pset

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February 2020

1. Define Vector Addition (mod 2) to be the component-wise addition of two vectors, reducing each component to 0 or 1 (mod 2). For example, $(1,0,1,0,0) + (1,1,0,1,1) = (0,1,1,1,1)$.

   (a) What is $\overrightarrow{v} + \overrightarrow{v}$, for any $\overrightarrow{v}$?

   (b) Let V denote the set of (mod 2) vectors that contain an even number of 1's. What is the size of V?

   (c) If you add two vectors in V, do you get another vector in V necessarily?

   (d) Non-zero vectors in V come in three colors:

      i. Red: those that contain exactly 2 1s, and the 2 1's are either consecutive or first and last;

      ii. Blue: those that contain exactly 2 1's and are NOT red;

      iii. Yellow: those that are not red or blue.

      How many vectors are red, blue, and yellow each have?

   (e) If you add together two vectors of the same color, do you ever get another vector of the same color?

   (f) Build a complete graph on the vertex set V. For a pair of distinct vectors $\overrightarrow{v}$ and $\overrightarrow{w}$, color the edge between them according to the color of $\overrightarrow{v} + \overrightarrow{w}$. Is there a monochromatic triangle in this coloring?

   (g) Consider the implications for Ramsey numbers.

2. You have a subset of size $n+1$ of $[2n]$. Prove that you can find some pair of them such that one divides the other.

3. n soccer teams play each other in some manner. Prove that some pair of the teams has played the same number of games.

4. You have $mn + 1$ numbers. Prove that there exists an increasing subsequence of size at least $m + 1$ or a decreasing subsequence of size at least $n + 1$. (Hint: Consider the pairs (a_i, b_i), where a_i is the length of the longest increasing subsequence that ends on index i of the sequence, and b_i is the length of the longest decreasing subsequence that ends on index i. Use the pigeonhole principle).