Geometric variational finite elements methods for fluids with application to MHD

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PLAN:

1. Geometric variational formulation of hydrodynamics

1.1 Lagrangian description of hydrodynamics:

Fluid dynamics in a compact manifold Ω with boundary.
Lagrangian motion $X \in \Omega \mapsto x = \varphi(t, X) \in \Omega$
Regarded as a dynamical system on a Lie group (Arnold [1965])

- Configuration Lie group: $G = \mathrm{Diff}(\Omega)$, compressible fluids;
  $G = \mathrm{Diff}_{\rm vol}(\Omega)$, incompressible fluids

- Lagrangian: $L : TG \to \mathbb{R}$,

$$L(\varphi, \dot\varphi) = \int_\Omega \frac{1}{2} \varrho_0 |\dot\varphi|^2 \mathrm{d}X - \int_\Omega E(\varphi, \nabla\varphi, \varrho_0, S_0) \mathrm{d}X$$

Depends on reference fields $\varrho_0(X)$, $S_0(X)$.

- Hamilton's principle: critical action principle for the flow $x = \varphi(t, X)$:

$$\delta \int_0^T L(\varphi, \dot\varphi) dt = 0, \quad \delta\varphi \text{ arbitrary variations } \longrightarrow \text{ Fluid equations in Lagr. variables.}$$

1.2 Eulerian (spatial) description of hydrodynamics

Invariance of L with respect to diffeomorphisms that preserve ϱ_0 and S_0

- Eulerian fields:

$$
\begin{aligned}
&u := \dot{\varphi} \circ \varphi^{-1} && \text{Eulerian velocity}\\
&\rho := (\varrho_0 \circ \varphi^{-1}) |\det D\varphi^{-1}| && \text{Eulerian mass density}\\
&s := (S_0 \circ \varphi^{-1}) |\det D\varphi^{-1}| && \text{Eulerian entropy density}
\end{aligned}
$$

- Lagrangian in Eulerian description:

$$\ell(u, \rho, s) = \int_\Omega \left[\frac{1}{2} \rho |u|^2 - \epsilon(\rho, s) \right] \mathrm{d}x$$

- Hamilton's principle in Eulerian form:
(Euler-Poincaré reduction, Holm, Marsden, Ratiu [1998])

$$\delta \int_0^T \ell(u, \rho, s) dt = 0, \quad \delta u = \partial_t \zeta + [u, \zeta], \quad \delta\rho = -\operatorname{div}(\rho\zeta), \quad \delta s = -\operatorname{div}(s\zeta).$$

- Equations of motion:

$$
\begin{cases}
\partial_t \dfrac{\delta \ell}{\delta u} + \mathcal{L}_u \dfrac{\delta \ell}{\delta u} = \rho \nabla \dfrac{\delta \ell}{\delta \rho} + s \nabla \dfrac{\delta \ell}{\delta s}\\
\partial_t \rho + \operatorname{div}(\rho u) = 0, \quad \partial_t s + \operatorname{div}(s u) = 0.
\end{cases}
$$

Goal: carry out the numerical discretization in a geometry preserving way by respecting the geometric variational formulation.

Main idea:
- "replace" this group by a finite dimensional Lie group approximation
- apply the variational principles on this finite dimensional Lie group
- temporal discretization in a structure preserving way

– Original idea & incompressible ideal case:
Pavlov, Mullen, Tong, Kanso, Marsden, Desbrun [2010]

– Several developments (motivated by GFD):
Rotating Boussinesq GFD equations: Desbrun, Gawlik, FGB, Zeitlin [2014]
Various generalizations of discrete group: Liu, Mason, Hodgson, Tong, Desbrun [2015]
Finite elements for incompressible: Natale and Cotter [2018]
Anelastic and pseudo-incompressible GFD & unstructured grids: Bauer and FGB [2017]
Compressible fluids & rotating shallow water: Bauer and FGB [2018]
On the sphere: Brecht, Bauer, Bihlo, FGB, MacLachlan [2019]

2. Discrete Lie group setting

SUMMARY OF THE APPROACH:

- STEP 1: Define the discrete diffeomorphism group
- STEP 2: Relate the Lie algebra of this group with discrete velocities
- STEP 3: Show that the space of discrete velocities is a Raviart-Thomas space (it is NOT a Lie subalgebra)
- STEP 4: Build a "Lie algebra-to-vector fields" map

STEP 1: Discrete diffeomorphism groups

- Discrete functions: finite element space $V_h \subset L^2(\Omega)$ associated to $\mathcal{T}_h$ (shape-regular and quasi-uniform family $\{\mathcal{T}_h\}$)

- Finite dimensional version of $\mathrm{Diff}(\Omega)$: chosen as

$$G_h = \{q \in GL(V_h) \mid q\mathbf{1} = \mathbf{1}\},$$

- Lie algebra

$$\mathfrak{g}_h = \{A \in L(V_h, V_h) \mid A\mathbf{1} = 0\}$$

$\leadsto$ Potential candidates to be discrete vector fields;
$\leadsto$ As linear maps these discrete vector fields act as discrete derivations on V_h;
$\leadsto$ Natural to choose them as discrete distributional directional derivatives.

STEP 2: Link with discrete velocities

$H(\mathrm{div}, \Omega) = \{u \in L^2(\Omega)^n \mid \mathrm{div}\, u \in L^2(\Omega)\}$.
$H_0(\mathrm{div}, \Omega) = \{u \in H(\mathrm{div}, \Omega) \mid u \cdot n = 0 \text{ on } \partial\Omega\}$.
$V_h = V_h^r$.

Proposition (Gawlik and FGB)

- To each velocity $u \in H_0(\mathrm{div}, \Omega) \cap L^p(\Omega)^n$, $p > 2$, $r \geq 0$ integer, we can associate the Lie algebra element $A_u \in \mathfrak{gl}(V_h^r)$ given by

$$\langle A_u f, g\rangle := \sum_{K \in \mathcal{T}_h} \int_K (\nabla_u f) g \, \mathrm{d}x - \sum_{e \in \mathcal{E}_h^0} \int_e u \cdot [\![f]\!] \{g\} \, \mathrm{d}s, \quad \forall\, f, g \in V_h^r.$$

- A_u is consistent approximation of the distributional derivative in direction u.

STEP 3: Relation with Raviart-Thomas finite element spaces

Not all the Lie algebra elements are of the form

$$A_u \quad \text{for some} \quad u \in H_0(\mathrm{div}, \Omega) \cap L^p(\Omega)^n, p > 2.$$

However:

Theorem (Gawlik and FGB)

The space $S_h^r := \{A_u \mid u \in H_0(\mathrm{div}, \Omega) \cap L^p(\Omega)^n, p > 2\}$ is isomorphic to the Raviart-Thomas space of order $2r$

$$RT_{2r}(\mathcal{T}_h) = \left\{ u \in H_0(\mathrm{div}, \Omega) \mid u|_K \in (P_{2r}(K))^n + xP_{2r}(K), \forall K \in \mathcal{T}_h \right\}.$$

Link between the geometric variational discretization and finite element methods.

STEP 4: The "Lie algebra-to-vector fields" map

- Construct a method valid for a large class of Lagrangians
  $\rightsquigarrow$ Lie algebra-to-vector fields map
- Use of Lagrange-d'Alembert principle of nonholonomic mechanics (e.g. Bloch [2003])
  $\rightsquigarrow$ Lie algebra-to-vector fields map at least defined on $S_h^r + [S_h^r, S_h^r]$

(cannot use $A_u \mapsto u$!!)

Definition

For $r \geq 0$ define **Lie algebra-to-vector field map** $\widehat{\ }: L(V_h^r, V_h^r) \to [V_h^r]^n$

$$\widehat{A} := \sum_{k=1}^{n} A(I_h^r(x^k))e_k,$$

$I_h^r : L^2(\Omega) \to V_h^r$: L^2-orthogonal projector onto V_h^r.

SUMMARY

$$
\begin{tikzcd}
H_0(\mathrm{div}, \Omega) \cap L^p(\Omega)^n \arrow[r, "A"] \arrow[d, bend right, "\Pi_{2r}"'] & S_h^r \arrow[r, hook] & \mathfrak{g}_h^r \arrow[r, "\widehat{\ }"] & {[V_h^r]^n} \\
RT_{2r}(\mathcal{T}_h) \arrow[u, hook] \arrow[ur, <->] & \Delta_h \arrow[u, hook] & & \\
R_h \arrow[u, hook] \arrow[ur, <->] & & &
\end{tikzcd}
$$

Not yet complete....

3. Variational integrator

3.1 Semidiscrete Euler-Poincaré-d'Alembert equations

Continuous Lagrangian $\ell(u, \rho) \rightsquigarrow$ Discrete Lagrangian $\ell_d(A, \rho) := \ell(\widehat{A}, \rho)$.

- Euler-Poincaré-d'Alembert variational principle

Given $g(t) \in G_h^r$ define $A(t) = \dot{g}(t) g(t)^{-1}$ and $\rho(t) = \rho_0 \cdot g(t)^{-1}$

The following are equivalent for $A(t) \in \Delta_h$ and $\rho(t) \in V_h^r$:

(i) $\delta \int_0^T \ell_d(A, \rho) \mathrm{d}t = 0, \qquad \delta A = \partial_t B + [B, A]$ and $\delta \rho = -\rho \cdot B$,

for all $B(t) \in \Delta_h$ with $B(0) = B(T) = 0$.

(ii) $\left\langle\!\!\left\langle \partial_t \frac{\delta \ell_d}{\delta A}, B \right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \frac{\delta \ell_d}{\delta A}, [A, B] \right\rangle\!\!\right\rangle + \left\langle \frac{\delta \ell_d}{\delta \rho}, \rho \cdot B \right\rangle = 0, \quad \forall t \in (0, T), \quad \forall\, B \in \Delta_h.$

equivalently $\partial_t \frac{\delta \ell_d}{\delta A} + \mathrm{ad}_A^* \frac{\delta \ell_d}{\delta A} - \frac{\delta \ell_d}{\delta \rho} \diamond \rho \in \Delta_h^\circ, \quad \forall t \in (0, T)$

Differential equation for $\delta(t)$:

$$\langle \partial_t \rho, \sigma \rangle + \langle \rho, A\sigma \rangle = 0, \quad \forall t \in (0, T), \quad \forall\, \sigma \in V_h^r.$$

- Choice of Δ_h

$\Delta_h \subset S_h^r$ such that

$$A \in \Delta_h \to \frac{\delta \ell_d}{\delta A}(A, \rho) \in (\mathfrak{g}_h^r)^* / \Delta_h^\circ$$

is a diffeomorphism for all $\rho \in V_h^r$ strictly positive.

$$\begin{tikzcd}
H_0(\mathrm{div}, \Omega) \cap L^p(\Omega)^n \arrow[r, "A"] \arrow[d, bend right, "\Pi_{2r}"'] & S_h^r \arrow[r, hook] \arrow[dl, leftrightarrow] & \mathfrak{g}_h^r \arrow[r, "\widehat{\ }"] & {[V_h^r]^n} \\
RT_{2r}(\mathcal{T}_h) \arrow[u, hook] & \Delta_h \arrow[u, hook] & & \\
R_h \arrow[u, hook] \arrow[ur, leftrightarrow] & & &
\end{tikzcd}$$

4. Compressible fluids

Focus on the barotropic fluid for simplicity

$$\ell(u,\rho) = \int_\Omega \left[\frac{1}{2}\rho|u|^2 - \rho e(\rho)\right]\mathrm{d}x$$

All results below can be naturally extended to more general Lagrangians (Gawlik and FGB [2020])

4.1 Discrete Lagrangian

$$\ell_d(A,\rho) := \ell(\widehat{A},\rho) = \int_\Omega \left[\frac{1}{2}\rho|\widehat{A}|^2 - \rho e(\rho)\right]\mathrm{d}x.$$

$$\frac{\delta \ell_d}{\delta A} = I_h^r(\rho\widehat{A})^\flat, \quad \frac{\delta \ell_d}{\delta \rho} = I_h^r\left(\frac{1}{2}|\widehat{A}|^2 - e(\rho) - \rho\frac{\partial e}{\partial \rho}\right).$$

4.2 Choice of Δ_h (R_h)

Lemma

$$\ker\left(A \in \Delta_h \to I_h^r(\rho\widehat{A})^\flat \in (\mathfrak{g}_h^r)^*/\Delta_h^\circ\right) = \{0\} \iff R_h \subset BDM_r(\mathcal{T}_h)$$

4.3 Geometric variational element scheme for compressible fluids

- Equations of motion

$$\begin{cases} \left\langle \partial_t(\rho\widehat{A}), \widehat{B}\right\rangle + \left\langle \rho\widehat{A}, \widehat{[A,B]}\right\rangle + \left\langle I_h^r\left(\frac{1}{2}|\widehat{A}|^2 - e(\rho) - \rho\frac{\partial e}{\partial \rho}\right), \rho\cdot B\right\rangle = 0, \quad \forall B \in \Delta_h \\ \left\langle \partial_t\rho, \sigma\right\rangle + \left\langle \rho, A\sigma\right\rangle = 0, \quad \forall \sigma \in V_h^r. \end{cases}$$

- Equivalently, in terms of ρ_h, $u_h = -\widehat{A}$, σ_h, and $v_h = -\widehat{B}$:
  Seek $u_h \in R_h$ and $\rho_h \in V_h^r$ such that

$$\begin{cases} \left\langle \partial_t(\rho_h u_h), v_h\right\rangle + a_h(w_h, u_h, v_h) - b_h(v_h, f_h, \rho_h) = 0, \quad \forall v_h \in R_h \\ \left\langle \partial_t\rho_h, \sigma_h\right\rangle - b_h(u_h, \sigma_h, \rho_h) = 0, \quad \forall \sigma_h \in V_h^r, \end{cases}$$

- $w_h = I_h^r(\rho_h u_h)$, $f_h = I_h^r\left(\frac{1}{2}|u_h|^2 - e(\rho_h) - \rho_h\frac{\partial e}{\partial \rho_h}\right)$, and

- $a_h(w,u,v) = \displaystyle\sum_{K\in\mathcal{T}_h}\int_K w\cdot(v\cdot\nabla u - u\cdot\nabla v)\,dx + \sum_{e\in\mathcal{E}_h^0}\int_e (v\cdot n[u] - u\cdot n[v])\cdot\{w\}\mathrm{d}s$

- $b_h(w,f,g) = \displaystyle\sum_{K\in\mathcal{T}_h}\int_K (w\cdot\nabla f)g\,dx - \sum_{e\in\mathcal{E}_h^0} w\cdot[[f]]\{g\}\mathrm{d}s.$

4.4 Temporal discretization

OPTION 1: variational discretization

OPTION 2: energy preserving discretization

$$\left\langle\!\left\langle \frac{1}{\Delta t}\left(\frac{\delta \ell_d}{\delta A_k}-\frac{\delta \ell_d}{\delta A_{k-1}}\right), B_k \right\rangle\!\right\rangle + \frac{1}{2}\left\langle\!\left\langle \frac{\delta \ell_d}{\delta A_{k-1}}+\frac{\delta \ell_d}{\delta A_k}, [A_{k-1/2}, B_k] \right\rangle\!\right\rangle + \langle F_{k-1/2}, \rho_{k-1/2}\cdot B_k\rangle = 0, \quad \forall\, B_k \in \Delta_h,$$

$$\left\langle \frac{\rho_k-\rho_{k-1}}{\Delta t}, E_k \right\rangle + \langle \rho_{k-1/2}\cdot A_{k-1/2}, E_k\rangle = 0, \quad \forall E_k \in V_h^r.$$

where

$$F_{k-1/2} = \frac{1}{2}\widehat{A_{k-1}}\cdot \widehat{A_k} - f(\rho_{k-1},\rho_k), \qquad f(x,y) = \frac{ye(y)-xe(x)}{y-x}$$

(reminiscent of a discrete gradient method Hairer, Lubich, Wanner [2006])

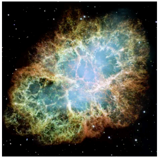
4.5 Rayleigh-Taylor instability

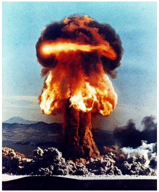
$$\ell(u,\rho,s) = \int_\Omega \left[\frac{1}{2}\rho|u|^2 - \rho e(\rho,\eta) - \rho\phi\right]\mathrm{d}x, \qquad e(\rho,\eta) = Ke^{\eta/C_v}\rho^{\gamma-1}$$

where $\gamma = 5/3$, $K = C_v = 1$, $\phi = -y$

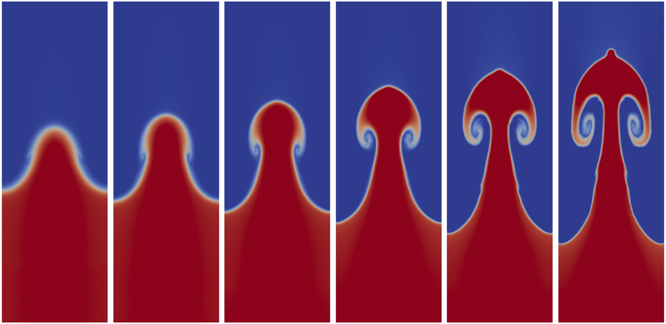
$\Omega = (0,1/4)\times(0,1)$, $R_h = RT_0(\mathcal{T}_h)$ and V_h^1 on uniform $\mathcal{T}_h$, $h = 2^{-8}$, with upwinding (later), $\Delta t = 0.01$.

$$\rho(x,y,0) = 1.5 - 0.5\tanh\left(\frac{y-0.5}{0.02}\right)$$





Contours of the mass density at $t = 1.0, 1.2, 1.4, 1.6, 1.8, 2.0$ in the Rayleigh-Taylor instability simulation with the energy-preserving time discretization



Energy preserved exactly up to roundoff errors.

5.1 Equations and variational principle

- Lagrangian version (Hamilton's principle): seek $\varphi : [0, T] \to \mathrm{Diff}_{\rm vol}(\Omega)$ such that

$$\delta \int_0^T L(\varphi, \dot\varphi)\mathrm{d}t = 0, \qquad L(\varphi, \dot\varphi) = \int_\Omega \frac{1}{2}\varrho_0 |\dot\varphi|^2 \mathrm{d}X$$

for all $\delta\varphi$ vanishing at the endpoints.

- Eulerian version (Euler-Poincaré)

$$\delta \int_0^T \ell(u, \rho)\mathrm{d}t = 0, \qquad \ell(u, \rho) = \int_\Omega \frac{1}{2}\rho |u|^2 \mathrm{d}x$$

for all variations δu, $\delta\rho$ of the form

$$\delta u = \partial_t \zeta + [u, \zeta], \qquad \delta\rho = -\operatorname{div}(\rho\zeta)$$

- Incompressible fluids with variable density

$$\begin{cases} \rho(\partial_t u + u \cdot \nabla u) = -\nabla p, \\ \partial_t \rho + \operatorname{div}(\rho u) = 0, \qquad \operatorname{div} u = 0. \end{cases}$$

5.2 Discrete setting

$$G_h^r = \{q \in GL(V_h^r) \mid q\mathbf{1} = \mathbf{1},\ \langle qf, qg\rangle = \langle f, g\rangle\},$$

$$\mathfrak{g}_h^r = \{A \in L(V_h^r, V_h^r) \mid A\mathbf{1} = 0,\ \langle Af, g\rangle + \langle f, Ag\rangle = 0,\ \forall\, f, g \in V_h^r\}$$

$R_h = \{u \in BDM_r(\mathcal{T}_h), \operatorname{div} u = 0\} \rightsquigarrow \Delta_h \subset \mathfrak{g}_h^r$

$$\ell_d(A, \rho) := \ell(\widehat{A}, \rho) = \int_\Omega \frac{1}{2}\rho|\widehat{A}|^2 \mathrm{d}x.$$

$$\frac{\delta \ell_d}{\delta A} = I_h^r(\rho\widehat{A})^\flat, \quad \frac{\delta \ell_d}{\delta \rho} = I_h^r\left(\frac{1}{2}|\widehat{A}|^2\right).$$

5.3 Spatial discretization

Proposition (Gawlik & FGB)

The semidiscrete solution satisfies

$$\frac{d}{dt}\int_\Omega \rho_h \mathrm{d}x = 0, \qquad \frac{d}{dt}\int_\Omega \rho_h^2 \mathrm{d}x = 0, \qquad \frac{d}{dt}\int_\Omega \rho_h |u_h|^2 \mathrm{d}x = 0, \qquad \operatorname{div} u_h = 0$$

5.4 Temporal discretization

Proposition (Gawlik & FGB)

The fully discrete solution of incompressible fluid with variable density satisfies

$$\int_\Omega \rho_{k+1} \mathrm{d}x = \int_\Omega \rho_k \mathrm{d}x, \qquad \int_\Omega \rho_{k+1}^2 \mathrm{d}x = \int_\Omega \rho_k^2 \mathrm{d}x,$$

$$\int_\Omega \frac{1}{2}\rho_{k+1}|u_{k+1}|^2 \mathrm{d}x = \int_\Omega \frac{1}{2}\rho_k |u_k|^2 \mathrm{d}x, \qquad \operatorname{div} u_k = 0$$

(upwind version exists which preserves all equalities except the 2^{nd} which becomes $\leq$)

5.5 Rayleigh-Taylor instability

Finite element spaces $u_k \in R_h = RT_0(\mathcal{T}_h)$, $\rho_k \in DG_1(\mathcal{T}_h)$, and $p_k \in DG_0(\mathcal{T}_h) \cap L^2_{\int=0}(\Omega)$ on uniform $\mathcal{T}_h$, $h = 2^{-j}$, $j = 4, 5, 6$, $\Delta t = 0.01$, with upwind.

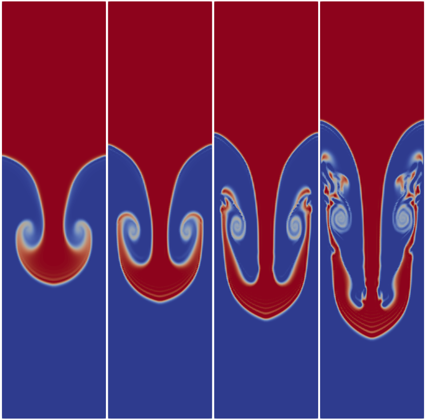


Figure: Density contours at $t = 0.8, 0.95, 1.1, 1.25$ in the Rayleigh-Taylor instability simulation with $h = 2^{-6}$.

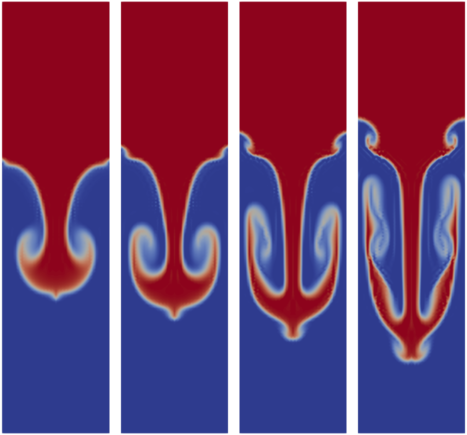


Figure: Density contours at $t = 0.8, 0.95, 1.1, 1.25$ in the Rayleigh-Taylor instability simulation, obtained using Guermond and Quartapelle [2000] with $h = 2^{-5}$, $\Delta t = 0.01$ (preserves the 3 invariants only before temporal discretization, preserves incompressibility in a weak sense).

The two methods under comparison produce qualitatively similar results for $t < 1$, and begin to deviate somewhat as t increases.

6. MHD

Same variational philosophy applies to MHD.

- Eulerian fields:

$$u := \dot{\varphi} \circ \varphi^{-1} \qquad \text{Eulerian velocity}$$

$$\rho := (\varrho_0 \circ \varphi^{-1}) |\det D\varphi^{-1}| \qquad \text{Eulerian mass density}$$

$$B := \varphi_* B_0 \qquad \text{Eulerian magnetic field}$$

- Lagrangian in Eulerian description:

$$\ell(u, \rho, B) = \int_\Omega \left[\frac{1}{2} \rho |u|^2 - \frac{1}{2} |B|^2 \right] \mathrm{d}x$$

- Hamilton's principle in Eulerian form (Euler-Poincaré):

$$\delta \int_0^T \ell(u, \rho, B) dt = 0, \quad \delta u = \partial_t \zeta + [u, \zeta], \quad \delta \rho = -\operatorname{div}(\rho \zeta), \quad \delta B = \nabla \times (\zeta \times B).$$

- Equations of motion:

$$\begin{cases} \partial_t \dfrac{\delta \ell}{\delta u} + \mathcal{L}_u \dfrac{\delta \ell}{\delta u} = \rho \nabla \dfrac{\delta \ell}{\delta \rho} + B \times (\nabla \times \dfrac{\delta \ell}{\delta B}) \\ \partial_t \rho + \operatorname{div}(\rho u) = 0, \quad \partial_t B - \nabla \times (u \times B) = 0. \end{cases}$$

- Boundary conditions:

$$u \cdot n = B \cdot n = 0.$$

Preserving magnetic helicity, cross-helicity, div $u = 0$, div $B = 0$ at the discrete level, turns out to be a very delicate issue!

Appropriate choice of the finite element spaces leads to

Proposition (Gawlik & FGB)

The fully discrete solution of incompressible MHD with variable density satisfies

$$\int_\Omega \rho_{k+1} \mathrm{d}x = \int_\Omega \rho_k \mathrm{d}x, \qquad \int_\Omega \rho_{k+1}^2 \mathrm{d}x = \int_\Omega \rho_k^2 \mathrm{d}x, \qquad \mathrm{div}\, u_k = 0, \qquad \mathrm{div}\, B_k = 0,$$

$$\int_\Omega \frac{1}{2}\rho_{k+1}|u_{k+1}|^2 + B_{k+1}\cdot B_{k+1} \mathrm{d}x = \int_\Omega \frac{1}{2}\rho_k |u_k|^2 + B_k \cdot B_k \mathrm{d}x$$

$$\int_\Omega A_{k+1}\cdot B_{k+1}\mathrm{d}x = \int_\Omega A_k \cdot B_k \mathrm{d}x$$

$$(\text{if } \rho_0 \equiv 1) \quad \int_\Omega u_{k+1}\cdot B_{k+1}\mathrm{d}x = \int_\Omega u_k \cdot B_k \mathrm{d}x$$

with A_k any vector field satisfying $\nabla \times A_k = B_k$ and $A_k \times n|_{\partial\Omega} = 0$.

– upwind version exists which preserves all equalities except the 2^{nd} which becomes $\leq$.

– compressible MHD (Gawlik and FGB [2021]): more involved, preserves exactly total energy, magnetic helicity, mass, $\operatorname{div} B = 0$.

– much of the literature focuses on the setting of constant density.
Some previous works in that setting
energy-stable schemes that preserve $\operatorname{div} B = 0$ Hu, Ma, Xu [2017];
energy-stable schemes that preserve $\operatorname{div} u = \operatorname{div} B = 0$ Hiptmair, Mao, Zhen [2018];
schemes that preserve energy, cross-helicity, and $\operatorname{div} u = \operatorname{div} B = 0$ Liu, Wang [2001], Gawlik, Mullen, Pavlov, Marsden, Desbrun [2011];
schemes that preserve energy, cross-helicity, $\int_\Omega A\,dx$, and $\operatorname{div} u = \operatorname{div} B = 0$ in two dimensions Kraus and Maj [2017];
scheme that preserves energy, cross-helicity, magnetic helicity, and $\operatorname{div} B = 0$ Hu, Lee, and Xu [2020] (not the same boundary conditions for u, weak divergence free condition, ρ constant).

Illustration of the structure-preserving properties in 3D:
$\Omega = [-1, 1]^3$ with initial conditions

$$
\begin{aligned}
u(x, y, z, 0) &= \left(ye^{-4(x^2+y^2)}, -xe^{-4(x^2+y^2)}, 0\right)^1, \\
B(x, y, z, 0) &= \nabla \times \left((1-x^2)(1-y^2)(1-z^2)v(x, y, z)\right), \\
\rho(x, y, z, 0) &= 2 + \sin(xy),
\end{aligned}
$$

with $v(x, y, z) = \frac{1}{2}(\sin \pi x, \sin \pi y, \sin \pi z)$.

- uniform triangulation $\mathcal{T}_h$ of Ω with maximum element diameter $h \approx 0.433$
- $u_h, B_h \in U_h^{\mathrm{div}} = RT_0(\mathcal{T}_h)$, $\rho_h \in DG_0(\mathcal{T}_h)$, $p_h \in DG_0(\mathcal{T}_h) \cap L^2_{\int=0}(\Omega)$.
- $\Delta t = 0.02$

[1] This vector field does not satisfy $u \cdot n = 0$ on $\partial\Omega$; hence, we used the nearest (in the L^2-norm) element of $U_h^{\mathrm{div}} \cap \mathring{H}(\mathrm{div}, \Omega)$ to $u(x, y, z, 0)$ as our initial condition for u.

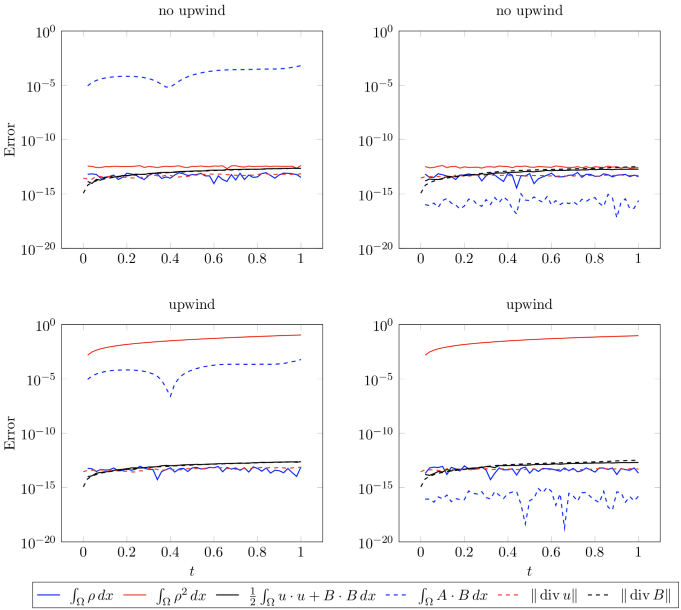


Figure: Error $|F(t) - F(0)|$ in conserved quantities during a three-dimensional simulation with variable density (left: a simpler version that does not preserve $\int A \cdot B \mathrm{d}x$ exactly).

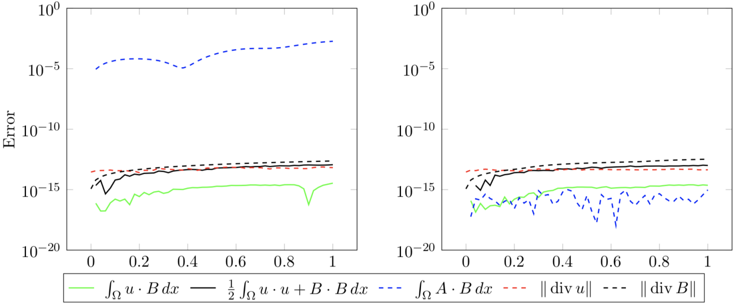


Figure: Error $|F(t) - F(0)|$ in conserved quantities during a three-dimensional simulation with constant density (left: a simpler version that does not preserve $\int A \cdot B \mathrm{d}x$ exactly).

7. Conclusion

- Used the geometric formulation of hydrodynamics to design structure preserving spatial and temporal discretization of fluid flow valid in 2D and 3D.
- Common approach to a large class of fluid models (GFD: SWE, QG, full compressible fluids, anelastic, etc).
- All the steps are guided by geometry (including choice of finite element space).
- The geometric approach yields new schemes via a constructive approach and more conservation properties.
- This geometric framework allows for several natural extensions to other fluid models - under investigation.

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THANK YOU FOR YOUR ATTENTION!