Computer Lab 3

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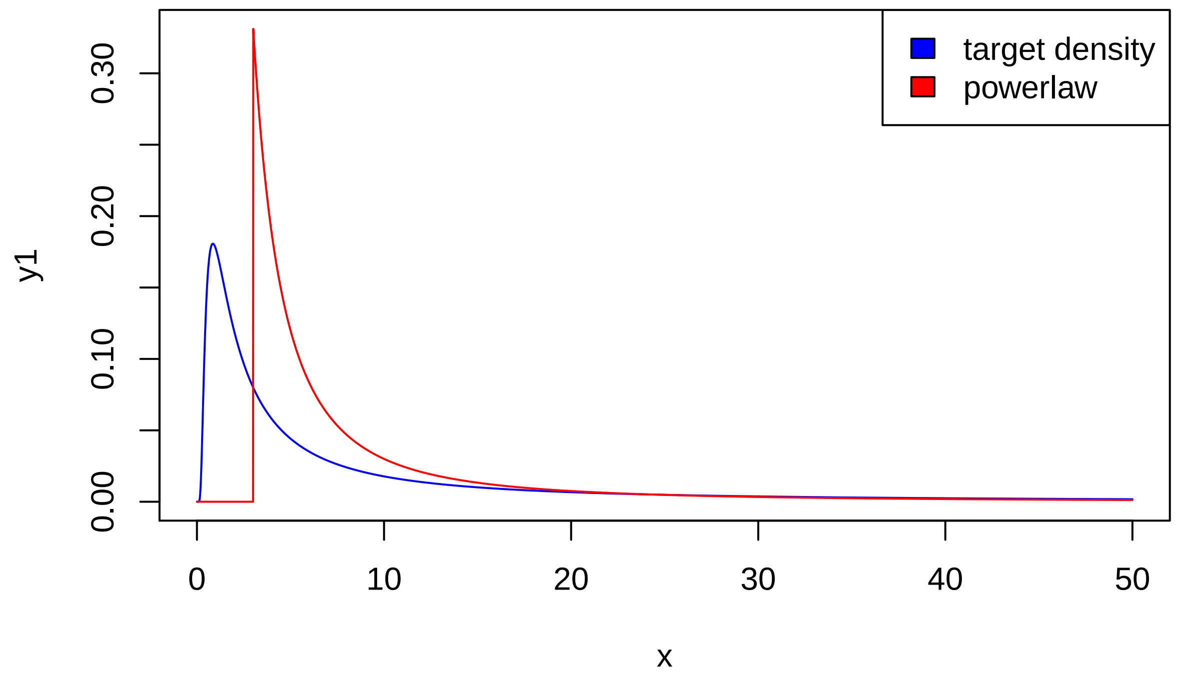
Question 1

```
distrib1 <- function(x,c){
  if(x>0){
    return(c*(sqrt(2*pi)^(-1))*(exp(-(c^2)/(2*x))) * (x^(-3/2)))
  }
  else return(0)
}

powerlaw <- function(x,a,t){
  if(x>t){
    return(((a-1)/t) * ((x/t)^(-a)))
  }
  else return(0)

}

x <- seq(0,50, by=0.01)
c <- 1.6
#n <- length(x)
#a= 1 + n*(sum(log(x/t)))^(-1)
y1 <- sapply(x, distrib1, c=c)
y2 <- sapply(x, powerlaw, a = 2,t = 3)
ymax <- max(y1,y2)
b <- min(c(y1,y2))
e <- max(c(y1,y2))
ax <- seq(b,e,by=(e-b)/200)

plot(x,y1, type="l", col = "blue", ylim = c(0,ymax))
lines(x,y2, col = "red")
legend("topright", c("target density", "powerlaw"), fill=c("blue", "red"))
```



```
# hist(y1, breaks = ax,
#      col = "red",
#      main = "Comparison of rnorm()  with our rNorm()",
#      xlab = "values",
#      xlim = range(y1,y2))
#
# hist(y2, breaks = ax, col = "blue",  xlim =range(y1,y2), add = TRUE)
```

Power-law distribution cannot be used just by itself because it does not support range from 0 to T_{min}. Because of this, we need to use mixture distribution. To support x from 0 to T_{min} we choose uniform distribution $Unif(0, T_{min})$. As we can see, powerlaw distribution is monotonically decreasing, so T_{min} should be equal to x where target density has maximum value. Aside from this, we will fix $\alpha = 2$ to ensure the powerlaw function goes pretty much over the target function for the most of its support. Afterwards, the majorizing constant will ensure it gets above of the target.

Let's find maximum of target density function:

$$\frac{\partial}{\partial x} \frac{ce^{-\frac{c^2}{2x}} x^{-\frac{3}{2}}}{\sqrt{2\pi}} = \frac{ce^{-\frac{c^2}{2x}} \left(c^2 - 3x\right)}{2\sqrt{2\pi} x^{7/2}}$$

$$\frac{c\frac{e^{-c^2}}{2x} \left(c^2 - 3x\right)}{2\sqrt{2\pi} x^{7/2}} = 0$$

$$x = \frac{c^2}{3}$$

$$T_{min} = \frac{c^2}{3}$$

To make a mixture model we need to know the probability of taking uniform distribution and powerlaw distribution probability. That number will be in the range $[0, T_{min}]$:

$$\int_0^{T_{min}} cx^{-\frac{3}{2}} e^{-\frac{c^2}{2x}} \sqrt{2\pi}^{-1} dx = \frac{\Gamma\left(\frac{1}{2}, \frac{c^2}{2T_{min}}\right)}{\sqrt{\pi}}$$

as $T_{min} = \frac{c^2}{3}$ - $\frac{\Gamma\left(\frac{1}{2}, \frac{c^2}{2T_{min}}\right)}{\sqrt{\pi}} = \frac{\Gamma\left(\frac{1}{2}, \frac{3}{2}\right)}{\sqrt{\pi}} \sim 0.08326451666$

As a result, majorising density function is:

$$g(x) = \frac{2 * 0.08326}{c^2} 1_{[0,T_{min}]} + (1 - 0.08326) * \frac{2^{1-a}(a-1)\left(\frac{x}{c^2}\right)^{-a}}{c^2} * 1_{(T_{min},\infty)}$$

2.

Target density:

$$f(x) = c(\sqrt{2\pi})^{-1} e^{\frac{-c^2}{2x}} x^{\frac{-3}{2}} 1_{(0,\infty)}(x)$$

We need to find c_{maj}

$$c_{maj} > 0; sup_x(f(x)/g(x)) \leq c_{maj}$$

$$h(x) = \frac{f(x)}{g(x)}$$

$c_{maj} = h(x_{maj})$

if x $< T_{min}$

$$\frac{\partial}{\partial x} \frac{f(x)}{\frac{0.16652}{c^2}} = \frac{e^{-\frac{c^2}{2x}}\left(1.19788c^5 - 3.59364c^3 x\right)}{x^{7/2}}$$

$$\frac{e^{-\frac{c^2}{2x}}\left(1.19788c^5 - 3.59364c^3 x\right)}{x^{7/2}} = 0$$

$$x_{maj} = \frac{c^2}{3}$$

x for cmaj $= x = 0.333333c^2 = T_{min}$

if x> Tmin

$$\frac{\partial}{\partial x} \frac{f(x)}{g(x)} = \frac{3^{a-1} c^3 \frac{e^{-c^2}}{2x} \left(\frac{x}{c^2}\right)^a \left((2a-3)x + c^2\right)}{2\sqrt{2\pi}(a-1)x^{7/2}}$$

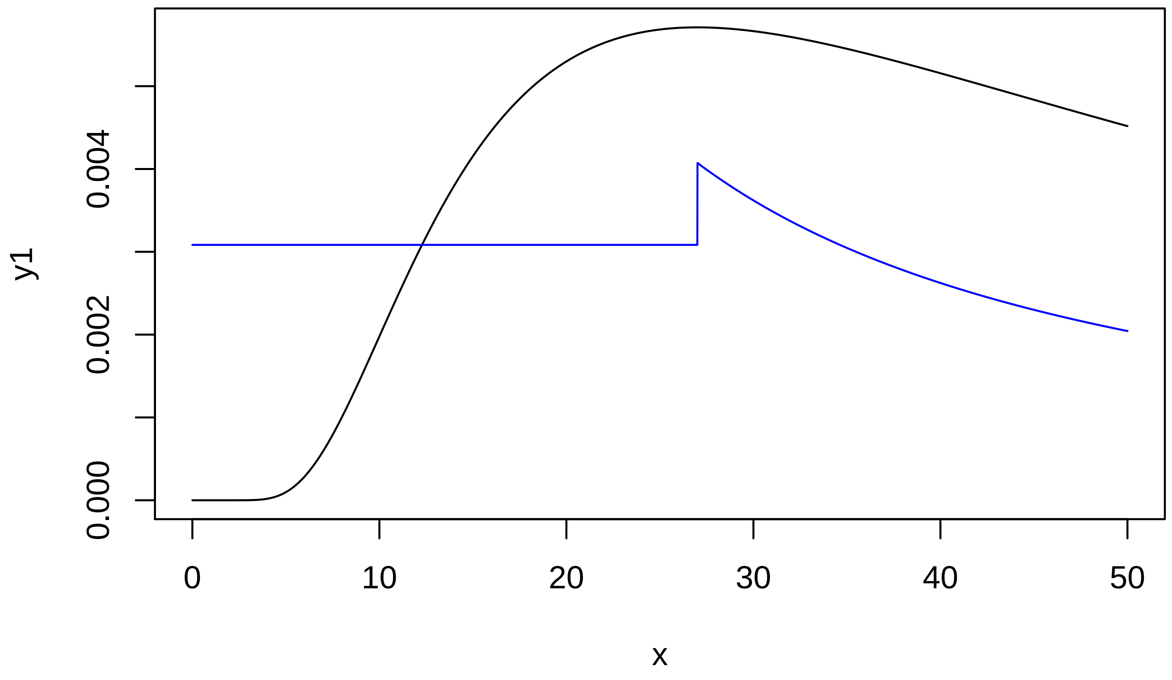
$$\frac{3^{a-1}c^3\,\frac{e^{-c^2}}{2x}\,\left(\frac{x}{c^2}\right)^a\left((2a-3)x+c^2\right)}{2\sqrt{2\pi}(a-1)x^{7/2}}=0$$

$$x_{maj}=\frac{c^2}{3-2a}$$

when $1 < a < 1.5$

$$c_{maj}=h(x_{maj})$$

```r
majDensity <- function(x, c,a){
  Tmin <- (c^2)/3

  if(x>Tmin){
    return((1-0.083264) * powerlaw(x, a, Tmin))
  }
  else {return(0.083264 * dunif(x,0,Tmin))}
}

c <- 9
y3 <- sapply(x, majDensity, c = c, a = 1.12)
y1 <- sapply(x, distrib1, c = c)
plot(x,y1, type = "l", ylim = c(0,max(y3,y1)))
#lines(x,dnorm(x,3,1.2), col ="red")

lines(x,y3, col ="blue")
```

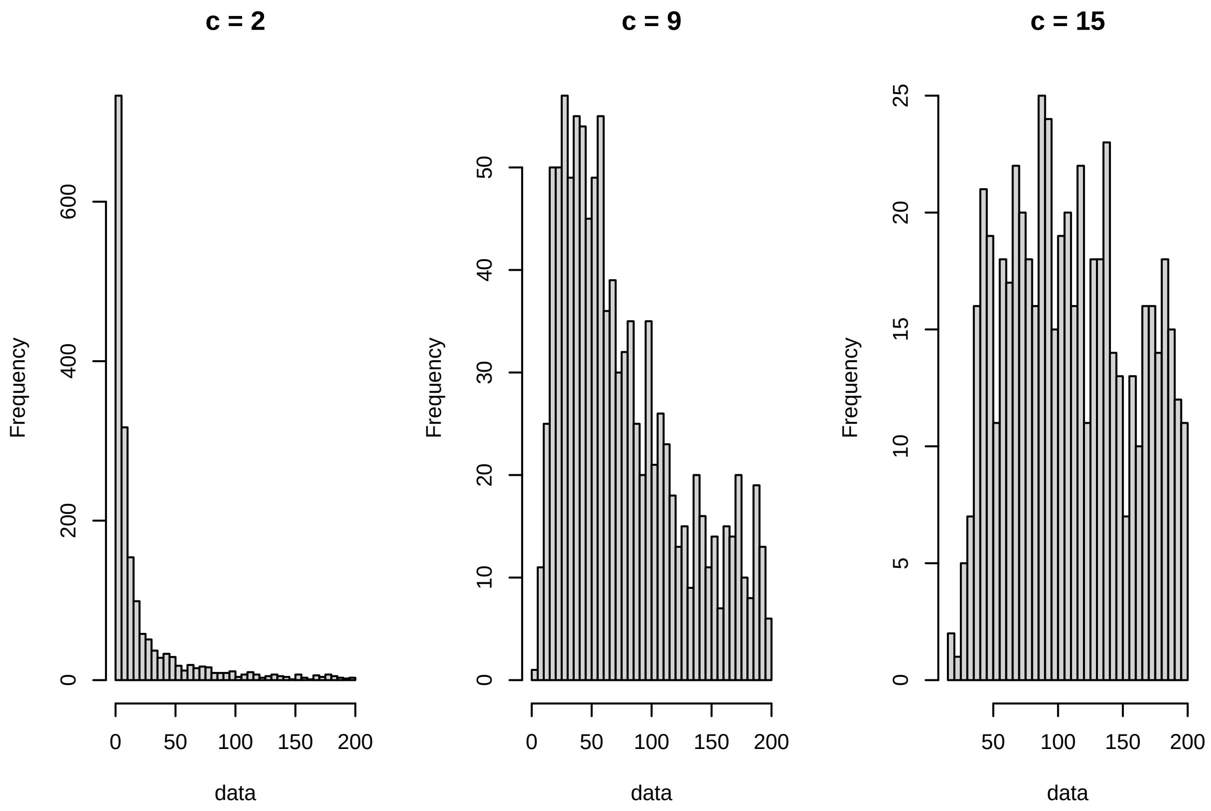


```r
#lines(x,y1, col = "red")
```

2

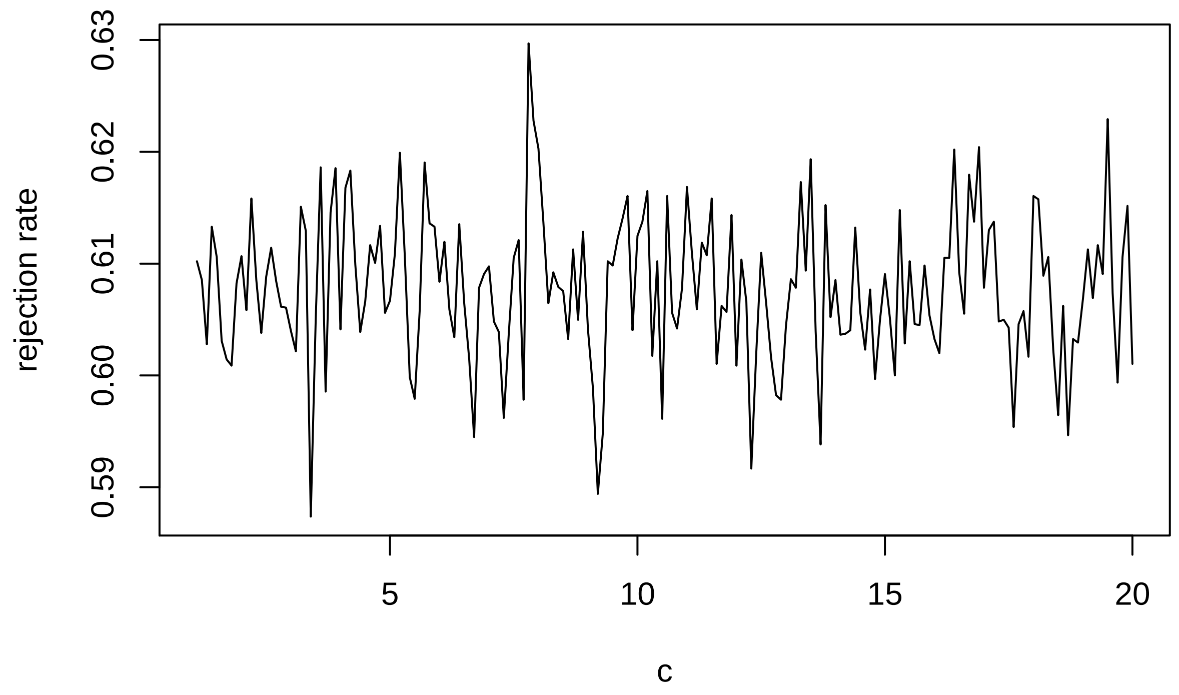
```r
library(poweRlaw)
randomnumber <- function(t,a){
  numb <- runif(1)
  if(numb<= 0.08326){
    return(runif(1,0,t))
  }
  else{
    return(rplcon(1,t,a))
  }
}

CompleteDist <- function(c, a, rej){
  z <- TRUE
  res <- 0

  Tmin <- (c^2)/3
  xmaj <- (c^2)/(3-2*a)
  cmaj <- distrib1(xmaj,c)/majDensity(xmaj,c, a)

  while (z == TRUE) {
    y <- randomnumber(Tmin,a)
    u <- runif(1)
    if(u <= distrib1(y, c) / (cmaj*majDensity(y,c,a))){
      res <- y
      z <- FALSE
    }
    if(rej){
    rejected <<- rejected + 1
    }
  }
  return(res)
}

rDist <- function(n,c,a ,rej = FALSE){
 return(replicate(n, CompleteDist(c, a, rej)))
}
```

3.



	c=2	c=9	c=15
mean	9.913881e+03	5.419880e+04	3.591555e+05
variance	3.997831e+10	1.444985e+12	4.047938e+13

Variance and mean is increase due to c



From plot we can see that rejection rate does not depend on the constant c of the target density.

Question 2

1.

$$DE(\mu, \alpha) = \frac{\alpha}{2} e^{-\alpha|x-\mu|}$$

- μ - location parameter
- $b > 0$ - scale parameter

inverse CDF of DE:

Source - https://en.wikipedia.org/wiki/Laplace_distribution

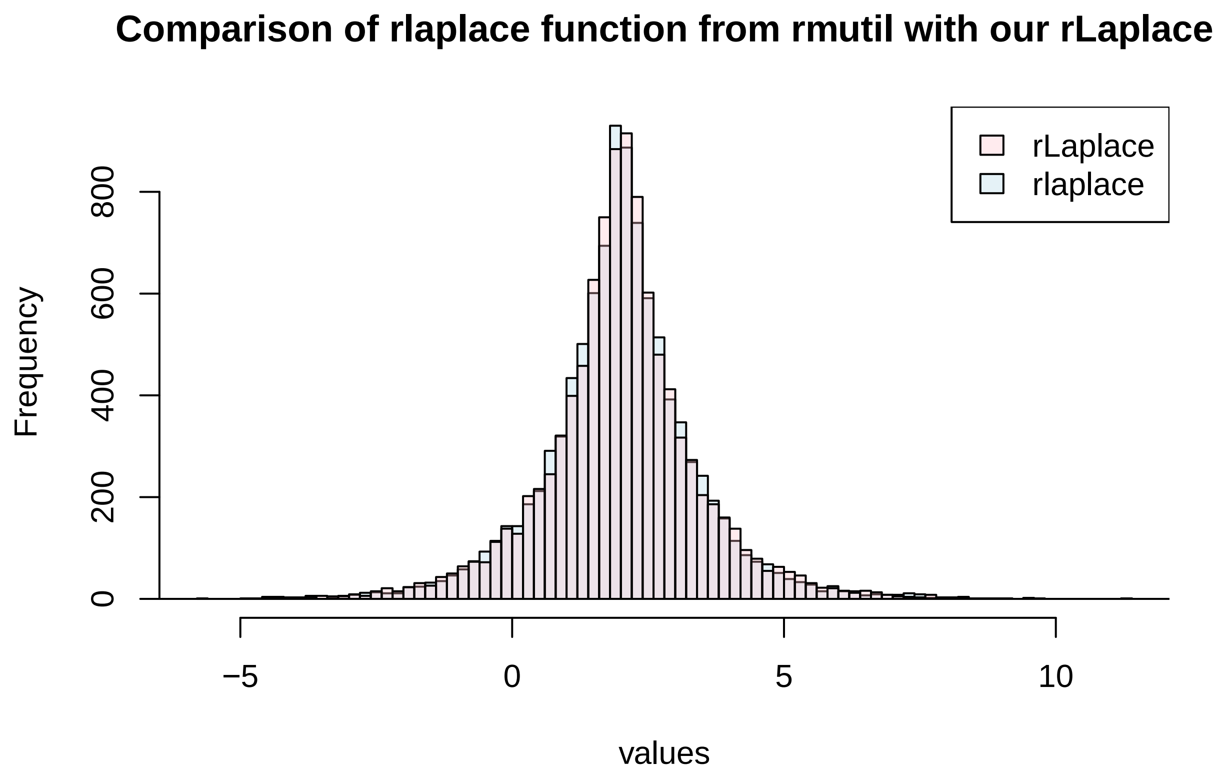
$$F^{-1}(p) = \mu - bsgn(p - 0.5)ln(1 - 2|p - 0.5|)$$

where $b = \frac{1}{\alpha}$

```r
rLaplace <- function(n, mean = 0, alpha = 1){
  b <- 1/alpha
  u <- runif(n)
  res <- mean - (b*sign(u-0.5) * log(1-(2*abs(u-0.5))))
  return(res)
}
```

meaning:

1. calculate b.
2. take n random variables from uniform distribution [0,1]
3. calculate random numbers from inverse CDF of laplace distribution where x is a random variable from uniform distribution



From histogram we can see that inverse CDF method approximated Laplace distribution reasonably, mean and variance similarly, in turn.

2.

```
DE <- function(x, mean = 0,alpha = 1){
  return((0.5*alpha)*exp((-alpha)*abs(x-mean)))
}

genNorm <- function(c, rej){
  z <- TRUE
  res <- 0
  while (z == TRUE) {
    y <- rLaplace(1)
    u <- runif(1)
    if(u <= pnorm(y) / (c*DE(y))){
      res <- y
      z <- FALSE
    }
    if(rej){
    rejected <<- rejected + 1
    }
  }
  return(res)
}

rNorm <- function(n,c,rej = FALSE){
 return(replicate(n, genNorm(c, rej)))
}
```

algorithm:

1. write Laplace probability function
2. assign 0 to result value res
3. generate random number y from rLaplace function
4. generate random number u from uniform distribution
5. check if u is less or equal to probability of y in normal distribution / c * probability of y in laplace distribution
    a) if yes, return y
    b) repeat steps from 3

$$c > 0; sup_x(f(x)/g(x)) \leq c$$

$$h(x) = \frac{f(x)}{g(x)}$$

$$f(x) = N(0,1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$g(x) = \frac{1}{2} e^{-|x|}$$

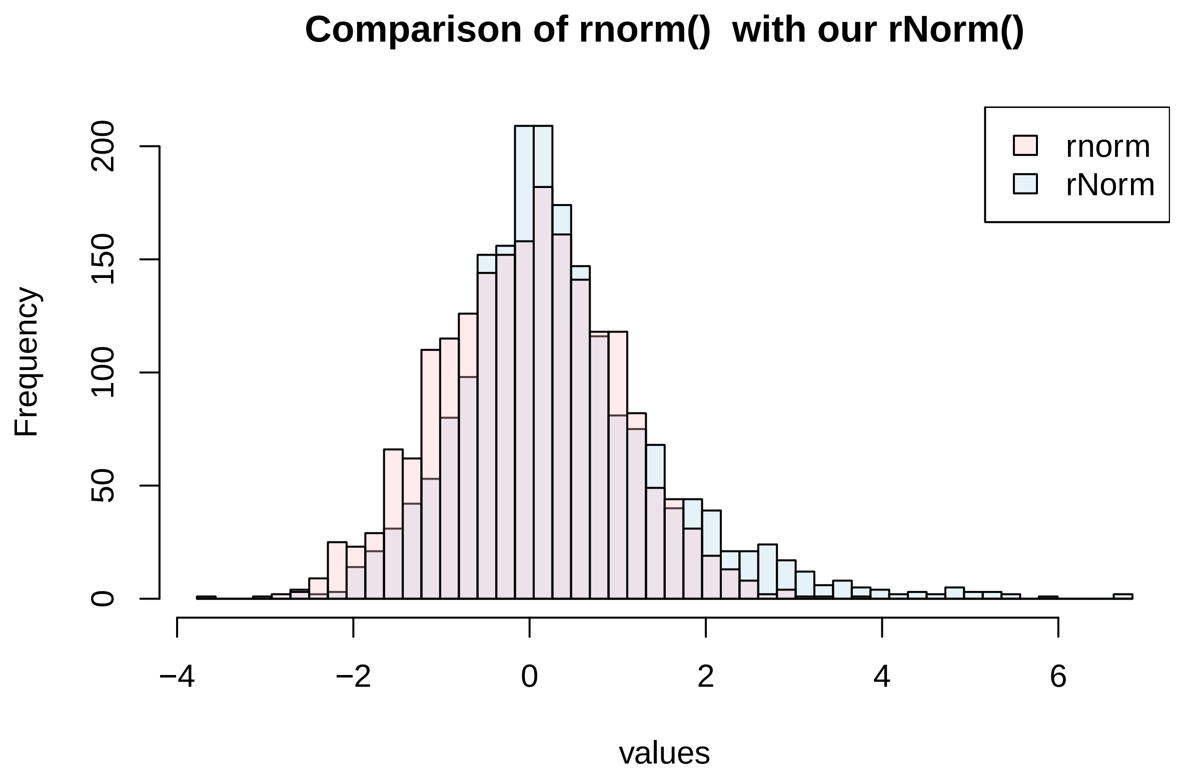
$$h(x) = \sqrt{\frac{2}{\pi}} e^{|x| - \frac{x^2}{2}}$$

$$\frac{d}{dx} \sqrt{\frac{2}{\pi}} e^{|x| - \frac{x^2}{2}} = \sqrt{\frac{2}{\pi}} x e^{|x| - \frac{x^2}{2}} \left(\frac{1}{|x|} - 1 \right)$$

$$\frac{\sqrt{2} e^{x - \frac{x^2}{2}} (x-1)}{\pi} = 0$$

$$x = \pm 1$$

$$c = h(1) = 1.3154892$$



	mean	variance
rNorm()	0.3477306	1.3947951
rnorm()	-0.0069392	0.9925606

rejection rate: 0.2079208, expected rejection rate = c = 0.2398265, difference - -0.0529892

We can see that mean and variance of generated distribution slightly differs.

Appendix

```
knitr::opts_chunk$set(echo = TRUE)


distrib1 <- function(x,c){
  if(x>0){
    return(c*(sqrt(2*pi)^(-1))*(exp(-(c^2)/(2*x))) * (x^(-3/2)))
  }
  else return(0)
}

powerlaw <- function(x,a,t){
  if(x>t){
    return(((a-1)/t) * ((x/t)^(-a)))
  }
  else return(0)

}

x <- seq(0,50, by=0.01)
c <- 1.6
#n <- length(x)
#a= 1 + n*(sum(log(x/t)))^(-1)
y1 <- sapply(x, distrib1, c=c)
y2 <- sapply(x, powerlaw, a = 2,t = 3)
ymax <- max(y1,y2)
b <- min(c(y1,y2))
e <- max(c(y1,y2))
ax <- seq(b,e,by=(e-b)/200)

plot(x,y1, type="l", col = "blue", ylim = c(0,ymax))
lines(x,y2, col = "red")
legend("topright", c("target density", "powerlaw"), fill=c("blue", "red"))
# hist(y1, breaks = ax,
#      col = "red",
#      main = "Comparison of rnorm()  with our rNorm()",
#      xlab = "values",
#      xlim = range(y1,y2))
#
# hist(y2, breaks = ax, col = "blue",  xlim =range(y1,y2), add = TRUE)

majDensity <- function(x, c,a){
  Tmin <- (c^2)/3

  if(x>Tmin){
    return((1-0.083264) * powerlaw(x, a, Tmin))
  }
  else {return(0.083264 * dunif(x,0,Tmin))}
}

c <- 9
y3 <- sapply(x, majDensity, c = c, a = 1.12)
y1 <- sapply(x, distrib1, c = c)
plot(x,y1, type = "l", ylim = c(0,max(y3,y1)))
#lines(x,dnorm(x,3,1.2), col ="red")

lines(x,y3, col ="blue")
#lines(x,y1, col = "red")
library(poweRlaw)
randomnumber <- function(t,a){
  numb <- runif(1)
  if(numb<= 0.08326){
    return(runif(1,0,t))
  }
  else{
    return(rplcon(1,t,a))
```

```r
  }
}

CompleteDist <- function(c, a, rej){
  z <- TRUE
  res <- 0

  Tmin <- (c^2)/3
  xmaj <- (c^2)/(3-2*a)
  cmaj <- distrib1(xmaj,c)/majDensity(xmaj,c, a)

  while (z == TRUE) {
    y <- randomnumber(Tmin,a)
    u <- runif(1)
    if(u <= distrib1(y, c) / (cmaj*majDensity(y,c,a))){
      res <- y
      z <- FALSE
    }
    if(rej){
    rejected <<- rejected + 1
    }
  }
  return(res)
}

rDist <- function(n,c,a ,rej = FALSE){
 return(replicate(n, CompleteDist(c, a, rej)))
}




rejected <- 0
par(mfrow=c(1,3))
datarDist1 <- rDist(2000, 2, 1.12, TRUE)
datarDist12 <- datarDist1[datarDist1 < 200]
rejected1 <- rejected

rejected <- 0
datarDist2 <- rDist(2000, 9, 1.12, TRUE)
datarDist22 <- datarDist2[datarDist2 < 200]
rejected2 <- rejected

rejected <- 0
datarDist3 <- rDist(2000, 15, 1.12, TRUE)
datarDist32 <- datarDist3[datarDist3 < 200]
rejected3 <- rejected

hist(datarDist12, breaks = 30, main = "c = 2", xlab = "data")
hist(datarDist22, breaks = 30, main = "c = 9", xlab = "data")
hist(datarDist32, breaks = 30, main = "c = 15", xlab = "data")
```

```r
results <- matrix(c(mean(datarDist1), var(datarDist1), mean(datarDist2), var(datarDist2), mean(datarDis
colnames(results) <- c("c=2", "c=9", "c=15")
row.names(results) <- c("mean","variance")

knitr::kable(results)

resrejected <- numeric()
c <- seq(1.1,20, by = 0.1)
for (ci in c) {
  rejected <- 0
  rDist(2000, ci, 1.12, TRUE)
  RR <- (rejected / 2000)
  resrejected <- append(resrejected ,1-(1/RR))
}

plot(c,resrejected, type="l", ylab= "rejection rate")
rLaplace <- function(n, mean = 0, alpha = 1){
  b <- 1/alpha
  u <- runif(n)
  res <- mean - (b*sign(u-0.5) * log(1-(2*abs(u-0.5))))
  return(res)
}
library(rmutil)
c1 <- rgb(173,216,230,max = 255, alpha = 80, names = "lt.blue")
c2 <- rgb(255,192,203, max = 255, alpha = 80, names = "lt.pink")

hist(rlaplace(10000, 2,1), 100,
     col = c1,
     main = "Comparison of rlaplace function from rmutil with our rLaplace",
     xlab = "values")
hist(rLaplace(10000, 2,1), 100, col = c2, add = TRUE)
legend("topright", c("rLaplace", "rlaplace"), fill=c(c2, c1))
DE <- function(x, mean = 0,alpha = 1){
  return((0.5*alpha)*exp((-alpha)*abs(x-mean)))
}

genNorm <- function(c, rej){
  z <- TRUE
  res <- 0
  while (z == TRUE) {
    y <- rLaplace(1)
    u <- runif(1)
    if(u <= pnorm(y) / (c*DE(y))){
      res <- y
      z <- FALSE
    }
    if(rej){
    rejected <<- rejected + 1
    }
  }
  return(res)
}
```

```r
rNorm <- function(n,c,rej = FALSE){
 return(replicate(n, genNorm(c, rej)))
}
# x <- seq(-5,5, by=0.1)
#
# test <- function(x){ return((dnorm(x)/DE(x)))}
# cde <- sapply(x, test)
# cnorm <- dnorm(x)
#c <- max(cnorm/cde)

## why it calculates 1,35 ???

c <- 1.3154892

rejected <- 0
datarNorm <- rNorm(2000, c, TRUE)
datarnorm <- rnorm(2000)

b <- min(c(datarNorm,datarnorm))
e <- max(c(datarNorm,datarnorm))
ax <- seq(b,e,by=(e-b)/50)



hist(datarNorm, breaks = ax,
     col = c1,
     main = "Comparison of rnorm()  with our rNorm()",
     xlab = "values",
     xlim = range(datarNorm,datarnorm))

hist(datarnorm, breaks = ax, col = c2,  xlim = range(datarNorm,datarnorm),  ylim = c(0,250), add = TRUE
legend("topright", c("rnorm", "rNorm"), fill=c(c2, c1))
compare <- matrix(c(mean(datarNorm), var(datarNorm), mean(datarnorm), var(datarnorm)), nrow = 2, byrow =
row.names(compare) <- c("rNorm()", "rnorm()")
colnames(compare) <- c("mean", "variance")
knitr::kable(compare)
RR <- (rejected / 2000)
```