# Midterm Report: Quantum Fourier Transform

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*Fourier Transform* decomposes or analysis a function of time or signal into its constituent frequencies. *Quantum Fourier Transform*QFT is a critical part of *Shor*'s Algorithm and many other algorithms as well. The main idea of this report is to explain the *Quantum Fourier Transform*, that can help to understand the QFT. This document is a midterm report of Quantum Computing class, that is why most the information comes from class lecture, documents and from web sites [1].

## 1 Quantum Fourier Transform(QFT)

Fourier analysis is a system or method that can help to describe the internal frequencies of a function. *Quantum Fourier Transform* is a quantum implementation of the discrete Fourier transform. Let say, you have a $n$ qubit state vector $|\alpha\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle + \cdots + \alpha_n |n\rangle$, now QFT algorithm will transform it into $|\beta\rangle = \beta_0 |0\rangle + \beta_1 |1\rangle + \cdots + \beta_n |n\rangle$ that a measurement on $|\beta\rangle$ will only return one of its $n$ components.

The main difference between *Hadamard* transform and QFT is that it introduces phase, that is called primitive roots of unity $\omega$. It is important to understand the concept of primitive roots of unity $\omega$ before going for the *Quantum Fourier Transform*. We know that $x^n = 1$ has exactly $n$ solutions. Let say $n = 2$ then $x$ could be 1 or -1. Now, if $n = 4$ then $x$ could be $1, i, -1$ *or* $-i$. Here $i$ is a complex number and these are the roots of the equation. The interesting fact is that you can write these roots by the powers of $\omega = e^{2\pi i/n}$ and $\omega$ is called a primitive $n^{th}$ root of unity.

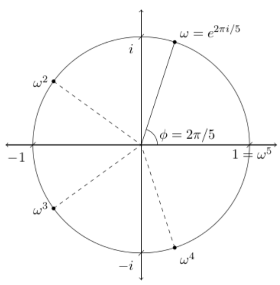


Figure 1: The 5 complex $5^{th}$ roots of 1. [1]

We can explain $\omega$ by the unit circle, figure 1 represents the roots of unity for $n = 5$. It shows the different phase of $\omega$ along with the other complex roots. We can see $\omega$ lies on the unit circle that means $|\omega|$ represent the absolute value of the radius, $|\omega| = 1$. On the other hand, the line from the center to the $\omega$ makes an angle with the horizontal original line, $\phi = 2\pi/M$. Here, M is the M$th$ root of unity. Now the relation between $\omega$ and $\phi$ is that if you square the $\omega$ then the angle $\phi$ will be double. In general, if you apply j$th$ power on the $\omega$ that means $\omega^j$ then the phase angle turn into $\phi = 2j\pi/M$. We can represent M as $2^m$ then we can get $\phi = 2j\pi/2^m$.

So far we get some knowledge about the roots of unity and phase angle, now we can move forward towards the *Quantum Fourier Transform.* A matrix is the best way to represent a *Quantum Fourier Transform* system, that is why to represent a $2^m$ dimensional QFT you need $2^m \times 2^m$ matrix $F_{2^m}$. QFT matrix function can be defined by the

$$F_{2^m}[i,j] = \omega^{ij}/\sqrt{2^m}$$

where $\omega = e^{2\pi i/2^m}$ is the $2^m$-*th* root of unity. So we will get a matrix like,

$$QFT_{2^m} = \frac{1}{\sqrt{2^m}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{2^m-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(2^m-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{2^m-1} & \omega^{2(2^m-1)} & \cdots & \omega^{(2^m-1)^2} \end{pmatrix}$$

The main idea of this transformation is to take the vector $\begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{n-1} \end{pmatrix}$ and transform them into $\begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{n-1} \end{pmatrix}$ Now, if we choose $M = 2$ that means, $2^m = 2$, $\omega = 2^{\pi i}$ and phase angle $\phi = \pi$. So, from the unity circle $\omega^0 = 1$ and $\omega^1 = -1$ because of the angle shift $\pi$.

$$QFT_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & \omega \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Similarly, if we choose $M = 4$ that means $2^m = 4$ then phase angle will be $\phi = \pi/2$. So, from the unity of circle we can get $\omega^0 = 1,\ \omega^1 = \omega = i,\ \omega^2 = -1,\ \omega^3 = -i,\ \omega^4 = 1,\ \omega^5 = i,\ \omega^6 = -1,\ \omega^7 = -i,\ \omega^8 = 1,\ \omega^9 = i$

$$F_4 = {}_QFT_{2^2} = QFT_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega^9 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

Now, if we move column(0 and 2) to the left then we will get,

$$F'_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & i & -i \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -i & i \end{pmatrix} = \begin{pmatrix} H & AH \\ H & -AH \end{pmatrix}$$

Where $A = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$ is a phase shift operation. At the beginning of the article, we mentioned that the main difference between QFT and *Hadamard* is the phase. In general, we can write,

$$F_{2^m} = \frac{1}{\sqrt{2}} \begin{pmatrix} F_{2^{m-1}} & AF_{2^{m-1}} \\ F_{2^{m-1}} & -AF_{2^{m-1}} \end{pmatrix}$$

where

$$A = \begin{pmatrix} 1 & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega^{2^{m-1}-1} \end{pmatrix}$$

Now, m qubits require $2^m$ operations to compute $F\,|\emptyset\rangle$and a *Fast Fourier Transform* can compute $A\,|X\rangle$ in $O(m2^m)$ steps. But, with the *Quantum* circuit speed up, we can implement $F_{2^m}\,|\emptyset\rangle$ with a *quantum circuit* of $O(m^2)$ gates. As earlier we mentioned, this calculation has a great significance on the *Shor*'s algorithm that is why this enhancement is also important.

# References


[1] Berkeley Quantum Computing. `https://courses.edx.org/c4x/BerkeleyX/CS191x/asset/chap5.pdf`.