Numerical Computing II

Eigenvalues, ODE's, and Control

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Exercise 19.1

Since A is an upper triangular matrix, we can simply pull the eigenvalues off the diagonal. Then, the characteristic polynomial is

$$p_A(z) = (1-z)(2-z)(3-z).$$

Exercise 19.2

A is not upper triangular, so we need to solve

$$\begin{aligned} p_A(z) &= det(A - zI) \\ &= \begin{vmatrix} 0.5 - z & 1 & 1 \\ 1 & 2 - z & 3 \\ 0 & 0 & 3 - z \end{vmatrix} \\ &= (.5 - z)(2 - z)(3 - z) - (3 - z). \end{aligned}$$

Exercise 19.3

Using

$$\begin{aligned} AV &= VD \\ A &= VDV^{-1}, \end{aligned}$$

we can write A^2 as

$$\begin{aligned} A^2 &= (VDV^{-1})^2 \\ &= (VDV^{-1})(VDV^{-1}). \end{aligned}$$

Recognizing that since $VV^{-1} = I$, we can simplify this expression as

$$\begin{aligned} A^2 &= (VDV^{-1})(VDV^{-1}) \\ &= (VD)(V^{-1}V)(DV^{-1}) \\ &= (VD)(DV) \\ &= VD^2V. \end{aligned}$$

In general, $A^k = VD^kV$ when $k \in \mathbb{N}$.

Then, when $k \in \mathbb{Z}$ and $k < 0$, and A is nonsingular, then A^{-1} exists, and we can say,

$$\begin{aligned} A^{-1} &= (VDV^{-1})^{-1} \\ &= VD^{-1}V^{-1} \\ (A^{-1})^k &= (VD^{-1}V^{-1})^k \\ A^{-k} &= VD^{-k}V^{-1}. \end{aligned}$$

Exercise 19.4

We'll again use the spectral factorization of A:

$$A = VDV^{-1}$$

The main question here is, "Does $A^{1/2} = VD^{1/2}V^{-1}$?" Let's start by defining the n x n matrix M such that $A = MM$. Then,

$$\begin{aligned} Ax &= \lambda x \\ MMx &= \lambda x. \end{aligned}$$

So A and MM have the same eigenvalues and eigenvectors. That means,

$$\begin{aligned} M &= \hat{V}\hat{D}\hat{V}^{-1} \\ MM &= \hat{V}\hat{D}^2\hat{V}^{-1} = VDV^{-1} = A. \end{aligned}$$

Since MM and A have the same eigenvectors and eigenvalues, then $\hat{V} = V$ and $\hat{D}^2 = D$. This means, $MM = VDV^{-1}$. Now, to reduce this equation to just M, we need to consider only the expression $\hat{D}^2 = D$. Since $\hat{D}$ and D are both diagonal matrices, we can simply allow $\hat{D} = D^{1/2}$ and take the square root of each entry of D to form $\hat{D}$. Then, we have

$$\begin{aligned} MM &= V\hat{D}^2V^{-1} \\ M &= VD^{1/2}V^{-1}. \end{aligned}$$

And since $M = A^{1/2}$, we finish with $A^{1/2} = VD^{1/2}V^{-1}$.
$\square$

Exercise 19.5

Let c be some scalar such that $c \in \mathbb{R}$ with $c \neq 0$. We'll start by using the fact that

$$Av = \lambda v.$$

We can create any nonzero multiple of v by simply multiplying it by c. Let $\hat{v}$ be a nonzero multiple of v. That is, let $\hat{v} = cv$ It necessarily follows that

$$\begin{aligned} Av &= \lambda v \\ cAv &= c\lambda v \\ A(cv) &= \lambda(cv) \\ A\hat{v} &= \lambda\hat{v}. \end{aligned}$$

$\square$

Exercise 19.7

If A =diag$[1,1,3]$, then we can pull the eigenvalues of A off the diagonal by using the characteristic equation $p_A(z) = (1-z)(1-z)(1-3)$. Then, to verify that $\lambda = 1, 3$, we can show that $\det(A - \lambda I) = 0)$.

First, we'll verify that when $\lambda = 1$, $\det(A - \lambda I) = 0)$.

$$\begin{aligned} det(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}) \\ = det(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}) \\ = 0. \end{aligned}$$

Next, let's verify that when $\lambda = 3$, $\det(A - \lambda I) = 0$.

$$\begin{aligned} det(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}) \\ = det(\begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}) \\ = 0. \end{aligned}$$

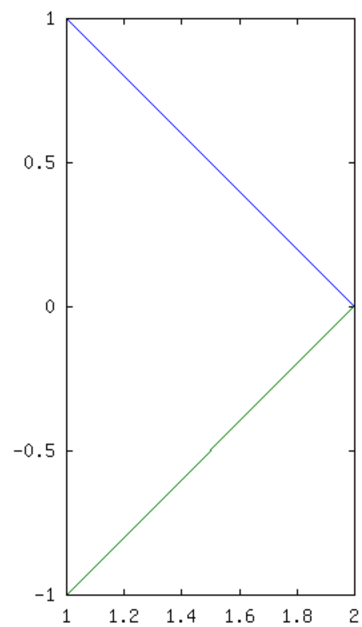
Exercise 19.8

Again, this is an upper triangular matrix, so we can pull the eigenvalues off the diagonal. We see that $\lambda = 1.01, 1$. Using the eig() command from an Octave shell, we can get corresponding eigenvectors from A:

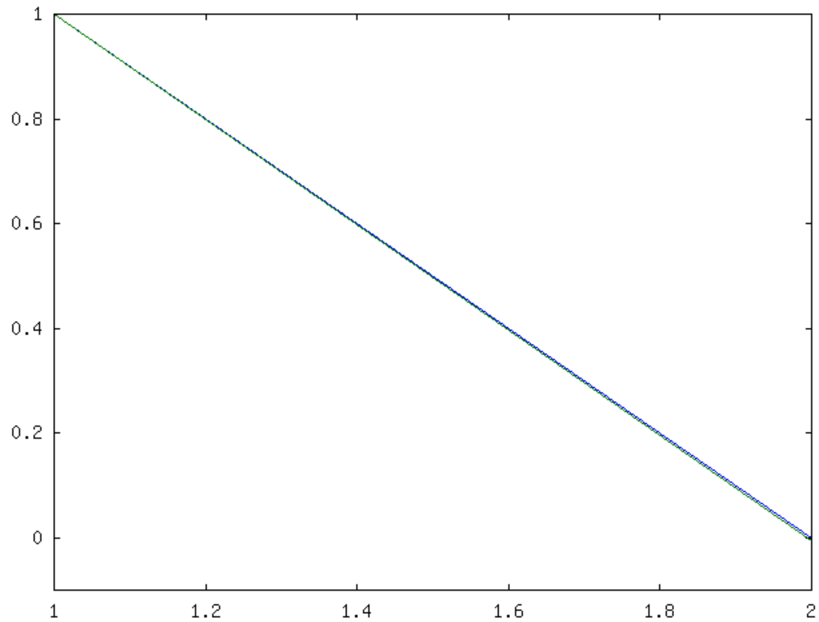
$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$v_2 = \begin{bmatrix} -0.999987500234370 \\ 0.004999937501172 \end{bmatrix}$$

We can see that v_1 and v_2 are "nearly" linearly dependent, but they are, in fact, linearly independent. We can also observe that had the matrix A used the value 1 instead of 1.01, we would see linearly dependent v_1 and v_2.

We also note that v_1 and v_2 are nearly perpendicular. Observe the graph of v_1 and v_2.



Though, if we were to "negate" one of the eigenvectors, we would get a picture of nearly parallel vectors. Let's see what happens when we multiply v_2 by -1.



These are nearly on top of each other. However, if we were to examine very closely, we would see them diverge ever so slightly. They are NOT parallel, but very nearly.

Exercise 19.9

The magic matrix is shaped like:

$$\begin{bmatrix} 16 & 2 & 3 & 13 \\ 5 & 11 & 10 & 8 \\ 9 & 7 & 6 & 12 \\ 4 & 14 & 15 & 1 \end{bmatrix}.$$

Looking at the eigenvectors through the eig() command in Octave, we see $\lambda = 34, \pm 8.94427, 0$. The only curious eigenvalue in the set is 34. We can add up each element of each row and obtain the value 34. Additionally, we can add up each element of each column and obtain the value 34. Clearly, 34 has some significance in this problem.

Let's examine our eigenvectors. These correspond to $\lambda = 34, 8.94427, -8.94427, 0$ in that order.

$$v_1 = \begin{bmatrix} -.5 \\ -.5 \\ -.5 \\ -.5 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} -0.823607 \\ 0.423607 \\ 0.023607 \\ 0.376393 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} 0.376393 \\ 0.023607 \\ 0.423607 \\ -0.823607 \end{bmatrix}, \qquad v_4 = \begin{bmatrix} -0.22361 \\ -0.67082 \\ 0.67082 \\ 0.22361 \end{bmatrix}$$

The eigenvector v_1, which corresponds to the eigenvalue $\lambda = 34$ consists of identical entries. This is because of the identical nature of each row and column adding up to become the same number, which happens to be the λ value corresponding to this eigenvector.

The eigenvectors are all normalized to have a norm of 1. This is probably because of Octave. The eigenvectors could have any discernable norm greater than zero because there are infinite number of them. Octave just chose to normalize them before returning them. Or perhaps this normalization is inherent in the numerical method used to produce said eigenvectors. Regardless, the only "important" eigenvalue to take away from this problem is probably 34.

Exercise 19.10

The $\exp(n)$ function in Octave will compute the value e^n. The $\text{expm}(A)$ function in Octave will compute the exponential of a matrix A.

> `> help exp`
>
> Compute the exponential of X.
>
> `> help expm`
>
> Return the exponential of a matrix, defined as the infinite Taylor series:
>
> $\text{expm}(A) = I + a + a^2/2! + a^3/3! + \cdots$

-From the Octave help command

Exercise 19.12

Solve $\max_{1 \leq j \leq n} \{e^{\lambda_j t}\}$, where $\lambda_j < 0$, $\forall j$. Choose $\lambda^\star = \min_{1 \leq j \leq n} \lambda_j$. We know $\lambda^\star < 0$ by hypothesis.

$$e^{\lambda^\star t} = \frac{1}{e^{-\lambda^\star t}}.$$

Clearly, $-\lambda^\star \geq 0$. Then,

$$\lim_{x \to \infty} \frac{1}{e^{-\lambda^\star t}} = 0.$$

So, as t increases without bound, $e^{\lambda^\star t}$ decreases to zero.

$\square$

Exercise 19.14

Using $\lambda = -1, -1, 0$, we can see that choosing $\lambda^\star = \max_{1 \leq j \leq 3} \lambda_j = 0$. We then bound $||x||$ by $e^{\lambda^\star t} = e^0 t = 1$, which does not depend on t. $||x||$, then, may decrease due to do the other eigenvalues, but will certainly not increase without bound, for it is bounded by a constant value, which is 1. At worst, it will stay constant or oscillate with bounded amplitude $\forall t$.

Exercise 19.17

Since we've defined

$$x(t) := \begin{bmatrix} \hat{h}(t) \\ \frac{d\hat{h}}{dt}(t) \\ \hat{i}(t) \end{bmatrix}$$

we can derive

$$\frac{d}{dt}x(t) := \begin{bmatrix} \frac{d}{dt}\hat{h}(t) \\ \frac{d^2\hat{h}}{dt^2}(t) \\ \frac{d}{dt}\hat{i}(t) \end{bmatrix}.$$

Then, we construct the system of equations which relate $\frac{d}{dt}x(t)$ to $x(t)$. This has been done in the notes, so I'll just use

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & -10 \end{bmatrix}.$$

We can solve for the eigenvalues of A by using the eig() command in Octave, or we can derive them quite simply from the characteristic equation

$$\begin{aligned} p_A(z) &= (-\lambda)(-\lambda)(-10-\lambda) - (-10-\lambda) \\ &= -\lambda^3 - 10\lambda^2 + \lambda + 10. \end{aligned}$$

Using Octave, we obtain eigenvectors v_1, v_2, v_3 corresponding to eigenvalues:

$$\begin{array}{lll} \lambda_1 = -10, & \lambda_2 = -1, & \lambda_3 = 1 \\ v_1 = \begin{bmatrix} -0.01005 \\ 0.10049 \\ 0.99489 \end{bmatrix}, & v_2 = \begin{bmatrix} -0.70711 \\ 0.70711 \\ 0 \end{bmatrix}, & v_3 = \begin{bmatrix} 0.70711 \\ 0.70711 \\ 0 \end{bmatrix}. \end{array}$$

To solve our system of linear equations, we consider the solution to the homogenous system of equations $\frac{d}{dt}x(t) = Ax(t)$. We recognize the solution $x(t) = \eta e^{rt}$ such that $\frac{d}{dt}x(t) = r\eta e^{rt}$, where η is some vector of constants and r is some constant. Then, we solve

$$\begin{aligned} Ax(t) &= \frac{d}{dt}x(t) \\ Ax(t) - \frac{d}{dt}x(t) &= 0 \\ A\eta e^{rt} - r\eta e^{rt} &= 0 \\ (A - rI)\eta e^{rt} &= 0. \end{aligned}$$

We see that solutions to this equation are clearly when r is an eigenvalue and η is the corresponding eigenvalue of A.

Since we know three seperate solutions to the equation $x(t)$:

$$\begin{aligned} x_1(t) &= \eta_1 e^{r_1 t} &= \begin{bmatrix} -0.01005 \\ 0.10049 \\ 0.99489 \end{bmatrix} e^{-10t} \\ x_2(t) &= \eta_2 e^{r_2 t} &= \begin{bmatrix} -0.70711 \\ 0.70711 \\ 0 \end{bmatrix} e^{-t} \\ x_3(t) &= \eta_3 e^{r_3 t} &= \begin{bmatrix} 0.70711 \\ 0.70711 \\ 0 \end{bmatrix} e^{t} \end{aligned}$$

Our general solution is obtained by adding each particular solution and solving for the initial value x_0. We need to be sure the Wronskian of these solutions is not zero.

$$\begin{aligned} W[x_1, x_2, x_3](t) =& \begin{vmatrix} 0 & -0.70711e^{-t} & 0.70711e^{t} \\ 0 & 0.70711e^{-t} & 0.70711e^{t} \\ e^{-10t} & 0 & 0 \end{vmatrix} \\ =&(-0.70711e^{-t})(0.70711e^{t})(e^{-10t}) \\ &-(e^{-10t})(0.70711e^{-t})(0.70711e^{t}) \\ =&-2e^{-10t}(0.70711)^2. \end{aligned}$$

$W[x_1, w_2, w_3](t)$ is never zero $\forall t$, so the solutions x_1, x_2, x_3 form a fundamental set of solutions. We'll use the method of undetermined coeffients to solve this Assume $x(t)$ is of the form $x(t) = ae^{10t} + be^{-t} + ce^{t} + d$ where a, b, c, and d are constants to be determined. Then, $\frac{d}{dt}x(t) = -10ae^{-10t} - be^{-t} + ce^{t}$.

$$\frac{d}{dt}x(t) = Ax(t) \tag{1}$$

$$-10ae^{-10t} - be^{-t} + ce^{t} = A(ae^{-10t} + be^{-t} + ce^{t} + d) \tag{2}$$

We then determine coefficient vectors a, b, c,and d by matching up coefficients from .

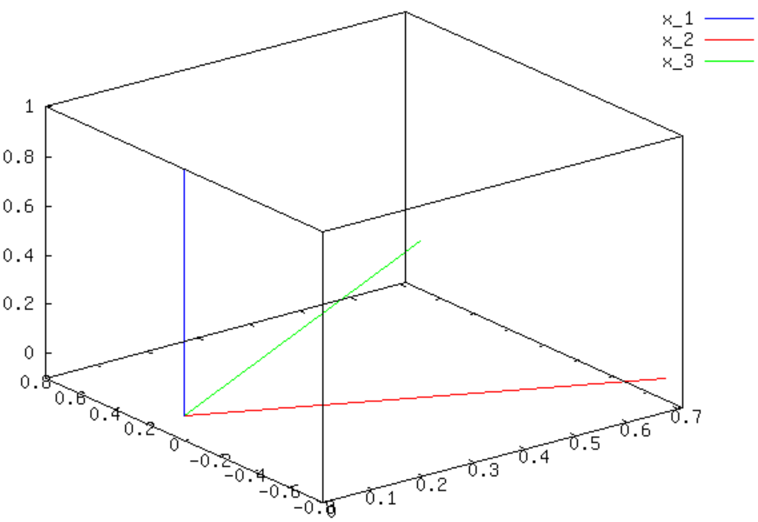
$$\begin{aligned} -10 * a &= Aa \\ -1 * b &= Ab \\ 1 * c &= Ac \\ 0 &= Ad \end{aligned}$$

Each of these solutions is the eigenvalue and eigenvector corresponding to the specific variables. Then $a = x_1$, $b = x_2$, $c = x_3$, and $d = [0, 0, 0]^T$ since there was no zero eigenvalue.

The general solution is determined by adding together the fundamental set of solutions with their appropriate coefficients.

$$\begin{aligned} x(t) =& x_1 + x_2 + x_3 \\ =& \begin{bmatrix} -0.01005 \\ 0.10049 \\ 0.99489 \end{bmatrix} e^{-10t} + \begin{bmatrix} -0.70711 \\ 0.70711 \\ 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 0.70711 \\ 0.70711 \\ 0 \end{bmatrix} e^{t}. \end{aligned}$$

Plotting the solution of our solution yields this 3-D graph.



We can see that from our solutions, as t grows without bound, $x_3 e^t$ will increase to infinity.

We should ook at the solution of $y(t)$, too.

$$y(t) = [1,0,0]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$
$$= x_1$$