Computer Lab 3

Martynas Lukosevicius

17/11/2020

Question 1

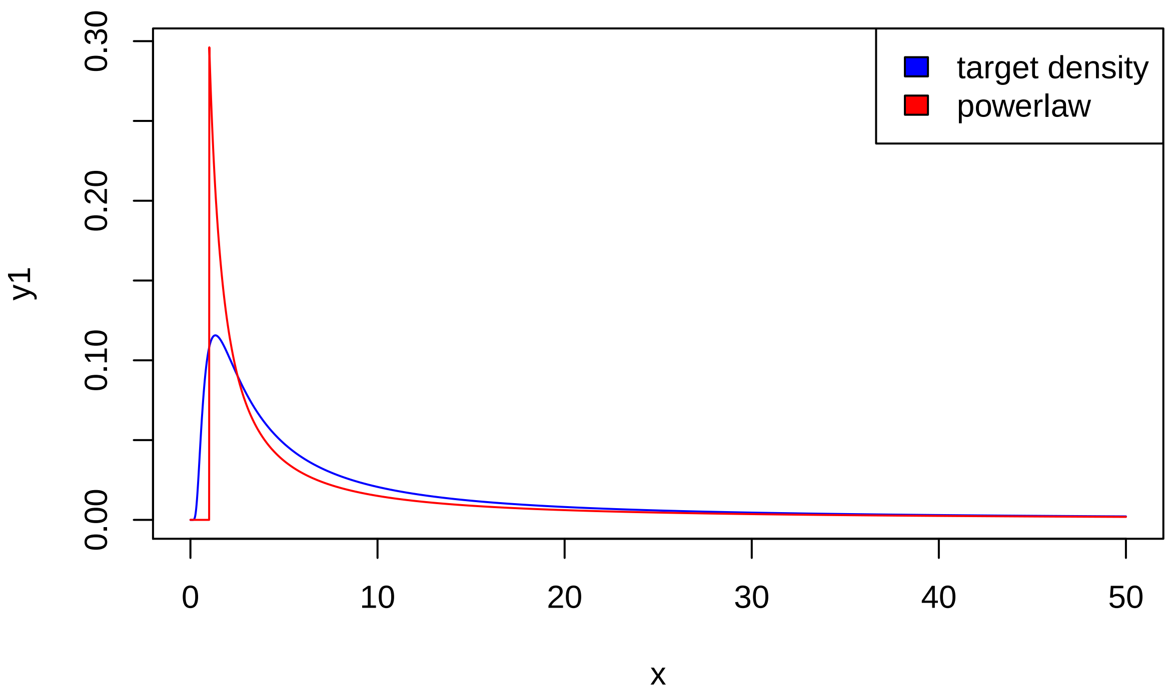
```r
distrib1 <- function(x,c){
  if(x>0){
    return(c*(sqrt(2*pi)^(-1))*(exp(-(c^2)/(2*x))) * (x^(-3/2)))
  }
  else return(0)
}

powerlaw <- function(x,a,t){
  if(x>t){
    return(((a-1)/t) * ((x/t)^(-a)))
  }
  else return(0)

}

x <- seq(0,50, by=0.01)
c <- 2
n <- length(x)
#a= 1 + n*(sum(log(x/t)))^(-1)
y1 <- sapply(x, distrib1, c=c)
y2 <- sapply(x, powerlaw, a = 1.3,t = 1)
ymax <- max(y1,y2)
b <- min(c(y1,y2))
e <- max(c(y1,y2))
ax <- seq(b,e,by=(e-b)/200)

plot(x,y1, type="l", col = "blue", ylim = c(0,ymax))
lines(x,y2, col = "red")
legend("topright", c("target density", "powerlaw"), fill=c("blue", "red"))
```




```
# hist(y1, breaks = ax,
#      col = "red",
#      main = "Comparison of rnorm()  with our rNorm()",
#      xlab = "values",
#      xlim = range(y1,y2))
#
# hist(y2, breaks = ax, col = "blue",  xlim =range(y1,y2), add = TRUE)
```

Power-law distribution cannot be used just by itself because it doesnt support range from 0 to T_{min}. Because of this we need to use mixture distribution. Tio support x from 0 to T_{min} we choose uniform distribution $Unif(0, T_{min})$. AS we can see power-law distribution is monotonically decreasing, so we T_{min} should be equal to x where target density has maximum value.

Lets find maximum of target density function:

$$\frac{\partial}{\partial x}\frac{ce^{-\frac{c^2}{2x}}x^{-\frac{3}{2}}}{\sqrt{2\pi}} = \frac{ce^{-\frac{c^2}{2x}}\left(c^2-3x\right)}{2\sqrt{2\pi}x^{7/2}}$$

$$\frac{c\frac{e^{-c^2}}{2x}\left(c^2-3x\right)}{2\sqrt{2\pi}x^{7/2}} = 0$$

$$x = \frac{c^2}{3}$$

$$T_{min} = \frac{c^2}{3}$$

To make a mixture model we need to know the probability of taking uniform distribution and powerlaw distribution probability that number will be in 0- T_{min} region is:

$$\int_0^{T_{min}} cx^{-\frac{3}{2}} e^{-\frac{c^2}{2x}} \sqrt{2\pi}^{-1} dx = \frac{\Gamma\left(\frac{1}{2}, \frac{c^2}{2T_{min}}\right)}{\sqrt{\pi}}$$

as $T_{min} = \frac{c^2}{3}$ - $\frac{\Gamma\left(\frac{1}{2}, \frac{c^2}{2T_{min}}\right)}{\sqrt{\pi}} = \frac{\Gamma\left(\frac{1}{2}, \frac{3}{2}\right)}{\sqrt{\pi}} \sim 0.08326451666$

As the result: majorising density function is:

$$g(x) = \frac{2 * 0.08326}{c^2} 1_{[0,T_{min}]} + (1 - 0.08326) * \frac{2^{1-a}(a-1)\left(\frac{x}{c^2}\right)^{-a}}{c^2} * 1_{(T_{min},\infty)}$$

2.

Target density:

$$f(x) = c(\sqrt{2\pi})^{-1} e^{\frac{-c^2}{2x}} x^{\frac{-3}{2}} 1_{(0,\infty)}(x)$$

We need to find c_{maj}

$$c_{maj} > 0; sup_x(f(x)/g(x)) \leq c_{maj}$$

$$h(x) = \frac{f(x)}{g(x)}$$

$c_{maj} = h(x_{maj})$

if x $< T_{min}$

$$\frac{\partial}{\partial x} \frac{f(x)}{\frac{0.16652}{c^2}} = \frac{e^{-\frac{c^2}{2x}}\left(1.19788c^5 - 3.59364c^3 x\right)}{x^{7/2}}$$

$$\frac{e^{-\frac{c^2}{2x}}\left(1.19788c^5 - 3.59364c^3 x\right)}{x^{7/2}} = 0$$

$$x_{maj} = \frac{c^2}{3}$$

x for cmaj $= x = 0.333333c^2 = T_{min}$

if x$>$ Tmin

$$\frac{\partial}{\partial x} \frac{f(x)}{g(x)} = \frac{3^{a-1}c^3 \frac{e^{-c^2}}{2x} \left(\frac{x}{c^2}\right)^a \left((2a-3)x + c^2\right)}{2\sqrt{2\pi}(a-1)x^{7/2}}$$

$$\frac{3^{a-1}c^3 \frac{e^{-c^2}}{2x} \left(\frac{x}{c^2}\right)^a \left((2a-3)x + c^2\right)}{2\sqrt{2\pi}(a-1)x^{7/2}} = 0$$

$$x_{maj} = \frac{c^2}{3 - 2a}$$

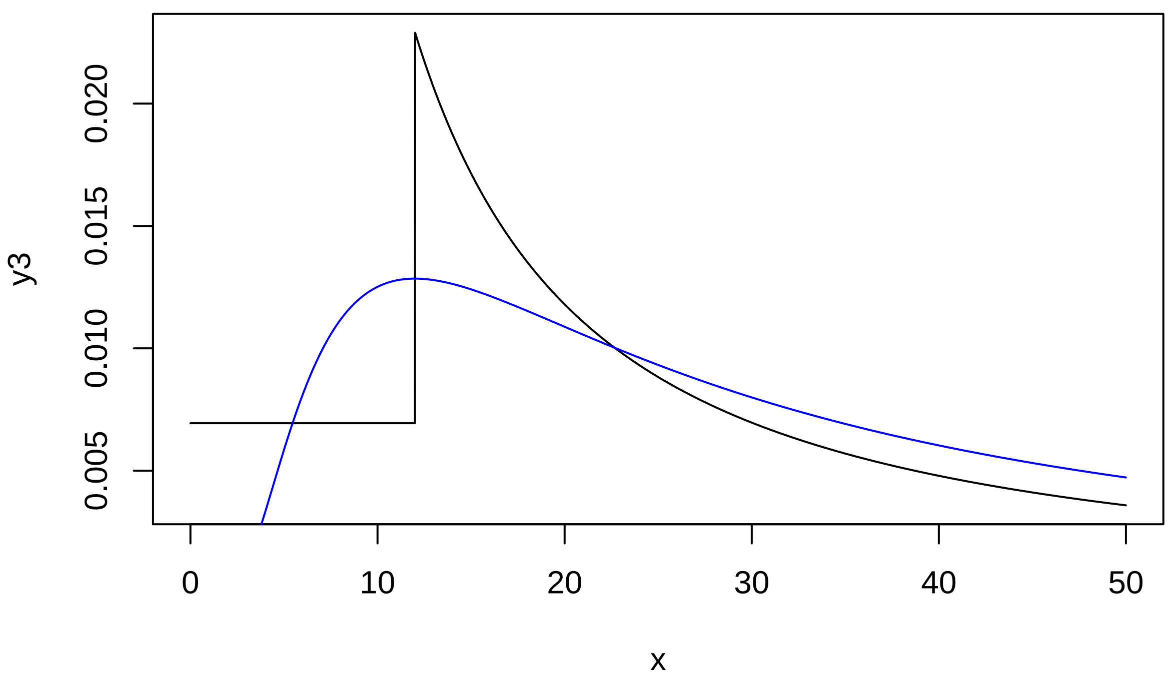
when $1 < a < 1.5$

$$c_{maj} = h(x_{maj})$$

```
majDensity <- function(x, c,a){
  Tmin <- (c^2)/3

  if(x>Tmin){
    return((1-0.083264) * powerlaw(x, a, Tmin))
  }
  else {return(0.083264 * dunif(x,0,Tmin))}
}

c <- 6
y3 <- sapply(x, majDensity, c = c, a = 1.3)
plot(x,y3, type = "l")
#lines(x,dnorm(x,3,1.2), col ="red")
y1 <- sapply(x, distrib1, c = c)
lines(x,y1, col ="blue")
```




```
#lines(x,y1, col = "red")
```

2

```r
library(poweRlaw)
```

```
## Warning: package 'poweRlaw' was built under R version 4.0.3
```

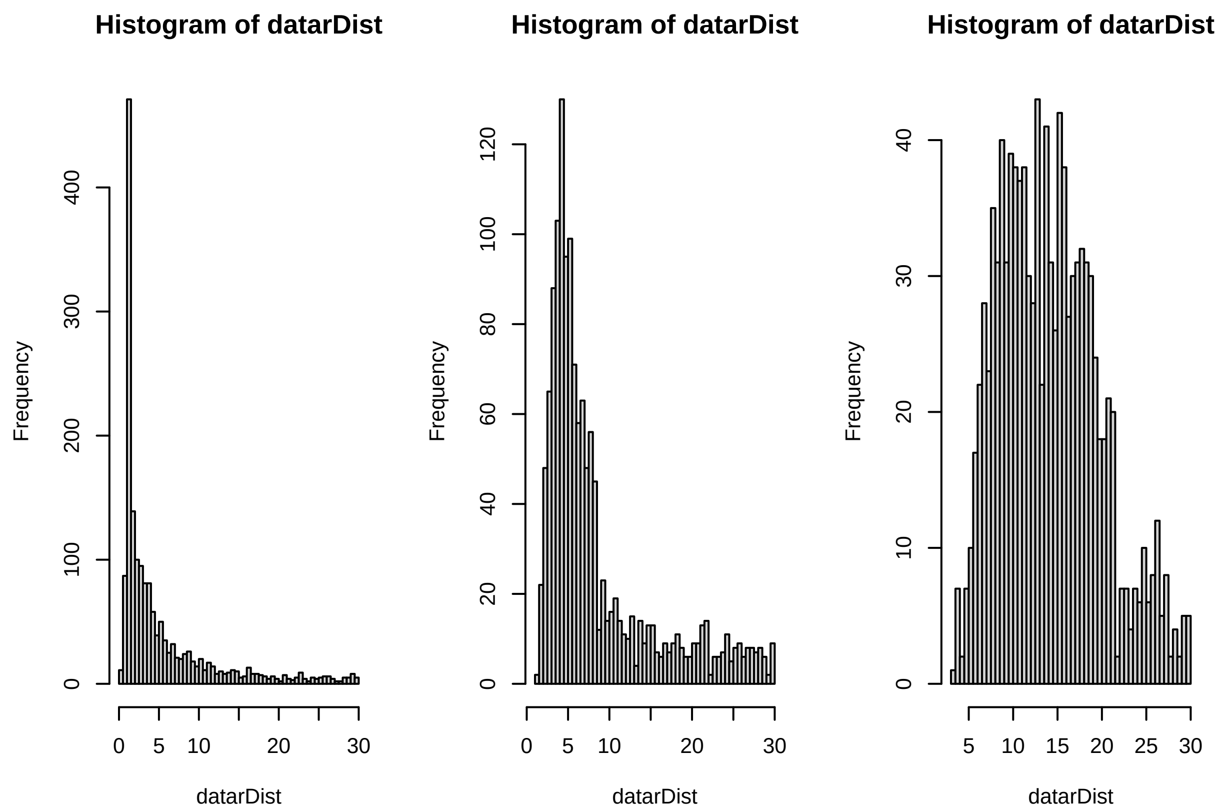
```r
randomnumber <- function(){
  numb <- runif(1)
  if(numb<= 0.08326){
    return(runif(1,0,1))
  }
  else{
    return(rplcon(1,1,1.5))
  }
}

CompleteDist <- function(c, a, rej){
  z <- TRUE
  res <- 0

  Tmin <- (c^2)/3
  xmaj <- Tmin
  cmaj <- distrib1(xmaj,c)/majDensity(xmaj,c, a)

  while (z == TRUE) {
    y <- randomnumber()
    u <- runif(1)
    if(u <= distrib1(y, c) / (cmaj*majDensity(y,c,a))){
      res <- y
      z <- FALSE
    }
    if(rej){
    rejected <<- rejected + 1
    }
  }
  return(res)
}

rDist <- function(n,c,a ,rej = FALSE){
 return(replicate(n, CompleteDist(c, a, rej)))
}
```



Question 2

1.

$$DE(\mu, \alpha) = \frac{\alpha}{2} e^{-\alpha|x-\mu|}$$

- μ - location parameter
- $b > 0$ - scale parameter

inverse CDF of DE:

Source - https://en.wikipedia.org/wiki/Laplace_distribution

$$F^{-1}(p) = \mu - bsgn(p - 0.5)ln(1 - 2|p - 0.5|)$$

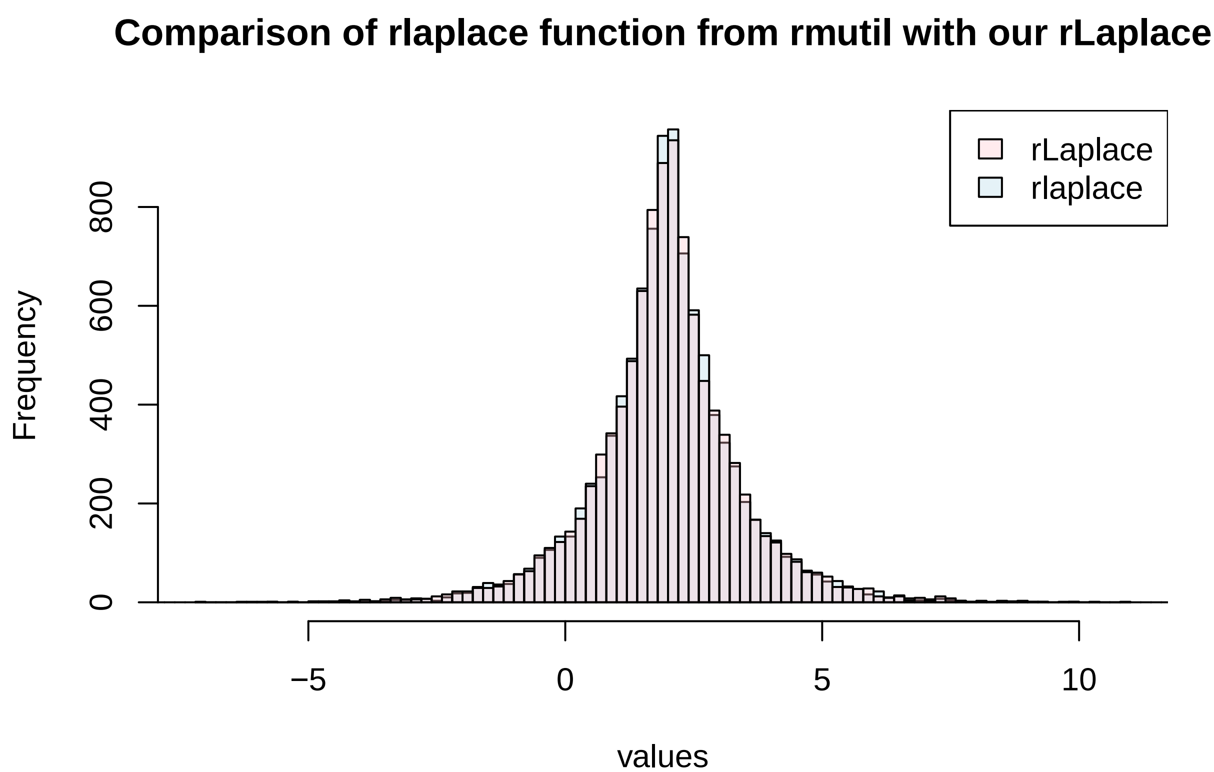
where $b = \frac{1}{\alpha}$

```
rLaplace <- function(n, mean = 0, alpha = 1){
  b <- 1/alpha
  u <- runif(n)
  res <- mean - (b*sign(u-0.5) * log(1-(2*abs(u-0.5))))
  return(res)
}
```

meaning:

1. calculate b.
2. take n random variables from uniform distribution [0,1]

3. calculate random numbers from inverse CDF of laplace distribution where x is a random variable from uniform distribution



2.

```
DE <- function(x, mean = 0,alpha = 1){
  return((0.5*alpha)*exp((-alpha)*abs(x-mean)))
}

genNorm <- function(c, rej){
  z <- TRUE
  res <- 0
  while (z == TRUE) {
    y <- rLaplace(1)
    u <- runif(1)
    if(u <= pnorm(y) / (c*DE(y))){
      res <- y
      z <- FALSE
    }
    if(rej){
    rejected <<- rejected + 1
    }
  }
  return(res)
}
```

```
rNorm <- function(n,c,rej = FALSE){
 return(replicate(n, genNorm(c, rej)))
}
```

algorithm:

1. write Laplace probability function
2. assign 0 to result value res
3. generate random number y from rLaplace function
4. generate random number u from uniform distribution
5. check if u is less or equal to probability of y in normal distribution / c * probability of y in laplace distribution
   a) if yes, return y
   b) repeat steps from 3

$$c > 0; sup_x(f(x)/g(x)) \leq c$$

$$h(x) = \frac{f(x)}{g(x)}$$

$$f(x) = N(0,1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$g(x) = \frac{1}{2} e^{-|x|}$$

$$h(x) = \sqrt{\frac{2}{\pi}} e^{|x| - \frac{x^2}{2}}$$

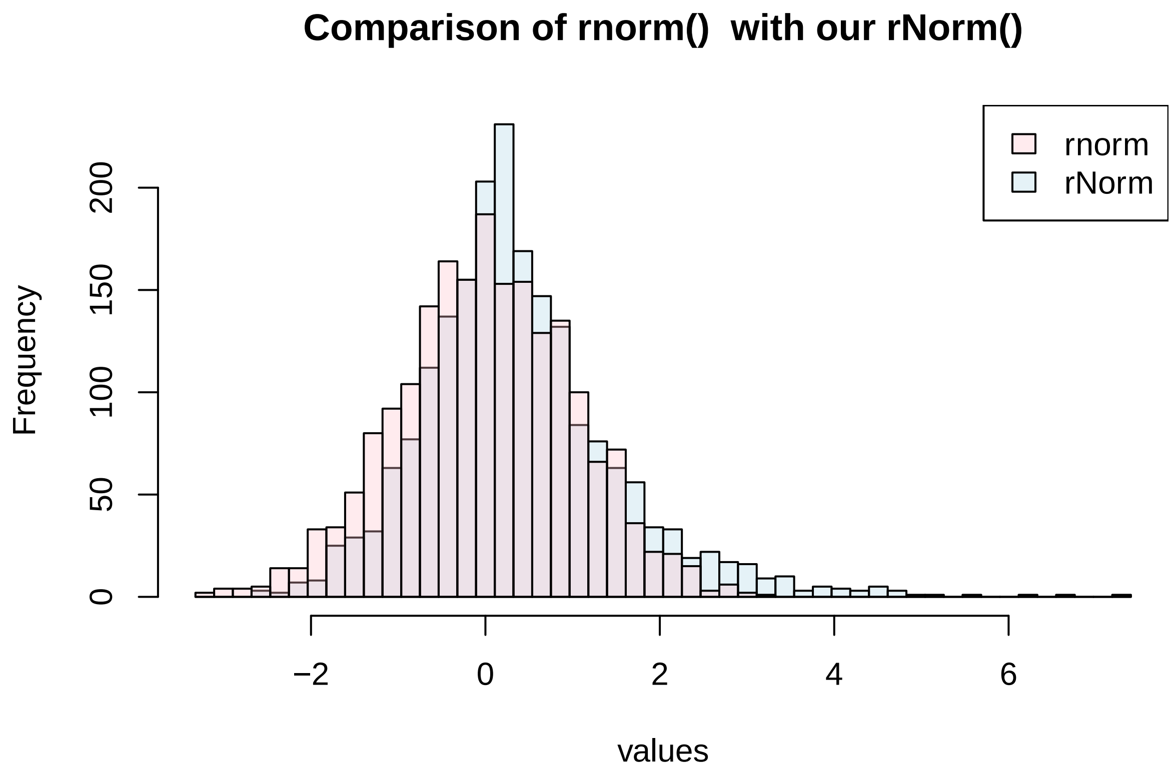
$$\frac{d}{dx} \sqrt{\frac{2}{\pi}} e^{|x| - \frac{x^2}{2}} = \sqrt{\frac{2}{\pi}} x e^{|x| - \frac{x^2}{2}} \left(\frac{1}{|x|} - 1 \right)$$

$$\frac{\sqrt{2} e^{x - \frac{x^2}{2}} (x-1)}{\pi} = 0$$

$$x = \pm 1$$

$$c = h(1) = 1.3154892$$

source - http://www.columbia.edu/~ks20/4703-Sigman/4703-07-Notes-ARM.pdf



	mean	variance
rNorm()	0.3875328	1.274826
rnorm()	0.0150809	1.020806

rejection rate: 0.188641, expected rejection rate = c = 0.2398265, difference - -0.0829892