Topological waves in magnetized cold plasma

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Outline

- Prelude: waves in plasmas
- Topological properties of waves
- Topological plasma waves
- Conclusions and discussions

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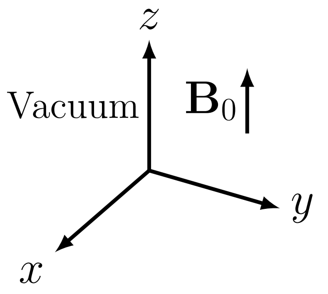
Bulk waves: EM waves in vacuum

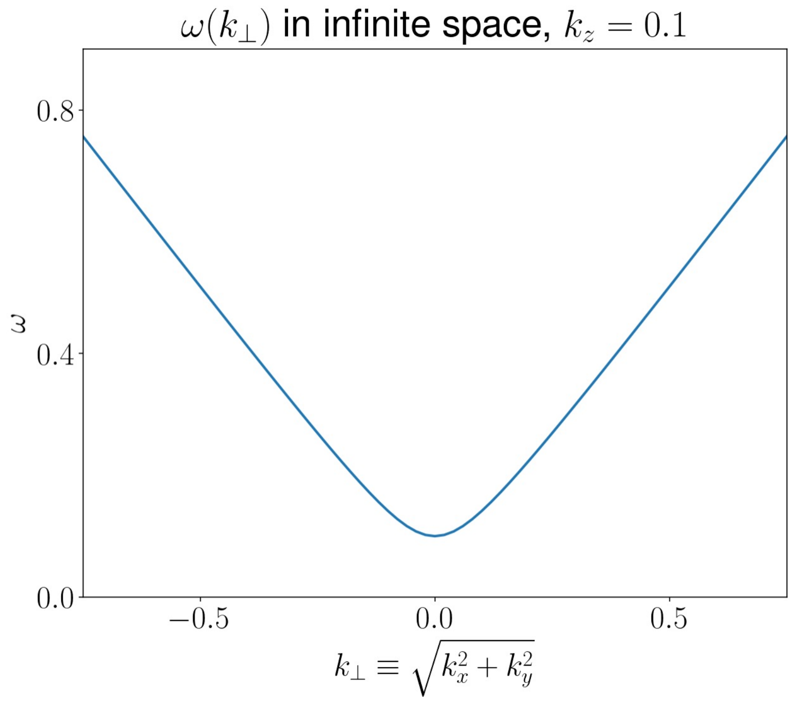


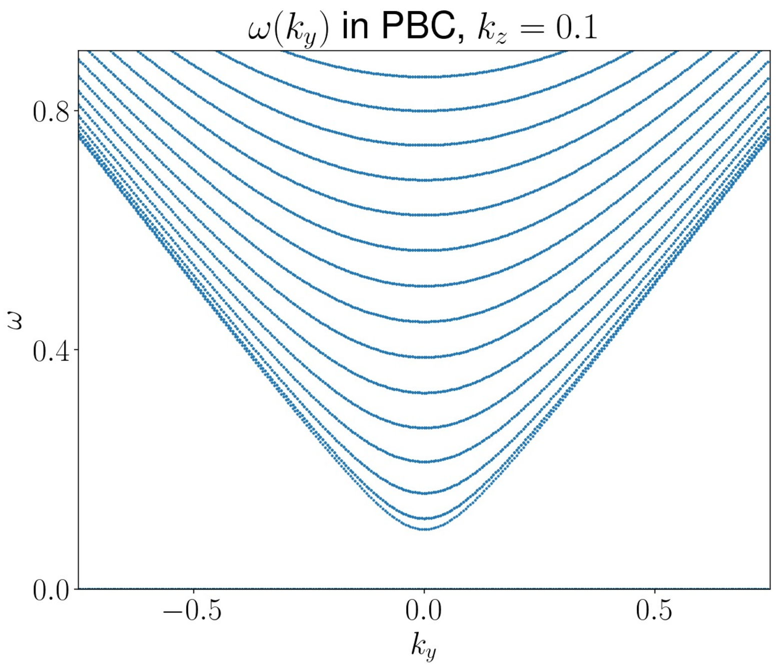
- In an infinitely large space:

$$\omega^2 = c^2\left(k_\perp^2 + k_z^2\right)$$

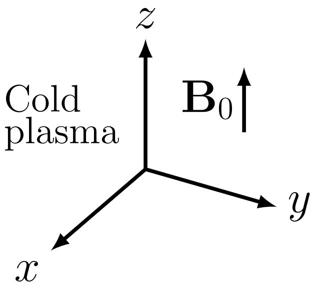
- With PBC: $f(x+L) = f(x)$

$$\omega^2 = c^2\left(k_x^2 + k_y^2 + k_z^2\right), \; k_x = \frac{2\pi n}{L}, \; n \in \mathbb{Z}$$





Bulk waves: waves in cold plasmas

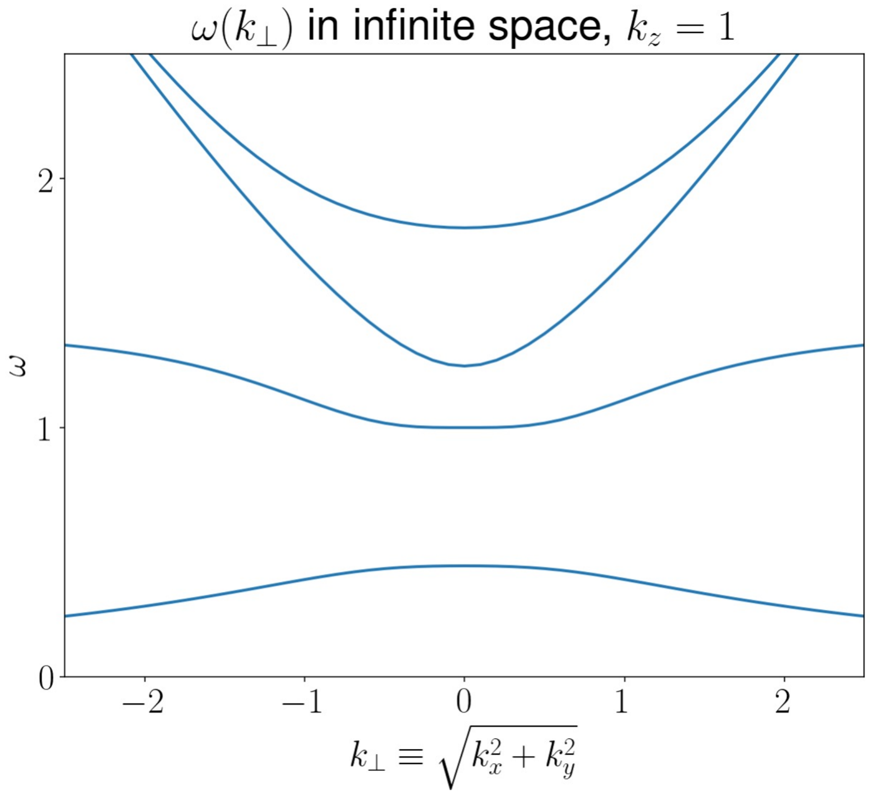


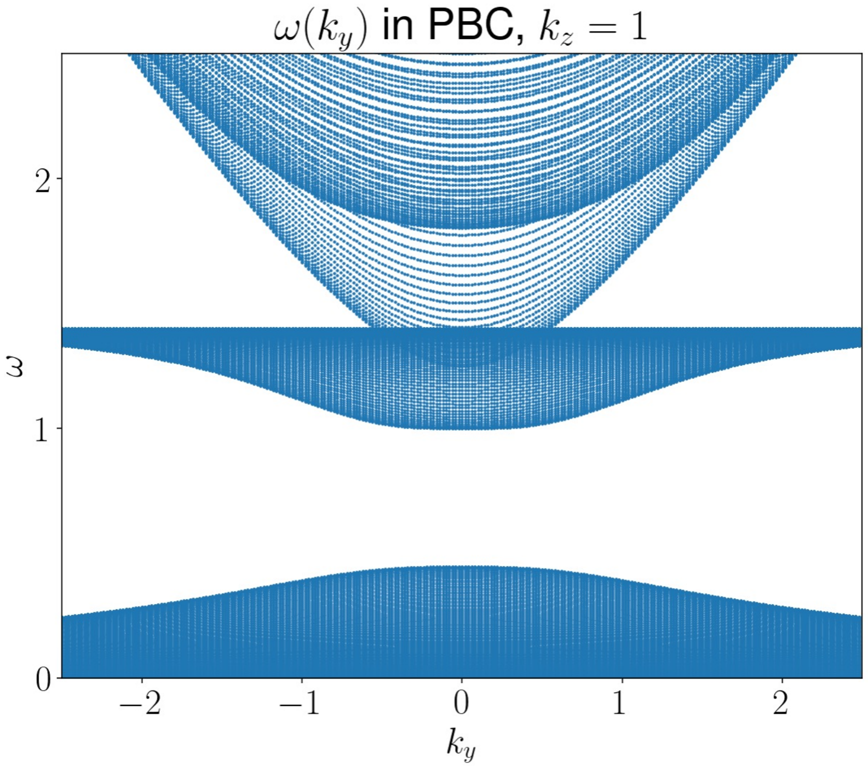
- In an infinitely large space:

$$\omega = \omega(k_\perp, k_z)$$

- With PBC: $f(x+L) = f(x)$

$$\omega = \omega\left(\sqrt{k_x^2 + k_y^2}, k_z\right), \; k_x = \frac{2\pi n}{L}, \; n \in \mathbb{Z}$$

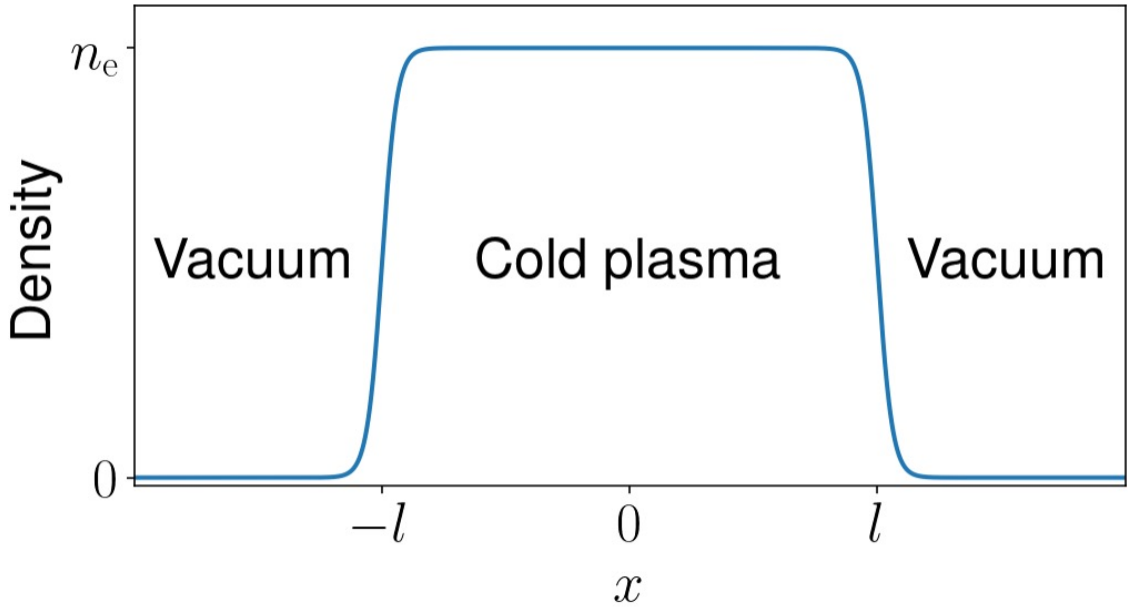


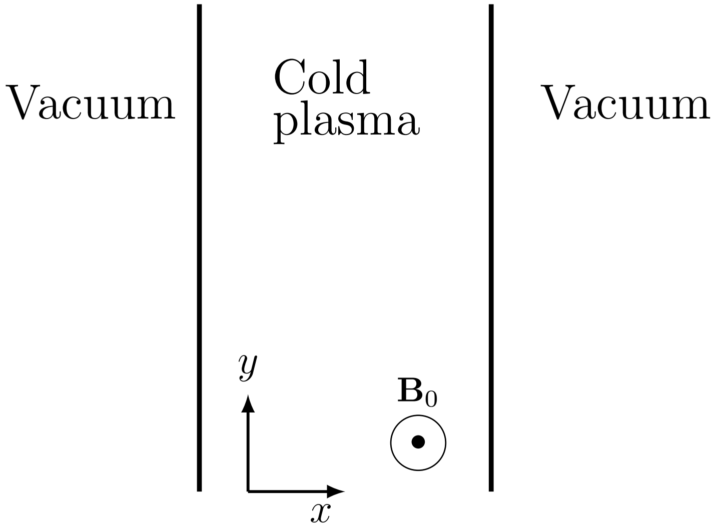


Systems with both plasma and vacuum

- **Question**: when we have a ‘realistic’ boundary between plasma and vacuum, what spectrum shall we expect?

- Assume the plasma is non-uniform in x direction, and infinite in other directions.



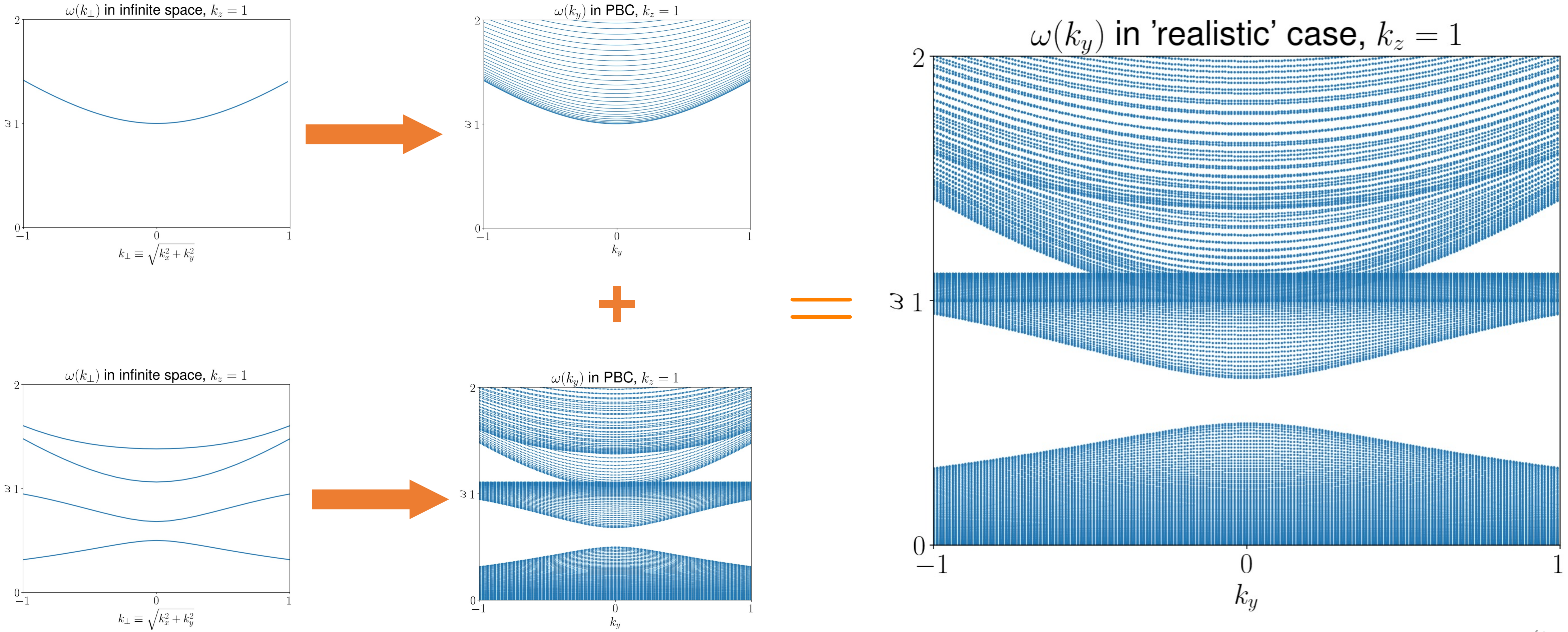


Spectrum in ‘low’ density case

Let $k_z = B_0 = m_\mathrm{e} = q = c = 1$, and we take $n_\mathrm{e} = 0.25$.

Vacuum part

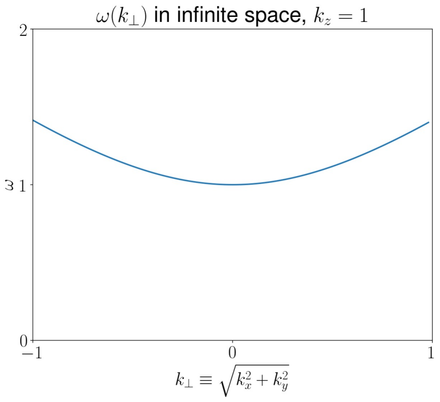
Plasma part

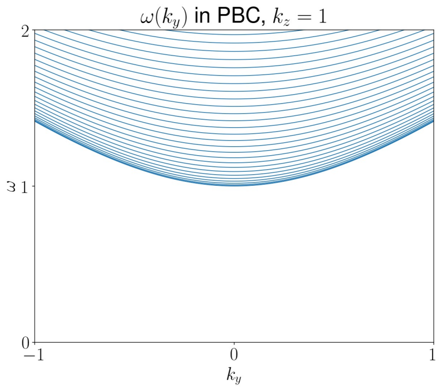


Spectrum in ‘high’ density case

Let $k_z = B_0 = m_\mathrm{e} = q = c = 1$, but now we take $n_\mathrm{e} = 0.6$.

Vacuum part





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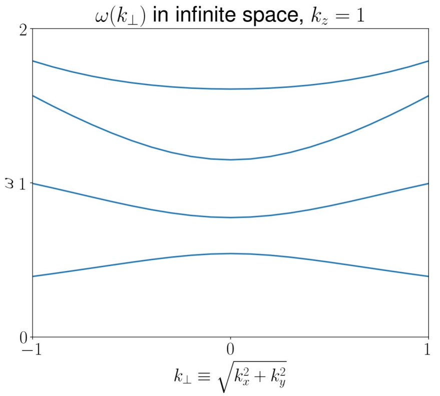
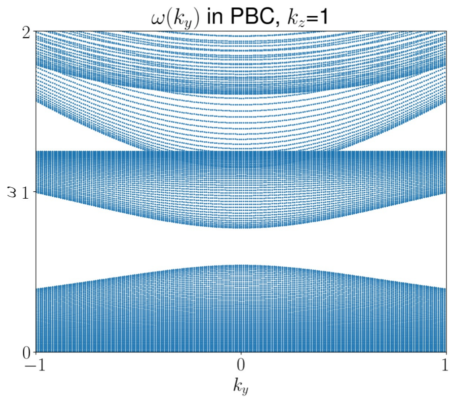
Plasma part





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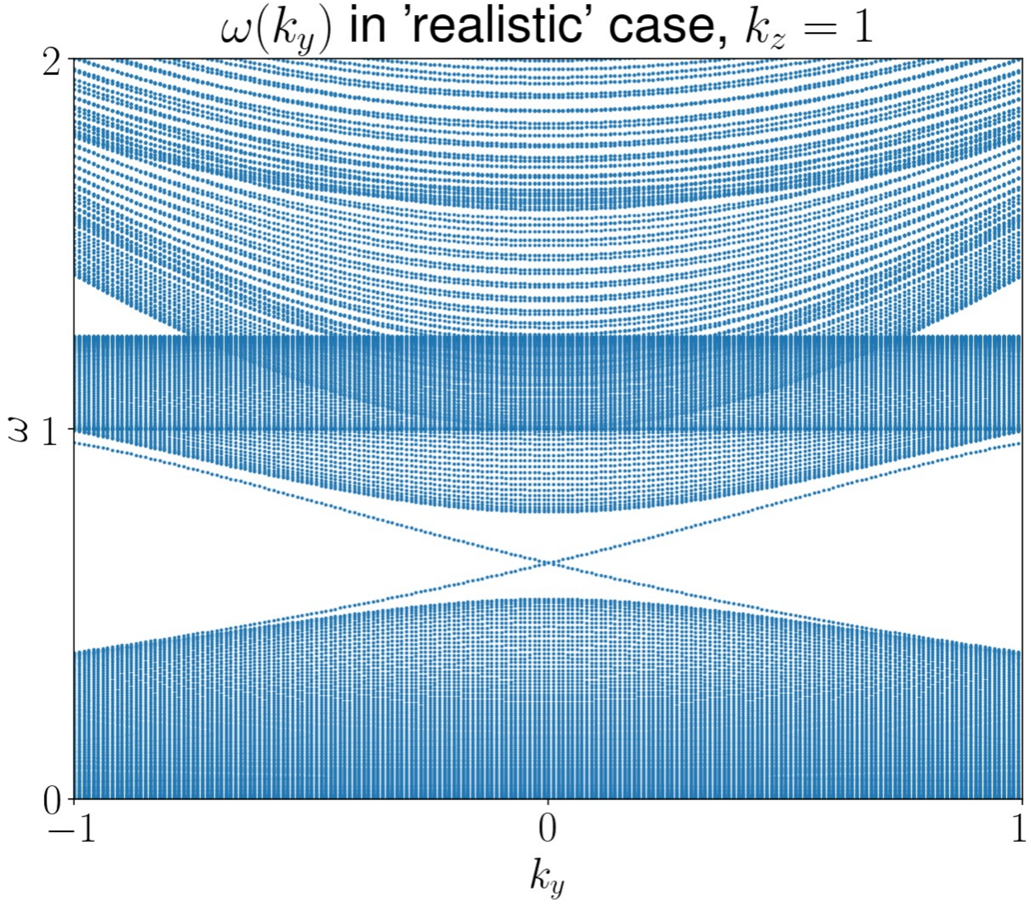
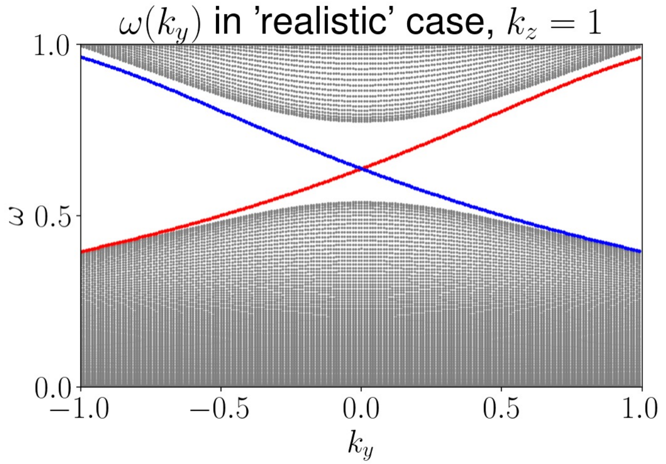
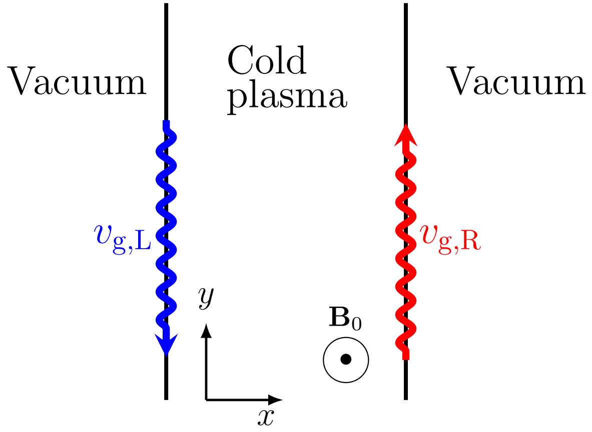
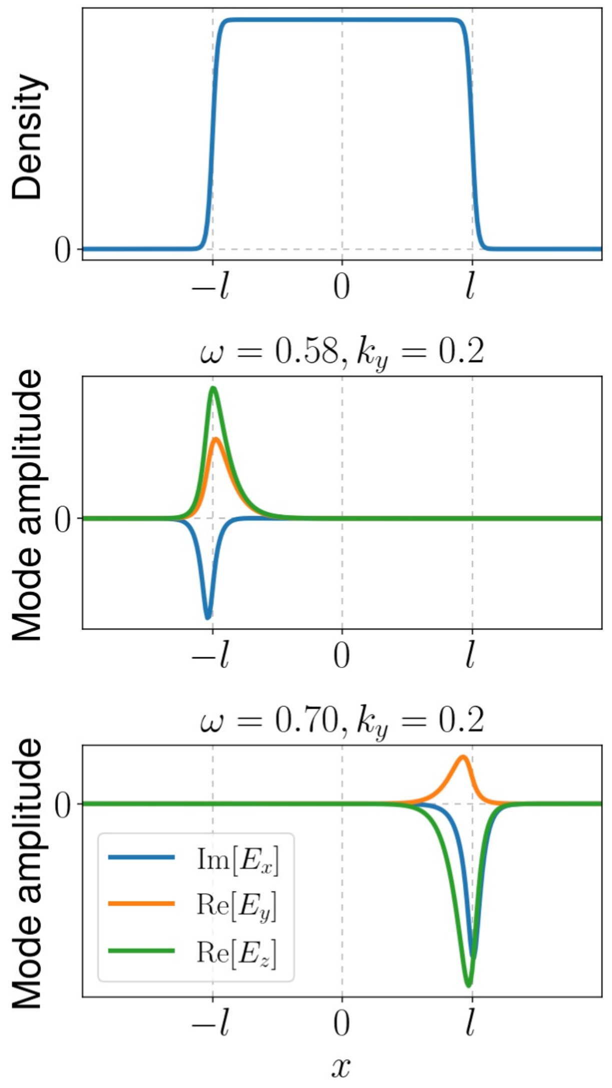


Details of these modes

- These waves are localized at plasma-vacuum interfaces. Each interface has one edge wave.
- The group velocities of the waves at each interface are opposite. This wave is the so-called chiral edge mode.
- The existence of edge modes does not rely on the length scale of density decay.







How to understand this phenomenon?

- For given k_z and $\Omega = qB_0/m_\mathrm{e}$, the critical density is given by:

$$\omega_{\mathrm{p,c}} = \frac{|\Omega|}{2}\left[\sqrt{\left(\frac{ck_z}{\Omega}\right)^4 + 4\left(\frac{ck_z}{\Omega}\right)^2} - \left(\frac{ck_z}{\Omega}\right)^2\right]$$

- **Traditional methods**: solve the waves on both sides and match them with boundary conditions.

- **New methods**: study the 'topology' of the waves on each side.

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Topology of eigenvectors

- Assume we have a **uniform** system in an **infinite** space. After Fourier transform, we get an N-dimensional Hermitian eigenvalue problem in momentum space: $\mathcal{H}(\mathbf{k})\,|\psi_n(\mathbf{k})\rangle = E_n(\mathbf{k})\,|\psi_n(\mathbf{k})\rangle$

- Let $\mathbf{k} \in \mathcal{M}_1, |\psi_n\rangle \in \mathcal{M}_2 \subset \left\{z \in \mathbb{C}^N \mid \|z\| = 1\right\}$, then each eigenvector defines a map: $|\psi_n\rangle : \mathcal{M}_1 \to \mathcal{M}_2$.

- Now, we can discuss the topological equivalent classes of the maps, characterized by various topological invariances.

- Let $\mathcal{M}_1$ be 2D, define $P_n(\mathbf{k}) := |\psi_n\rangle\langle\psi_n|$, the (first) **Chern number** can be defined as:

$$
\begin{aligned}
C_n &= -\frac{\mathrm{i}}{2\pi}\int_{\mathcal{M}_1} \mathrm{tr}\left(P_n\ \mathrm{d}P_n \wedge \mathrm{d}P_n\right) \\
&= -\frac{\mathrm{i}}{2\pi}\int_{\mathcal{M}_1} \left(\langle\partial_{k_x}\psi_n|\partial_{k_y}\psi_n\rangle - \langle\partial_{k_y}\psi_n|\partial_{k_x}\psi_n\rangle\right)\mathrm{d}^2\mathbf{k} \qquad \text{(TKNN)}
\end{aligned}
$$

Example: two-level system

- Let the momentum space be 2D and periodic in both directions, i.e., the 2D Brillouin zone, $\mathbf{k} \in \mathrm{T}^2$.

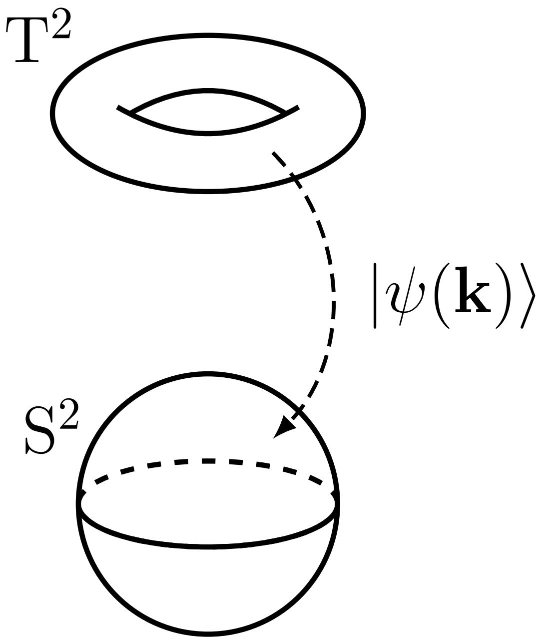
- The Hamiltonian is: $\mathcal{H}(\mathbf{k}) = \mathbf{d}(\mathbf{k}) \cdot \boldsymbol{\sigma}$, $\mathbf{d} = (d_x, d_y, d_z)$, $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$.

  Its eigen vectors can be mapped to the Bloch sphere S^2, e. g.,

$$|\psi^+\rangle = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) e^{-\mathrm{i}\varphi} \\ \sin\left(\frac{\theta}{2}\right) \end{pmatrix}, \theta(\mathbf{k}) \in [0, \pi], \varphi(\mathbf{k}) \in [0, 2\pi).$$

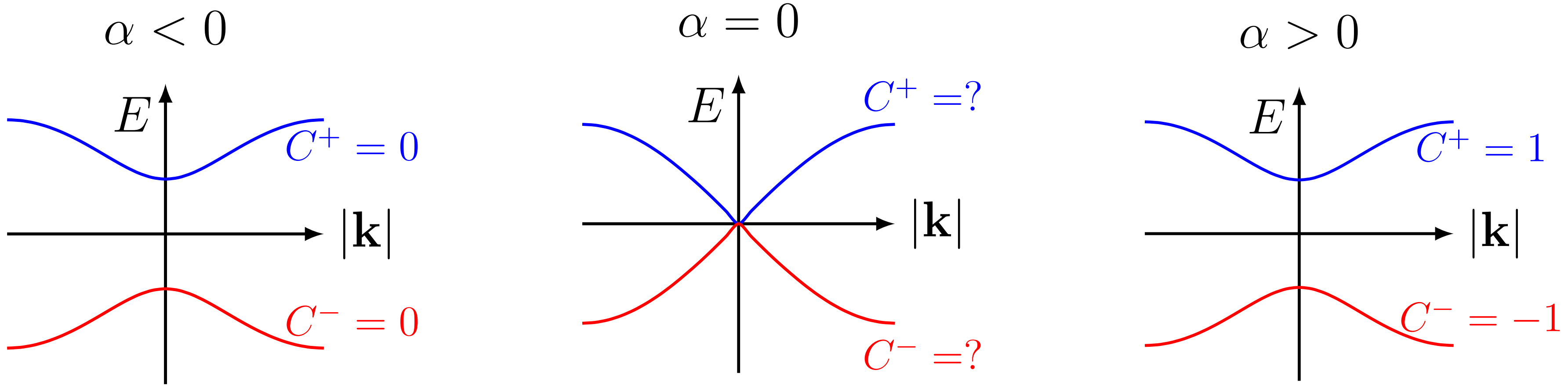
- Thus, $|\psi(\mathbf{k})\rangle$ defines a map $\mathrm{T}^2 \to \mathrm{S}^2$, which is classified by $\pi_2(\mathrm{S}^2) = \mathbb{Z}$.

- The Chern number is: $C^+ = \frac{1}{4\pi} \int_{\mathrm{T}^2} \hat{\mathbf{d}} \cdot \left(\partial_{k_x} \hat{\mathbf{d}} \times \partial_{k_y} \hat{\mathbf{d}} \right) \mathrm{d}^2\mathbf{k}.$



Topological phase transition

- Assume $\mathcal{H}(\mathbf{k};\alpha)$ and $|\psi_n(\mathbf{k};\alpha)\rangle$ depends on parameter α . The Chern numbers of each band may be different for different α.



- A **topological phase** is a parameter region where all bands' topological invariance(s) keep constant.

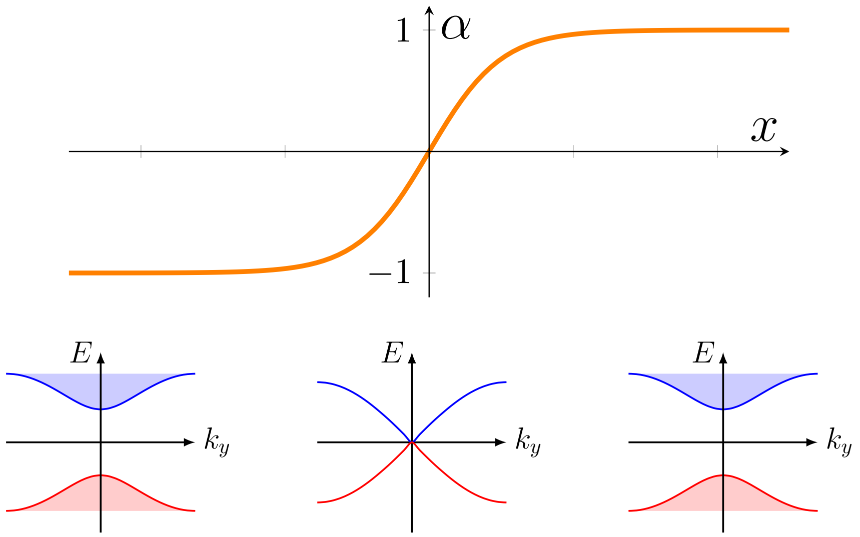
- When parameter changes cross different phases, the system will undergo a **topological phase transition**, usually through band touching.

Topological edge modes

- If parameter α changes in the space, the location where the topological phase changes will have a gapless edge mode.

The bulk-edge correspondence

When two materials with differing topological phases and a common gapped spectrum are brought next to each other, modes localized to the interface and crossing the gap must appear at the interface.



- Physical intuition: topologically different bands can not continuously change to each other without band touching.

- Mathematically rigorous statement is possible by using the index theorem.

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Topological plasma waves

- Topological edge modes have been discovered in various plasma models:
  - Cold magnetized plasma[1]
  - Ideal MHD plasma with constant magnetic shear[2]

- Recently, our study[3] has mapped out the topological phase diagram in cold magnetized plasmas, and thus gives the condition when topological edge modes exist.

- We also show that such edge modes can exist in a simple plasma-plasma interface.

[1] Parker, et al., Phys. Rev. Lett., 2020., [2] Parker, et al., Phys. Rev. Res., 2020., [3] Fu, et al., accepted by Nature Comm.

Waves in cold magnetized plasma

- We consider the electromagnetic waves in cold electron-ion plasma in 3D. Without base flow, $\mathbf{B}_0 = B_0\hat{z}$, the linearized equations are:

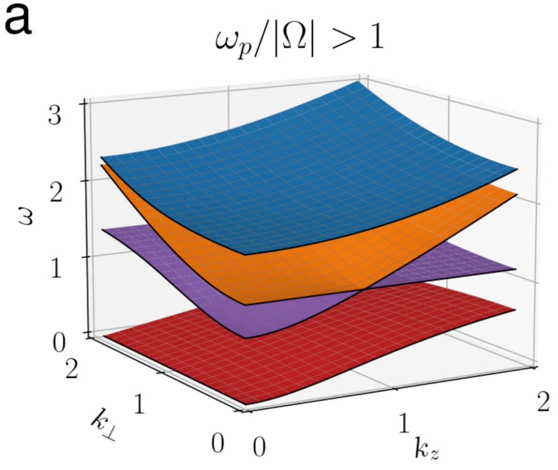
$$
\begin{aligned}
\partial_t \mathbf{v}_1 &= \frac{e}{m_\mathrm{e}}\left(\mathbf{E}_1 + \mathbf{v}_1 \times \mathbf{B}_0\right) \\
\partial_t \mathbf{E}_1 &= c^2 \nabla \times \mathbf{B}_1 - \frac{e n_\mathrm{e}}{\epsilon_0}\mathbf{v}_1 \\
\partial_t \mathbf{B}_1 &= -\nabla \times \mathbf{E}_1
\end{aligned}
\quad \Rightarrow \quad
\mathrm{i}\frac{\partial}{\partial t}\begin{bmatrix}\mathbf{v}_1 \\ \mathbf{E}_1 \\ \mathbf{B}_1\end{bmatrix} = \begin{bmatrix}\mathrm{i}\Omega\hat{z}\times & \mathrm{i}\omega_\mathrm{p} & 0 \\ -\mathrm{i}\omega_\mathrm{p} & 0 & \mathrm{i}\nabla\times \\ 0 & -\mathrm{i}\nabla\times & 0\end{bmatrix}\begin{bmatrix}\mathbf{v}_1 \\ \mathbf{E}_1 \\ \mathbf{B}_1\end{bmatrix}
$$

- Fourier transform gives: $\partial_t \to -\mathrm{i}\omega, \nabla \to \mathrm{i}\mathbf{k}$.
  The Hamiltonian is $\mathcal{H}(\mathbf{k}; \omega_\mathrm{p}, \Omega)$.
  The dispersion relations $\omega(k_\perp, k_z)$ are:

- The bands touch each other only when:

$$|k_z| = k^\pm := \frac{\omega_\mathrm{p}/c}{\sqrt{1 \pm \omega_\mathrm{p}/\Omega}}$$

a



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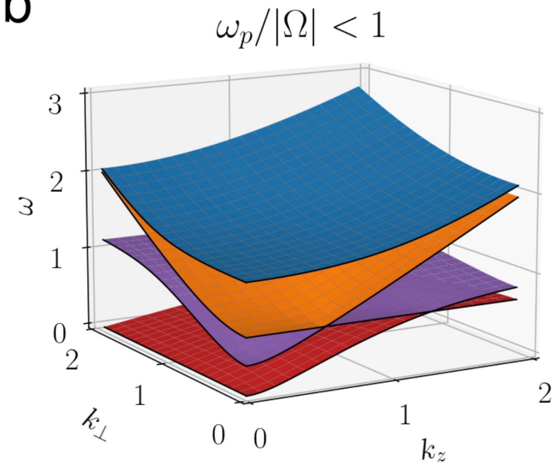


Reduce to 2D

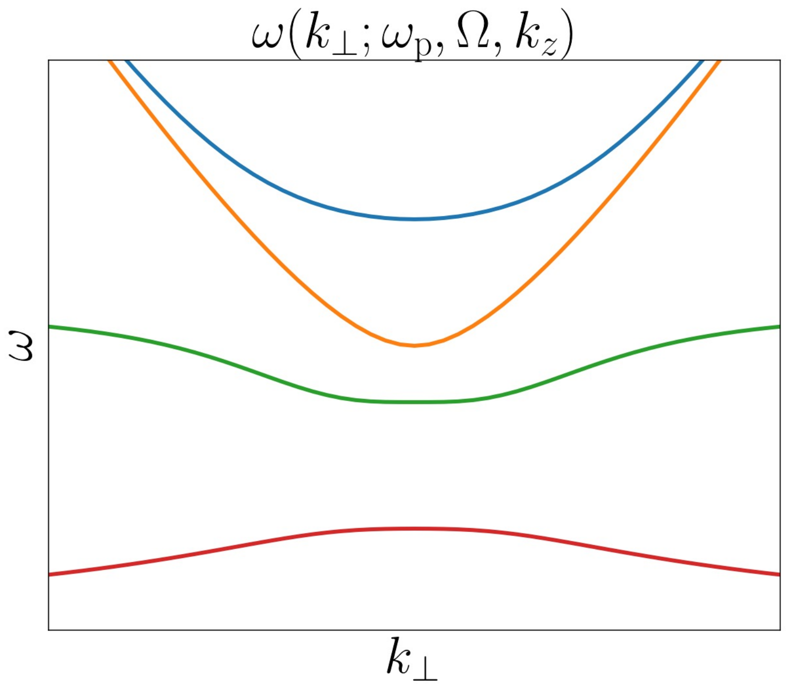
- To characterize each band using Chern number, we further fixed k_z as a parameter.

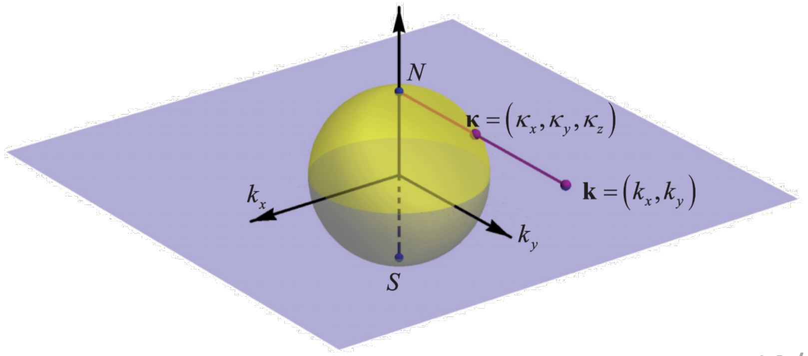
- Notice here $(k_x, k_y) \in \mathbb{R}^2$, which is not a compact space. The Chern number may not be integers. However, the space can be compactified if:

$$k_\perp \to \infty,\ |\psi_n(k_x, k_y)\rangle \to |\tilde{\psi}_n\rangle.$$

  It is equivalent to compactify $\mathbb{R}^2$ into S^2.

- If $|\psi_n\rangle$ does not satisfy the condition, a cut-off can be added artificially[1]. This method is valid if the $k_\perp \to \infty$ behavior can be ignored in the system.

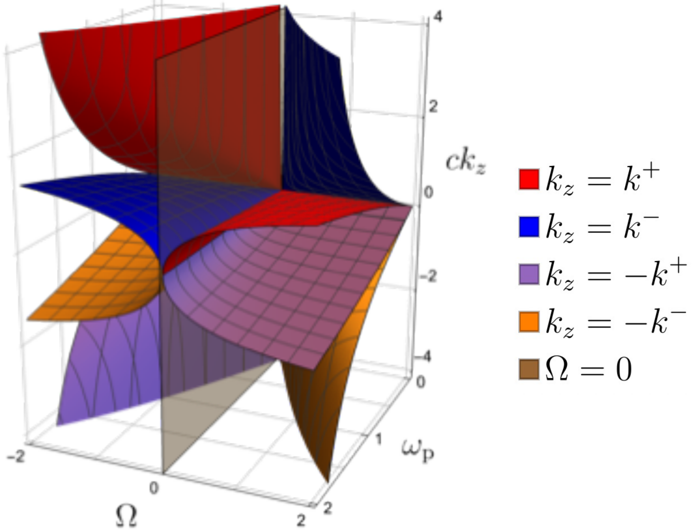


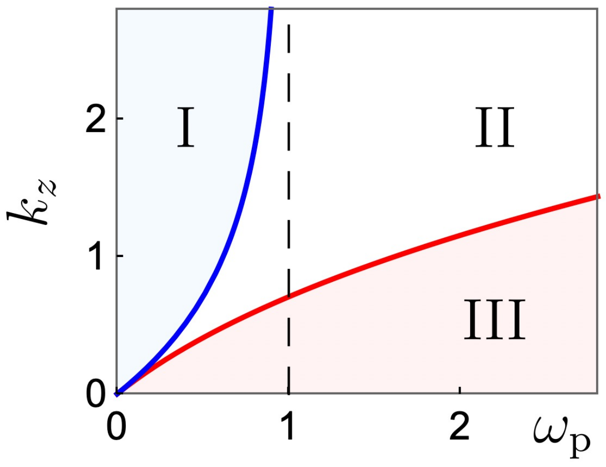


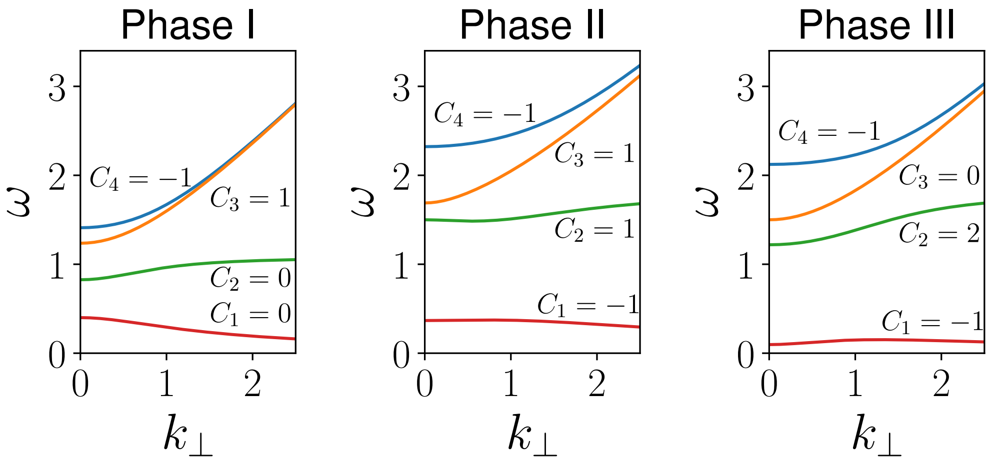
[1] Silveirinha, Phys. Rev. B, 2015.

Topological phase diagram

- At the band-touch points, Chern numbers may change. Therefore, the locations of $k^{\pm}$ give the boundary of different (10) topological phases in the parameter space $(\omega_{\mathrm{p}}, \Omega, k_z)$.





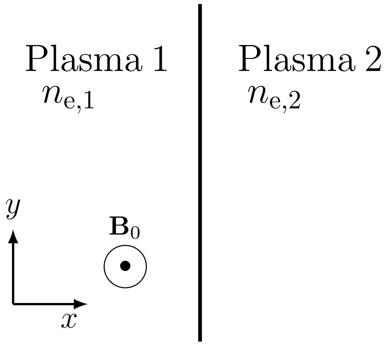


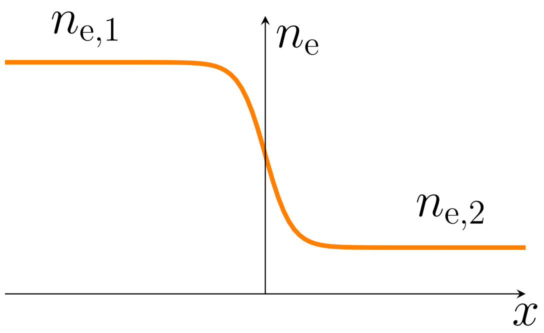
- For example, when $\Omega = 1$, the phase diagram (at $k_z > 0$) and Chern numbers shown above.

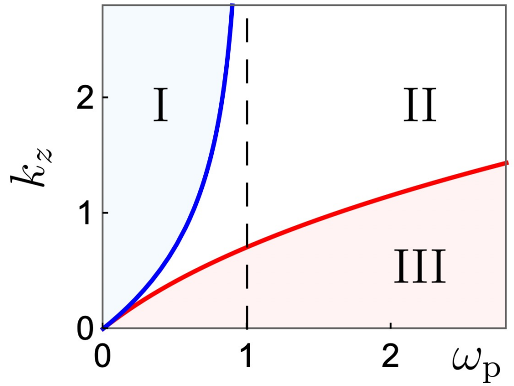
- Notice that phase I includes the case $\omega_{\mathrm{p}} \to 0$, which is the vacuum.

Edge modes in non-uniform plasma

- Consider the following non-uniform plasma:







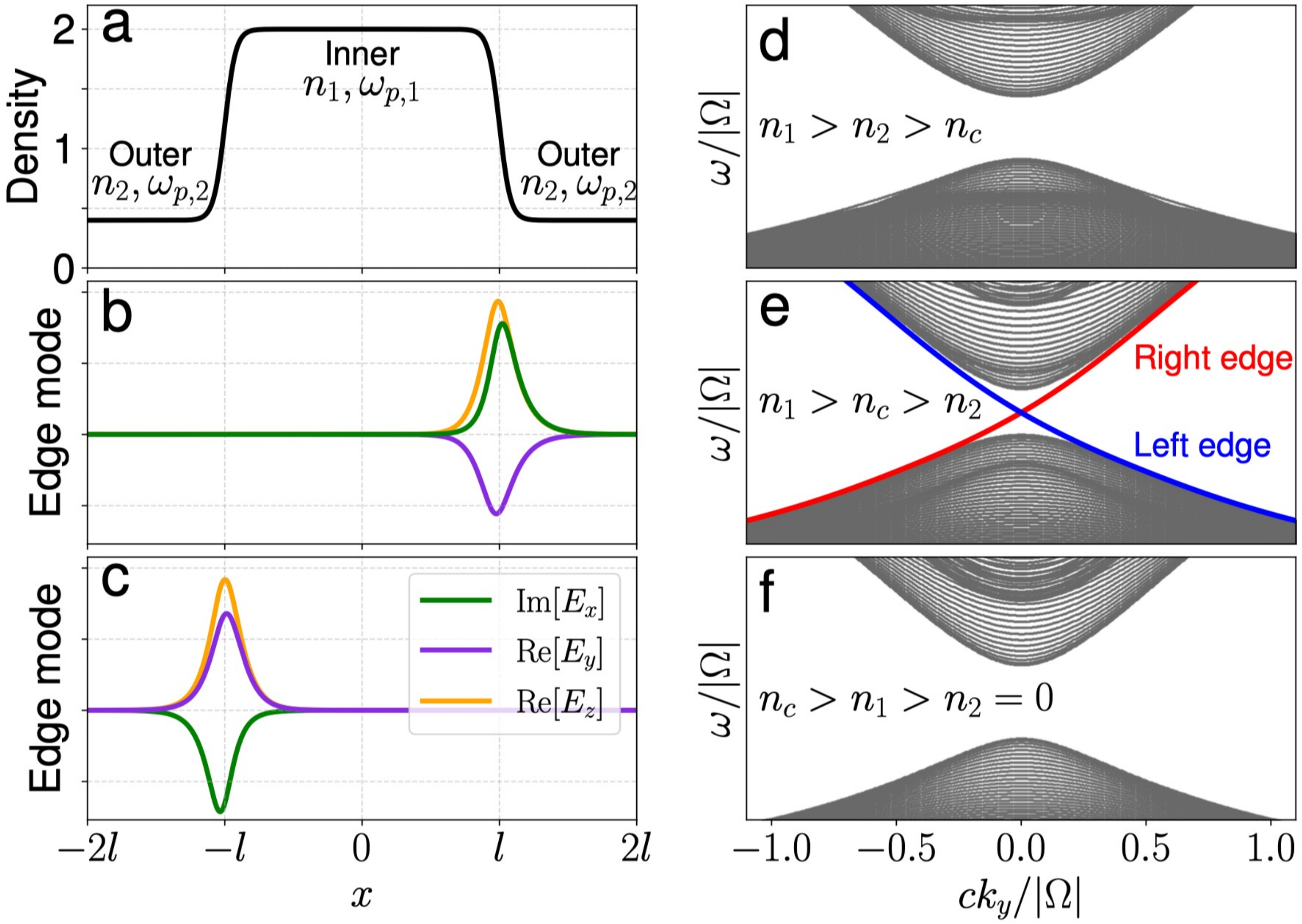
- For each given Ω, k_z. If and only if two sides of plasma are in different topological phases can the edge modes exist. The boundary of different phases gives a critical density n_c. The condition of the existence of edge modes is:

$$n_1 > n_c > n_2 \quad \Rightarrow \quad \frac{\omega_{p,1}}{|\Omega|} + \frac{\omega_{p,1}^2}{c^2 k_z^2} > 1 > \frac{\omega_{p,2}}{|\Omega|} + \frac{\omega_{p,2}^2}{c^2 k_z^2}$$

- The edge modes can exist not only in plasma-vacuum interface, but also inside the plasma at the interface between different density regions.

Numerical calculation of edge modes

- For convenience, in x-direction the density profile is chosen to be periodic.
- Edge modes only exist when the previous condition is satisfied.
- This type of edge mode localized at density gradient is a unique feature of gaseous systems.



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Conclusions

- A Hamiltonian $\mathcal{H}(\mathbf{k})$ maps a wave vector $\mathbf{k}$ to its eigenvectors $|\psi_n(\mathbf{k})\rangle$. These maps define the topological phase of the system, which can be classified by various topological invariants.

- When two materials with different topological phases are put together, the topological edge modes may be found at the interface.

- In the cold magnetized plasma, there exist 10 topological phases in the parameter space $(\omega_{\mathrm{p}}, \Omega, k_z)$. Topological edge modes can be found if plasmas with different topological phases are physically put together.

- These topological methods provide a new perspective to understand various waves in uniform and nonuniform plasmas.

Discussions and future researches

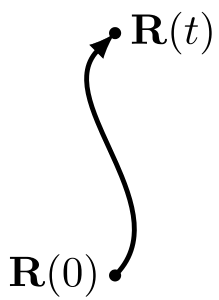
- The topological methods in fluids and plasmas were only developed recently. Some fundamental problems remain unsolved.

- Different plasma models usually have different topologies. The topology of more complicated and realistic plasma models is still unknown.

- Extending the topological ideas to non-Hermitian or non-linear systems is possible. However, the topologies are much more complicated and less understood.

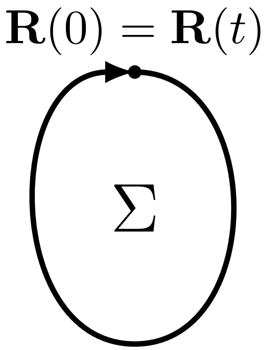
Appendix: Chern numbers as Berry phases

- Assume we Hamiltonian $\mathcal{H}(\mathbf{R})$ and eigenvectors $|\psi_n(\mathbf{R})\rangle$. When $\mathrm{d}\mathbf{R}(t)/\mathrm{d}t \to 0$, the initial state $|\Psi_n(0)\rangle = |\psi_n(\mathbf{R}(0))\rangle$ will obtain a geometric phase $e^{\mathrm{i}\gamma_n(t)}$:



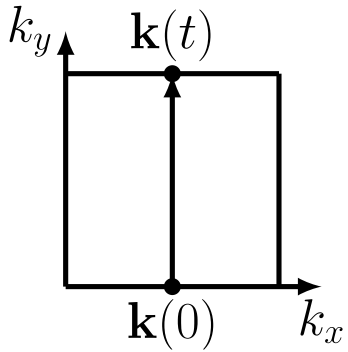
$$\gamma_n(t) = \mathrm{i}\int_0^t \mathrm{d}t' \, \langle \psi_n(\mathbf{R}(t'))|\frac{\mathrm{d}}{\mathrm{d}t'}|\psi_n(\mathbf{R}(t'))\rangle = \mathrm{i}\int_{\mathbf{R}(0)}^{\mathbf{R}(t)} \mathrm{d}\mathbf{R} \, \langle \psi_n(\mathbf{R})|\nabla_{\mathbf{R}}|\psi_n(\mathbf{R})\rangle.$$

- Define Berry connection $\mathcal{A}_n(\mathbf{R}) = \langle \psi_n(\mathbf{R})|\nabla_{\mathbf{R}}|\psi_n(\mathbf{R})\rangle$, if $\mathbf{R}(t)$ goes around a closed loop, this phase becomes:



$$\gamma_n = \oint_{\partial\Sigma} \mathcal{A}_n(\mathbf{R}) \cdot \mathrm{d}\mathbf{R} = \int_{\Sigma} (\nabla_{\mathbf{R}} \times \mathcal{A}_n) \cdot \mathrm{d}\mathbf{S} = \int_{\Sigma} \mathbf{\Omega}_n \cdot \mathrm{d}\mathbf{S}.$$

- We can choose $\mathbf{R}$ to be the wave vector $\mathbf{k}$, then the Chern numbers becomes the total Berry phase in $\mathbf{k}$-space.



$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \frac{\mathrm{d}(\hbar\mathbf{k})}{\mathrm{d}t} = q\mathbf{E} \quad \Rightarrow \quad \mathbf{k} = \mathbf{k}(t).$$