb) = (1-t)f(a) + tf(b).$$

Polar forms for polynomials

A polar form $F[u_1, \ldots, u_d]$ for a polynomial $f(t)$ in P_d is a multivariable multi-valued function satisfying the following properties:

- (symmetry) $F[u_{\sigma(1)}, \ldots, u_{\sigma(d)}] = F[u_1, \ldots, u_d]$ for any permutation σ of the set $\{1, 2, \ldots, d\}$.
- (substitution) $F[t, \ldots, t] = f(t)$
- (affine) $F[u_1, \ldots, x, \ldots, u_d]$ is an affine function of x, where x replaces the variable u_i, for any i, and the other u_j are treated as constant. The coefficients in the affine function then depend on the the u_j for $j \neq i$.

Examples:

- The polar form of $f(t) = 1$ in P_d is $F[u_1, \ldots, u_d] = 1$.
- The polar form of $f(t) = t$ in P_d is

$$F[u_1, \ldots, u_d] = \frac{u_1 + u_2 + \cdots + u_d}{d}.$$

- The polar form of $f(t) = t^d$ in P_d is

$$F[u_1, \ldots, u_d] = u_1 u_2 \cdots u_d.$$

- The polar form of $f(t) = t^2$ in P_3 is

$$F[u_1, u_2, u_3] = \frac{u_1u_2 + u_1u_3 + u_2u_3}{3}.$$

- The polar form of $f(t) = t^3$ in P_4 is

$$F[u_1, u_2, u_3, u_4] = \frac{u_1u_2u_3 + u_1u_2u_4 + u_1u_3u_4 + u_2u_3u_4}{4}.$$

Polar forms for the standard basis

We can find polar forms for any polynomial in the standard basis of P_d as indicated in the above examples. For $f(t) = t^j$ in P_d, we simply form the sum of all products of exactly j variables from the list $u_1, \ldots, u_d$, and then divide by $\binom{d}{j}$. We can write this as:

$$F[u_1, \ldots, u_d] = \frac{u_1u_2 \cdots u_j + \cdots + u_{d-j+1}u_{d-j} \cdots u_d}{\binom{d}{j}}.$$

Polar forms for Bernstein basis

For $f(t) = B_i^d(t) = \binom{d}{i}(1-t)^{d-i}t^i$, we can form the polar form $F[u_1, \ldots, u_d]$ which consists of the sum of all terms which are products of $d - i$ factors of the type $(1 - u_j)$ and i factors of the type u_j. We can write this as:

$$\begin{aligned} F[u_1, \ldots, u_d] \quad &= \quad (1-u_1)(1-u_2)\cdots(1-u_{d-i})u_{d-i+1}\cdots u_d \\ &\quad\;\; \vdots \\ &+ \quad (1-u_{i+1})(1-u_{i+2}\cdots(1-u_d)u_1u_2\cdots u_i. \end{aligned}$$

Examples:

- The polar form of $f(t) = B_1^2(t) = 2(1-t)t$ in P_2 is

$$F[u_1, u_2] = (1-u_1)u_2 + (1-u_2)u_1.$$

- The polar form of $f(t) = B_1^3(t) = 3(1-t)^2t$ in P_3 is

$$F[u_1, u_2] = (1-u_1)(1-u_2)u_3 + (1-u_1)(1-u_3)u_2(1-u_2)(1-u_3)u_1.$$

- The polar form of $f(t) = B_2^4(t) = 6(1-t)^2t^2$ in P_4 is

$$\begin{aligned} F[u_1, u_2] = (1-u_1)(1-u_2)u_3u_4 + (1-u_1)(1-u_3)u_2u_4 + (1-u_1)(1-u_4)u_2u_3 \\ +(1-u_2)(1-u_3)u_1u_4 + (1-u_2)(1-u_4)u_1u_3 + (1-u_3)(1-u_4)u_1u_2. \end{aligned}$$

Polar forms for any polynomial

We define a polar form for a general polynomial to be the sum of polar forms with coefficients with respect to a basis. Specifically, if

$$f(t) = a_0 + a_1t + a_2t^2 + \cdots + a_dt^d,$$

then a polar form is:

$$F[u_1, \ldots, u_d] = a_0F_0 + a_1F_1 + \cdots + a_dF_d,$$

where each F_i is a polar form for t^i in d variables $u_1, \ldots, u_d$. Similarly,

Examples:

- A polar form for $f(t) = 2 - 3t + 4t^2$ is:

$$F[u_1, u_2] = 2 - 3\left(\frac{u_1 + u_2}{2}\right) + 4u_1u_2.$$

Existence and Uniqueness Theorem for Polar Forms of polynomials

For any polynomial $f(t)$ in P_d, there is exactly one multi-variable function $F[u_1, \ldots, u_d]$, called the polar form of f, which satisfies the three defining properties.

Lecture 15, Th Feb.28, 2013

Main Points:

- Polar forms for Bezier curves

Polar forms for parametric curves

For $\gamma(t) = (p_1(t), \ldots, p_n(t))$ a parametric polynomial curve, we define a polar form for $\gamma(t)$ to be given by the polar forms for each component polynomial. It is convenient to put the curve into point coefficient form to reduce the repetition of functions on the coordinates.

Examples:

- Let $\gamma(t)$ be defined in parametric form, and point-coefficient form with respect to the standard basis:

$$\gamma(t) = (1 - t^2, 3 + t + 2t^2) = (1, 3) + (0, 1)t + (-1, 2)t^2.$$

  Then the polar form of $\gamma(t)$ is:

$$F[u_1, u_2] = (1, 3) + (0, 1)\frac{u_1 + u_2}{2} + (-1, 2)u_1u_2$$

- Let $\gamma(t)$ be defined in BB-form, which is point-coefficient form with respect to the Bernstein basis:

$$\gamma(t) = (1 - t)^2(2, -1) + 2(1 - t)t(3, 4) + t^2(1, 3).$$

  Then the polar form of $\gamma(t)$ is:

$$F[u_1, u_2] = ((1 - u_1)(1 - u_2))\,(2, -1) + ((1 - u_1)u_2 + (1 - u_2)u_1)\,(3, 4) + u_1u_2(1, 3).$$

Control Point Property for a parametric polynomial curve

If $\gamma(t)$ is a parametric polynomial curve with polar form $F[u_1, \ldots, u_d]$, then the control points of $\gamma(t)$ are given by:

$$P_i = F[0, 0, \ldots, 0, 1, 1, \ldots, 1],$$

where the number of 1's is equal to the subscript i, for $i = 0, \ldots, d$.

Reparametrization Property for a parametric polynomial curve

If $\gamma(t)$ is a parametric polynomial curve with polar form $F[u_1, \ldots, u_d]$, and $\alpha(t) = \gamma((1-t)a+tb)$ is a reparametrization of $\gamma(t)$, then the control points of $\alpha(t)$ are given by:

$$P_i = F[a, a, \ldots, a, b, b, \ldots, b],$$

where the number of b's is equal to the subscript i, for $i = 0, \ldots, d$.

Examples:

- Polar Form of a quadratic Bezier curve $\gamma_{[P_0,P_1,P_2]}(t)$:

$$F[u_1, u_2] = (1 - u_1)(1 - u_2)P_0 + [(1 - u_1)u_2 + (1 - u_2)u_1]P_1 + u_1u_2P_2.$$

- Control point property: $F[0,0] = P_0$, $F[0,1] = P_1$, $F[1,1] = P_2$.
- Reparamatrization property: $F[a,a] = Q_0$, $F[a,b] = Q_1$, $F[b,b] = Q_2$, where Q_0, Q_1, and Q_2 are the control points of $\alpha(t) = \gamma((1-t)a + tb)$.
- Polar Form of a cubic Bezier curve $\gamma_{[P_0,P_1,P_2,P_3]}(t)$:

$$F[u_1, u_2, u_3] = (1-u_1)(1-u_2)(1-u_3)P_0$$
$$+[(1-u_1)(1-u_2)u_3 + (1-u_1)(1-u_3)u_2 + (1-u_2)(1-u_3)u_1]P_1$$
$$+[(1-u_1)u_2u_3 + (1-u_2)u_1u_3 + (1-u_3)u_1u_2]P_2 + u_1u_2u_3P_3.$$

- Control point property: $F[0,0,0] = P_0$, $F[0,0,1] = P_1$, $F[0,1,1] = P_2$, $F[1,1,1] = P_3$.
- Reparamatrization property: $F[a,a,a] = Q_0$, $F[a,a,b] = Q_1$, $F[a,b,b] = Q_2$, $F[b,b,b] = Q_3$, where Q_0, Q_1, Q_2 and Q_3 are the control points of $\alpha(t) = \gamma((1-t)a + tb)$.

Main Theorem on Polar Forms for parametric curves (Existence and Uniqueness)

Every parametric polynomial curve $\gamma(t)$ has a unique polar form $F[u_1, \ldots, u_d]$ which satisfies the defining properties above, and further satisfies the control point property and reparametrization property.

Polar forms can be evaluated by Nested Linear Interpolation

In order to prove the uniqueness property, as well as the control point and reparametrization properties, of polar forms, we first need to show that they can be evaluated by Nested Linear Interpolation.

First recall the Nested Linear Interpolation diagram of Bezier points for a degree 2 Bezier curve $\gamma(t)$:

$$\begin{array}{ccc} P_0^0 & & \\ & P_0^1 & \\ P_1^0 & & P_0^2 = \gamma(t) \\ & P_1^1 & \\ P_2^0 & & \end{array}$$

Any small triangle of points in this diagram such as:

$$\begin{array}{cc} P_0^0 & \\ & P_0^1 \\ P_1^0 & \end{array}$$

is meant to imply that

$$(1-t)P_0^0 + tP_1^0 = P_0^1,$$

for example. Now we will construct a similar table of values of $F[u_1, u_2]$, assuming F is a polar form for $\gamma(t)$:

$$\begin{array}{ccc} F[0,0] & & \\ & F[u_1,0] & \\ F[0,1] & & F[u_1,u_2] \\ & F[u_1,1] & \\ F[1,1] & & \end{array}$$

In this diagram, each small triangle of points represents a nested linear interpolation, but not with parameter t. In this case, the values in the second column are obtained with parameter u_1, and the value in the third column is obtained with parameter u_2. Specifically:

$$(1-u_1)F[0,0] + u_1F[0,1] = F[u_1,0], \qquad (1-u_1)F[0,1] + u_1F[1,1] = F[u_1,1],$$

$$\text{and} \qquad (1-u_2)F[u_1,0] + u_2F[u_1,1] = F[u_1,u_2].$$

To see that these relationships are valid, we use the properties of the polar form. For instance, since a polar form is affine in each coordinate, we know that it respects affine sums in either coordinate in the following way:

$$F[(1-s)a+sb,c]=(1-s)F[a,c]+sF[b,c], \qquad \text{and} \qquad F[c,(1-s)a+sb]=(1-s)F[c,a]+sF[c,b].$$

Using the first of these with $s=u_1$, $a=0$, $b=1$, and $c=1$, we get:

$$(1-u_1)F[0,1]+u_1F[1,1]=F[u_1,1].$$

The other two can be obtained similarly, also using the symmetry property: $F[u_1,u_2]=F[u_2.u_1]$. For instance, with $s=u_1$, $a=0$, $b=1$, and $c=0$, in the second equation, we get:

$$(1-u_1)F[0,0]+u_1F[0,1]=F[0,u_1]=F[u_1,0].$$

Finally, with $s=u_2$, $a=0$, $b=1$, and $c=u_1$, in the second equation, we get:

$$(1-u_2)F[u_1,0]+u_2F[u_1,1]=F[u_1,u_2].$$

More generally, it follows for any polar form $F[u_1,\ldots,u_d]$, that F can be evaluated by nested linear interpolation in stages with parameters $u_1,u_2\ldots u_d$, from the starting values

$$F[0,0,\ldots,0],F[0,0,\ldots,0,1],F[0,0,\ldots,0,1,1],\ldots,F[0,1,\ldots,1,1],F[1,1,\ldots,1,1].$$

By combining the affine and symmetry properties, as in the above example for $d=2$, we arrive at the same conclusion.

Equivalence of Nested Linear Interpolation (NLI) and BB-form for Bezier curves

For degree $d=1$, the NLI form for a Bezier curve is:

$$\gamma(t)=(1-t)P_0+tP_1.$$

This is identical to the BB-form, since the Bernstein polynomials for degree $d=1$ are simply:

$$B_0^1(t)=1-t, \qquad \text{and} \qquad B_1^1(t0=t.$$

For degree $d=2$, the NLI form for a Bezier curve is:

$$\begin{aligned}
\gamma(t) &= (1-t)\left[(1-t)P_0+tP_1\right]+t\left[(1-t)P_1+tP_2\right] \\
&= (1-t)^2P_0+(1-t)tP_1+t(1-t)P_1+t^2P_2 \\
&= (1-t)^2P_0+2(1-t)tP_1+t^2P_2 \\
&= B_0^2(t)P_0+B_1^2(t)P_1+B_2^2(t)P_2.
\end{aligned}$$

which is also identical to the BB-form.

In general, we can write the NLI form for a curve $\gamma(t)$ with control points $P_0,\ldots,P_d$ recursively as:

$$\gamma(t)=\gamma_{[P_0,\ldots,P_d]}(t)=(1-t)\gamma_{[P_0,\ldots,P_{d-1}]}(t)+t\gamma_{[P_1,\ldots,P_d]}(t).$$

This form has the same content as the Bezier point table, where the parameter t is implied. In fact, the final triangle in that table:

$$\begin{array}{ccc}
P_0^{d-1} & & \\
 & & P_0^d \\
P_1^{d-1} & &
\end{array}$$

is equivalent to the statement:

$$\gamma(t)=P_0^d=(1-t)P_0^{d-1}+tP_1^{d-1}=(1-t)\gamma_{[P_0,\ldots,P_{d-1}]}(t)+t\gamma_{[P_1,\ldots,P_d]}(t).$$

We also have the BB-form with the same control points as:

$$\gamma(t) = B_0^d(t)P_0 + B_1^d(t)P_1 + \cdots + B_d^d(t)P_d = \sum_{i=0}^{d} B_i^d(t)P_i.$$

To see that these are the same, we use induction the recursive formula for the Bernstein polynomials. For the induction hypothesis we assume that these are equivalent for degree k with $0 \le k \le d-1$:

$$\gamma_{[P_0,\ldots,P_k]}(t) = \sum_{i=0}^{d} B_i^k(t)P_i.$$

Then we have:

$$\begin{aligned}
\gamma_{[P_0,\ldots,P_d]}(t) &= (1-t)\gamma_{[P_0,\ldots,P_{d-1}]}(t) + t\gamma_{[P_1,\ldots,P_d]}(t) \\
&= (1-t)\sum_{i=0}^{d-1} B_i^{d-1}(t)P_i + t\sum_{i=0}^{d-1} B_i^{d-1}(t)P_{i+1} \\
&= (1-t)^d P_0 + \sum_{i=1}^{d-1} B_i^{d-1}(t)P_i + t\sum_{i=0}^{d-2} B_i^{d-1}(t)P_{i+1} + t^d P_d \\
&= (1-t)^d P_0 + (1-t)\sum_{i=1}^{d-1} B_i^{d-1}(t)P_i + t\sum_{i=1}^{d-1} B_{i-1}^{d-1}(t)P_i + t^d P_d \\
&= (1-t)^d P_0 + \sum_{i=1}^{d-1} \left[(1-t)B_i^{d-1}(t) + tB_{i-1}^{d-1}(t)\right] P_i + t^d P_d \\
&= (1-t)^d P_0 + \sum_{i=1}^{d-1} \left[(1-t)B_i^{d-1}(t) + tB_{i-1}^{d-1}(t)\right] P_i + t^d P_d \\
&= (1-t)^d P_0 + \sum_{i=1}^{d-1} \left[B_i^d(t)\right] P_i + t^d P_d \\
&= \sum_{i=0}^{d} B_i^d(t).
\end{aligned}$$

Proof of the Control Point Property for Polar forms

In the previous section we showed that a polar form can be evaluated with nested linear interpolation, and we did this using the affine and symmetry properties. If we also now use the substitution property, we can arrive at the control point property. For $d = 2$, we can set $u_1 = u_2 = t$ in the table:

$$\begin{array}{ccc}
F[0,0] & & \\
 & F[u_1,0] & \\
F[0,1] & & F[u_1,u_2] \\
 & F[u_1,1] & \\
F[1,1] & &
\end{array}$$

to obtain the new table:

$$\begin{array}{ccc}
F[0,0] & & \\
 & F[t,0] & \\
F[0,1] & & F[t,t] = \gamma(t) \\
 & F[t,1] & \\
F[1,1] & &
\end{array}$$

Note that in this table all nested linear interpolation is done with parameter t, just as in the case for the Bezier curve. Moreover, we arrive on the right at the function $\gamma(t)$. Since this table behaves identically to the Bezier point table:

$$\begin{array}{ccc} P_0^0 & & \\ & P_0^1 & \\ P_1^0 & & P_0^2 = \gamma(t) \\ & P_1^1 & \\ P_2^0 & & \end{array}$$

we would like to conclude that the starting values must be the same. To see that this must be true, we can use the labels:

$$Q_0 = F[0,0], Q_1 = F[0,1], \quad \text{and} \quad Q_2 = F[1,1].$$

Now suppose that at least one of the equalities $P_0 = Q_0$, $P_1 = Q_1$, and $P_2 = Q_2$ is false. Then we have two sets of control points which give the same result through nested linear interpolation:

$$\gamma_{[P_0,P_1,P_2]}(t) = \gamma_{[Q_0,Q_1,Q_2]}(t).$$

But from the previous section, we know that this is equivalent to the BB-forms being equal:

$$\sum_{i=0}^{d} B_i^d(t) P_i = \sum_{i=0}^{d} B_i^d(t) Q_i.$$

But then if $P_i = (a_i, b_i)$ and $Q_i = (c_i, d_i)$, we can focus on the first coordinate and obtain:

$$\sum_{i=0}^{d} a_i B_i^d(t) = \sum_{i=0}^{d} c_i B_i^d(t).$$

But since the Bernstein polynomials are a basis of P_d, any polynomial is uniquely represented by a choice of coefficients. So the only way for the last sums to be equal is if $a_i = c_i$ for each i. The same applies to the second coordinate, and so we must have $P_i = Q_i$ for all i, which shows that the control point property is true.

Proof of Uniqueness of Polar Forms

In order to show that there is only one polar form for any polynomial, or for any parametric polynomial curve, we use the facts from the previous section.

Suppose that a parametric polynomial curve $\gamma(t)$ has two polar forms $F[u_1, \ldots, u_d]$ and $G[u_1, \ldots, u_d]$. Each of these functions must then satisfy the three defining properties of a polar form. We have also shown in the previous section that each polar form is computable by nested linear interpolation from the special values, with 0's or 1's as arguments, which are in turn equal to the control points of $\gamma(t)$. But then F and G are both computing exactly the same output, and so are equal as functions.

Proof of Reparametrization Property of Polar Forms

The proof of the reparametrization property follows the same ideas used in the proof of the control point property.

For the reparametrization property we assume that we have a curve $\gamma(t)$ and also a reparametrization:

$$\alpha(t) = \gamma((1-t)a + tb),$$

for some constants a and b. We also assume that we have a polar form for $\gamma(t)$, called $F[u_1, \ldots, u_d]$. Assume again, for simplicity, that $d = 2$. Then we can construct a table for nested linear interpolation, starting from the values $F[a,a]$, $F[a,b]$, and $F[b,b]$. But first we work out a few linear interpolations:

$$(1-t)F[a,a] + tF[a,b] = F[a,(1-t)a + tb],$$

and also:

$$(1-t)F[a,b] + tF[b,b] = F[(1-t)a+tb, b] = F[b, (1-t)a+tb].$$

Combining these two we have:

$$\begin{aligned}(1-t)F[a, (1-t)a+tb] &+ tF[b, (1-t)a+tb] \\ &= F[(1-t)a+tb, (1-t)a+tb] \\ &= \gamma((1-t)a+tb) = \alpha(t).\end{aligned}$$

The table then looks like:

$$\begin{array}{lll} F[a,a] & & \\ & F[a,(1-t)a+tb] & \\ F[a,b] & & F[(1-t)a+tb,(1-t)a+tb] = \alpha(t) \\ & F[b,(1-t)a+tb] & \\ F[b,b] & & \end{array}$$

This shows that $\alpha(t)$ can be computed by nested linear interpolation from the values $F[a,a]$, $F[a,b]$ and $F[b,b]$. As before, we conclude that these must be the control points of $\alpha(t)$. This proves the reparametrization property for $d=2$. The cases for higher d follow the same argument.

Lecture 16, T Mar.5, 2013

Main Points:

- Derivatives of Bezier Curves
- Implicit forms for quadratic Bezier Curves

Derivatives of parametric polynomial curves

The derivative of a parametric curve

$$\gamma(t) = (x(t), y(t))$$

is simply:

$$\gamma'(t) = (x'(t), y'(t)).$$

The curve is called differentiable at t_0 if this derivative exists at t_0. The curve is called *smooth* at t_0 if this derivative exists and is nonzero at t_0.

The derivative can be interpreted as a velocity vector in the direction of increasing t if the curve $\gamma(t)$ is traversed by a particle for time t.

The tangent line at a point $\gamma(t_0) = (x_0, y_0)$ on a differentiable curve $\gamma(t)$ is defined to be the line through (x_0, y_0) with direction vector $\gamma'(t_0)$.

Cumulative form for Bezier curves

A Bezier curve can be written in BB-form:

$$\gamma(t) = \sum_{i=0}^{d} B_i^d(t) P_i,$$

where $B_i^d(t)$, for $i = 0, \ldots, d$ are the Bernstein polynomials. We can also use the cumulative Bernstein polynomials to write $\gamma(t)$ in a new form which is convenient for derivatives. First, we recall the definition of the cumulative Bernstein polynomials:

$$C_i^d(t) = \sum_{j=i}^{d} B_j^d(t),$$

which are also defined for $i = 0, \ldots, d$. Since each summation for $C_i^d(t)$ starts with $B_i^d(t)$ and continues to add the higher indexed Bernstein polynomials, we can write:

$$C_i^d(t) = B_i^d(t) + \sum_{j=i+1}^{d} B_j^d(t) = B_i^d(t) + C_{i+1}^d(t),$$

except for the case $i = d$, in which case $C_i^d(t) = B_i^d(t)$. This then leads to:

$$\sum_{i=0}^{d} C_i^d(t) = \sum_{i=0}^{d} B_i^d(t) + \sum_{i=0}^{d-1} C_{i+1}^d(t).$$

We can then write the sum with control points:

$$\begin{aligned} \sum_{i=0}^{d} C_i^d(t) P_i &= \sum_{i=0}^{d} B_i^d(t) P_i + \sum_{i=0}^{d-1} C_{i+1}^d(t) P_i \\ &= \sum_{i=0}^{d} B_i^d(t) P_i + \sum_{i=1}^{d} C_i^d(t) P_{i-1}, \end{aligned}$$

which allows us to solve for:

$$
\begin{aligned}
\gamma(t) &= \sum_{i=0}^{d} B_i^d(t) P_i \\
&= \sum_{i=0}^{d} C_i^d(t) P_i - \sum_{i=1}^{d} C_i^d(t) P_{i-1} \\
&= C_0^d(t) P_0 + \sum_{i=1}^{d} C_i^d(t) \left(P_i - P_{i-1}\right) \\
&= P_0 + \sum_{i=1}^{d} C_i^d(t) \mathbf{v}_i,
\end{aligned}
$$

where $\mathbf{v}_i$ is the vector from P_{i-1} to P_i.

Derivatives of Bezier curves with Cumulative form

Recall the derivative of the cumulative Bernstein polynomials:

$$\frac{d}{dt} C_i^d(t) = d B_{i-1}^{d-1}(t),$$

for $i = 0, \ldots, d$. This can be used to get the derivative of the cumulative form of the Bezier curve:

$$
\begin{aligned}
\gamma'(t) &= \frac{d}{dt} \left[P_0 + \sum_{i=1}^{d} C_i^d(t) \mathbf{v}_i \right] \\
&= \sum_{i=1}^{d} \frac{d}{dt} \left[C_i^d(t) \right] \mathbf{v}_i \\
&= \sum_{i=1}^{d} d B_{i-1}^{d-1}(t) \mathbf{v}_i \\
&= d \sum_{i=0}^{d-1} B_i^{d-1}(t) \mathbf{v}_{i+1}.
\end{aligned}
$$

Examples:

- From the above formula we can deduce:

  $$\gamma'(0) = d\mathbf{v}_1.$$

  For example, if $P_0 = (0,0)$, $P_1 = (2,3)$, and $P_2 = (5,7)$, then we can say that the quadratic Bezier curve $\gamma(t)$ with these control points must have tangent vector at $t = 0$ given by two times the vector between the first two control points:

  $$\gamma'(0) = 2\mathbf{v}_1 = 2\left(P_1 - P_0\right) = 2\left((2,3) - (0,0)\right) = \!\!\rightarrow (4,6).$$

- From the above formula we also can deduce:

  $$\gamma'(1) = d\mathbf{v}_d.$$

  Again, if $P_0 = (0,0)$, $P_1 = (2,3)$, and $P_2 = (5,7)$, then we can say that the quadratic Bezier curve $\gamma(t)$ with these control points must have tangent vector at $t = 1$ given by two times the vector between the last two control points:

  $$\gamma'(1) = 2\mathbf{v}_2 = 2\left(P_2 - P_1\right) = 2\left((5,7) - (2,3)\right) = \!\!\rightarrow (3,4).$$

Degenerate and non-degenerate quadratic Bezier curves

A quadratic Bezier curve $\gamma(t)$ can be define by its BB-form with 3 control points P_0, P_1, and P_2. If all of these control points are collinear, then we call $\gamma(t)$ a degenerate quadratic Bezier curve. In this case all of the points of $\gamma(t)$ lie on the same line. This must be the case since any point $\gamma(t)$ can be computed by nested linear interpolation, starting with the control points, and thus can never be off the line. If the control points are not all collinear, then we call the curve non-degenerate.

Examples:

- For example, if we define:

$$\gamma(t) = (1-t)^2(2,-1) + 2(1-t)t(0,1) + t^2(-1,2),$$

  then since the control points all lie on the line $y = -x + 1$, we must have all the points of $\gamma(t)$ on this line.

- We can also reverse this process and define a quadratic parametric curve which clearly must have all its points on a line. For example:

$$\gamma(t) = (t^2 - 2t + 3, t^2 - 2t)$$

  satisfies the linear relationship $y = x - 3$ since:

$$x - 3 = (t^2 - 2t + 3) - 3 = t^2 - 2t = y.$$

  So if we write this curve in BB-form with three control points, it must be the case that these control points are collinear. We can find the control points with the polar form:

$$F[u_1, u_2] = (3,0) + (-2,-2)\frac{1}{2}[u_1 + u_2] + (1,1)u_1u_2.$$

  This yields:

$$P_0 = F[0,0] = (3,0), \qquad P_1 = F[0,1] = (1,-2), \qquad P_2 = F[1,1] = (2,-1),$$

  which are all on the line $y = x - 3$.

Every non-degenerate quadratic Bezier curve has all points lying on a parabola

A quadratic Bezier curve is one form of a quadratic polynomial parametric curve. In the standard basis, such a curve could be defined as:

$$\gamma(t) = (x, y) = (a_0 + a_1t + a_2t^2, b_0 + b_1t + b_2t^2).$$

Suppose that $\gamma(t)$ is non-degenerate. In order to show that such curves must have all points lying on a parabola, we will first treat the case where at least one of a_2 or b_2 is zero.

Suppose $a_2 = 0$. Then we can solve for t in terms of x:

$$t = \frac{1}{a_1}\left(x - a_0\right).$$

Note: We can divide by a_1 since we cannot have both a_2 and a_1 equal to zero, otherwise the x-coordinate of $\gamma(t)$ would be constant and all its points would lie on a vertical line, but we are assuming that $\gamma(t)$ is non-degnerate. Substituting, we have:

$$y = b_0 + b_1t + b_2t^2 = b_0 + b_1\frac{1}{a_1}\left(x - a_0\right) + b_2\frac{1}{a_1^2}\left(x - a_0\right)^2.$$

Collecting terms and completing the square, we can then write such a quadratic as:

$$y = a(x - b) + c^2,$$

which is a standard form for a parabola with vertex at (b, c) and axis of symmetry parallel to the y-axis.

The case where $b_2 = 0$ produces a standard form for a parabola with axis of symmetry parallel to the x-axis.

Now suppose that both a_2 and b_2 are nonzero. We will show that there is a linear change of coordinates which is in fact simply a rotation of coordinates, which represents $\gamma(t)$ as a parabola.

We can represent any linear change of coordinates as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

In other words:

$$x' = a_{1,1}x + a_{1,2}y = a_{1,1}(a_0 + a_1 t + a_2 t^2) + a_{1,2}(b_0 + b_1 t + b_2 t^2),$$

and

$$y' = a_{2,1}x + a_{2,2}y = a_{2,1}(a_0 + a_1 t + a_2 t^2) + a_{2,2}(b_0 + b_1 t + b_2 t^2).$$

In order for such a coordinate system to represent $\gamma(t)$ as a parabola, we would need to have the t^2 coefficient in x' or in y' be equal to zero. For instance, we can force this in x' if we take

$$a_{1,1} = b_2, \qquad \text{and} \qquad a_{1,2} = -a_2.$$

The transformation matrix then becomes:

$$A = \begin{pmatrix} b_2 & -a_2 \\ a_{2,1} & a_{2,2} \end{pmatrix}.$$

But we wanted to produce a rotation, so we will need to have also $a_{2,1} = a_2$, and $a_{2,2} = b_2$. Additionally, we need to have determinant one, which can be achieved by multiplying the matrix by the constant

$$\frac{1}{\delta} = \frac{1}{\sqrt{a_2^2 + b_2^2}}$$

producing:

$$A = \frac{1}{\delta} \begin{pmatrix} b_2 & -a_2 \\ a_2 & b_2 \end{pmatrix}.$$

This is a rotation matrix and thus the equation for $\gamma(t)$ in the new coordinates x' and y' will be a parabola. Since a rotated parabola is still a parabola, we must conclude that the original graph is also a parabola.

Implicit quadratic equations of conics and the discriminant

The general equation of a conic, or quadratic, in x and y is:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

The discriminant of this equation is:

$$\Delta = B^2 - 4AC.$$

The graph of the conic is called non-degenerate if it is an ellipse, a parabola, or a hyperbola. Other cases, such as a single point, a line, or a pair of lines, are called degenerate. If the graph is non-degenerate, then the type can be determined from the discriminant by the following:

$$\begin{array}{ccc} \Delta < 0 & \longleftrightarrow & \text{Ellipse} \\ \Delta = 0 & \longleftrightarrow & \text{Parabola} \\ \Delta > 0 & \longleftrightarrow & \text{Hyperbola} \end{array}$$

Examples:

- $x^2 + y^2 = 1$ has $A = C = 1$ and $B = 0$, so $\Delta = B^2 - 4AC = -1$, which confirms that the unit circle is an ellipse.
- $y = x^2$ has $A = -1$ and $B = C = 0$, so $\Delta = B^2 - 4AC = 0$, which confirms that this is a parabola.
- $xy = 1$ has $A = C = 0$ and $B = 1$, so $\Delta = B^2 - 4AC = 1$, which confirms that this is a hyperbola.

Five point construction of conics

In order to construct the implicit equation for a quadratic Bezier curve, we will use a geometric technique which starts with the five point construction. This construction allows us to find the equation of a conic, or quadratic polynomial in x and y, which passes through any collection of five points.

Suppose we are given 5 points P_0, P_1, P_2, P_3 and P_4 in the plane. We would like to find an equation of the type

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

which satisfies all five points.

We start by writing two pairs of lines through the first 4 points. Suppose the first pair of lines is

$$L_{0,1}(x, y) = 0 \qquad \text{and} \qquad L_{2,3}(x, y) = 0$$

where $L_{0,1}$ passes through P_0 and P_1, and $L_{2,3}$ passes through P_2 and P_3; and the second pair is

$$L_{0,2}(x, y) = 0 \qquad \text{and} \qquad L_{1,3}(x, y) = 0$$

where $L_{0,2}$ passes through P_0 and P_2, and $L_{1,3}$ passes through P_1 and P_3.

Next, we write the quadratic equation

$$f_c(x, y) = L_{0,1}(x, y)L_{2,3}(x, y) + cL_{0,2}(x, y)L_{1,3}(x, y) = 0,$$

which represents a family of conics each of which passes through the four points P_0, P_1, P_2, and P_3.

The final step is to solve for c by plugging the coordinates of P_4 into both sides of the equation for $f_c(x, y)$. This will guarantee that $f_c(x, y) = 0$ also passes through the fifth point.

Examples:

- Find the conic which passes through the points: $P_0 = (0, 0)$, $P_1 = (1, 0)$, $P_2 = (2, 1)$, $P_3 = (0, 1)$, and $P_4 = (1, 4)$. Note: the points are chosen to be suggestive of an ellipse, and indeed are inconsistent with the shape of a parabola or hyperbola. So we expect that the equation will have negative discriminant.

  We form the lines:

$$L_{0,1}(x, y) = y = 0, \qquad L_{2,3}(x, y) = y - 1 = 0,$$
$$L_{0,2}(x, y) = x - 2y = 0, \qquad L_{1,3}(x, y) = x + y - 1 = 0.$$

  Next we form $f_c(x, y)$:

$$\begin{aligned} f_c(x, y) &= L_{0,1}(x, y)L_{2,3}(x, y) + cL_{0,2}(x, y)L_{1,3}(x, y) \\ &= y(y - 1) + c(x - 2y)(x + y - 1) \\ &= 0. \end{aligned}$$

  Now we insert the coordinates of $P_4 = (1, 4)$:

$$\begin{aligned} f_c(1, 4) &= 4(4 - 1) + c(1 - 2 \cdot 4)(1 + 4 - 1) \\ &= 12 + c(-7)(4) \\ &= 12 - 28c \\ &= 0 \end{aligned}$$

  which means that $c = \frac{12}{28} = \frac{3}{7}$, and the equation of the desired conic is:

$$f(x, y) = y(y - 1) + \frac{3}{7}(x - 2y)(x + y - 1) = 0,$$

which can also be written as:

$$7y(y-1)+3(x-2y)(x+y-1)=0.$$

Expanding and simplifying we get:

$$3x^2-3xy+y^2-3x-y=0.$$

Finally, we check the discriminant and get:

$$\Delta = B^2-4AC = 9-12 = -3 < 0$$

and we know that we have an ellipse.

Tangent construction of conics

Suppose that in the five point construction we let the point P_1 approach P_0 along the line between them until they become very close together and finally they meet at the point P_0. By considering the conics that pass through this sequence of points, we can guess that the limiting case, when P_1 meets P_0, will have tangent line equal to the original line through those two points. This is in fact true, and we can also see that the equation for $f_c(x,y)$ does not change. The role of the line $L_{0,1}$ simply changes to being a tangent line.

We can do the same with the points P_2 and P_3, supposing that P_3 approaches and becomes equal to P_2. If this happens, in addition to P_1 becoming P_0, we now have an effect on the equation for $f_c(x,y)$, which is that the two lines $L_{0,2}$ and $L_{1,3}$ have become the same line, which we will call simply L. This line then appears twice, or with multiplicity two, in the equation. We also relabel the lines $L_{0,1}$ to L_0 and $L_{2,3}$ to L_2, since these are now the tangent lines at P_0 and P_2. We then have:

$$f_c(x,y) = L_0(x,y)L_2(x,y)+cL(x,y)^2 = 0.$$

We can solve for c just as we did in the five point construction, by inserting P_4 into the equation. The resulting equation will then pass through P_0, P_2 and P_4, and have tangent lines L_0 at P_0 and L_2 at P_2.

The above discussion gives an intuitive idea of why the constructions work. In order to verify them completely rigorously we would use linear systems and some calculus, but we will not do this here.

Implicit form of a quadratic Bezier curve

Lecture 17, Th Mar.7, 2013

Main Points:

- Non-degenerate quadratic Bezier curves as parabolas
- Five point construction of conics

Every non-degenerate quadratic Bezier curve has all points lying on a parabola

A quadratic Bezier curve is one form of a quadratic polynomial parametric curve. In the standard basis, such a curve could be defined as:

$$\gamma(t) = (x, y) = (a_0 + a_1 t + a_2 t^2, b_0 + b_1 t + b_2 t^2).$$

Suppose that $\gamma(t)$ is non-degenerate. In order to show that such curves must have all points lying on a parabola, we will first treat the case where at least one of a_2 or b_2 is zero.

Suppose $a_2 = 0$. Then we can solve for t in terms of x:

$$t = \frac{1}{a_1}\left(x - a_0\right).$$

Note: We can divide by a_1 since we cannot have both a_2 and a_1 equal to zero, otherwise the x-coordinate of $\gamma(t)$ would be constant and all its points would lie on a vertical line, but we are assuming that $\gamma(t)$ is non-degenerate. Substituting, we have:

$$y = b_0 + b_1 t + b_2 t^2 = b_0 + b_1 \frac{1}{a_1}\left(x - a_0\right) + b_2 \frac{1}{a_1^2}\left(x - a_0\right)^2.$$

Collecting terms and completing the square, we can then write such a quadratic as:

$$y = a(x - b) + c^2,$$

which is a standard form for a parabola with vertex at (b, c) and axis of symmetry parallel to the y-axis.

The case where $b_2 = 0$ produces a standard form for a parabola with axis of symmetry parallel to the x-axis.

Now suppose that both a_2 and b_2 are nonzero. We will show that there is a linear change of coordinates which is in fact simply a rotation of coordinates, which represents $\gamma(t)$ as a parabola.

We can represent any linear change of coordinates as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

In other words:

$$x' = a_{1,1}x + a_{1,2}y = a_{1,1}(a_0 + a_1 t + a_2 t^2) + a_{1,2}(b_0 + b_1 t + b_2 t^2),$$

and

$$y' = a_{2,1}x + a_{2,2}y = a_{2,1}(a_0 + a_1 t + a_2 t^2) + a_{2,2}(b_0 + b_1 t + b_2 t^2).$$

In order for such a coordinate system to represent $\gamma(t)$ as a parabola, we would need to have the t^2 coefficient in x' or in y' be equal to zero. For instance, we can force this in x' if we take

$$a_{1,1} = b_2, \qquad \text{and} \qquad a_{1,2} = -a_2.$$

The transformation matrix then becomes:

$$A = \begin{pmatrix} b_2 & -a_2 \\ a_{2,1} & a_{2,2} \end{pmatrix}.$$

But we wanted to produce a rotation, so we will need to have also $a_{2,1} = a_2$, and $a_{2,2} = b_2$. Additionally, we need to have determinant one, which can be achieved by multiplying the matrix by the constant

$$\frac{1}{\delta} = \frac{1}{\sqrt{a_2^2 + b_2^2}}$$

producing:

$$A = \frac{1}{\delta} \begin{pmatrix} b_2 & -a_2 \\ a_2 & b_2 \end{pmatrix}.$$

This is a rotation matrix and thus the equation for $\gamma(t)$ in the new coordinates x' and y' will be a parabola. Since a rotated parabola is still a parabola, we must conclude that the original graph is also a parabola.

Five point construction of conics

In order to construct the implicit equation for a quadratic Bezier curve, we will use a geometric technique which starts with the five point construction. This construction allows us to find the equation of a conic, or quadratic polynomial in x and y, which passes through any collection of five points.

Suppose we are given 5 points P_0, P_1, P_2, P_3 and P_4 in the plane. We would like to find an equation of the type

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

which satisfies all five points.

We start by writing two pairs of lines through the first 4 points. Suppose the first pair of lines is

$$L_{0,1}(x, y) = 0 \quad \text{and} \quad L_{2,3}(x, y) = 0$$

where $L_{0,1}$ passes through P_0 and P_1, and $L_{2,3}$ passes through P_2 and P_3; and the second pair is

$$L_{0,2}(x, y) = 0 \quad \text{and} \quad L_{1,3}(x, y) = 0$$

where $L_{0,2}$ passes through P_0 and P_2, and $L_{1,3}$ passes through P_1 and P_3.

Next, we write the quadratic equation

$$f_c(x, y) = L_{0,1}(x, y)L_{2,3}(x, y) + L_{0,2}(x, y)L_{1,3}(x, y) = 0,$$

which represents a family of conics, each of which passes through the four points P_0, P_1, P_2, and P_3.

The final step is to solve for c after plugging in the coordinates of P_4 into the equation for $f_c(x, y)$. This will guarantee that $f_c(x, y)$ also passes through the fifth point.

Examples:

- Find the conic which passes through the points: $P_0 = (0, 0)$, $P_1 = (1, 0)$, $P_2 = (2, 1)$, $P_3 = (0, 1)$, and $P_4 = (1, 4)$. Note: the points are chosen to be suggestive of the shape of an ellipse, and are indeed inconsistent with the shape of a parabola or hyperbola. So we expect that the equation will have negative discriminant.

  We form the lines:

  $$\begin{aligned} L_{0,1}(x, y) = y = 0, &\quad \text{and} \quad L_{2,3}(x, y) = y - 1 = 0, \\ L_{0,2}(x, y) = x - 2y = 0, &\quad \text{and} \quad L_{1,3}(x, y) = x + y - 1 = 0. \end{aligned}$$

  Next, we form $f_c(x, y)$:

  $$\begin{aligned} f_c(x, y) &= L_{0,1}(x, y)L_{2,3}(x, y) + L_{0,2}(x, y)L_{1,3}(x, y) = 0 \\ &= y(y - 1) + c(x - 2y)(x + y - 1) \\ &= 0. \end{aligned}$$

Now we insert the coordinates of $P_4 = (1, 4)$:

$$\begin{aligned} f_c(1,4) &= 4(4-1) + c(1-2\cdot 4)(1+4-1) \\ &= 12 - 28c \\ &= 0, \end{aligned}$$

which means that $c = \frac{12}{28} = \frac{3}{7}$, and the equation of the desired concic is:

$$f_c(x,y) = y(y-1) + \frac{3}{7}(x-2y)(x+y-1) = 0,$$

which can also be written as:

$$7y(y-1) + 3(x-2y)(x+y-1) = 0.$$

Expanding and simplifying, we get:

$$3x^2 - 3xy + y^2 - 3x - y = 0.$$

Finally, we check the discriminant and get:

$$\Delta = B^2 - 4AC = 9 - 12 = -3 < 0$$

and we know that we indeed have an ellipse.

- If we choose five points in certain configurations, we expect to get a hyperbola. For example, suppose $P_0 = (0, 3)$, $P_1 = (3, 0)$, $P_2 = (0, -3)$ and $P_3 = (-3, 0)$. If we now choose $P_4 = (0, 0)$, then we clearly cannot have a parabola nor an ellipse passing through these five points, so we expect to get a hyperbola. In this case, however, there is already a degenerate conic given by the product of the lines $x = 0$ and $y = 0$ which passes through all five points. So, we choose a slightly less symmetric fifth point: $P_4 = (1, 1)$. It seems that in this case, we should get a hyperbola which has one branch passing through P_0, P_1 and P_4, and another symmetric branch passing through P_2, P_3, and $(-1, -1)$.

  We form the lines:

$$L_{0,1}(x,y) = x + y - 3 = 0, \qquad \text{and} \qquad L_{2,3}(x,y) = x + y + 3 = 0,$$
$$L_{0,2}(x,y) = x = 0, \qquad \text{and} \qquad L_{1,3}(x,y) = y = 0.$$

  Next, we form $f_c(x, y)$:

$$\begin{aligned} f_c(x,y) &= L_{0,1}(x,y)L_{2,3}(x,y) + L_{0,2}(x,y)L_{1,3}(x,y) = 0 \\ &= (x+y-3)(x+y+3) + cxy \\ &= 0. \end{aligned}$$

  Now we insert the coordinates of $P_4 = (1, 1)$:

$$\begin{aligned} f_c(1,1) &= (-1)(5) + c\cdot 1 \\ &= -5 + c \\ &= 0, \end{aligned}$$

  which means that $c = 5$ and the equation of the desired concic is:

$$f_c(x,y) = (x+y-3)(x+y+3) + 5xy.$$

  Expanding and simplifying, we get:

$$x^2 + 7xy + y^2 - 9 = 0.$$

  Finally, we check the discriminant and get:

$$\Delta = B^2 - 4AC = 49 - 4 = 45 > 0$$

  and we know that we indeed have a hyperbola. We also see that the hyperbola passes through $(-1, -1)$ as expected.

Lecture 18, T Mar.12, 2013

Main Points:

- Tangent construction of conics
- Implicit forms for quadratic Bezier Curves

Tangent construction of conics

Suppose that in the five point construction we let the point P_1 approach P_0 along the line between them until they finally meet at the point P_0. If we could watch the conics smoothly deform as we perform this transition, we would see that the limit as P_1 approaches P_0 is in fact the conic which has tangent line at P_0 given by the initial line between P_0 and P_1.

We can do the same with the points P_2 and P_3, allowing P_3 to approach P_2, and obtaining a conic in the limit which has tangent line at P_2 given by the initial line through P_2 and P_3. In this process we can also note that the two lines $L_{0,2}$ and $L_{2,3}$ have become the same line, which we call simply L. We also relabel the lines $L_{0,2}$ to L_0 and $L_{2,3}$ to L_2, since these are now the tangent lines at P_0 and P_2. We then have:

$$f_c(x,y) = L_0(x,y)L_2(x,y) + cL(x,y)^2 = 0.$$

We can solve for c just as we did in the five point construction, by inserting P_4 into the equation. The resulting equation will then pass through P_0, P_2, and P_4, and have tangent lines L_0 at P_0 and L_2 at P_2.

The above discussion gives an intuitive idea of how these constructions work. A full verification requires techniques in algebraic geometry, which we will not pursue here.

Implicit form of a quadratic Bezier curve

The quadratic Bezier curve with control points P_0, P_1, and P_2 has an equation of the form

$$f_c(x,y) = L_0(x,y)L_2(x,y) + cL(x,y)^2 = 0,$$

where L_0 is the line containing P_0 and P_1, and is tangent to the curve at P_0, L_2 is the line containing P_1 and P_2, and is tangent to the curve at P_2, and L is the line containing P_0 and P_2.

To solve for c we can use a third point on the the curve, such as $\gamma(\frac{1}{2})$.

Examples:

- Find the implicit equation for the curve $\gamma(t)$ with control points $P_0 = (0,0)$, $P_1 = (1,0)$ and $P_2 = (2,4)$. We find the linear equations:

  $$L_0(x,y) = y = 0, \qquad L_2(x,y) = 4x - y - 4 = 0, \qquad \text{and} \qquad L(x,y) = y - 2x = 0,$$

  which gives:
  $$f_c(x,y) = L_0(x,y)L_2(x,y) + cL(x,y)^2 = y(4x - y - 4) + c(y - 2x)^2 = 0.$$

  Next, we compute $\gamma(\frac{1}{2})$ with the Bezier point array:

  $$\begin{array}{lll} P_0 = (0,0) & & \\ & P_0^1 = (\frac{1}{2}, 0) & \\ P_1 = (1,0) & & P_0^2 = (1,1) = \gamma(\frac{1}{2}) \\ & P_1^1 = (\frac{3}{2}, 2) & \\ P_2 = (2,4) & & \end{array}$$

Finally, we plug in $(1,1)$ to $f_c(x,y)=0$ and solve for c:

$$\begin{aligned} f_c(1,1) &= 1(4-1-4)+c(1-2)^2 \\ &= -1+c \\ &= 0, \end{aligned}$$

which means that $c=1$, and the equation for $f_c = f$ is:

$$\begin{aligned} f(x,y) &= y(4x-y-4)+(y-2x)^2 \\ &= 4xy-y^2-4y+y^2-4xy+4x^2 \\ &= 4x^2-4y \\ &= 0, \end{aligned}$$

and this is equivalent to:

$$4x^2-4y=0,$$

which means that this Bezier curve has the simple implicit form:

$$y=x^2.$$

- We can also reverse this procedure and assume that a Bezier curve has implicit form $y=x^2$, and specify some of the control points and ask for the remaining control points. For example, we could leave $P_2=(2,4)$, and change P_0 to $P_0=(-1,1)$. How do we find P_1?

  Since P_1 determines the tangent line to the curve at P_0, because

  $$\gamma'(0)=2\mathbf{v}_1=2(P_1-P_0),$$

  and since P_1 also determines the tangent line to the curve at P_2, because

  $$\gamma'(1)=2\mathbf{v}_2=2(P_2-P_1),$$

  we see that P_1 must be the intersection point of the two tangent lines at P_0 and P_2. But the tangent lines are easily found using the implicit form. We find the derivative $y'=2x$, and thus the tangent slope at $(2,4)$ is $y'(2)=4$ and the tangent slope at $(-1,1)$ is $y'(-1)=-2$. Thus the tangent line at $(2,4)$ is:

  $$y=4(x-2)+4=4x-4,$$

  and the tangent line at $(-1,1)$ is:

  $$y=-2(x+1)+1=-2x-1.$$

  Then to find the intersection we set:

  $$4x-4=y=-2x-1,$$

  which means $6x=3$, or $x=\frac{1}{2}$, and $y=4\frac{1}{2}-4=-2$. So we have:

  $$P_1=\left(\frac{1}{2},-2\right).$$

  We can perform a small consistency check by computing the point $\gamma(\frac{1}{2})$ with these control points, which should be a point on the curve $y=x^2$. Again, we use the Bezier point array:

  $$\begin{array}{ccccc} P_0=(-1,1) & & & & \\ & & P_0^1=(-\frac{1}{4},-\frac{1}{2}) & & \\ P_1=(\frac{1}{2},-2) & & & & P_0^2=(\frac{1}{2},\frac{1}{4})=\gamma(\frac{1}{2}) \\ & & P_1^1=(\frac{5}{4},1) & & \\ P_2=(2,4) & & & & \end{array}$$

  and indeed $\gamma(\frac{1}{2})=(\frac{1}{2},\frac{1}{4})$ is a point on $y=x^2$.

- Find the implicit equation for the curve $\gamma(t)$ with control points $P_0 = (0, 2)$, $P_1 = (0, 0)$ and $P_2 = (2, 0)$. We expect this one to be a parabola with axis of symmetry along the line $y = x$. This means that the equation should have a nonzero 'cross term' xy. We find the linear equations:

$$L_0(x, y) = x = 0, \qquad L_2(x, y) = y = 0, \qquad \text{and} \qquad L(x, y) = x + y - 2 = 0,$$

which gives:

$$f_c(x, y) = L_0(x, y)L_2(x, y) + cL(x, y)^2 = xy + c(x + y - 2)^2 = 0.$$

Next, we compute $\gamma(\frac{1}{2})$ with the Bezier point array:

$$\begin{array}{ccc} P_0 = (0,2) & & \\ & P_0^1 = (0,1) & \\ P_1 = (0,0) & & P_0^2 = (\frac{1}{2}, \frac{1}{2}) = \gamma(\frac{1}{2}) \\ & P_1^1 = (1,0) & \\ P_2 = (2,0) & & \end{array}$$

Finally, we plug in $(\frac{1}{2}, \frac{1}{2})$ to $f_c(x, y) = 0$ and solve for c:

$$\begin{aligned} f_c\left(\frac{1}{2}, \frac{1}{2}\right) &= \frac{1}{2} \cdot \frac{1}{2} + c\left(\frac{1}{2} + \frac{1}{2} - 2\right)^2 \\ &= \frac{1}{4} + c \\ &= 0, \end{aligned}$$

which means that $c = -\frac{1}{4}$, and the equation for $f_c = f$ is:

$$\begin{aligned} f(x, y) &= xy + c(x + y - 2)^2 \\ &= xy - \frac{1}{4}(x + y - 2)^2 \\ &= 0. \end{aligned}$$

This is equivalent to:

$$4xy - (x + y - 2)^2 = 0,$$

or

$$4xy - (x^2 + 2xy + y^2 - 4x - 4y + 4) = 0,$$

and in standard form we have:

$$x^2 + y^2 - 2xy - 4x - 4y + 4 = 0.$$

Finally, we can check the discriminant $\Delta = B^2 - 4AC = (-2)^2 - 4 = 0$, which confirms that we have a parabola.

Lecture 19, Th Mar.14, 2013

Main Points:

- Further examples of quadratic Bezier Curves
- Knot sequences for spline bases

More Examples:

- Assume that a Bezier curve has implicit form $y = x^2$, and we know that $P_0 = (-2, 4)$ and $P_1 = (-\frac{1}{2}, -2)$. We will find the third control point P_2.

  Note: Once we specify $P_0 = (-2, 4)$, we cannot choose P_1 arbitrarily since it must lie on the tangent line to $y = x^2$ at the point $(-2, 4)$. Since the derivative $y' = 2x$ has value -4 at $x = -2$, we see that this tangent line has equation $y = -4x - 4$ and indeed $(-\frac{1}{2}, -2)$ is on this line.

  Next, we need to find a point P_2, on $y = x^2$, which must have the property that the tangent line at P_2 passes through P_1. Suppose that the point P_2 is written:

  $$P_2 = (a, b) = (a, a^2).$$

  Then the tangent line at $P_2 = (a, a^2)$ has slope $2a$ and can be written:

  $$y = 2a(x - a) + a^2.$$

  Now we can plug in $P_1 = (-\frac{1}{2}, -2)$ to this line to solve for a:

  $$-2 = 2a(-\frac{1}{2} - a) + a^2,$$

  or

  $$-2 = -a - a^2,$$

  or

  $$a^2 + a - 2 = (a + 2)(a - 1) = 0.$$

  We expect to find two solutions, since we already know that the tangent line at $P_0 = (-2, 4)$ does pass through P_1. This corresponds to $a = -2$ in the above quadratic equation. The new solution is $a = 1$, which gives us the point

  $$P_2 = (a, a^2) = (1, 1).$$

  Once again, we can perform a consistency check by computing the point $\gamma(\frac{1}{2})$ with these control points, which should be a point on the curve $y = x^2$.

  $$\begin{array}{ccc} P_0 = (-2,4) & & \\ & P_0^1 = (-\frac{5}{4}, 1) & \\ P_1 = (-\frac{1}{2}, -2) & & P_0^2 = (-\frac{1}{2}, \frac{1}{4}) = \gamma(\frac{1}{2}) \\ & P_1^1 = (\frac{1}{4}, -\frac{1}{2}) & \\ P_2 = (1, 1) & & \end{array}$$

  and indeed $\gamma(\frac{1}{2}) = (-\frac{1}{2}, \frac{1}{4})$ is a point on $y = x^2$.

- What happens if we have collinear control points in the tangent construction? Suppose $P_0 = (0, 0)$, $P_1 = (1, 0)$ and $P_2 = (3, 0)$.

  In this case, since all the control points are on the x-axis, we have

  $$L_0(x, y) = y = 0, \qquad L_2(x, y) = y = 0, \qquad \text{and} \qquad L(x, y) = y = 0,$$

which gives:

$$f_c(x,y) = L_0(x,y)L_2(x,y) + cL(x,y)^2 = y^2 + cy^2 = 0.$$

Now, since any other point on the Bezier curve with these control points can be obtained by nested linear interpolation, it must also be on the same line. So any constant c will do, and we have simply:

$$f_c(x,y) = y^2 = 0.$$

This is just a "double line", which is technically a quadratic equation, whose graph is however simply a line.

The Bezier curve with these control points:

$$\begin{aligned} \gamma(t) &= (1-t)^2 P_0 + 2(1-t)tP_1 + t^2 P_2 \\ &= (1-t)^2(0,0) + 2(1-t)t(1,0) + t^2(3,0) \\ &= (2(1-t)t + 3t^2, 0) \\ &= (2t + t^2, 0) \end{aligned}$$

gives a parametrization which lies on the line $y = 0$, however it does not travel the line in a typical way. Instead, it proceeds in one direction, stops and turns around, and then proceeds in the opposite direction. To see where it stops, we can consider the derivative:

$$\gamma'(t) = 2\left[(1-t)\mathbf{v}_1 + t\mathbf{v}_2\right] = 2\left[(1-t)(1,0) + t(2,0)\right] = 2(1+t, 0).$$

A particle moving with this velocity vector will stop when the vector is zero, and we see that $\gamma'(t) = (0,0)$ exactly for $t = -1$. At $t = 0$ it is at $P_0 = (0,0)$, with velocity vector $(2,0)$, and at $t = 1$ it is at $P_2 = (2,0)$, with velocity vector $(4,0)$.

So we see that the particle must come from $+\infty$ on the x-axis, as t comes from $-\infty$, then pass through P_2 and P_0 for some negative values of t, then when $t = -1$ it reaches the point $(-1,0)$, turns around, and heads back towards $+\infty$ on the x-axis. We can think of this trajectory as a "squashed parabola", which has been flattened so that its vertex is now at $(-1,0)$.

- Find the intersection points of two Bezier curves. Let $\gamma(t)$ have control points $P_0 = (0,2)$, $P_1 = (0,0)$, and $P_2 = (2,0)$. We found the implicit equation to be:

$$x^2 + y^2 - 2xy - 4x - 4y + 4 = 0.$$

Now let $\alpha(t)$ have control points $Q_0 = (0,2)$, $Q_1 = (-1,1)$, and $Q_2 = (2,0)$. Then clearly $\gamma(t)$ and $\alpha(t)$ have at least the two points $P_0 = Q_0$ and $P_2 = Q_2$ in common. From the graphs we can see that they also must have another point in common, with coordinates between 0 and 1.

First, we need the implicit form for $\alpha(t)$. We find the linear equations:

$$L_0(x,y) = x - y + 2 = 0, \qquad L_2(x,y) = x + 3y - 2 = 0, \qquad \text{and} \qquad L(x,y) = x + y - 2 = 0,$$

which gives:

$$f_c(x,y) = L_0(x,y)L_2(x,y) + cL(x,y)^2 = (x - y + 2)(x + 3y - 2) + c(x + y - 2)^2 = 0.$$

Next, we compute $\gamma(\frac{1}{2})$ with the Bezier point array:

$$\begin{array}{ccccc} P_0 = (0,2) & & & & \\ & & P_0^1 = (-\frac{1}{2}, \frac{3}{2}) & & \\ P_1 = (-1,1) & & & & P_0^2 = (0,1) = \alpha(\frac{1}{2}) \\ & & P_1^1 = (\frac{1}{2}, \frac{1}{2}) & & \\ P_2 = (2,0) & & & & \end{array}$$

Finally, we plug in $(0,1)$ to $f_c(x,y)=0$ and solve for c:

$$\begin{aligned} f_c\,(0,1) &= (0-1+2)(0+3-2)+c\,(0+1-2)^2 \\ &= 1+c \\ &= 0, \end{aligned}$$

which means that $c=-1$, and the equation for $f_c=f$ is:

$$f(x,y)=(x-y+2)(x+3y-2)-(x+y-2)^2=0.$$

This is equivalent to:

$$4x-4y^2+12y-8=0,$$

or

$$x=y^2-3y+2=(y-\frac{3}{2})^2-\frac{1}{4}.$$

Now to find the intersection points, we substitute $x=y^2-3y+2$ into the equation of the first parabola

$$x^2+y^2-2xy-4x-4y+4=0$$

to get:

$$(y^2-3y+2)^2+y^2-2(y^2-3y+2)y-4(y^2-3y+2)-4y+4=0$$

and simplifying, we have:

$$y^4-8y^3+16y^2-8y=0.$$

Since we know that these two parabolas do intersect at $P_0=(0,2)$ and $P_2=(2,0)$, we know that the y-coordinates of these points must be solutions of this fourth degree polynomial. Indeed, we can factor out $y(y-2)$ to get:

$$y(y-2)(y^2-6y+4)=0$$

which also has the solutions:

$$y=3\pm\sqrt{5},$$

which gives the approximate points of intersection:

$$(13.7, 5.2) \qquad \text{and} \qquad (0.29, 0.76).$$

Resultants

To intersect to general quadratic equations, we can use the resultant. Suppose that

$$f(x,y)=a_0(x)+a_1(x)y+a_2(x)y^2,$$

and

$$g(x,y)=b_0(x)+b_1(x)y+b_2(x)y^2.$$

Then the resultant is defined as a determinant:

$$R(x)=\begin{vmatrix} a_0(x) & 0 & b_0(x) & 0 \\ a_1(x) & a_0(x) & b_1(x) & b_0(x) \\ a_2(x) & a_1(x) & b_2(x) & b_1(x) \\ 0 & a_2(x) & 0 & b_2(x) \end{vmatrix}.$$

The x-coordinates of the intersection points of $f(x,y)=0$ and $g(x,y)=0$ are zeros of the resultant $R(x)$. Similarly, we can define a resultant $R(y)$, whose zeros are the y-coordinates of the intersection points.

Splines with different orders of continuity at each break point

$P^k_{d,\mathbf{r}}[u_0,\ldots,u_k]$, with $\mathbf{r}=(r_1,\ldots,r_{k-1})$ is the subset of $P^k_d[u_0,\ldots,u_k]$ of functions which have r_i continuous derivatives at the break point u_i, for $i=1,\ldots,k-1$. This is equivalent to requiring that $p_i^{(j)}(u_i)=p_{i+1}^{(j)}(u_i)$ for $j=0,\ldots,r_i$ and $i=1,\ldots,k-1$. In other words, the polynomials must match in their function values (zeroth derivative) and their first r_i derivatives at each break point u_i, for $i=1,\ldots,k-1$.

$P^k_{d,\mathbf{r}}[u_0,\ldots,u_k]$, with $\mathbf{r}=(r_1,\ldots,r_{k-1})$ is a vector space with standard basis:

$$\{1,t,t^2,\ldots,t^d,(t-u_1)_+^{r_1+1},\ldots,(t-u_1)_+^d,\ldots,(t-u_{k-1})_+^{r_{k-1}+1},\ldots,(t-u_{k-1})_+^d\}.$$

Note: This basis is formed in the same way that we did for $P^k_{d,r}[u_0,\ldots,u_k]$ when the order of continuity was the same at each break point. In particular, we can simply throw out those elements from the standard basis of $P^k_d[u_0,\ldots,u_k]$ which are not continuous to order r.

Lecture 20, T Mar.19, 2013

Main Points:

- Vector spaces of splines with various orders of continuity
- Knot sequences for spline bases

Vector spaces of splines with various orders of continuity

Recall that a standard basis for polynomial splines can be formed by including first a basis of the polynomials of degree at most d, and then including at each break point u_i $(1 \leq i \leq k-1)$ the shifted power functions $(t-u_i)_+^j$ where $r+1 \leq j \leq d$.

This gave us the basis for $P_{d,r}^k[u_0, \ldots, u_k]$:

$$\{1, t, t^2, \ldots, t^d, (t-u_1)^{r+1}, \ldots, (t-u_1)^d, \ldots$$

$$\ldots (t-u_{k-1})^{r+1}, \ldots, (t-u_{k-1})^d\}.$$

The same can be done if r is allowed to vary. In particular, we can give a vector of continuity conditions $\mathbf{r} = (r_1, \ldots, r_{k-1})$ and define the vector space of splines with continuity of order at least r_i at each u_i, $1 \leq i \leq k-1$. We call this vector space:

$$P_{d,\mathbf{r}}^k[u_0, \ldots, u_k],$$

where $\mathbf{r} = (r_1, \ldots, r_{k-1})$.

In summary, we define $P_{d,\mathbf{r}}^k[u_0, \ldots, u_k]$, with $\mathbf{r} = (r_1, \ldots, r_{k-1})$ to be the subset of $P_d^k[u_0, \ldots, u_k]$ consisting of functions which have r_i continuous derivatives at the break point u_i, for $i = 1, \ldots, k-1$. This is equivalent to requiring that $p_i^{(j)}(u_i) = p_{i+1}^{(j)}(u_i)$ for $j = 0, \ldots, r_i$ and $i = 1, \ldots, k-1$. In other words, the polynomials must match in their function values (zeroth derivative) and their first r_i derivatives at each break point u_i, for $i = 1, \ldots, k-1$.

A basis for $P_{d,\mathbf{r}}^k[u_0, \ldots, u_k]$, with $\mathbf{r} = (r_1, \ldots, r_{k-1})$ is:

$$\{1, t, t^2, \ldots, t^d, (t-u_1)^{r_1+1}, \ldots, (t-u_1)^d, \ldots$$

$$\ldots (t-u_{k-1})^{r_{k-1}+1}, \ldots, (t-u_{k-1})^d\}.$$

Examples:

- A basis for $P_{2,\mathbf{r}}^4[0,1,2,3,4]$ with $\mathbf{r} = (1, 0, -1)$ is:

$$\{1, t, t^2, (t-1)_+^2, (t-2)_+^1, (t-2)_+^2, (t-3)_+^0, (t-3)_+^1, (t-3)_+^2\}.$$

- A basis for $P_{3,\mathbf{r}}^4[0,1,2,3,4]$ with $\mathbf{r} = (1, 2, 0)$ is:

$$\{1, t, t^2, t^3, (t-1)_+^2, (t-1)_+^3, (t-3)_+^3, (t-3)_+^1, (t-3)_+^2, (t-3)_+^3\}.$$

Knot Sequences:

A knot sequence $\mathbf{t}$ is defined as a non-decreasing sequence of real numbers $\{t_0, t_1, \ldots, t_N\}$, ie. $t_0 \leq t_1 \leq \cdots \leq t_N$. The multipicity of $t_i = c$ in the knot sequence is the greatest number of consecutive values in the knot sequence equal to c.

$sp_d(\mathbf{t})$ is a set of shifted power functions determined by the knot sequence $\mathbf{t}$ so that whenever $t_i = t_{i+1} = \cdots = t_{i+q}$ then we have $(t-t_i)_+^d, (t-t_i)_+^{d-1}, \ldots, (t-t_i)_+^{d-q}$ is part of $sp_d(\mathbf{t})$. If the multiplicities are all equal to 1, then $sp_d(\mathbf{t})$ is just the sequence of shifted power functions $(t-t_i)_+^d$, $i = 0, \ldots, d$.

Examples:

- The knot sequence $\mathbf{t} = \{0, 1, 1, 2, 3, 3, 3\}$ can be used to form the sets of shifted power functions:

$$sp_2(\mathbf{t}) = \{(t-0)_+^2, (t-1)_+^2, (t-1)_+^1, \{(t-2)_+^2, (t-3)_+^2, (t-3)_+^1, (t-3)_+^0\},$$

  and

$$sp_3(\mathbf{t}) = \{(t-0)_+^3, (t-1)_+^3, (t-1)_+^2, \{(t-2)_+^3, (t-3)_+^3, (t-3)_+^2, (t-3)_+^1\}.$$

  We can also create other sets of functions for higher d.

- The knot sequence $\mathbf{t} = \{1, 1, 2\}$ can be used to form the set of shifted power functions:

$$sp_2(\mathbf{t}) = \{(t-1)_+^2, (t-1)_+^1, (t-2)_+^2\}.$$

  This set of functions looks a lot like a top-down basis of P_2. But these functions are not polynomials. They are in fact piecewise polynomial functions. However, if we restrict them to an interval such as $[3, 7]$, then they become simply:

$$\{(t-1)^2, t-1, (t-2)^2\},$$

  which is indeed a top-down basis of P_2 restricted to the interval $[3, 7]$. We will make use of this observation in describing bases for spline vector spaces.

- A basis for $P_2^3[1, 2, 3, 4]$ can be given as:

$$sp_2(\mathbf{t}) = \{(t-1)_+^2, (t-1)_+^1, (t-1)_+^0, (t-2)_+^2, (t-2)_+^1, (t-2)_+^0, (t-3)_+^2, (t-3)_+^1, (t-3)_+^0\},$$

  with $\mathbf{t} = \{1, 1, 1, 2, 2, 2, 3, 3, 3\}$. Note: We have written the terms with decreasing orders. This is convenient since the default case, for knot sequences, is to start with the highest power and only list lower powers if there are multiple copies of the same number. In the correspondence to triples in P_2^3, this basis corresponds to:

$$\{(1,1,1), (t-1, t-1, t-1), ((t-1)^2, (t-1)^2, (t-1)^2),$$
$$(0,1,1), (0, t-2, t-2), (0, (t-2)^2, (t-2)^2),$$
$$(0,0,1), (0,0,t-3), (0,0,(t-3)^2)\}.$$

  Note: these terms are written in the usual degree increasing order, as we have done before for k-tuples. (Can you see which triple corresponds to which function?)

- A basis for $P_{2,\mathbf{r}}^3[1, 2, 3, 4]$, with $\mathbf{r} = (1, 0)$, can be given as:

$$sp_2(\mathbf{t}) = \{(t-0)_+^2, (t-0)_+^1, (t-0)_+^0, (t-2)_+^2, (t-3)_+^2, (t-3)_+^1\},$$

  with $\mathbf{t} = \{0, 0, 0, 2, 3, 3\}$. Note that on the interval $[1, 4]$ the functions $(t-0)_+^2, (t-0)_+^1$ and $(t-0)_+^0$ are exactly the same as the standard basis t^2, t and 1.

Multiplicity Vector

Related to the continuity vector $\mathbf{r} = (r_1, \ldots, r_{k-1})$ is the multiplicity vector:

$$\mathbf{m} = (m-1, \ldots, m_{k-1}).$$

Each m_i is defined to be:

$$m_i = d - r_i,$$

which also coincides with the number of functions in the list:

$$(t-u_i)_+^{r_i+1}, \ldots, (t-u_i)_+^d.$$

Also, since the orders of continuity satisfy $-1 \leq r_i \leq d-1$, we have:

$$1 \leq m_i \leq d+1.$$

Dimension of spline vector spaces with different orders of continuity at each break point

The dimension of $P_{d,\mathbf{r}}^k[u_0, \ldots, u_k]$ is given as:

$$d+1+\sum_{i=1}^{k-1}(d-r_i) = d+1+\sum_{i=1}^{k-1} m_i.$$

Note: This basis consists of a basis of P_d (the standard basis) and also $d-r_i$ shifted power functions at u_i, for $i=1,\ldots,k-1$.

General shifted power bases

The vector space $V = P_{d,\mathbf{r}}^k[u_0, \ldots, u_k]$, with $\mathbf{r} = (r_1, \ldots, r_{k-1})$ has many different shifted power bases, which can be given by specifying different knot sequences

$$\mathbf{t} = \{t_0, t_1, \ldots, t_N\},$$

where

$$N+1 = d+1+\sum_{i=1}^{k-1} m_i$$

is the dimension of V. The first $d+1$ values in the knot sequence must correspond to a basis of polynomials which replaces the standard basis $\{1, t, t^2, \ldots, t^d\}$. We can construct such a set by choosing shifted power functions $(t-c)_+^j$ whose value c is less than or equal to u_0. This way, any such function is simply a polynomial on the interval $[u_0, u_k]$. If we choose

$$t_0 \leq t_1 \leq \cdots \leq t_d \leq u_0,$$

then we find that the set of shifted power functions associated to this set

$$sp_d(\{t_0, t_1, \ldots, t_d\})$$

is in fact a top down basis of P_d, when restricted to the interval $[u_0, u_k]$. Note: the right half of the sequence:

$$\{t_{d+1}, t_{d+2}, \ldots, t_d\} = \{u_1, \ldots, u_1, \ldots, u_{k-1}, \ldots, u_{k-1}\}$$

is made up of the break points with their multiplicities given by the m_i.

Examples: The vector space $P_{2,\mathbf{r}}^3[1,2,3,4]$, with $\mathbf{r} = (1,0)$, has multiplicity vector $\mathbf{m} = (1,2)$, and has the bases $sp_2(\mathbf{t})$ for all of the following knot sequences:

- $\mathbf{t} = \{1,1,1,2,3,3\}$, with basis:

  $$\{(t-1)_+^2, (t-1)_+^1, (t-1)_+^0, (t-2)_+^2, (t-3)_+^2, (t-3)_+^2\}.$$

  (Note: the first three functions give a shifted basis of polynomials.)

- $\mathbf{t} = \{-3,-2,-1,2,3,3\}$, with basis:

  $$\{(t+3)_+^2, (t+2)_+^2, (t+1)_+^2, (t-2)_+^2, (t-3)_+^2, (t-3)_+^2\}.$$

  (Note: the first three functions give a Vandermonde basis of polynomials.)

- $\mathbf{t} = \{-2, -2, -1, 2, 3, 3\}$, with basis:

$$\{(t+2)_+^2, (t+2)_+^1, (t+1)_+^2, (t-2)_+^2, (t-3)_+^2, (t-3)_+^2\}.$$

  (Note: the first three functions give a top-down basis of polynomials.)

Definition of B-Splines with divided differences

The definition of B-splines with the divided difference formulation: Given a knot sequence $\mathbf{t}$ with $t_0 \le t_1 \le \ldots \le t_N$, we define for all $d \ge 0$ and for $0 \le i \le N-d-1$:

$$\mathcal{B}_i^d(t) = (-1)^{d+1}(t_{i+d+1} - t_i)[t_i, t_{i+1}, \ldots, t_{i+d+1}](t-x)_+^d.$$

Note: The divided difference is computed with t as constant and x as the variable. We can think of $g(x) = (t-x)_+^d$ as playing the role of the data function.

Examples:

- We will compute the piecewise polynomial form for $\mathcal{B}_i^0(t)$. First, we write:

$$\mathcal{B}_i^0(t) = (-1)(t_{i+1} - t_i)[t_i, t_{i+1}](t-x)_+^0.$$

  This definition uses the left continuous shifted power function

$$(t-x)_+^0 = \begin{cases} 1, & x \le t \\ 0, & x > t \end{cases}$$

  Now suppose that

$$t_i \le t < t_{i+1}.$$

  Then the data function $g(x)$ has values:

$$g(t_i) = 1, \qquad \text{and} \qquad g(t_{i+1}) = 0.$$

  Now recall that the divided difference $[t_i, t_{i+1}]g$ is equal to the coefficient of x in the interpolating polynomial $p(x)$ which matches g at the values $x = t_i$ and $x = t_{i+1}$. But this is just the slope of the line through the two points $(t_i, 1)$ and $(t_{i+1}, 0)$ which we can compute as

$$[t_i, t_{i+1}]((t-x)_+^0 = \frac{-1}{t_{i+1} - t_i}.$$

  So, when $t_i \le t < t_{i+1}$, then

$$\begin{aligned} \mathcal{B}_i^0(t) &= (-1)(t_{i+1} - t_i)[t_i, t_{i+1}]((t-x)_+^0 \\ &= (-1)(t_{i+1} - t_i)\frac{-1}{t_{i+1} - t_i} \\ &= 1 \end{aligned}$$

  Now if $t < t_i$ we can see from the graph of $g(x) = (t-x)_+^0$ that $g(t_i) = g(t_{i+1}) = 0$ and thus the slope of the line through $(t_i, 0)$ and $(t_{i+1}, 0)$ is zero, and thus the divided difference is also zero.

  Similarly, if $t < t_i$ we can see from the graph of $g(x) = (t-x)_+^0$ that $g(t_i) = g(t_{i+1}) = 0$ and thus the slope of the line through $(t_i, 0)$ and $(t_{i+1}, 0)$ is zero, and thus the divided difference is also zero. So we have:

$$\mathcal{B}_i^0(t) = \begin{cases} 1, & t_i \le t < t_{i+1} \\ 0, & t < t_i \ \text{ or } \ t \ge t_{i+1} \end{cases}$$

Lecture 21, Th March 21, 2013

Main Points:

- Quiz 8
- General discussion of B-splines and B-spline curves
- Order of continuity of B-splines based on knot sequence
- Project Part IV Overview

Definition of B-splines of degree d for a knot sequence $\mathbf{t}$

Given a knot sequence $\mathbf{t}$ with $t_0 \le t_1 \le \ldots \le t_N$, we define for any degree $d \ge 0$, and for $0 \le i \le N-d-1$:

$$\mathcal{B}_i^d(t) = (-1)^{d+1}(t_{i+d+1} - t_i)[t_i, t_{i+1}, \ldots, t_{i+d+1}](t-x)_+^d$$

Some points regarding B-splines:

- The set $\mathcal{B}_d(\mathbf{t})$ or $\mathcal{B}_{d,\mathbf{t}}$ is the set of B-splines associated to the knot sequence $\mathbf{t}$: $\mathcal{B}_d(\mathbf{t}) = \{\mathcal{B}_0^d(t), \ldots, \mathcal{B}_{N-d-1}^d(t)\}$
- The interval of support of a nonzero B-spline $\mathcal{B}_i^d(t)$ is the interval where it is nonzero: (t_i, t_{i+d+1}).
- Note: From the definition of B-spline, if $t_i = t_{i+d+1}$ (which means that all t_j with $i < j < i+d+1$ are also equal to t_i) then the B-spline $\mathcal{B}_i^d(t)$ is just zero for all t, since it starts with the factor $(t_{i+d+1} - t_i) = 0$.
- Positivity of nonzero B-splines: $\mathcal{B}_i^d(t) > 0$ for $t_i < t < t_{i+d+1}$.

Orders of continuity for a B-spline

The possible orders of continuity at a given breakpoint c in a spline space with order of continuity r are: $r, r+1, \ldots, d$ for $r = d - m$, where m is the multiplicity of the knots equal to c. Given the exact descriptions of bases, and the notes above, it is possible to say exactly which functions in a basis have which exact orders of continuity. We say that a function f has exact order of continuity r at c if $f^{(j)}(c)$ exists for $j = 0, \ldots r$, but $f^{(r+1)}(c)$ does not exist.

The exact order of continuity of a B-spline $\mathcal{B}_i^d(t)$ at each of the knot values $t_i, \ldots, t_{i+d+1}$ is given by $d - m(t_i)$ where $m(t_i)$ is the multiplicity of the value t_i in the *subsequence* $t_i, \ldots, t_{i+d+1}$. (Note: This is NOT necessarily the same as the multiplicity in the whole knot sequence $\mathbf{t}$.)

Examples:

- A degree $d = 0$ B-spline has two knot values, say t_i and t_{i+1}. As we saw before, if $t_i < t_{i+1}$, then $\mathcal{B}_i^0(t) = 1$, for $t_i \le t < t_{i+1}$ and is zero otherwise. Since each of t_i and t_{i+1} has multiplicity one, this B-spline must have order of continuity $d - m = 0 - 1 = -1$ at each value, which means a *discontinuity.*
- A degree $d = 1$ B-spline has three knot values, say t_i, t_{i+1}, and t_{i+2}. We can guess the shape of such a B-spline depending on the multiplicities of the knot values. For example, if the multiplicities are all one, so that $t_i < t_{i+1} < t_{i+2}$, then the $\mathcal{B}_i^1(t)$, has order of continuity $d = m = 1 - 1 = 0$ at each knot values, which means that it is continuous there. Since it is also positive (by the property above) and piecewise linear for $t_i < t < t_{i+2}$, and equal to zero at t_i and t_{i+2}, we see that the shape of the function is a hat shape.
- Now suppose $d = 1$ and $t_i = t_{i+1} < t_{i+2}$. Then $\mathcal{B}_i^1(t)$ has order of continuity $d - m = 1 - 2 = -1$ at $t = t_i$ and order of continuity 0 at $t = t_{i+2}$.
- A degree $d = 2$ B-spline has four knot values t_i, t_{i+1}, t_{i+2}, and t_{i+3}. We can list some examples with orders of continuity and multiplicities:

degree	knot values	multiplicities	orders of continuity
2	0,1,2,3	1,1,1,1	1,1,1,1
2	0,0,1,2	2,1,1	0,1,1
2	0,1,1,2	1,2,1	1,0,1
2	0,1,2,2	1,1,2	1,1,0
2	0,2,2,2	1,3	1,-1

- With the knot sequence $\mathbf{t} = \{0, 1, 2, 2, 3, 3, 3, 4, 5, 6\}$, the spline $\mathcal{B}_1^2(t)$ is based on the subsequence $\{1, 2, 2, 3\}$. The order of continuity at 3 will be 1 since the multiplicity of 3 in this subsequence is 1 and $r = d - m = 2 - 1 = 1$.

  Here is a table of the B-splines associated to this knot sequence, with their subsequence, multiplicities and orders of continuity. Note: the index i of each B-spline determines the starting knot value t_i.

index i	B-spline	knot values	multiplicities	orders of continuity
0	$\mathcal{B}_0^2(t)$	0,1,2,2	1,1,2	1,1,0
1	$\mathcal{B}_1^2(t)$	1,2,2,3	1,2,1	1,0,1
2	$\mathcal{B}_2^2(t)$	2,2,3,3	2,2	0,0
3	$\mathcal{B}_3^2(t)$	2,3,3,3	1,3	1,-1
4	$\mathcal{B}_4^2(t)$	3,3,3,4	3,1	-1,1
5	$\mathcal{B}_5^2(t)$	3,3,4,5	2,1,1	0,1,1
6	$\mathcal{B}_6^2(t)$	3,4,5,6	1,1,1,1	1,1,1,1

  The graphs of each of the above B-splines can be drawn, up to a scaling factor, based on the orders of continuity at each of the knot values.

Curry-Schoenberg Theorem for B-spline bases.

The Curry-Schoeberg Theorem states that $P_{d,\mathbf{r}}^k[u_0, \dots, u_k]$, with $\mathbf{r} = (r_1, \dots, r_{k-1})$ and $\mathbf{m} = (m_1, \dots, m_{k-1})$, has a basis consisting of B-splines associated to a knot sequence $\mathbf{t} = \{t_0, \dots, t_N\}$ given by $t_0 \le t_1 \le \cdots \le t_d \le u_0$, and $u_k \le t_{N-d} \le \cdots \le t_N$. The middle part of the knot sequence $t_{d+1}, \dots, t_{N-d-1}$ corresponds exactly to the sequence of breakpoints $u_1, \dots, u_1, u_2, \dots, u_2, \quad \dots \quad, u_{k-1}, \dots, u_{k-1}$ where the multiplicity of each u_i is $m_i = d - r_i$.

B-spline curves

A B-spline curve can be written in the form:

$$\gamma(t) = \sum_{i=0}^{N-d-1} \mathcal{B}_i^d(t) P_i,$$

where the points P_i are called the control points. This form is analogous to the BB-form form Bezier curves, where the functions of t are the Bernstein polynomials, and the index runs from $i = 0$ to d since P_d has dimension $d+1$. In the case of the B-splines, the dimension of the vector space of splines has dimension $N - d$. By the Curry-Schoenberg Theorem, this number aligns with the dimension of the vector space $P_{d,\mathbf{r}}^k[u_0, \dots, u_k]$, for some orders of continuity $(r_1, \dots, r_{k-1})$, which is $d + 1 + \sum_{i=1}^{k-1} d - r_i$. So we have:

$$dim\left(P_{d,\mathbf{r}}^k[u_0, \dots, u_k]\right) = d + 1 + \sum_{i=1}^{k-1} d - r_i = N - d.$$

Since there are many different bases for this vector space of splines, the curve $\gamma(t)$ can be written in different forms without changing the function. This particular form also shares the property that control points affect the shape of the curve, just as with the BB-form, and that the curve can be evaluated by nested linear interpolation. The NLI algorithm for evaluating B-spline curves is called the DeBoor Algorithm.

Lecture 22, T April 2, 2013

Main Points:

- B-spline Recursion formula
- Sums of shifted power functions

B-splines of degree d for a knot sequence

Given a knot sequence $\mathbf{t}$ with $t_0 \leq t_1 \leq \ldots \leq t_N$, we define for any degree $d \geq 0$, and for $0 \leq i \leq N-d-1$:

$$\mathcal{B}_i^d(t) = (-1)^{d+1}(t_{i+d+1} - t_i)[t_i, t_{i+1}, \ldots, t_{i+d+1}](t-x)_+^d$$

The set $\mathcal{B}_d(\mathbf{t})$ or $\mathcal{B}_{d,\mathbf{t}}$ is the set of B-splines associated to the knot sequence $\mathbf{t}$: $\mathcal{B}_d(\mathbf{t}) = \{\mathcal{B}_0^d(t), \ldots, \mathcal{B}_{N-d-1}^d(t)\}$

Examples:

- We will compute the piecewise polynomial form for $\mathcal{B}_i^1(t)$. First, we write:

$$\mathcal{B}_i^1(t) = (-1)^2(t_{i+2} - t_i)[t_i, t_{i+1}, t_{i+2}](t-x)_+^1.$$

This definition uses the continuous shifted power function

$$g(x) = (t-x)_+^1 = \begin{cases} t-x, & x \leq t \\ 0, & x > t \end{cases}$$

If $t < t_i$ then we have $g(t_i) = g(t_{i+1}) = g(t_{i+2}) = 0$ and thus the divided difference

$$[t_i, t_{i+1}, t_{i+2}](t-x)_+^1 = 0, \quad \text{for} \quad t < t_i.$$

Also, if $t \geq t_{i+2}$ then $g(x)$ takes the values of the straight line $t-x$ at each of these inputs, which means that the interpolating polynomial which matches g at those values is in fact a line. But then the coefficient of x^2 in this polynomial must be zero. So again we have:

$$[t_i, t_{i+1}, t_{i+2}](t-x)_+^1 = 0, \quad \text{for} \quad t \geq t_{i+2}.$$

Now suppose that

$$t_i \leq t < t_{i+1}.$$

Then we compute $[t_i, t_{i+1}, t_{i+2}]g$ from the divided difference table:

$$\begin{array}{cccc}
t_i & t - t_i & & \\
 & & -\frac{t-t_i}{t_{i+1}-t_i} & \\
t_{i+1} & 0 & & \frac{t-t_i}{(t_{i+2}-t_i)(t_{i+1}-t_i)} = [t_i, t_{i+1}, t_{i+2}]g \\
 & & 0 & \\
t_{i+2} & 0 & &
\end{array}$$

Then we have:

$$\mathcal{B}_i^1(t) = (t_{i+2} - t_i)\frac{t - t_i}{(t_{i+2} - t_i)(t_{i+1} - t_i)} = \frac{t - t_i}{t_{i+1} - t_i}$$

for $t_i \leq t < t_{i+1}$. Note that this is a line segment which starts at $(t_i, 0)$ and ends by approaching the point $(t_{i+1}, 1)$.

Now suppose that

$$t_{i+1} \leq t \leq t_{i+2}.$$

Again we compute $[t_i, t_{i+1}, t_{i+2}]g$ from the divided difference table:

$$
\begin{array}{cccc}
t_i & t - t_i & & \\
 & & -1 & \\
t_{i+1} & t - t_{i+1} & & \frac{1}{t_{i+2}-t_i}\left(1 - \frac{t-t_{i+1}}{t_{i+2}-t_{i+1}}\right) = [t_i, t_{i+1}, t_{i+2}]g \\
 & & -\frac{t-t_{i+1}}{t_{i+2}-t_{i+1}} & \\
t_{i+2} & 0 & &
\end{array}
$$

Then we have:

$$\mathcal{B}_i^1(t) = (t_{i+2} - t_i)\frac{1}{t_{i+2} - t_i}\left(1 - \frac{t - t_{i+1}}{t_{i+2} - t_{i+1}}\right) = 1 - \frac{t - t_{i+1}}{t_{i+2} - t_{i+1}}$$

for $t_{i+1} \le t \le t_{i+2}$. Note that this is a line segment which starts at $(t_{i+1}, 1)$ and ends at the point $(t_{i+2}, 0)$.

We can now put the whole function together, to get:

$$\mathcal{B}_i^1(t) = \begin{cases} 0, & t < t_i \\ \frac{t-t_i}{t_{i+1}-t_i}, & t_i \le t < t_{i+1} \\ 1 - \frac{t-t_{i+1}}{t_{i+2}-t_{i+1}}, & t_{i+1} \le t \le t_{i+2} \\ 0, & t > t_{i+2} \end{cases}$$

The graph of this B-spline is a 'hat function' which is zero outside and at the endpoints of the interval $[t_i, t_{i+2}]$, and reaches the value 1 at the value t_{i+1} in between.

The B-spline recursion formula

The (DeBoor-Cox) recursion formula for B-splines of degree d associated to a knot sequence $\mathbf{t} = \{t_0, \dots, t_N\}$ is:

$$\mathcal{B}_i^d(t) = \frac{t - t_i}{t_{i+d} - t_i}\mathcal{B}_i^{d-1}(t) + \frac{t_{i+d+1} - t}{t_{i+d+1} - t_{i+1}}\mathcal{B}_{i+1}^{d-1}(t)$$

The base case is:

$$\mathcal{B}_i^0(t) = \begin{cases} 1, & t_i \le t < t_{i+1} \\ 0, & elsewhere \end{cases}$$

Special case: if $t_{i+d+1} = t_i$ then the B-spline $\mathcal{B}_i^d(t)$ is defined to be zero.

Note: The base case and special case above are necessary if this is taken to be the definition of B-splines, which it is in many texts. However, the recursion follows directly from the definition using the divided differences. We have already worked out the base case $d = 0$ with the definition. Next we will prove the recursion using the same definition.

Proof of the B-spline recursion formula

First, we write the function $(t - x)_+^d$ as a product:

$$(t - x)_+^d = g(x)h(x) = (t - x)(t - x)_+^{d-1}.$$

We will use the Leibniz formula for divided difference of a product applied to the divided difference in the B-spline definition:

$$[t_i, t_{i+1}, \dots, t_{i+d+1}](t - x)_+^d = [t_i, t_{i+1}, \dots, t_{i+d+1}]g \cdot h.$$

Recalling the Leibniz formula, we have:

$$[t_i, t_{i+1}, \ldots, t_{i+d+1}]g \cdot h = [t_i]g \cdot [t_i, \ldots, t_{i+d+1}]h + [t_i, t_{i+1}]g \cdot [t_{i+1}, \ldots, t_{i+d+1}]h + \cdots$$

where the terms described by the three dots will shortly be seen to be all equal to zero. First we describe the two terms above. In the first term we have

$$[t_i]g = g(t_i) = t - t_i$$

from the definition of the function $g(x) = t - x$. In the second term we have

$$[t_i, t_{i+1}]g = \frac{g(t_{i+1}) - g(t_i)}{t_{i+1} - t_i} = \frac{t - t_{i+1} - (t - t_i)}{t_{i+1} - t_i} = \frac{t_i - t_{i+1}}{t_{i+1} - t_i} = -1.$$

This can also be deduced directly from the definition of the divided difference as the coefficient of x in the interpolating polynomial since in this case that polynomial must simply be the line $t - x$ with slope -1.

In order to see that the higher terms are all zero, we use the same interpretation of the divided difference for $[t_i, t_{i+1}, t_{i+2}]g$. In this case, it is equal to the coefficient of x^2 in the interpolating polynomial which matches g at those three values. But g is still a linear polynomial $t - x$, so $p(x) = t - x$, and thus the coefficient of x^2 is zero. This holds for all higher terms as well.

So far we have:

$$[t_i, t_{i+1}, \ldots, t_{i+d+1}]g \cdot h = (t - t_i) \cdot [t_i, \ldots, t_{i+d+1}]h + (-1) \cdot [t_{i+1}, \ldots, t_{i+d+1}]h.$$

Next, we apply the recursive form of the divided difference to $[t_i, \ldots, t_{i+d+1}]h$, and simplify:

$$\begin{aligned}
[t_i, t_{i+1}, \ldots, t_{i+d+1}]g \cdot h &= (t - t_i) \cdot [t_i, \ldots, t_{i+d+1}]h + (-1) \cdot [t_{i+1}, \ldots, t_{i+d+1}]h \\
&= (t - t_i) \left(\frac{[t_{i+1}, \ldots, t_{i+d+1}]h - [t_i, \ldots, t_{i+d}]h}{t_{i+d+1} - t_i} \right) - [t_{i+1}, \ldots, t_{i+d+1}]h \\
&= \left(\frac{t - t_i}{t_{i+d+1} - t_i} - 1 \right) [t_{i+1}, \ldots, t_{i+d+1}]h - \frac{(t - t_i)}{t_{i+d+1} - t_i} [t_i, \ldots, t_{i+d}]h \\
&= \frac{t - t_{i+d+1}}{t_{i+d+1} - t_i} [t_{i+1}, \ldots, t_{i+d+1}]h - \frac{(t - t_i)}{t_{i+d+1} - t_i} [t_i, \ldots, t_{i+d}]h
\end{aligned}$$

Next, recall the definitions of two lower degree B-splines:

$$\mathcal{B}_i^{d-1}(t) = (-1)^d (t_{i+d} - t_i) [t_i, t_{i+1}, \ldots, t_{i+d}](t - x)_+^{d-1}$$

and

$$\mathcal{B}_{i+1}^{d-1}(t) = (-1)^d (t_{i+d+1} - t_{i+1}) [t_{i+1}, t_{i+2}, \ldots, t_{i+d+1}](t - x)_+^{d-1}.$$

These allow us to solve for the divided differences in the above formula:

$$[t_i, \ldots, t_{i+d}]h = \frac{(-1)^d}{(t_{i+d} - t_i)} \mathcal{B}_i^{d-1}(t),$$

and

$$[t_{i+1},\dots,t_{i+d+1}]h = \frac{(-1)^d}{(t_{i+d+1}-t_{i+1})}\mathcal{B}_{i+1}^{d-1}(t).$$

Putting all of this together, we then have:

$$\begin{aligned}
\mathcal{B}_i^d(t) &= (-1)^{d+1}(t_{i+d+1}-t_i)[t_i,t_{i+1},\dots,t_{i+d+1}]g\cdot h \\
&= (-1)^{d+1}(t_{i+d+1}-t_i)\left(\frac{t-t_{i+d+1}}{t_{i+d+1}-t_i}[t_{i+1},\dots,t_{i+d+1}]h - \frac{(t-t_i)}{t_{i+d+1}-t_i}[t_i,\dots,t_{i+d}]h\right) \\
&= (-1)^{d+1}\left((t-t_{i+d+1})[t_{i+1},\dots,t_{i+d+1}]h - (t-t_i)[t_i,\dots,t_{i+d}]h\right) \\
&= (-1)^{d+1}\left((t-t_{i+d+1})\frac{(-1)^d}{(t_{i+d+1}-t_{i+1})}\mathcal{B}_{i+1}^{d-1}(t) - (t-t_i)\frac{(-1)^d}{(t_{i+d}-t_i)}\mathcal{B}_i^{d-1}(t)\right) \\
&= (-1)^{2d+1}\left(\frac{(t-t_{i+d+1})}{(t_{i+d+1}-t_{i+1})}\mathcal{B}_{i+1}^{d-1}(t) - \frac{(t-t_i)}{(t_{i+d}-t_i)}\mathcal{B}_i^{d-1}(t)\right) \\
&= -\frac{(t-t_{i+d+1})}{(t_{i+d+1}-t_{i+1})}\mathcal{B}_{i+1}^{d-1}(t) + \frac{(t-t_i)}{(t_{i+d}-t_i)}\mathcal{B}_i^{d-1}(t) \\
&= \frac{(t-t_i)}{(t_{i+d}-t_i)}\mathcal{B}_i^{d-1}(t) + \frac{(t_{i+d+1}-t)}{(t_{i+d+1}-t_{i+1})}\mathcal{B}_{i+1}^{d-1}(t)
\end{aligned}$$

This establishes the Recursion formula for B-splines.

Examples:

- Let $\mathbf{t} = \{0,1,2,3,4,5\}$. Then we can write the degree 1 B-spline $\mathcal{B}_0^1(t)$ in terms of degree 0 B-splines according to the recursion formula: (using the first three knot values 0, 1, 2)

  $$\mathcal{B}_0^1(t) = \frac{t-0}{1-0}\mathcal{B}_0^0(t) + \frac{2-t}{2-1}\mathcal{B}_1^0(t) = t\mathcal{B}_0^0(t) + (2-t)\mathcal{B}_1^0(t).$$

  We can also obtain the piecewise form from this recursive form:

  $$\begin{aligned}
  \mathcal{B}_0^1(t) &= t\mathcal{B}_0^0(t) + (2-t)\mathcal{B}_1^0(t) \\
  &= t\begin{cases} 0, & t<0 \\ 1, & 0\le t<1 \\ 0, & t\ge 1\end{cases} + (2-t)\begin{cases} 0, & t<1 \\ 1, & 1\le t<2 \\ 0, & t\ge 2\end{cases}
  \end{aligned}$$

$$= t\begin{cases} 0, & t<0 \\ 1, & 0\le t<1 \\ 0, & 1\le t<2 \\ 0, & t\ge 2 \end{cases} + (2-t)\begin{cases} 0, & t<0 \\ 0, & 0\le t<1 \\ 1, & 1\le t<2 \\ 0, & t\ge 2 \end{cases}$$

$$= \begin{cases} 0, & t<0 \\ t, & 0\le t<1 \\ 0, & 1\le t<2 \\ 0, & t\ge 2 \end{cases} + \begin{cases} 0, & t<0 \\ 0, & 0\le t<1 \\ 2-t, & 1\le t<2 \\ 0, & t\ge 2 \end{cases}$$

$$= \begin{cases} 0, & t<0 \\ t, & 0\le t<1 \\ 2-t, & 1\le t<2 \\ 0, & t\ge 2 \end{cases}$$

- By a similar calculation, or by simply shifting $\mathcal{B}_0^1(t)$ to the right one unit, we can obtain:

$$\mathcal{B}_1^1(t) = \begin{cases} 0, & t<1 \\ t-1, & 1\le t<2 \\ 3-t, & 2\le t<3 \\ 0, & t\ge 3 \end{cases}$$

- With the same knot sequence $\mathbf{t} = \{0,1,2,3,4,5\}$, we can write the degree 2 B-spline $\mathcal{B}_0^2(t)$ in terms of degree 1 B-splines according to the recursion formula: (using the first four knot values 0, 1, 2, 3)

$$\mathcal{B}_0^2(t) = \frac{t-0}{3-1}\mathcal{B}_0^1(t) + \frac{3-t}{2-0}\mathcal{B}_1^1(t) = \frac{1}{2}t\mathcal{B}_0^1(t) + \frac{1}{2}(3-t)\mathcal{B}_1^1(t).$$

We can also obtain the piecewise form from this recursive form:

$$\begin{aligned}
\mathcal{B}_0^2(t) &= \frac{1}{2}t\mathcal{B}_0^1(t) + \frac{1}{2}(3-t)\mathcal{B}_1^1(t) \\
&= \frac{1}{2}t\begin{cases} 0, & t<0 \\ t, & 0\le t<1 \\ 2-t, & 1\le t<2 \\ 0, & t\ge 2 \end{cases} + \frac{1}{2}(3-t)\begin{cases} 0, & t<1 \\ t-1, & 1\le t<2 \\ 3-t, & 2\le t<3 \\ 0, & t\ge 3 \end{cases} \\
&= \frac{1}{2}t\begin{cases} 0, & t<0 \\ t, & 0\le t<1 \\ 2-t, & 1\le t<2 \\ 0, & 2\le t<3 \\ 0, & t\ge 3 \end{cases} + \frac{1}{2}(3-t)\begin{cases} 0, & t<0 \\ 0, & 0\le t<1 \\ t-1, & 1\le t<2 \\ 3-t, & 2\le t<3 \\ 0, & t\ge 3 \end{cases} \\
&= \begin{cases} 0, & t<0 \\ \frac{1}{2}t^2, & 0\le t<1 \\ \frac{1}{2}t(2-t), & 1\le t<2 \\ 0, & 2\le t<3 \\ 0, & t\ge 3 \end{cases} + \begin{cases} 0, & t<0 \\ 0, & 0\le t<1 \\ \frac{1}{2}(3-t)(t-1), & 1\le t<2 \\ \frac{1}{2}(3-t)^2, & 2\le t<3 \\ 0, & t\ge 3 \end{cases}
\end{aligned}$$

$$
= \begin{cases} 0, & t < 0 \\ \frac{1}{2}t^2, & 0 \le t < 1 \\ \frac{1}{2}t(2-t) + \frac{1}{2}(3-t)(t-1), & 1 \le t < 2 \\ \frac{1}{2}(3-t)^2, & 2 \le t < 3 \\ 0, & t \ge 3 \end{cases}
$$

$$
= \begin{cases} 0, & t < 0 \\ \frac{1}{2}t^2, & 0 \le t < 1 \\ -t^2 + 3t - \frac{3}{2}, & 1 \le t < 2 \\ \frac{1}{2}(3-t)^2, & 2 \le t < 3 \\ 0, & t \ge 3 \end{cases}
$$

Lecture 23, Th April 4, 2013

Main Points:

- Writing B-splines in terms of shifted power functions
- Orders of continuity for sums of shifted power functions
- Orders of continuity for B-splines

Writing a B-spline in terms of shifted power functions

To write a B-spline as a sum of shifted power functions, we use the definition of the divided difference. This says that, for instance, the divided difference

$$[t_i, \ldots, t_{i+d+1}]g$$

is equal to the coefficient of t^d in the interpolating polynomial

$$p(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_d t^d,$$

which matches the function g at the data values $t_i, \ldots, t_{i+d+1}$. This definition allows us to compute this divided difference by solving for a_d by any method. One method was to use the recursion formula for divided differences. Another method is Cramer's Rule.

Recall Cramer's Rule for a linear system with variables x_1, x_2, and x_3, and augmented matrix:

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & | & b_1 \\ a_{2,1} & a_{2,2} & a_{2,3} & | & b_2 \\ a_{3,1} & a_{3,2} & a_{3,3} & | & b_3 \end{pmatrix}$$

This system can also be written:

$$A\mathbf{x} = \mathbf{b}.$$

If $\det(A) \neq 0$ then the solution can be given by Cramer's Rule as:

$$x_1 = \frac{1}{\det(A)} \begin{vmatrix} b_1 & a_{1,2} & a_{1,3} \\ b_2 & a_{2,2} & a_{2,3} \\ b_3 & a_{3,2} & a_{3,3} \end{vmatrix}, \qquad x_2 = \frac{1}{\det(A)} \begin{vmatrix} a_{1,1} & b_1 & a_{1,3} \\ a_{2,1} & b_2 & a_{2,3} \\ a_{3,1} & b_3 & a_{3,3} \end{vmatrix}, \qquad x_3 = \frac{1}{\det(A)} \begin{vmatrix} a_{1,1} & a_{1,3} & b_3 \\ a_{2,1} & a_{2,3} & b_2 \\ a_{3,1} & a_{3,3} & b_3 \end{vmatrix}.$$

We can apply this to solve for the coefficients of an interpolating polynomial

$$p(x) = a_0 + a_1 x + a_2 x^2.$$

We have changed the variable to be x since our application to the definition of B-splines uses x as the dummy variable. If we suppose that the interpolating polynomial fits the data values $x = t_0$, $x = t_1$, and $x = t_2$ for a function $g(x)$, with $t_0 < t - 1 < t_2$, then the linear system, with variables a_0, a_1, and a_2 becomes:

$$\begin{pmatrix} 1 & t_0 & t_0^2 & | & g(t_0) \\ 1 & t_1 & t_1^2 & | & g(t_1) \\ 1 & t_2 & t_2^2 & | & g(t_2) \end{pmatrix}$$

If we call the determinant of the coefficient matrix

$$D = \begin{vmatrix} 1 & t_0 & t_0^2 \\ 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \end{vmatrix} = (t_1 - t_0)(t_2 - t_0)(t_2 - t_1),$$

then by Cramer's Rule, the solution for the variables is:

$$a_0 = \frac{1}{D}\begin{vmatrix} g(t_0) & t_0 & t_0^2 \\ g(t_1) & t_1 & t_1^2 \\ g(t_2) & t_2 & t_2^2 \end{vmatrix}, \qquad a_1 = \frac{1}{D}\begin{vmatrix} 1 & g(t_0) & t_0^2 \\ 1 & g(t_1) & t_1^2 \\ 1 & g(t_2) & t_2^2 \end{vmatrix}, \qquad a_2 = \frac{1}{D}\begin{vmatrix} 1 & t_0 & g(t_0) \\ 1 & t_1 & g(t_1) \\ 1 & t_1 & g(t_2) \end{vmatrix}.$$

So we see that the divided difference for degree $d = 2$ can be written:

$$[t_0, t_1, t_2]g = a_2 = \frac{1}{D}\begin{vmatrix} 1 & t_0 & g(t_0) \\ 1 & t_1 & g(t_1) \\ 1 & t_2 & g(t_2) \end{vmatrix}.$$

Finally, we can use this to write a degree $d = 1$ B-spline as a sum of shifted power functions, with $g(x) = (t - x)_+^1$:

$$\begin{aligned}
\mathcal{B}_0^1(t) &= (-1)^2(t_2 - t_0)[t_0, t_1, t_2](t - x)_+^1 \\
&= (t_2 - t_0)\frac{1}{D}\begin{vmatrix} 1 & t_0 & g(t_0) \\ 1 & t_1 & g(t_1) \\ 1 & t_2 & g(t_2) \end{vmatrix} \\
&= \frac{(t_2 - t_0)}{(t_1 - t_0)(t_2 - t_0)(t_2 - t_1)}\left[g(t_0)\begin{vmatrix} 1 & t_1 \\ 1 & t_2 \end{vmatrix} - g(t_1)\begin{vmatrix} 1 & t_0 \\ 1 & t_2 \end{vmatrix} + g(t_2)\begin{vmatrix} 1 & t_0 \\ 1 & t_1 \end{vmatrix}\right] \\
&= \frac{1}{(t_1 - t_0)(t_2 - t_1)}\left[(t_2 - t_1)g(t_0) - (t_2 - t_0)g(t_1) + (t_1 - t_0)g(t_2)\right] \\
&= \frac{1}{(t_1 - t_0)}g(t_0) - \frac{(t_2 - t_0)}{(t_1 - t_0)(t_2 - t_1)}g(t_1) + \frac{1}{(t_2 - t_1)}g(t_2) \\
&= \frac{1}{(t_1 - t_0)}(t - t_0)_+^1 - \frac{(t_2 - t_0)}{(t_1 - t_0)(t_2 - t_1)}(t - t_1)_+^1 + \frac{1}{(t_2 - t_1)}(t - t_2)_+^1
\end{aligned}$$

We can verify that this last expression has the shape of the hat function, by checking that the value at $t = t_0$ is zero, and at $t = t_1$ is 1, and at $t = t_2$ is zero. Also, it is clear that for $t < t_0$ the value is zero. To check that the function is also zero for $t > t_2$, we can use a property of determinants. Recall that a determinant is linear as a function of any single row or column. Since the functions lose their piecewise nature for $t > t_2$, the determinant becomes:

$$\begin{vmatrix} 1 & t_0 & t - t_0 \\ 1 & t_1 & t - t_1 \\ 1 & t_2 & t - t_2 \end{vmatrix} = \begin{vmatrix} 1 & t_0 & t \\ 1 & t_1 & t \\ 1 & t_2 & t \end{vmatrix} - \begin{vmatrix} 1 & t_0 & t_0 \\ 1 & t_1 & t_1 \\ 1 & t_2 & t_2 \end{vmatrix} = t\begin{vmatrix} 1 & t_0 & 1 \\ 1 & t_1 & 1 \\ 1 & t_2 & 1 \end{vmatrix} - 0 = 0 - 0 = 0.$$

Examples:

- Suppose that t_0, t_1, and t_2 are consecutive integers, such as $1, 2, 3$. Then the function $\mathcal{B}_0^1(t)$ can be written as:

$$\mathcal{B}_0^1(t) = (t - 1)_+^1 - 2(t - 2)_+^1 + (t - 3)_+^1.$$

Next, we do the same for a degree 2 B-spline $\mathcal{B}_i^2(t)$, with simple knot values $t_i = a$, $t_{i+1} = b$, $t_{i+2} = c$, $t_{i+3} = d$, and $a < b < c < d$.

As in the previous case, $[a, b, c, d]g$, with $g(x) = (t-x)_+^2$, can be calculated as the coefficient a_3 in the interpolating polynomial $p(x)$ matching g for the data values a, b, c, d, where $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$. By Cramer's Rule with D given by

$$D = \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} = (b-a)(c-a)(d-a)(c-b)(d-b)(d-c)$$

we have:

$$[a, b, c, d]g(x) = a_3 = \frac{1}{D} \begin{vmatrix} 1 & a & a^2 & g(a) \\ 1 & b & b^2 & g(b) \\ 1 & c & c^2 & g(c) \\ 1 & d & d^2 & g(d) \end{vmatrix}.$$

So if $t_i = a$, $t_{i+1} = b$, $t_{i+2} = c$, $t_{i+3} = d$, then we have

$$\begin{aligned}
\mathcal{B}_i^2(t) &= (-1)^{2+1}(d-a)[a, b, c, d](t-x)_+^2 \\
&= \frac{-(d-a)}{D} \begin{vmatrix} 1 & a & a^2 & g(a) \\ 1 & b & b^2 & g(b) \\ 1 & c & c^2 & g(c) \\ 1 & d & d^2 & g(d) \end{vmatrix} \\
&= \frac{-(d-a)}{D} \left[-g(a) \begin{vmatrix} 1 & b & b^2 \\ 1 & c & c^2 \\ 1 & d & d^2 \end{vmatrix} + g(b) \begin{vmatrix} 1 & a & a^2 \\ 1 & c & c^2 \\ 1 & d & d^2 \end{vmatrix} - g(c) \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & d & d^2 \end{vmatrix} + g(d) \begin{vmatrix} 1 & b & b^2 \\ 1 & c & c^2 \\ 1 & d & d^2 \end{vmatrix} \right] \\
&= (d-a) \left[\frac{g(a)}{D} \begin{vmatrix} 1 & b & b^2 \\ 1 & c & c^2 \\ 1 & d & d^2 \end{vmatrix} - \frac{g(b)}{D} \begin{vmatrix} 1 & a & a^2 \\ 1 & c & c^2 \\ 1 & d & d^2 \end{vmatrix} + \frac{g(c)}{D} \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & d & d^2 \end{vmatrix} - \frac{g(d)}{D} \begin{vmatrix} 1 & b & b^2 \\ 1 & c & c^2 \\ 1 & d & d^2 \end{vmatrix} \right] \\
&= (d-a) \left[\frac{g(a)}{(b-a)(c-a)(d-a)} - \frac{g(b)}{(b-a)(c-b)(d-b)} + \frac{g(c)}{(c-a)(c-b)(d-c)} - \frac{g(d)}{(d-a)(d-b)(d-c)} \right] \\
&= \frac{g(a)}{(b-a)(c-a)} - \frac{(d-a)g(b)}{(b-a)(c-b)(d-b)} + \frac{(d-a)g(c)}{(c-a)(c-b)(d-c)} - \frac{g(d)}{(d-b)(d-c)} \\
&= \frac{(t-a)_+^2}{(b-a)(c-a)} - \frac{(d-a)(t-b)_+^2}{(b-a)(c-b)(d-b)} + \frac{(d-a)(t-c)_+^2}{(c-a)(c-b)(d-c)} - \frac{(t-d)_+^2}{(d-b)(d-c)}
\end{aligned}$$

Examples:

- Let $a = 1$, $b = 2$, $c = 3$, $d = 4$. Then we have:

$$\begin{aligned}
\mathcal{B}_i^2(t) &= (-1)^{2+1}(4-1)[1, 2, 3, 4](t-x)_+^2 \\
&= \frac{1}{2}(t-1)_+^2 - \frac{3}{2}(t-2)_+^2 + \frac{3}{2}(t-3)_+^2 - \frac{1}{2}(t-4)_+^2.
\end{aligned}$$

Now consider the case of non-simple knot sequence. For instance, we could let $t_i = t_{i+1} = a$, $t_{i+2} = c$, and $t_{i+3} = d$. Then we can compute $[a, a, c, d]g$, for $g(x) = (t - x)_+^2$, again using Cramer's Rule with D given by

$$D = D(aacd) = \begin{vmatrix} 1 & a & a^2 & a^3 \\ 0 & 1 & 2a & 3a^2 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} = (c-a)^2(d-a)^2(d-c).$$

Then we have:

$$[a, a, c, d]g = a_3 = \frac{1}{D}\begin{vmatrix} 1 & a & a^2 & g(a) \\ 0 & 1 & 2a & g'(a) \\ 1 & c & c^2 & g(c) \\ 1 & d & d^2 & g(d) \end{vmatrix} = \frac{1}{D}\begin{vmatrix} 1 & a & a^2 & (t-a)_+^2 \\ 0 & 1 & 2a & 2(t-a)_+^1 \\ 1 & c & c^2 & (t-c)_+^2 \\ 1 & d & d^2 & (t-d)_+^2 \end{vmatrix}.$$

Applying this to the B-spline $\mathcal{B}_i^2(t)$, with knot sequence a, a, c, d, we get:

$$\begin{aligned}
\mathcal{B}_i^2(t) &= (-1)^{2+1}(d-a)[a, a, c, d](t-x)_+^2 \\
&= -\frac{d-a}{D}\begin{vmatrix} 1 & a & a^2 & g(a) \\ 0 & 1 & 2a & g'(a) \\ 1 & c & c^2 & g(c) \\ 1 & d & d^2 & g(d) \end{vmatrix} \\
&= \frac{d-a}{D}\left[g(a)\begin{vmatrix} 0 & 1 & 2a \\ 1 & c & c^2 \\ 1 & d & d^2 \end{vmatrix} - g'(a)\begin{vmatrix} 1 & a & a^2 \\ 1 & c & c^2 \\ 1 & d & d^2 \end{vmatrix} + g(c)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & 2a \\ 1 & d & d^2 \end{vmatrix} - g(d)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & 2a \\ 1 & c & c^2 \end{vmatrix}\right] \qquad (1) \\
&= \frac{d-a}{D}\left[g(a)\left[-(d^2-c^2) + 2a(d-c)\right] - g'(a)(c-a)(d-a)(d-c) + g(c)(d-a)^2 - g(d)(c-a)^2\right] \\
&= \frac{d-a}{D}\left[\left[-(d^2-c^2) + 2a(d-c)\right](t-a)_+^2 - (c-a)(d-a)(d-c)2(t-a)_+^1 + (d-a)^2(t-c)_+^2 - (c-a)^2(t-d)_+^2\right].
\end{aligned}$$

Note: In line (1) above, the determinants (which become the coefficients of the shifted power functions) can sometimes be recognized as Vandermonde or Confluent Vandermonde. In this line, the second one is Vandermonde, and the last two are Confluent Vandermonde, but the first one is neither of these types, so it is simply evaluated by the cofactor expansion formula for determinants.

Examples:

- Let $a = 1$, $c = 2$, $d = 3$. Then we have: $D = 4$, and we can write the B-spline:

$$\begin{aligned}
\mathcal{B}_i^2(t) &= (-1)^{2+1}(3-1)[1, 1, 2, 3](t-x)_+^2 \\
&= \frac{1}{2}\left[-3(t-1)_+^2 - 4(t-1)_+^1 + 4(t-2)_+^2 - (t-3)_+^2\right].
\end{aligned}$$

Lowest degree shifted power function in a B-spline has non-zero coefficient

The lowest degree shifted power function in a B-spline expansion, given by the above determinant formula coming from Cramer's Rule, must have non-zero coefficient. The reason for this is simply that the coefficient comes from a determinant which is either Vandermonde or Confluent Vandermonde. This can be seen by deleting the row corresponding to the lowest degree shifted power function, say $a(t-t_j)_+^k$, which must be the highest derivative of some shifted power function $(t-t_j)_+^n$. The remaining rows corresponding to t_j have all the lower order derivatives, and thus the determinant must be Confluent Vandermonde (or regular Vandermonde, which is of course a sub-case).

Orders of continuity for sums of shifted power functions

We have seen that the shifted power function $(t-c)_+^k$ is continuous to all orders at all points not equal to c, and can be seen to have exact order of continuity $k-1$ at $t=c$.

It easy to extend this fact to sums of such functions. In particular, if $f(t)$ is a sum of shifted power functions, then the order of continuity of f is simply the minimum of all orders of continuity of the summands. If the function $(t-c)_+^k$ is the one that achieves this minimum, then clearly that lowest order of continuity is $k-1$ and it occurs for the value $t=c$.

Examples:

- The function
$$f(t) = 7(t-4)_+^4 + 3(t-4)_+^6 - 5(t-7)_+^5$$
has exact order of continuity 3 which is attained by the summand $7(t-4)_+^4$ at the value $t=4$.
- The function
$$f(t) = 3(t-4)_+^3 + 2(t-4)_+^4 - 5(t-7)_+^5 + 6(t-8)_+^4 - (t-8)_+^7$$
has exact order of continuity 0 which is attained by the summand $2(t-3)_+^1$ at the value $t=3$.

Further Details on orders of continuity for sums of shifted power functions

Recall that a function $f(t)$ has order of continuity r at $t=c$ if f is continuous at $t=c$ and each of the derivative functions $f', f'', \ldots, f^{(r)}$ are continuous at $t=c$. If, in addition, the function $f^{(r+1)}$ is *not* continuous at $t=c$, then we say that f has *exact order of continuity* r at $t=c$. If f and all of its derivatives are continuous at $t=c$ then we say f has infinite order of continuity, or simply f is continuous to all orders at $t=c$.

If we let $f(t)=(t-c)_+^k$ then the derivatives of f are:

$$f'(t) = k(t-c)_+^{k-1}, f''(t) = k(k-1)(t-c)_+^{k-2}, \ldots, f^{(k-1)}(t) = k!(t-c)_+^1.$$

Note: The function $f(t)=(t-c)_+^1$ is not differentiable at $t=c$, although it is continuous there. The function $f(t)=(t-c)_+^0$ is neither continuous nor differentiable at $t=c$.

Let $f(t)$ be defined as a sum:

$$f(t) = a_1(t-u)_+^{j_1} + a_2(t-u)_+^{j_2} + \cdots + a_n(t-u)_+^{j_n},$$

with all $a_i \neq 0$, and $j_1 < j_2 < \cdots < j_n$. Then the exact order of continuity of f at u is simply $j_1 - 1$. This follows from the above, since the higher powers are differentiable to higher orders.

Now let $f(t)$ be defined as a sum:

$$f(t) = a_1(t-u_1)_+^{j_1} + a_2(t-u_2)_+^{j_2} + \cdots + a_n(t-u_n)_+^{j_n},$$

with all $a_i \neq 0$, and $u_1 < u_2 < \cdots < u_n$. Then the exact order of continuity of f at u_i is simply $j_i - 1$. It is then a fact that f is a member of any vector space of piecewise polynomial functions of the form:

$$f \in P_{d,\mathbf{r}}^{n+1}[u_0, u_1, \ldots, u_n, u_{n+1}],$$

where $u_0 < u_1$ and $u_{n+1} > u_n$, and d is the maximum of the j_i, $i = 1, \dots, n$, and $\mathbf{r}$ is the vector of continuity conditions:

$$\mathbf{r} = \{r_1, r_2, \dots, r_n\}, \quad \text{with} \quad r_i = j_i - 1.$$

Finally, suppose we mix the above two cases by adding higher degree shifted power functions at each u_i, and now define f as:

$$\begin{aligned}
f(t) &= a_{1,1}(t-u_1)_+^{j_{1,1}} + a_{1,2}(t-u_1)_+^{j_{1,2}} + \cdots + a_{1,n_1}(t-u_1)_+^{j_{1,n_1}} \\
&+ a_{2,1}(t-u_2)_+^{j_{2,1}} + a_{2,2}(t-u_2)_+^{j_{2,2}} + \cdots + a_{2,n_2}(t-u_2)_+^{j_{2,n_2}} \\
&\vdots \\
&+ a_{k,1}(t-u_k)_+^{j_{k,1}} + a_{k,2}(t-u_k)_+^{j_{k,2}} + \cdots + a_{k,n_k}(t-u_k)_+^{j_{k,n_k}}
\end{aligned}$$

Then if each row is written with increasing powers $j_{i,1} < j_{i,2} < \cdots < j_{i,n_i}$, we can choose a minimal value $j_{m,1}$ from the first column and we have the order of continuity of f is $j_{m,1} - 1$.

Proof of orders of continuity of a B-spline

We can prove the claim that the order of continuity of a B-spline $\mathcal{B}_i^d(t)$ at t_j is given by the multiplicity of t_j in the subsequence $t_i, \dots, t_{i+d+1}$.

This fact follows from two previous statements above. First, the coefficient of the lowest degree shifted power function of type $(t - t_j)_+^k$ in the expansion of B-spline $\mathcal{B}_i^d(t)$ must be nonzero, for t_j in the sequence $t_i, \dots, t_{i+d+1}$. Then, by the above, the order of continuity of $\mathcal{B}_i^d(t)$ at t_j must be $d - m$, where m is the multiplicity of t_j in the sequence $t_i, \dots, t_{i+d+1}$.

Lecture 24, T April 9, 2013

Main Points:

- Quiz 9
- Equivalence of DeBoor Algorithm and B-spline summation

Equivalence of DeBoor Algorithm and B-spline summation

Let $\gamma(t)$ be a degree d B-spline curve defined by:

$$\gamma(t) = \sum_{i=0}^{N-d-1} P_i \mathcal{B}_i^d(t)$$

for some knot sequence $\mathbf{t} = \{t_0, t_1, \ldots, t_N\}$ and control points $P_0, P_1, \ldots, P_{N-d-1}$, and t-values in the interval $[t_d, t_{N-d})$.

For the same knot sequence, control points and degree d, we can define, with the DeBoor algorithm, the point $P_J^{[d]}$ obtained through nested linear interpolation.

We will show that these two methods produce the same point:

$$\gamma(t) = \sum_{i=0}^{N-d-1} P_i \mathcal{B}_i^d(t) = P_J^{[d]}.$$

Some ingredients in the proof:

The B-spline recursion formula

The (DeBoor-Cox) recursion formula for B-splines of degree d associated to a knot sequence $\mathbf{t} = \{t_0, \ldots, t_N\}$ is:

$$\mathcal{B}_i^d(t) = \frac{t - t_i}{t_{i+d} - t_i} \mathcal{B}_i^{d-1}(t) + \frac{t_{i+d+1} - t}{t_{i+d+1} - t_{i+1}} \mathcal{B}_{i+1}^{d-1}(t)$$

The DeBoor Algorithm

The nested linear interpolation formula for B-splines is called the DeBoor Algorithm. For a knot sequence $\mathbf{t} = \{t_0, \ldots, t_N\}$, and control points $P_0, \ldots, P_{N-d-1}$, and $t \in [t_d, t_{N-d})$, we first define the index J to be the unique value such that $t \in [t_J, t_{J+1})$. Then the DeBoor points can be computed according to the nested linear interpolation scheme given by

$$P_i^{[k]} = \frac{t_{i+d-(k-1)} - t}{t_{i+d-(k-1)} - t_i} P_{i-1}^{[k-1]} + \frac{t - t_i}{t_{i+d-(k-1)} - t_i} P_i^{[k-1]},$$

where $k = 1, \ldots, d$ and $i = J - d + k, \ldots, J$ and the final point is $P_J^{[d]}$. Since this process produces a point for each value $t \in [t_d, t_{N-d})$, we can call it a curve $\gamma(t)$ and write

$$\gamma(t) = P_J^{[d]}.$$

Proof of the equivalence of the B-spline summation and the DeBoor Algorithm

Define $\gamma(t)$ as above according to the B-spline summation formula and also the DeBoor algorithm. Then we claim:

$$\gamma(t) = \sum_{i=0}^{N-d-1} P_i \mathcal{B}_i^d(t) = P_J^{[d]}.$$

Proof:

The first step is to note that some of the B-splines in the the above summation are zero for $t \in [t_J, t_{J+1})$. In particular, each B-spline $\mathcal{B}_i^d(t)$ has support equal to the interval (t_i, t_{i+d+1}) and thus $\mathcal{B}_i^d(t) = 0$ if $i + d + 1 \leq J$, or if $i \geq J + 1$. This leaves only the B-splines with indices $J - d, \ldots, J$.

Below, we make this adjustment to the summation indices, then insert the B-spline recursive form and adjust the index i in the second term in order to recombine terms with the same B-spline.

$$\begin{aligned}
\gamma(t) &= \sum_{i=0}^{N-d-1} P_i \mathcal{B}_i^d(t) \\
&= \sum_{i=J-d}^{J} P_i \mathcal{B}_i^d(t) \\
&= \sum_{i=J-d}^{J} P_i \left[\frac{t - t_i}{t_{i+d} - t_i} \mathcal{B}_i^{d-1}(t) + \frac{t_{i+d+1} - t}{t_{i+d+1} - t_{i+1}} \mathcal{B}_{i+1}^{d-1}(t) \right] \\
&= \sum_{i=J-d}^{J} P_i \frac{t - t_i}{t_{i+d} - t_i} \mathcal{B}_i^{d-1}(t) \quad + \quad \sum_{i=J-d}^{J} P_i \frac{t_{i+d+1} - t}{t_{i+d+1} - t_{i+1}} \mathcal{B}_{i+1}^{d-1}(t) \\
&= \sum_{i=J-d}^{J} P_i \frac{t - t_i}{t_{i+d} - t_i} \mathcal{B}_i^{d-1}(t) \quad + \quad \sum_{i=J-d+1}^{J+1} P_{i-1} \frac{t_{i+d} - t}{t_{i+d} - t_i} \mathcal{B}_i^{d-1}(t)
\end{aligned}$$

Now, since $t \in [t_J, t_{J+1})$, we have $\mathcal{B}_{J+1}^{d-1}(t) = 0$ and $\mathcal{B}_{J-d}^{d-1}(t) = 0$, which allows us to eliminate the first term in the first summation, and the last term in the second summation, in the above line. Continuing, we can now recognize the affine sum of points coming from the DeBoor algorithm, and we have:

$$\begin{aligned}
\gamma(t) &= \sum_{i=J-d+1}^{J} P_i \frac{t - t_i}{t_{i+d} - t_i} \mathcal{B}_i^{d-1}(t) \quad + \quad \sum_{i=J-d+1}^{J} P_{i-1} \frac{t_{i+d} - t}{t_{i+d} - t_i} \mathcal{B}_i^{d-1}(t) \\
&= \sum_{i=J-d+1}^{J} \left[P_i \frac{t - t_i}{t_{i+d} - t_i} \ + \ P_{i-1} \frac{t_{i+d} - t}{t_{i+d} - t_i} \right] \mathcal{B}_i^{d-1}(t) \\
&= \sum_{i=J-d+1}^{J} \left[\frac{t - t_i}{t_{i+d} - t_i} P_i^{[0]} \ + \ \frac{t_{i+d} - t}{t_{i+d} - t_i} P_{i-1}^{[0]} \right] \mathcal{B}_i^{d-1}(t) \\
&= \sum_{i=J-d+1}^{J} P_i^{[1]} \mathcal{B}_i^{d-1}(t).
\end{aligned}$$

We can now apply the entire above process to this summation recursively to get:

$$
\begin{aligned}
\gamma(t) &= \sum_{i=J-d}^{J} P_i^{[0]} \mathcal{B}_i^d(t) \\
&= \sum_{i=J-d+1}^{J} P_i^{[1]} \mathcal{B}_i^{d-1}(t) \\
&= \sum_{i=J-d+2}^{J} P_i^{[2]} \mathcal{B}_i^{d-2}(t) \\
&\vdots \\
&= \sum_{i=J}^{J} P_J^{[d]} \quad \mathcal{B}_d^0(t) \\
&= P_J^{[d]}.
\end{aligned}
$$

Lecture 25, Th April 11, 2013

Main Points:

- Proof of Curry Schoenberg Theorem

Writing B-splines as sums of shifted power functions

We showed that it is possible to write a B-spline function $\mathcal{B}_i^d(t)$ in terms of shifted power functions, specifically those which are indicated by the subsequence $t_i, \ldots, t_{i+d+1}$ which is used to define the B-spline.

For instance, in the case where the knots $t_i, \ldots, t_{i+d+1}$ are all simple (of multiplicity one), we can write $\mathcal{B}_i^d(t)$ as:

$$
\begin{aligned}
\mathcal{B}_i^d(t) &= (-1)^{d+1}(t_{i+d+1} - t_i)[t_i, \ldots, t_{i+d+1}](t-x)_+^d \\
&= (-1)^{d+1}(t_{i+d+1} - t_i)\frac{1}{D}\begin{vmatrix} 1 & t_i & t_i^2 & \cdots & t_i^{d-1} & (t-t_i)_+^d \\ 1 & t_{i+1} & t_{i+1}^2 & \cdots & t_{i+1}^{d-1} & (t-t_{i+1})_+^d \\ \vdots & \vdots & \cdots & \vdots & \vdots & \\ 1 & t_{i+d+1} & t_{i+d+1}^2 & \cdots & t_{i+d+1}^{d-1} & (t-t_{i+d+1})_+^d \end{vmatrix}
\end{aligned}
$$

where D is the Vandermonde determinant:

$$
D = \begin{vmatrix} 1 & t_i & t_i^2 & \cdots & t_i^{d-1} & t_i^d \\ 1 & t_{i+1} & t_{i+1}^2 & \cdots & t_{i+1}^{d-1} & t_{i+1}^d \\ \vdots & \vdots & \cdots & \vdots & \vdots & \\ 1 & t_{i+d+1} & t_{i+d+1}^2 & \cdots & t_{i+d+1}^{d-1} & t_{i+d+1}^d \end{vmatrix}.
$$

By performing a cofactor expansion of the previous determinant along the last column, we can write the B-spline as a sum:

$$
\mathcal{B}_i^d(t) = a_i(t-t_i)_+^d + a_{i+1}(t-t_{i+1})_+^d + \cdots + a_{i+d+1}(t-t_{i+d+1})_+^d.
$$

To allow for multiplicities higher than one, we could have for instance knots $t_i = t_{i+1} < t_{i+2} < \cdots < t_{i+d+1}$. In this case we have:

$$
\mathcal{B}_i^d(t) = (-1)^{d+1}(t_{i+d+1} - t_i)[t_i, \ldots, t_{i+d+1}](t-x)_+^d
$$

$$
= \quad (-1)^{d+1}(t_{i+d+1}-t_i)\frac{1}{D}\begin{vmatrix}
1 & t_i & t_i^2 & \cdots & t_i^{d-1} & (t-t_i)_+^d \\
0 & 1 & 2t_i & \cdots & (d-1)t_{i+1}^{d-2} & -d(t-t_{i+1})_+^{d-1} \\
1 & t_{i+2} & t_{i+2}^2 & \cdots & t_{i+2}^{d-1} & (t-t_{i+2})_+^d \\
\vdots & \vdots & \cdots & \vdots & \vdots & \\
1 & t_{i+d+1} & t_{i+d+1}^2 & \cdots & t_{i+d+1}^{d-1} & (t-t_{i+d+1})_+^d
\end{vmatrix}
$$

where D is now the Confluent Vandermonde determinant:

$$
D = \begin{vmatrix}
1 & t_i & t_i^2 & \cdots & t_i^{d-1} & t_i^d \\
0 & 1 & 2t_i & \cdots & (d-1)t_{i+1}^{d-2} & dt_{i+1}^{d-1} \\
1 & t_{i+2} & t_{i+2}^2 & \cdots & t_{i+2}^{d-1} & t_{i+2}^d \\
\vdots & \vdots & \cdots & \vdots & \vdots & \\
1 & t_{i+d+1} & t_{i+d+1}^2 & \cdots & t_{i+d+1}^{d-1} & t_{i+d+1}^d
\end{vmatrix}.
$$

Again by performing a cofactor expansion of the previous determinant along the last column, we can write the B-spline as a sum:

$$
\mathcal{B}_i^d(t) = a_i(t-t_i)_+^d + a_{i+1}(t-t_i)_+^{d-1} + a_{i+2}(t-t_{i+2})_+^d + \cdots + a_{i+d+1}(t-t_{i+d+1})_+^d.
$$

Note that in this case the sum involves one shifted power function of degree $d-1$, corresponding to the second row in the determinant which gives the B-spline coefficients.

Writing a shifted power basis with increasing degree order

It is an important (but trivial) point, to note that we can write a shifted power function basis with increasing degrees. Of course, we can write a basis in any order since it is a set of functions and is independent of order. However, the order can sometimes be a crucial point, especially when the goal is to find a simple change of basis matrix. This is precisely why we choose to write the shifted power basis with increasing degree order. We will see that the matrix of coordinate vectors of the B-splines, with respect to the shifted power basis, is then lower triangular with nonzero diagonal entries, and thus has nonzero determinant. This matrix then gives the change of basis from B-splines to shifted power basis.

Examples:

- Let $V = P^3_{2,\mathbf{r}}[0,1,2,3,4]$, with $\mathbf{r} = (1,0,1)$. The multiplicity sequence is then $\mathbf{m} = (1,2,1)$. Then a shifted power basis can be written in terms of the knot sequence: $\mathbf{t} = \{0,0,0,1,2,2,3\}$. The basis, in degree-increasing order, is then:
$$\{(t-0)_+^0, (t-0)_+^1, (t-0)_+^2, (t-1)_+^2, (t-2)_+^1, (t-2)_+^2, (t-3)_+^2\}.$$

- Let $V = P^4_{3,\mathbf{r}}[0,1,2,3,4]$, with $\mathbf{r} = (2,2,1)$. The multiplicity sequence is then $\mathbf{m} = (1,1,2)$. Then a shifted power basis can be written in terms of the knot sequence: $\mathbf{t} = \{0,0,0,0,1,2,3,3,4,4,4,4\}$. The basis, in degree-increasing order, is then:
$$\{(t-0)_+^0, (t-0)_+^1, (t-0)_+^2, (t-0)_+^3, (t-1)_+^3, (t-2)_+^3, (t-3)_+^2, (t-3)_+^3\}.$$

Proof of Curry Schoenberg Theorem (B-spline basis theorem)

As indicated in the above paragraph, it is enough to show that the matrix of coordinate vectors of B-splines has nonzero determinant.

Examples:

- Let $V = P^3_{2,\mathbf{r}}[0,1,2,3,4]$, with $\mathbf{r} = (1,0,1)$. The shifted power basis was written above. We can also write a knot sequence whose associated B-splines are a basis of V. Such a knot sequence is: $\mathbf{t}' = \{0,0,0,1,2,2,3,4,4,4\}$. The B-splines of degree 2 associated to $\mathbf{t}'$ are:

$$\mathcal{B}_2(\mathbf{t}') = \{\mathcal{B}^2_0(t), \mathcal{B}^2_1(t), \mathcal{B}^2_2(t), \mathcal{B}^2_3(t), \mathcal{B}^2_4(t), \mathcal{B}^2_5(t), \mathcal{B}^2_6(t)\}.$$

Each of these B-splines can be written in terms of the shifted power basis as follows. First we compute the relevant confluent Vandermonde determinants:

$$D_0 = D(0,0,0,1) = \begin{vmatrix} 1 & 0 & 0^2 & 0^3 \\ 0 & 1 & 2\cdot 0 & 3\cdot 0^2 \\ 0 & 0 & 2 & 6\cdot 0 \\ 1 & 1 & 1^2 & 1^3 \end{vmatrix} = 2, \qquad D_1 = D(0,0,1,2) = \begin{vmatrix} 1 & 0 & 0^2 & 0^3 \\ 0 & 1 & 2\cdot 0 & 3\cdot 0^2 \\ 1 & 1 & 1^2 & 1^3 \\ 1 & 2 & 2^2 & 2^3 \end{vmatrix} = 4,$$

$$D_2 = D(0,1,2,2) = \begin{vmatrix} 1 & 0 & 0^2 & 0^3 \\ 1 & 1 & 1^2 & 1^3 \\ 1 & 2 & 2^2 & 2^3 \\ 0 & 1 & 2\cdot 2 & 3\cdot 2^2 \end{vmatrix} = 4, \qquad D_3 = D(1,2,2,3) = \begin{vmatrix} 1 & 1 & 1^2 & 1^3 \\ 1 & 2 & 2^2 & 2^3 \\ 0 & 1 & 2\cdot 2 & 3\cdot 2^2 \\ 1 & 3 & 3^2 & 3^3 \end{vmatrix} = 2,$$

$$D_4 = D(2,2,3,4) = \begin{vmatrix} 1 & 2 & 2^2 & 2^3 \\ 0 & 1 & 2\cdot 2 & 3\cdot 2^2 \\ 1 & 3 & 3^2 & 3^3 \\ 1 & 4 & 4^2 & 4^3 \end{vmatrix} = 4, \qquad D_5 = D(2,3,4,4) = \begin{vmatrix} 1 & 2 & 2^2 & 2^3 \\ 1 & 3 & 3^2 & 3^3 \\ 1 & 4 & 4^2 & 4^3 \\ 0 & 1 & 2\cdot 4 & 3\cdot 4^2 \end{vmatrix} = 4,$$

$$D_6 = D(3,4,4,4) = \begin{vmatrix} 1 & 3 & 3^2 & 3^3 \\ 1 & 4 & 4^2 & 4^3 \\ 0 & 1 & 2\cdot 4 & 3\cdot 4^2 \\ 0 & 0 & 2 & 6\cdot 4 \end{vmatrix} = 2.$$

Next, we compute the B-splines as sums of shifted power functions. Since each of these can be easily done in a symbolic algebra package, such as PARI, we include the PARI commands which show the input and output. The determinant always has last column with symbols `a,b,c,d` instead of the shifted power functions, for ease of input. The coefficients are then assigned to the correct functions in the output. We also give the coordinate vector of the B-spline with respect to the degree-increasing ordered basis:

$$\{(t-0)^0_+, (t-0)^1_+, (t-0)^2_+, (t-1)^2_+, (t-2)^1_+, (t-2)^2_+, (t-3)^2_+\}.$$

$$\begin{aligned} \mathcal{B}^2_0(t) &= (-1)^{2+1}(1-0)[0,0,0,1](t-x)^2_+ \\ &= \frac{-1}{D_0} \begin{vmatrix} 1 & 0 & 0^2 & (t-0)^2_+ \\ 0 & 1 & 2\cdot 0 & -2(t-0)^1_+ \\ 0 & 0 & 2 & 2(t-0)^0_+ \\ 1 & 1 & 1^2 & (t-1)^2_+ \end{vmatrix} \\ &= (t-0)^2_+ - 2(t-0)^1_+ + (t-0)^0_+ - (t-1)^1_+. \end{aligned}$$

PARI input: $(-1/2)*\mathtt{matdet}([1,0,0,\mathtt{a};0,1,0,-2*\mathtt{b};0,0,2,2*\mathtt{c};1,1,1,\mathtt{d}])$

PARI output: $\mathtt{a}-2*\mathtt{b}+\mathtt{c}-\mathtt{d}$

Coordinate vector: $(1,-2,1,-1,0,0,0)$

$$\begin{aligned}
\mathcal{B}_1^2(t) &= (-1)^{2+1}(1-0)[0,0,1,2](t-x)_+^2 \\
&= \frac{-1}{D_1}\begin{vmatrix} 1 & 0 & 0^2 & (t-0)_+^2 \\ 0 & 1 & 2\cdot 0 & -2(t-0)_+^1 \\ 1 & 1 & 1^2 & (t-1)_+^2 \\ 1 & 2 & 2^2 & (t-2)_+^2 \end{vmatrix} \\
&= -\frac{3}{4}(t-0)_+^2+(t-0)_+^1+(t-1)_+^2-\frac{1}{4}(t-2)_+^1.
\end{aligned}$$

PARI input: $(-1/4)*\mathtt{matdet}([1,0,0,\mathtt{a};0,1,0,-2*\mathtt{b};1,1,1,\mathtt{c};1,2,4,\mathtt{d}])$

PARI output: $-3/4*\mathtt{a}+\mathtt{b}+\mathtt{c}-1/4*\mathtt{d}$

Coordinate vector: $(0,1,-\frac{3}{4},1,-\frac{1}{4},0,0)$

$$\begin{aligned}
\mathcal{B}_2^2(t) &= (-1)^{2+1}(1-0)[0,1,2,2](t-x)_+^2 \\
&= \frac{-1}{D_2}\begin{vmatrix} 1 & 0 & 0^2 & (t-0)_+^2 \\ 1 & 1 & 1^2 & (t-1)_+^2 \\ 1 & 2 & 2^2 & (t-2)_+^2 \\ 0 & 1 & 2\cdot 2 & -2(t-2)_+^1 \end{vmatrix} \\
&= \frac{1}{4}(t-0)_+^2-(t-1)_+^2+\frac{3}{4}(t-2)_+^2+(t-2)_+^1.
\end{aligned}$$

PARI input: $(-1/4)*\mathtt{matdet}([1,0,0,\mathtt{a};1,1,1,\mathtt{b};1,2,4,\mathtt{c};0,1,4,-2*\mathtt{d}])$

PARI output: $1/4*\mathtt{a}-\mathtt{b}+3/4*\mathtt{c}+\mathtt{d}$

Coordinate vector: $(0,0,\frac{1}{4},-1,1,\frac{3}{4},0)$

$$\begin{aligned}
\mathcal{B}_3^2(t) &= (-1)^{2+1}(1-0)[1,2,2,3](t-x)_+^2 \\
&= \frac{-1}{D_3}\begin{vmatrix} 1 & 1 & 1^2 & (t-1)_+^2 \\ 1 & 2 & 2^2 & (t-2)_+^2 \\ 0 & 1 & 2\cdot 2 & -2(t-2)_+^1 \\ 1 & 3 & 3^2 & (t-3)_+^2 \end{vmatrix} \\
&= \frac{1}{2}(t-1)_+^2+0\cdot(t-2)_+^2-2(t-2)_+^1-\frac{1}{2}(t-3)_+^2.
\end{aligned}$$

PARI input: $(-1/2)*\mathtt{matdet}([1,1,1,\mathtt{a};1,2,4,\mathtt{b};0,1,4,-2*\mathtt{c};1,3,9,\mathtt{d}])$

PARI output: $1/2*\mathtt{a}-2*\mathtt{c}-1/2*\mathtt{d}$

Coordinate vector: $(0,0,0,\frac{1}{2},-2,0,-1)$

$$\begin{aligned}
\mathcal{B}_4^2(t) &= (-1)^{2+1}(1-0)[2,2,3,4](t-x)_+^2 \\
&= \frac{-1}{D_3}\begin{vmatrix} 1 & 2 & 2^2 & (t-2)_+^2 \\ 0 & 1 & 2\cdot 2 & -2(t-2)_+^1 \\ 1 & 3 & 3^2 & (t-3)_+^2 \\ 1 & 4 & 4^2 & (t-4)_+^2 \end{vmatrix} \\
&= -\frac{5}{4}(t-2)_+^2 + (t-2)_+^1 + 2(t-3)_+^2 - \frac{3}{4}(t-4)_+^2.
\end{aligned}$$

`PARI input:` $(-1/4)*\mathtt{matdet}([1,2,4,a;0,1,2,-2*b;1,3,9,c;1,4,16,d])$

`PARI output:` $-5/4*a+b+2*c-3/4*d$

Coordinate vector: $(0,0,0,0,1,-\frac{5}{4},2)$

$$\begin{aligned}
\mathcal{B}_5^2(t) &= (-1)^{2+1}(1-0)[2,3,4,4](t-x)_+^2 \\
&= \frac{-1}{D_5}\begin{vmatrix} 1 & 2 & 2^2 & (t-2)_+^2 \\ 1 & 3 & 3^2 & (t-3)_+^2 \\ 1 & 4 & 4^2 & (t-4)_+^2 \\ 0 & 1 & 2\cdot 4 & -2(t-4)_+^1 \end{vmatrix} \\
&= \frac{1}{4}(t-2)_+^2 - (t-3)_+^2 + \frac{3}{4}(t-4)_+^2 + (t-4)_+^1.
\end{aligned}$$

`PARI input:` $(-1/4)*\mathtt{matdet}([1,2,4,a;1,3,9,b;1,4,16,c;0,1,8,-2*d])$

`PARI output:` $1/4*a-b+3/4*c+d$

Coordinate vector: $(0,0,0,0,0,\frac{1}{4},-1)$

$$\begin{aligned}
\mathcal{B}_6^2(t) &= (-1)^{2+1}(1-0)[3,4,4,4](t-x)_+^2 \\
&= \frac{-1}{D_6}\begin{vmatrix} 1 & 3 & 3^2 & (t-3)_+^2 \\ 1 & 4 & 4^2 & (t-4)_+^2 \\ 0 & 1 & 2\cdot 4 & -2(t-4)_+^1 \\ 0 & 0 & 2 & 2(t-4)_+^0 \end{vmatrix} \\
&= (t-3)_+^2 - (t-4)_+^2 - 2(t-4)_+^1 - (t-4)_+^0.
\end{aligned}$$

`PARI input:` $(-1/2)*\mathtt{matdet}([1,3,9,a;1,4,16,b;0,1,8,-2*c;0,0,2,2*d])$

`PARI output:` $a-b-2*c-d$

Coordinate vector: $(0,0,0,0,0,0,1)$

Finally, we can write the matrix of coordinate vectors and see that it indeed is lower triangular, with nonzero diagonal entries, hence has nonzero determinant. This matrix, called A below, is then the change of basis matrix from the B-spline to the shifted power basis, with respect to the chosen knot sequences $\mathbf{t}$ and $\mathbf{t}'$.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -\frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{4} & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & \frac{3}{4} & 0 & -\frac{5}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 1 \end{pmatrix}$$

The inverse of A is then the change of basis matrix from the shifted power basis, with knot sequence $\mathbf{t}$, to the B-spline basis, with knot sequence $\mathbf{t}'$.

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 3 & 4 & 0 & 0 & 0 & 0 \\ 2 & 4 & 8 & 2 & 0 & 0 & 0 \\ \frac{5}{2} & \frac{21}{4} & \frac{1}{2} & 4 & 1 & 0 & 0 \\ \frac{13}{2} & \frac{69}{4} & 48 & 20 & 5 & 4 & 0 \\ \frac{7}{2} & \frac{43}{4} & 32 & 14 & 3 & 4 & 1 \end{pmatrix}$$

- Let $V = P^5_{3,\mathbf{r}}[0,1,2,3,4,5]$, with $\mathbf{r} = (0,1,2,1)$. The associated multiplicity sequence is then: $\mathbf{m} = (3,2,1,2)$. A shifted power basis can be written with knot sequence: $\mathbf{t} = \{0,0,0,0,1,1,1,2,2,3,4,4\}$. The shifted power basis $sp_3(\mathbf{t})$, written in degree-increasing order, is then:

$$\{(t-0)^0_+, (t-0)^1_+, (t-0)^2_+, (t-0)^3_+, (t-1)^1_+, (t-1)^2_+, (t-1)^3_+, (t-2)^2_+,$$
$$(t-2)^3_+, (t-3)^3_+, (t-4)^2_+, (t-4)^3_+\}.$$

We can also write a knot sequence whose associated B-splines are a basis of V. Such a knot sequence is: $\mathbf{t}' = \{0,0,0,0,1,1,1,2,2,3,4,4,5,5,5,5\}$. The B-splines of degree 3 associated to $\mathbf{t}'$ are:

$$\mathcal{B}_3(\mathbf{t}') = \{\mathcal{B}^3_0(t), \mathcal{B}^3_1(t), \ldots, \mathcal{B}^3_{10}(t), \mathcal{B}^3_{11}(t)\}.$$

Without working out the numerical values of the shifted power basis coordinate vectors for each B-spline, we can still see the general shape of the change of basis matrix, called B below, and verify that indeed it has nonzero determinant. This follows from the fact that for each B-spline, the coefficient of the first shifted power function is always nonzero. This is due to the fact that the coefficient comes from a confluent Vandermonde determinant.

In the change of basis matrix B below, the stars represent nonzero values which are computed from confluent Vandermonde determinants, and the dots represent the other coefficients which contribute to the B-spline and which may or may not be zero. This illustrates the proof of the Curry-Schoenberg Theorem, which is simply the claim that the matrix has nonzero determinant.

The vector to the left shows the shifted power basis which corresponds to the columns of the matrix.

$$
\mathbf{v} = \begin{pmatrix} (t-0)_+^0 \\ (t-0)_+^1 \\ (t-0)_+^2 \\ (t-0)_+^3 \\ (t-1)_+^1 \\ (t-1)_+^2 \\ (t-1)_+^3 \\ (t-2)_+^2 \\ (t-2)_+^3 \\ (t-3)_+^3 \\ (t-4)_+^2 \\ (t-4)_+^3 \end{pmatrix} \qquad
B = \begin{pmatrix}
* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
. & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
. & . & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
. & . & . & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & 0 & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & * & . & . & . & * & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & . & . & . & . & . & * & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & * & * & * & * & * & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & . & . & . & . & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & * & * & * & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & . & . & . & . & *
\end{pmatrix}
$$

Lecture 26, T April 16, 2013

Main Points:

- Interpolation with B-splines
- Schoenberg-Whitney Theorem
- Polar forms for splines

General Interpolation with a vector space of functions

If we have a vector space V of functions, with dimension n, then we can ask questions about interpolation using functions in V.

For instance, given data values $s_1, \ldots, s_n$ and some data function $g(t)$, can we find a function $f(t) \in V$ such that:

$$f(s_i) = g(s_i), \qquad i = 1, \ldots, n?$$

Note: This question is independent of the basis chosen for V. This means that if the interpolation problem has a solution, then we can write the function $f(t)$ in any basis that we like.

We know that in the case that $V = P_d$, and thus $n = d+1$, we have the existence and uniqueness of the interpolating polynomial, which says that this interpolation problem always has a unique solution.

We will see that this property does *not* always hold in other vector spaces of functions, in particular for splines.

Interpolation with splines

If we have a vector space of polynomial spline functions such as

$$V = P^k_{d,\mathbf{r}}[u_0, \ldots, u_k]$$

with $\mathbf{r} = (r_1, \ldots, r_k)$, and dimension

$$n = d + 1 \sum_{i=1}^{k-1} d - r_i$$

then we can ask questions about interpolation using functions in V.

For instance, given data values $s_1, \ldots, s_n$ and some data function $g(t)$, can we find a function $f(t) \in V$ such that:

$$f(s_i) = g(s_i), \qquad i = 1, \ldots, n?$$

The problem that comes up immediately is that since spline functions are 'locally polynomial', we cannot put 'too many' of the data values 'too close' together.

More precisely, if we put $d+2$ or more data values in the interval $[u_i, u_{i+1}]$, then we cannot expect to find a solution $f(t)$. This is because on the interval $[u_i, u_{i+1}]$ the function $f(t)$ is given by a polynomial $p(t)$ in P_d which must have degree at most d. But we cannot match such a polynomial to g at more than $d+1$ points, except if we have the coincidence that all the points lie on a polynomial of degree at most $d+1$ already.

So, a necessary condition to have a solution f to the above interpolation problem is that: Not more than $d+1$ of the data values lie in any interval of the form $[u_i, u_{i+1}]$.

Examples:

- Let $V = P^4_{3,1}[0,1,2,3,4]$ with dimension 10, and choose the data sequence:

$$\begin{array}{ll} s_1 = 0.0 & s_6 = 1.5 \\ s_2 = 0.5 & s_7 = 2.5 \\ s_3 = 0.6 & s_8 = 3.5 \\ s_4 = 0.7 & s_9 = 3.7 \\ s_5 = 0.8 & s_{10} = 3.8 \end{array}$$

  Since there are five data values in the interval $[0,1]$, and the dimension of the space of cubic polynomials is only four, we know that we cannot choose an arbitrary data function $g(t)$ and expect to find a unique solution to the interpolation problem in V. In other words, given $g(t)$, we cannot expect to find a solution $f(t)$ in V such that $f(s_i) = g(s_i)$, for $i = 1, \ldots, 10$. Of course, it is always possible that there could exist a solution simply by chance, or by design, for instance if we chose g to be in V. But in general the interpolation problem will not have a solution at all.

- Let $V = P^4_{3,1}[0,1,2,3,4]$ with dimension 10, and choose the data sequence:

$$\begin{array}{ll} s_1 = 0.0 & s_6 = 1.5 \\ s_2 = 0.5 & s_7 = 2.5 \\ s_3 = 0.6 & s_8 = 3.5 \\ s_4 = 0.7 & s_9 = 3.7 \\ s_5 = 1.2 & s_{10} = 3.8 \end{array}$$

  Now we see that there are at most four data values in each of the sub-intervals, so at least there is a chance that when we choose an arbitrary data function $g(t)$ we might be able to find a solution to the interpolation problem in V. In other words, given $g(t)$, we might be able to find a solution $f(t)$ in V such that $f(s_i) = g(s_i)$, for $i = 1, \ldots, 10$.

The Schoenberg-Whitney Theorem gives a precise criterion for the existence and uniqueness of a solution to the interpolation problem for spline vector spaces.

Schoenberg-Whitney Theorem

Let $\mathbf{t} = \{t_0, \ldots, t_N\}$ be a knot sequence such that the B-splines of degree d associated to $\mathbf{t}$ are a basis of $V = P^k_{d,\mathbf{r}}[u_0, \ldots, u_k]$, with $dim(V) = N - d = n$. Now suppose we have a sequence of data values $u_0 \leq s_0 < s_1 < \cdots < s_{n-1} \leq u_k$, and a data function $g(t)$. Then there exists a unique function $f(t) \in V$, satisfying $f(s_i) = g(s_i)$, for $i = 0, \ldots, n-1$ if and only if:

$$\mathcal{B}^d_i(s_i) > 0, \quad i = 0, \ldots, n-1.$$

Note: This last condition is equivalent to $t_i < s_i < t_{i+d+1}$ for the case of continuous B-splines, and $t_i \leq s_i < t_{i+d+1}$ for a discontinuous B-spline. So, we can only have $s_i = t_i$ when the knot t_i has multiplicity $m = d+1$ for the B-spline $\mathcal{B}^d_i(t)$.

Examples:

- Let $V = P^4_{3,1}[0,1,2,3,4]$ with dimension 10, and choose the data sequence:

$$\begin{array}{ll} s_1 = 0.0 & s_6 = 1.5 \\ s_2 = 0.5 & s_7 = 2.5 \\ s_3 = 0.6 & s_8 = 3.5 \\ s_4 = 0.7 & s_9 = 3.7 \\ s_5 = 1.2 & s_{10} = 3.8 \end{array}$$

  Now let $\mathbf{t} = \{0,0,0,0,1,1,2,2,3,3,4,4,4,4\}$ be a knot sequence such that the B-splines of degree 3 for $\mathbf{t}$ are a basis of V. Then we can check the inequalities in the Schoenberg-Whitney Theorem:

$0 < 0.0 < 1$	$1 < 1.5 < 3$
$0 < 0.5 < 1$	$2 < 2.5 < 4$
$0 < 0.6 < 2$	$2 < 3.5 < 4$
$0 < 0.7 < 2$	$3 < 3.7 < 4$
$1 < 1.2 < 3$	$3 < 3.8 < 4$

Thus we can conclude that given any data function $g(t)$, there exists a unique solution $f(t)$ in V such that $f(s_i) = g(s_i)$, for $i = 1, \ldots, 10$.

Other Properties of B-splines

- **Partition of Unity**

  Let $\mathbf{t} = \{t_0, \ldots, t_N\}$ be a knot sequence such that the B-splines of degree d associated to $\mathbf{t}$ are a basis of $V = P^k_{d,\mathbf{r}}[u_0, \ldots, u_k]$. Then for $t \in [u_0, u_k]$ we have:

$$\sum_{i=0}^{N-d-1} \mathcal{B}_i^d(t) = 1.$$

- **Bernstein knot sequence**

  Let $\mathbf{t} = \{0, 0, \ldots, 0, 1, 1, \ldots, 1\} = \{t_0, \ldots, t_{2(d+1)}\}$ be a knot sequence consisting of $d+1$ zeros followed by $d+1$ ones. Then the B-splines associated to this knot sequence $\mathbf{t}$, restricted to the interval $[0, 1]$, are precisely the Bernstein polynomials.

- **Derivative of a B-spline function**

  The following derivative is similar to the derivative of the Bernstein polynomials:

$$\frac{d}{dt}\mathcal{B}_i^d(t) = d\left[\frac{\mathcal{B}_i^{d-1}(t)}{t_{i+d} - t_i} - \frac{\mathcal{B}_{i+1}^{d-1}(t)}{t_{i+d+1} - t_{i+1}}\right].$$

Polar forms for splines

We can write a polar form for each polynomial piece of a spline function. Together, all of these polar forms are related to the nested linear interpolation properties, and hence to the Bernstein basis coefficients, or control points, of the polynomials. It turns out that we can also relate these polar forms to the B-spline coefficients, or DeBoor control points, of the spline function. This can also be used to form a change of basis matrix from the B-spline basis to a shifted power basis.

In particular, suppose that a spline function $f(t)$ in $V = P^k_{d,\mathbf{r}}[u_0, \ldots, u_k]$ has polar forms assigned to each interval, so that if $f(t) = p_i(t)$ on the interval $[u_{i-1}, u_i)$, then $p_i(t)$ has polar form $F_i[x_1, \ldots, x_d]$. Suppose also that $\mathbf{t} = \{t_0, \ldots, t_N\}$ is a knot sequence such that the B-splines of degree d associated to $\mathbf{t}$ are a basis of V. Then the B-spline coefficients of $f(t)$ can be obtained from the polar forms F_i for any $t \in [t_d, t_{N-d})$. In particular, suppose that $t \in [t_J, t_{J+1})$. Then the knot t_J corresponds to some break point u_{i-1}, with $t_J = u_{i-1} < u_i = t_{J+1}$. We can then use the polar form $F_i[x_1, \ldots, x_d]$ on this interval to obtain the coefficients:

$$b_q = F_i[t_{q+1}, t_{q+2}, \ldots, t_{q+d}], \qquad q = J - d, \ldots, J.$$

This means that if f is given in terms of a shifted power basis, then we can write f in terms of the B-spline basis without actually doing the change of basis. This is the same situation that we had in changing from any polynomial basis to the Bernstein basis using the polar form.

We can also extend this to spline curves $\gamma(t)$, where the coordinate functions, say $x(t)$ and $y(t)$, are now in V. As we saw before, we can write such a curve with point coefficients with respect to some basis of V. The DeBoor control points can then be extracted using the polar forms just as we did for the coefficients above, except that now the polar forms are with respect to the function $\gamma(t)$.

Again, suppose that $t \in [t_J, t_{J+1})$, and $t_J = u_{i-1} < u_i = t_{J+1}$. We can then use the polar form $F_i[x_1, \ldots, x_d]$ for $\gamma(t)$ on this interval to obtain the DeBoor control points:

$$P_q^{[0]} = F_i[t_{q+1}, \ldots, t_{q+d}], \qquad q = J-d, \ldots, J.$$

Lecture 27, Th April 18, 2013

Main Points:

- Introduction to Surfaces
- Review, Practice Quiz 10

Total degree polynomial surfaces

Let $V = P_{d,\{x,y\}}$ denote the vector space of polynomials in the variables x and y with total degree at most d. A basis of V is:

$$\{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, \ldots, x^d, x^{d-1}y, \ldots, xy^{d-1}, y^d\}.$$

The dimension of V is

$$dim(V) = \binom{d+2}{2} = \frac{(d+2)(d+1)}{2}.$$

Tensor Product Surfaces

The tensor product of two vector spaces of functions V and W, with bases $B_V = \{f_1, \ldots, f_n\}$ and $B_W = \{g_1, \ldots, g_m\}$ respectively, is defined as the vector space $V \otimes W$ with basis $\{h_{i,j} = f_i g_j : 1 \le i \le n, 1 \le j \le m\}$, where the basis functions $h_{i,j}(x,y) = f_i(x)g_j(y)$. Any function h in $V \otimes W$ can be expressed a sum:

$$h = \sum_{i=1}^{n} \sum_{j=1}^{m} c_{i,j} h_{i,j}.$$

So, the definition of tensor product space is:

$$V \otimes W = \{\sum_{i=1}^{n} \sum_{j=1}^{m} c_{i,j} h_{i,j} \ : \ h_{i,j}(x,y) = f_i(x)g_j(y), f_i \in B_V, g_j \in B_W\}.$$

The dimension of a tensor product space is:

$$dim(V \otimes W) = m \cdot n = dim(V) \cdot dim(W).$$

A tensor product surface is the graph of $z = h(x,y)$ with $h \in V \otimes W$.

The space $P_d \otimes P_d$ is a subspace of $P_{2d,\{x,y\}}$.

Similarly, the space $P_m \otimes P_n$ is a subspace of $P_{m+n,\{x,y\}}$.

Examples:

- Let $V = P_2$ with variable x, and let $W = P_3$ with variable y. Then the tensor product $V \otimes W$ has basis:

  $$\{1, x, x^2, y, xy, x^2y, y^2, xy^2, x^2y^2, y^3, xy^3, x^2y^3\}.$$

  This is a subset of the basis of of $P_{5,\{x,y\}}$:

  $$\{1, x, x^2, x^3, x^5, y, xy, x^2y, x^3y, x^4y, y^2, xy^2, x^2y^2, x^3y^2, y^3, xy^3, x^2y^3, y^4, xy^4, y^5\}.$$

Points to be able to work out for the Final Exam:

- Find the polar form of a polynomial in Standard basis
- Find the polar form of a polynomial in Bernstein, or other, bases
- Identify the three defining properties of polar forms
- Verify if a given function satisfies one or more of the defining properties of polar forms
- Write the polar form for a parametric polynomial curve from the standard form
- Write the polar form for a Bezier curve from the BB-form
- Apply the control point property to find control points of a parametric polynomial curve in standard basis form
- Apply the reparametrization property to find control points of a reparametrized curve using the polar form of the original
- Use Nested Linear Interpolation to evaluate a polar form
- Verify equivalence of BB-form and NLI-form
- Write a Bezier curve in Cumulative form
- Find the derivative of a Bezier curve using Cumulative form
- Recognize a degenerate Bezier curve
- Classify the type of an implicit quadratic by using the Discriminant
- Use the five point construction to find a quadratic equation
- Use the tangent construction to find an implicit equation of a Bezier curve
- Use the tangent line properties of quadratic Bezier curves to solve for one control point given the other two
- Write the standard basis for a vector space of splines with continuity and multiplicity vectors
- Use a knot sequence to write a shifted power basis
- Work out a B-spline from the divided difference formula
- Write the knot subsequence for a B-spline based on a knot sequence and starting index
- Find the support of a B-spline (where it is nonzero)
- Find the exact order of continuity of a B-spline at one of its knot values using the multiplicity
- Use the Curry-Schoenberg Theorem to find a knot sequence for a basis of B-splines for a given vector space of splines
- Find the dimension of a vector space of splines given a knot sequence for a shifted power basis or a B-spline basis
- Write a B-spline as a sum of two lower degree B-splines using the recursion formula
- Write a B-spline as a sum of shifted power functions by using Cramer's rule and determinants
- Identify the lowest degree shifted power function coefficient in a B-spline as a nonzero confluent Vandermonde determinant
- Identify the index J used in the DeBoor algorithm to compute a B-spline curve at some value t
- Write a shifted power basis with increasing degree order
- Identify the nonzero values in the coordinate vector of a B-spline with respect to the shifted power basis

- Apply the Schoenberg-Whitney Theorem to determine if a set of points is correctly chosen to give a unique interpolant in a vector space of splines
- Write a basis of a total degree polynomial vector space in two variables
- Write a basis of a tensor product of function spaces in two variables
- Identify tensor products of polynomials inside the larger vector space of total degree polynomials