DigiPen Institute of Technology Europe-Bilbao

MAT300 Curves & Surfaces

Spring 2020. Homework 3: Deadline: 26-2-2020

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1. (10%) Use a system of linear equations to find an interpolant polynomial in standard basis satisfying $p(1) = 6$, $p'(1) = 8$, $p''(1) = 42$, $p^{(3)}(1) = 168$, $p(2) = 91$, $p'(2) = 255$, and $p''(2) = 620$.

2. (20%) Compute the divided differences for a polynomial satisfying $p(1) = 6$, $p'(1) = 8$, $p''(1) = 42$, $p^{(3)}(1) = 168$, $p(2) = 91$, $p'(2) = 255$, and $p''(2) = 620$. Give the Newton basis and the vector of coordinates of the polynomial in that basis.

3. (10%) Obtain the change of basis transformation from Newton to Standard basis, and verify that the polynomials in exercises 1 and 2 are the same.

4. (30%) Given the polynomial

$$p(x) = \begin{cases} 1 + 2x + x^2, & x \in [-2, 0) \\ 1 + 2x - x^3, & x \in [0, 1) \\ 3 - x, & x \in [1, 3) \\ 12 - 7x + x^2, & x \in [3, 5] \end{cases}$$

   a) (5%) Determine to which polynomial vector space belongs p, **taking into account orders of continuity**.

   b) (10%) Construct a right shifted basis for that space.

   c) (15%) Give the vector of coordinates of p in that basis.

5. (30%) Consider a cubic spline such that $p(0) = 1$, $p(1) = 0$, $p(2) = -1$ and $p(5) = 1$.

   a) (15%) Give the vector of coordinates of such a spline in the right shifted basis.

   b) (15%) Give the piecewise expression.

① $p(1)=6$, $p'(1)=8$, $p''(1)=42$, $p^{(3)}(1)=168$, $p(2)=91$, $p'(2)=255$, $p''(2)=620$

7 conditions $\Rightarrow$ 7 equations

In order to have a unique solution we want 7 unknowns, so $p \in P_6$

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6$$
$$p'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + 6a_6 x^5$$
$$p''(x) = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + 30a_6 x^4$$
$$p'''(x) = 6a_3 + 24a_4 x + 60a_5 x^2 + 120a_6 x^4$$

Substituting the 7 conditions in p and its derivatives we create a linear system of 7 equations

$$\left.\begin{array}{ll}
p(1)=6 \to & a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 6 \\
p'(1)=8 \to & a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5 + 6a_6 = 8 \\
p''(1)=42 \to & 2a_2 + 6a_3 + 12a_4 + 20a_5 + 30a_6 = 42 \\
p'''(1)=168 \to & 6a_3 + 24a_4 + 60a_5 + 120a_6 = 168 \\
p(2)=91 \to & a_0 + 2a_1 + 4a_2 + 8a_3 + 16a_4 + 32a_5 + 64a_6 = 91 \\
p'(2)=255 \to & a_1 + 4a_2 + 12a_3 + 32a_4 + 80a_5 + 192a_6 = 255 \\
p''(2)=620 \to & 2a_2 + 12a_3 + 48a_4 + 160a_5 + 480a_6 = 620
\end{array}\right\}$$

We build the augmented matrix and compute its RREF

$$\left(\begin{array}{ccccccc|c}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 6 \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 8 \\
0 & 0 & 2 & 6 & 12 & 20 & 30 & 42 \\
0 & 0 & 0 & 6 & 24 & 60 & 120 & 168 \\
1 & 2 & 4 & 8 & 16 & 32 & 64 & 91 \\
0 & 1 & 4 & 12 & 32 & 80 & 192 & 255 \\
0 & 0 & 2 & 12 & 48 & 160 & 480 & 620
\end{array}\right) \xrightarrow{RREF} \left(\begin{array}{c|c}
 & 5 \\
 & -1 \\
 & 2 \\
I_7 & -2 \\
 & 0 \\
 & 1 \\
 & 1
\end{array}\right)$$

So the polynomial is $p(x) = 5 - x + 2x^2 - 2x^3 + x^5 + x^6$

② Divided differences:

$p(1)=6 \quad p'(1)=8 \quad p''(1)=42 \quad p'''(1)=168$

$p(2)=91 \quad p'(2)=255 \quad p''(2)=620$

x_i	$f[x_i]_0$	$f[x_i,x_j]_1$	$f[x_i,x_j,x_k]_2$	$f[x_i,x_j,x_k,x_l]_3$	$f[x_i\dots]_4$	$f[x_i\dots]_5$	$f[x_i\dots]_6$
1	6						
		8					
1	6		21				
		8		28			
1	6		21		28		
		8		56		9	
1	6		77		37		1
		85		93		10	
2	91		170		47		
		255		140			
2	91		310				
		255					
2	91						

$f[1,1]=\frac{p'(1)}{1!}=8 \qquad f[1,2]=\frac{91-6}{2-1}=85 \qquad f[2,2]=\frac{p'(2)}{1!}=255$

$f[1,1,1]=\frac{p''(1)}{2!}=21 \qquad f[1,1,2]=\frac{85-8}{2-1}=77 \qquad f[1,2,2]=\frac{255-85}{2-1}=170$

$f[2,2,2]=\frac{p''(2)}{2!}=310 \qquad f[1,1,1,1]=\frac{p'''(1)}{3!}=28 \qquad f[1,1,1,2]=\frac{77-21}{2-1}=56$

$f[1,1,2,2]=\frac{170-77}{2-1}=93 \qquad f[1,2,2,2]=\frac{310-170}{2-1}=140$

$f[1,1,1,1,2]=\frac{56-28}{2-1}=28 \qquad f[1,1,1,2,2]=\frac{93-56}{2-1}=37$

$f[1,1,2,2,2]=\frac{140-93}{2-1}=47 \qquad f[1,1,1,1,2,2]=\frac{37-28}{2-1}=9$

$f[1,1,1,2,2,2]=\frac{47-37}{2-1}=10 \qquad f[1,1,1,1,2,2,2]=\frac{10-9}{2-1}=1$

Newton basis $B_N=\{1,(x-1),(x-1)^2,(x-1)^3,(x-1)^4,(x-1)^4(x-2),(x-1)^4(x-2)^2\}$

The vector of coordinates of the polynomial in the Newton basis is

$$(6,8,21,28,28,9,1)_{BN}$$

③ Change of basis transformation from Newton to standard

$T: \mathbb{R}^7 \longrightarrow \mathbb{R}^7 \qquad T(\vec{x}) = M\vec{x}$ where M is given by

$$M = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -2 & 4 \\ 0 & 1 & -2 & 3 & -4 & 9 & -20 \\ 0 & 0 & 1 & -3 & 6 & -16 & 41 \\ 0 & 0 & 0 & 1 & -4 & 14 & -44 \\ 0 & 0 & 0 & 0 & 1 & -6 & 26 \\ 0 & 0 & 0 & 0 & 0 & 1 & -8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$1 \to (1,0,0,0,0,0,0)_S$

$(x-1) \longrightarrow (-1,1,0,0,0,0,0)_S$

$(x-1)^2 = x^2-2x+1 \longrightarrow (1,-2,1,0,0,0,0)_S$

$(x-1)^3 = x^3-3x^2+3x-1 \rightarrow (-1,3,-3,1,0,0,0)_S$

$(x-1)^4 = x^4-4x^3+6x^2-4x+1 \longrightarrow (1,-4,6,-4,1,0,0)_S$

$(x-1)^4(x-2) = x^5-6x^4+14x^3-16x^2+9x-2 \longrightarrow (-2,9,-16,14,-6,1,0)_S$

$(x-1)^4(x-2)^2 = x^6-8x^5+26x^4-44x^3+41x^2-20x+4 \rightarrow (4,-20,41,-44,26,-8,1)_S$

To verify that both polynomials in exercises 1 and 2 are the same we need to check $P_S = M P_N$

$$M \cdot \begin{pmatrix} 6 \\ 8 \\ 21 \\ 28 \\ 28 \\ 9 \\ 1 \end{pmatrix}_{B_N} = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -2 & 4 \\ 0 & 1 & -2 & 3 & -4 & 9 & -20 \\ 0 & 0 & 1 & -3 & 6 & -16 & 41 \\ 0 & 0 & 0 & 1 & -4 & 14 & -44 \\ 0 & 0 & 0 & 0 & 1 & -6 & 26 \\ 0 & 0 & 0 & 0 & 0 & 1 & -8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 6 \\ 8 \\ 21 \\ 28 \\ 28 \\ 9 \\ 1 \end{pmatrix}_{B_N} = \begin{pmatrix} 5 \\ -1 \\ 2 \\ -2 \\ 0 \\ 1 \\ 1 \end{pmatrix}_S$$

As it is satisfied they are the same polynomials.

④ $p(x) = \begin{cases} 1+2x+x^2 & x \in [-2,0) \\ 1+2x-x^3 & x \in [0,1) \\ 3-x & x \in [1,3) \\ 12-7x+x^2 & x \in [3,5] \end{cases}$

a) $p \in P_3^4[-2,0,1,3,5]$ Now we have to check to which subspace belongs considering continuity conditions.

<u>Verify continuity:</u> $p_1(0) = 1 = p_2(0)$ continuous at 0

$p_2(1) = 1+2-1 = 2$ $p_3(1) = 3-1 = 2$ ✓ Continuous at 1

$p_3(3) = 3-3 = 0$ $p_4(3) = 12-21+9 = 0$ ✓ Continuous at 3

→ p is continuous $p \in P_{3,0}^4[-2,0,1,3,5]$

<u>Verify differentiability:</u>

$p'(x) = \begin{cases} 2+2x & x \in [-2,0) \\ 2-3x^2 & x \in [0,1) \\ -1 & x \in [1,3) \\ -7+2x & x \in [3,5) \end{cases}$

$p_1'(0) = 2 = p_2'(0)$ ✓

$p_2'(1) = 2-3 = -1 = p_3'(1)$ ✓

$p_3'(3) = -1 = -7+6 = p_4'(1)$ ✓

→ p is differentiable $p \in P_{3,1}^4[-2,0,1,3,5]$

<u>Verify twice differentiability:</u>

$p''(x) = \begin{cases} 2 & x \in [-2,0) \\ -6x & x \in [0,1) \\ 0 & x \in [1,3) \\ 2 & x \in [3,5) \end{cases}$

$p_1''(0) = 2 \neq 0 = p_2''(0)$

→ p'' is not twice differentiable, therefore

$p \notin P_{3,2}^4[-2,0,1,3,5]$

The space is $P_{3,1}^{4}\,[-2,0,1,3,5]$

b) Right shifted basis.

Starting with basis for $P_3^4[-2,0,1,3,5]$

$$B=\{1,x,x^2,x^3,(x-0)_+^0,(x-0)_+^1,(x-0)_+^2,(x-0)_+^3,(x-1)_+^0,$$
$$(x-1)_+^1,(x-1)_+^2,(x-1)_+^3,(x-3)_+^0,(x-3)_+^1,(x-3)_+^2,(x-3)_+^3\}$$

Now we delete the elements that break continuity and differentiability

$$B=\{1,x,x^2,x^3,\cancel{(x-0)_+^0},\cancel{(x-0)_+^1},(x-0)_+^2,(x-0)_+^3,\cancel{(x-1)_+^0},$$
$$\cancel{(x-1)_+^1},(x-1)_+^2,(x-1)_+^3,\cancel{(x-3)_+^0},\cancel{(x-3)_+^1},(x-3)_+^2,(x-3)_+^3\}$$

so we get

$$B=\{1,x,x^2,x^3,(x-0)_+^2,(x-0)_+^3,(x-1)_+^2,(x-1)_+^3,(x-3)_+^2,(x-3)_+^3\}$$

$$p=a_0+a_1x+a_2x^2+a_3x^3+a_4(x-0)_+^2+a_5(x-0)_+^3+a_6(x-1)_+^2+a_7(x-1)_+^3$$
$$+a_8(x-3)_+^2+a_9(x-3)_+^3$$

For $x\in[-2,0)$: $(x-0)_+^2=(x-0)_+^3=(x-1)_+^2=(x-1)_+^3=(x-3)_+^2=(x-3)_+^3=0$

and so $p(x)=a_0+a_1x+a_2x^2+a_3x^3$

then $1+2x+x^2=a_0+a_1x+a_2x^2+a_3x^3$ and

$$\boxed{a_0=1,\quad a_1=2,\quad a_2=1,\quad a_3=0}$$

For $x\in[0,1)$: $(x-0)_+^2=x^2$, $(x-0)_+^3=x^3$ and

$(x-1)_+^2=(x-1)_+^3=(x-3)_+^2=(x-3)_+^3=0$ therefore

$p(x)=1+2x+x^2+a_4x^2+a_5x^3$ then

$$\cancel{1}+\cancel{2x}-x^3=\cancel{1}+\cancel{2x}+x^2+a_4x^2+a_5x^3\longrightarrow\boxed{\begin{array}{c}-1=a_4\\-1=a_5\end{array}}$$

For $x \in [1,3)$: $(x-0)_+^2 = x^2$, $(x-0)_+^3 = x^3$, $(x-1)_+^2 = (x-1)^2$, $(x-1)_+^3 = (x-1)^3$

and $(x-3)_+^2 = 0$, $(x-3)_+^3 = 0$ therefore

$$p(x) = 1 + 2x + x^2 - x^2 - x^3 + a_6 (x-1)^2 + a_7 (x-1)^3$$

$$3 - x = 1 + 2x - x^3 + a_6 (x^2 - 2x + 1) + a_7 (x^3 - 3x^2 + 3x - 1)$$

$$\begin{cases} 3 = 1 + a_6 - a_7 \\ -1 = 2 - 2a_6 + 3a_7 \\ 0 = a_6 - 3a_7 \\ 0 = -1 + a_7 \end{cases} \longrightarrow \boxed{a_7 = 1, \quad a_6 = 3}$$

For $x \in [3,5]$: $(x-0)_+^2 = x^2$, $(x-0)_+^3 = x^3$, $(x-1)_+^2 = (x-1)^2$, $(x-1)_+^3 = (x-1)^3$,

$(x-3)_+^2 = (x-3)^2$ and $(x-3)_+^3 = (x-3)^3$ therefore

$$p(x) = 1 + 2x + \cancel{x^2} + \cancel{0x^3} - \cancel{x^2} - x^3 + 3(x-1)^2 + (x-1)^3 + a_8 (x-3)^2 + a_9 (x-3)^3$$

$$12 - 7x + x^2 = 1 + 2x - x^3 + 3(x^2 - 2x + 1) + (x^3 - 3x^2 + 3x - 1) + a_8 (x^2 - 6x + 9)$$
$$+ a_9 (x^3 - 9x^2 + 27x - 27)$$

$$9 - 6x + x^2 = a_8 (x^2 - 6x + 9) + a_9 (x^3 - 9x^2 + 27x - 27)$$

$$\longrightarrow \boxed{a_8 = 1 \quad \text{and} \quad a_9 = 0}$$

The vector of coordinates of p is then

$$p = (1, 2, 1, 0, -1, -1, 3, 1, 1, 0)_B$$

⑤ Cubic spline s.th. $p(0)=1$, $p(1)=0$, $p(2)=-1$

a) $p(5)=1$

$$p \in P^3_{3,2}[0,1,2,5]$$

$$B = \{1, x, x^2, x^3, (x-1)_+^3, (x-2)_+^3\}$$

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 (x-1)_+^3 + a_5 (x-2)_+^3$$

$$p'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 3a_4 (x-1)_+^2 + 3a_5 (x-2)_+^2$$

$$p''(x) = 2a_2 + 6a_3 x + 6a_4 (x-1)_+ + 6a_5 (x-2)_+$$

We impose $p''(0)=0$ and $p''(5)=0$ then we get

$$\begin{array}{ll}
p(0)=1 & \left\{\begin{array}{l}
a_0 = 1 \\
a_0 + a_1 + a_2 + a_3 = 0 \\
a_0 + 2a_1 + 4a_2 + 8a_3 + a_4 = -1 \\
a_0 + 5a_1 + 25a_2 + 125a_3 + 64a_4 + 27a_5 = 1 \\
2a_2 = 0 \\
2a_2 + 30a_3 + 24a_4 + 18a_5 = 0
\end{array}\right. \\
p(1)=0 & \\
p(2)=-1 & \\
p(5)=1 & \\
p''(0)=0 & \\
p''(5)=0 &
\end{array}$$

and solve the system

$$\left(\begin{array}{cccccc|c}
1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 2 & 4 & 8 & 1 & 0 & -1 \\
1 & 5 & 25 & 125 & 64 & 27 & 1 \\
0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 30 & 24 & 18 & 0
\end{array}\right) \xrightarrow{RREF} \left(\begin{array}{c|c}
 & 279/279 \\
 & -264/279 \\
I & 0 \\
 & -15/279 \\
 & 90/279 \\
 & -95/279
\end{array}\right)$$

So the vector of coordinates is

$$\left(1, \frac{-88}{93}, 0, \frac{-5}{93}, +\frac{10}{31}, \frac{-95}{279}\right)$$

b) Piecewise expression

$$P(x) = 1 - \frac{88}{93}x - \frac{5}{93}x^3 + \frac{10}{31}(x-1)_+^3 - \frac{95}{279}(x-2)_+^3$$

for $x \in [0,1)$: $(x-1)_+^3 = 0$ and $(x-2)_+^3 = 0$

so $$P_1(x) = 1 - \frac{88}{93}x - \frac{5}{93}x^3 \quad x \in [0,1)$$

for $x \in [1,2)$: $(x-1)_+^3 = (x-1)^3$ and $(x-2)_+^3 = 0$

so $$P_2(x) = 1 - \frac{88}{93}x - \frac{5}{93}x^3 + \frac{10}{31}(x^3 - 3x^2 + 3x - 1)$$

$$P_2(x) = \frac{21}{31} + \frac{2}{93}x - \frac{30}{31}x^2 + \frac{25}{93}x^3 \quad x \in [1,2)$$

for $x \in [2,5]$: $(x-1)_+^3 = (x-1)^3$ and $(x-2)_+^3 = (x-2)^3$

so $$P_3(x) = 1 - \frac{88}{93}x - \frac{5}{93}x^3 + \frac{10}{31}(x^3 - 3x^2 + 3x - 1) - \frac{95}{279}(x^3 - 6x^2 + 12x - 8)$$

$$P_3(x) = \frac{949}{279} - \frac{126}{31}x + \frac{100}{93}x^2 - \frac{20}{279}x^3 \quad x \in [2,5]$$

So the piewise expression for the polynomial is

$$P(x) = \begin{cases} 1 - \frac{88}{93}x - \frac{5}{93}x^3 & x \in [0,1) \\ \frac{21}{31} + \frac{2}{93}x - \frac{30}{31}x^2 + \frac{25}{93}x^3 & x \in [1,2) \\ \frac{949}{279} - \frac{126}{31}x + \frac{100}{93}x^2 - \frac{20}{279}x^3 & x \in [2,5] \end{cases}$$