The Riemann Hypothesis for Hypersurfaces over Finite Fields

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Introduction

Let $X/\mathbb{F}_q$ be a projective, smooth, geometrically irreducible variety of dimension d over the finite field of $q = p^r$ elements. Define the **geometric zeta function**

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- *(Rationality)* $Z(X, T) \in \mathbb{Q}(T)$.
- *(Functional Equation)* $Z(X, \frac{1}{q^d T}) = \pm q^{-\frac{d \cdot \chi}{2}} T^{\chi} Z(X, T)$, where χ is the self intersection of the diagonal of X in $X \times X$, i.e. the "Poincaré-Euler characteristic" of X.

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A polynomial $P(T) = \prod_{i=0}^{k}(1 - \alpha_i T) \in \mathbb{Z}[T]$ is a q-**Weil polynomial pure of weight** n if for every complex embedding $\mathbb{Q}(\alpha_1, \ldots, \alpha_k) \hookrightarrow \mathbb{C}$, $|\alpha_i| = q^{n/2}$. An number α which is a root of such a polynomial is called a q-**Weil number of weight** n.

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- In this talk we sketch a proof of the case where X is a smooth projective hypersurface (Katz 2014). It was shown (Scholl 2011) that by a deformation argument one can reduce the general case to that of hypersurfaces.Together these two papers constitute a new proof which does not use the theory of Lefschetz pencils or the ℓ-adic Fourier transform.

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- For any closed point $Spec(\mathbb{F}_{q^n}) = \{\wp\} \to X_0$ "below $\bar{x}$", we have by functoriality a map $Gal(\overline{\mathbb{F}}_{q^n}/\mathbb{F}_{q^n}) \simeq \pi_1^{ét}(\wp, \bar{x}) \to \pi_1^{ét}(X_0, \bar{x})$.

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- We denote by $Frob_\wp$ the image of the inverse of the Frobenius substitution (rather, a representative of the conjugacy class), and we define the **L-function** of a lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ as

$$L(X_0, \mathcal{F}, T) = \prod_{\wp \in |X|} det(1 - T^{deg(\wp)} Frob_\wp | \mathcal{F}_{\bar{x}})^{-1}$$

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- Example: $L(X_0, \mathbb{Q}_\ell, q^{-s})$ is the classical zeta function of X_0, since $\mathbb{Q}_\ell$ is the local system corresponding to the 1-dimensional trivial rep.

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- Grothendieck's trace formula gives the following cohomological interpretation of the L-function:

$$L(X_0, \mathcal{F}, T) = \prod_{i=0}^{2d} det(1 - TFrob_q | H_c^i(X, \mathcal{F}))^{(-1)^{i+1}}$$

where $Frob_q$ is the so-called geometric Frobenious arising from the pullback on cohomology by the endomorphism on $X = X_0 \times Spec(\overline{\mathbb{F}}_q)$ given by $(id_{X_0} \times f^{-1})$.

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- It also follows from the trace formula that $Z(X_0, T) = L(X_0, \mathbb{Q}_\ell, T)$, and by an elementary argument one can show that the Riemann hypothesis is equivalent to the statement that $Frob_q | H_c^i(X, \mathbb{Q}_\ell)$ has as eigenvalues q-Weil numbers of weight i (Weil I.1.6).

The Key Lemma

For a fixed embedding $\iota : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$, we say that a local system $\mathcal{F}$ is ι-**real** if each Euler factor $det(1 - TFrob_{\wp}|\mathcal{F})^{-1}$ lies in $\mathbb{R}[[T]]$ via ι.

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Lemma (Key Lemma)

Let U_0 be a smooth geometrically connected affine curve, $\mathcal{F}$ an ℓ-adic local system on U_0 which is ι-real. Suppose that for some closed point $\wp_0$ that every eigenvalue $\alpha_{\wp_0,i}$ of $Frob_{\wp_0}|\mathcal{F}$ has $|\iota(\alpha_{\wp_0,i})| = 1$. Then for every closed point $\wp$ of U_0, every eigenvalue $\alpha_{\wp,i}$ of $Frob_\wp|\mathcal{F}$ has $|\iota(\alpha_{\wp,i})| = 1$.

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Theorem (Smooth and Proper Base Change)

Let $f : \mathcal{X} \to U_0$ be a smooth and proper morphism, and $\mathcal{F}$ a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on $\mathcal{X}$. Then the sheaves $R^i f_* \mathcal{F}$ are lisse $\overline{\mathbb{Q}}_\ell$-sheaves on U_0. Furthermore, the Frobenius action is compatible in the sense that for any closed point $\wp$ of U_0, the fibre $X_{\wp,0}$ is smooth and proper over $\mathbb{F}_{\mathbb{N}(\wp)}$ and

$$det(1 - TFrob_{\mathbb{N}(\wp)} | H^i(X_\wp, \mathcal{F})) = det(1 - TFrob_\wp | R^i f_* \mathcal{F})$$

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$$det(1 - TFrob_{\mathbb{N}(\wp)} | H^i(X_\wp, \mathcal{F})) = det(1 - TFrob_\wp | R^i f_* \mathcal{F})$$

Remember that $R^i f_* \mathcal{F}$ is just the sheaf on U_0 associated to the presheaf $V \mapsto H^i(\mathcal{X} \times_{U_0} V, \mathcal{F})$, and the stalk of $R^i f_* \mathcal{F}$ at a geometric point is just the i-th cohomology group of the fibre over that point.

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Hence for any inclusion of balls $V \subset U_i$, the inclusion $f^{-1}(V) \to f^{-1}(U_i)$ is a homotopy equivalence. This means that the presheaf $U \mapsto H^i(f^{-1}(U), \underline{\mathbb{R}})$ is locally constant, hence so are the $R^i f_* \underline{\mathbb{R}}$.

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Letting $P(T) = det(1 - Frob_q | Ker)^{-1}$, it follows that the zeta function of a hypersurface takes the following shape:

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By rationality, it follows that $P(T) \in \mathbb{Q}[T]$, hence that $det(1 - TFrob_q|H^i(X, \mathbb{Q}_\ell)) \in \mathbb{Q}[T]$ for all i.

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Let $X_0'/\mathbb{F}_p$ be a smooth hypersurface of degree d and dimension n which *does* satisfy RH. Then extending by scalars, $X_0'/\mathbb{F}_q$ also satisfies RH. Let X_0 be defined by F and X_0' by G. Consider the one parameter family $\mathcal{X}$ defined by $tG + (1-t)F$:

$$\begin{tikzcd} \mathcal{X} \arrow[r, hook, "i"] \arrow[dr, "f"'] & \mathbb{P}^{n+1}_{\mathbb{F}_q} \times \mathbb{A}^1_{\mathbb{F}_q} \arrow[d] \\ & \mathbb{A}^1_{\mathbb{F}_q} \end{tikzcd}$$

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By smooth and proper base change, $R^i f_* \mathbb{Q}_\ell$ is a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on U_0. Furthermore, it is ι-real for any ι since

$$det(1 - TFrob_{\wp} | R^i f_* \mathbb{Q}_\ell) = det(1 - TFrob_{\mathbb{N}(\wp)} | H^i(X_{\wp}, \mathbb{Q}_\ell))$$

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and we just proved that the factors in the cohomological formulation of the zeta function of a smooth projective hypersurface are polynomials with rational coefficients.

Letting $\wp_1$ be the prime $t = 1$, we know that $Frob_{\wp_1}|R^n f_* \mathbb{Q}_\ell(n/2)_{\bar{u}_1}$ has eigenvalues of absolute value 1 via any ι, where $R^n f_* \mathbb{Q}(n/2)$ is the $(n/2)$ Tate-twist of $R^n f_* \mathbb{Q}_\ell$.

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Hence $R^n f_* \mathbb{Q}_\ell$ is pure of weight n. This is implies the result.

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where $q = p^k$.

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  where $q = p^k$.

Hence we just need to find enough hypersurfaces which satisfy this equality.

- In the case of $(p, d) = 1$, one can take the so-called Fermat hypersurfaces given by equations of the form $\sum_{i=1}^{n+2} X_i^d$. The fact that they satisfy the equality was proven by Weil in his famous 1949 paper "Number of Solutions of Equations in Finite Fields". That proof is simple and uses elementary properties of Gauss and Jacobi sums.

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- If $d = 2$, then $p = 2$ is the only problematic prime and even this isn't so bad to work out. It depends on whether n is even or odd.
- If $d > 2$ and $p|d$, Katz proves that Gabber's hypersurface $X_1^d + \sum_{i=1}^{n+2} X_i X_{i+1}^{d-1}$ is a smooth model which satisfies RH. The argument given in [Delsarte 1951] is simple, using only elementary manipulations of Gauss and Jacobi sums.

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Definition

Let $N \geq 1$ be an integer, and $W = (w_1, \dots, w_N)$ be an N-tuple of non-negative integers. Write X^W for the monomial $X_1^{w_1} \cdots X_N^{w_N}$. We say that a set of monomials $\{X^{W_\nu}\}_\nu$ is **linearly independent** if the set of integer vectors W_ν are linearly independent in $\mathbb{Q}^N$.

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Examples: Fermat hypersurfaces, Gabber's hypersurface.

We have the following theorem which implies our result ($N := n+2$):

Theorem

Let $N \geq 1$, and let $X^{W_1}, \ldots, X^{W_N}$ be N linearly independent monomials in N variables. Suppose that each variable occurs in at most 2 of these monomials. Then for the affine hypersurface V defined by $\sum_i X^{W_i} = 0$ in $\mathbb{A}^N$, and for various finite fields $\mathbb{F}_q$ we have

$$|V(\mathbb{F}_q)| = q^{N-1} + O(q^{N/2})$$

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Theorem (Delsarte's Theorem)

Let $N > k \geq 0$, and let $\phi : \mathbb{G}_m^N \to \mathbb{G}_m^{N-k}$ be a surjective morphism of split tori. Denote by $\sigma : G^{N-k} \to \mathbb{A}^1$ the function which "sums coordinates". Then for various finite extensions $\mathbb{F}_q/\mathbb{F}_p$ we have the estimate:

$$|\{x \in \mathbb{G}_m^N(\mathbb{F}_q) | \sigma(\phi(x)) = 0\}| = \frac{(q-1)^N}{q} + O(q^{(N+k)/2})$$

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- Reduce to the case $k = 0$.

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$$0 \to \mu(\mathbb{F}_q) \to \mathbb{G}_m^N(\mathbb{F}_q) \xrightarrow{\phi} \mathbb{G}_m^N(\mathbb{F}_q) \to H^1_{fppf}(Spec(\mathbb{F}_q), \mu) \to 0$$

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- Writing $t \in \mathbb{G}_m^N(\mathbb{F}_q) = (\mathbb{F}_q^\times)^N$ as $(t_1, \ldots, t_n)$, we see that

$$|\{t \in (\mathbb{F}_q^\times)^N | \sigma(\phi(t)) = 0\}| = |ker| \cdot |\{t \in (\mathbb{F}_q^\times)^N | \sum t_i = 0, t \in im(\phi)\}|.$$

- Thus we want to count $|ker| \cdot \{t \in (\mathbb{F}_q^\times)^N | \sum t_i = 0, t \in im(\phi)\}$.

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- Identifying characters of *coker* with those of $(\mathbb{F}_q^\times)^N$ vanishing on the image of ϕ, for t in $(\mathbb{F}_q^\times)^N$ we have:

$$\sum_{\chi \in coker^\vee} \chi(t) = \begin{cases} |coker| & \text{if } t \in im(\phi) \\ 0 & else \end{cases}$$

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- Since $|coker| = |ker|$, we derive that

$$|ker| \cdot |\{t \in (\mathbb{F}_q^\times)^N | \sum t_i = 0, t \in im(\phi)\}| = \sum_{\{t | \sum t_i = 0\}} \sum_{\chi \in coker^\vee} \chi(t)$$

- For $t \in (\mathbb{F}_q^\times)^N$, to determine if $\sum t_i = 0$ we choose a non-trivial additive character ψ of $\mathbb{F}_q$ and use that

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- It's now clear how we can use Gauss sums to finish this computation, which is really good since we can say something about their absolute values.

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Not really a problem– it arises as a group scheme from the group algebra of a finite abelian group.

The End