Numerical Computing II

Homework 22: Initial Value Problems for Ordinary Differential Equations

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Exercise 22.2

To show that the Backwards Euler's Method is $O(h)$, we begin by showing the following:

$$\begin{aligned} u_n &= (\frac{1}{1-\lambda h})^n u_0 \\ &= (1+\lambda h + O(h^2))^n u_0 \\ e^{\lambda nh + nO(h^2)} u_0 &= (1+\lambda h + O(h^2))^n u_0. \end{aligned}$$

So we have equality with $O(h^2)$ between u_n and $e^{n\lambda h + nO(h^2)}$. Use this equality to solve the global error. Note that $t_n - t_0 = nh$.

$$\begin{aligned} u_n - u(t_n) &= u_n - u_0 e^{\lambda(t_n - t_0)} \\ &= u_n - u_0 e^{\lambda nh} \\ &= u_0 e^{\lambda nh + nO(h^2)} - u_0 e^{\lambda nh}. \end{aligned}$$

We know that $nh = t_n - t_0$ is a constant, so $nO(h^2) = O(nh^2) = (t_n - t_0)O(h) = O(h)$. Then,

$$e^{nO(h^2)} = e^{O(h)} = 1 + O(h).$$

Substituting this in for our global error yields:

$$\begin{aligned} u_n - u(t_n) &= u_0 e^{\lambda nh} e^{nO(h^2)} - u_0 e^{\lambda nh} \\ &= u_0 e^{\lambda nh} e^{O(h)} - u_0 e^{\lambda nh} \\ &= u_0 (1 + O(h)) e^{\lambda nh} - u_0 e^{\lambda nh} \\ &= u_0 e^{\lambda nh} + u_0 O(h) - u_0 e^{\lambda nh} \\ &= u_0 O(h) \\ &= O(h) \end{aligned}$$

□

Exercise 22.3

We need to explicitly solve for u_1 in the trapezoidal method $u_1 = 1/2(u_0 + hf(t_0, u_0) + u_1 + hf(t_1, u_1))$. Multiplying this out and solving u_1 gives us:

$$\begin{aligned}
u_1 &= u_0 + \frac{h}{2}(f(t_0, u_0) + f(t_1, u_1)) \\
&= u_0 + \frac{h}{2}(\lambda u_0 + \lambda u_1) \\
&= u_0 + \frac{h\lambda}{2}u_0 + \frac{h\lambda}{2}u_1 \\
u_1 - \frac{h\lambda}{2}u_1 &= u_0 + \frac{h\lambda}{2}u_0 \\
u_1(1 - \frac{h\lambda}{2}) &= u_0(1 + \frac{h\lambda}{2}) \\
u_1 &= u_0\frac{1 + \frac{h\lambda}{2}}{1 - \frac{h\lambda}{2}}.
\end{aligned}$$

Now, we need to manipulate u_1 by solving it as a geometric series:

$$\begin{aligned}
u_1 &= u_0(1 + \frac{h\lambda}{2})(\frac{1}{1 - h\lambda}) \\
&= u_0(1 + \frac{h\lambda}{2})(1 + \frac{h\lambda}{2} + (\frac{h\lambda}{2})^2 + (\frac{h\lambda}{2})^3 + O(h^4)) \\
&= u_0(1 + h\lambda + \frac{(h\lambda)^2}{2} + \frac{2(h\lambda)^3}{8} + \frac{(h\lambda)^3}{8} + O(h^4)) \\
&= u_0(1 + h\lambda + \frac{(h\lambda)^2}{2} + \frac{3(h\lambda)^3}{8} + O(h^4)).
\end{aligned}$$

Now, using the exponential expansion:

$$u_0e^{h\lambda} = u_0(1 + h\lambda + \frac{(h\lambda)^2}{2} + \frac{(h\lambda)^3}{6} + O(h^4))$$

we can finally solve the local error:

$$\begin{aligned}
u_1 - u(t_1) =&u_0(1 + h\lambda + \frac{(h\lambda)^2}{2} + \frac{3(h\lambda)^3}{8} + O(h^4)) - \\
&u_0(1 + h\lambda + \frac{(h\lambda)^2}{2} + \frac{(h\lambda)^3}{6} + O(h^4)) \\
=&O(h^3).
\end{aligned}$$

□

Exercise 22.5

$$u_{j+1} = u_j + h(\alpha u'_{j+1} + \beta u'_j)$$

We use this method with the conditions $u'_l = f(t_l, u_l)$. We begin with the function $u(t) = 1$, which means that $f(t_l, u_l) = 0 \forall t_i, u_i$. Choose $t_0 = 0, t_1 = h$ and then $u(t_0) = 0, u(t_1) = 0$.

$$\begin{aligned}
u'_0 &= f(t_0, u_0) = 0 \\
u'_1 &= f(t_1, u_1) = 0 \\
u_1 &= u_0 + h(\alpha u'_1 + \beta u'_0) \\
&= u_0 + h(\alpha 0 + \beta 0) \\
u_1 &= u_0 = 0.
\end{aligned}$$

Now we try $u(t) = t$, which means that $f(t_l, u_l) = 1 \forall t_i, u_i$. Choose $t_0 = 0, t_1 = h$ and then $u(t_0) = 0, u(t_1) = h$.

$$\begin{aligned}
u'_0 &= f(t_0, u_0) = 1 \\
u'_1 &= f(t_1, u_1) = 1 \\
u_1 &= u_0 + h(\alpha f(t_1, u_1) + \beta f(t_0, u_0)) \\
&= u_0 + h(\alpha 1 + \beta 1) \\
h &= h(\alpha + \beta) \\
1 &= \alpha + \beta
\end{aligned}$$

We have our first equation, then. Now we try $u(t) = t^2$, which means that $f(t_l, u_l) = 2t$. Choose $t_0 = 0, t_1 = h$ and then $u(t_0) = 0, u(t_1) = h^2$.

$$\begin{aligned}
u'_0 &= f(t_0, u_0) = 0 \\
u'_1 &= f(t_1, u_1) = 2h \\
u_1 &= u_0 + h(\alpha u'_1 + \beta u'_0) \\
h^2 &= 0 + h(\alpha 2h + \beta 0) \\
h^2 &= \alpha 2h^2 \\
\frac{1}{2} &= \alpha = \beta.
\end{aligned}$$

Then, try $u(t) = t^3$, which means that $f(t_l, u_l) = 3t^2$. Choose $t_0 = 0, t_1 = h$ and then $u(t_0) = 0, u(t_1) = h^3$.

$$
\begin{aligned}
u_0' &= f(t_0, u_0) = 0 \\
u_1' &= f(t_1, u_1) = 3h^2 \\
u_1 &= u_0 + h(\alpha u_1' + \beta u_0') \\
h^3 &= 0 + h(\alpha 3h^2 + \beta 0) \\
h^3 &= \alpha 3h^3 \\
\frac{1}{3} &= \alpha \implies \beta = \frac{2}{3}.
\end{aligned}
$$

But this doesn't agree with our other equations. $\alpha = \beta = 1/2$ works for all polynomials up to and including t^2. When we reach the polynomial t^3, we need $\alpha = 1/3$, which conflicts for polynomials of lower degree.

Exercise 22.6

Exercise 22.6

Exercise 22.8