Numerical Computing II

Homework 20: Properties of Eigenvalues and Eigenvectors

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Exercise 20.1

To show that an $n \times n$ matrix A with n distinct eigenvectors has n linearly independent eigenvectors, we assume the contrary. That is, assume that some eigenvector of A, v_i can be represented as a linear combination of the other eigenvectors. Then,

$$\begin{aligned} 0 &= c_1 v_1 + \cdots + c_i v_i + \cdots + c_n v_n \\ &= A(c_1 v_1 + \cdots + c_i v_i + \cdots + c_n v_n) \\ &= c_1 A v_1 + \cdots + c_i A v_i + \cdots + c_n A v_n \\ &= c_1 \lambda_1 v_1 + \cdots + c_i \lambda_i + \cdots + c_n \lambda_n v_n. \end{aligned}$$

We multiply the first line here with λ_i to obtain

$$0 = c_1 \lambda_i v_1 + \cdots + c_i \lambda_i v_i + \cdots + c_n \lambda_i v_n.$$

Then, we can subtract this equation from the second one to remove the $c_i \lambda_i v_i$ term completely and maintain a constant 0 as the sum.

$$\begin{aligned} 0 &= c_1(\lambda_1 - \lambda_i) v_1 + \cdots + c_i(\lambda_i - \lambda_i) v_i + \cdots + c_n(\lambda_n - \lambda_i) v_n. \\ &= c_1(\lambda_1 - \lambda_i) v_1 + \cdots + 0 + \cdots + c_n(\lambda_n - \lambda_i) v_n. \end{aligned}$$

But each $\lambda_j \neq \lambda_i$, so they cannot evaluate to 0. $\Rightarrow\Leftarrow$

Exercise 20.2

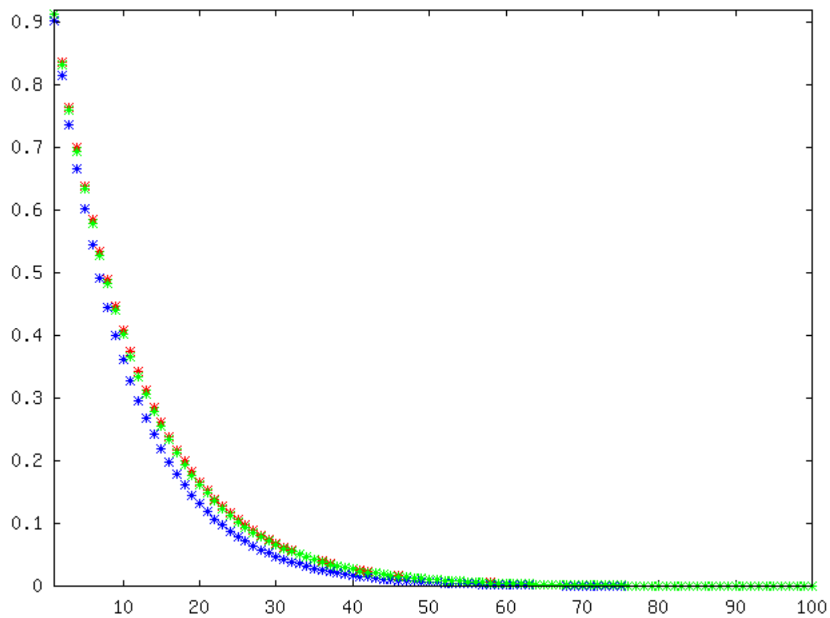
A matrix A can be expressed purely in terms of its eigenvalues and eigenvectors

$$A = V \Lambda V^{-1}$$

where V is the matrix composed of A's linearly independent eigenvectors and Λ is a diagonal matrix of A's eigenvalues. V is necessarily orthogonal; that is,

$V^T = V^{-1}$. What we did in this problem was construct an orthogonal matrix of "made up" eigenvectors and a diagonal matrix of "made up" eigenvalues and then constructed the matrix A using these. We went "backwards" in the process and then gave the matrix A to Octave and had it determine the eigenvalues using the `eig` command.

If we plot the solutions of the system of ordinary differential equations using `expm(A*t)` (since the `exp(A*t)` will give divergent solutions, I figured it was a typo...), we get this graph:



Exercise 20.4

The Gershgorin disks can be shown as

$$l_i = \sum_{k=1, k\neq j}^{n} |a_{jk}|,$$

where l_i is the size of the disk for the i'th eigenvalue.

$$\begin{array}{rcccl} l_1 & = & .1 + .1 & = & .2 \\ l_2 & = & .01 + .01 & = & .02 \\ l_3 & = & 0 + .1 & = & .1 \end{array}$$

Then, we know our first eigenvalue, λ_1 is within l_1 of 1. $\lambda_1 \in [.8, 1.2]$ Similarly, $\lambda_2 \in [.48, .52]$, and $\lambda_3 \in [.09, .11]$. If 0 is not an eigenvalue, then the matrix is nonsingular. We note that none of the Gershgorin disks include 0, so none of the eigenvalues are 0, meaning that the matrix must be nonsingular.

Exercise 20.5

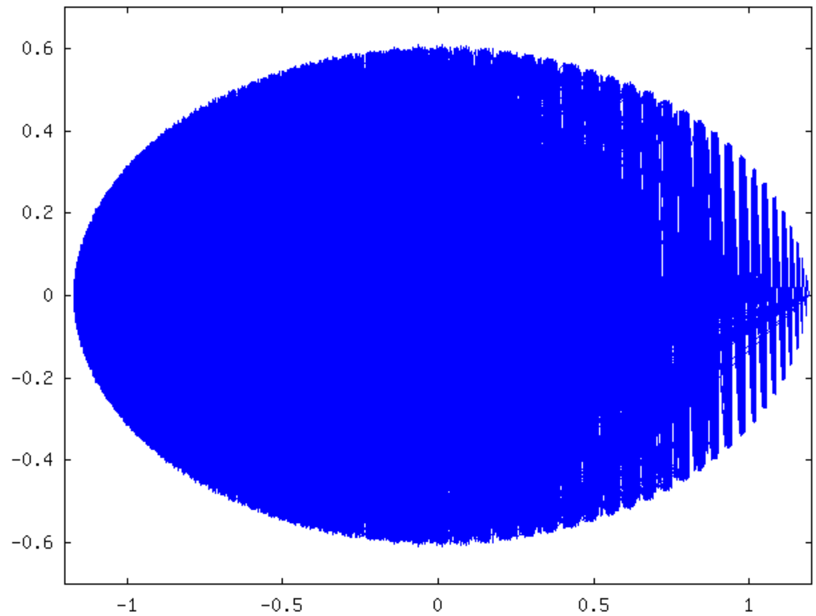
Similarly, we can build disks D_i, where $\lambda_i \in D_i$:

$$\begin{array}{rcccc} D_1 & = & [.1 - 1, .1 + 1] & = & [-.9, 1.1] \\ D_2 & = & [.1 - 1, .1 + 1] & = & [-.9, 1.1] \\ D_3 & = & [.1 - 1 \cdot 10^{-4}, .1 + 1 \cdot 10^{-4}] & = & [.0999, .1001] \end{array}$$

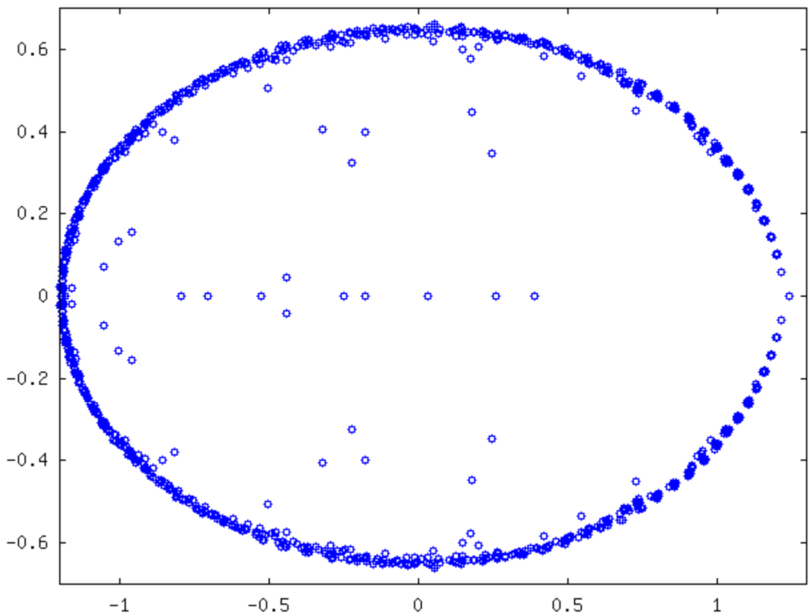
We have overlapping disks, and most are pretty big, so the eigenvalues of this matrix are sensitive to small perturbations.

Exercise 20.6

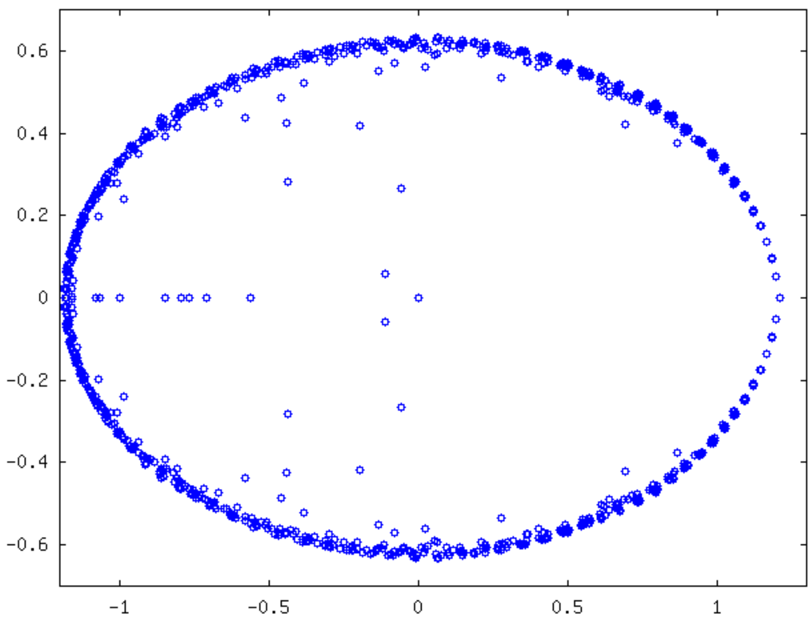
The Gershgorin disks formed by this matrix are identical in size and location with radius 1.25 located about 0. Here is a 'pseudo' Gershgorin disk made with 1000 matrices and $\epsilon = 1 \cdot 10^{-4}$.



Here is the ϵ-pseudospectrum for $\epsilon = 0.01$.



Here is the ϵ-pseudospectrum for $\epsilon = 0.001$.

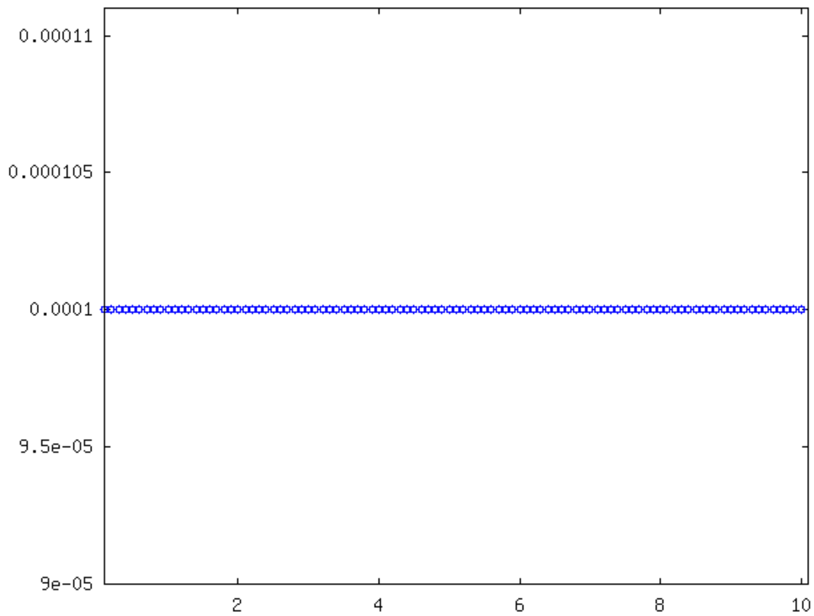


These were made using 10 matrices and the Octave `eig` command.

The condition number of the eigenvalue matrix of A is about $5.3028 \cdot 10^{31}$ according to the `eig` command in Octave.

Exercise 20.7

This is a pretty perfect looking matrix with eigenvalues on the diagonal. Here's the ϵ-pseudospectrum for $\epsilon = 0.01$.



The condition number of the eigenvalue vector of this matrix according to the `eig` command in Octave is 1.