4.6 Variation of Parameters

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Variation of Parameters

The method of undetermined coefficients has two inherent weaknesses that limit its wider application to linear equations: The DE must have constant coefficients and the input function must be of the type listed in Table 4.4.1.

TABLE 4.4.1 Trial Particular Solutions

$g(x)$	Form of y_p
1. 1 (any constant)	A
2. $5x + 7$	$Ax + B$
3. $3x^2 - 2$	$Ax^2 + Bx + C$
4. $x^3 - x + 1$	$Ax^3 + Bx^2 + Cx + E$
5. $\sin 4x$	$A \cos 4x + B \sin 4x$
6. $\cos 4x$	$A \cos 4x + B \sin 4x$
7. e^{5x}	Ae^{5x}
8. $(9x - 2)e^{5x}$	$(Ax + B)e^{5x}$
9. x^2e^{5x}	$(Ax^2 + Bx + C)e^{5x}$
10. $e^{3x} \sin 4x$	$Ae^{3x} \cos 4x + Be^{3x} \sin 4x$
11. $5x^2 \sin 4x$	$(Ax^2 + Bx + C) \cos 4x + (Ex^2 + Fx + G) \sin 4x$
12. $xe^{3x} \cos 4x$	$(Ax + B)e^{3x} \cos 4x + (Cx + E)e^{3x} \sin 4x$

In this section we examine a method for determining a particular solution of a nonhomogeneous linear DE that has, in theory, no such restrictions on it. This method, due to the eminent astronomer and mathematician **Joseph Louis Lagrange**, is known as **variation of parameters**.

Lecture continued

Linear Second-Order DEs Next we consider the case of a linear second-order equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x), \tag{5}$$

although, as we shall see, variation of parameters extends to higher-order equations. The method again begins by putting (5) into the standard form

$$y'' + P(x)y' + Q(x)y = f(x) \tag{6}$$

by dividing by the leading coefficient $a_2(x)$. In (6) we suppose that coefficient functions $P(x)$, $Q(x)$, and $f(x)$ are continuous on some common interval I. As we have already seen in Section 4.3, there is no difficulty in obtaining the complementary solution $y_c = c_1y_1(x) + c_2y_2(x)$, the general solution of the associated homogeneous equation of (6), when the coefficients are constants. Analogous to the preceding discussion, we now ask: Can the parameters c_1 and c_2 in y_c can be replaced with functions u_1 and u_2, or "variable parameters," so that

Lecture continued

$$y = u_1(x)y_1(x) + u_2(x)y_2(x) \tag{7}$$

is a particular solution of (6)? To answer this question we substitute (7) into (6). Using the Product Rule to differentiate y_p twice, we get

$$y_p' = u_1y_1' + y_1u_1' + u_2y_2' + y_2u_2'$$

$$y_p'' = u_1y_1'' + y_1'u_1' + y_1u_1'' + u_1'y_1' + u_2y_2'' + y_2'u_2' + y_2u_2'' + u_2'y_2'.$$

Substituting (7) and the foregoing derivatives into (6) and grouping terms yields

$$\begin{aligned} y_p'' + P(x)y_p' + Q(x)y_p &= u_1[\overbrace{y_1'' + Py_1' + Qy_1}^{\text{zero}}] + u_2[\overbrace{y_2'' + Py_2' + Qy_2}^{\text{zero}}] + y_1u_1'' + u_1'y_1' \\ &\quad + y_2u_2'' + u_2'y_2' + P[y_1u_1' + y_2u_2'] + y_1'u_1' + y_2'u_2' \\ &= \frac{d}{dx}[y_1u_1'] + \frac{d}{dx}[y_2u_2'] + P[y_1u_1' + y_2u_2'] + y_1'u_1' + y_2'u_2' \\ &= \frac{d}{dx}[y_1u_1' + y_2u_2'] + P[y_1u_1' + y_2u_2'] + y_1'u_1' + y_2'u_2' = f(x). \end{aligned} \tag{8}$$

Lecture continued

Because we seek to determine two unknown functions u_1 and u_2, reason dictates that we need two equations. We can obtain these equations by making the further assumption that the functions u_1 and u_2 satisfy $y_1u_1' + y_2u_2' = 0$. This assumption does not come out of the blue but is prompted by the first two terms in (8), since if we demand that $y_1u_1' + y_2u_2' = 0$, then (8) reduces to $y_1'u_1' + y_2'u_2' = f(x)$. We now have our desired two equations, albeit two equations for determining the derivatives u_1' and u_2'. By Cramer's Rule, the solution of the system

$$\begin{aligned} y_1u_1' + y_2u_2' &= 0 \\ y_1'u_1' + y_2'u_2' &= f(x) \end{aligned}$$

can be expressed in terms of determinants:

$$u_1' = \frac{W_1}{W} = -\frac{y_2 f(x)}{W} \quad \text{and} \quad u_2' = \frac{W_2}{W} = \frac{y_1 f(x)}{W}, \tag{9}$$

where

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \quad W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}. \tag{10}$$

The functions u_1 and u_2 are found by integrating the results in (9). The determinant W is recognized as the Wronskian of y_1 and y_2. By linear independence of y_1 and y_2 on I, we know that $W(y_1(x), y_2(x)) \neq 0$ for every x in the interval.

Lecture continued

Summary of the Method Usually, it is not a good idea to memorize formulas in lieu of understanding a procedure. However, the foregoing procedure is too long and complicated to use each time we wish to solve a differential equation. In this case it is more efficient to simply use the formulas in (9). Thus to solve $a_2y'' + a_1y' + a_0y = g(x)$, first find the complementary function $y_c = c_1y_1 + c_2y_2$ and then compute the Wronskian $W(y_1(x), y_2(x))$. By dividing by a_2, we put the equation into the standard form $y'' + Py' + Qy = f(x)$ to determine $f(x)$. We find u_1 and u_2 by integrating $u_1' = W_1/W$ and $u_2' = W_2/W$, where W_1 and W_2 are defined as in (10). A particular solution is $y_p = u_1y_1 + u_2y_2$. The general solution of the equation is then $y = y_c + y_p$.

EXAMPLE 1 General Solution Using Variation of Parameters

Solve $y'' - 4y' + 4y = (x + 1)e^{2x}$.

SOLUTION From the auxiliary equation $m^2 - 4m + 4 = (m - 2)^2 = 0$ we have $y_c = c_1e^{2x} + c_2xe^{2x}$. With the identifications $y_1 = e^{2x}$ and $y_2 = xe^{2x}$, we next compute the Wronskian:

Lecture continued

$$W(e^{2x}, xe^{2x}) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = e^{4x}.$$

Since the given differential equation is already in form (6) (that is, the coefficient of y'' is 1), we identify $f(x) = (x + 1)e^{2x}$. From (10) we obtain

$$W_1 = \begin{vmatrix} 0 & xe^{2x} \\ (x+1)e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = -(x + 1)xe^{4x}, \qquad W_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & (x+1)e^{2x} \end{vmatrix} = (x + 1)e^{4x},$$

and so from (9)

$$u_1' = -\frac{(x+1)xe^{4x}}{e^{4x}} = -x^2 - x, \qquad u_2' = \frac{(x+1)e^{4x}}{e^{4x}} = x + 1.$$

It follows that $u_1 = -\frac{1}{3}x^3 - \frac{1}{2}x^2$ and $u_2 = \frac{1}{2}x^2 + x$. Hence

$$y_p = \left(-\frac{1}{3}x^3 - \frac{1}{2}x^2\right)e^{2x} + \left(\frac{1}{2}x^2 + x\right)xe^{2x} = \frac{1}{6}x^3e^{2x} + \frac{1}{2}x^2e^{2x}$$

and
$$y = y_c + y_p = c_1e^{2x} + c_2xe^{2x} + \frac{1}{6}x^3e^{2x} + \frac{1}{2}x^2e^{2x}.$$

Lecture continued

EXAMPLE 2 **General Solution Using Variation of Parameters**

Solve $4y'' + 36y = \csc 3x$.

SOLUTION We first put the equation in the standard form (6) by dividing by 4:

$$y'' + 9y = \frac{1}{4}\csc 3x.$$

Because the roots of the auxiliary equation $m^2 + 9 = 0$ are $m_1 = 3i$ and $m_2 = -3i$, the complementary function is $y_c = c_1 \cos 3x + c_2 \sin 3x$. Using $y_1 = \cos 3x$, $y_2 = \sin 3x$, and $f(x) = \frac{1}{4}\csc 3x$, we obtain

$$W(\cos 3x, \sin 3x) = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3,$$

$$W_1 = \begin{vmatrix} 0 & \sin 3x \\ \frac{1}{4}\csc 3x & 3\cos 3x \end{vmatrix} = -\frac{1}{4}, \qquad W_2 = \begin{vmatrix} \cos 3x & 0 \\ -3\sin 3x & \frac{1}{4}\csc 3x \end{vmatrix} = \frac{1}{4}\frac{\cos 3x}{\sin 3x}.$$

Lecture continued

Integrating $\quad u_1' = \dfrac{W_1}{W} = -\dfrac{1}{12} \quad$ and $\quad u_2' = \dfrac{W_2}{W} = \dfrac{1}{12}\dfrac{\cos 3x}{\sin 3x}$

gives $u_1 = -\frac{1}{12}x$ and $u_2 = \frac{1}{36}\ln|\sin 3x|$. Thus a particular solution is

$$y_p = -\frac{1}{12}x\cos 3x + \frac{1}{36}(\sin 3x)\ln|\sin 3x|.$$

The general solution of the equation is

$$y = y_c + y_p = c_1\cos 3x + c_2\sin 3x - \frac{1}{12}x\cos 3x + \frac{1}{36}(\sin 3x)\ln|\sin 3x|. \quad (11)$$

Equation (11) represents the general solution of the differential equation on, say, the interval $(0, \pi/6)$.

Constants of Integration When computing the indefinite integrals of u_1' and u_2', we need not introduce any constants. This is because

$$\begin{aligned} y = y_c + y_p &= c_1y_1 + c_2y_2 + (u_1 + a_1)y_1 + (u_2 + b_1)y_2 \\ &= (c_1 + a_1)y_1 + (c_2 + b_1)y_2 + u_1y_1 + u_2y_2 \\ &= C_1y_1 + C_2y_2 + u_1y_1 + u_2y_2. \end{aligned}$$

Lecture continued

EXAMPLE 3 General Solution Using Variation of Parameters

Solve $y'' - y = \dfrac{1}{x}$.

SOLUTION The auxiliary equation $m^2 - 1 = 0$ yields $m_1 = -1$ and $m_2 = 1$. Therefore $y_c = c_1e^x + c_2e^{-x}$. Now $W(e^x, e^{-x}) = -2$, and

$$u_1' = -\frac{e^{-x}(1/x)}{-2}, \qquad u_1 = \frac{1}{2}\int_{x_0}^{x}\frac{e^{-t}}{t}\,dt,$$

$$u_2' = \frac{e^{x}(1/x)}{-2}, \qquad u_2 = -\frac{1}{2}\int_{x_0}^{x}\frac{e^{t}}{t}\,dt.$$

Since the foregoing integrals are nonelementary, we are forced to write

Lecture continued

$$y_p = \frac{1}{2} e^x \int_{x_0}^{x} \frac{e^{-t}}{t}\, dt - \frac{1}{2} e^{-x} \int_{x_0}^{x} \frac{e^t}{t}\, dt,$$

and so $$y = y_c + y_p = c_1 e^x + c_2 e^{-x} + \frac{1}{2} e^x \int_{x_0}^{x} \frac{e^{-t}}{t}\, dt - \frac{1}{2} e^{-x} \int_{x_0}^{x} \frac{e^t}{t}\, dt. \quad (12)$$

Higher-Order Equations The method that we have just examined for non homogeneous second-order differential equations can be generalized to linear nth-order equations that have been put into the standard form

$$y^{(n)} + P_{n-1}(x) y^{(n-1)} + \cdots + P_1(x) y' + P_0(x) y = f(x). \quad (13)$$

If $y_c = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$ is the complementary function for (13), then a particular solution is

$$y_p = u_1(x) y_1(x) + u_2(x) y_2(x) + \cdots + u_n(x) y_n(x),$$

where the u_k', $k = 1, 2, \ldots, n$ are determined by the n equations

Lecture continued

$$
\begin{aligned}
y_1u_1' + \quad & y_2u_2' + \cdots + \quad y_nu_n' = 0 \\
y_1'u_1' + \quad & y_2'u_2' + \cdots + \quad y_n'u_n' = 0 \\
\vdots \qquad & \qquad\qquad\qquad\qquad \vdots \\
y_1^{(n-1)}u_1' + & y_2^{(n-1)}u_2' + \cdots + y_n^{(n-1)}u_n' = f(x).
\end{aligned} \tag{14}
$$

The first $n - 1$ equations in this system, like $y_1u_1' + y_2u_2' = 0$ in (8), are assumptions that are made to simplify the resulting equation after $y_p = u_1(x)y_1(x) + \cdots + u_n(x)y_n(x)$ is substituted in (13). In this case Cramer's Rule gives

$$u_k' = \frac{W_k}{W}, \quad k = 1, 2, \ldots, n,$$

where W is the Wronskian of $y_1, y_2, \ldots, y_n$ and W_k is the determinant obtained by replacing the kth column of the Wronskian by the column consisting of the right-hand side of (14)—that is, the column consisting of $(0, 0, \ldots, f(x))$. When $n = 2$, we get (9). When $n = 3$, the particular solution is $y_p = u_1y_1 + u_2y_2 + u_3y_3$, where y_1, y_2, and y_3 constitute a linearly independent set of solutions of the associated homogeneous DE and u_1, u_2, u_3 are determined from

Lecture continued

$$u_1' = \frac{W_1}{W}, \qquad u_2' = \frac{W_2}{W}, \qquad u_3' = \frac{W_3}{W}, \tag{15}$$

$$W_1 = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ f(x) & y_2'' & y_3'' \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & f(x) & y_3'' \end{vmatrix}, \quad W_3 = \begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & f(x) \end{vmatrix}, \quad \text{and} \quad W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

REMARKS

(*i*) Variation of parameters has a distinct advantage over the method of undetermined coefficients in that it will *always* yield a particular solution y_p provided that the associated homogeneous equation can be solved. The present method is not limited to a function $f(x)$ that is a combination of the four types listed on page 140. As we shall see in the next section, variation of parameters, unlike undetermined coefficients, is applicable to linear DEs with variable coefficients.

(*ii*) In the problems that follow, do not hesitate to simplify the form of y_p. Depending on how the antiderivatives of u_1' and u_2' are found, you might not obtain the same y_p as given in the answer section. For example, in Problem 3 in Exercises 4.6 both $y_p = \frac{1}{2}\sin x - \frac{1}{2}x\cos x$ and $y_p = \frac{1}{4}\sin x - \frac{1}{2}x\cos x$ are valid answers. In either case the general solution $y = y_c + y_p$ simplifies to $y = c_1\cos x + c_2\sin x - \frac{1}{2}x\cos x$. Why?

Lecture continued

In Problems 1 – 18 solve each differential equation by variation of parameters.

1. $y'' + y = \sec x$
2. $y'' + y = \tan x$
3. $y'' + y = \sin x$
4. $y'' + y = \sec\theta\tan\theta$
5. $y'' + y = \cos^2 x$
6. $y'' + y = \sec^2 x$
7. $y'' - y = \cosh x$
8. $y'' - y = \sinh 2x$
9. $y'' - 4y = \dfrac{e^{2x}}{x}$
10. $y'' - 9y = \dfrac{9x}{e^{3x}}$
11. $y'' + 3y' + 2y = \dfrac{1}{1 + e^x}$
12. $y'' - 2y' + y = \dfrac{e^x}{1 + x^2}$
13. $y'' + 3y' + 2y = \sin e^x$

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Lecture continued

14. $y'' - 2y' + y = e^t \arctan t$

15. $y'' + 2y' + y = e^{-t} \ln t$

16. $2y'' + 2y' + y = 4\sqrt{x}$

17. $3y'' - 6y' + 6y = e^x \sec x$

18. $4y'' - 4y' + y = e^{x/2}\sqrt{1 - x^2}$

In Problems 19–22 solve each differential equation by variation of parameters, subject to the initial conditions $y(0) = 1$, $y'(0) = 0$.

19. $4y'' - y = xe^{x/2}$

20. $2y'' + y' - y = x + 1$

21. $y'' + 2y' - 8y = 2e^{-2x} - e^{-x}$

22. $y'' - 4y' + 4y = (12x^2 - 6x)e^{2x}$

Solved Exercise Problems

Exercises: Solve each of the following differential equation by variation of parameters

Ex-4: Solve $y'' + y = \sec\theta \tan\theta$

Solution: Let $y'' + y = \sec\theta\tan\theta ---(1)$

$AE: m^2 + 1 = 0 \Rightarrow m = \pm i$

$\therefore y_c = c_1 \cos\theta + c_2 \sin\theta ---------(2)$

$\Rightarrow \boxed{y_1 = \cos\theta}; \boxed{y_2 = \sin\theta} --------(3)$

$$w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = 1. \left[\text{By } (3)\right]$$

$$w_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} = \begin{vmatrix} 0 & \sin\theta \\ \sec\theta\tan\theta & \cos\theta \end{vmatrix} = -\tan^2\theta = 1 - \sec^2\theta$$

$\Rightarrow \boxed{w_1 = 1 - \sec^2\theta}$

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Solved Exercise Problems

$$w_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix} = \begin{vmatrix} \cos\theta & 0 \\ -\sin\theta & \sec\theta\tan\theta \end{vmatrix} = \tan\theta$$

$$\Rightarrow \boxed{w_2 = \tan\theta}$$

$$\therefore u_1' = \frac{w_1}{w} = \frac{-\tan^2\theta}{1} = 1-\sec^2\theta$$

$$\Rightarrow u_1 = \int\left(1-\sec^2\theta\right)d\theta \Rightarrow \boxed{u_1 = \theta-\tan\theta}$$

$$u_2' = \frac{w_2}{w} = \frac{\tan\theta}{1} = \tan\theta$$

$$\Rightarrow u_2 = \int \tan\theta\, d\theta \Rightarrow \boxed{u_2 = -ln\left(\cos\theta\right)}$$

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Solved Exercise Problems

$$\therefore y_p = u_1 y_1 + u_2 y_2$$

$$\Rightarrow y_p = \cos\theta(\theta - \tan\theta) + \sin\theta(-ln(\cos\theta))$$

$$\therefore \boxed{y_p = \theta\cos\theta - \sin\theta - \sin\theta\ ln(\cos\theta)} ---------(4)$$

Combining (2) and (4), the General Solution (GS) is given by

$$y = y_c + y_p$$

$$\therefore GS: y = y_c + y_p = c_1\cos\theta + c_2\sin\theta + \theta\cos\theta - \sin\theta - \sin\theta\ ln(\cos\theta)$$

$$= c_1\cos\theta + (c_2 - 1)\sin\theta + \theta\cos\theta - \sin\theta\ln(\cos\theta)$$

$$= c_1\cos\theta + c_2\sin\theta + \theta\cos\theta - \sin\theta\ln(\cos) \left[\text{Letting the new constant to be } c_2\right]$$

$$\Rightarrow \boxed{GS: y = c_1\cos\theta + c_2\sin\theta + \theta\cos\theta - \sin\theta\ln(\cos\theta)}\ \square$$

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Solved Exercise Problems

Ex-7:Solve $y'' - y = \cosh x = \dfrac{e^x + e^{-x}}{2} ----(1)$

AE:$m^2 - 1 = 0 = m = \pm 1$

$y_c = c_1 e^x + c_2 e^{-x} ----------------------(2)$

$\Rightarrow \boxed{y_1 = e^x}; \boxed{y_2 = e^{-x}}$ and $\boxed{f(x) = \dfrac{e^x + e^{-x}}{2}}$

$$\therefore w = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2.$$

$$u_1' = -\frac{y_2 f(x)}{w} = \frac{-e^{-x}\left(e^x + e^{-x}\right)}{2\times(-2)} = \frac{1+e^{-2x}}{4}$$

$$\Rightarrow u_1 = \int \frac{1+e^{-2x}}{4}\,dx = \boxed{u_1 = \frac{x}{4} - \frac{e^{-2x}}{8}}$$

Solved Exercise Problems

$$u_2' = \frac{y_1 f(x)}{w} = \frac{e^x\left(e^x + e^{-x}\right)}{-4} = \frac{1+e^{2x}}{-4}$$

$$\Rightarrow u_2 = \int \frac{1+e^{2x}}{-4}\,dx = \boxed{u_2 = \frac{-x}{4} - \frac{e^{2x}}{8}}$$

$$\therefore y_p = u_1 y_1 + u_2 y_2$$

$$\Rightarrow \therefore y_p = \left(\frac{x}{4} - \frac{e^{-2x}}{8}\right)e^x + \left(\frac{-x}{4} - \frac{e^{2x}}{8}\right)e^{-x} \text{-------------} (3)$$

The General Solution is given by

$$y = y_c + y_p$$

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Solved Exercise Problems

$$\Rightarrow y = c_1e^x + c_2e^{-x} + \left(\frac{x}{4} - \frac{e^{-2x}}{8}\right)e^x - \left(\frac{x}{4} + \frac{e^{2x}}{8}\right)e^{-x} \text{ [Combining (2) and (3)]}$$

$$= c_1e^x + c_2e^{-x} + \frac{x}{4}\left(e^x - e^{-x}\right) - \frac{e^{-x}}{8} - \frac{e^x}{8}$$

$$= \left(c_1 - \frac{1}{8}\right)e^x + \left(c_2 - \frac{1}{8}\right)e^{-x} + \frac{1}{2}x\sinh x = c_3e^x + c_4e^{-x} + \frac{x}{2}\sinh x$$

$$\therefore \boxed{y = c_1e^x + c_2e^{-x} + \frac{x}{2}\sinh x} \quad \square$$

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Solved Exercise Problems

Ex-18: Solve $4y'' - 4y' + y = e^{x/2}\sqrt{1-x^2}$

Solution: Dividing both sides by 4 to get the standard form, we have

$$\Rightarrow y'' - y' + \frac{1}{4}y = \frac{e^{x/2}\sqrt{1-x^2}}{4}$$

$$\text{AE}: m^2 - m + \frac{1}{4} = 0 \Rightarrow \left(m - \frac{1}{2}\right)^2 = 0 \Rightarrow m = \frac{1}{2}, \frac{1}{2}$$

$$\therefore y_c = c_1 e^{x/2} + c_2 x e^{x/2} ---(2)$$

$$\therefore w = \begin{vmatrix} e^{x/2} & xe^{x/2} \\ \frac{1}{2}e^{x/2} & e^{x/2} + \frac{1}{2}xe^{x/2} \end{vmatrix} = e^x + \frac{1}{2}xe^x - \frac{1}{2}xe^x = e^x$$

$$\Rightarrow \boxed{w = e^x}$$

Solved Exercise Problems

$$u_1' = \frac{-y_2 f(x)}{w} = \frac{-xe^{x/2}\left(e^{\frac{x}{2}}\sqrt{1-x^2}\right)}{4e^x} = \frac{-xe^x\sqrt{1-x^2}}{4e^x} = \frac{-x\sqrt{1-x^2}}{4}$$

$$\therefore u_1 = -\frac{1}{4}\int x\sqrt{1-x^2}\,dx \quad \Big|\text{Let } 1-x^2 = t \Rightarrow -2xdx = dt \Rightarrow -xdx = \left(\frac{1}{2}\right)dt$$

$$\therefore u_1 = \frac{1}{4}\int \frac{\sqrt{t}}{2}\,dt = \frac{1}{8}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right)$$

$$= \frac{1}{8}\left[\frac{\left(1-x^2\right)^{3/2}}{3/2}\right]$$

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Solved Exercise Problems

$$\Rightarrow \boxed{u_1 = \frac{\left(1-x^2\right)^{3/2}}{12}} ---(3)$$

$$u_2' = \frac{y_1 f(x)}{w} = e^{x/2} \frac{\left(e^{x/2}\sqrt{1-x^2}\right)}{4e^x} = \frac{1}{4}\sqrt{1-x^2}$$

$$\therefore u_2 = \frac{1}{4}\int \sqrt{1-x^2}\,dx \quad \boxed{\text{let } x = \sin\theta \Rightarrow dx = \cos d\theta \text{ and } \theta = \sin^{-1} x}$$

$$= \frac{1}{4}\int \cos^2\theta\, d\theta$$

$$= \frac{1}{8}\int (1+\cos 2\theta)\, d\theta = \frac{1}{8}\left(\theta + \frac{\sin 2\theta}{2}\right)$$

Solved Exercise Problems

$$=\frac{1}{8}\left(\sin^{-1}x+\frac{2\sin\theta\cos\theta}{2}\right)=\frac{1}{8}\left(\sin^{1}x+x\sqrt{1-x^{2}}\right)$$

$$\Rightarrow \boxed{u_2=\frac{1}{8}\left(\sin^{1}x+x\sqrt{1-x^{2}}\right)}$$

$$\therefore y=c_1e^{x/2}+c_2xe^{x/2}+\frac{e^{x/2}}{12}\left(1-x^{2}\right)^{3/2}+\frac{1}{8}\left[xe^{x/2}\sin^{-1}x+x^{2}e^{x/2}\sqrt{1-x^{2}}\right]$$

Thanks a lot …