Exact Differential Equations

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Exact Differential Equations

- For first-order differential equations there are a number of integration methods.
- A class of equations known as Exact Differential Equations (EDE) for which there is a well-defined method of solution is discussed next.

DEFINITION 2.4.1 Exact Equation

A differential expression $M(x, y)\,dx + N(x, y)\,dy$ is an **exact differential** in a region R of the xy-plane if it corresponds to the differential of some function $f(x, y)$ defined in R. A first-order differential equation of the form

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is said to be an **exact equation** if the expression on the left-hand side is an exact differential.

Exact Differential Equations

For example, $x^2y^3\,dx + x^3y^2\,dy = 0$ is an exact equation, because its left-hand side is an exact differential:

$$d\left(\frac{1}{3}x^3y^3\right) = x^2y^3dx + x^3y^2dy$$

THEOREM 2.4.1 Criterion for an Exact Differential

Let $M(x, y)$ and $N(x, y)$ be continuous and have continuous first partial derivatives in a rectangular region R defined by $a \quad x \quad b$, $c \quad y \quad d$. Then a necessary and sufficient condition that $M(x, y)\,dx + N(x, y)\,dy$ be an exact differential is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{4}$$

Exmaples

EXAMPLE 1 **Solving an Exact DE**

Solve $2xy\,dx + (x^2 - 1)\,dy = 0$.

SOLUTION With $M(x, y) = 2xy$ and $N(x, y) = x^2 - 1$ we have

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}.$$

Thus the equation is exact, and so by Theorem 2.4.1 there exists a function $f(x, y)$ such that

$$\frac{\partial f}{\partial x} = 2xy \qquad \text{and} \qquad \frac{\partial f}{\partial y} = x^2 - 1.$$

From the first of these equations we obtain, after integrating,

$$f(x, y) = x^2y + g(y).$$

Example continues

Taking the partial derivative of the last expression with respect to y and setting the result equal to $N(x, y)$ gives

$$\frac{\partial f}{\partial y} = x^2 + g'(y) = x^2 - 1. \qquad \leftarrow N(x, y)$$

It follows that $g'(y) = -1$ and $g(y) = -y$. Hence $f(x, y) = x^2y - y$, so the solution of the equation in implicit form is $x^2y - y = c$. The explicit form of the solution is easily seen to be $y = c/(1 - x^2)$ and is defined on any interval not containing either $x = 1$ or $x = -1$.

Note The solution of the DE in Example 1 is *not* $f(x, y) = x^2y - y$. Rather, it is $f(x, y) = c$; if a constant is used in the integration of $g'(y)$, we can then write the solution as $f(x, y) = 0$. Note, too, that the equation could be solved by separation of variables.

Example continues

EXAMPLE 2 **Solving an Exact DE**

Solve $(e^{2y} - y\cos xy)\,dx + (2xe^{2y} - x\cos xy + 2y)\,dy = 0$.

SOLUTION The equation is exact because

$$\frac{\partial M}{\partial y} = 2e^{2y} + xy\sin xy - \cos xy = \frac{\partial N}{\partial x}.$$

Hence a function $f(x, y)$ exists for which

$$M(x, y) = \frac{\partial f}{\partial x} \quad \text{and} \quad N(x, y) = \frac{\partial f}{\partial y}.$$

Now, for variety, we shall start with the assumption that $\partial f/\partial y = N(x, y)$; that is,

$$\frac{\partial f}{\partial y} = 2xe^{2y} - x\cos xy + 2y$$

Example continues

$$f(x, y) = 2x \int e^{2y}\,dy - x \int \cos xy\,dy + 2 \int y\,dy.$$

Remember, the reason x can come out in front of the symbol $\int$ is that in the integration with respect to y, x is treated as an ordinary constant. It follows that

$$f(x, y) = xe^{2y} - \sin xy + y^2 + h(x)$$

$$\frac{\partial f}{\partial x} = e^{2y} - y\cos xy + h'(x) = e^{2y} - y\cos xy, \quad \leftarrow M(x, y)$$

and so $h'(x) = 0$ or $h(x) = c$. Hence a family of solutions is

$$xe^{2y} - \sin xy + y^2 + c = 0.$$

Example continues

EXAMPLE 3 An Initial-Value Problem

Solve $\dfrac{dy}{dx} = \dfrac{xy^2 - \cos x \sin x}{y(1 - x^2)}, \quad y(0) = 2.$

SOLUTION By writing the differential equation in the form

$$(\cos x \sin x - xy^2)\,dx + y(1 - x^2)\,dy = 0,$$

we recognize that the equation is exact because

$$\frac{\partial M}{\partial y} = -2xy = \frac{\partial N}{\partial x}.$$

Now $\qquad \dfrac{\partial f}{\partial y} = y(1 - x^2)$

Example continues

$$f(x, y) = \frac{y^2}{2}(1 - x^2) + h(x)$$

$$\frac{\partial f}{\partial x} = -xy^2 + h'(x) = \cos x \sin x - xy^2.$$

The last equation implies that $h'(x) = \cos x \sin x$. Integrating gives

$$h(x) = -\int (\cos x)(-\sin x\, dx) = -\frac{1}{2}\cos^2 x.$$

Thus $\quad \dfrac{y^2}{2}(1 - x^2) - \dfrac{1}{2}\cos^2 x = c_1 \quad$ or $\quad y^2(1 - x^2) - \cos^2 x = c, \quad (7)$

where $2c_1$ has been replaced by c. The initial condition $y = 2$ when $x = 0$ demands that $4(1) - \cos^2(0) = c$, and so $c = 3$. An implicit solution of the problem is then $y^2(1 - x^2) - \cos^2 x = 3$.

Exercises

In Problems 1–20 determine whether the given differential equation is exact. If it is exact, solve it.

19. $(4t^3y - 15t^2 - y)\,dt + (t^4 + 3y^2 - t)\,dy = 0$

20. $\left(\dfrac{1}{t} + \dfrac{1}{t^2} - \dfrac{y}{t^2 + y^2}\right) dt + \left(ye^y + \dfrac{t}{t^2 + y^2}\right) dy = 0$

In Problems 21–26 solve the given initial-value problem.

21. $(x + y)^2\,dx + (2xy + x^2 - 1)\,dy = 0, \quad y(1) = 1$

22. $(e^x + y)\,dx + (2 + x + ye^y)\,dy = 0, \quad y(0) = 1$

23. $(4y + 2t - 5)\,dt + (6y + 4t - 1)\,dy = 0, \quad y(-1) = 2$

24. $\left(\dfrac{3y^2 - t^2}{y^5}\right)\dfrac{dy}{dt} + \dfrac{t}{2y^4} = 0, \quad y(1) = 1$

25. $(y^2 \cos x - 3x^2y - 2x)\,dx$
$+ (2y \sin x - x^3 + \ln y)\,dy = 0, \quad y(0) = e$

26. $\left(\dfrac{1}{1 + y^2} + \cos x - 2xy\right)\dfrac{dy}{dx} = y(y + \sin x), \; y(0) = 1$

Short cut method for solving Exact DE

The differential equation $M(x,y)dx + N(x,y)dy = 0$ is an exact equation if : $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Working Rule: If $M\,dx + N\,dy = 0$ is an exact DE, then the method of solution is as follows:

Step-1: Integrate M with respect to x keeping y constant.

Step-2: Find out the terms in N which are independent of x and Integrate those terms with respect to y.

Step-3: Add the two terms obtained and equate the sum of these two integrals to an arbitrary constant. This gives the general solution of the required exact differential equation.

Short cut method for solving Exact DE

Example: Show that the differential equation $\left(y^2 e^{xy^2} + 4x^3\right)dx + \left(2xye^{xy^2} - 3y^2\right)dy = 0$ is exact and solve it.

Solution Let

$$M = \left(y^2 e^{xy^2} + 4x^3\right) \qquad N = \left(2xye^{xy^2} - 3y^2\right)$$

$$\frac{\partial M}{\partial y} = \left(2xy^3 e^{xy^2} + 2ye^{xy^2}\right) \qquad \frac{\partial N}{\partial x} = \left(2xy^3 e^{xy^2} + 2ye^{xy^2}\right)$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

$\therefore$ The given equation is exact. Then the solution is

$$\int_{y \text{ constant}} M\,dx + \int (\text{terms in N free from } x)\,dy = C$$

$$\int_{y \text{ constant}} (y^2 e^{xy^2} + 4x^3)\,dx + \int (-3y^2)\,dy = C$$

$$\Rightarrow e^{xy^2} + x^4 - y^3 = C.$$

$\therefore$ the solution is $e^{xy^2} + x^4 - y^3 = C$. Ans.

Short cut method for solving Exact DE

Example Solve:

$$\left(ye^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x\right)dx + \left(xe^{xy}\cos 2x - 3\right)dy = 0$$

Solution

Here $M = ye^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x,\quad N = xe^{xy}\cos 2x - 3$

$$\frac{\partial M}{\partial y} = yxe^{xy}\cos 2x + e^{xy}\cos 2x - 2xe^{xy}\sin 2x$$

$$\frac{\partial N}{\partial x} = x(-2e^{xy}\sin 2x + ye^{xy}\cos 2x) + e^{xy}\cos 2x$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

$\therefore$ The equation is exact.

Short cut method for solving Exact DE

Now $$\int_x (ye^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x)dx$$

$$= \int d(e^{xy}\cos 2x)dx + \int 2xdx$$

$$= e^{xy}\cos 2x + x^2 .$$

The term in N free from x is -3.

Now $$-3\int dy = -3y$$

$\therefore$ the solution is $$e^{xy}\cos 2x + x^2 - 3y = c .$$

Exercises

1. Solve $(e^{y}+1)\cos x\,dx + e^{y}\sin x\,dy = 0$

   Ans. $(e^{y}+1)\sin x = c$

2. Solve

$$(y^{3}-y^{2}\sec^{2}x-x)\,dx+(3xy^{2}-2y\tan x+\frac{1}{y})\,dy=0$$

Ans $$xy^{3}-y^{2}\tan x-\frac{x^{2}}{2}+\ln y=c$$

3. Solve $(e^{2y}-y\cos xy)\,dx+(2xe^{2y}-x\cos xy+2y)\,dy=0$

Ans. $xe^{2y}-\sin xy+y^{2}+c=0$

Non-exact equations reducible to exact

The DE $M(x,y)dx + N(x,y)dy = 0$ is a non exact equation if :

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

The solutions are obtained by using integrating factor to change the equation into exact equation.

Integrating factor: A non-exact DE can be made exact by multiplying it by an Integrating Factor (IF).

Working Rule: Multiply the non-exact DE with IF which then will reduce to an EDE and could be solved easily.

There are certain rules for determining integrating factors for various types of equations.

Example continues

We summarize the results for the differential equation

$$M(x, y)\,dx + N(x, y)\,dy = 0. \tag{12}$$

- If $(M_y - N_x)/N$ is a function of x alone, then an integrating factor for (12) is

$$\mu(x) = e^{\int \frac{M_y - N_x}{N}\,dx}. \tag{13}$$

- If $(N_x - M_y)/M$ is a function of y alone, then an integrating factor for (12) is

$$\mu(y) = e^{\int \frac{N_x - M_y}{M}\,dy}. \tag{14}$$

Example continues

EXAMPLE 4 **A Nonexact DE Made Exact**

The nonlinear first-order differential equation

$$xy\,dx + (2x^2 + 3y^2 - 20)\,dy = 0$$

is not exact. With the identifications $M = xy$, $N = 2x^2 + 3y^2 - 20$, we find the partial derivatives $M_y = x$ and $N_x = 4x$. The first quotient from (13) gets us nowhere, since

$$\frac{M_y - N_x}{N} = \frac{x - 4x}{2x^2 + 3y^2 - 20} = \frac{-3x}{2x^2 + 3y^2 - 20}$$

depends on x and y. However, (14) yields a quotient that depends only on y:

$$\frac{N_x - M_y}{M} = \frac{4x - x}{xy} = \frac{3x}{xy} = \frac{3}{y}.$$

Example continues

The integrating factor is then $e^{\int 3dy/y} = e^{3\ln y} = e^{\ln y^3} = y^3$. After we multiply the given DE by $\mu(y) = y^3$, the resulting equation is

$$xy^4\,dx + (2x^2y^3 + 3y^5 - 20y^3)\,dy = 0.$$

You should verify that the last equation is now exact as well as show, using the method of this section, that a family of solutions is $\frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4 = c$.

REMARKS

(*i*) When testing an equation for exactness, make sure it is of the precise form $M(x, y)\;dx + N(x, y)\;dy = 0$. Sometimes a differential equation is written $G(x, y)\;dx = H(x, y)\;dy$. In this case, first rewrite it as $G(x, y)\;dx - H(x, y)\;dy = 0$ and then identify $M(x, y) = G(x, y)$ and $N(x, y) = -H(x, y)$ before using (4).

Exercises

In Problems 27 and 28 find the value of k so that the given differential equation is exact.

27. $(y^3 + kxy^4 - 2x)\,dx + (3xy^2 + 20x^2y^3)\,dy = 0$

28. $(6xy^3 + \cos y)\,dx + (2kx^2y^2 - x\sin y)\,dy = 0$

In Problems 29 and 30 verify that the given differential equation is not exact. Multiply the given differential equation by the indicated integrating factor $\mu(x, y)$ and verify that the new equation is exact. Solve.

29. $(-xy\sin x + 2y\cos x)\,dx + 2x\cos x\,dy = 0$;
$\mu(x, y) = xy$

30. $(x^2 + 2xy - y^2)\,dx + (y^2 + 2xy - x^2)\,dy = 0$;
$\mu(x, y) = (x + y)^{-2}$

Exercises

In Problems 37 and 38 solve the given initial-value problem by finding, as in Example 4, an appropriate integrating factor.

37. $x\,dx + (x^2y + 4y)\,dy = 0, \quad y(4) = 0$

38. $(x^2 + y^2 - 5)\,dx = (y + xy)\,dy, \quad y(0) = 1$

39. **(a)** Show that a one-parameter family of solutions of the equation

$$(4xy + 3x^2)\,dx + (2y + 2x^2)\,dy = 0$$

is $x^3 + 2x^2y + y^2 = c$.

(b) Show that the initial conditions $y(0) = -2$ and $y(1) = 1$ determine the same implicit solution.

(c) Find explicit solutions $y_1(x)$ and $y_2(x)$ of the differential equation in part (a) such that $y_1(0) = -2$ and $y_2(1) = 1$. Use a graphing utility to graph $y_1(x)$ and $y_2(x)$.

Example: Solve $(2x\ln x - xy)dy + 2y\,dx = 0$

Solution Let $N = 2x\ln x - xy$, $M = 2y$

$\dfrac{\partial M}{\partial y} = 2$, $\dfrac{\partial N}{\partial x} = 2(1+\ln x) - y$. So the equation is not exact.

However
$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = \frac{2-2-2\ln x + y}{2x\ln x - xy} = -\frac{1}{x} = f(x)$$

$$\therefore\ I.\ F. = e^{\int f(x)dx} = e^{\int -\frac{dx}{x}} = e^{-\ln x} = \frac{1}{x}$$

So, DE reduces to $\frac{2y}{x}dx + \frac{(2x\ln x - xy)}{x}dy = 0$

$\Rightarrow \frac{2y}{x}dx + (2\ln x - y)dy = 0$ which is EDE.

Hence solution is

$$\int_{y\ \text{constant}} M dx + \int (\text{terms in N free from x})\, dy = C$$

$$\int_{y\ \text{constant}} \frac{2y}{x}dx + \int (-y)\, dy = C$$

$\Rightarrow 2y\ln x - \frac{y^2}{2} = C$ Ans

Exercises

1. Solve $(x^2 + y^2 + 2x)dx + 2ydy = 0$

Ans. $(x^2 + y^2)e^x = c$

2. Solve $(x^2 + y^2 + 1)dx + x(x - 2y)y = 0$

Ans. $(x^2 + xy) - (y^2 + 1) = cx$

3 Solve $(x^2 + y^2 + x)dx + xydy = 0$

Ans. $3x^4 + 6x^2y^2 + 4x^3 = c$

Solve: $(3x^2y^4+2xy)dx+(2x^3y^3-x^2)dy=0.$

Solution: Let $M=3x^2y^4+2xy$, $N=2x^3y^3-x^2$

$$\frac{\partial M}{\partial y}=12x^2y^3+2x \quad \frac{\partial N}{\partial x}=6x^2y^3-2x$$

$$\frac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{M}=\frac{12x^2y^3+2x-6x^2y^3+2x}{xy(3xy^3+2)}=\frac{2x(3xy^3+2)}{xy(3xy^3+2)}=\frac{2}{y}$$

$$\therefore \quad I.F=e^{-\int\frac{2}{y}dy}=e^{-\ln y^2}=\frac{1}{y^2}$$

Multiplying by the integrating factor, then given equation becomes

Non-exact equation now reduces to exact

$$\frac{(3x^2y^4+2xy)}{y^2}dx+\frac{(2x^3y^3-x^2)}{y^2}dy=0.$$

Integrating $M_1=3x^2y^2+\dfrac{2x}{y}$ w.r.t x , keeping y as constant

$$x^3y^2+\frac{x^2}{y}$$

There is no term in $N_1=\dfrac{(2x^3y^3-x^2)}{y^2}$ independence of x.

$\therefore$ The solution is $x^3y^3+x^2=cy$ *Ans.*

Example **Solve**

$$(y^4+2y)dx+(xy^3+2y^4-4x)dy=0$$

Here $M=y^4+2y \quad N=xy^3+2y^4-4x$

$$\frac{\partial M}{\partial y}=4y^3+2 \ ; \quad \frac{\partial N}{\partial x}=y^3-4.$$

Now $$\frac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{M}=\frac{4y^3+2-y^3+4}{y^4+2y}=\frac{3}{y}.$$

$$\text{I. F.}=e^{-\int\frac{3}{y}dy}=e^{-3\ln y}=\frac{1}{y^3}$$

Multiplying the given differential equation by $\frac{1}{y^3}$, we get

$$\frac{y^4+2y}{y^3}dx+\frac{xy^3+2y^4-4x}{y^3}dy=0$$

$$\Rightarrow (y+\frac{2}{y^2})dx+(x+2y-\frac{4x}{y^3})dy=0 --(1)$$

$$Let \quad M_1=y+\frac{2}{y^2}, \quad N_1=x+2y-\frac{4x}{y^3}$$

$$\frac{\partial M_1}{\partial y} = 1 - \frac{4}{y^3}, \quad \frac{\partial N_1}{\partial x} = 1 - \frac{4}{y^3}$$

$\therefore$ The equation (1) is exact . Hence solution is

$$\int\limits_{y \ \text{constant}} M dx + \int (\text{terms } in \text{ N free from } \text{x})\, dy = C$$

$$\int\limits_{y \ \text{constant}} (y + \frac{2}{y^2}) dx + \int 2\, y \ dy = C$$

$$\Rightarrow x(y + \frac{2}{y^2}) + y^2 = C \qquad \text{Ans.}$$

Exercises

1. Solve: $(2xy^4e^y + 2xy^3 + y)\,dx + (x^2y^4e^y - x^2y^2 - 3x)\,dy = 0$

   Ans $x^2e^y + \dfrac{x^2}{y} + \dfrac{x}{y^3} = c$

2. Solve: $(xy^3 + y)\,dx + 2(x^2y^2 + x + y^4)\,dy = 0$

   Ans $3x^2y^4 + 6xy^2 + 2y = c$

3. Solve: $(x^2y^4 + 2xy)\,dx + (2x^3y^3 - x^2)\,dy = 0$

   Ans $x^3y^3 + x^2 = cy$

Thanks a lot …