Minimum Cut and Minimum *k*-Cut in Hypergraphs via Branching Contractions

Kyle Fox

joint with

Debmalya Panigrahi and Fred Zhang
Duke University

Talk Outline

- A (short) lecture, some recent work, and a new result…

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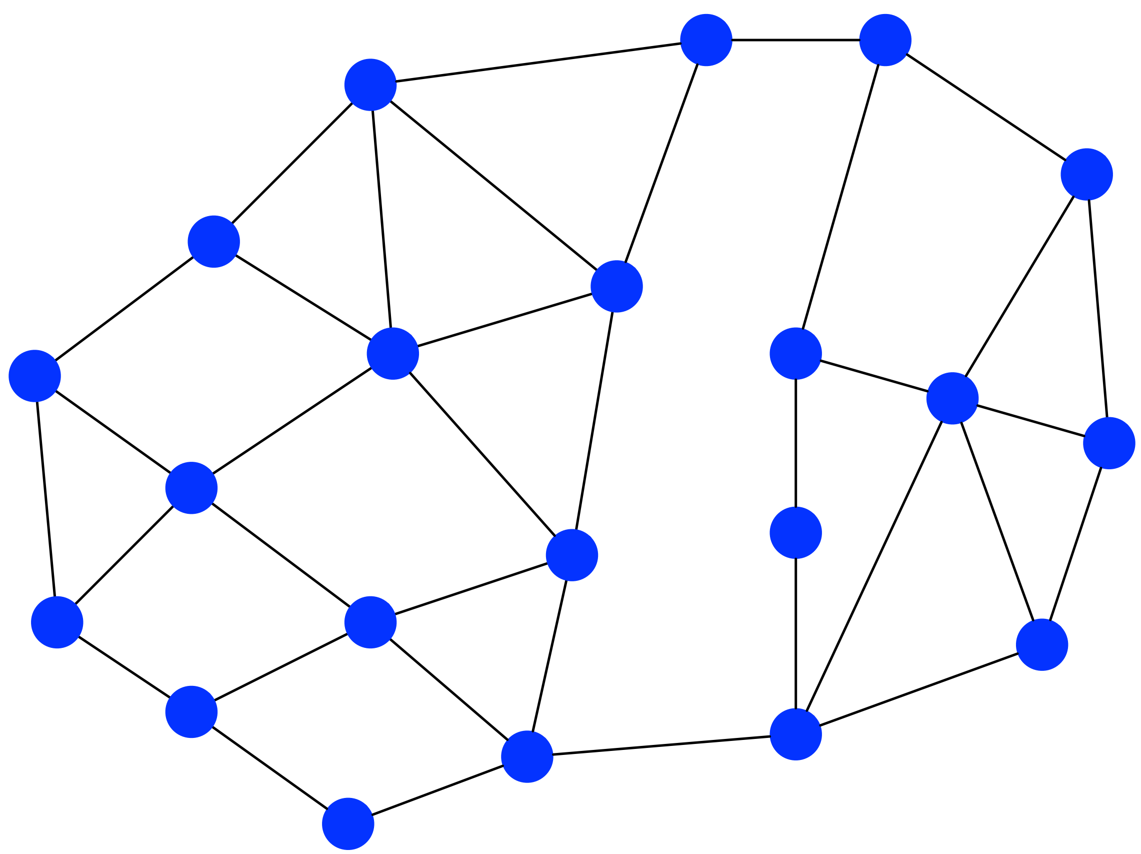
2. Minimum cuts in hypergraphs via *serial* random contractions [Ghaffari *et al.* ’17; Chandrasekharan *et al.* ’18]

3. Faster minimum cuts in hypergraphs via *branching* contractions [**F**PZ]

Minimum Cuts

- A ***cut*** is a set of edges whose removal creates at least 2 connected components; a ***k-cut*** creates at least *k* components



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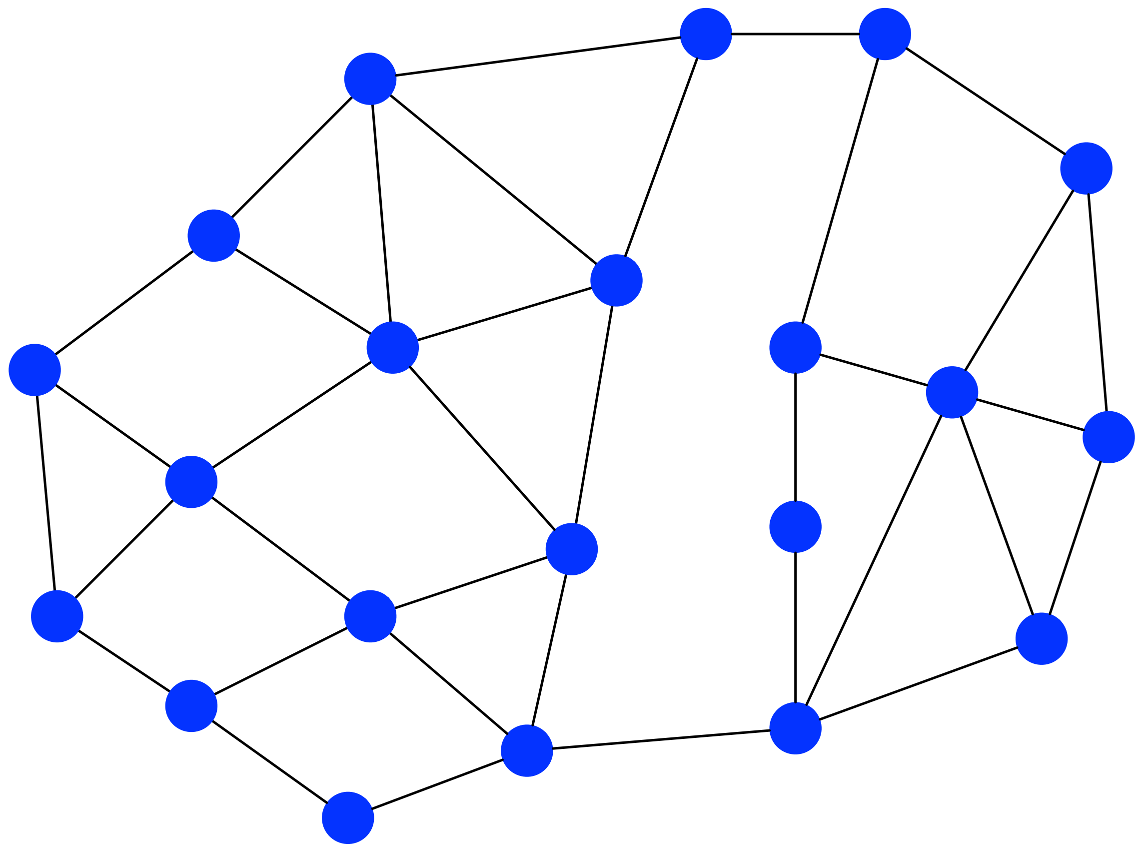

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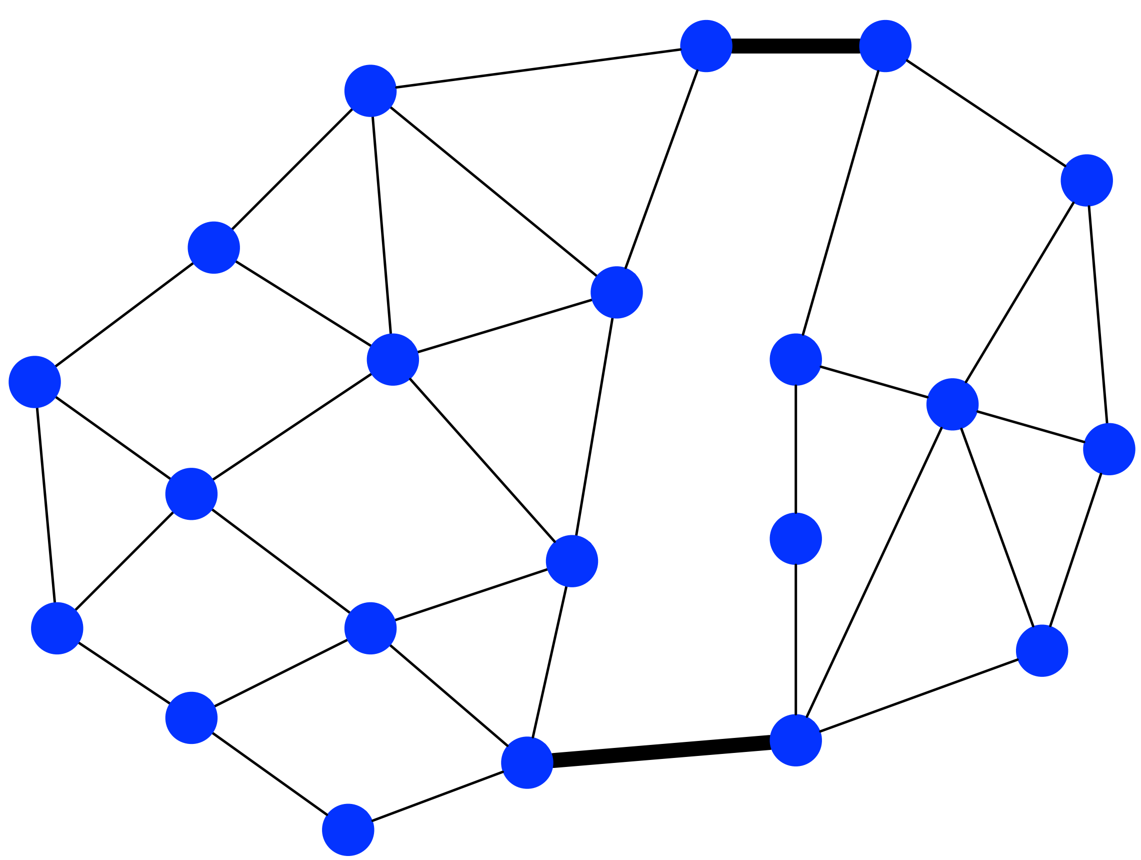

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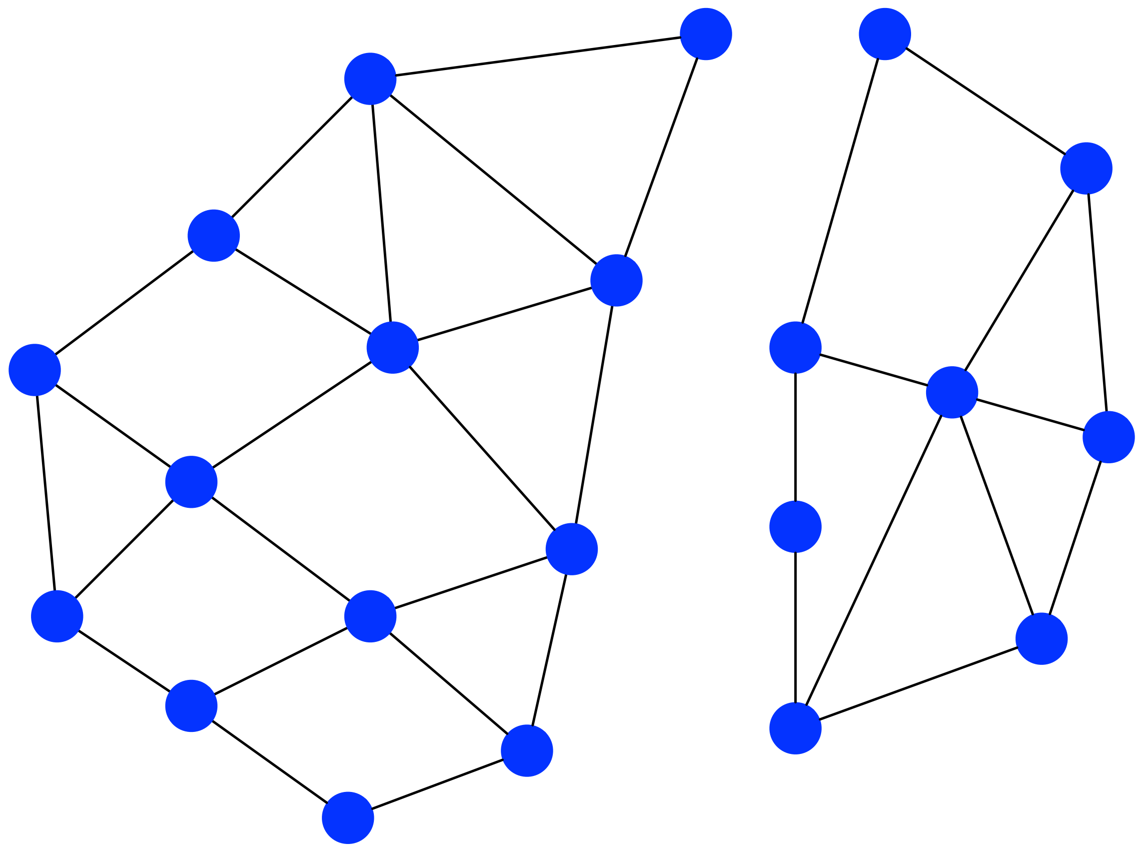

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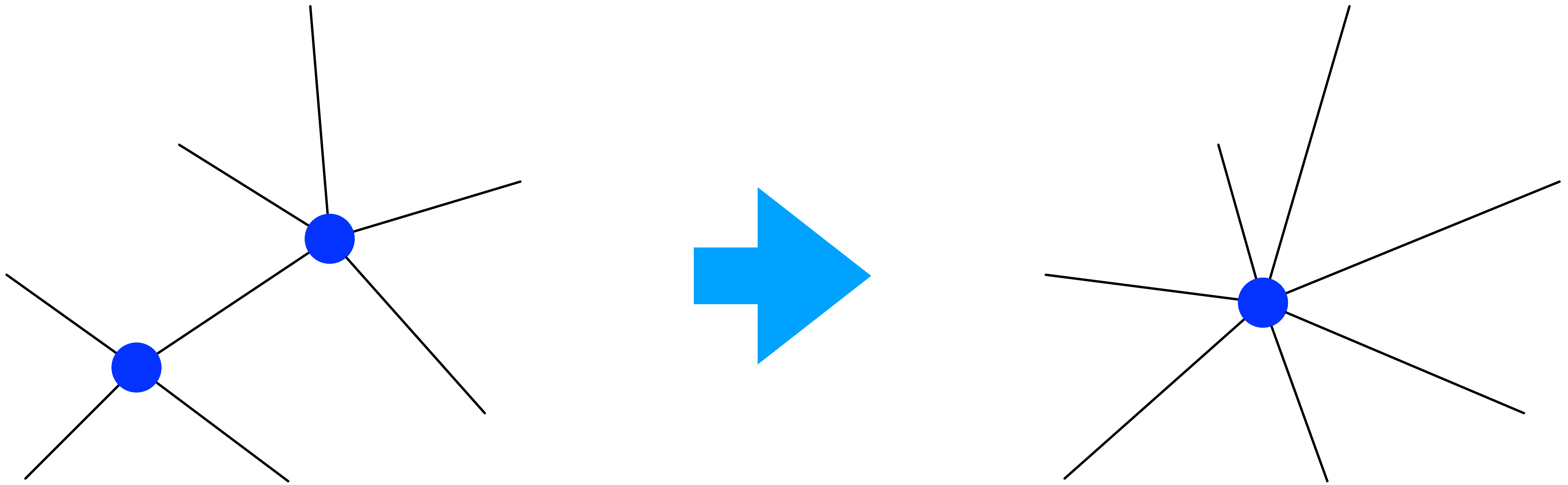
- We'll focus on finding minimum (2-)cuts (*i.e.*, minimum cuts)

- Given a graph with n vertices and m edges…
  - Compute $n - 1$ minimum s,t-cuts in $O(mn^2)$ time
  - Guess an edge outside a minimum cut (uniformly at random) and ***contract***…

Edge Contraction



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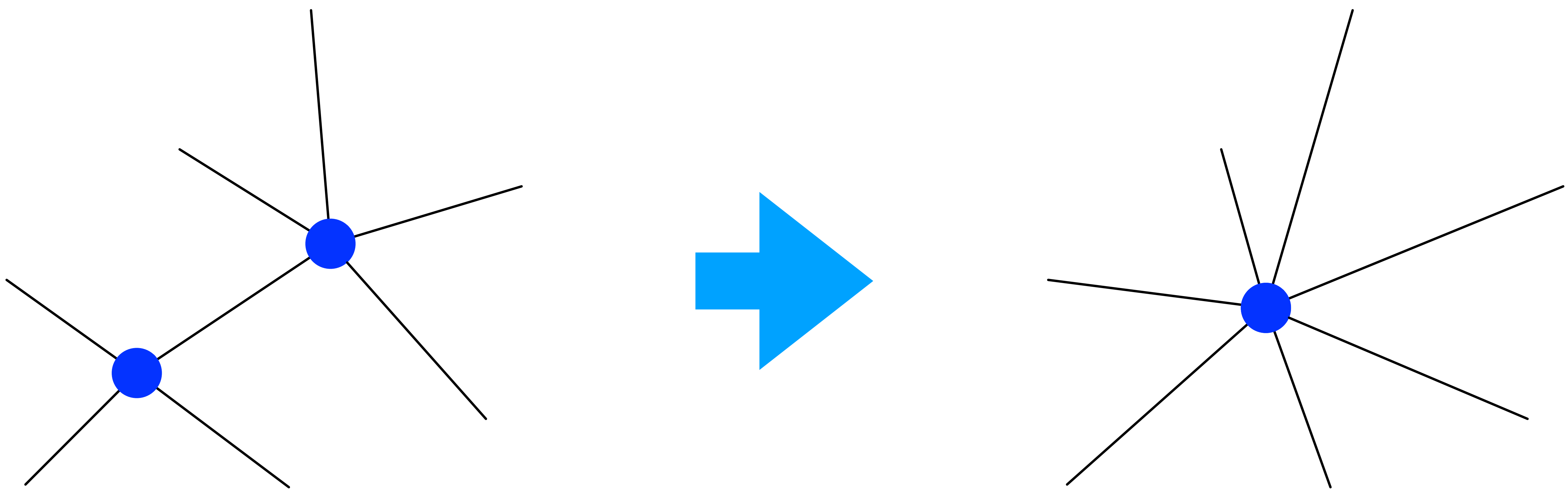

- Identify endpoints and remove loops
- Like committing to *not* separating the endpoints with a cut

Contractions Preserve Cuts (Probably)

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- **Lemma:** For *any* minimum cut C, we contract an edge of C with probability $\leq 2 / n$
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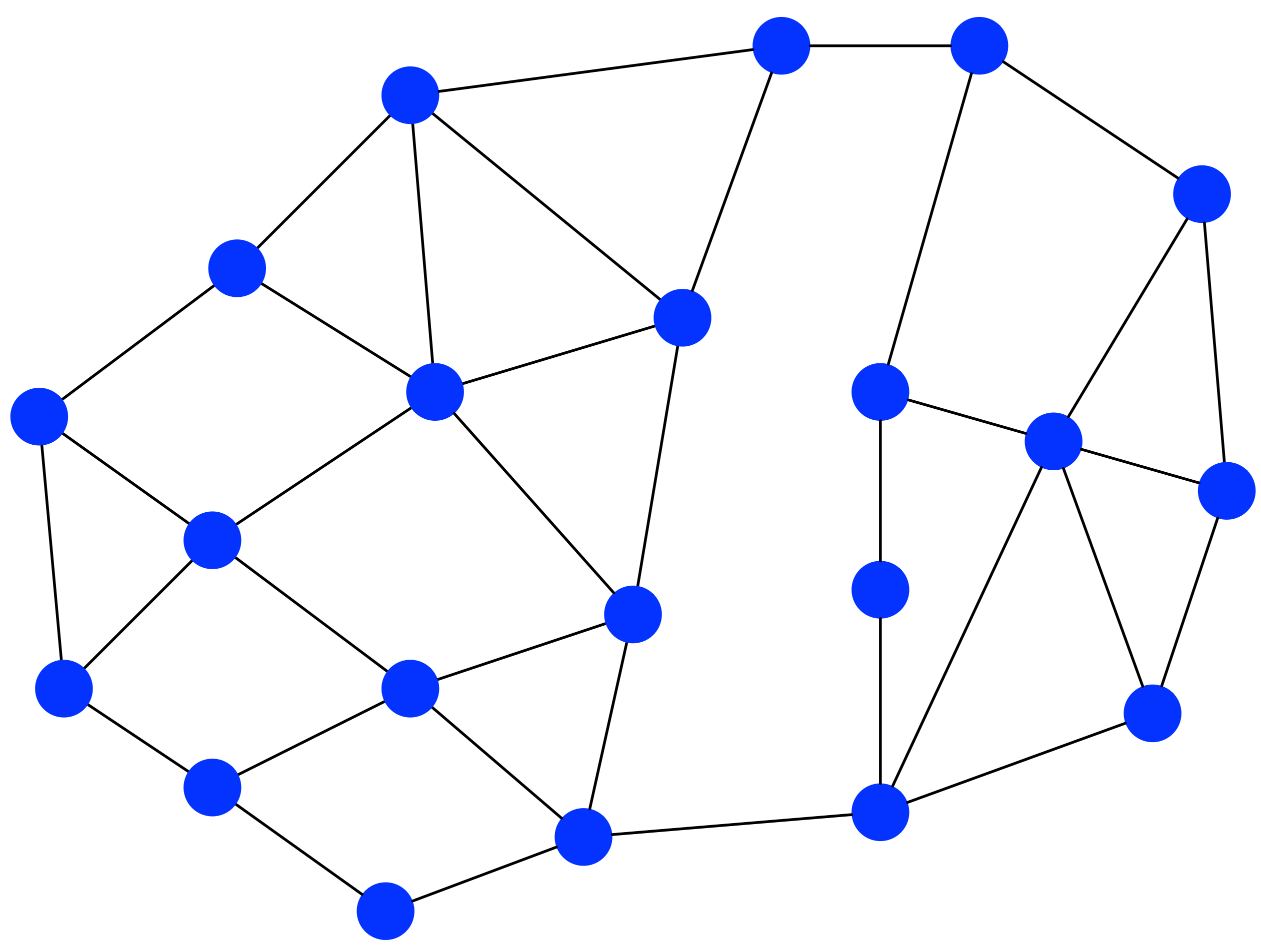
  - So there are at least $|C| (n / 2)$ edges

Repeating Contractions [Karger '93]

- Repeatedly contract until two vertices remain



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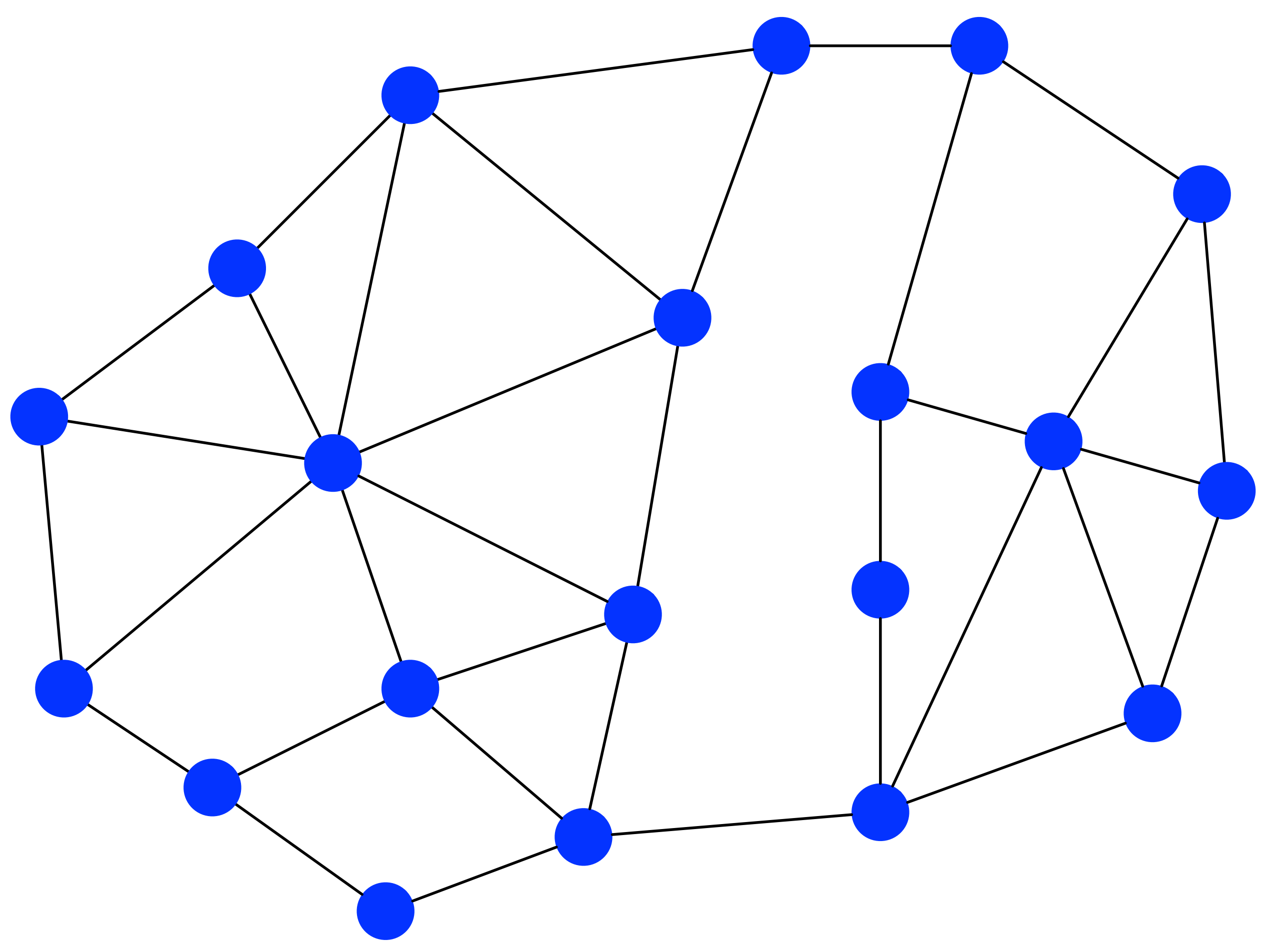

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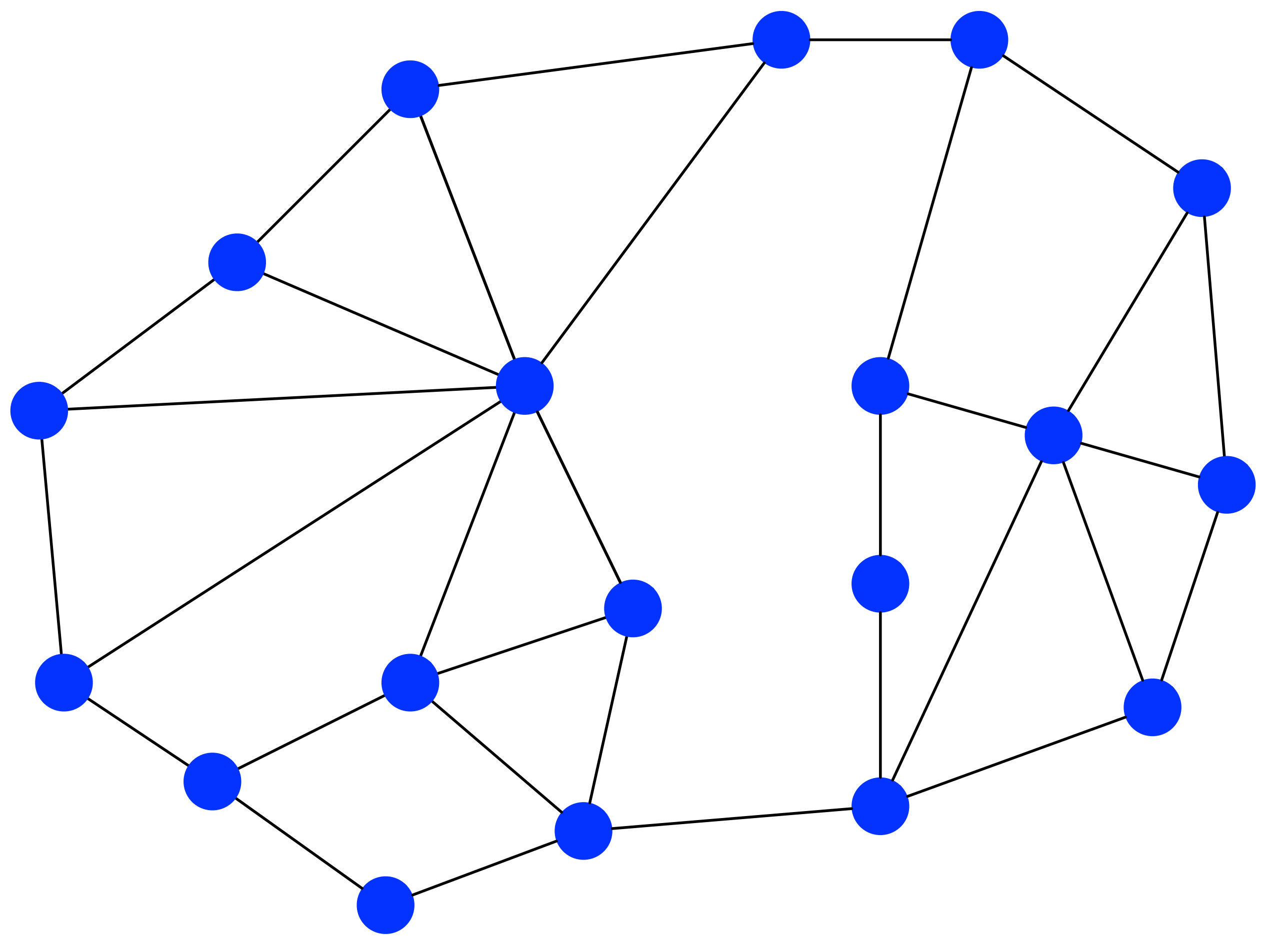

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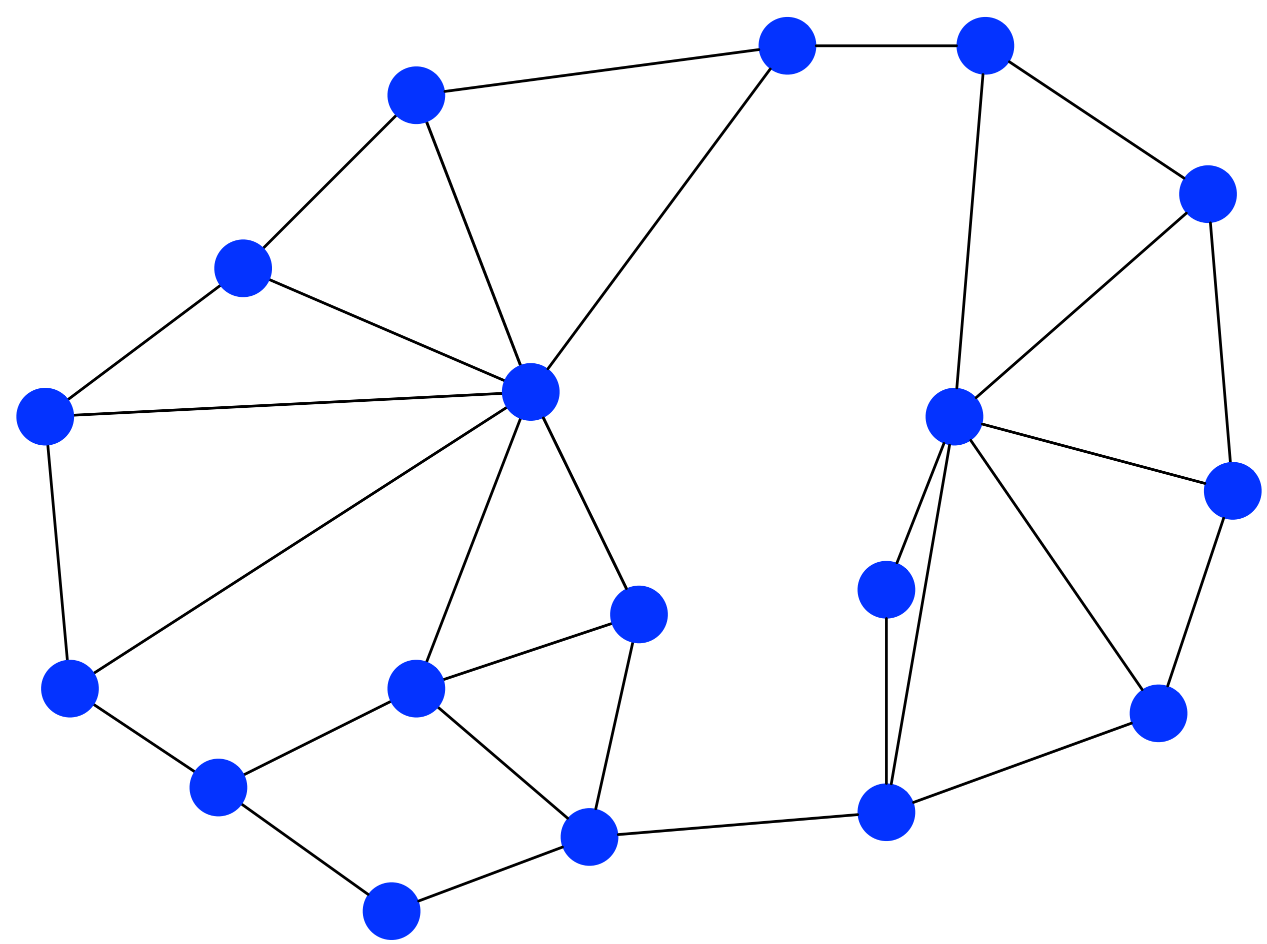

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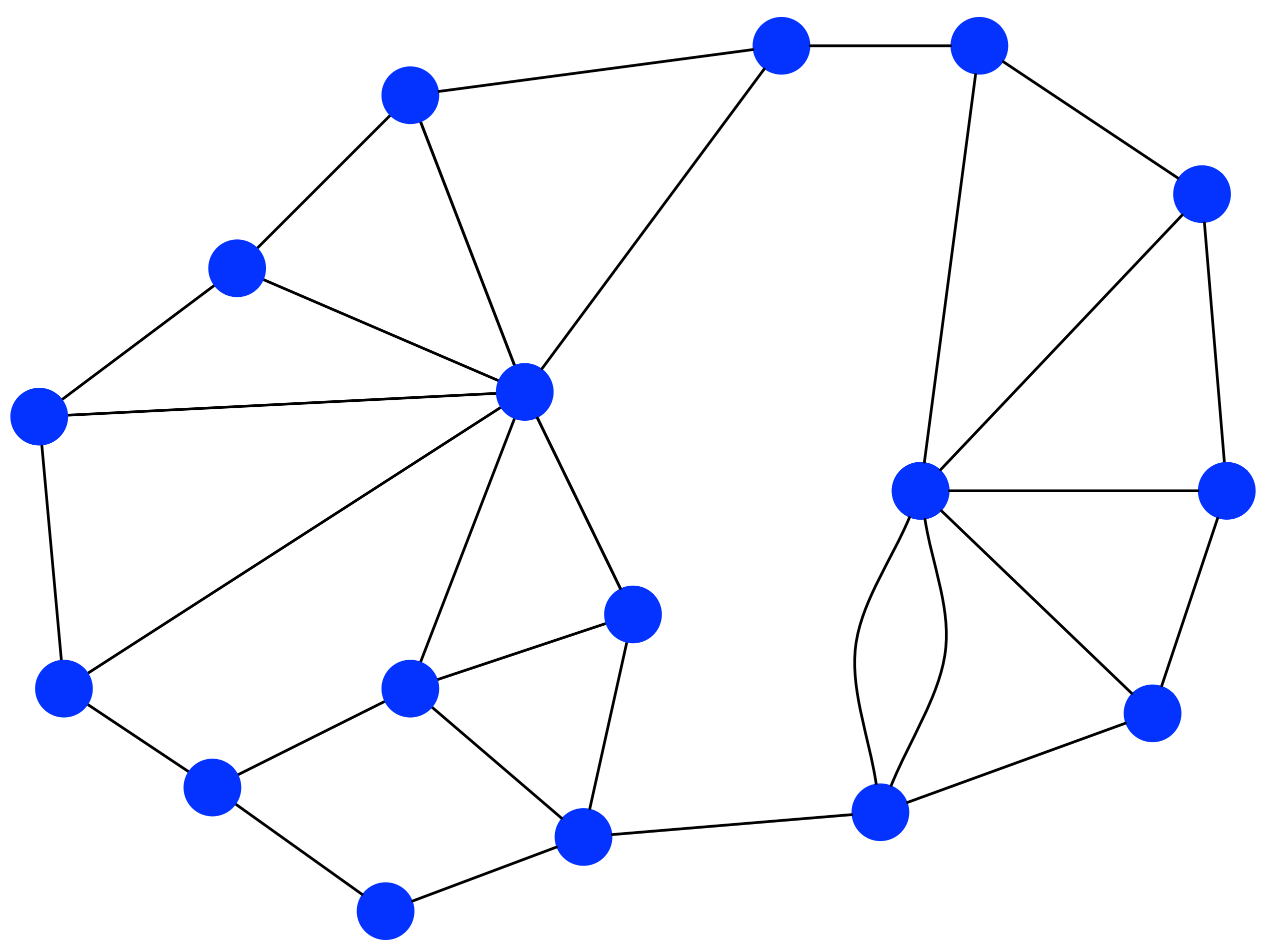

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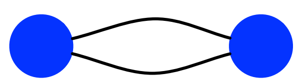
- Repeatedly contract until two vertices remain
- Surviving edges form a cut



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- Minimum cut C survives the entire process with probability $\Omega(1 / n^2)$

Try, Try Again

- One run of the repeated contraction process finds a minimum cut with probability $\Omega(1 / n^2)$

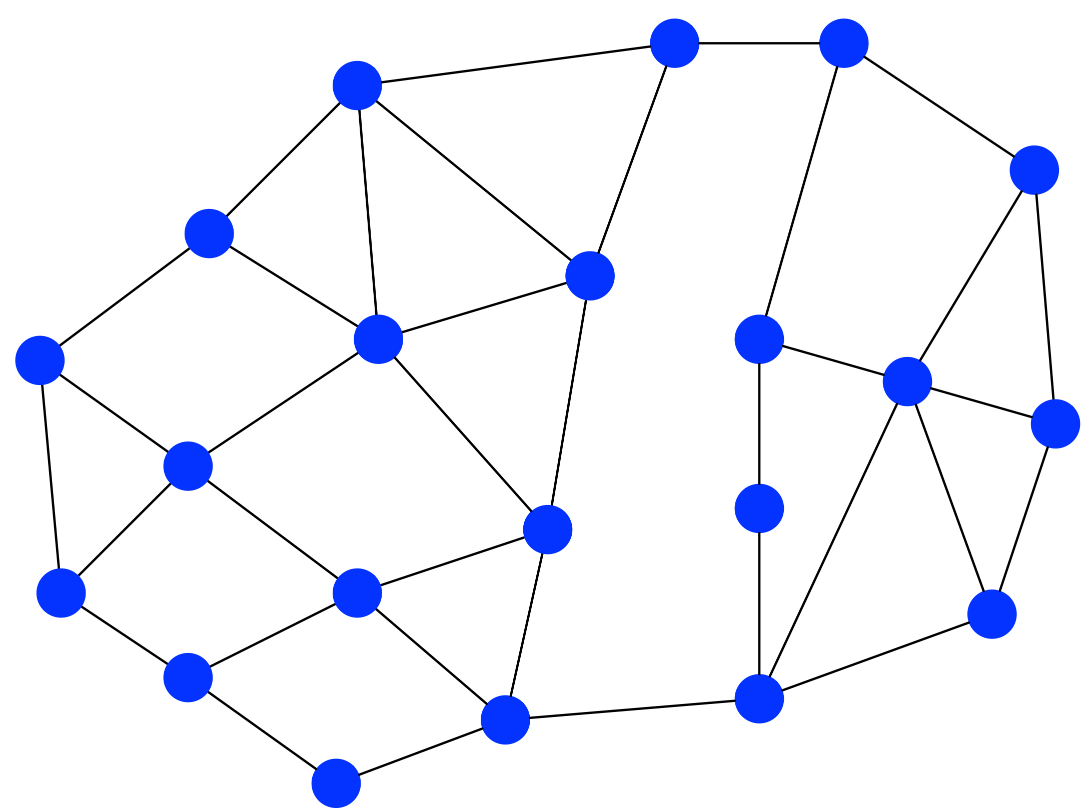
- Run it $O(n^2)$ times to succeed with constant probability

- Or $O(n^2 \log n)$ times to succeed with ***high probability*** (probability $\geq 1 - 1 / n$)

- Takes $O(n^2)$ time to fully contract once, so $O(n^4 \log n)$ time total

Branching Contractions [Karger, Stein '96]

- Probably for one run to succeed is too low!



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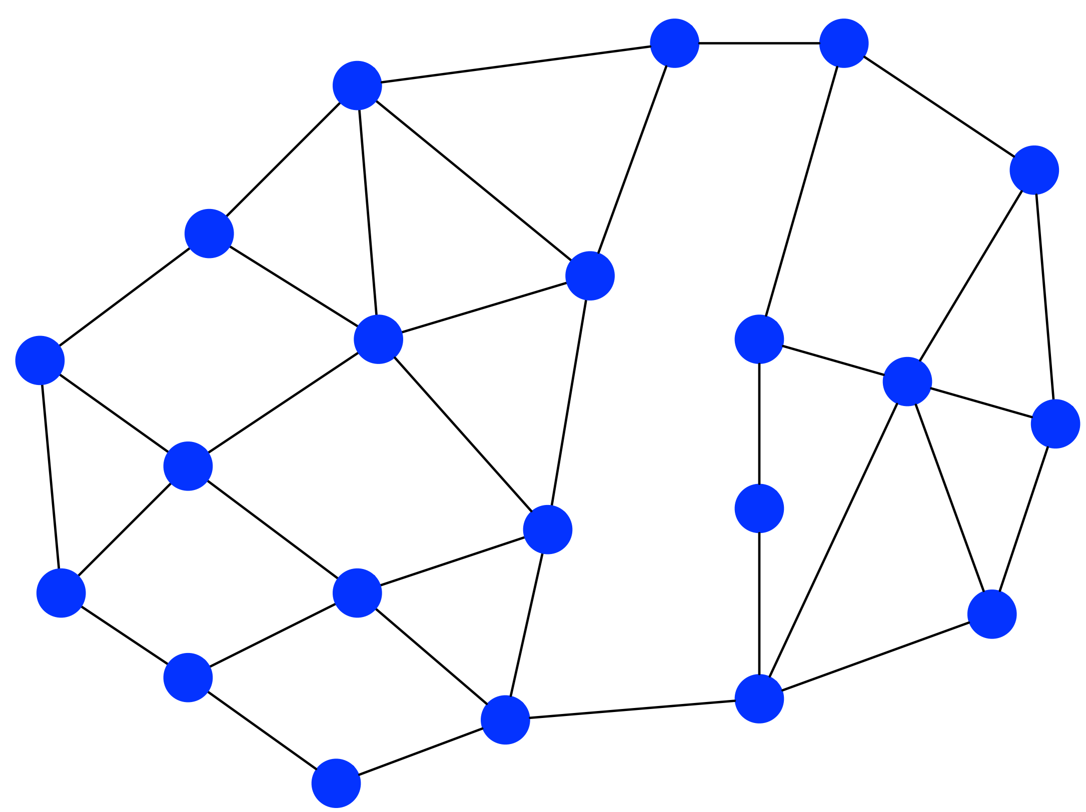

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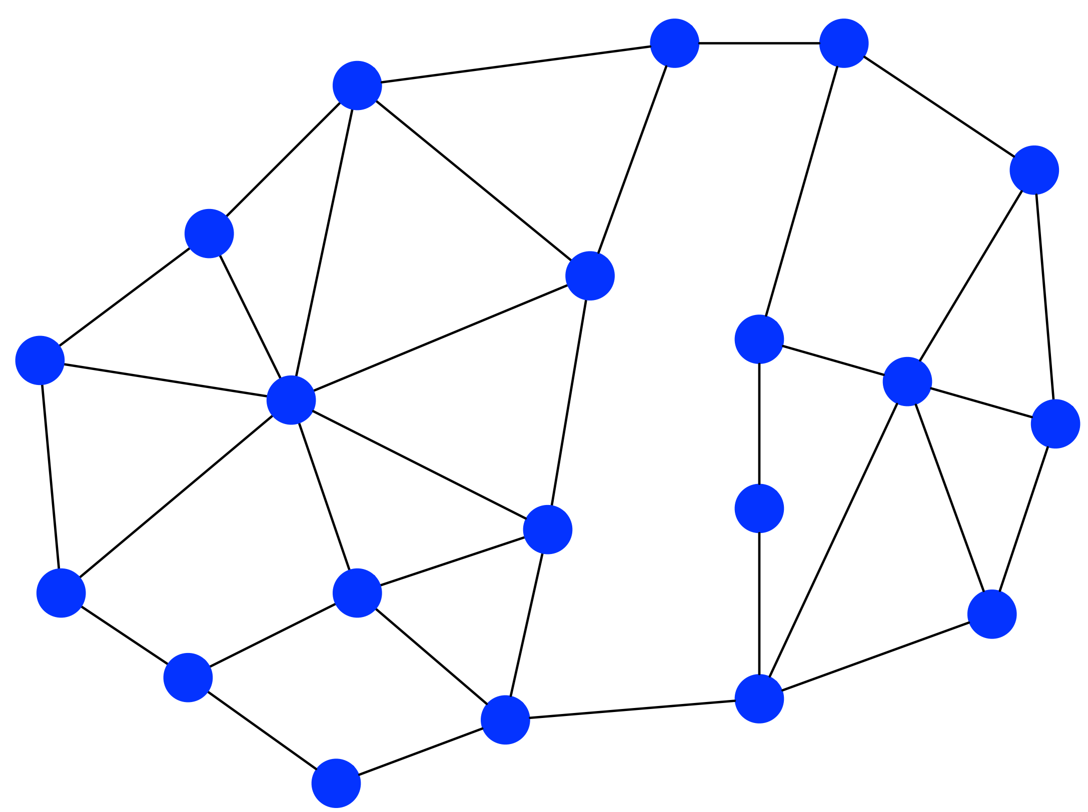

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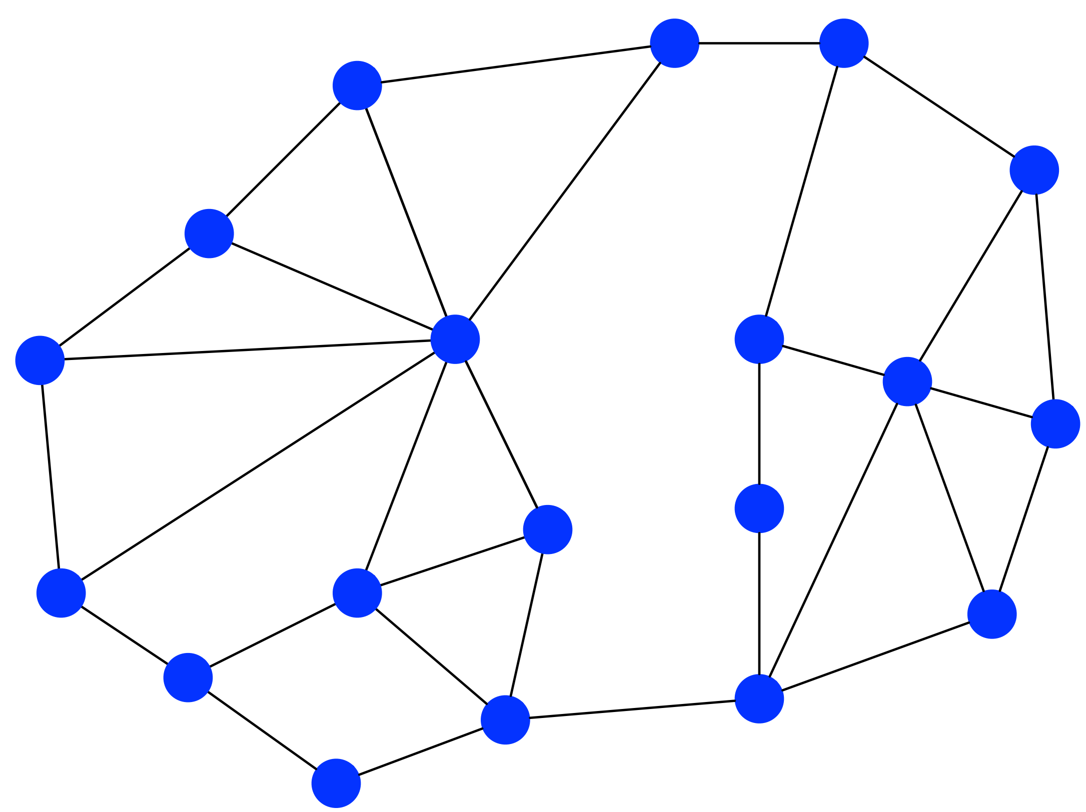

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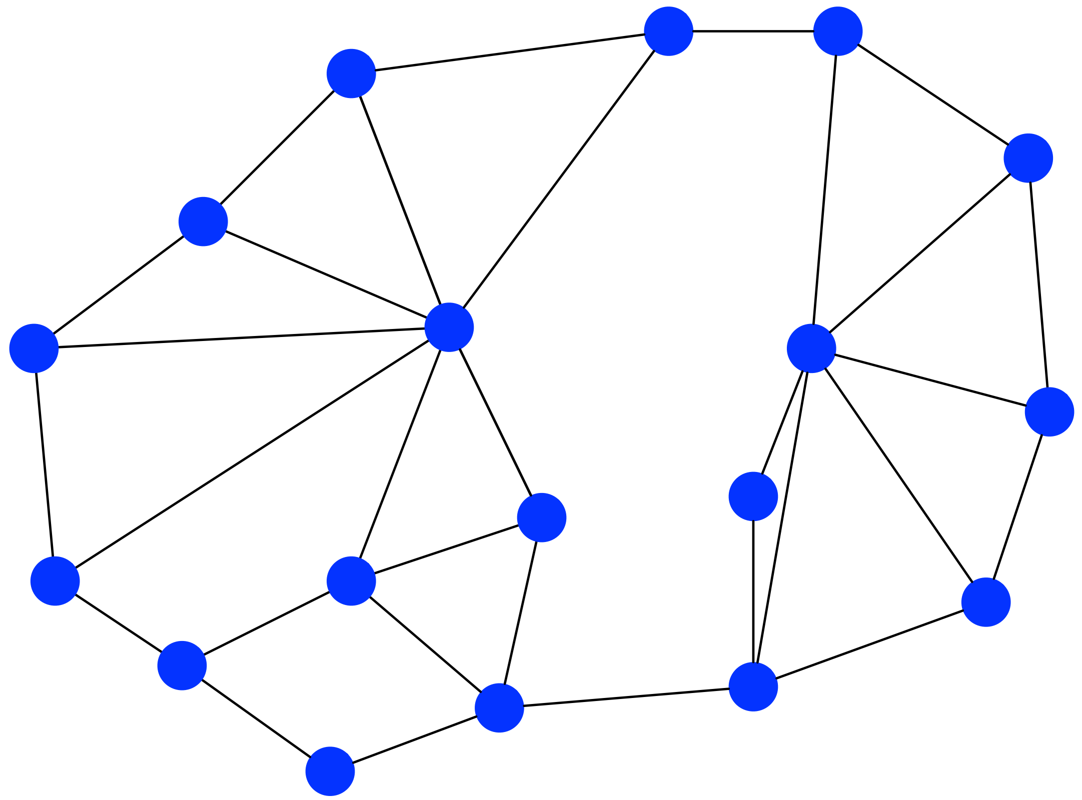

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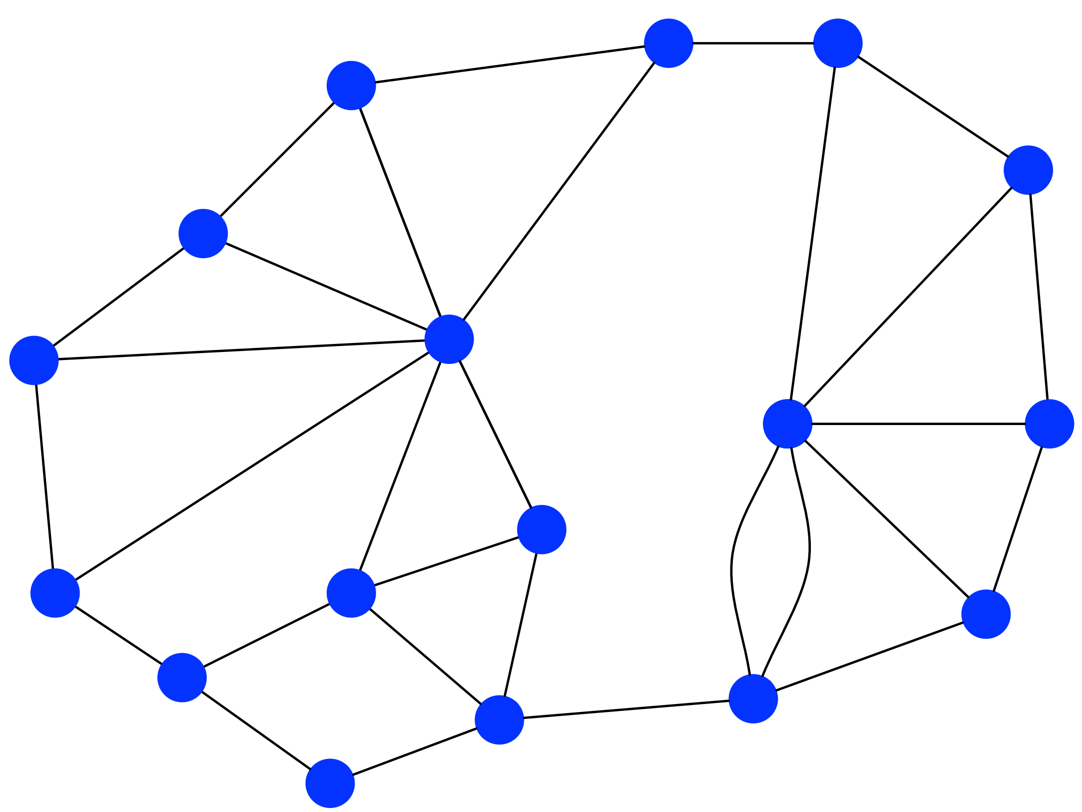
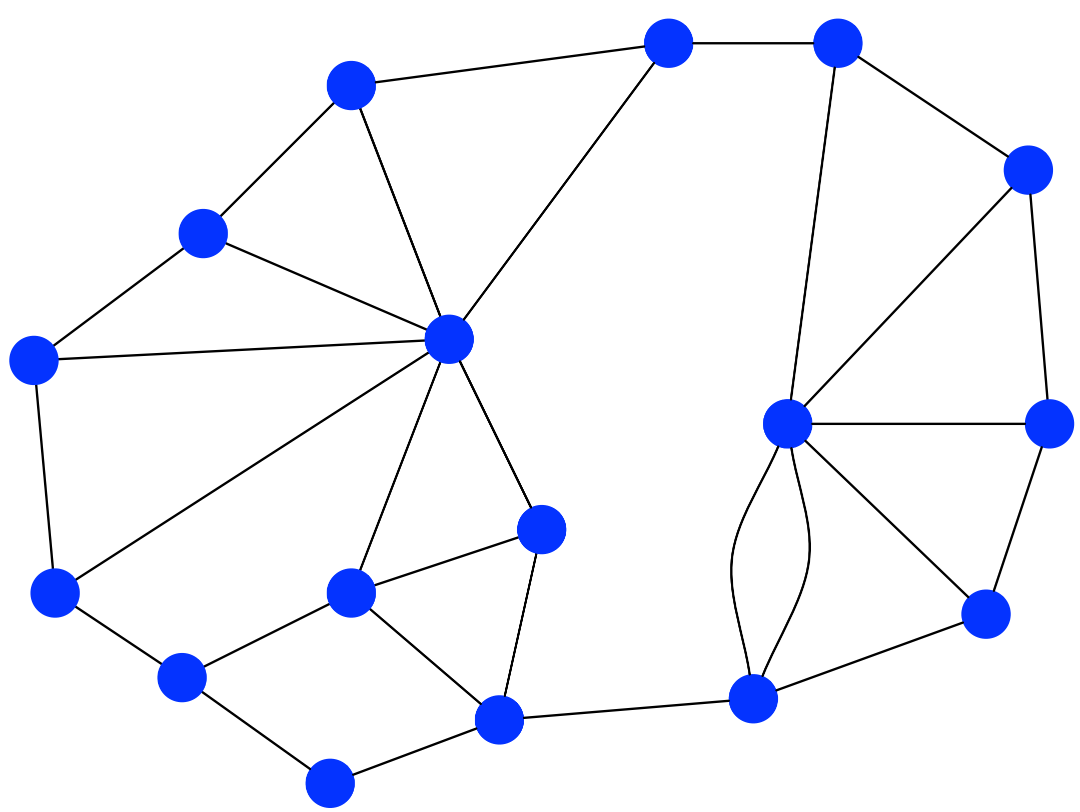


Branching Contractions [Karger, Stein '96]

- Probably for one run to succeed is too low!
- To boost probability, contract until $\lceil n/\sqrt{2} \rceil$ vertices remain
- Make two copies of contracted graph, recurse, and return the smaller cut



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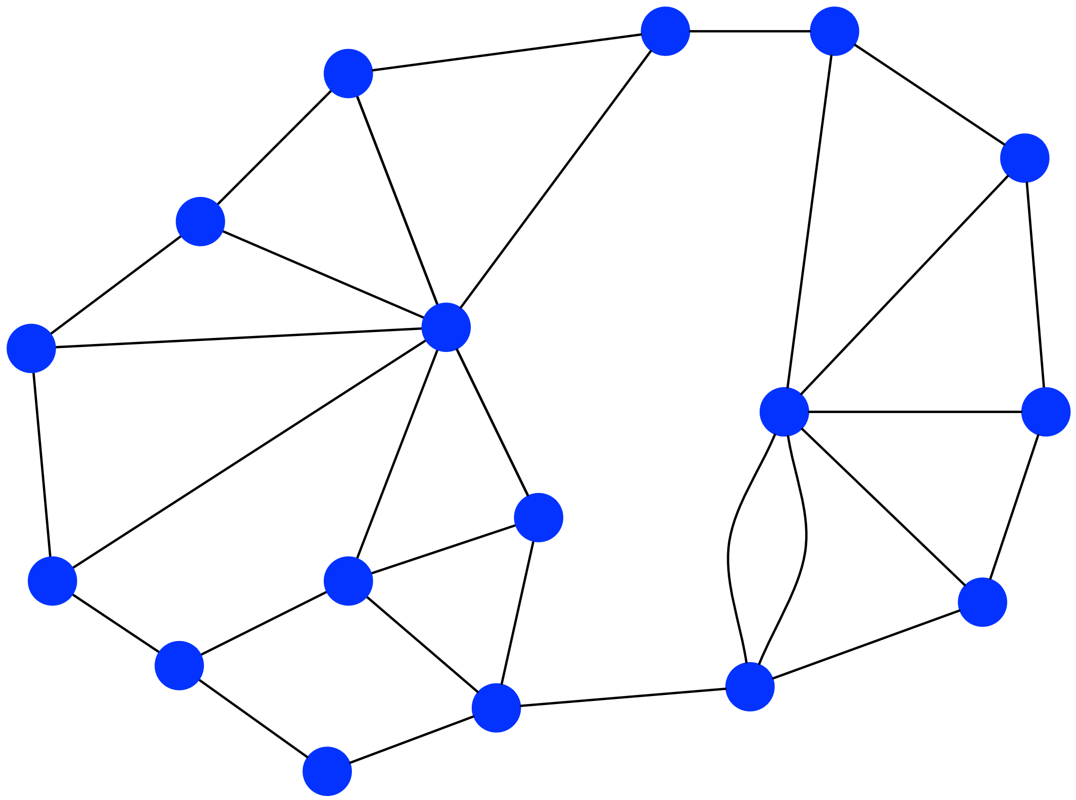

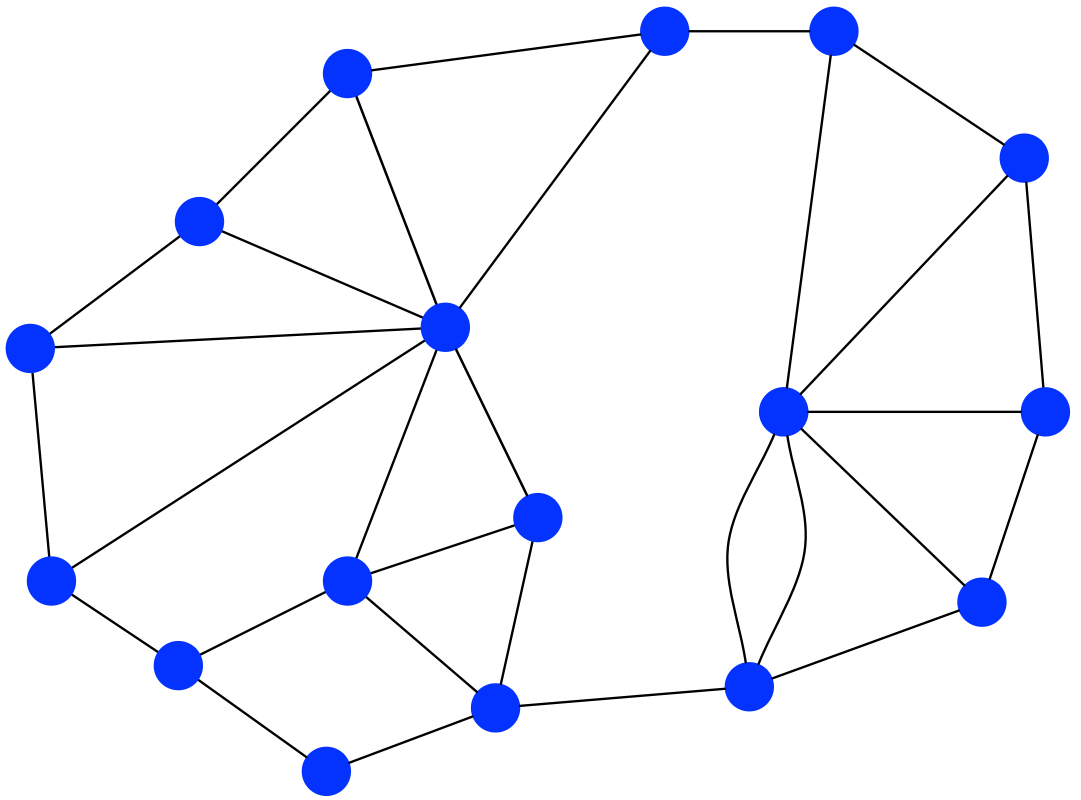

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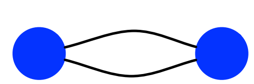

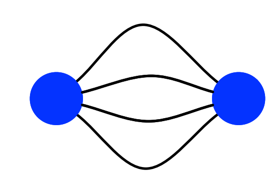

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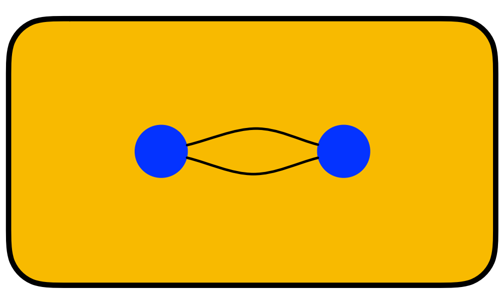

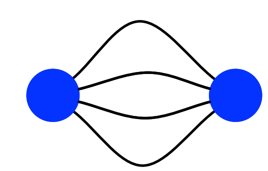

Still Fast

- A single try has the runtime recurrence $T(n) = 2T(n/\sqrt{2}) + O(n^2)$
- So $O(n^2 \log n)$ time per run

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- We return a minimum cut with probability $\Omega(1 / \log n)$

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- Can easily modify algorithm to find minimum k-cuts in $O(n^{2k - 2} \log^2 n)$ time (best known algorithm)

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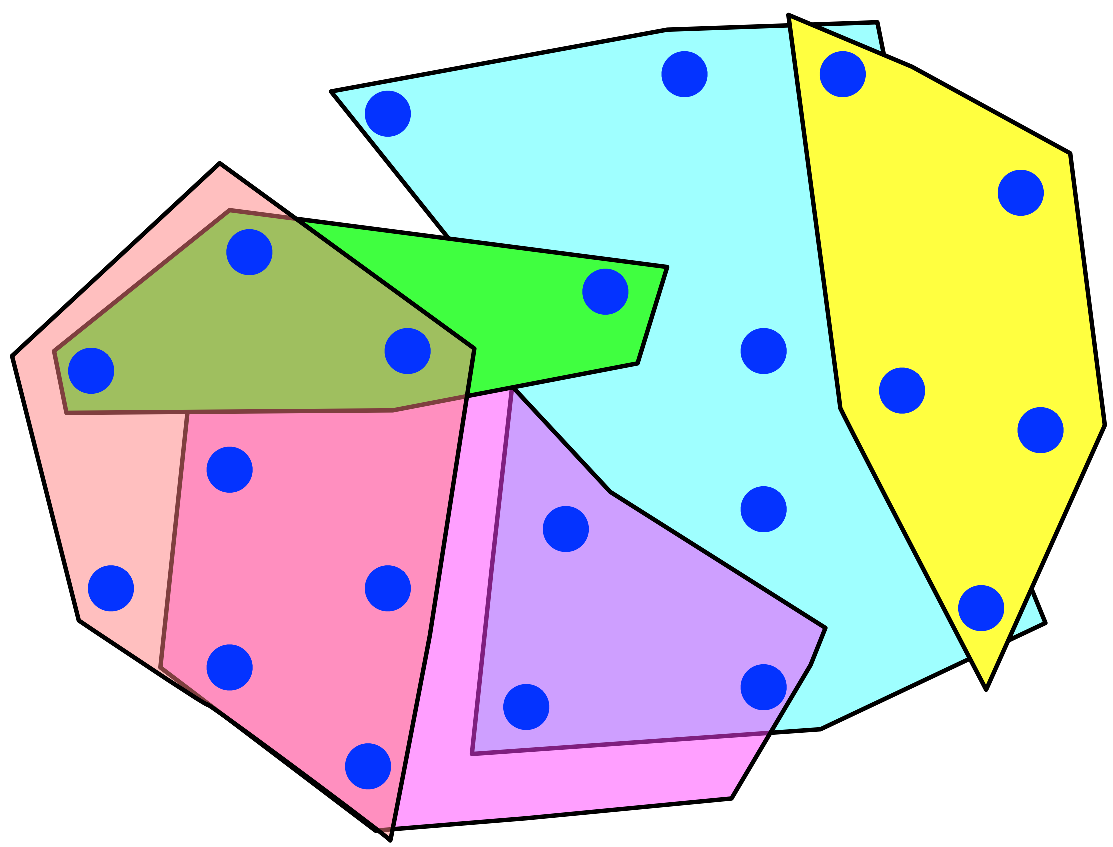
- A ***hypergraph*** is a collection of vertices and arbitrary subsets of vertices called ***hyperedges***



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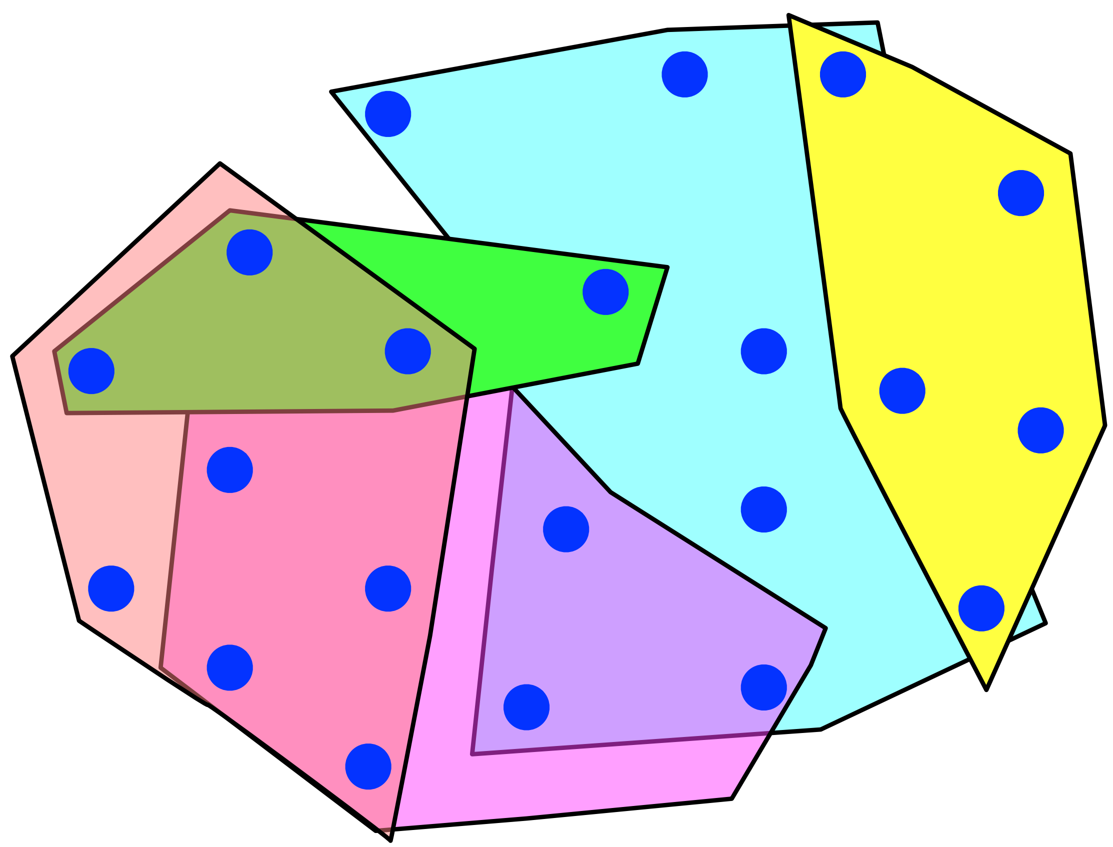

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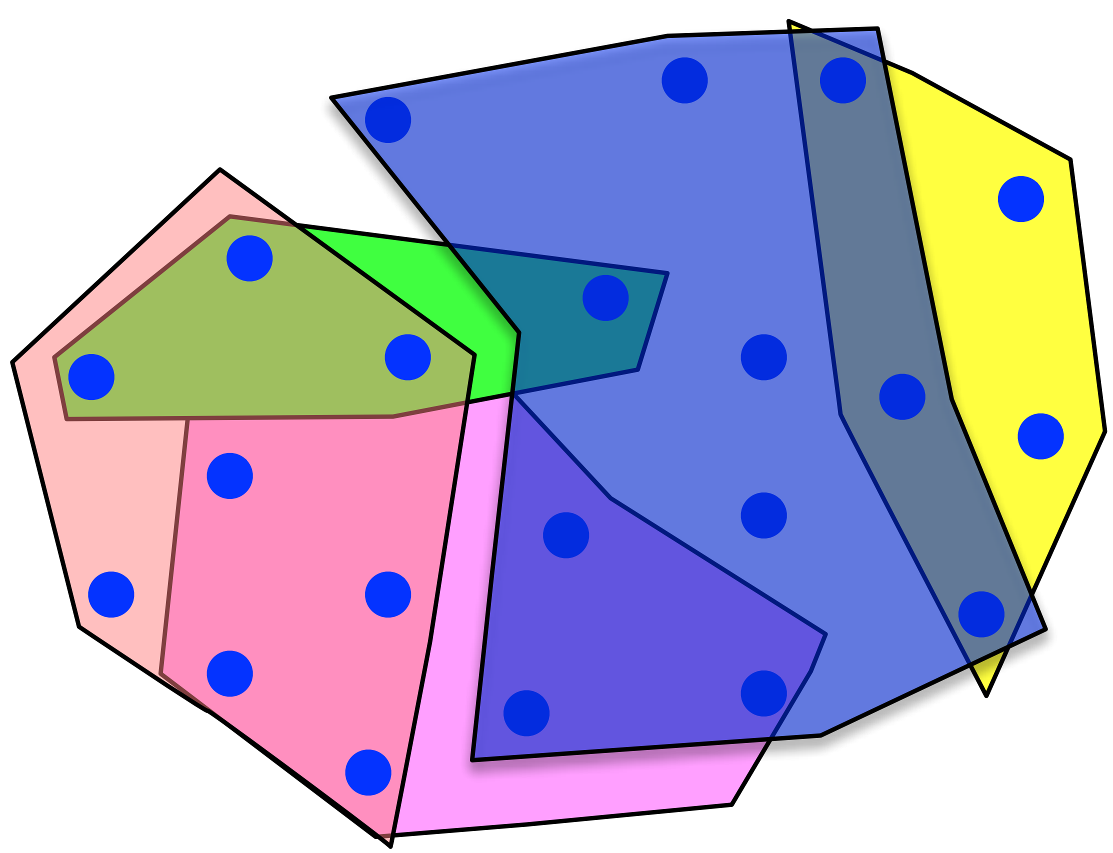

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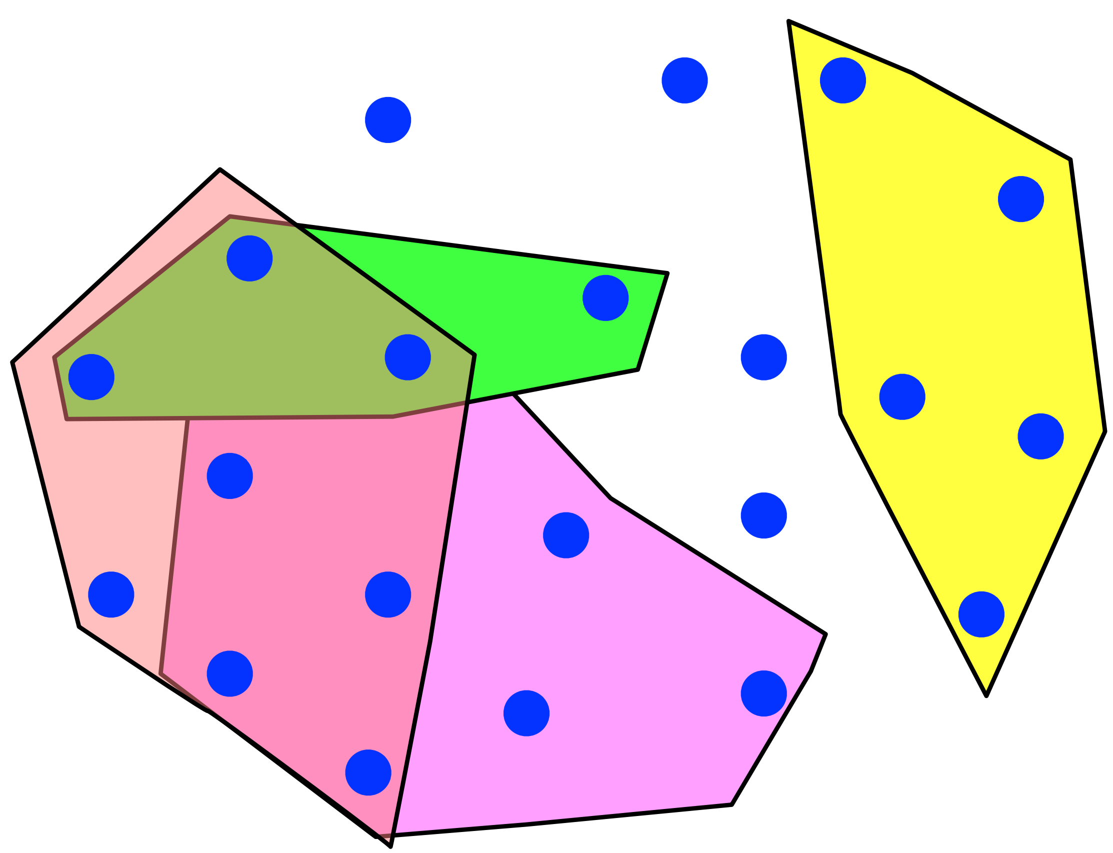

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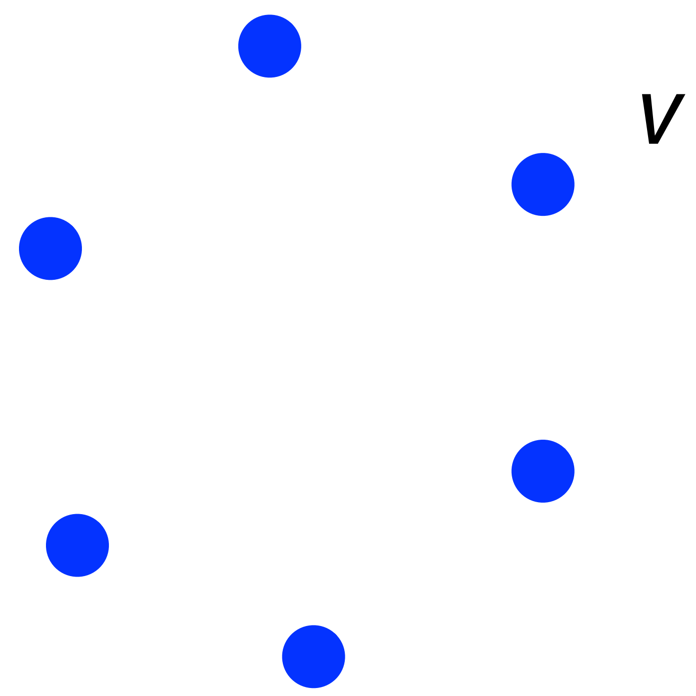
- Given a hypergraph with n vertices, m hyperedges, and total hyperedge size p
  - Guess a hyperedge outside a minimum cut and ***contract***…?

Uniform Contracts are Bad

- n - 1 distinct edges of size n - 1, all containing vertex v



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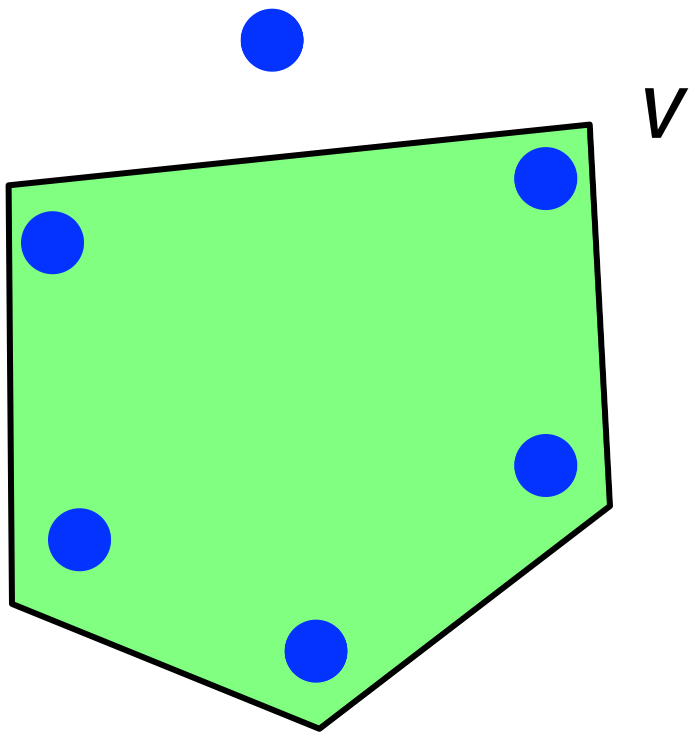

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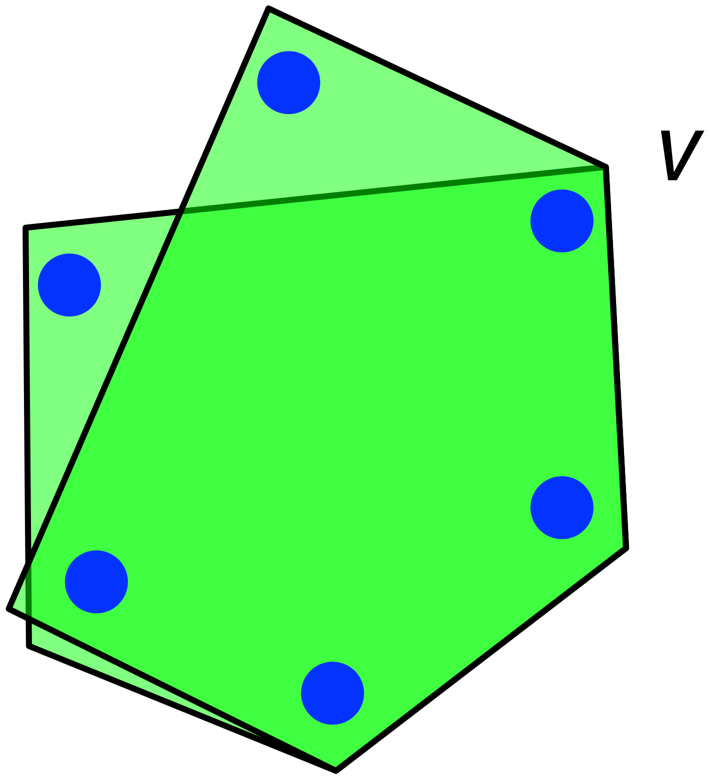

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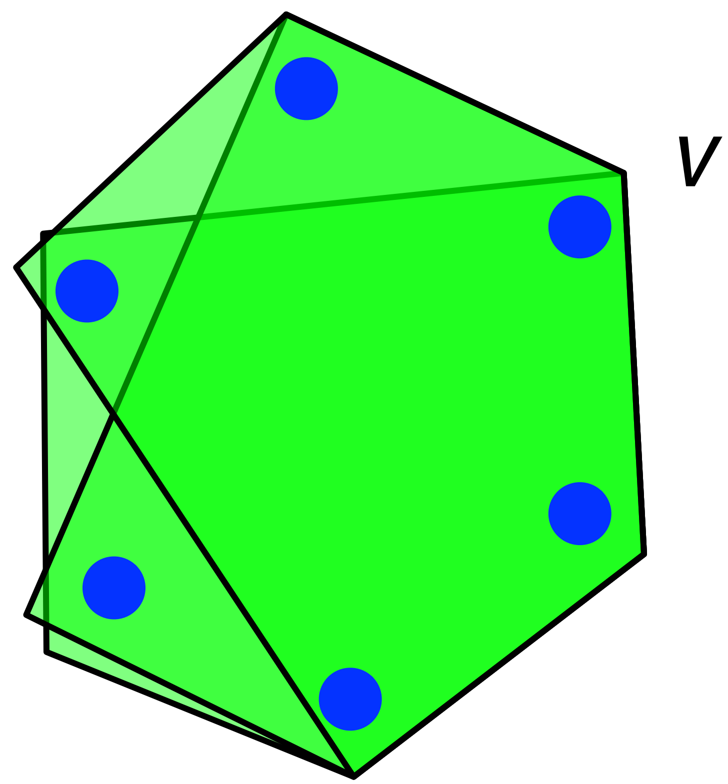

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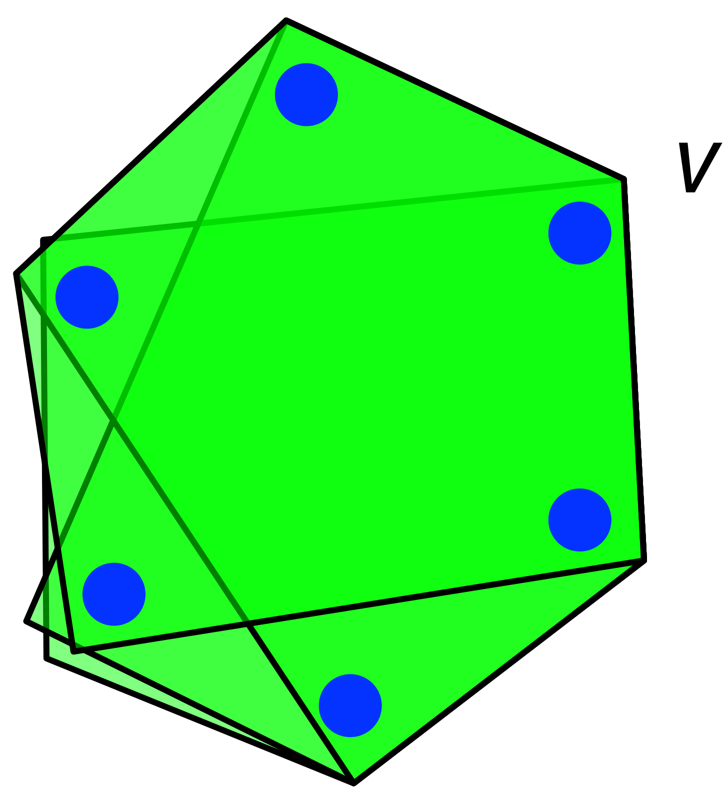

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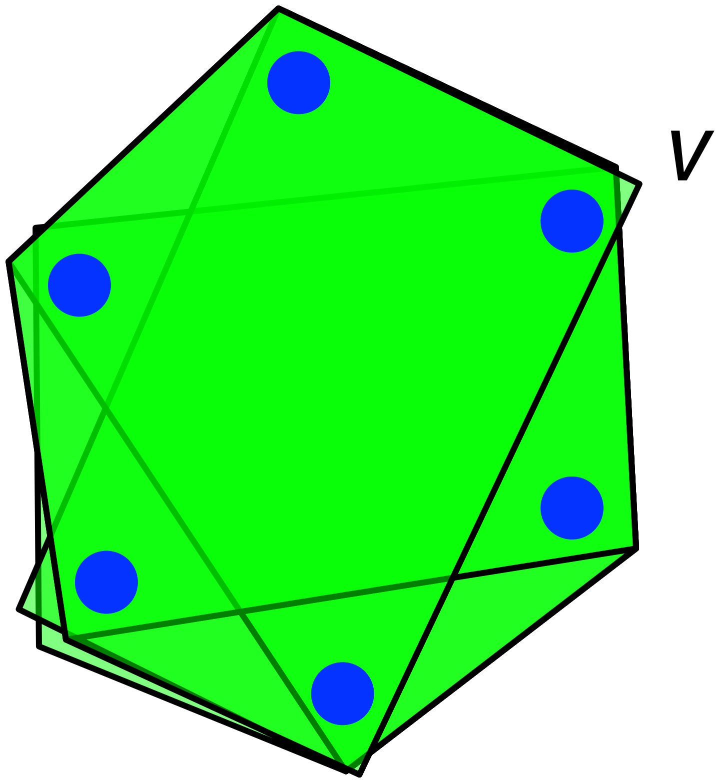

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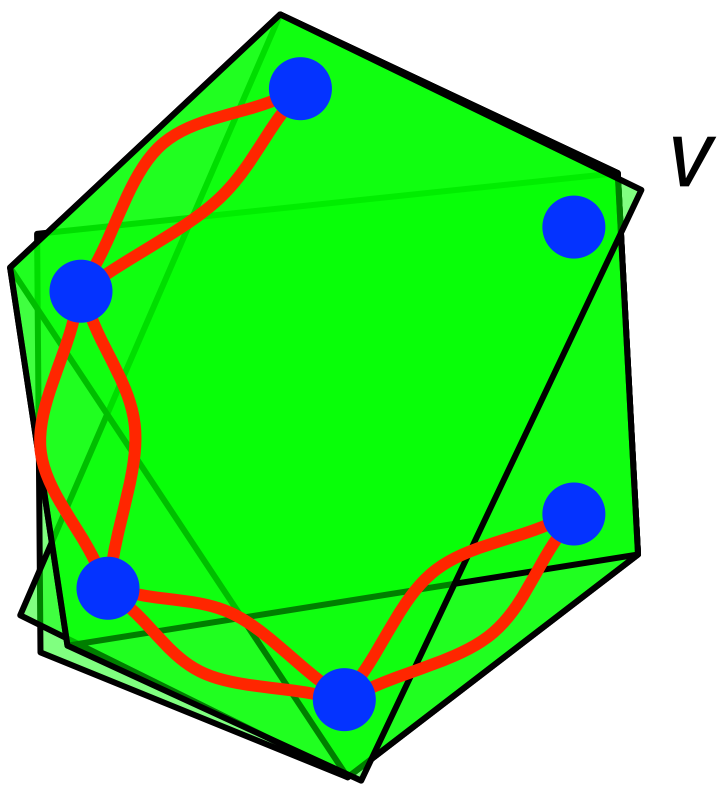
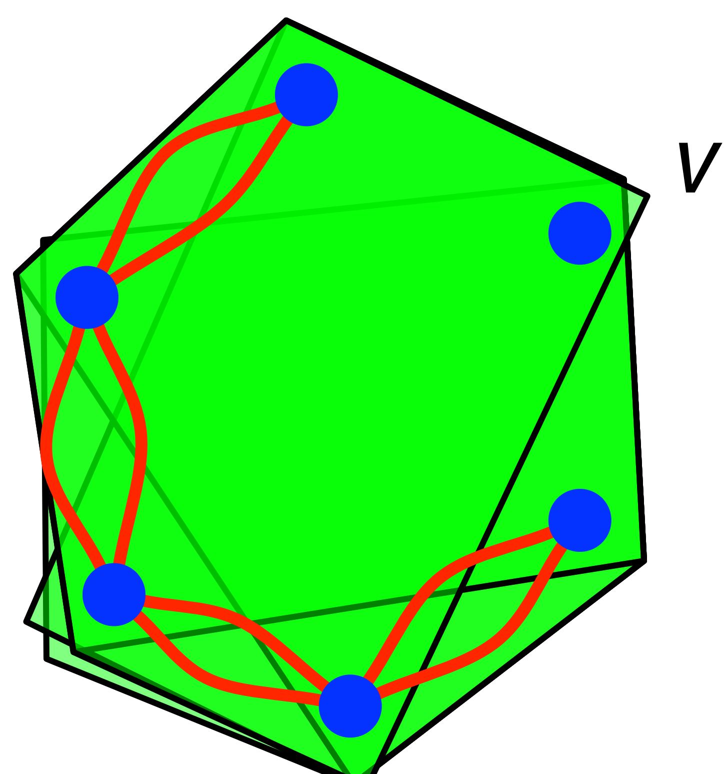


Uniform Contracts are Bad

- n - 1 distinct edges of size n - 1, all containing vertex v
- Two copies of a path spanning over every vertex except v
- Minimum cut contains every hyperedge, but there is a ≥ 1/3 probability of contracting a hyperedge each round



Really Bad

- Minimum cut survives all contractions with probability ***exponentially*** low in maximum hyperedge size [Kogan, Krauthgamer '15]

- Need to bias selection away from large hyperedges

Biased Selection

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- Minimum k-cut ($k \geq 3$) with high probability in $O(p\, n^{2k-1} \log n) = O(mn^{2k} \log n)$ time

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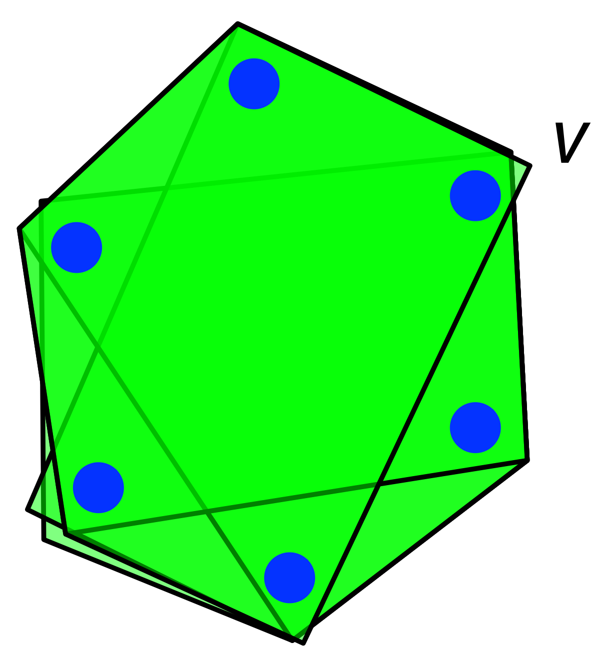
- Large hyperedges make it impossible to branch at the same time they do

Many Branches Needed

- Same example as before, but replace the paths with two hyperedges containing every vertex except *v*



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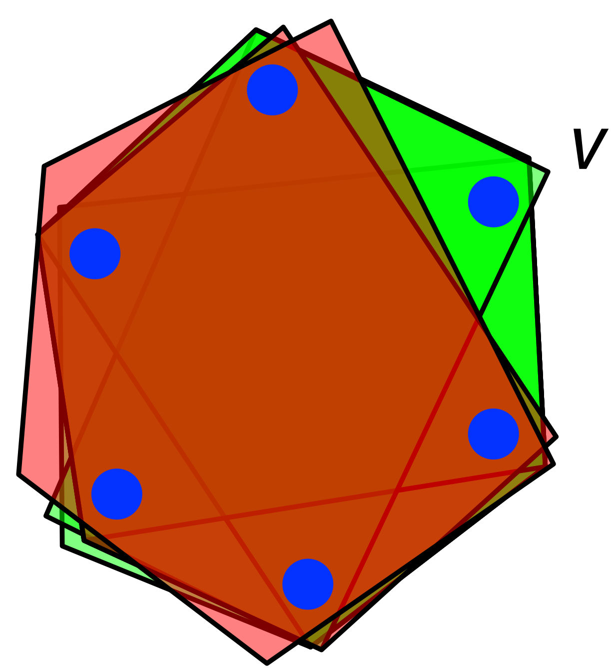

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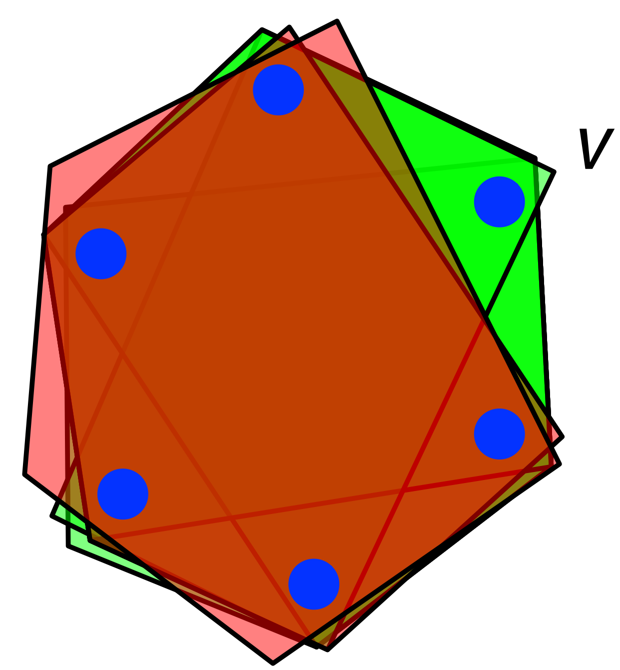

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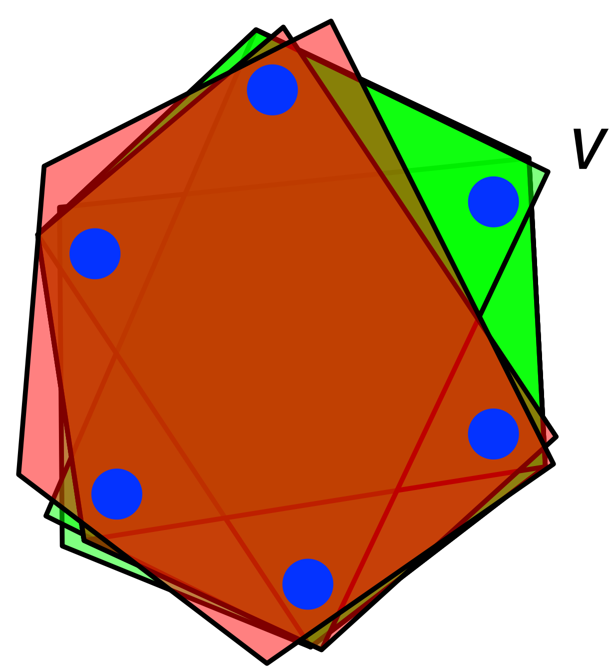
- Minimum cut contains every green hyperedge; contraction preserves cut with probability $\leq 2 / (n + 1)$

- Need $\Omega(n)$ independent contractions



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- Math works out easiest if we could do a fractional number of branches before each contraction
- Can achieve this ideal *in expectation* by branching with a probability dependent on the size of present hyperedges

Probabilistic Branching [FPZ]

- Before *every* contraction, we flip coins to decide whether or not to copy the current hypergraph and recursively call our algorithm

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- The larger the hyperedges, the more likely we are to branch

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- But before contraction, we copy the hypergraph with probability based on the selected hyperedge's size

Probability of Branching

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- Then, *in expectation*, we preserve one copy of the minimum cut in either the contracted hypergraph or the copy we create before contraction

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- So if we select hyperedge e, we branch with probability $|e| / n$

That Looks Familiar…

- We branch with probability $|e| / n$; Chandrasekaran *et al*. would have *selected* hyperedge e with probability proportional to $(1 - |e| / n)$

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- We branch with probability $|e| / n$; Chandrasekaran *et al*. would have *selected* hyperedge e with probability proportional to $(1 - |e| / n)$

- Their algorithm is the same as selecting a hyperedge e uniformly at random, and then ***redoing*** the selection with probability $|e| / n$

The Algorithm

- Maintain a set S of hyperedges we believe belong to the a minimum cut in hypergraph H

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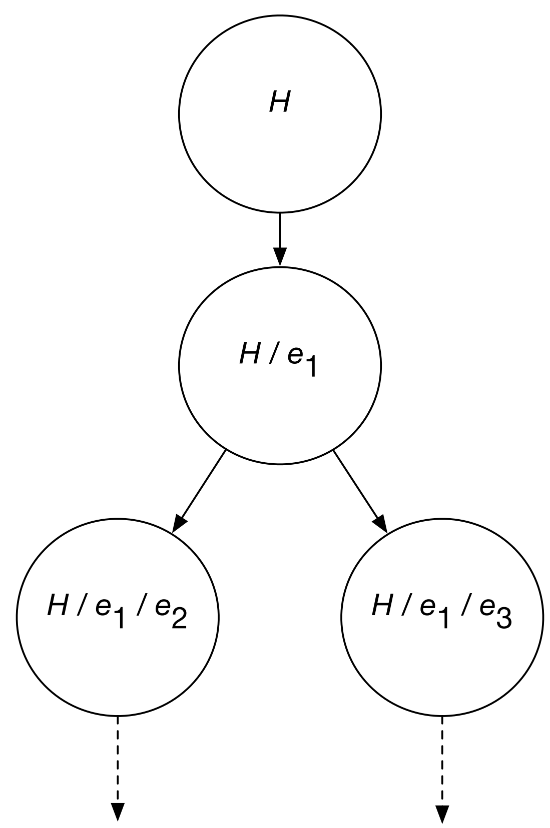
BranchingContract(H, S):

Add each spanning hyperedge to S and remove it from H
If H has no edges, return S

Select hyperedge e uniformly at random
With probability $|e| / n$, return the smaller of the cuts
BranchingContract(H / e, S) and
BranchingContract(H, S)
Otherwise, return BranchingContract(H / e, S)

Computation Tree

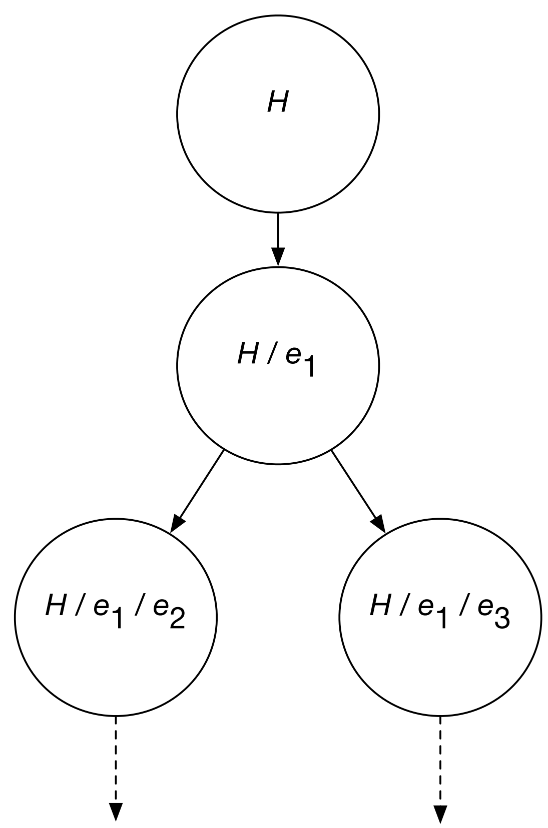
- Visualize the algorithm's execution as a rooted tree over hypergraphs; input hypergraph *H* is the root, each time we perform a contraction, that node gets a child



our algorithm

Computation Tree

- Our computation tree is probabilistic while Karger-Stein's is deterministic



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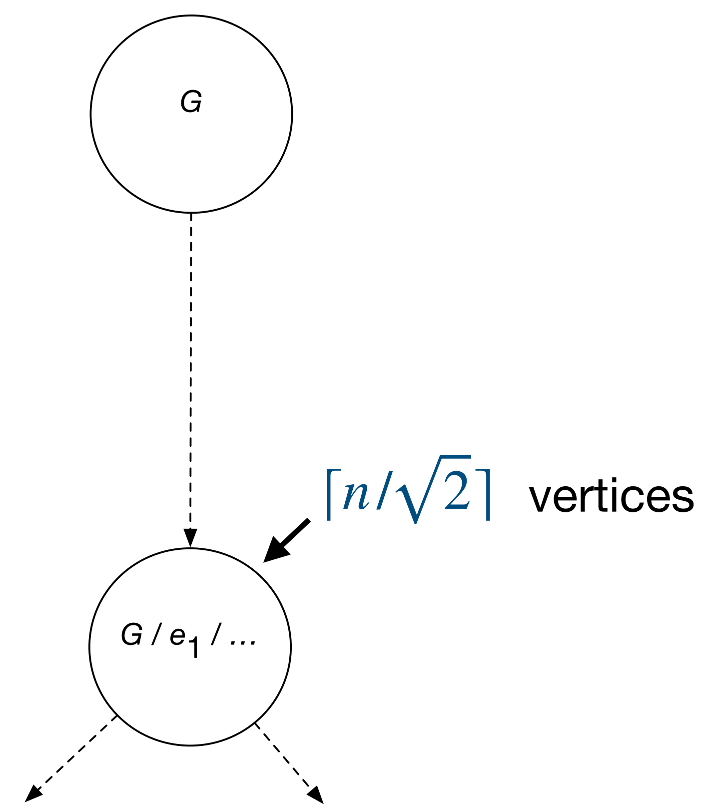
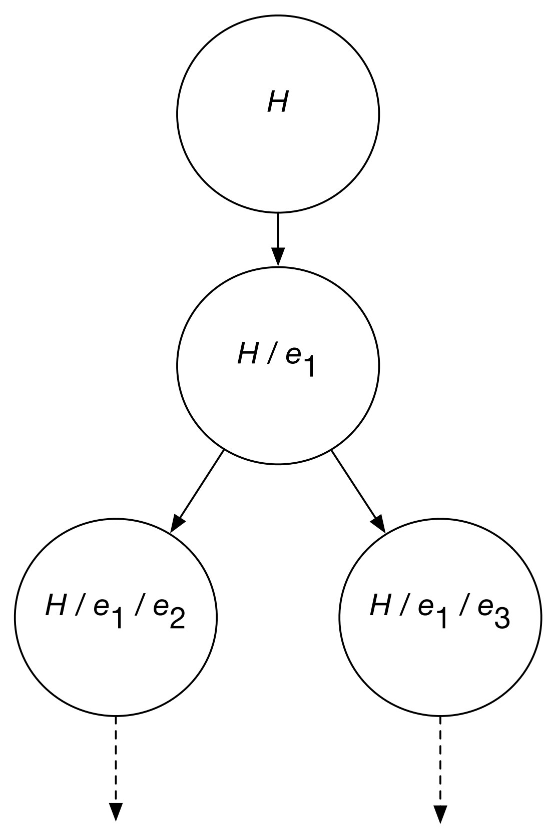


Karger-Stein

Computation Tree

- Our computation tree is probabilistic while Karger-Stein's is deterministic

- They have more branches per level *in exception*



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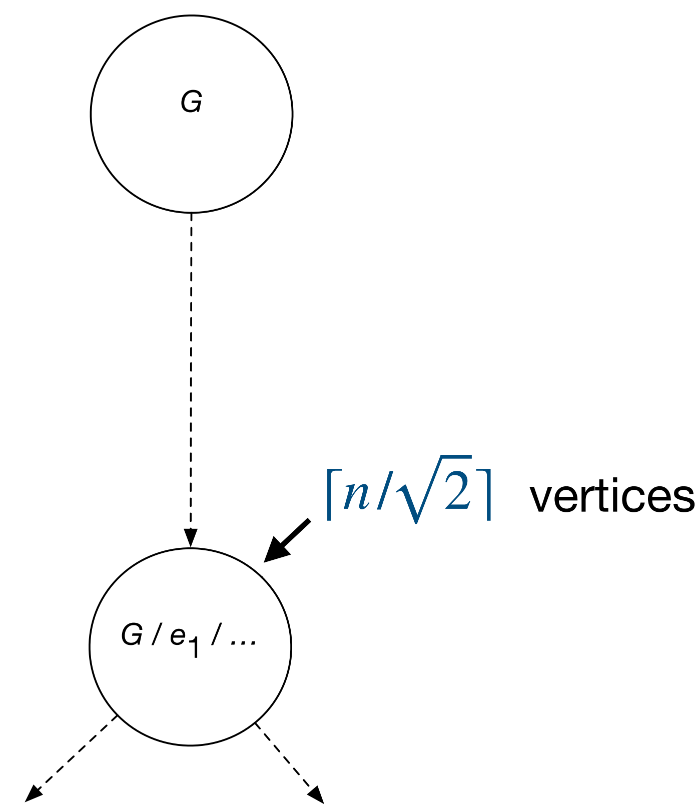

Karger-Stein

Just As Accurate

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- We return a minimum cut with probability $\geq 1 / (2H_n - 2)$ where $H_n = 1 + 1/2 + \ldots + 1 / n = \Theta(\log n)$

And Nearly as Fast

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- Can compute a minimum cut with high probability in $O(mn^2 \log^3 n)$ time, a nearly $\Omega(n)$ improvement

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- Nearly optimal for dense hypergraphs (and we match Karger-Stein when $r = 2$)

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- Branch more often when $k \geq 3$

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- Minimum k-cut with high probability in $O(n^{2k-2} \log^3 n)$ time if $r = 2k - 2$ and $O(n^{\max\{r,2k-2\}} \log^2 n)$ time otherwise for any constant k, $r \geq 2$

Thanks!