Problem Setup

Let $g \in C^n[a,b]$, and consider $n+1$ distinct points $x_0, x_1, \ldots, x_n \in [a,b]$ such that
$$x_0 < x_1 < \cdots < x_n.$$
Assume further that
$$g(x_0) = g(x_1) = \cdots = g(x_n) = 0.$$
We aim to prove that there exists a point $\xi \in [x_0, x_n]$ such that $g^{(n)}(\xi) = 0$.

Auxiliary Function

Define the polynomial
$$h(x) = \prod_{i=0}^{n} (x - x_i),$$
which is of degree $n+1$ and satisfies $h(x_i) = 0$ for all $i = 0, 1, \ldots, n$.
Next, define the quotient function
$$f(x) = \frac{g(x)}{h(x)}.$$

Step 1: First Application

Since $g(x_0) = g(x_1) = 0$, by Rolle's Theorem there exists a point $c_1 \in (x_0, x_1)$ such that
$$g'(c_1) = 0.$$
Similarly, $g(x_1) = g(x_2) = 0$ implies there exists a point $c_2 \in (x_1, x_2)$ such that
$$g'(c_2) = 0.$$
Continuing this, we find that $g'(x)$ has zeros in $(x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n)$.

Step 2: Second Application

For each interval where $g'(x)$ has a zero, apply Rolle's Theorem again to $g'(x)$.
For example, in (x_0, x_1), there exists a point $d_1 \in (c_1, c_2)$ such that
$$g''(d_1) = 0.$$
Similarly, zeros of $g''(x)$ can be found in $(c_2, c_3), (c_3, c_4), \ldots, (c_{n-1}, c_n)$.

Step 3: Repeated Applications

By iteratively applying Rolle's Theorem, we find that:

- $g'(x)$ has zeros in $(x_0, x_1), \ldots, (x_{n-1}, x_n)$,
- $g''(x)$ has zeros in the intervals formed by zeros of $g'(x)$,
- This process continues until the n-th derivative.

After n applications of Rolle's Theorem, there exists a point $\xi \in [x_0, x_n]$ such that

$$g^{(n)}(\xi) = 0.$$