Computer Lab 4

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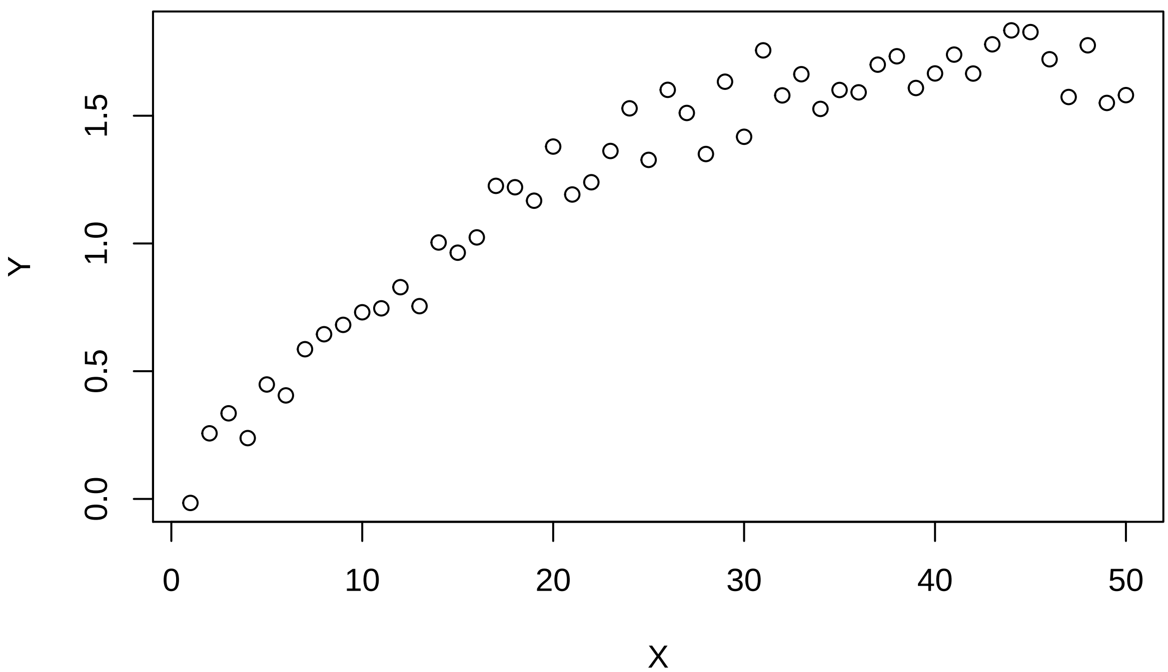
Question 2. Gibbs sampling

1.

```r
load("chemical.RData")  # read csv file
plot(X,Y)
```



The scatter imaginary line traced by the scatter plot resembles a logarithm function. Therefore, a logarithmic model would probably fit the data.

2.

We know

$$Y_i \sim N(\mu_i, 0.2)$$

The likelihood of $p(\vec{Y}|\vec{\mu})$ is the probability of observing our $\vec{Y}$ data given a set of parameters $\vec{\mu}$. It is defined like the product of our probability function all over the observed data:

$$\mathcal{L}(\vec{Y}|\vec{\mu}) = \prod_{i=1}^{n} p(\vec{Y}|\vec{\mu})$$

$$\mathcal{L}(\vec{Y}|\vec{\mu}) = (2\pi\sigma^2)^{-\frac{n}{2}} exp[-\frac{\sum_{i=2}^{n}(y_i - \mu_i)^2}{2\sigma^2}]$$

Our prior probability is defined like the following expression according to the chain rule:

$$p(\vec{\mu}) = (2\pi\sigma^2)^{-\frac{n}{2}} exp[-\frac{\sum_{i=2}^{n}(\mu_i - \mu_{i-1})^2}{2\sigma^2}]p(\mu_1)$$

where: $\sigma = 0.2$ and $p(\mu_1) = 1$

The posterior probability according to Bayes theorem follows the next equation:

$$p(\vec{\mu}|\vec{Y}) \propto \mathcal{L}(\vec{Y}|\vec{\mu}) * p(\vec{\mu})$$

$$p(\vec{\mu}|\vec{Y}) \propto exp[-\frac{\sum_{i=1}^{n}(y_i - \mu_i)^2}{2\sigma^2}] * exp[-\frac{\sum_{i=2}^{n}(\mu_i - \mu_{i-1})^2}{2\sigma^2}]$$

$$p(\vec{\mu}|\vec{Y}) \propto exp[-\frac{(y_1 - \mu_1)^2 + \sum_{i=2}^{n}[(\mu_i - \mu_{i-1})^2 + (y_i - \mu_i)^2]}{2\sigma^2}]$$

Now we will develop a expression for $p(\mu_i|\vec{\mu}_{-i})$ first splitting between $p(\mu_1|\vec{\mu}_{-1,},$ $p(\mu_n|\vec{\mu}_{-n,}, \vec{Y})$ and the middle points. We leverage the property of conditional probability assuming independent events:

$$p(A|B) = \frac{p(A) \cap (B)}{p(B)} = \frac{p(A)(B)}{p(B)}$$

$$p(\mu_1|\vec{\mu}_{-1,}, \vec{Y}) \propto exp[-\frac{1}{2\sigma^2}[(\mu_1 - \mu_2)^2 + (\mu_1 - y_1)^2]]$$

$$p(\mu_1|\vec{\mu}_{-1,}, \vec{Y}) \propto exp[-\frac{1}{\sigma^2}[\mu_1 - \frac{\mu_2 + y_1}{2}]^2] \sim N(\frac{\mu_2 + y_1}{2}, \frac{\sigma^2}{2})$$

Let the procedure for the last point be applied:

$$p(\mu_n|\vec{\mu}_{-n,}, \vec{Y}) \propto exp[-\frac{1}{2\sigma^2}[(\mu_{n-1} - \mu_n)^2 + (\mu_n - y_n)^2]]$$

$$p(\mu_n|\vec{\mu}_{-n}, \vec{Y}) \propto exp[-\frac{1}{\sigma^2}[\mu_n - \frac{\mu_{n-1} + y_n}{2}]^2] \sim N(\frac{\mu_{n-1} + y_n}{2}, \frac{\sigma^2}{2})$$

Finally, for the middle points:

$$p(\mu_i|\vec{\mu}_{-i}, \vec{Y}) \propto exp[-\frac{1}{2\sigma^2}[(\mu_{i-1} - \mu_i)^2 + (\mu_i - \mu_{i+1})^2 + (\mu_i - y_i)^2]$$

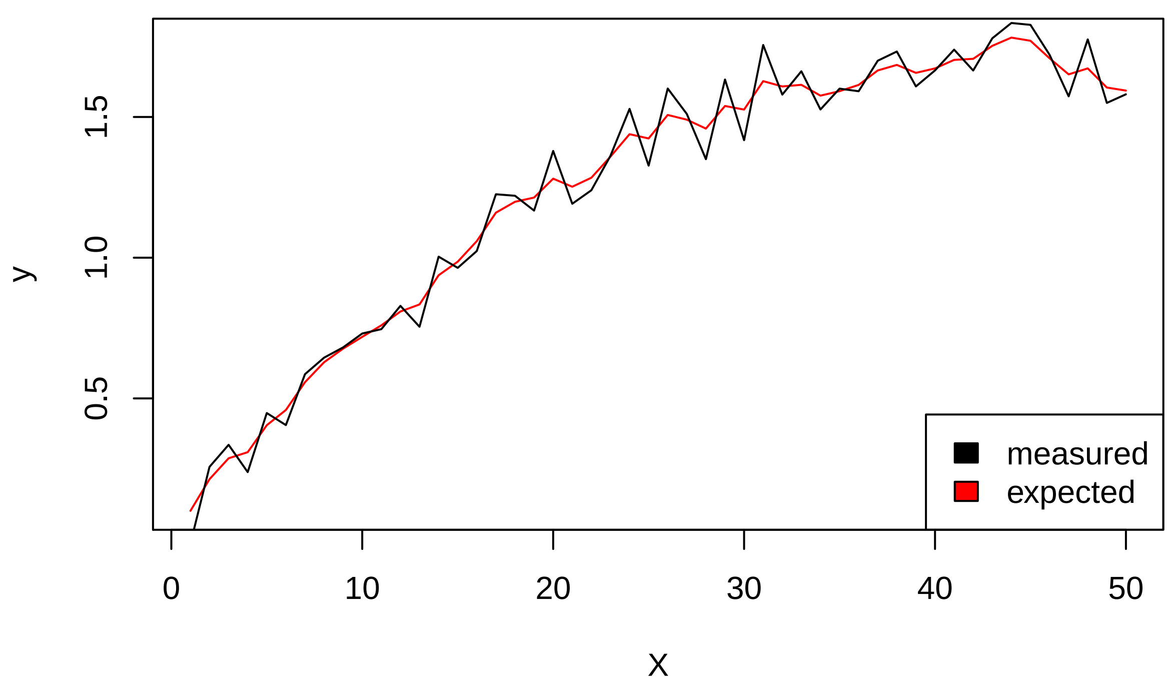
$$p(\mu_i|\vec{\mu}_{-i}, \vec{Y}) \propto exp[-\frac{3}{2\sigma^2}[\mu_i - \frac{\mu_{i-1} + \mu_{i+1} + y_i}{3}]^2] \sim N(\frac{\mu_{i-1} + \mu_{i+1} + y_i}{3}, \frac{\sigma^2}{3})$$

4.

```r
GibsSampler <- function(n,k){
  initx <- as.data.frame(t(rep(0,k)))
  for (i in 2:n) {
    mu <- unlist(initx[i-1, ])
    mu[1] <- rnorm(1,(mu[2]+Y[1])/2,0.2/2)
    for (j in 2:(k-1)) {
      mu[j] <- rnorm(1,(mu[j-1]+mu[j+1]+Y[j])/3,0.2/3)
    }
    mu[k] <- rnorm(1,(mu[k-1]+Y[k])/2,0.2/2)
    initx <- rbind(initx,mu)
  }
  return(initx)
}
```

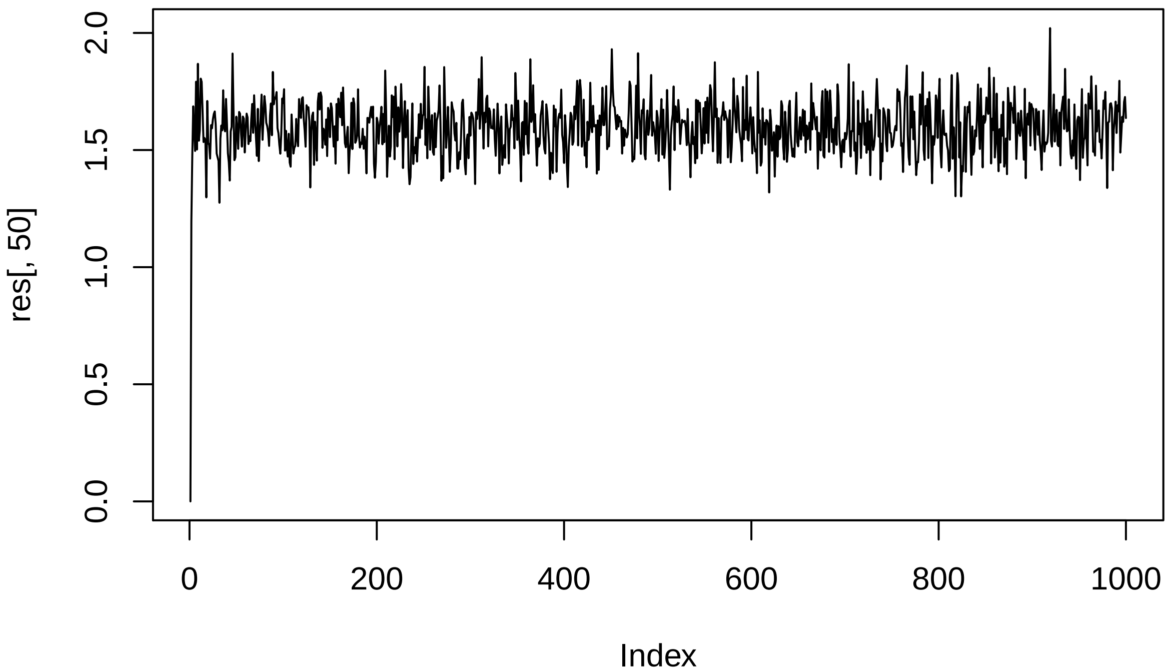
```r
res <- GibsSampler(1000, length(X))

expectedmean <- unname(unlist(colSums(res/nrow(res))))
```

```r
plot(X,expectedmean,type = "l", col="red",ylab = "y")
lines(X,Y)
legend("bottomright", c("measured", "expected"), fill=c("black", "red"))
```



Using this method it looks like the expected values plotted smooth the noise compared to the observed ones, leading us to observe with more clarity the correlation variables have underlying.

```r
plot(res[, 50], type="l")
```



The plot does not present a clear burn out period since it goes directly from value 0 to the span it begins oscillating around. Then, convergence is not notoriously achieved as values of mu fluctuate around 1.5.