Differential Equation

Dr. Md. Manirul Alam Sarker
Professor
Department of Mathematics, BUET

Introduction to Differential Equation

❑ Definitions and Terminology

A Differential Equation is an equation containing the derivatives of one or more dependent variables with respect to one or more independent variables.

Examples:

unknown function
or dependent variable

$$\frac{d^2x}{dt^2} + 16x = 0$$

independent variable

an ODE can contain more
than one unknown function

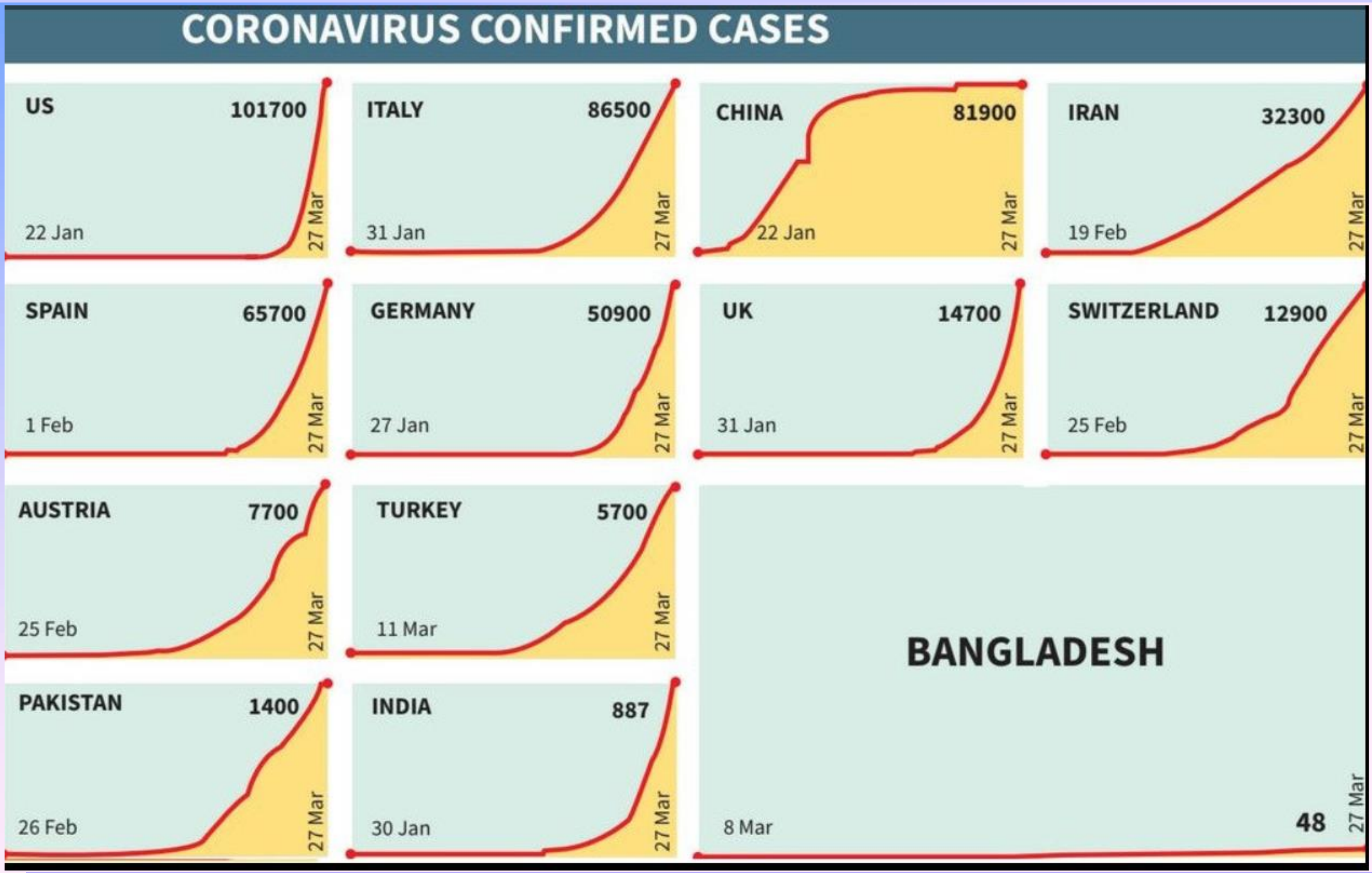
$$\frac{dy}{dx} + 5y = e^x, \quad \frac{d^2y}{dx^2} - \frac{dy}{dx} + 6y = 0, \quad \text{and} \quad \frac{dx}{dt} + \frac{dy}{dt} = 2x + y$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

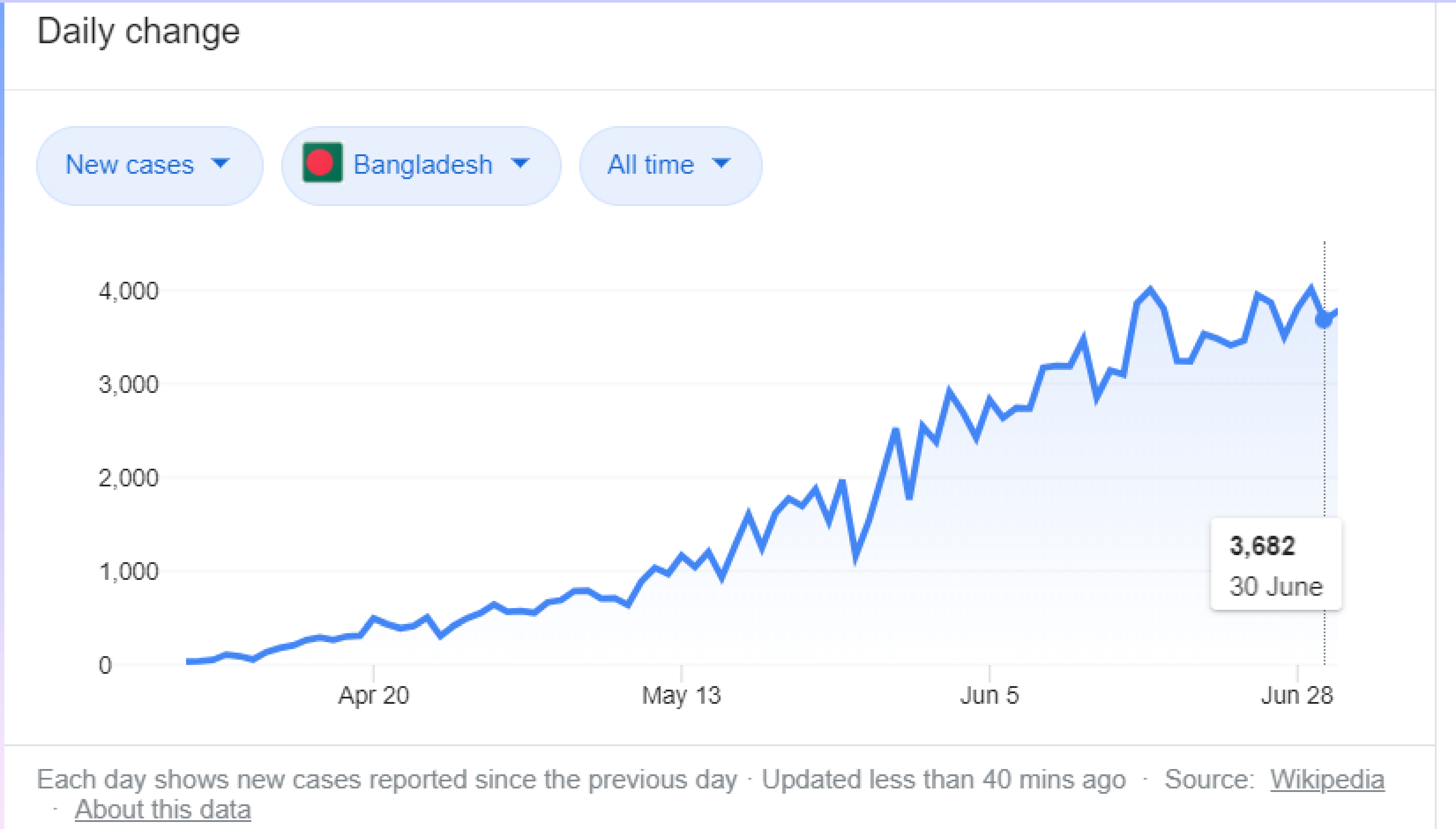
Introduction to Differential Equation

- **CALCULUS is widely used for Mathematical Modeling. The model often takes the form of a differential equation.**
- **Perhaps the most important of all the applications of calculus is to differential equations.**
- **When physical or social scientists use calculus, more often than not, it is to analyze a differential equation that has arisen in the process of modeling some phenomenon they are studying.**
- **In a real-world problem, we often notice that changes occur, and we want to predict future behavior on the basis of how current values change.**
- **Let's see the graph of current contemporary issue: COVID 19**

Introduction to DE continues



Introduction to DE continues



Introduction to DE continues

Total ▾ | Bangladesh ▾

Confirmed	Recovered	Deaths
149K	62,108	1,888
+3,775		+41

Location	Confirmed ↓	Recovered	Deaths
Dhaka Division	15,790	-	-
Chittagong Division	4,731	-	-
Mymensingh Division	992	-	-
Rangpur Division	857	-	-
Sylhet Division	663	-	-
Rajshahi Division	656	-	-
Khulna Division	505	-	-
Barisal Division	231	-	-

'+' shows new cases reported yesterday · Updated less than 19 hours ago · Source: Wikipedia · About this data

Introduction to DE continues

The history of differential equations is usually linked with Newton, Leibniz, and the development of calculus in the seventeenth century, and with other scientists who lived at that period of time, such as those belonging to the Bernoulli family.

Introduction to DE continues

Differential equations are introduce in different fields and its importance appears not only in mathematics but also in Engineering, Natural science, Chemical science, Medicine, Ecology and Economy.

Mechanics

Engineering

Biology

Chemistry

Economics

Introduction to DE continues

Classification

Differential Equations are classified by - Type, Order & Linearity

Classification by Type

Ordinary Differential Equation:

If a Differential Equation contains only ordinary derivatives of one or more dependent variables with respect to a single independent variable, it is said to be an Ordinary Differential Equation (ODE).

For Example:

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0 \qquad \frac{dy}{dx} = 2xy \qquad x\frac{dy}{dx} = y - 1$$

Classification by Type continues

Partial Differential Equation

If a Differential Equation contains partial derivatives of one or more dependent variables of two or more independent variables, it is said to be a Partial Differential Equation or (PDE)

For Example:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Introduction to DE continues

LANGUAGE OF DIFFERENTIAL EQUATIONS

- Degree of ODE
- Order of ODE
- Solutions of ODE
  - General Solution
  - Particular Solution
  - Trivial Solution
  - Singular Solution
  - Explicit And Implicit Solution
- Homogeneous Equations
- Non-homogeneous Equtions
- Integrating Factor

Classification by Order

The order of the differential equation is the order/index/exponent of the highest derivative involved in it. Degree of a DE is the power/index of the highest derivative after the equation has been put in the form free from radicals and fractions.

$$\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} = 0 \qquad \text{Order} = 3$$

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0 \qquad \text{Order} = 2$$

$$x\frac{dy}{dx} = y - 1 \qquad \text{Order} = 1$$

General form of nth Order ODE is

$$\frac{d^ny}{dx^n} = f(x, y, y_1, y_2, \ldots, y_{(n)})$$

where f is a real valued continuous function.

Determining Order and Degree

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}=\rho------(A)$$

$$(A)\Rightarrow\left[1+\left(\frac{dy}{dx}\right)^2\right]^3=\rho^2\left(\frac{d^2y}{dx^2}\right)^2$$

It's an equation of second degree & 2nd order

Ex: Determine the degree of

$$\frac{d^2y}{dx^2}=k\left[1+\left(\frac{dy}{dx}\right)^2\right]^{5/2}$$

Classification **by Linearity** continues

Linear DE

An *n*th-order ordinary differential equation

$$F\left(x, y, y', y'', \ldots y^{(n)}\right) = 0$$

is said to be **linear** if F is linear in $y, y', y'', \ldots y^{(n)}$. In other words, it has the following general form:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots\ldots + a_2(x)\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

now for n = 1, $$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

and for n = 2, $$a_2(x)\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Classification **by Linearity** continues

Properties of a linear ODE:

- The dependent variable y and all its derivatives are of the first degree, that is, the power of each term involving y is 1.
- The coefficients $a_0, a_1, a_2, \ldots a_n$ of $y, y', y'', \ldots y^{(n)}$ depend at most on the independent variable x.

Example:

$$(y - x)\,dx + 4x\,dy = 0$$

$$y'' - 2y' + y = 0$$

$$x^3 \frac{d^3y}{dx^3} + x\frac{dy}{dx} - 5y = e^x$$

Classification **by Linearity** continues

Non-Linear DE

A nonlinear ODE is simply one that is not linear. It contains nonlinear functions of one of the dependent variable or its derivatives.

Examples:

nonlinear term: coefficient depends on y ↓

nonlinear term: nonlinear function of y ↓

nonlinear term: power not 1 ↓

$$(1-y)y' + 2y = e^x, \qquad \frac{d^2y}{dx^2} + \sin y = 0, \qquad \text{and} \qquad \frac{d^4y}{dx^4} + y^2 = 0$$

Formation of ordinary differential equation

An ordinary differential equation is formed by differentiating the equation and eliminating arbitrary constant(s).

Procedure

Let $f(x, y, c_1) = 0$ ---- (1)

be a solution of a differential equation with a single arbitrary constant c_1. differentiating (1) w.r.t. 'x' we have

$$\phi\left(x, y, c_1, \frac{dy}{dx} \right) = 0 \qquad (2)$$

Eliminating c_1 from (1) and (2), we get a lowest order ODE

$$\phi\left(x,\ y,\ \frac{dy}{dx}\right)=0$$

is the differential equation of (1)

Example: Form the differential equation of the lowest order by eliminating arbitrary constants, in the following cases and also write down the order and degree of the differential equations obtained. $y=ax+a^2$

Solution:

$$y=ax+a^2 \;--(1)\quad \frac{dy}{dx}=a\;--(2), \text{eliminating a}$$

$$\text{from(1)and(2), we get DE } y=\frac{dy}{dx}x+\left(\frac{dy}{dx}\right)^2$$

If $f(x, y, c_1, c_2) = 0 - - - - -(1)$

be a solution of a differential equation with two arbitrary constants c_1 and c_2 we require two more relations to eliminate c_1 and c_2. Differentiating (1) twice successively we have

$$f\left(x, y, c_1, c_2, \frac{dy}{dx}\right) = 0 \qquad (2)$$

$$f\left(x, y, c_1, c_2, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0 \qquad (3)$$

Eliminating c_1 and c_2 from (1) , (2)and (3), we get a lowest order ODE

$$F\left(x,\ y,\ \frac{dy}{dx},\ \frac{d^2y}{dx^2}\right)=0$$

In this way, we obtain *n*th order ODE by eliminating '*n*' arbitrary constants

Example: Find the differential equation by eliminating arbitrary constants *a* and *b* from the equation $y=ae^{3x}+be^{x}$

Solution:

$$y=ae^{3x}+be^{x}---(1)$$

$$\frac{dy}{dx}=3ae^{3x}+be^{x}---(2)$$

$$\frac{d^2y}{dx^2}=9ae^{3x}+be^{x}---(3)$$

from (2) and (3), we get

$$a = \frac{1}{6} e^{-3x} \left(\frac{d^2 y}{dx^2} - \frac{dy}{dx} \right) -- (4)$$

Putting the value of a in (3) we get

$$b = e^{-x} \left[\frac{d^2 y}{dx^2} - \frac{9}{6} e^{-3x} \left(\frac{d^2 y}{dx^2} - \frac{dy}{dx} \right) \right] -- \quad (5)$$

Substitute the values of *a* and *b* in (1)

$$y = \frac{1}{6} \left(\frac{d^2 y}{dx^2} - \frac{dy}{dx} \right) - \frac{1}{2} \frac{d^2 y}{dx^2} + \frac{3}{2} \frac{dy}{dx}$$

$$\frac{d^2 y}{dx^2} - \frac{4dy}{dx} + 3y = 0$$

is the required differential equation

Formation of ODE contd.

Ex: Find the DE of the family of circles of fixed radius r with centers on y axis.

Let the centers be (0, k). The equation of the circle is

$$x^2+(y-k)^2=r^2, ------(1)$$

$$\Rightarrow 2x+2(y-k)y_1=0$$

$$\Rightarrow y-k=-\frac{x}{y_1}-------(2)$$

$$(1) \text{ and } (2) \Rightarrow x^2-r^2+\frac{x^2}{y_1^2}=0$$

$$\Rightarrow (x^2-r^2)\left(\frac{dy}{dx}\right)^2+x^2=0 \square$$

Formation of ODE contd.

Ex: Find the differential equation of

$$y = Ae^{3x} + Be^{5x} -----(1)$$

for the different values of A & B.

$$y_1 = 3Ae^3x + 5Be^{5x} ------(2)$$

$$y_2 = 9Ae^{3x} + 25Be^{5x} ------(3)$$

Formation of ODE contd.

$$(2)-(3)\times 1 \Rightarrow y_1 - 3y = 2Be^{5x}$$

$$\Rightarrow Be^{5x} = \frac{1}{2}(y_1 - 3y) \text{ in to } (1) \text{ to get}$$

$$Ae^{3x} = y - Be^{5x} = y - \frac{1}{2}(y_1 - 3y)$$

$$= \frac{1}{2}(5y - y_1)$$

Putting Ae^{3x} & Be^{5x} in (3) we get

$$y_2 = \frac{9}{2}(5y - y_1) + \frac{25}{2}(y_1 - 3y)$$

$$\Rightarrow 2y_2 = 45y - 9y_1 + 25y_1 - 75y$$

$$\Rightarrow 2y_2 = 16y_1 - 30y$$

$$\Rightarrow y_2 = 8y_1 - 16y$$

$$\Rightarrow \frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 15y = 0 \;\square$$

Exercises on Formation of ODE

Exercises-1: Form the differential equation of the lowest order by eliminating arbitrary constants, in the following cases and also write down the order and degree of the differential equations obtained.

$(a)\, y = e^{2}\left(a\cos x + b\sin x\right).$

$(b)\, y = e^{mx}\left(A\cos nx + B\sin nx\right)$

$(c)\, y = e^{\frac{x}{a}} - e^{-\frac{x}{a}}$

$(d)\, x = c\cosh\dfrac{x}{c}$

Ex-2: Form an ODE corresponding to $x = 2t + c,\ y = ct + 3,$ **where t is a parameter.**

Exercises on Formation of ODE

Ex-3: Find the differential equation of all circles passing through the origin and having their centres on the y-axis.

Ex-4: Form the differential equation of all parabolas whose axes are parabolas whose the axis of y.

Ex-5: Consider the equation of simple harmonic motion $x = A\cos(pt - \alpha)$. Form the DE by eliminating A, α and p.

Ex-6: Find the differential equation of all parabolas whose axis is the axis of x.

Thanks a lot …