DigiPen Institute of Technology, Bilbao

MAT300 Curves & Surfaces

Spring 2018. Homework 5 SOLUTIONS: Deadline: 15-4-2020

Consider the parametrized polynomial curve $\gamma : [0,1] \to \mathbb{R}^2$ given by

$$\gamma(t) = (1 - 4t + 8t^2 - 3t^3, 2 + 4t - 5t^2 + 2t^3) \tag{1}$$

1. (20%) Compute its polar form.

   - We first express the curve as a combination of points and polynomials of the standard basis.

   $$\gamma(t) = (1,2) + t(-4,4) + t^2(8,-5) + t^3(-3,2)$$

   - We now compute the polar forms for the polynomials in the standard basis for P_3.

   $$\begin{aligned} F_0[u_1,u_2,u_3] &= 1 \\ F_1[u_1,u_2,u_3] &= \frac{u_1+u_2+u_3}{3} \\ F_2[u_1,u_2,u_3] &= \frac{u_1u_2+u_1u_3+u_2u_3}{3} \\ F_3[u_1,u_2,u_3] &= u_1u_2u_3 \end{aligned}$$

   - The polar form of γ is

   $$F[u_1,u_2,u_3] = F_0 \cdot (1,2) + F_1 \cdot (-4,4) + F_2 \cdot (8,-5) + F_3 \cdot (-3,2) =$$
   $$(1,2) + \left(\frac{u_1+u_2+u_3}{3}\right)(-4,4) + \left(\frac{u_1u_2+u_1u_3+u_2u_3}{3}\right)(8,-5) + u_1u_2u_3 \cdot (-3,2)$$
   $$= \left(\tfrac{3-4u_1-4u_2-4u_3+8u_1u_2+8u_1u_3+8u_2u_3-9u_1u_2u_3}{3} \ , \ \tfrac{6+4u_1+4u_2+4u_3-5u_1u_2-5u_1u_3-5u_2u_3+6u_1u_2u_3}{3} \right)$$

2. (15%) Use the polar form to obtain the control points of its Bezier representation and give the Bezier representation of the curve.

   - The curve is cubic so its Bezier representation has four control points P_0, P_1, P_2 and P_3. The points are computed as follows:

   $$\begin{aligned} P_0 = F[0,0,0] &= \left(\tfrac{3-0-0-0+0+0+0-0}{3} \ , \ \tfrac{6+0+0+0-0-0-0+0}{3} \right) = (1,2) \\ P_1 = F[0,0,1] &= \left(\tfrac{3-0-0-4+0+0+0-0}{3} \ , \ \tfrac{6+0+0+4-0-0-0+0}{3} \right) = \left(-\frac{1}{3}, \frac{10}{3}\right) \\ P_2 = F[0,1,1] &= \left(\tfrac{3-0-4-4+0+0+8-0}{3} \ , \ \tfrac{6+0+4+4-0-0-5+0}{3} \right) = (1,3) \\ P_3 = F[1,1,1] &= \left(\tfrac{3-4-4-4+8+8+8-9}{3} \ , \ \tfrac{6+4+4+4-5-5-5+6}{3} \right) = (2,3) \end{aligned}$$

- Once we have the control points, the Bezier representation is $\gamma : [0,1] \to \mathbb{R}^2$ given by

$$\gamma(t) = \sum_{i=0}^{3} B_i^3(t)P_i = B_0^3(t)(1,2) + B_1^3(t)\left(-\frac{1}{3}, \frac{10}{3}\right) + B_2^3(t)(1,3) + B_3^3(t)(2,3)$$

where the Bernstein polynomial are

$$\begin{aligned}
B_0^3(t) &= \begin{pmatrix} 3 \\ 0 \end{pmatrix}(1-t)^3 t^0 = 1 - 3t + 3t^2 - t^3 \\
B_1^3(t) &= \begin{pmatrix} 3 \\ 1 \end{pmatrix}(1-t)^2 t^1 = 3t - 6t^2 + 3t^3 \\
B_2^3(t) &= \begin{pmatrix} 3 \\ 2 \end{pmatrix}(1-t)^1 t^2 = 3t^2 - 3t^3 \\
B_3^3(t) &= \begin{pmatrix} 3 \\ 3 \end{pmatrix}(1-t)^0 t^3 = t^3
\end{aligned}$$

3. (15%) Now consider the curve $\gamma : [-2,-1] \to \mathbb{R}^2$ given with the above formula (1). Obtain the control points of its Bezier representation and give the Bezier representation of the curve.

- The polar form for a polynomial is unique, therefore we will use the polar form obtained in exercise 1 with the entries corresponding to the interval $[-2,-1]$ to find the control points of the curve.

$$\begin{aligned}
P_0 &= F[-2,-2,-2] = \left(\tfrac{3+8+8+8+32+32+32+72}{3} , \tfrac{6-8-8-8-20-20-20-48}{3} \right) = (65,-42) \\
P_1 &= F[-2,-2,-1] = \left(\tfrac{3+8+8+4+32+16+16+36}{3} , \tfrac{6-8-8-4-20-10-10-24}{3} \right) = (41,-26) \\
P_2 &= F[-2,-1,-1] = \left(\tfrac{3+8+4+4+16+16+8+18}{3} , \tfrac{6-8-4-4-10-10-5-12}{3} \right) = \left(\frac{77}{3}, -\frac{47}{3}\right) \\
P_3 &= F[-1,-1,-1] = \left(\tfrac{3+4+4+4+8+8+8+9}{3} , \tfrac{6-4-4-4-5-5-5-6}{3} \right) = (16,-9)
\end{aligned}$$

- Once we have the control points, the Bezier representation is $\gamma : [0,1] \to \mathbb{R}^2$ given by

$$\begin{aligned}
\gamma(t) &= \sum_{i=0}^{3} B_i^3(t)P_i = B_0^3(t)(65,-42) + B_1^3(t)(41,-26) + B_2^3(t)\left(\frac{77}{3}, -\frac{47}{3}\right) \\
&+ B_3^3(t)(16,-9)
\end{aligned}$$

where the Bernstein polynomial are those in exercise 2.

4. (20%) Compute the derivative of the Bezier curve in exercise 3.

- The derivative of a cubic Bezier curve is given by

$$\gamma'(t) = 3\sum_{i=0}^{2} \vec{v_{i+1}} B_i^2(t), \quad t \in (0,1)$$

for $\vec{v_{i+1}} = P_{i+1} - P_i$ and B_i^2 the Bernstein polynomials of degree 2.

- We compute the vectors $\vec{v_{i+1}} = P_{i+1} - P_i$ for $i = 0, 1, 2$.

$$\begin{aligned}
\vec{v_1} &= P_1 - P_0 = (41, -26) - (65, -42) = (-24, 16) \\
\vec{v_2} &= P_2 - P_1 = \left(\frac{77}{3}, -\frac{47}{3}\right) - (41, -26) = \left(-\frac{46}{3}, \frac{31}{3}\right) \\
\vec{v_3} &= P_3 - P_2 = (16, -9) - \left(\frac{77}{3}, -\frac{47}{3}\right) = \left(-\frac{29}{3}, \frac{20}{3}\right)
\end{aligned}$$

- We compute the Bernstein polynomials of degree 2.

$$\begin{aligned}
B_0^2(t) &= \begin{pmatrix} 2 \\ 0 \end{pmatrix} (1-t)^2 t^0 = 1 - 2t + t^2 \\
B_1^2(t) &= \begin{pmatrix} 2 \\ 1 \end{pmatrix} (1-t)^1 t^1 = 2t - 2t^2 \\
B_2^2(t) &= \begin{pmatrix} 2 \\ 2 \end{pmatrix} (1-t)^0 t^2 = t^2
\end{aligned}$$

- Substituting in the above formula we get

$$\gamma'(t) = 3\sum_{i=0}^{2} \vec{v_{i+1}} B_i^2(t) =$$
$$3\left(B_0^2(t)(-24, 16) + B_1^2(t)\left(-\frac{46}{3}, \frac{31}{3}\right) + B_2^2(t)\left(-\frac{29}{3}, \frac{20}{3}\right)\right) =$$
$$3\left((1 - 2t + t^2)(-24, 16) + (2t - 2t^2)\left(-\frac{46}{3}, \frac{31}{3}\right) + t^2\left(-\frac{29}{3}, \frac{20}{3}\right)\right), \quad t \in (0,1)$$

5. (10%) Compute the tangent line to the curve in exercise 3 at $t = \frac{3}{4}$.

- The tangent line to the curve at $t = \frac{3}{4}$ in vector form is given as follows

$$(x, y) = \gamma\left(\frac{3}{4}\right) + \lambda\gamma'\left(\frac{3}{4}\right), \quad \lambda \in \mathbb{R}$$

- We evaluate γ at $t = \frac{3}{4}$

$$\gamma\left(\frac{3}{4}\right) = B_0^3\left(\frac{3}{4}\right)(65, -42) + B_1^3\left(\frac{3}{4}\right)(41, -26) + B_2^3\left(\frac{3}{4}\right)\left(\frac{77}{3}, -\frac{47}{3}\right)$$
$$+ B_3^3\left(\frac{3}{4}\right)(16, -9) = \frac{1}{64}(65, -42) + \frac{9}{64}(41, -26) + \frac{27}{64}\left(\frac{77}{3}, -\frac{47}{3}\right) + \frac{27}{64}(16, -9)$$
$$= \left(\frac{1559}{64}, \frac{-942}{64}\right)$$

- We evaluate γ' at $t = \frac{3}{4}$

$$\gamma'(t) = 3\left(B_0^2\left(\frac{3}{4}\right)(-24,16) + B_1^2\left(\frac{3}{4}\right)\left(-\frac{46}{3},\frac{31}{3}\right) + B_2^2\left(\frac{3}{4}\right)\left(-\frac{29}{3},\frac{20}{3}\right)\right) =$$
$$3\left(\frac{1}{16}(-24,16) + \frac{6}{16}\left(-\frac{46}{3},\frac{31}{3}\right) + \frac{9}{16}\left(-\frac{29}{3},\frac{20}{3}\right)\right) = \left(-\frac{609}{16},\frac{414}{16}\right)$$

- Substituting in the equation of the line we get

$$(x,y) = \left(\frac{1559}{64},\frac{-942}{64}\right) + \lambda\left(-\frac{609}{16},\frac{414}{16}\right), \quad \lambda \in \mathbb{R}$$

6. (20%) Let $P_0 = (1,4)$, $P_1 = (2,3)$ and $P_2 = (-1,-1)$ be the control points of a quadratic Bezier curve. Give the implicit expression $f(x,y) = 0$ of the quadratic curve on which it lies.

   - First of all we check if the curve is degenerated or not.

$$\vec{v_1} = P_1 - P_0 = (2,3) - (1,4) = (1,-1)$$
$$\vec{v_2} = P_2 - P_1 = (-1,-1) - (2,3) = (-3,-4)$$

   As the vectors are not parallel the points are not aligned and so the curve is not degenerated.

   - The implicit expression of a not degenerated quadratic Bezier curve is

$$L_0(x,y)L_2(x,y) + kL_1(x,y)^2 = 0$$

   where $L_0(x,y)$ is the left side of the normal equation of the line through P_0 and P_1, $L_1(x,y)$ the left side of the normal equation of the line through P_0 and P_2, $L_2(x,y)$ the left side of the normal equation of the line through P_1 and P_2, and k a parameter to be determined.

   - We obtain $L_0(x,y)$. The normal equation of a line in 2D is

$$ax + by + c = 0$$

   The line passes through P_0 and P_1 so it has vector director $\vec{v_1} = (1,-1)$ and normal vector $\vec{n} = (1,1)$ so our line is

$$x + y + c = 0$$

   and we obtain c by substituting a point, for instance $P_0 = (1,4)$

$$1 + 4 + c = 0 \quad \rightarrow \quad c = -5 \quad \rightarrow \quad x + y - 5 = 0$$

   therefore $L_0(x,y) = x + y - 5$.

- We obtain $L_1(x, y)$. The line passes through P_0 and P_2 so it has vector director $\vec{v} = P_2 - P_0 = (-2, -5)$ and normal vector $\vec{n} = (5, -2)$ so our line is
$$5x - 2y + c = 0$$
and we obtain c by substituting a point, for instance $P_0 = (1, 4)$
$$5 - 8 + c = 0 \quad \rightarrow \quad c = 3 \quad \rightarrow \quad 5x - 2y + 3 = 0$$
therefore $L_1(x, y) = 5x - 2y + 3$.
- We obtain $L_2(x, y)$. The line passes through P_1 and P_2 so it has vector director $\vec{v_2} = (-3, -4)$ and normal vector $\vec{n} = (4, -3)$ so our line is
$$4x - 3y + c = 0$$
and we obtain c by substituting a point, for instance $P_2 = (-1, -1)$
$$-4 + 3 + c = 0 \quad \rightarrow \quad c = 1 \quad \rightarrow \quad 4x - 3y + 1 = 0$$
therefore $L_2(x, y) = 4x - 3y + 1$.
- Substituting in the above equation we get that the implicit expression for the curve is
$$(x + y - 5)(4x - 3y + 1) + k(5x - 2y + 3)^2 = 0$$
$$4x^2 - 3xy + x + 4xy - 3y^2 + y - 20x + 15y - 5 + k(5x - 2y + 3)^2 = 0$$
$$4x^2 - 3y^2 + xy - 19x + 16y - 5 + k(5x - 2y + 3)(5x - 2y + 3) = 0$$
$$4x^2 - 3y^2 + xy - 19x + 16y - 5 + k(25x^2 + 4y^2 - 20xy + 30x - 12y + 9) = 0$$
where k is a parameter to be determined through a point in the curve.
- Applying the midpoint subdivision algorithm to find the evaluation of the curve at $t = \frac{1}{2}$ we get
$$\begin{array}{lll} P_0 = (1,4) & & \\ & P_0^1 = \frac{1}{2}P_1 + \frac{1}{2}P_0 = (\frac{3}{2}, \frac{7}{2}) & \\ P_1 = (2,3) & & P_0^2 = \frac{1}{2}P_1^1 + \frac{1}{2}P_0^1 = (1, \frac{9}{4}) \\ & P_1^1 = \frac{1}{2}P_2 + \frac{1}{2}P_1 = (\frac{1}{2}, 1) & \\ P_2 = (-1,-1) & & \end{array}$$
so the point $\left(1, \frac{9}{4}\right)$ belongs to the curve.
- We substitute in the implicit expression to find the value of k.
$$4 - 3\left(\frac{9}{4}\right)^2 + \frac{9}{4} - 19 + 16 \cdot \frac{9}{4} - 5 + k\left(25 + 4\left(\frac{9}{4}\right)^2 - 20 \cdot \frac{9}{4} + 30 - 12 \cdot \frac{9}{4} + 9\right) = 0$$
$$\frac{49}{16} + k \cdot \frac{49}{4} = 0 \quad \rightarrow \quad k = -\frac{1}{4}$$
So the expression becomes by substitution
$$4x^2 - 3y^2 + xy - 19x + 16y - 5 - \frac{1}{4}(25x^2 + 4y^2 - 20xy + 30x - 12y + 9) = 0$$
$$16x^2 - 12y^2 + 4xy - 76x + 64y - 20 - 25x^2 - 4y^2 + 20xy - 30x + 12y - 9 = 0$$
$$-9x^2 - 16y^2 + 24xy - 106x + 76y - 29 = 0$$