Linear Differential Equation

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Linear Differential Equations
Method of Integrating Factors

Linear DEs are friendly to be solved.

DEFINITION

A first-order DE of the form

$$a_1(x)\frac{dy}{dx}+a_0(x)y=g(x) \quad \dots \quad (1)$$

is said to be a ***linear equation*** in y. When $g(x) = 0$, (1) is said to be ***homogeneous***; otherwise it is ***nonhomogeneous***.

Method of Integrating Factors

- ***Standard Form***

The standard for of a first-order LDE can be written as

$$dy/dx + P(x)y = f(x) \qquad (2)$$

- ***METHOD OF SOLUTION***

Let, left part of DE (2) be integrable, if we multiply both sides by $\mu(x)$, then

$$\left[\frac{dy}{dx} + P\left(x\right)y\right]\mu(x) = f\left(x\right)\mu\left(x\right)$$

$$\frac{dy}{dx}\mu(x) + \mu(x)P\left(x\right)y = f\left(x\right)\mu\left(x\right)... \quad (3)$$

Method of Integrating Factors

If we can make the LHS of (3)

$$\frac{dy}{dx}\mu(x)+\mu(x)P\left(x\right)y=\frac{d}{dx}\left[\mu(x)y\right] \ldots \quad (4)$$

a whole derivative, (3) can be solved easily in terms of $\mu(x)y$.

Equation (4) can be written as

$$\frac{dy}{dx}\mu(x)+\mu(x)P\left(x\right)y=\frac{dy}{dx}\mu(x)+y\frac{d\left[\mu(x)\right]}{dx} \ldots \quad (5)$$

Cancelling $\frac{dy}{dx}\mu(x)$ on both sides of DE (5), we have

Method of Integrating Factors

$$\mu(x)P\left(x\right)=\frac{d\left[\mu(x)\right]}{dx}$$

$$or,\ \frac{d\,\mu(x)}{\mu(x)}=P\left(x\right)dx$$

$$or,\ \ln\mu(x)=\int P\left(x\right)$$

$$or,\ \mu(x)=e^{\int P\left(x\right)dx}\ \ldots(6)$$

Eq.(6) is called an ***integrating factor*** for equation (2).

Method of Integrating Factors

Here is what we have so far: We multiplied both sides of (2) by (6) and, by construction, the left-hand side is the derivative of a product of the integrating factor and *y*:

$$e^{\int P(x)\,dx}\frac{dy}{dx} + P(x)e^{\int P(x)\,dx}y = e^{\int P(x)\,dx}f(x)$$

$$\frac{d}{dx}\left[e^{\int P(x)\,dx}y\right] = e^{\int P(x)\,dx}f(x).$$

Finally, we discover why (6) is called an *integrating factor.* We can integrate both sides of the last equation,

Method of Integrating Factors

$$e^{\int P(x)\,dx}\, y = \int e^{\int P(x)\,dx} f(x)\,dx + c$$

We emphasize that you should **not memorize** formula (6). The following procedure should be worked through each time.

2.1: Method of Integrating Factors

Solving a Linear First-Order Equation

(*i*) Remember to put a linear equation into the standard form (2).

(*ii*) From the standard form of the equation identify $P(x)$ and then find the integrating factor $e^{\int P(x)\,dx}$. No constant need be used in evaluating the indefinite integral $\int P(x)\,dx$.

(*iii*) Multiply the both sides of the standard form equation by the integrating factor. The left-hand side of the resulting equation is automatically the derivative of the product of the integrating factor $e^{\int P(x)\,dx}$ and y:

$$\frac{d}{dx}\left[e^{\int P(x)\,dx}y\right] = e^{\int P(x)\,dx} f(x). \tag{5}$$

(*iv*) Integrate both sides of the last equation and solve for y.

Workedout Examples

Solve $x\dfrac{dy}{dx} - 4y = x^6e^x$.

SOLUTION Dividing by x, the standard form of the given DE is

$$\frac{dy}{dx} - \frac{4}{x}y = x^5e^x. \tag{6}$$

From this form we identify $P(x) = -4/x$ and $f(x) = x^5e^x$ and further observe that P and f are continuous on $(0, \infty)$. Hence the integrating factor is

$$\overset{\text{we can use } \ln x \text{ instead of } \ln|x| \text{ since } x > 0}{\downarrow}$$

$$e^{-4\int dx/x} = e^{-4\ln x} = e^{\ln x^{-4}} = x^{-4}.$$

Here we have used the basic identity $b^{\log_b N} = N$, $N > 0$. Now we multiply (6) by x^{-4} and rewrite

$$x^{-4}\frac{dy}{dx} - 4x^{-5}y = xe^x \qquad \text{as} \qquad \frac{d}{dx}[x^{-4}y] = xe^x.$$

It follows from integration by parts that the general solution defined on the interval $(0, \infty)$ is $x^{-4}y = xe^x - e^x + c$ or $y = x^5e^x - x^4e^x + cx^4$.

Workedout Examples continued

Find the general solution of $(x^2 - 9)\dfrac{dy}{dx} + xy = 0$.

SOLUTION We write the differential equation in standard form

$$\frac{dy}{dx} + \frac{x}{x^2 - 9}y = 0 \tag{7}$$

and identify $P(x) = x/(x^2 - 9)$. Although P is continuous on $(-\infty, -3)$, $(-3, 3)$, and $(3, \infty)$, we shall solve the equation on the first and third intervals. On these intervals the integrating factor is

$$e^{\int x\,dx/(x^2-9)} = e^{\frac{1}{2}\int 2x\,dx/(x^2-9)} = e^{\frac{1}{2}\ln|x^2-9|} = \sqrt{x^2 - 9}.$$

After multiplying the standard form (7) by this factor, we get

$$\frac{d}{dx}\left[\sqrt{x^2 - 9}\,y\right] = 0.$$

Integrating both sides of the last equation gives $\sqrt{x^2 - 9}\,y = c$. Thus for either $x > 3$ or $x < -3$ the general solution of the equation is $y = \dfrac{c}{\sqrt{x^2 - 9}}$.

Workedout Examples continued

Solve $\dfrac{dy}{dx} + y = x, \quad y(0) = 4.$

SOLUTION The equation is in standard form, and $P(x) = 1$ and $f(x) = x$ are continuous on $(-\infty, \infty)$. The integrating factor is $e^{\int dx} = e^x$, so integrating

$$\frac{d}{dx}[e^x y] = xe^x$$

gives $e^x y = xe^x - e^x + c$. Solving this last equation for y yields the general solution $y = x - 1 + ce^{-x}$. But from the initial condition we know that $y = 4$ when $x = 0$. Substituting these values into the general solution implies that $c = 5$. Hence the solution of the problem is

$$y = x - 1 + 5e^{-x}, \quad -\infty < x < \infty. \tag{8}$$

Figure 2.3.2, obtained with the aid of a graphing utility, shows the graph of the solution (8) in dark blue along with the graphs of other members of the one-parameter family of solutions $y = x - 1 + ce^{-x}$. It is interesting to observe that as x increases, the graphs of *all* members of this family are close to the graph of the solution $y = x - 1$. The last solution corresponds to $c = 0$ in the family and is shown in

Workedout Examples continued

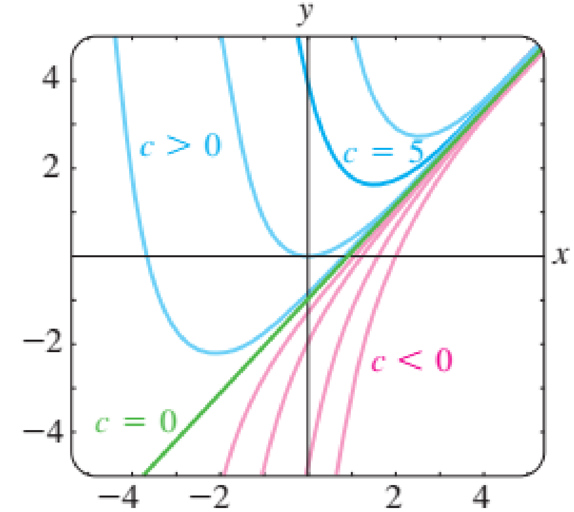


FIGURE 2.3.2 Solution curves of DE in Example 5

dark green in Figure 2.3.2. This asymptotic behavior of solutions is due to the fact that the contribution of ce^{-x}, $c \neq 0$, becomes negligible for increasing values of x. We say that ce^{-x} is a **transient term,** since $e^{-x} \to 0$ as $x \to \infty$. While this behavior is not characteristic of all general solutions of linear equations (see Example 2), the notion of a transient is often important in applied problems.

Workedout Examples continued

Solve $\dfrac{dy}{dx} + y = f(x), \quad y(0) = 0$ where $f(x) = \begin{cases} 1, & 0 \le x \le 1, \\ 0, & x > 1. \end{cases}$

SOLUTION The graph of the discontinuous function f is shown in Figure 2.3.3. We solve the DE for $y(x)$ first on the interval $[0, 1]$ and then on the interval $(1, \infty)$. For $0 \le x \le 1$ we have

$$\frac{dy}{dx} + y = 1 \qquad \text{or, equivalently,} \qquad \frac{d}{dx}[e^x y] = e^x.$$

Integrating this last equation and solving for y gives $y = 1 + c_1 e^{-x}$. Since $y(0) = 0$, we must have $c_1 = -1$, and therefore $y = 1 - e^{-x}$, $0 \le x \le 1$. Then for $x > 1$ the equation

$$\frac{dy}{dx} + y = 0$$

leads to $y = c_2 e^{-x}$. Hence we can write

$$y = \begin{cases} 1 - e^{-x}, & 0 \le x \le 1, \\ c_2 e^{-x}, & x > 1. \end{cases}$$

Workedout Examples continued

By appealing to the definition of continuity at a point, it is possible to determine c_2 so that the foregoing function is continuous at $x = 1$. The requirement that $\lim_{x\to 1^+} y(x) = y(1)$ implies that $c_2 e^{-1} = 1 - e^{-1}$ or $c_2 = e - 1$. As seen in Figure 2.3.4, the function

$$y = \begin{cases} 1 - e^{-x}, & 0 \le x \le 1, \\ (e-1)e^{-x}, & x > 1 \end{cases} \tag{9}$$

is continuous on $(0, \infty)$. ≡

It is worthwhile to think about (9) and Figure 2.3.4 a little bit; you are urged to read and answer Problem 48 in Exercises 2.3.

Workedout Examples continued

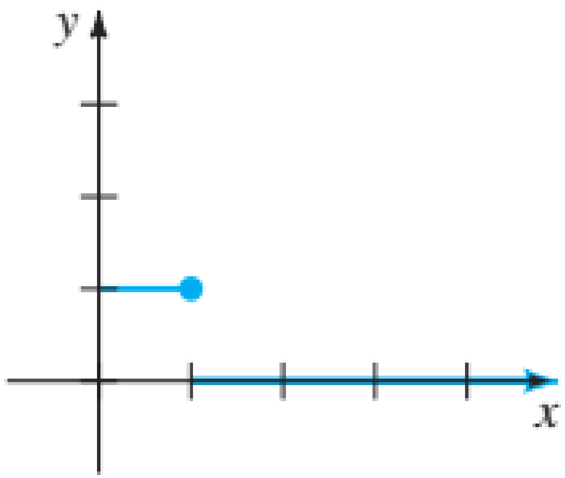


FIGURE 2.3.3 Discontinuous $f(x)$ in Example 6

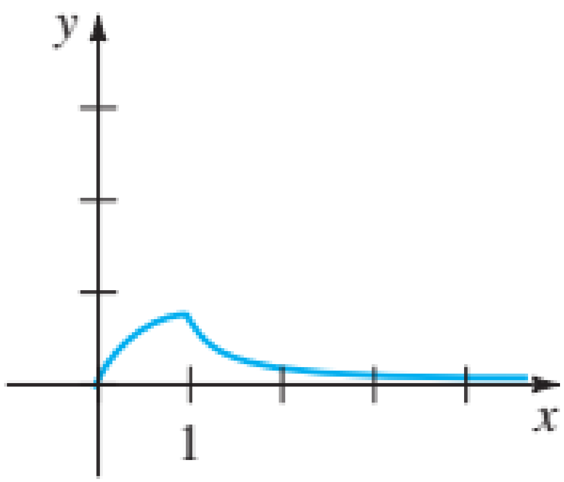


FIGURE 2.3.4 Graph of (9) in Example 6

Workedout Examples continued

Solve the initial value problem

$$t y' + 2y = 4t^2, \tag{34}$$

$$y(1) = 2. \tag{35}$$

Solution:

In order to determine $p(t)$ and $g(t)$ correctly, we must first rewrite equation (34) in the standard form (3). Thus we have

$$y' + \frac{2}{t} y = 4t, \tag{36}$$

so $p(t) = 2/t$ and $g(t) = 4t$. To solve equation (36), we first compute the integrating factor $\mu(t)$:

$$\mu(t) = \exp\left(\int \frac{2}{t}\, dt\right) = e^{2\ln|t|} = t^2.$$

Workedout Examples continued

On multiplying equation (36) by $\mu(t) = t^2$, we obtain

$$t^2y' + 2ty = (t^2y)' = 4t^3,$$

and therefore

$$t^2y = \int 4t^3\,dt = t^4 + c,$$

where c is an arbitrary constant. It follows that, for $t > 0$,

$$y = t^2 + \frac{c}{t^2} \tag{37}$$

is the general solution of equation (34). Integral curves of equation (34) for several values of c are shown in Figure 2.1.3.

To satisfy initial condition (35), set $t = 1$ and $y = 2$ in equation (37): $2 = 1 + c$, so $c = 1$; thus

$$y = t^2 + \frac{1}{t^2}, \quad t > 0 \tag{38}$$

Workedout Examples continued

Solve: $y' + \tan(x)y = \cos^2(x), \quad y(0) = 2$

Solution:

$$I.F. = e^{\int \tan x dx} = e^{\ln \sec x} = \sec x$$

The solution $y \sec x = \int \sec x \cos^2 x \, dx + c = \int \cos x \, dx + c$

$$\Rightarrow y \sec x = \sin x + c$$

When x=0 , then 2 = c

$$\therefore \; y \sec x = \sin x + 2 \quad Ans.$$

Solve: $x\dfrac{dy}{dx}=y(1-x\tan x)+x^2(\cos x+scex)$

Soln: $x\dfrac{dy}{dx}=y(1-x\tan x)+x^2(\cos x+\sec x)$

$$\Rightarrow \frac{dy}{dx}-\frac{y(1-x\tan x)}{x}=x\cos x+x\sec x$$

$$I.F=e^{\int -\frac{(1-x\tan x)}{x}dx}=e^{\int -\frac{1}{x}dx}\times e^{\int \tan x\ dx}$$

$$=e^{-\ln x}\times e^{\ln\sec x}=\frac{\sec x}{x}$$

Workedout examples continued

The solution is

$$\frac{y\sec x}{x} = \int \frac{\sec x}{x} x\cos x\,dx + \int \frac{\sec x}{x} x\sec x\,dx + c$$

$$\Rightarrow \frac{y\sec x}{x} = \int dx + \int \sec^2 x\,dx + c$$

$$\Rightarrow \frac{y\sec x}{x} = x + \tan x + c$$

$$\Rightarrow y = x^2\cos x + x\sin x + cx\cos x \qquad Ans$$

Exercises

In Problems 1–24 find the general solution of the given differential equation. Give the largest interval I over which the general solution is defined. Determine whether there are any transient terms in the general solution.

1. $\dfrac{dy}{dx} = 5y$

2. $\dfrac{dy}{dx} + 2y = 0$

3. $\dfrac{dy}{dx} + y = e^{3x}$

4. $3\dfrac{dy}{dx} + 12y = 4$

5. $y' + 3x^2y = x^2$

6. $y' + 2xy = x^3$

7. $x^2y' + xy = 1$

8. $y' = 2y + x^2 + 5$

9. $x\dfrac{dy}{dx} - y = x^2 \sin x$

10. $x\dfrac{dy}{dx} + 2y = 3$

Workedout examples continued

23. $x\dfrac{dy}{dx} + (3x + 1)y = e^{-3x}$

24. $(x^2 - 1)\dfrac{dy}{dx} + 2y = (x + 1)^2$

In Problems 25–36 solve the given initial-value problem. Give the largest interval I over which the solution is defined.

25. $\dfrac{dy}{dx} = x + 5y, \quad y(0) = 3$

26. $\dfrac{dy}{dx} = 2x - 3y, \quad y(0) = \frac{1}{3}$

27. $xy' + y = e^x, \quad y(1) = 2$

Workedout examples continued

29. $L\dfrac{di}{dt} + Ri = E, \quad i(0) = i_0, \quad L, R, E, i_0$ constants

30. $\dfrac{dT}{dt} = k(T - T_m), \quad T(0) = T_0, \quad k, T_m, T_0$ constants

31. $x\dfrac{dy}{dx} + y = 4x + 1, \quad y(1) = 8$

32. $y' + 4xy = x^3 e^{x^2}, \quad y(0) = -1$

33. $(x + 1)\dfrac{dy}{dx} + y = \ln x, \quad y(1) = 10$

34. $x(x + 1)\dfrac{dy}{dx} + xy = 1, \quad y(e) = 1$

35. $y' - (\sin x)y = 2 \sin x, \quad y(\ /2) = 1$

36. $y' + (\tan x)y = \cos^2 x, \quad y(0) = -1$

Workedout examples continued

In Problems 37–40 proceed as in Example 6 to solve the given initial-value problem. Use a graphing utility to graph the continuous function $y(x)$.

37. $\dfrac{dy}{dx} + 2y = f(x),\ y(0) = 0,$ where

$$f(x) = \begin{cases} 1, & 0 \le x \le 3 \\ 0, & x > 3 \end{cases}$$

38. $\dfrac{dy}{dx} + y = f(x),\ y(0) = 1,$ where

$$f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ -1, & x > 1 \end{cases}$$

39. $\dfrac{dy}{dx} + 2xy = f(x),\ y(0) = 2,$ where

$$f(x) = \begin{cases} x, & 0 \le x \quad 1 \\ 0, & x \ge 1 \end{cases}$$

40. $(1 + x^2)\dfrac{dy}{dx} + 2xy = f(x),\ y(0) = 0,$ where

$$f(x) = \begin{cases} x, & 0 \le x \quad 1 \\ -x, & x \ge 1 \end{cases}$$

Workedout examples continued

EXAMPLE 5 **Mixture of Two Salt Solutions**

Recall that the large tank considered in Section 1.3 held 300 gallons of a brine solution. Salt was entering and leaving the tank; a brine solution was being pumped into the tank at the rate of 3 gal/min; it mixed with the solution there, and then the mixture was pumped out at the rate of 3 gal/min. The concentration of the salt in the inflow, or solution entering, was 2 lb/gal, so salt was entering the tank at the rate $R_{in} = (2 \text{ lb/gal}) \cdot (3 \text{ gal/min}) = 6 \text{ lb/min}$ and leaving the tank at the rate $R_{out} = (A/300 \text{ lb/gal}) \cdot (3 \text{ gal/min}) = A/100 \text{ lb/min}$. From this data and (5) we get equation (8) of Section 1.3. Let us pose the question: If 50 pounds of salt were dissolved initially in the 300 gallons, how much salt is in the tank after a long time?

SOLUTION To find the amount of salt $A(t)$ in the tank at time t, we solve the initial-value problem

$$\frac{dA}{dt} + \frac{1}{100}A = 6, \quad A(0) = 50.$$

Workedout examples continued

Note here that the side condition is the initial amount of salt $A(0) = 50$ in the tank and *not* the initial amount of liquid in the tank. Now since the integrating factor of the linear differential equation is $e^{t/100}$, we can write the equation as

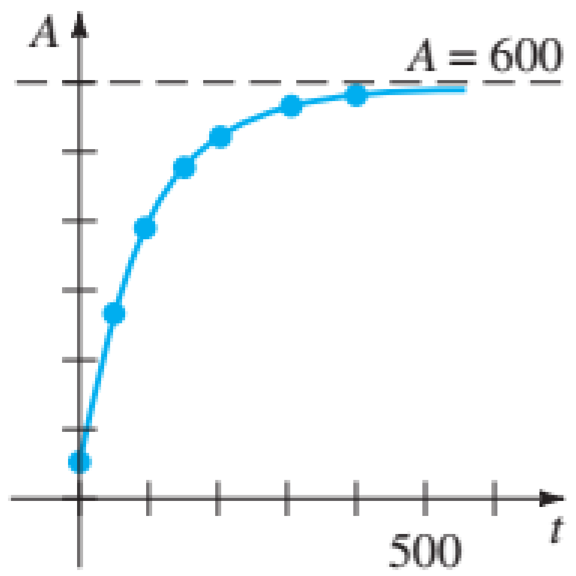
$$\frac{d}{dt}[e^{t/100}A] = 6e^{t/100}.$$

Integrating the last equation and solving for A gives the general solution $A(t) = 600 + ce^{-t/100}$. When $t = 0$, $A = 50$, so we find that $c = -550$. Thus the amount of salt in the tank at time t is given by

$$A(t) = 600 - 550e^{-t/100}. \tag{6}$$

The solution (6) was used to construct the table in Figure 3.1.5(b). Also, it can be seen from (6) and Figure 3.1.5(a) that $A(t) \rightarrow 600$ as $t \rightarrow \infty$. Of course, this is what we would intuitively expect; over a long time the number of pounds of salt in the solution must be (300 gal)(2 lb/gal) = 600 lb.

Workedout examples continued



(a)

t (min)	A (lb)
50	266.41
100	397.67
150	477.27
200	525.57
300	572.62
400	589.93

(b)

FIGURE 3.1.5 Pounds of salt in the tank in Example 5

Workedout examples continued

Series Circuits For a series circuit containing only a resistor and an inductor, Kirchhoff's second law states that the sum of the voltage drop across the inductor ($L(di/dt)$) and the voltage drop across the resistor (iR) is the same as the impressed voltage ($E(t)$) on the circuit. See Figure 3.1.7.

Thus we obtain the linear differential equation for the current $i(t)$,

$$L\frac{di}{dt} + Ri = E(t), \tag{7}$$

where L and R are constants known as the inductance and the resistance, respectively. The current $i(t)$ is also called the **response** of the system.

The voltage drop across a capacitor with capacitance C is given by $q(t)/C$, where q is the charge on the capacitor. Hence, for the series circuit shown in Figure 3.1.8, Kirchhoff's second law gives

$$Ri + \frac{1}{C}q = E(t). \tag{8}$$

But current i and charge q are related by $i = dq/dt$, so (8) becomes the linear differential equation

$$R\frac{dq}{dt} + \frac{1}{C}q = E(t). \tag{9}$$

Workedout examples continued

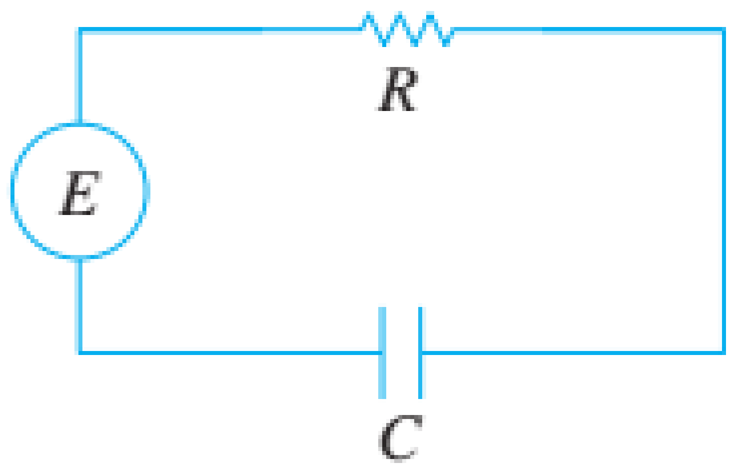


FIGURE 3.1.8 *RC*-series circuit

Workedout examples continued

EXAMPLE 7 **Series Circuit**

A 12-volt battery is connected to a series circuit in which the inductance is $\frac{1}{2}$ henry and the resistance is 10 ohms. Determine the current i if the initial current is zero.

SOLUTION From (7) we see that we must solve

$$\frac{1}{2}\frac{di}{dt} + 10i = 12,$$

subject to $i(0) = 0$. First, we multiply the differential equation by 2 and read off the integrating factor e^{20t}. We then obtain

$$\frac{d}{dt}[e^{20t}i] = 24e^{20t}.$$

Integrating each side of the last equation and solving for i gives $i(t) = \frac{6}{5} + ce^{-20t}$. Now $i(0) = 0$ implies that $0 = \frac{6}{5} + c$ or $c = -\frac{6}{5}$. Therefore the response is $i(t) = \frac{6}{5} - \frac{6}{5}e^{-20t}$.

Workedout examples continued

Bernoulli's Equation The differential equation

$$\frac{dy}{dx} + P(x)y = f(x)y^n, \tag{4}$$

where n is any real number, is called **Bernoulli's equation.** Note that for $n = 0$ and $n = 1$, equation (4) is linear. For $n \neq 0$ and $n \neq 1$ the substitution $u = y^{1-n}$ reduces any equation of form (4) to a linear equation.

EXAMPLE 2 **Solving a Bernoulli DE**

Solve $x\dfrac{dy}{dx} + y = x^2y^2$.

SOLUTION We first rewrite the equation as

$$\frac{dy}{dx} + \frac{1}{x}y = xy^2$$

Workedout examples continued

by dividing by x. With $n = 2$ we have $u = y^{-1}$ or $y = u^{-1}$. We then substitute

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -u^{-2}\frac{du}{dx} \qquad \leftarrow \text{Chain Rule}$$

into the given equation and simplify. The result is

$$\frac{du}{dx} - \frac{1}{x}u = -x.$$

The integrating factor for this linear equation on, say, $(0, \infty)$ is

$$e^{-\int dx/x} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}.$$

Integrating $$\frac{d}{dx}[x^{-1}u] = -1$$

gives $x^{-1}u = -x + c$ or $u = -x^2 + cx$. Since $u = y^{-1}$, we have $y = 1/u$, so a solution of the given equation is $y = 1/(-x^2 + cx)$.

Workedout examples continued

Solve: $x\dfrac{dy}{dx}+y=(xy)^{\frac{3}{2}}$

Solution: The given equation can be written as

$$y^{-\frac{3}{2}}\frac{dy}{dx}+\frac{1}{xy^{\frac{1}{2}}}=x^{\frac{1}{2}}$$

putting $y^{-\frac{1}{2}}=v \Rightarrow y^{-\frac{3}{2}}\dfrac{dy}{dx}=-2\dfrac{dv}{dx}$

Workedout examples continued

The above equation reduces to

$$-2\frac{dv}{dx}+\frac{v}{x}=x^{\frac{1}{2}} \Rightarrow \frac{dv}{dx}-\frac{v}{2x}=-\frac{x^{\frac{1}{2}}}{2}$$

$$\text{I.F} \quad e^{-\int\frac{dx}{2x}}=e^{-\frac{1}{2}\log x}=\frac{1}{\sqrt{x}}$$

Therefore, the solution is

$$\frac{v}{\sqrt{x}}=-\int\frac{\sqrt{x}}{2\sqrt{x}}\,dx \Rightarrow \frac{1}{\sqrt{xy}}=-\frac{x}{2}+c \quad Ans$$

Exercises

Each DE in Problems 15–22 is a Bernoulli equation.

In Problems 15–20 solve the given differential equation by using an appropriate substitution.

15. $x\dfrac{dy}{dx} + y = \dfrac{1}{y^2}$

16. $\dfrac{dy}{dx} - y = e^x y^2$

17. $\dfrac{dy}{dx} = y(xy^3 - 1)$

18. $x\dfrac{dy}{dx} - (1 + x)y = xy^2$

19. $t^2\dfrac{dy}{dt} + y^2 = ty$

20. $3(1 + t^2)\dfrac{dy}{dt} = 2ty(y^3 - 1)$

In Problems 21 and 22 solve the given initial-value problem.

21. $x^2\dfrac{dy}{dx} - 2xy = 3y^4, \quad y(1) = \frac{1}{2}$

22. $y^{1/2}\dfrac{dy}{dx} + y^{3/2} = 1, \quad y(0) = 4$

Thanks a lot …