Numerical Computing II

Homework 22: Initial Value Problems for Ordinary Differential Equations

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Exercise 22.2

To show that the Backwards Euler's Method is $O(h)$, we begin by showing the following:

$$\begin{aligned} u_n &= (\frac{1}{1-\lambda h})^n u_0 \\ &= (1+\lambda h + O(h^2))^n u_0 \\ e^{\lambda n h + nO(h^2)} u_0 &= (1+\lambda h + O(h^2))^n u_0. \end{aligned}$$

So we have equality with $O(h^2)$ between u_n and $e^{n\lambda h + nO(h^2)}$. Use this equality to solve the global error. Note that $t_n - t_0 = nh$.

$$\begin{aligned} u_n - u(t_n) &= u_n - u_0 e^{\lambda(t_n - t_0)} \\ &= u_n - u_0 e^{\lambda n h} \\ &= u_0 e^{\lambda n h + nO(h^2)} - u_0 e^{\lambda n h}. \end{aligned}$$

We know that $nh = t_n - t_0$ is a constant, so $nO(h^2) = O(nh^2) = (t_n - t_0)O(h) = O(h)$. Then,

$$e^{nO(h^2)} = e^{O(h)} = 1 + O(h).$$

Substituting this in for our global error yields:

$$\begin{aligned} u_n - u(t_n) &= u_0 e^{\lambda n h} e^{nO(h^2)} - u_0 e^{\lambda n h} \\ &= u_0 e^{\lambda n h} e^{O(h)} - u_0 e^{\lambda n h} \\ &= u_0 (1 + O(h)) e^{\lambda n h} - u_0 e^{\lambda n h} \\ &= u_0 e^{\lambda n h} + u_0 O(h) - u_0 e^{\lambda n h} \\ &= u_0 O(h) \\ &= O(h) \end{aligned}$$

□

Exercise 22.3

We need to explicitly solve for u_1 in the trapezoidal method $u_1 = 1/2(u_0 + hf(t_0, u_0) + u_1 + hf(t_1, u_1))$. Multiplying this out and solving u_1 gives us:

$$\begin{aligned}
u_1 &= u_0 + \frac{h}{2}(f(t_0,u_0) + f(t_1,u_1)) \\
&= u_0 + \frac{h}{2}(\lambda u_0 + \lambda u_1) \\
&= u_0 + \frac{h\lambda}{2}u_0 + \frac{h\lambda}{2}u_1 \\
u_1 - \frac{h\lambda}{2}u_1 &= u_0 + \frac{h\lambda}{2}u_0 \\
u_1(1 - \frac{h\lambda}{2}) &= u_0(1 + \frac{h\lambda}{2}) \\
u_1 &= u_0\frac{1+\frac{h\lambda}{2}}{1-\frac{h\lambda}{2}}.
\end{aligned}$$

Now, we need to manipulate u_1 by solving it as a geometric series:

$$\begin{aligned}
u_1 &= u_0(1 + \frac{h\lambda}{2})(\frac{1}{1-h\lambda}) \\
&= u_0(1 + \frac{h\lambda}{2})(1 + \frac{h\lambda}{2} + (\frac{h\lambda}{2})^2 + (\frac{h\lambda}{2})^3 + O(h^4)) \\
&= u_0(1 + h\lambda + \frac{(h\lambda)^2}{2} + \frac{2(h\lambda)^3}{8} + \frac{(h\lambda)^3}{8} + O(h^4)) \\
&= u_0(1 + h\lambda + \frac{(h\lambda)^2}{2} + \frac{3(h\lambda)^3}{8} + O(h^4)).
\end{aligned}$$

Now, using the exponential expansion:

$$u_0e^{h\lambda} = u_0(1 + h\lambda + \frac{(h\lambda)^2}{2} + \frac{(h\lambda)^3}{6} + O(h^4))$$

we can finally solve the local error:

$$\begin{aligned}
u_1 - u(t_1) =&u_0(1 + h\lambda + \frac{(h\lambda)^2}{2} + \frac{3(h\lambda)^3}{8} + O(h^4)) - \\
&u_0(1 + h\lambda + \frac{(h\lambda)^2}{2} + \frac{(h\lambda)^3}{6} + O(h^4)) \\
=&O(h^3).
\end{aligned}$$

□

Exercise 22.5

$$u_{j+1} = u_j + h(\alpha u'_{j+1} + \beta u'_j)$$

We use this method with the conditions $u'_l = f(t_l, u_l)$. We begin with the function $u(t) = 1$, which means that $f(t_l, u_l) = 0 \forall t_i, u_i$. Choose $t_0 = 0, t_1 = h$ and then $u(t_0) = 0, u(t_1) = 0$.

$$\begin{aligned}
u'_0 &= f(t_0, u_0) = 0 \\
u'_1 &= f(t_1, u_1) = 0 \\
u_1 &= u_0 + h(\alpha u'_1 + \beta u'_0) \\
&= u_0 + h(\alpha 0 + \beta 0) \\
u_1 &= u_0 = 0.
\end{aligned}$$

Now we try $u(t) = t$, which means that $f(t_l, u_l) = 1 \forall t_i, u_i$. Choose $t_0 = 0, t_1 = h$ and then $u(t_0) = 0, u(t_1) = h$.

$$\begin{aligned}
u'_0 &= f(t_0, u_0) = 1 \\
u'_1 &= f(t_1, u_1) = 1 \\
u_1 &= u_0 + h(\alpha f(t_1, u_1) + \beta f(t_0, u_0)) \\
&= u_0 + h(\alpha 1 + \beta 1) \\
h &= h(\alpha + \beta) \\
1 &= \alpha + \beta
\end{aligned}$$

We have our first equation, then. Now we try $u(t) = t^2$, which means that $f(t_l, u_l) = 2t$. Choose $t_0 = 0, t_1 = h$ and then $u(t_0) = 0, u(t_1) = h^2$.

$$\begin{aligned}
u'_0 &= f(t_0, u_0) = 0 \\
u'_1 &= f(t_1, u_1) = 2h \\
u_1 &= u_0 + h(\alpha u'_1 + \beta u'_0) \\
h^2 &= 0 + h(\alpha 2h + \beta 0) \\
h^2 &= \alpha 2h^2 \\
\frac{1}{2} &= \alpha = \beta.
\end{aligned}$$

Then, try $u(t) = t^3$, which means that $f(t_l, u_l) = 3t^2$. Choose $t_0 = 0, t_1 = h$ and then $u(t_0) = 0, u(t_1) = h^3$.

$$
\begin{aligned}
&u_0' = f(t_0, u_0) = 0 \\
&u_1' = f(t_1, u_1) = 3h^2 \\
&u_1 = u_0 + h(\alpha u_1' + \beta u_0') \\
&h^3 = 0 + h(\alpha 3h^2 + \beta 0) \\
&h^3 = \alpha 3h^3 \\
&\frac{1}{3} = \alpha \implies \beta = \frac{2}{3}.
\end{aligned}
$$

But this doesn't agree with our other equations. $\alpha = \beta = 1/2$ works for all polynomials up to and including t^2. When we reach the polynomial t^3, we need $\alpha = 1/3$, which conflicts for polynomials of lower degree.

Exercise 22.6

Apply each method to:

$$
\begin{aligned}
u'(t) &= -25u(t) \\
t &\geq t_0 \\
u(t_0) &= u_0.
\end{aligned}
$$

Trapezoidal Method

The trapezoidal method is given by:

$$u_{j+1} = u_j + \frac{h}{2}(f(t_j, u_j) + f(t_{j+1}, u_{j+1})).$$

We apply this rule for the function $f(t, u) = \lambda u$ and observe:

$$\begin{aligned}
u_{j+1} &= u_j + \frac{h}{2}(f(t_j, u_j) + f(t_{j+1}, u_{j+1})) \\
&= u_j + \frac{h}{2}(\lambda u_j + \lambda u_{j+1}) \\
&= u_j + \frac{h}{2}\lambda u_j + \frac{h}{2}\lambda u_{j+1}) \\
u_{j+1} - \frac{h}{2}\lambda u_{j+1} &= u_j + \frac{h}{2}\lambda u_j \\
u_{j+1}(1 - \frac{h}{2}\lambda) &= u_j + \frac{h}{2}\lambda u_j \\
u_{j+1} &= u_j \frac{1 + \frac{h}{2}\lambda}{1 - \frac{h}{2}\lambda} \\
u_{j+1} &= u_0 (\frac{1 + \frac{h}{2}\lambda}{1 - \frac{h}{2}\lambda})^j.
\end{aligned}$$

This converges if and only if

$$\frac{1 + \frac{h}{2}\lambda}{1 - \frac{h}{2}\lambda} < 1. \tag{1}$$

Since $h > 0$ and $\lambda < 0$, (1) is equivalent to

$$\begin{aligned}
1 + \frac{h}{2}\lambda < 1 - \frac{h}{2}\lambda \\
h\lambda < 0 \\
-25h < 0.
\end{aligned}$$

In this case, it works for all positive h.

Heun's Method

Heun's method is given by:

$$u_{j+1} = u_j + \frac{h}{2}(f(t_j, u_j) + f(t_{j+1}, \hat{u_{j+1}})).$$

Note that $\hat{u_{j+1}}$ is taken from Euler's method

$$\hat{u_{j+1}} = u_j + hf(t_j, u_j).$$

Rewriting Heun's method, we get:

$$u_{j+1} = u_j + \frac{h}{2}(f(t_j, u_j) + f(t_{j+1}, u_j + hf(t_j, u_j))).$$

We apply this rule for our function given by $f(t, u) = \lambda u$ and observe:

$$\begin{aligned}
u_{j+1} &= u_j + \frac{h}{2}(f(t_j, u_j) + f(t_{j+1}, u_j + hf(t_j, u_j))) \\
&= u_j + \frac{h}{2}(f(t_j, u_j) + f(t_{j+1}, u_j + h\lambda u_j)) \\
&= u_j + \frac{h}{2}(\lambda u_j + \lambda(u_j + h\lambda u_j)) \\
&= u_j + \frac{h}{2}(2\lambda u_j + h\lambda^2 u_j)) \\
&= u_j + h\lambda u_j + \frac{h^2}{2}\lambda^2 u_j)) \\
&= u_j(1 + h\lambda + \frac{h^2}{2}\lambda^2) \\
&= u_0(1 + h\lambda + \frac{h^2}{2}\lambda^2)^j.
\end{aligned}$$

This converges if and only if

$$\begin{aligned}
1 + h\lambda + \frac{h^2}{2}\lambda^2 &< 1 \\
2h\lambda + h^2\lambda^2 &< 0 \\
h\lambda(2 + h\lambda) &< 0.
\end{aligned}$$

Since $h\lambda < 0$, we need $2 + h\lambda > 0$ to satisfy the inequality. Then,

$$\begin{aligned}
2 + h\lambda &> 0 \\
h\lambda &> -2 \\
h &< \frac{2}{25}.
\end{aligned}$$

We get this from $\lambda = -25$. So, when $h < \frac{2}{25}$, Heun's method converges in this case.

Example 22.7: Explicit Multistep Method

The Explicit Multistep Method from Example 22.7 is given by

$$u_{j+1} = u_j + h(\alpha u'_j + \beta u'_{j-1}).$$

Use the fact that $u'_l = f(t_l, u_l) = \lambda u_l$, and apply this rule for $f(t, u) = \lambda u$.

$$\begin{aligned} u_{j+1} &= u_j + h(\alpha u'_j + \beta u'_{j-1}) \\ &= u_j + h(\alpha\lambda u_j + \beta\lambda u_{j-1}) \\ &= u_j + h\alpha\lambda u_j + h\beta\lambda u_{j-1}. \end{aligned}$$

We can't solve explicitly for this function since it is implicit, but we can use our knowledge of the values of u_j and u_{j-1} to show that since $\lambda = -25$, we have a decreasing function. Therefore, $u_j < u_{j-1}$. Now, we have an upper bound on our u_{j+1} value.

$$\begin{aligned} u_{j+1} &= u_j + h\alpha\lambda u_j + h\beta\lambda u_{j-1} \\ &< u_{j-1}(1 + h\alpha\lambda + h\beta\lambda). \end{aligned}$$

This inequality is equal to

$$u_0(1 + h\alpha\lambda + h\beta\lambda)^{j-1}$$

which converges if and only if

$$\begin{aligned} 1 + h\alpha\lambda + h\beta\lambda) &< 1 \\ h\alpha\lambda + h\beta\lambda) &< 0 \\ h\lambda(\alpha + \beta) &< 0 \\ h(\alpha + \beta) &> 0. \end{aligned}$$

Notice the inequality was flipped since $\lambda = -25$. So this converges if h is positive and the sum of the coefficients $\alpha + \beta$ is positive.

Conclussions

The only method which required h to be small was Heun's Method. Both the Trapezoidal Method and the Explicit Multistep Method from Example 22.7 converged for all positive values of h (so long as the sum of the coefficients $\alpha + \beta$ was positive for the Explicit Multistep Method). Therefore, both the Trapezoidal Method and the Explicit Multistep Method allowed the largest step length by not setting any bounds on the size of h.

Exercise 22.8

$$u'' - \epsilon(1 - u^2)u' + u = 0 \tag{2}$$

We begin by setting these up into a system of linear ordinary differential equations. Let $v = u'$, making $v' = u''$. Then,

$$v' = \epsilon(1 - u^2)v - u.$$

We can build the following system of equations based on (**??**) and **??**:

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} v \\ \epsilon(1-u^2)v - u \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & \epsilon(1-u^2) \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

We can use the `lsode()` solver in Octave to solve the following function with $\epsilon = 1$.

```
function ydot = f228(y,t)
    eps = .01;
    ydot = [0, 1; -1, eps*(1-y(1)^2)];
    ydot = ydot*y;
end
```

Run the following commands.

```
> y0 = [1/2 1/2]';
> t  = linspace(0,25,500);
> y  = lsode("f", y0, t);
```

This makes a pretty nice looking graph.

Figure 1: Graph of (2) with $\epsilon = 1$

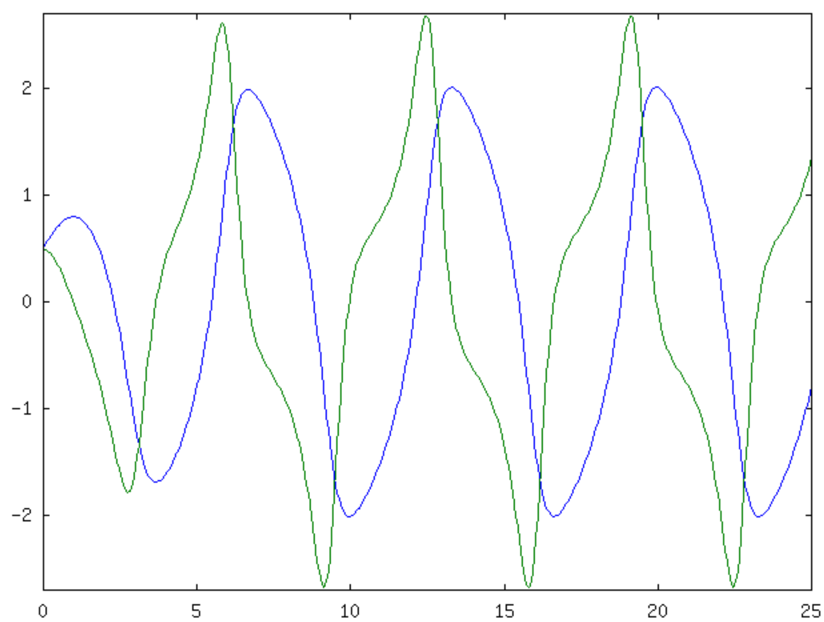

Figure 2: Graph of (2) with $\epsilon = .1$

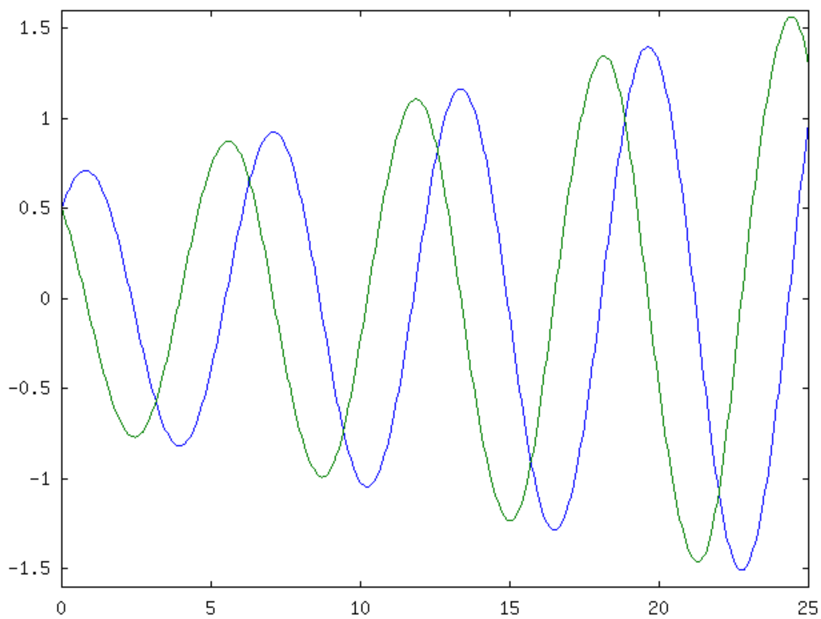

Figure 3: Graph of (2) with $\epsilon = .01$