Solving First Order Differential Equations

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Solving First Order Differential Equations

- Introduction: Some important analytical methods for solving first-order DE $dy/dx = f(x, y)$:

➢ Variables Separable Equation

➢ Exact Differential Equation

➢ Linear Differential Equation

➢ Linear Differential Equation, by Suitable Substitution

➢ Bernoulli's Differential Equation

➢ Homogeneous Differential Equation

Separable Differential Equations

- Introduction: Consider $dy/dx = f(x, y) = g(x)$. The DE

$$dy/dx = g(x) \quad \ldots \qquad (1)$$

can be solved by integration.

Integrating both sides to get

$$y = \int g(x)\, dx = G(x) + c.$$

Ex: $dy/dx = 1 + e^{2x}$, then

$$y = \int (1 + e^{2x})\, dx = x + \tfrac{1}{2} e^{2x} + c$$

Separable Differential Equations continues

DEFINITION:

A first-order DE of the form

$$dy/dx = g(x)h(y)$$

is said to be separable or to have separable variables.

- Rewrite the above equation as

$$\frac{dy}{h(y)} = g(x)\,dx \qquad .... \qquad (2)$$

Now, it is easy to integrate

$$\int \frac{dy}{h(y)} = \int g(x)\,dx \quad .... \ (3)$$

Example 1

Solve $(1 + x)\, dy - y\, dx = 0$.

Solution: Since $dy/y = dx/(1 + x)$, we have

$$\int \frac{dy}{y} = \int \frac{dx}{1+x}$$

$$\ln|y| = \ln|1+x| + c_1$$

$$y = e^{\ln|1+x|+c_1} = e^{\ln|1+x|} \cdot e^{c_1} = |1+x|e^{c_1}$$

$$= \pm e^{c_1}(1+x)$$

Replacing $\pm e^{c_1}$ by c, gives $y = c(1 + x)$.

EXAMPLE 2 Solution Curve

Solve the initial-value problem $\dfrac{dy}{dx} = -\dfrac{x}{y}, \quad y(4) = -3.$

SOLUTION Rewriting the equation as $y\,dy = -x\,dx$, we get

$$\int y\,dy = -\int x\,dx \qquad \text{and} \qquad \frac{y^2}{2} = -\frac{x^2}{2} + c_1.$$

We can write the result of the integration as $x^2 + y^2 = c^2$ by replacing the constant $2c_1$ by c^2. This solution of the differential equation represents a family of concentric circles centered at the origin.

Now when $x = 4$, $y = -3$, so $16 + 9 = 25 = c^2$. Thus the initial-value problem determines the circle $x^2 + y^2 = 25$ with radius 5. Because of its simplicity we can solve this implicit solution for an explicit solution that satisfies the initial condition. We saw this solution as $y = \phi_2(x)$ or $y = -\sqrt{25 - x^2}$, $-5 \quad x \quad 5$ in Example 3 of Section 1.1. A solution curve is the graph of a differentiable function. In this case the solution curve is the lower semicircle, shown in dark blue in Figure 2.2.1 containing the point $(4, -3)$.

Example 2 continues

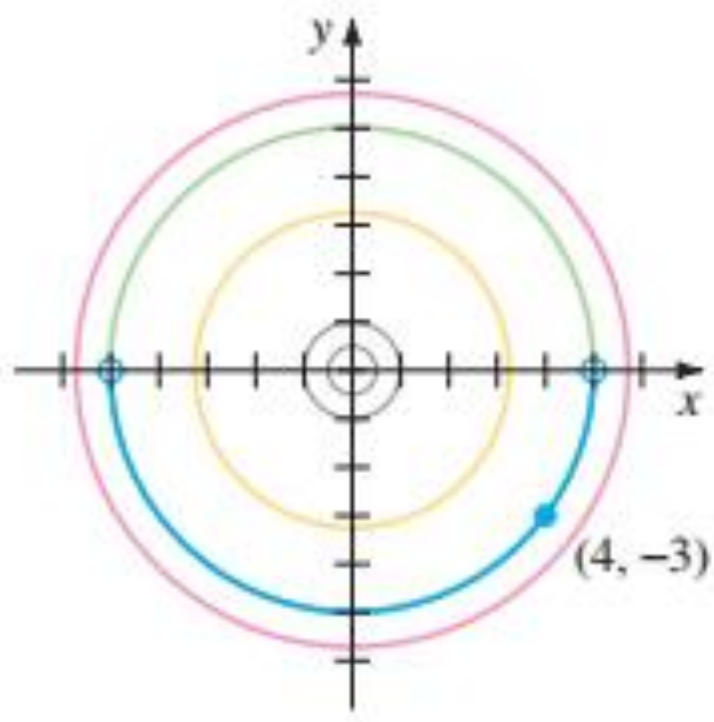

FIGURE 2.2.1 Solution curve for the IVP in Example 2

EXAMPLE 4 An Initial-Value Problem

Solve $(e^{2y} - y) \cos x \dfrac{dy}{dx} = e^y \sin 2x, \quad y(0) = 0.$

SOLUTION Dividing the equation by $e^y \cos x$ gives

$$\frac{e^{2y} - y}{e^y}\, dy = \frac{\sin 2x}{\cos x}\, dx.$$

Before integrating, we use termwise division on the left-hand side and the trigonometric identity $\sin 2x = 2 \sin x \cos x$ on the right-hand side. Then

integration by parts →
$$\int (e^y - ye^{-y})\, dy = 2 \int \sin x\, dx$$

yields
$$e^y + ye^{-y} + e^{-y} = -2 \cos x + c. \tag{7}$$

The initial condition $y = 0$ when $x = 0$ implies $c = 4$. Thus a solution of the initial-value problem is

$$e^y + ye^{-y} + e^{-y} = 4 - 2 \cos x. \tag{8}$$

Example Continues

Solve: $$y\sqrt{1+x^2}\,dy - x\sqrt{1+y^2}\,dx = 0$$

Solution : $$y\sqrt{1+x^2}\,dy - x\sqrt{1+y^2}\,dx = 0$$

$$\Rightarrow \frac{ydy}{\sqrt{1+y^2}} = \frac{xdx}{\sqrt{1+x^2}}$$

$$\Rightarrow \frac{d(1+y^2)}{\sqrt{1+y^2}} = \frac{d(1+x^2)}{\sqrt{1+x^2}}$$

Example Continues

$$\Rightarrow \left(1+y^2\right)^{-\frac{1}{2}} d\left(1+y^2\right) = \left(1+x^2\right)^{-\frac{1}{2}} d\left(1+x^2\right)$$

$$\Rightarrow \frac{\left(1+y^2\right)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = \frac{\left(1+x^2\right)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c_1$$

$$\Rightarrow 2\sqrt{1+y^2} = 2\sqrt{1+x^2} + c_1$$

$$\Rightarrow \sqrt{1+y^2} = \sqrt{1+x^2} + c \quad Ans$$

Example Continues

Solve: $$y\left(1+e^{\frac{x}{y}}\right)dx+e^{\frac{x}{y}}(y-x)dy=0$$

Solution: Rearranging, we have

$$\frac{dx}{dy}+\frac{e^{\frac{x}{y}}(y-x)}{y\left(1+e^{\frac{x}{y}}\right)}=0$$

Example Continues

Put $\dfrac{x}{y} = v$ then

Then the given equation can be written as

$$v + y\frac{dv}{dy} + \frac{e^{v}(y - yv)}{y(1+e^{v})} = 0$$

$$y\frac{dv}{dy} + \frac{e^{v}(1 - v)}{(1+e^{v})} = 0$$

Example Continues

or, $$y\frac{dv}{dy}+\frac{v+ve^{v}+e^{v}-ve^{v}}{1+e^{v}}=0$$

or, $$y\frac{dv}{dy}+\frac{v+ve^{v}+e^{v}-ve^{v}}{1+e^{v}}=0$$

or, $$y\frac{dv}{dy}+\frac{v+e^{v}}{1+e^{v}}=0$$

or, $$\frac{dy}{y}+\frac{1+e^{v}}{v+e^{v}}dv=0$$

Example Continues

Or, $$\frac{dy}{y} + \frac{d(v + e^{v})}{v + e^{v}} = 0$$

integrating

$$\ln y + \ln(v + e^{v}) = \ln c$$

$$or,\ \ln y(v + e^{v}) = \ln c$$

$$or,\ y(v + e^{v}) = c \ or,\ y(\frac{x}{y} + e^{\frac{x}{y}}) = c \ \ Ans.$$

Home work

1 Solve $(x+y)^2\left(x\dfrac{dy}{dx}+y\right)=xy\left(1+\dfrac{dy}{dx}\right)$

Ans. $\log xy=-\dfrac{1}{x+y}+c$

2 Solve $\left(y-x\dfrac{dy}{dx}\right)=3\left(1+x^2\dfrac{dy}{dx}\right)$

Ans. $(y\,\text{-}3)(\text{y}+3\text{x})=cx$

4 Solve $(x+y)^2\dfrac{dy}{dx}=a^2$

Ans. $y-a\tan^{-1}\dfrac{x+y}{a}=c$

Exercises

In Problems 1–22 solve the given differential equation by separation of variables.

1. $\dfrac{dy}{dx} = \sin 5x$

2. $\dfrac{dy}{dx} = (x + 1)^2$

3. $dx + e^{3x}dy = 0$

4. $dy - (y - 1)^2 dx = 0$

5. $x\dfrac{dy}{dx} = 4y$

6. $\dfrac{dy}{dx} + 2xy^2 = 0$

7. $\dfrac{dy}{dx} = e^{3x+2y}$

8. $e^x y\dfrac{dy}{dx} = e^{-y} + e^{-2x-y}$

9. $y \ln x \dfrac{dx}{dy} = \left(\dfrac{y + 1}{x}\right)^2$

10. $\dfrac{dy}{dx} = \left(\dfrac{2y + 3}{4x + 5}\right)^2$

11. $\csc y\, dx + \sec^2 x\, dy = 0$

12. $\sin 3x\, dx + 2y \cos^3 3x\, dy = 0$

13. $(e^y + 1)^2 e^{-y}\, dx + (e^x + 1)^3 e^{-x}\, dy = 0$

14. $x(1 + y^2)^{1/2}\, dx = y(1 + x^2)^{1/2}\, dy$

15. $\dfrac{dS}{dr} = kS$

16. $\dfrac{dQ}{dt} = k(Q - 70)$

17. $\dfrac{dP}{dt} = P - P^2$

18. $\dfrac{dN}{dt} + N = Nte^{t+2}$

19. $\dfrac{dy}{dx} = \dfrac{xy + 3x - y - 3}{xy - 2x + 4y - 8}$

20. $\dfrac{dy}{dx} = \dfrac{xy + 2y - x - 2}{xy - 3y + x - 3}$

21. $\dfrac{dy}{dx} = x\sqrt{1 - y^2}$

22. $(e^x + e^{-x})\dfrac{dy}{dx} = y^2$

In Problems 23–28 find an explicit solution of the given initial-value problem.

23. $\dfrac{dx}{dt} = 4(x^2 + 1),\quad x(\ \ /4) = 1$

24. $\dfrac{dy}{dx} = \dfrac{y^2 - 1}{x^2 - 1},\quad y(2) = 2$

25. $x^2\dfrac{dy}{dx} = y - xy,\quad y(-1) = -1$

Example 2: Field Mice and Owls

- Consider the equation

$$\frac{dp}{dt} = 0.5p - 450$$

which describes the interaction of certain populations of field mice and owls. Find solutions of this equation.

Solution:

To solve equation (4), we need to find functions $p(t)$ that, when substituted into the equation, reduce it to an obvious identity. Here is one way to proceed. First, rewrite equation (4) in the form

$$\frac{dp}{dt} = \frac{p - 900}{2}, \tag{5}$$

or, if $p \neq 900$,

$$\frac{dp/dt}{p - 900} = \frac{1}{2}. \tag{6}$$

Example 2-Solution continues

By the chain rule the left-hand side of equation (6) is the derivative of $\ln|p-900|$ with respect to t, so we have

$$\frac{d}{dt}\ln|p-900| = \frac{1}{2}. \tag{7}$$

Then, by integrating both sides of equation (7), we obtain

$$\ln|p-900| = \frac{t}{2} + C, \tag{8}$$

where C is an arbitrary constant of integration. Therefore, by taking the exponential of both sides of equation (8), we find that

$$|p-900| = e^{t/2+C} = e^C e^{t/2}, \tag{9}$$

or

$$p - 900 = \pm e^C e^{t/2}, \tag{10}$$

and finally

$$p = 900 + ce^{t/2}, \tag{11}$$

where $c = \pm e^C$ is also an arbitrary (nonzero) constant. Note that the constant function $p = 900$ is also a solution of equation (5) and that it is contained in the expression (11) if we allow c to take the value zero. Graphs of equation (11) for several values of c are shown in Figure 1.2.1.

Example 2-Solution continues

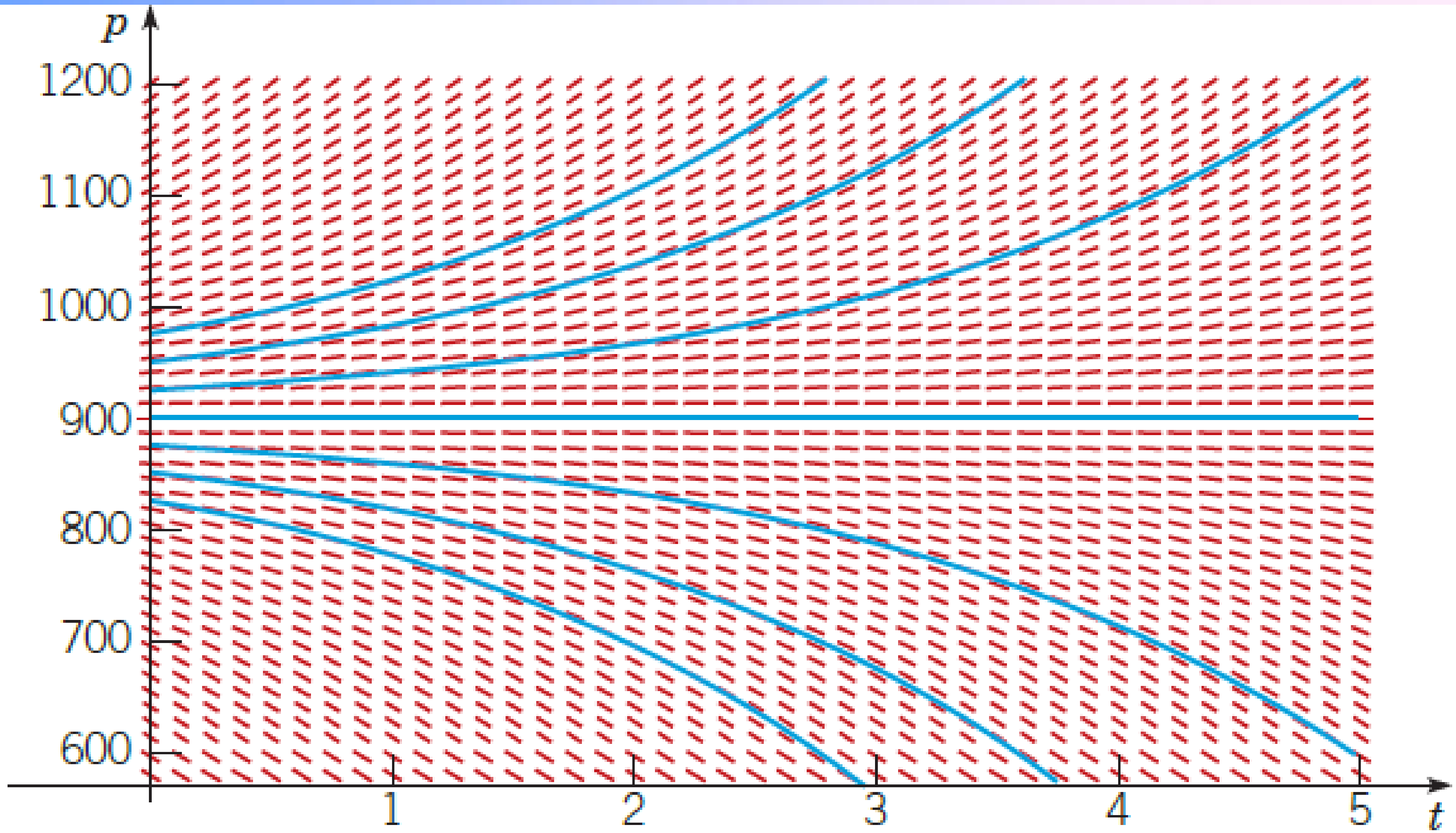

FIGURE 1.2.1 Graphs of $p = 900 + ce^{t/2}$ for several values of c. Each blue curve is a solution of $dp/dt = 0.5p - 450$.

Example 3:A Falling Object

Suppose that, as in Example 2 of Section 1.1, we consider a falling object of mass $m = 10$ kg and drag coefficient $\gamma = 2$ kg/s. Then the equation of motion (1) becomes

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}. \tag{21}$$

Suppose this object is dropped from a height of 300 m. Find its velocity at any time t. How long will it take to fall to the ground, and how fast will it be moving at the time of impact?

Solution:

The first step is to state an appropriate initial condition for equation (21). The word "dropped" in the statement of the problem suggests that the object starts from rest, that is, its initial velocity is zero, so we will use the initial condition

$$v(0) = 0. \tag{22}$$

The solution of equation (21) can be found by substituting the values of the coefficients into the solution (20), but we will proceed instead to solve equation (21) directly. First, rewrite the equation as

$$\frac{dv/dt}{v - 49} = -\frac{1}{5}. \tag{23}$$

By integrating both sides, we obtain

$$\ln|v(t) - 49| = -\frac{t}{5} + C, \tag{24}$$

Solution Continues…

and then the general solution of equation (21) is

$$v(t) = 49 + ce^{-t/5}, \tag{25}$$

where the constant c is arbitrary. To determine the particular value of c that corresponds to the initial condition (22), we substitute $t = 0$ and $v = 0$ into equation (25), with the result that $c = -49$. Then the solution of the initial value problem (21), (22) is

$$v(t) = 49\left(1 - e^{-t/5}\right). \tag{26}$$

Equation (26) gives the velocity of the falling object at any positive time after being dropped—until it hits the ground, of course.

Solution Continues…

Graphs of the solution (25) for several values of c are shown in Figure 1.2.2, with the solution (26) shown by the green curve. It is evident that, regardless of the initial velocity of the object, all solutions tend to approach the equilibrium solution $v(t) = 49$. This confirms the conclusions we reached in Section 1.1 on the basis of the direction fields in Figures 1.1.2 and 1.1.3.

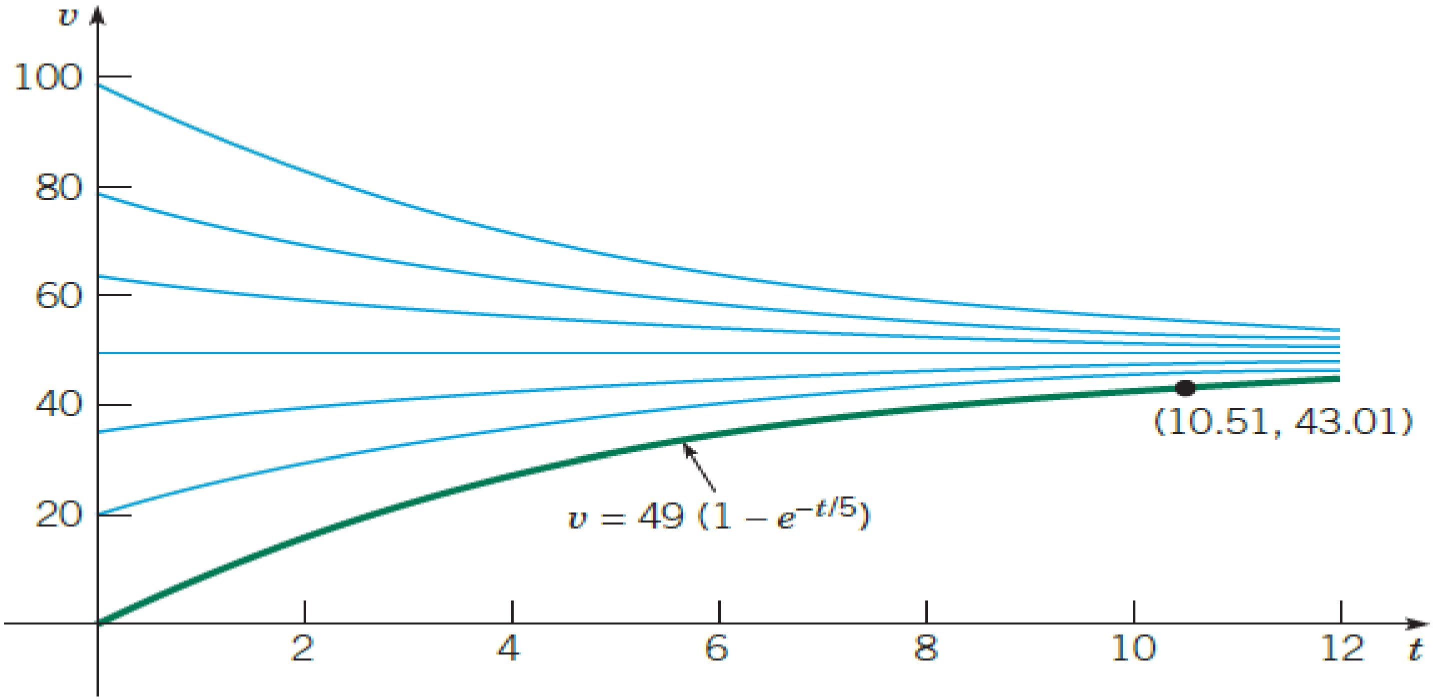

FIGURE 1.2.2 Graphs of the solution (25), $v = 49 + ce^{-t/5}$, for several values of c. The green curve corresponds to the initial condition $v(0) = 0$. The point $(10.51, 43.01)$ shows the velocity when the object hits the ground.

Example 3[Exercise 1.2:7]

- The field mouse population in Example 1 satisfies the differential equation

$$\frac{dp}{dt} = \frac{p}{2} - 450$$

1) Find the time at which the population becomes extinct if $p(0) = 850$.
2) Find the time of extinction if $p(0) = p_o$, where $0 < p_o < 900$.
3) Find the initial population p_o if the population is to become extinct in 1 year.

Solution:

(a) From the previous two problems (or with the technique from the Chapter), we can write down the solution:

$$\frac{dp}{dt} = \frac{1}{2}(p - 900) \qquad \frac{dp}{p - 900} = \frac{1}{2}\,dt$$

And integrate both sides:

$$\ln|p - 900| = \frac{1}{2}t + C \quad \Rightarrow \quad p(t) = Ae^{(1/2)t} + 900$$

Now, if $p(0) = 850$, we can get the particular solution (solve for A):

$$p(0) = A + 900 = 850 \quad \Rightarrow \quad A = -50$$

Therefore, $p(t) = -50e^{(1/2)t} + 900$. To say that the population became extinct means that the population is zero. Set $p(t) = 0$ and solve for t:

$$-50e^{(1/2)t} + 900 = 0 \quad \Rightarrow \quad e^{(1/2)t} = 18 \quad \Rightarrow t = 2\ln(18) \approx 5.78$$

Solution Continues…

(b) Similarly, if $p(0) = p_0$, with $0 < p_0 < 900$,

$$p_0 = A + 900 \Rightarrow A = p_0 - 900$$

and:

$$(p_0 - 900)e^{(1/2)t} + 900 = 0 \quad \Rightarrow \quad e^{(1/2)t} = \frac{-900}{p_0 - 900} = \frac{900}{900 - p_0}$$

(I wrote the last fraction like that so it would be clear that this is a positive number before we take the log of both sides)
Therefore, our conclusion is: Given $p' = \frac{1}{2}p - 450$, $p(0) = p_0$, where $0 < p_0 < 900$, then the time at which extinction occurs is:

$$t = 2\ln\left(\frac{900}{900 - p_0}\right)$$

(c) Find the initial population if the population becomes extinct in one year. Note that t is measured in months, so that would mean that we want to solve our general equation for p_0 if $p(12) = 0$. We can use our last result:

$$12 = 2\ln\left(\frac{900}{900 - p_0}\right)$$

Solve for p_0:

$$\frac{900}{900 - p_0} = e^6 \quad \Rightarrow \quad 900e^{-6} = 900 - p_0 \quad \Rightarrow \quad p_0 = 900 - 900e^{-6}$$

Solution Continues…

Part (a): $t = 2\ln(18) \approx 5.78$ months

Part (b): $t = 2\ln\left(\dfrac{900}{900 - p_0}\right)$ months

Part (c): $p_0 = 900 - 900 \cdot e^{-6} \approx 897.8$

Example 3[page 35]

- Solve the initial value problem

$$\frac{dy}{dx}=\frac{3x^2+4x+2}{2(y-1)},\quad y(0)=-1.$$

and determine the interval in which the solution exists.

Solution:

The differential equation can be written as

$$2(y-1)\,dy=(3x^2+4x+2)\,dx.$$

Integrating the left-hand side with respect to y and the right-hand side with respect to x gives

$$y^2-2y=x^3+2x^2+2x+c, \tag{18}$$

where c is an arbitrary constant. To determine the solution satisfying the prescribed initial condition, we substitute $x=0$ and $y=-1$ in equation (18), obtaining $c=3$. Hence the solution of the initial value problem is given implicitly by

$$y^2-2y=x^3+2x^2+2x+3. \tag{19}$$

Solution Continues…

To obtain the solution explicitly, we must solve equation (19) for y in terms of x. That is a simple matter in this case, since equation (19) is quadratic in y, and we obtain

$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4}. \tag{20}$$

Equation (20) gives two solutions of the differential equation, only one of which, however, satisfies the given initial condition. This is the solution corresponding to the minus sign in equation (20), so we finally obtain

$$y = \phi(x) = 1 - \sqrt{x^3 + 2x^2 + 2x + 4} \tag{21}$$

as the solution of the initial value problem (15). Note that if we choose the plus sign by mistake in equation (20), then we obtain the solution of the same differential equation that satisfies the initial condition $y(0) = 3$. Finally, to determine the interval in which the solution (21) is valid, we must find the interval in which the quantity under the radical is positive. The only real zero of this expression is $x = -2$, so the desired interval is $x > -2$. Some integral curves of the differential equation are shown in Figure 2.2.2. The green curve passes through the point $(0, -1)$ and thus is the solution of the initial value problem (15). Observe that the boundary of the interval of validity of the solution (21) is determined by the point $(-2, 1)$ at which the tangent line is vertical.

Solution Continues…

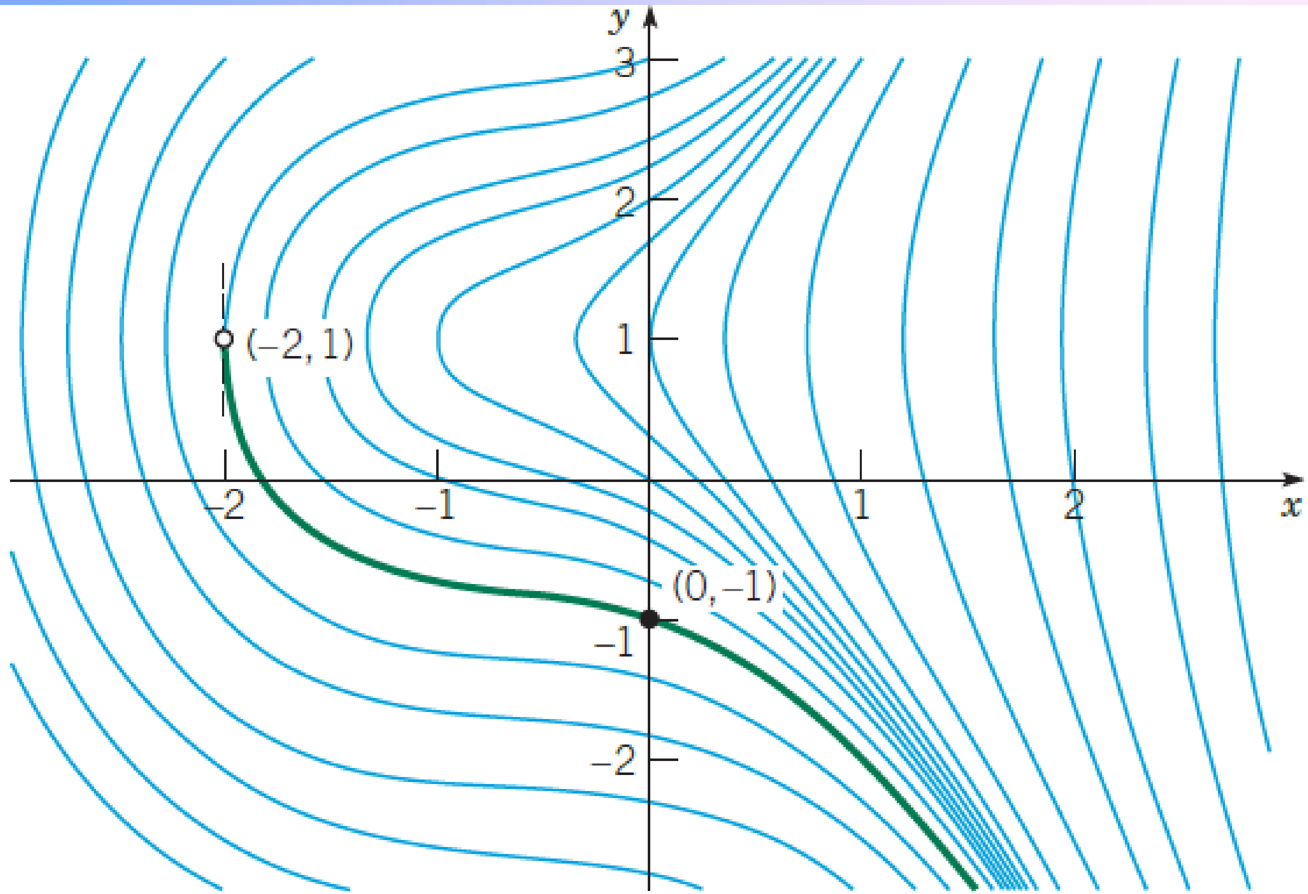

FIGURE 2.2.2 Integral curves of $y' = (3x^2 + 4x + 2)/2(y - 1)$; the solution satisfying $y(0) = -1$ is shown in green and is valid for $x > -2$.

Suggested Problems

1. Solve each of the following initial value problems and plot the solutions for several values of y_0. Then describe in a few words how the solutions resemble, and differ from, each other.

a. $dy/dt = -y + 5, \quad y(0) = y_0$

b. $dy/dt = -2y + 5, \quad y(0) = y_0$

c. $dy/dt = -2y + 10, \quad y(0) = y_0$

2. Follow the instructions for Problem 1 for the following initial-value problems:

a. $dy/dt = y - 5, \quad y(0) = y_0$

b. $dy/dt = 2y - 5, \quad y(0) = y_0$

c. $dy/dt = 2y - 10, \quad y(0) = y_0$

Suggested Problems continues

8. The falling object in Example 2 satisfies the initial value problem

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}, \quad v(0) = 0.$$

a. Find the time that must elapse for the object to reach 98% of its limiting velocity.

b. How far does the object fall in the time found in part a?

9. Consider the falling object of mass 10 kg in Example 2, but assume now that the drag force is proportional to the square of the velocity.

a. If the limiting velocity is 49 m/s (the same as in Example 2), show that the equation of motion can be written as

$$\frac{dv}{dt} = \frac{1}{245}\left(49^2 - v^2\right).$$

Also see Problem 21 of Section 1.1.

b. If $v(0) = 0$, find an expression for $v(t)$ at any time.

G **c.** Plot your solution from part b and the solution (26) from Example 2 on the same axes.

d. Based on your plots in part c, compare the effect of a quadratic drag force with that of a linear drag force.

e. Find the distance $x(t)$ that the object falls in time t.

N **f.** Find the time T it takes the object to fall 300 m.

Suggested Problems continues

10. A radioactive material, such as the isotope thorium-234, disintegrates at a rate proportional to the amount currently present. If $Q(t)$ is the amount present at time t, then $dQ/dt = -rQ$, where $r > 0$ is the decay rate.

- **a.** If 100 mg of thorium-234 decays to 82.04 mg in 1 week, determine the decay rate r.
- **b.** Find an expression for the amount of thorium-234 present at any time t.
- **c.** Find the time required for the thorium-234 to decay to one-half its original amount.

11. The **half-life** of a radioactive material is the time required for an amount of this material to decay to one-half its original value. Show that for any radioactive material that decays according to the equation $Q' = -rQ$, the half-life τ and the decay rate r satisfy the equation $r\tau = \ln 2$.

Suggested Problems

In each of Problems 1 through 8, solve the given differential equation.

1. $y' = \dfrac{x^2}{y}$
2. $y' + y^2 \sin x = 0$
3. $y' = \cos^2(x)\cos^2(2y)$
4. $xy' = (1 - y^2)^{1/2}$
5. $\dfrac{dy}{dx} = \dfrac{x - e^{-x}}{y + e^y}$
6. $\dfrac{dy}{dx} = \dfrac{x^2}{1 + y^2}$
7. $\dfrac{dy}{dx} = \dfrac{y}{x}$
8. $\dfrac{dy}{dx} = \dfrac{-x}{y}$

Suggested Problems

In each of Problems 9 through 16:

a. Find the solution of the given initial value problem in explicit form.

G b. Plot the graph of the solution.

c. Determine (at least approximately) the interval in which the solution is defined.

9. $y' = (1 - 2$

G 19. Solve the initial value problem

$$y' = 2y^2 + xy^2, \quad y(0) = 1$$

and determine where the solution attains its minimum value.

10. $y' = (1 - 2$

11. $x\,dx + ye^{-x}\,dy = 0, \quad y(0) = 1$

12. $dr/d\theta = r^2/\theta, \quad r(1) = 2$

13. $y' = xy^3(1 + x^2)^{-1/2}, \quad y(0) = 1$

14. $y' = 2x/(1 + 2y), \quad y(2) = 0$

15. $y' = (3x^2 - e^x)/(2y - 5), \quad y(0) = 1$

16. $\sin(2x)\,dx + \cos(3y)\,dy = 0, \quad y(\pi/2) = \pi/3$

Suggested Problems

G **19.** Solve the initial value problem

$$y' = 2y^2 + xy^2, \quad y(0) = 1$$

and determine where the solution attains its minimum value.

G **22.** Consider the initial value problem

$$y' = \frac{ty(4-y)}{1+t}, \quad y(0) = y_0 > 0.$$

a. Determine how the solution behaves as $t \to \infty$.
b. If $y_0 = 2$, find the time T at which the solution first reaches the value 3.99.
c. Find the range of initial values for which the solution lies in the interval $3.99 < y < 4.01$ by the time $t = 2$.

Thanks a lot …