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# On the Convergence of A Class of Adam-Type Algorithms for Non-Convex Optimization

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## Abstract

This paper studies a class of adaptive gradient based momentum algorithms that update the search directions and learning rates simultaneously using past gradients. This class, which we refer to as the "Adam-type", includes the popular algorithms such as the Adam [1], AMSGrad [2] and AdaGrad [3]. Despite their popularity in training deep neural networks, the convergence of these algorithms for solving nonconvex problems remains an *open* question. This paper provides a set of mild sufficient conditions that guarantee the convergence for the Adam-type methods. We prove that under our derived conditions, these methods can achieve the convergence rate of order  $O(\log T/\sqrt{T})$  for nonconvex stochastic optimization. We show the conditions are essential in the sense that violating them may make the algorithm diverge. Moreover, we propose and analyze a class of (deterministic) incremental adaptive gradient algorithms, which has the same  $O(\log T/\sqrt{T})$  convergence rate. Our study could also be extended to a broader class of adaptive gradient methods in machine learning and optimization.

## 1 Introduction

First-order optimization has witnessed tremendous progress in last decade, especially to solve machine learning problems [4]. Almost every first-order method obeys the following generic form [5],  $\mathbf{x}_{t+1} = \mathbf{x}_t - \alpha_t \Delta_t$ , where  $\mathbf{x}_t$  denotes the solution updated at the  $t$ th iteration for  $t = 1, 2, \dots, T$ ,  $T$  is the number of iterations,  $\Delta_t$  is a certain (approximate) descent direction, and  $\alpha_t > 0$  is some learning rate. The most well-known first-order algorithms are gradient descent (GD) for deterministic optimization [6, 7] and stochastic gradient descent (SGD) for stochastic optimization [8, 9], where the former determines  $\Delta_t$  using the full (batch) gradient of an objective function, and the latter uses a simpler but more computationally-efficient stochastic (unbiased) gradient estimate.

Recent works have proposed a variety of accelerated versions of GD and SGD [6, 10–20, 3, 21–24, 1, 2]. These achievements fall into three categories: a) momentum methods [6, 10–19] which carefully design the descent direction  $\Delta_t$ ; b) adaptive learning rate methods [20, 3, 21, 22] which determine good learning rates  $\alpha_t$ , and c) adaptive gradient methods [23, 24, 1, 2] that enjoy dual advantages of a) and b). In particular, Adam [1], belonging to the third type of methods, has become extremely popular to solve deep learning problems, e.g., to train deep neural networks. Despite its superior performance in practice, theoretical investigation of Adam-like methods for *nonconvex* optimization is still missing. Very recently, the work [2] pointed out the convergence issues of Adam even in the convex setting, and proposed AMSGrad, a corrected version of Adam. Although AMSGrad has made a positive step towards understanding the theoretical behavior of adaptive gradient methods, the convergence analysis of [2] was still very restrictive because it only works for convex problems, despite the fact that the most successful applications are for nonconvex problems.

Apparently, there still exists a large gap between theory and practice. To the best of our knowledge, the question that whether adaptive gradient methods such as Adam, AMSGrad, AdaGrad converge for *nonconvex* problems is still open in theory.

**Related work** *Momentum methods* take into account the history of first-order information [6, 10–19]. A well-known method, called Nesterov’s accelerated gradient (NAG) originally designed for convex deterministic optimization [6, 10, 11], constructs the descent direction  $\Delta_t$  using the difference between the current iterate and the previous iterate. A recent work [12] studied a generalization of NAG for nonconvex stochastic programming. Similar in spirit to NAG, heavy-ball (HB) methods [13–16] form the descent direction vector through a decaying sum of the previous gradient information. In addition to NAG and HB methods, stochastic variance reduced gradient (SVRG) methods integrate SGD with GD to acquire a hybrid descent direction of reduced variance [17–19]. Recently, certain accelerated version of perturbed gradient descent (PAGD) algorithm is also proposed in [25], which shows the fastest convergence rate among all Hessian free algorithms.

*Adaptive learning rate methods* accelerate ordinary SGD by using knowledge of the past gradients or second-order information into the current learning rate  $\alpha_t$  [20, 3, 21, 22]. In [20], the diagonal elements of the Hessian matrix were used to penalize a constant learning rate. However, acquiring the second-order information is computationally prohibitive. More recently, an adaptive subgradient method (i.e., AdaGrad) penalized the current gradient by dividing the square root of averaging of the squared gradient coordinates in earlier iterations [3]. Although AdaGrad works well when gradients are sparse, its convergence is only analyzed in the convex world. Other adaptive learning rate methods include Adadelta [21] and ESGD [22], which lacked theoretical investigation although some convergence improvement was shown in practice.

*adaptive gradient methods* update the descent direction and the learning rate simultaneously using knowledge in the past, and thus enjoy dual advantages of momentum and adaptive learning rate methods. Algorithms of this family include RMSProp [23], Nadam [24], and Adam [1]. Among these, Adam has become the most widely-used method to train deep neural networks (DNNs). Specifically, Adam adopts *exponential* moving averages (with decaying/forgetting factors) of the past gradients to update the descent direction. It also uses inverse of exponential moving average of squared past gradients to adjust the learning rate. The work [1] showed Adam converges with at most  $O(1/\sqrt{T})$  rate for convex problems. However, the recent work [2] pointed out the convergence issues of Adam even in the convex setting, and proposed a modified version of Adam (i.e., AMSGrad), which utilizes a non-increasing quadratic normalization and avoids the pitfalls of Adam. Although AMSGrad has made a significant progress toward understanding the theoretical behavior of adaptive gradient methods, the convergence analysis of [2] only works for convex problems.

After the non-convergence issue of Adam has been raised in [2], there are a few recent works on proposing new variants of Adam-type algorithms. In convex setting, reference [26] proposed to stabilize the coordinate-wise weighting factor to ensure convergence. Reference [27] developed an algorithm that changes the coordinate-wise weighting factor to achieve better generalization performance. Concurrent with this work, several works are trying to understand performance of Adam in nonconvex optimization problems. Reference [28] provided convergence rate of original Adam and RMSprop under full-batch (deterministic) setting, and [29] proved convergence rate of a modified version of AdaGrad where coordinate-wise weighting is removed.

**Contributions** Differently from the aforementioned works, our work aims to build the theory to understand convergence issues for a generic class of adaptive gradient methods for nonconvex optimization. In particular, we provide a set of mild sufficient conditions that guarantee the convergence for the Adam-type methods. We summarize our contribution as follows.

- We consider both stochastic version and deterministic incremental version of a class of generalized Adam, referred to as the “Adam-type”, which includes the recently proposed AMSGrad [2], AdaGrad [3], and stochastic heavy-ball methods as special cases. We show for the first time that under suitable conditions about the stepsizes and algorithm parameters, the aforementioned algorithms all converge to first-order stationary solutions of the original nonconvex problems, yielding  $O(\log T/\sqrt{T})$  convergence rate.

- Different from recently developed theoretical analysis of AMSGrad [2], which has been focused on diminishing momentum controlling parameter, our convergence analysis is applicable to the more popular constant momentum parameter setting.
- Our work provides theoretical support for a generic class of adaptive momentum based methods, including existing algorithms such as Adam, AMSGrad, AdaGrad, as well as their new variants such as AdaGrad with momentum.
- We show the conditions are essential in the sense that violating them may make an algorithm diverge. We also provide interpretations of the convergence conditions and show empirically when certain Adam-type algorithm can outperform SGD.

We emphasize that the main technical challenge in analyzing the nonconvex version of Adam-type adaptive gradient methods is that the update directions are no longer unbiased estimates of the true gradients. Further, additional difficulty is introduced by the involved form of the learning rate. Therefore the biased gradients have to be carefully analyzed together with the use of the inverse of exponential moving average while adjusting the learning rate. We further note that the existing convex analysis [2] does not apply to the nonconvex scenario. This is because nonconvex optimization requires a different convergence criterion, given by stationarity rather than global optimality, and we consider a more practical setting with *constant* momentum controlling parameter.

**Notations** We use  $z = x/y$  to denote element-wise division if  $x$  and  $y$  are both vectors of size  $\text{size}$ ;  $x \odot y$  is element-wise product,  $x^2$  is element-wise square if  $x$  is a vector,  $\sqrt{x}$  is element-wise square root if  $x$  is a vector,  $(x)_j$  denotes  $j$ th coordinate of  $x$ ,  $\|x\|$  is  $\|x\|_2$  if not otherwise specified. We use  $[N]$  to denote the set  $\{1, \dots, N\}$ .

## 2 Preliminaries and Considered Algorithms

Stochastic optimization is a popular framework for analyzing algorithms in machine learning due to the popularity of mini-batch gradient evaluation. We consider the following generic problem where we are minimizing a function  $f$ , expressed in the expectation form as follows

$$\min_{x \in \mathbb{R}^d} f(x) = \mathbb{E}_\xi[f(x; \xi)], \quad (1)$$

where  $\xi$  is a certain random variable representing randomly selected data sample or random noise.

In a generic first-order optimization algorithm, at given time  $t$  we have access to an unbiased noisy gradient  $g_t$  of  $f(x)$ , evaluated at the current iterate  $x_t$ . The noisy gradient is assumed to be bounded and the noise on gradient at different time  $t$  is assumed to be independent. An important assumption that we will make throughout this paper is that the function  $f(x)$  is continuously differentiable and has Lipschitz continuous gradient, but could otherwise be a *nonconvex* function. The nonconvex assumption represents a major departure from the convexity that has been assumed in recent papers while analyzing Adam-type methods, such as [1] and [2]. We argue that it is critical to be able to analyze problem (2) without convexity assumption, because after all, the most popular use cases for Adam and its variants are training neural networks, whose objective function is highly nonconvex.

Our work focuses on the following generic form of exponentially weighted stochastic gradient descent algorithm, for which we name as *generalized Adam* due to its resemblance to the original Adam algorithm and many of its variants.

### Algorithm 1. Generalized Adam

(S0). Initialize  $m_0 = 0$  and  $x_1$   
 For  $t = 1, \dots, T$ , do  
   (S1).  $m_t = \beta_{1,t}m_{t-1} + (1 - \beta_{1,t})g_t$   
   (S2).  $\hat{v}_t = f_t(g_1, g_2, \dots, g_t)$   
   (S3).  $x_{t+1} = x_t - \alpha_t m_t / \sqrt{\hat{v}_t}$   
 End

In Algorithm 1,  $\alpha_t$  is the step size at time  $t$ ,  $\beta_{1,t} > 0$  is a sequence of problem parameters;  $m_t \in \mathbb{R}^d$  denotes some (exponentially weighted) gradient estimate;  $\hat{v}_t = f_t(g_1, g_2, \dots, g_t) \in \mathbb{R}^d$  takes all the

past gradients as input and returns a vector of dimension  $d$  and is later used to inversely weight the gradient estimate  $m_t$ ; And note that  $m_t/\sqrt{\widehat{v}_t} \in \mathbb{R}^d$  represents element-wise division.

We highlight that the generalized Adam algorithm includes the following well-known algorithms as special cases:

- When  $\beta_{1,t} = 0, (\widehat{v}_t)_j = 1, \forall t > 0, j \in [d]$ , the algorithm becomes the classic SGD;
- When  $\beta_{1,t} = \beta_1, 0 < \beta_1 < 1, (\widehat{v}_t)_j = 1, \forall t > 0, j \in [d]$ , the algorithm becomes the stochastic gradient descent with heavy-ball momentum (i.e., SGD with Momentum);
- When  $\beta_{1,t} = \beta_1, 0 < \beta_1 < 1, \widehat{v}_t = \beta_2 \widehat{v}_{t-1} + (1 - \beta_2)g_t^2$  for some  $\beta_2 \in (0, 1)$ , it is the Adam algorithm proposed in [1];
- When  $\widehat{v}_t = \max(\widehat{v}_{t-1}, v_t)$  and  $v_t = \beta_2 v_{t-1} + (1 - \beta_2)g_t^2$ , this is AMSGrad algorithm recently proposed in [2];
- When  $\beta_{1,t} = 0$  and  $\widehat{v}_t = \sum_{i=1}^t g_i^2/t$ , i.e.,  $\widehat{v}_t = (1 - 1/t)\widehat{v}_{t-1} + (1/t)g_t^2$ , it becomes AdaGrad [3]. Such a mapping has also been discussed in [2].

We summarize some popular variants of the generalized Adam algorithm in Table 1.

$\widehat{v}_t \backslash \beta_{1,t}$	$\beta_{1,t} = 0$	$\beta_{1,t} \leq \beta_{1,t-1}, \beta_{1,t} \xrightarrow[t \rightarrow \infty]{} b$	$\beta_{1,t} = \beta_1$
$\widehat{v}_t = 1$	SGD	N/A	Heavy-ball method
$\widehat{v}_t = \frac{1}{t} \sum_{i=1}^t g_i^2$	AdaGrad	AdaGrad with momentum	AdaGrad with momentum
$v_t = \beta_2 v_{t-1} + (1 - \beta_2)g_t^2, \widehat{v}_t = \max(\widehat{v}_{t-1}, v_t)$	AMSGrad	AMSGrad	AMSGrad
$\widehat{v}_t = \beta_2 \widehat{v}_{t-1} + (1 - \beta_2)g_t^2$	RMSProp	N/A	Adam

**Table 1:** Variants of the generalized Adam algorithm, where N/A stands for an informal algorithm that was not defined in literature.

In Table 1, convergence of AdaGrad for non-convex optimization is unknown, and AdaGrad with momentum has not been formally considered in literature. The convergence of AMSGrad using a fast diminishing  $\beta_{1,t}$  under  $\beta_{1,t} \leq \beta_{1,t-1}, \beta_{1,t} \xrightarrow[t \rightarrow \infty]{} b, b = 0$  in convex optimization was studied in [2], however, its convergence using constant  $\beta_1$  or non-zero  $b$  or under nonconvex setting is unexplored. Algorithms in the last row of Table 1 was proven to be divergent in some cases by [2].

It is also worth mentioning that Algorithm 1 can be applied to solve the popular “finite-sum” problems whose objective is a sum of  $n$  individual cost functions. That is,

$$\min_{\mathbf{x} \in \mathbb{R}^d} \sum_{i=1}^n f_i(\mathbf{x}) := f(\mathbf{x}), \quad (2)$$

where each  $f_i : \mathbb{R}^d \rightarrow \mathbb{R}$  is a smooth and possibly nonconvex function. If at each time instance the index  $i$  is chosen randomly uniformly, then Algorithm 1 still applies, with  $g_t = \nabla f_t(\mathbf{x}_t)$ . It can also be extended to the mini-batch case with  $\mathbf{g}_t = \frac{1}{b} \sum_{i \in \mathcal{I}_t} \nabla f_i(\mathbf{x}_t)$ , where  $\mathcal{I}_t$  denotes the minibatch of size  $b$ . It is easy to show that  $g_t$  is an unbiased estimator for  $\nabla f(\mathbf{x})$ .

Additionally, for problem (2), Algorithm 1 can be slightly modified to take into account the possibility that the data point will be picked deterministically and incrementally; see Algorithm 2.

**Algorithm 2. Incremental Generalized Adam**

(S0). Initialize  $x_1$  and  $m_0^n = 0$ .  
For  $t = 1, \dots, T$ , do  
(S1). Let  $m_t^0 = m_{t-1}^n$ ,  $x_t^1 = x_t$   
For  $i = 1, \dots, n$ , do  
(S1-1).  $m_t^i = \beta_{1,t} m_t^{i-1} + (1 - \beta_{1,t}) \nabla f_i(x_t^i)$   
(S1-2).  $\hat{v}_t^i = f_{t,i}(\nabla f_i(x_1^1), \nabla f_i(x_1^2), \dots, \nabla f_i(x_t^i))$   
(S1-3).  $x_t^{i+1} = x_t^i - \alpha_t^i m_t^i / \sqrt{\hat{v}_t^i}$   
End  
(S2).  $x_{t+1} = x_t^{n+1}$   
End

In Algorithm 2,  $\nabla f_i(x_t^i)^2 \in \mathbb{R}^d$  takes the component-wise square of  $\nabla f_i(x_t^i)$ , and  $m_t^i / \sqrt{\hat{v}_t^i}$  denotes component-wise division. The superscript  $i$  here indicates the “inner” iteration in which each data point is processed once. Compared with Algorithm 1, Algorithm 2 is deterministic and it cycles through the data with a fixed order. Other than that, the main steps of the algorithm are the same as the previous one.

In the remainder of this paper, we will analyze Algorithm 1 and 2, and provide sufficient conditions under which these algorithms are convergent to first-order stationary solutions with global sublinear rate. We will also discuss how our results can be applied to special cases of generalized Adam.

### 3 Convergence Analysis for Generalized Adam

In the following, we formalize the assumptions we need to prove convergence.

**Assumptions**

A1:  $f$  is differentiable and has  $L$ -Lipschitz gradient, i.e.  $\forall x, y$ ,

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\| \quad (3)$$

It is also lower bounded, i.e.  $f(x^*) > -\infty$  where  $x^*$  is an optimal solution.

A2: At time  $t$ , the algorithm can access a bounded noisy gradient and the true gradient is bounded, i.e.

$$\|\nabla f(x_t)\| \leq H, \quad \|g_t\| \leq H, \quad \forall t > 1. \quad (4)$$

A3: The noisy gradient is unbiased and the noise is independent, i.e.  $g_t = \nabla f(x_t) + \zeta_t$ ,

$$E[\zeta_t] = 0 \quad (5)$$

and  $\zeta_i$  is independent of  $\zeta_j$  if  $i \neq j$ .

We note that reference [2] uses a similar assumption as in A2, i.e., the bounded elements of the gradient  $\|g_t\|_\infty \leq a$  for some finite  $a$ . And other assumptions are standard in stochastic optimization and are common in machine learning problems. Our main result shows that if the coordinate-wise weighting term  $\sqrt{\hat{v}_t}$  in Algorithm 1 is properly chosen, we can ensure the global convergence as well as the sublinear convergence rate of the algorithm (to a first-order stationary solution).

**Theorem 3.1.** Suppose that Assumption A is satisfied,  $\beta_1 \geq \beta_{1,t}$ ,  $\beta_{1,t} \in [0, 1)$  is non-increasing, and  $\|\alpha_t m_t / \sqrt{\hat{v}_t}\| \leq G$  for any  $t$ , then Algorithm 1 yields

$$\begin{aligned} & E \left[ \sum_{t=1}^T \alpha_t \langle \nabla f(x_t), \nabla f(x_t) / \sqrt{\hat{v}_t} \rangle \right] \\ & \leq E \left[ C_1 \sum_{t=1}^T \left\| \alpha_t g_t / \sqrt{\hat{v}_t} \right\|^2 + C_2 \sum_{t=2}^T \left\| \frac{\alpha_t}{\sqrt{\hat{v}_t}} - \frac{\alpha_{t-1}}{\sqrt{\hat{v}_{t-1}}} \right\|_1 + C_3 \sum_{t=2}^{T-1} \left\| \frac{\alpha_t}{\sqrt{\hat{v}_t}} - \frac{\alpha_{t-1}}{\sqrt{\hat{v}_{t-1}}} \right\|^2 \right] + C_4 \end{aligned} \quad (6)$$

where  $C_1, C_2, C_3$  are constants  $d$  and  $T$ ,  $C_4$  is a constant independent of  $T$ , the expectation is taken with respect to all the randomness corresponding to  $\{g_t\}$ .

**Proof sketch:** We first define an auxiliary sequence  $z_t$  that is close to  $x_t$ , the sequence of  $z_t$  is defined such that  $z_{t+1} - z_t = d_t$  and  $d_t$  is approximately  $-\alpha_t g_t / \sqrt{\widehat{v}_t}$ . Compared with  $x_{t+1} - x_t = -\alpha_t m_t / \sqrt{v_t}$ , the momentum  $m_t$  is replaced by stochastic  $g_t$  and the proof can be less convoluted. By smoothness of  $f$ , we can bound the difference between the first iteration and last iteration as

$$E[f(z_{T+1}) - f(z_1)] \leq E\left[\sum_{t=1}^T \langle \nabla f(z_t), d_t \rangle\right] + E\left[\sum_{t=1}^T \frac{L}{2} \|d_t\|^2\right]. \quad (7)$$

where

$$d_t = -\left(\frac{\beta_{1,t+1}}{1 - \beta_{1,t+1}} - \frac{\beta_{1,t}}{1 - \beta_{1,t}}\right) \alpha_t m_t / \sqrt{v_t} - \frac{\beta_1}{1 - \beta_1} \left(\frac{\alpha_t}{\sqrt{\widehat{v}_t}} - \frac{\alpha_{t-1}}{\sqrt{\widehat{v}_{t-1}}}\right) \odot m_{t-1} - \alpha_t g_t / \sqrt{\widehat{v}_t},$$

It can be seen from (7) that the term  $-\langle \nabla f(z_t), \alpha_t g_t / \sqrt{\widehat{v}_t} \rangle$  involved in  $\langle \nabla f(z_t), d_t \rangle$  is related to  $-\alpha_t \langle \nabla f(x_t), \nabla f(x_t) / \sqrt{\widehat{v}_t} \rangle$ , which gives us a desired descent. However, since  $\widehat{v}_t$  is dependent on  $g_t$ ,  $E[g_t / \sqrt{\widehat{v}_t}]$  is not in the same direction as  $\nabla f(x_t)$  and such a discrepancy yields the term weighted by  $C_2$  in (6). All other terms in  $d_t$  give rise to errors to be upper bounded and we translate them to the terms in RHS of (6). The term weighted by  $C_1$  is comparable to sum of squared step size in the analysis of the standard SGD. The term weighted by  $C_3$  can be easily bounded by the  $C_2$  term using  $\|\cdot\|_2 \leq \|\cdot\|_1$ . For AMSGrad and AdaGrad with momentum, the  $C_1$  term is the dominant term. For the original Adam, the  $C_2$  term also becomes dominant, which causes the algorithm not converging to critical points in certain cases. We refer readers to Appendix 7.2 for more detailed proof. ■

The convergence rate of Algorithm 1 can be derived from Theorem 3.1 under certain conditions. A typical characterization of convergence rate is given in following corollary.

**Corollary 3.1.** Define  $\gamma_t := \min_{j \in [d]} \min_{\{g_i\}_{i=1}^t} \alpha_t / (\sqrt{\widehat{v}_t})_j$ . Suppose the assumptions in Theorem 3.1 hold true and

$$E\left[\sum_{t=1}^T \left\|\alpha_t g_t / \sqrt{\widehat{v}_t}\right\|^2 + \sum_{t=2}^T \left\|\frac{\alpha_t}{\sqrt{\widehat{v}_t}} - \frac{\alpha_{t-1}}{\sqrt{\widehat{v}_{t-1}}}\right\|_1 + \sum_{t=2}^{T-1} \left\|\frac{\alpha_t}{\sqrt{\widehat{v}_t}} - \frac{\alpha_{t-1}}{\sqrt{\widehat{v}_{t-1}}}\right\|^2\right] = O(s_1(T)), \quad (8)$$

$$\sum_{t=1}^T \gamma_t = \Omega(s_2(T)). \quad (9)$$

Then we have the following rate estimate

$$\min_{t \in [T]} E[\|\nabla f(x_t)\|^2] = O\left(\frac{s_1(T)}{s_2(T)}\right). \quad (10)$$

This result is a direct implication of Theorem 3.1, and the following fact derived from the definition of  $\gamma_t$ :

$$E\left[\alpha_t \langle \nabla f(x_t), \nabla f(x_t) / \sqrt{\widehat{v}_t} \rangle\right] \geq E[\|\nabla f(x_t)\|^2] \gamma_t.$$

From Corollary 3.1, a sufficient condition for the algorithm to converge is that  $s_1(T)$  grows slower than  $s_2(T)$ . Instead of bounding the minimum norm of  $\nabla f$  in (10), we can also apply a probabilistic output (e.g., select an output  $x_R$  with probability  $p(R = t) = \frac{\gamma_t}{\sum_{t=1}^T \gamma_t}$ ) to bound  $E[\|\nabla f(x_R)\|^2]$  [9, 19].

**Remark (on the convergence conditions).** We now give some interpretation of the terms in Corollary 3.1. First, the bounds on the terms  $E[\sum_{t=1}^T \|\alpha_t g_t / \sqrt{\widehat{v}_t}\|^2]$  and  $\sum_{t=1}^T \gamma_t$  are two common conditions adapted from SGD. The former is a generalization of the well-known condition  $\sum_{t=1}^T \alpha_t^2 = O(s_1(T))$  for SGD, and it quantifies possible increase in objective brought by higher order curvature. The latter condition (9) is the lower bound on summation of effective stepsizes, which reduces to  $\sum_{t=1}^T \alpha_t = \Omega(s_4(T))$  when Algorithm 1 is simplified to SGD.

The other two terms in (8) characterize the oscillation of effective stepsizes  $\alpha_t / \sqrt{\widehat{v}_t}$ . In our analysis such an oscillation term upper bounds the expected possible ascent in objective induced by skewed update direction  $g_t / \sqrt{\widehat{v}_t}$  (“skewed” because  $E[g_t / \sqrt{\widehat{v}_t}]$  is not parallel with  $\nabla f(x_t)$ ), therefore it

cannot be too large. Bounding this term is critical, and to demonstrate this fact, in Section 7.1.2 we show that large oscillation can result in non-convergence of Adam for even simple unconstrained non-convex problems.

The benefit of adaptive gradient methods can be reflected in the term  $E[\sum_{t=1}^T \|\alpha_t g_t / \sqrt{\widehat{v}_t}\|^2]$ , since there are cases where the use of the weight vector  $\widehat{v}_t$  can help reduce this quantity compared with SGD. Adaptive gradient methods like AMSGrad can provide a flexible choice of stepsizes, since  $\widehat{v}_t$  has a normalization effect to reduce oscillation and overshoot. An example is provided in Appendix 7.1.1 to further illustrate this fact. ■

We highlight that Theorem 3.1 provides a general way to design the weighting sequence  $\{\widehat{v}_t\}$  and analyzes the convergence of Adam-type algorithms. For example, SGD specified by Table 1 with stepsizes  $\alpha_t = 1/\sqrt{t}$  yields  $O(\log T/\sqrt{T})$  convergence speed, since  $s_1(T) = O(\log T)$  and  $s_2(T) = \Omega(\sqrt{T})$ . Here the former conclusion on  $s_1(T)$  holds as the first term in RHS of (6) is upper bounded by  $\sum_{t=1}^T \|\alpha_t g_t\|^2 \leq H^2 \sum_{t=1}^T 1/t \leq H^2(1 + \log T)$ , and the other terms are two constants by applying the telescoping sum. Moreover, it has been shown in [2] that Adam could fail to converge even in some convex cases. The explanation in [2] is consistent with (10), where  $s_1(T)$  can not be bounded by a slow growing rate since when  $\widehat{v}_t$  can oscillate arbitrarily, the term weighted by  $C_2$  in (6) can grow very fast. The convergence rates of AdaGrad and AMSGrad can also be derived from Theorem 3.1, which will be given as corollaries later. In particular, our proposed convergence rate of AMSGrad matches the convergence rate of AMSGrad in [2] for stochastic convex optimization. Compared to the analysis of AMSGrad in [2], which assumed that  $\beta_{1,t}$  is diminishing, our analysis can work with *constant*  $\beta_1$  – a choice that is standard in practice.

In Theorem 3.1,  $\|\alpha_t m_t / \sqrt{\widehat{v}_t}\| \leq G$  is a mild condition. Roughly speaking, it implies that the change of  $x_t$  at each iteration should be finite. As will be evident later, with  $\|g_t\| \leq H$ , the condition  $\|\alpha_t m_t / \sqrt{\widehat{v}_t}\| \leq G$  is satisfied for both AdaGrad and AMSGrad.

Next, in Corollary 3.2 and Corollary 3.3, we prove convergence rates of AMSGrad (Algorithm 3) and AdaGrad with momentum (Algorithm 4 in Appendix 7.2.4), respectively. Note that AdaGrad with momentum is more general than AdaGrad since when  $\beta_{1,t} = 0$ , AdaGrad with momentum becomes AdaGrad.

### Algorithm 3. AMSGrad

(S0). Define  $m_0 = 0, v_0 = 0, \widehat{v}_0 = 0$ ;  
 For  $t = 1, \dots, T$ , do  
   (S1).  $m_t = \beta_{1,t} m_{t-1} + (1 - \beta_{1,t}) g_t$   
   (S2).  $v_t = \beta_2 v_{t-1} + (1 - \beta_2) g_t^2$   
   (S3).  $\widehat{v}_t = \max\{\widehat{v}_{t-1}, v_t\}$   
   (S4).  $x_{t+1} = x_t - \alpha_t m_t / \sqrt{\widehat{v}_t}$   
 End

**Corollary 3.2.** For AMSGrad (Algorithm 3) with  $\beta_{1,t} \leq \beta_1 \in [0, 1)$  and  $\beta_{1,t}$  is non-increasing,  $\alpha_t = 1/\sqrt{t}$ , we have for any  $T$ ,

$$\min_{t \in [T]} E [\|f(x_t)\|^2] \leq \frac{1}{\sqrt{T}} (Q_1 + Q_2 \log T) \quad (11)$$

where  $Q_1$  and  $Q_2$  are two constants independent of  $T$ .

**Proof sketch:** The proof follows Theorem 3.1 and (10) by proving  $s_1(T) = O(\log T)$  and  $s_2(T) = \Omega(\sqrt{T})$ . A detailed proof can be found in Appendix 7.2.3.

**Corollary 3.3.** For AdaGrad with momentum (Algorithm 5 in Appendix 7.2.4) with  $\beta_{1,t} \leq \beta_1 \in [0, 1)$  and  $\beta_{1,t}$  is non-increasing,  $\alpha_t = 1/\sqrt{t}$ , we have for any  $T$ ,

$$\min_{t \in [T]} E [\|f(x_t)\|^2] \leq \frac{1}{\sqrt{T}} (Q'_1 + Q'_2 \log T) \quad (12)$$

where  $Q'_1$  and  $Q'_2$  are two constants independent of  $T$ .

**Proof sketch:** Same as previous corollary, the proof follows Theorem 3.1; see details in Appendix 7.2.4.



## 4 Convergence Analysis for Incremental Generalized Adam

Reference [2] pointed out the non-convergence of Adam by constructing an online optimization problem with a periodic function sequence, this scenario is equivalent to adding a time-dependent noise to the gradient in stochastic optimization we analyzed above. However, since we have assumed independently distributed noise in the last section, it is interesting to see whether time dependent noise can make the algorithm diverge. Spurred by that, we consider the finite-sum problem (2), where each function can be viewed as a loss function evaluated on a data point or a mini-batch. To impose the temporal dependency on stochastic gradients, we sample  $\{f_i\}$  sequentially and periodically. This leads to the incremental generalized Adam algorithm (Algorithm 2). We will show that our convergence analysis is applicable to the incremental counterparts of Adam-type algorithms.

We begin by elaborating on assumptions made for problem (2).

### Assumption

B1. Each  $f_i$  has Lipschitz continuous gradient

$$\|\nabla f_i(x) - \nabla f_i(y)\| \leq L_i \|x - y\|, \forall i, \forall x. \quad (13)$$

Clearly we will have

$$\|\nabla f(x) - \nabla f(y)\| \leq L \|x - y\|, \forall x, \text{ where } L := \sum_i L_i. \quad (14)$$

It is also lower bounded, i.e.  $f(x^*) > -\infty$  where  $x^*$  is an optimal solution.

B2. Each  $\|\nabla f_i\|$  is bounded, i.e.,

$$\|\nabla f_i(x)\| \leq H_i, \forall i, \forall x, \text{ and } \|\nabla f(x)\| \leq \sum_i H_i := H.$$

Finite sum can be viewed as a special case of stochastic optimization when each function  $f_i$  is uniformly sampled to form a stochastic gradient. In this case, A1 = B1, and A2 = B2. However, our analysis will be extended to time-dependent incremental optimization in this section, where A3 is not satisfied.

Let us consider the incremental version of generalized Adam algorithm outlined in Algorithm 2 in Section 2. We illustrate its convergence rate in Theorem 4.1.

**Theorem 4.1.** *Suppose that Assumption B is satisfied,  $\beta_{1,t} = \beta_1$ ,  $\beta_1 \in [0, 1)$ , and  $\alpha_t^i / \sqrt{(\hat{v}_t^i)_j} \leq M$  for any  $t, i, j$ . Define  $\alpha_t^{n+1} \triangleq \alpha_{t+1}^1$ ,  $\hat{v}_t^{n+1} \triangleq \hat{v}_{t+1}^1$ , then Algorithm 2 yields*

$$\begin{aligned} & \sum_{t=1}^T \alpha_t^1 \langle \nabla f(x_t), \nabla f(x_t) / \sqrt{\hat{v}_t^1} \rangle \\ & \leq \sum_{t=1}^T \sum_{i=1}^n \left( C'_1 \left\| \alpha_t^i \nabla f(x_t^i) / \sqrt{\hat{v}_t^i} \right\|^2 + C'_2 \left\| \frac{\alpha_t^{i+1}}{\sqrt{\hat{v}_t^{i+1}}} - \frac{\alpha_t^i}{\sqrt{\hat{v}_t^i}} \right\| + C'_3 \left\| \frac{\alpha_t^{i+1}}{\sqrt{\hat{v}_t^{i+1}}} - \frac{\alpha_t^i}{\sqrt{\hat{v}_t^i}} \right\|^2 \right) + C'_4 \end{aligned} \quad (15)$$

where  $C'_1, C'_2, C'_3, C'_4$  are constants independent of  $T$  and  $d$ .

**Proof:** See Appendix 7.3.

We remark that Theorem 4.1 is a deterministic counterpart of Theorem 3.1. And thus, it implies a sufficient condition for the convergence of incremental Adam-type algorithms. In Corollaries 4.1 and 4.2, we specify the convergence rates of deterministic versions of AMSGrad and AdaGrad with momentum.

**Algorithm 4. Incremental AMSGrad**

(S0). Initialize  $x_1, m_0^n = 0, v_0^n = 0, \hat{v}_0^n = 0$ ;  
 For  $t = 1, \dots, T$ , do  
 (S1). Let  $m_t^0 = m_{t-1}^n, x_t^1 = x_t, v_t^0 = v_{t-1}^n, \hat{v}_t^0 = \hat{v}_{t-1}^n$   
 For  $i = 1, \dots, n$ , do  
 (S1-1).  $m_t^i = \beta_{1,t} m_{t-1}^{i-1} + (1 - \beta_{1,t}) \nabla f_i(x_t^i)$   
 (S1-2).  $v_t^i = \beta_{2,t} v_{t-1}^{i-1} + (1 - \beta_{2,t}) \nabla f_i(x_t^i)^2$   
 (S1-3).  $\hat{v}_t^i = \max\{\hat{v}_{t-1}^{i-1}, v_t^i\}$   
 (S1-4).  $x_t^{i+1} = x_t^i - \alpha_t m_t^i / \sqrt{\hat{v}_t^i}$   
 End  
 (S2).  $x_{t+1} = x_t^{n+1}$   
 End

**Corollary 4.1.** *For incremental AMSGrad (Algorithm 4), suppose that Assumptions B is satisfied and  $\beta_{1,t} = \beta_1, \beta_1 \in [0, 1), \alpha_t^i = 1/\sqrt{n(t-1) + i}$ . We have for any  $T$ ,*

$$\min_{t \in [T]} \|\nabla f(x_t)\|^2 \leq \frac{1}{\sqrt{T}} (R_1 + R_2 \log T) \quad (16)$$

for some constants  $R_1$  and  $R_2$  independent of  $T$ .

Proof: See Appendix 7.3.3

**Corollary 4.2.** *For incremental AdaGrad with momentum (Algorithm 6 in Appendix 7.3.4), suppose that Assumptions B is satisfied and  $\beta_{1,t} = \beta_1, \beta_1 \in [0, 1), \alpha_t^i = 1/\sqrt{n(t-1) + i}$ . We have for any  $T$ ,*

$$\min_{t \in [T]} \|\nabla f(x_t)\|^2 \leq \frac{1}{\sqrt{T}} (R'_1 + R'_2 \log T) \quad (17)$$

for some constants  $R'_1$  and  $R'_2$  independent of  $T$ .

Proof: See Appendix 7.3.4

## 5 Experiments

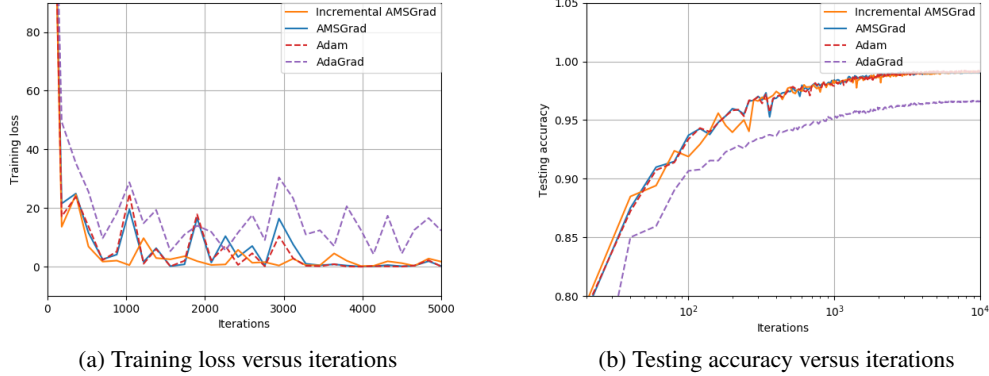
In this section, we present empirical results of our considered methods for solving the multi-classification problem on MNIST. We focus on Algorithm 3 (AMSGrad) and Algorithm 4 (incremental AMSGrad), two specializations of our generalized algorithmic frameworks Algorithm 1 and 2. For comparison, we also present the empirical performance of the commonly-used Adam algorithm in [1] and AdaGrad in [3] under the same parameter setting as stochastic AMSGrad.

We consider a convolutional neural network (CNN), which includes 3 convolutional layers and 2 fully-connected layers. In convolutional layers, we adopt filters of sizes  $6 \times 6 \times 1$  (with stride 1),  $5 \times 5 \times 6$  (with stride 2), and  $6 \times 6 \times 12$  (with stride 2), respectively. In both AMSGrad and incremental AMSGrad, we set  $\beta_1 = 0.9$  and  $\beta_2 = 0.99$ , and choose 50 as the mini-batch size. In AMSGrad<sup>1</sup>, we set  $T = 240000$ , and in incremental AMSGrad we set  $n = 1200$  and  $T = 20$ . Such a parameter setting guarantees the same total number of iterations (24000) for both methods.

Figure 1 shows the training loss and the classification accuracy versus the number of iterations. As we can see, AMSGrad, incremental AMSGrad, and Adam yield quite similar training and testing performance. In particular, AMSGrad achieves almost the same loss and classification accuracy as Adam. Our results confirm that the empirical performance of AMSGrad is comparable to Adam [2], but as we showed in our work, the former has the theoretical convergence guarantee for nonconvex optimization. The performance of AdaGrad is a little worse than other algorithms in the experiment, intuitively, this might be a result of its intrinsic diminishing effective step size and the lack of momentum.

<sup>1</sup>We customized our algorithms based on the open source code <https://github.com/taki0112/AMSGrad-Tensorflow>.

More experiments on comparison of different algorithms including AMSGrad, Adam, and SGD are performed in Appendix 7.1 with more detailed discussion. We empirically show that Adam can diverge due to high growth rate of  $\sum_{t=2}^T \left\| \frac{\alpha_t}{\sqrt{\hat{v}_t}} - \frac{\alpha_{t-1}}{\sqrt{\hat{v}_{t-1}}} \right\|_1$  and AMSGrad benefits from a flexible choice of stepsizes compared with SGD.



**Figure 1:** Comparison of AMSGrad, incremental AMSGrad and Adam under MNIST in training loss and testing accuracy.

## 6 Discussion

We provided some mild conditions to ensure convergence of a class of Adam-type algorithms, which include Adam, AMSGrad, AdaGrad, AdaGrad with momentum, SGD, SGD with momentum as special cases. To the best of our knowledge, the convergence of Adam-type algorithm for non-convex problems was unknown before.

The convergence rate of Adam-type method proven in this paper matches convergence rate of SGD. We also show empirically how certain Adam-type algorithm can performs better than SGD in certain cases. This paper focuses on unconstrained nonconvex optimization problems. The future work could relax this assumption and study a more general setting of constrained nonconvex optimization.

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## 7 Appendix

### 7.1 Additional Experiments

#### 7.1.1 Advantages and Disadvantages of Adaptive gradient method

In this section, we provide some additional experiments to demonstrate the following two facts:

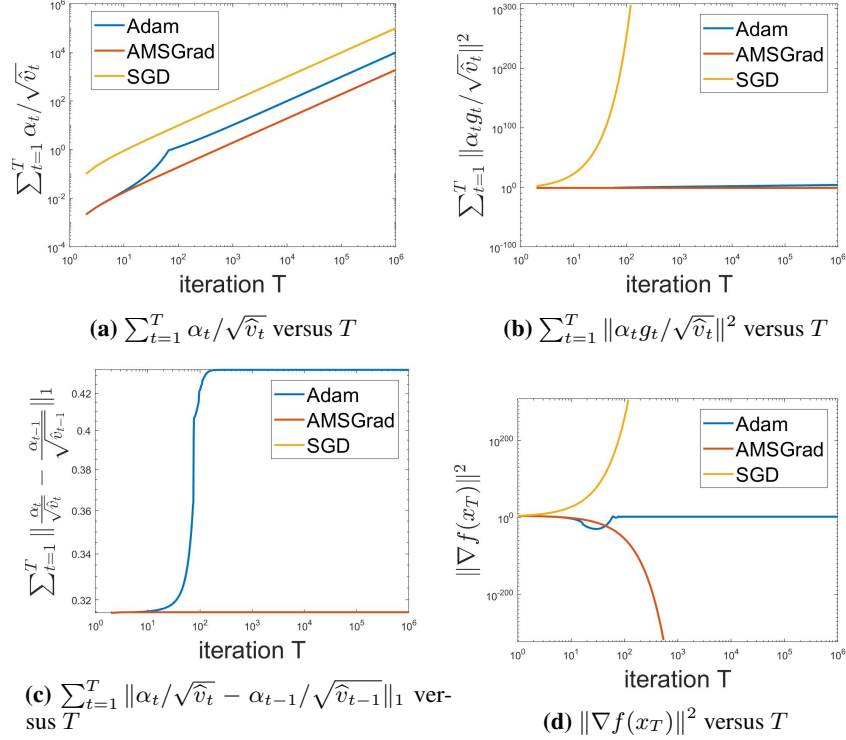
- For some problems, specific Adam-type algorithms can perform better than SGD;
- The original Adam algorithm can diverge due to high growth rate of the oscillation term

$$\sum_{t=2}^T \left\| \frac{\alpha_t}{\sqrt{\widehat{v}_t}} - \frac{\alpha_{t-1}}{\sqrt{\widehat{v}_{t-1}}} \right\|_1.$$

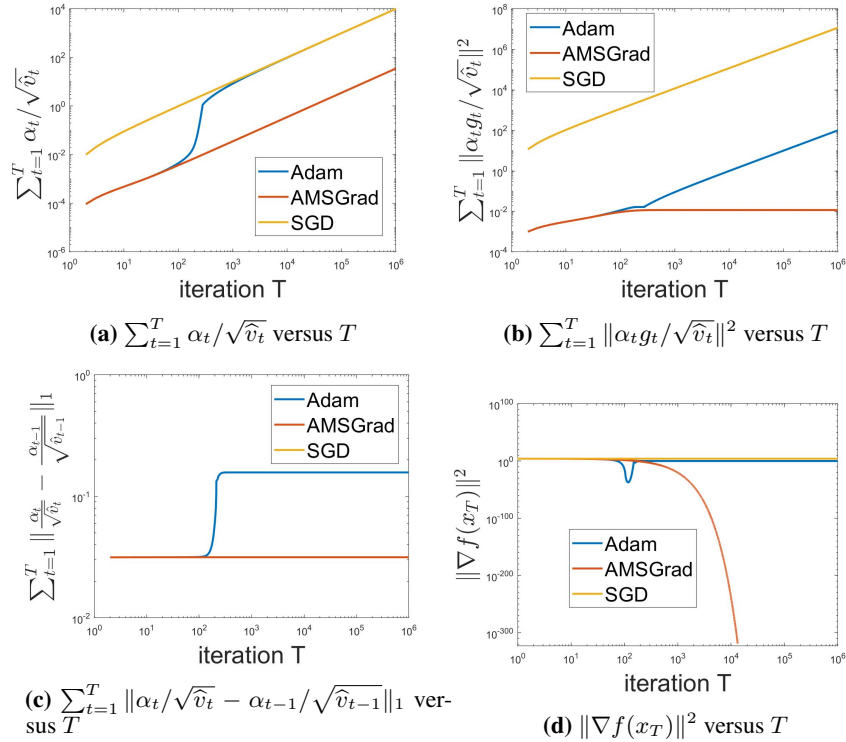
One benefit of AMSGrad compared with SGD is its flexible choice of stepsizes. First, in nonconvex problems there can be multiple valleys with different curvatures. When using fixed or diminishing stepsizes, SGD can only converge to local optima in valleys with small curvature while AMSGrad and its related adaptive gradient algorithms can potentially converge to optima in valleys with relative high curvature. Second, the flexible choice of stepsizes implies less hyperparameter tuning. Third, another benefit of adaptive gradient method is sparse noise reduction effect and this is illustrated in [30].

We empirically demonstrate the flexible stepsizes property of AMSGrad using a deterministic quadratic problem. Consider a toy optimization problem  $\min_x f(x)$ ,  $f(x) = 100x^2$ , the gradient is given by  $200x$ . For SGD (which reduces to gradient descent in this case) to converge, we must have  $\alpha_t < 0.01$ ; for AMSGrad,  $\widehat{v}_t$  has a strong normalization effect and it allows the algorithm to use larger  $\alpha_t$ 's. We show the growth rate of different terms given in Corollary 3.1 for different stepsizes in Figure 2 to Figure 5 (where we choose  $\beta_{1,t} = 0$ ,  $\beta_{2,t} = 0.9$  for both Adam and AMSGrad). In Figure 2,  $\alpha_t = 0.1$  and SGD diverges due to large  $\alpha_t$ , AMSGrad converges in this case, Adam is oscillating between two non-zero points. In Figure 3, stepsizes  $\alpha_t$  is set to 0.01, SGD and Adam are oscillating, AMSGrad converges to 0. For Figure 4, SGD converges to 0 and AMSGrad is converging slower than SGD due to its smaller effective stepsizes, Adam is oscillating. One may wonder how diminishing stepsizes affects performance of the algorithms, this is shown in Figure 5 where  $\alpha_t = 0.1/\sqrt{t}$ , we can see SGD is diverging until stepsizes is small, AMSGrad is converging all the time, Adam appears to get stuck but it is actually converging very slowly due to diminishing stepsizes. This example shows AMSGrad can converge with a larger range of stepsizes compared with SGD.

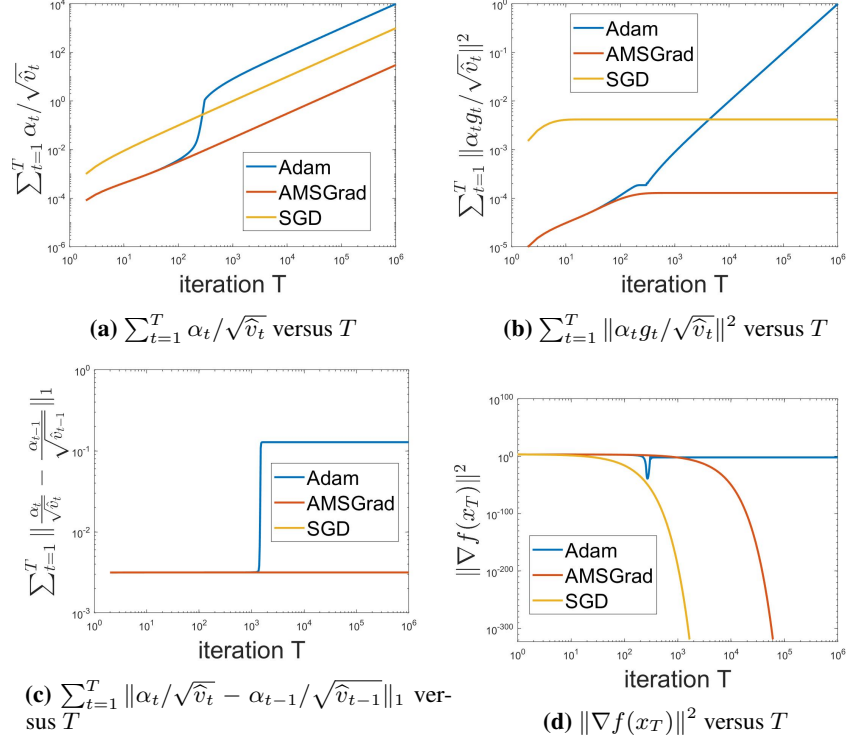
From the figures, we can see that the term  $\sum_{t=1}^T \|\alpha_t g_t / \sqrt{\widehat{v}_t}\|^2$  is the key quantity that limits the convergence speed of algorithms in this case. In Figure 2, Figure 3, and early stage of Figure 5, the quantity is obviously a good sign of convergence speed. In Figure 4, since the difference of quantity between AMSGrad and SGD is compensated by the larger effective stepsizes of SGD and some problem independent constant, SGD converges faster. In fact, Figure 4 provides a case where AMSGrad does not perform well. Note that the normalization factor  $\sqrt{\widehat{v}_t}$  can be understood as imitating the largest Lipschitz constant along the way of optimization, so generally speaking dividing by this number makes the algorithm converge easier. However when the Lipschitz constant becomes smaller locally around a local optimal point, the stepsizes choice of AMSGrad dictates that  $\sqrt{\widehat{v}_t}$  does not change, resulting a small effective stepsizes. This could be mitigated by AdaGrad and its momentum variants which allows  $\widehat{v}_t$  to decrease when  $g_t$  keeps decreasing.



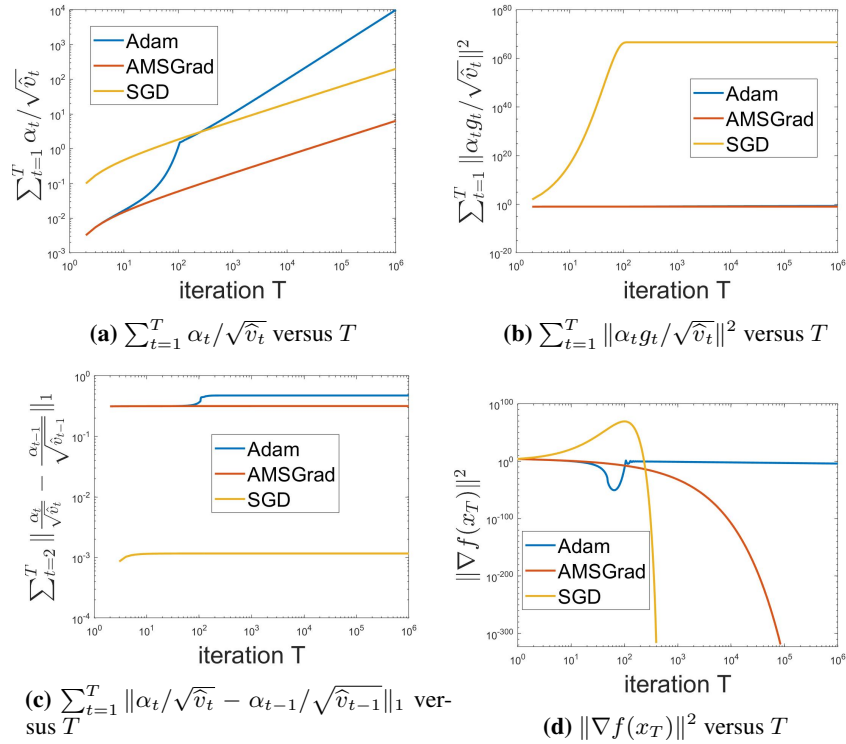
**Figure 2:** Comparison of algorithms with  $\alpha_t = 0.1$ , we defined  $\alpha_0 = 0$



**Figure 3:** Comparison of algorithms with  $\alpha_t = 0.01$ , we defined  $\alpha_0 = 0$



**Figure 4:** Comparison of algorithms with  $\alpha_t = 0.001$ , we defined  $\alpha_0 = 0$



**Figure 5:** Comparison of algorithms with  $\alpha_t = 0.1/\sqrt{t}$ , we defined  $\alpha_0 = 0$



### 7.1.2 The Effect of $\sum_{t=1}^T \|\alpha_t/\sqrt{\widehat{v}_t} - \alpha_{t-1}/\sqrt{\widehat{v}_{t-1}}\|_1$

Next, we use another example to demonstrate the importance of the term  $\sum_{t=1}^T \|\alpha_t/\sqrt{\widehat{v}_t} - \alpha_{t-1}/\sqrt{\widehat{v}_{t-1}}\|_1$  in Corollary 3.1 and the corresponding term in Theorem 4.1 for the convergence of Adam-type algorithms. Since one can trivially derive counterpart of Corollary 3.1 for Theorem 4.1, we use notations in Corollary 3.1 for the rest of this example.

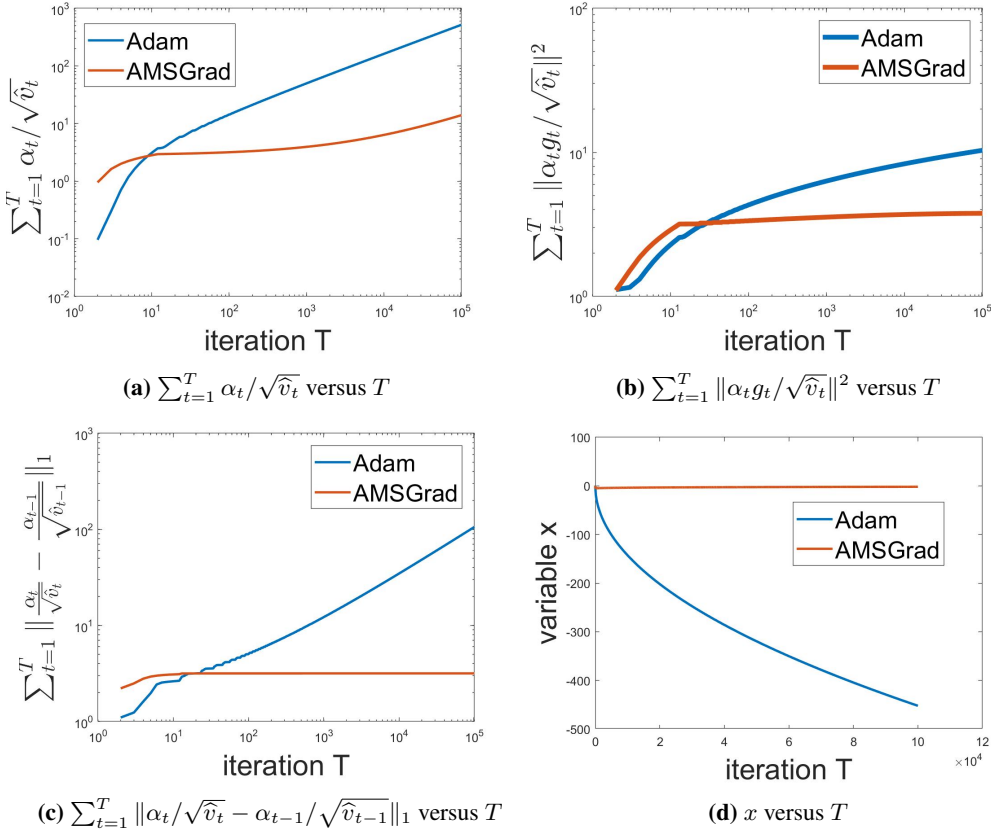
Consider incremental optimization problem  $\min_x \sum_{i=1}^{11} f_i(x)$  where

$$f_1(x) = \begin{cases} 5.5x^2, & \text{if } -1 \leq x \leq 1 \\ 11x - 5.5, & \text{if } x > 1 \\ -11x - 5.5, & \text{if } x < -1 \end{cases} \quad (18)$$

and for  $i \neq 1$ ,

$$f_i(x) = \begin{cases} -0.5x^2, & \text{if } -1 \leq x \leq 1 \\ -x + 0.5, & \text{if } x > 1 \\ x + 0.5, & \text{if } x < -1. \end{cases} \quad (19)$$

It is easy to verify that the problem satisfies the assumptions in Theorem 4.1. We now use the incremental version of AMSGrad and Adam to optimize  $x$ , the results are given in Figure 6 (we set  $\beta_{1,t} = 0, \beta_{2,t} = 0.1$  for both Adam and AMSGrad). We can see  $\sum_{t=1}^T \|\alpha_t/\sqrt{\widehat{v}_t} - \alpha_{t-1}/\sqrt{\widehat{v}_{t-1}}\|_1$  grows with same rate as  $\sum_{t=1}^T \alpha_t/\sqrt{\widehat{v}_t}$  for Adam. In Theorem 4.1, this implies the gradient will not converge to 0 and this is indeed the case for the example ( $x$  diverges due to non-convergence of gradient). AMSGrad has smaller oscillation in effective stepsizes and converges in this example. We note that the quantity  $\sum_{t=1}^T \|\alpha_t/\sqrt{\widehat{v}_t} - \alpha_{t-1}/\sqrt{\widehat{v}_{t-1}}\|_1$  is also mentioned in [26] and our analysis implies the algorithm in [26] can converge in non-convex problems.



**Figure 6:** Comparison of AMSGrad and Adam, we defined  $\alpha_0 = 0$

## 7.2 Convergence for Generalized Adam (Algorithm 1)

In this section, we present the convergence proof of Algorithm 1. We will first give several lemmas prior to proving Theorem 3.1.

### 7.2.1 Proof of Auxiliary Lemmas

**Lemma 7.1.** *Let  $x_0 \triangleq x_1$  in Algorithm 1, consider the sequence*

$$z_t = x_t + \frac{\beta_{1,t}}{1 - \beta_{1,t}}(x_t - x_{t-1}), \forall t \geq 1. \quad (20)$$

*Then the following holds true*

$$\begin{aligned} z_{t+1} - z_t = & - \left( \frac{\beta_{1,t+1}}{1 - \beta_{1,t+1}} - \frac{\beta_{1,t}}{1 - \beta_{1,t}} \right) \alpha_t m_t / \sqrt{\widehat{v}_t} \\ & - \frac{\beta_{1,t}}{1 - \beta_{1,t}} \left( \frac{\alpha_t}{\sqrt{\widehat{v}_t}} - \frac{\alpha_{t-1}}{\sqrt{\widehat{v}_{t-1}}} \right) \odot m_{t-1} - \alpha_t g_t / \sqrt{\widehat{v}_t}, \quad \forall t > 1 \end{aligned}$$

*and*

$$z_2 - z_1 = - \left( \frac{\beta_{1,2}}{1 - \beta_{1,2}} - \frac{\beta_{1,1}}{1 - \beta_{1,1}} \right) \alpha_1 m_1 / \sqrt{\widehat{v}_1} - \alpha_1 g_1 / \sqrt{\widehat{v}_1}.$$

**Proof.** [Proof of Lemma 7.1] By the update rules S1-S3 in Algorithm 1, we have when  $t > 1$ ,

$$\begin{aligned} x_{t+1} - x_t &= -\alpha_t m_t / \sqrt{\widehat{v}_t} \\ &\stackrel{\text{S1}}{=} -\alpha_t (\beta_{1,t} m_{t-1} + (1 - \beta_{1,t}) g_t) / \sqrt{\widehat{v}_t} \\ &\stackrel{\text{S3}}{=} \beta_{1,t} \frac{\alpha_t}{\alpha_{t-1}} \frac{\sqrt{\widehat{v}_{t-1}}}{\sqrt{\widehat{v}_t}} \odot (x_t - x_{t-1}) - \alpha_t (1 - \beta_{1,t}) g_t / \sqrt{\widehat{v}_t} \\ &= \beta_{1,t} (x_t - x_{t-1}) + \beta_{1,t} \left( \frac{\alpha_t}{\alpha_{t-1}} \frac{\sqrt{\widehat{v}_{t-1}}}{\sqrt{\widehat{v}_t}} - 1 \right) \odot (x_t - x_{t-1}) - \alpha_t (1 - \beta_{1,t}) g_t / \sqrt{\widehat{v}_t} \\ &\stackrel{\text{S3}}{=} \beta_{1,t} (x_t - x_{t-1}) - \beta_{1,t} \left( \frac{\alpha_t}{\sqrt{\widehat{v}_t}} - \frac{\alpha_{t-1}}{\sqrt{\widehat{v}_{t-1}}} \right) \odot m_{t-1} - \alpha_t (1 - \beta_{1,t}) g_t / \sqrt{\widehat{v}_t}. \quad (21) \end{aligned}$$

Since  $x_{t+1} - x_t = (1 - \beta_{1,t}) x_{t+1} + \beta_{1,t} (x_{t+1} - x_t) - (1 - \beta_{1,t}) x_t$ , based on (21) we have

$$\begin{aligned} & (1 - \beta_{1,t}) x_{t+1} + \beta_{1,t} (x_{t+1} - x_t) \\ &= (1 - \beta_{1,t}) x_t + \beta_{1,t} (x_t - x_{t-1}) - \beta_{1,t} \left( \frac{\alpha_t}{\sqrt{\widehat{v}_t}} - \frac{\alpha_{t-1}}{\sqrt{\widehat{v}_{t-1}}} \right) \odot m_{t-1} - \alpha_t (1 - \beta_{1,t}) g_t / \sqrt{\widehat{v}_t}. \end{aligned}$$

Divide both sides by  $1 - \beta_{1,t}$ , we have

$$\begin{aligned} & x_{t+1} + \frac{\beta_{1,t}}{1 - \beta_{1,t}} (x_{t+1} - x_t) \\ &= x_t + \frac{\beta_{1,t}}{1 - \beta_{1,t}} (x_t - x_{t-1}) - \frac{\beta_{1,t}}{1 - \beta_{1,t}} \left( \frac{\alpha_t}{\sqrt{\widehat{v}_t}} - \frac{\alpha_{t-1}}{\sqrt{\widehat{v}_{t-1}}} \right) \odot m_{t-1} - \alpha_t g_t / \sqrt{\widehat{v}_t}. \quad (22) \end{aligned}$$

Define the sequence

$$z_t = x_t + \frac{\beta_{1,t}}{1 - \beta_{1,t}} (x_t - x_{t-1}).$$

Then (22) can be written as

$$\begin{aligned}
z_{t+1} &= z_t + \left( \frac{\beta_{1,t+1}}{1 - \beta_{1,t+1}} - \frac{\beta_{1,t}}{1 - \beta_{1,t}} \right) (x_{t+1} - x_t) \\
&\quad - \frac{\beta_{1,t}}{1 - \beta_{1,t}} \left( \frac{\alpha_t}{\sqrt{\widehat{v}_t}} - \frac{\alpha_{t-1}}{\sqrt{\widehat{v}_{t-1}}} \right) \odot m_{t-1} - \alpha_t g_t / \sqrt{\widehat{v}_t} \\
&= z_t - \left( \frac{\beta_{1,t+1}}{1 - \beta_{1,t+1}} - \frac{\beta_{1,t}}{1 - \beta_{1,t}} \right) \alpha_t m_t \sqrt{\widehat{v}_t} \\
&\quad - \frac{\beta_{1,t}}{1 - \beta_{1,t}} \left( \frac{\alpha_t}{\sqrt{\widehat{v}_t}} - \frac{\alpha_{t-1}}{\sqrt{\widehat{v}_{t-1}}} \right) \odot m_{t-1} - \alpha_t g_t / \sqrt{\widehat{v}_t}, \quad \forall t > 1,
\end{aligned}$$

where the second equality is due to  $x_{t+1} - x_t = -\alpha_t m_t / \sqrt{\widehat{v}_t}$ .

For  $t = 1$ , we have  $z_1 = x_1$  (due to  $x_1 = x_0$ ), and

$$\begin{aligned}
z_2 - z_1 &= x_2 + \frac{\beta_{1,2}}{1 - \beta_{1,2}} (x_2 - x_1) - x_1 \\
&= x_2 + \left( \frac{\beta_{1,2}}{1 - \beta_{1,2}} - \frac{\beta_{1,1}}{1 - \beta_{1,1}} \right) (x_2 - x_1) + \frac{\beta_{1,1}}{1 - \beta_{1,1}} (x_2 - x_1) - x_1 \\
&= \left( \frac{\beta_{1,2}}{1 - \beta_{1,2}} - \frac{\beta_{1,1}}{1 - \beta_{1,1}} \right) (-\alpha_1 m_1 / \sqrt{\widehat{v}_1}) + \left( \frac{\beta_{1,1}}{1 - \beta_{1,1}} + 1 \right) (x_2 - x_1) \\
&= \left( \frac{\beta_{1,2}}{1 - \beta_{1,2}} - \frac{\beta_{1,1}}{1 - \beta_{1,1}} \right) (-\alpha_1 m_1 / \sqrt{\widehat{v}_1}) + \frac{1}{1 - \beta_{1,1}} (-\alpha_1 (1 - \beta_{1,1}) g_1 / \sqrt{\widehat{v}_1}) \\
&= - \left( \frac{\beta_{1,2}}{1 - \beta_{1,2}} - \frac{\beta_{1,1}}{1 - \beta_{1,1}} \right) (\alpha_1 m_1 / \sqrt{\widehat{v}_1}) - \alpha_1 g_1 / \sqrt{\widehat{v}_1},
\end{aligned}$$

where the forth equality holds due to (S1) and (S3) of Algorithm 1.

The proof is now complete. **Q.E.D.**

Without loss of generality, we initialize Algorithm 1 as below to simplify our analysis in what follows,

$$\left( \frac{\alpha_1}{\sqrt{\widehat{v}_1}} - \frac{\alpha_0}{\sqrt{\widehat{v}_0}} \right) \odot m_0 = 0. \tag{23}$$

**Lemma 7.2.** *Suppose that the conditions in Theorem 3.1 hold, then*

$$E[f(z_{t+1}) - f(z_1)] \leq \sum_{i=1}^6 T_i, \tag{24}$$

where

$$T_1 = -E \left[ \sum_{i=1}^t \langle \nabla f(z_i), \frac{\beta_{1,i}}{1 - \beta_{1,i}} \left( \frac{\alpha_i}{\sqrt{\widehat{v}_i}} - \frac{\alpha_{i-1}}{\sqrt{\widehat{v}_{i-1}}} \right) \odot m_{i-1} \rangle \right], \tag{25}$$

$$T_2 = -E \left[ \sum_{i=1}^t \alpha_i \langle \nabla f(z_i), g_i / \sqrt{\widehat{v}_i} \rangle \right], \tag{26}$$

$$T_3 = -E \left[ \sum_{i=1}^t \langle \nabla f(z_i), \left( \frac{\beta_{1,i+1}}{1 - \beta_{1,i+1}} - \frac{\beta_{1,i}}{1 - \beta_{1,i}} \right) \alpha_i m_i / \sqrt{\widehat{v}_i} \rangle \right], \tag{27}$$

$$T_4 = E \left[ \sum_{i=1}^t \frac{3}{2} L \left\| \left( \frac{\beta_{1,t+1}}{1 - \beta_{1,t+1}} - \frac{\beta_{1,t}}{1 - \beta_{1,t}} \right) \alpha_t m_t / \sqrt{v_t} \right\|^2 \right], \tag{28}$$

$$T_5 = E \left[ \sum_{i=1}^t \frac{3}{2} L \left\| \frac{\beta_{1,i}}{1 - \beta_{1,i}} \left( \frac{\alpha_t}{\sqrt{\widehat{v}_i}} - \frac{\alpha_{i-1}}{\sqrt{\widehat{v}_{i-1}}} \right) \odot m_{i-1} \right\|^2 \right], \tag{29}$$

$$T_6 = E \left[ \sum_{i=1}^t \frac{3}{2} L \left\| \alpha_i g_i / \sqrt{\widehat{v}_i} \right\|^2 \right]. \tag{30}$$

**Proof.** [Proof of Lemma 7.2] By the Lipschitz smoothness of  $\nabla f$ , we obtain

$$f(z_{t+1}) \leq f(z_t) + \langle \nabla f(z_t), d_t \rangle + \frac{L}{2} \|d_t\|^2, \quad (31)$$

where  $d_t = z_{t+1} - z_t$ , and Lemma 7.1 together with (23) yield

$$\begin{aligned} d_t = & - \left( \frac{\beta_{1,t+1}}{1 - \beta_{1,t+1}} - \frac{\beta_{1,t}}{1 - \beta_{1,t}} \right) \alpha_t m_t / \sqrt{\widehat{v}_t} \\ & - \frac{\beta_1}{1 - \beta_1} \left( \frac{\alpha_t}{\sqrt{\widehat{v}_t}} - \frac{\alpha_{t-1}}{\sqrt{\widehat{v}_{t-1}}} \right) \odot m_{t-1} - \alpha_t g_t / \sqrt{\widehat{v}_t}, \quad \forall t \geq 1. \end{aligned} \quad (32)$$

Based on (31) and (32), we then have

$$\begin{aligned} E[f(z_{t+1}) - f(z_1)] &= E \left[ \sum_{i=1}^t f(z_{i+1}) - f(z_i) \right] \\ &\leq E \left[ \sum_{i=1}^t \langle \nabla f(z_i), d_i \rangle + \frac{L}{2} \|d_i\|^2 \right] \\ &= -E \left[ \sum_{i=1}^t \langle \nabla f(z_i), \frac{\beta_{1,i}}{1 - \beta_{1,i}} \left( \frac{\alpha_i}{\sqrt{\widehat{v}_i}} - \frac{\alpha_{i-1}}{\sqrt{\widehat{v}_{i-1}}} \right) \odot m_{i-1} \rangle \right] \\ &\quad - E \left[ \sum_{i=1}^t \alpha_i \langle \nabla f(z_i), g_i / \sqrt{\widehat{v}_i} \rangle \right] \\ &\quad - E \left[ \sum_{i=1}^t \langle \nabla f(z_i), \left( \frac{\beta_{1,i+1}}{1 - \beta_{1,i+1}} - \frac{\beta_{1,i}}{1 - \beta_{1,i}} \right) \alpha_i m_i / \sqrt{\widehat{v}_i} \rangle \right] \\ &\quad + E \left[ \sum_{i=1}^t \frac{L}{2} \|d_i\|^2 \right] = T_1 + T_2 + T_3 + E \left[ \sum_{i=1}^t \frac{L}{2} \|d_i\|^2 \right], \end{aligned} \quad (33)$$

where  $\{T_i\}$  have been defined in (25)-(30). Further, using inequality  $\|a + b + c\|^2 \leq 3\|a\|^2 + 3\|b\|^2 + 3\|c\|^2$  and (33), we have

$$E \left[ \sum_{i=1}^t \|d_i\|^2 \right] \leq T_4 + T_5 + T_6.$$

Substituting the above inequality into (33), we then obtain (24). **Q.E.D.**

The next series of lemmas separately bound the terms on RHS of (24).

**Lemma 7.3.** Suppose that the conditions in Theorem 3.1 hold,  $T_1$  in (25) can be bounded as

$$\begin{aligned} T_1 &= -E \left[ \sum_{i=1}^t \langle \nabla f(z_i), \frac{\beta_{1,t}}{1 - \beta_{1,t}} \left( \frac{\alpha_i}{\sqrt{\widehat{v}_i}} - \frac{\alpha_{i-1}}{\sqrt{\widehat{v}_{i-1}}} \right) \odot m_{i-1} \rangle \right] \\ &\leq H^2 \frac{\beta_1}{1 - \beta_1} E \left[ \sum_{i=2}^t \sum_{j=1}^d \left| \left( \frac{\alpha_i}{\sqrt{\widehat{v}_i}} - \frac{\alpha_{i-1}}{\sqrt{\widehat{v}_{i-1}}} \right)_j \right| \right] \end{aligned}$$

**Proof.** [Proof of Lemma 7.3] Since  $\|g_t\| \leq H$ , by the update rule of  $m_t$ , we have  $\|m_t\| \leq H$ , this can be proved by induction as below.

Recall that  $m_t = \beta_{1,t} m_{t-1} + (1 - \beta_{1,t}) g_t$ , suppose  $\|m_{t-1}\| \leq H$ , we have

$$\|m_t\| \leq (\beta_{1,t} + (1 - \beta_{1,t})) \max(\|g_t\|, \|m_{t-1}\|) = \max(\|g_t\|, \|m_{t-1}\|) \leq H, \quad (34)$$

then since  $m_0 = 0$ , we have  $\|m_0\| \leq H$  which completes the induction.

Given  $\|m_t\| \leq H$ , we further have

$$\begin{aligned}
T_1 &= -E \left[ \sum_{i=2}^t \left\langle \nabla f(z_i), \frac{\beta_{1,t}}{1-\beta_{1,t}} \left( \frac{\alpha_i}{\sqrt{\widehat{v}_i}} - \frac{\alpha_{i-1}}{\sqrt{\widehat{v}_{i-1}}} \right) \odot m_{i-1} \right\rangle \right] \\
&\leq E \left[ \sum_{i=1}^t \|\nabla f(z_i)\| \|m_{i-1}\| \left( \frac{1}{1-\beta_{1,t}} - 1 \right) \sum_{j=1}^d \left| \left( \frac{\alpha_i}{\sqrt{\widehat{v}_i}} - \frac{\alpha_{i-1}}{\sqrt{\widehat{v}_{i-1}}} \right)_j \right| \right] \\
&\leq H^2 \frac{\beta_1}{1-\beta_1} E \left[ \sum_{i=1}^t \sum_{j=1}^d \left| \left( \frac{\alpha_i}{\sqrt{\widehat{v}_i}} - \frac{\alpha_{i-1}}{\sqrt{\widehat{v}_{i-1}}} \right)_j \right| \right]
\end{aligned}$$

where the first equality holds due to (23), and the last inequality is due to  $\beta_1 \geq \beta_{1,i}$ .

The proof is now complete. **Q.E.D.**

**Lemma 7.4.** *Suppose the conditions in Theorem 3.1 hold. For  $T_3$  in (27), we have*

$$\begin{aligned}
T_3 &= -E \left[ \sum_{i=1}^t \left\langle \nabla f(z_i), \left( \frac{\beta_{1,i+1}}{1-\beta_{1,i+1}} - \frac{\beta_{1,i}}{1-\beta_{1,i}} \right) \alpha_i m_i / \sqrt{\widehat{v}_i} \right\rangle \right] \\
&\leq \left( \frac{\beta_1}{1-\beta_1} - \frac{\beta_{1,t+1}}{1-\beta_{1,t+1}} \right) (H^2 + G^2)
\end{aligned}$$

**Proof.** [Proof of Lemma 7.4]

$$\begin{aligned}
T_3 &\leq E \left[ \sum_{i=1}^t \left| \frac{\beta_{1,i+1}}{1-\beta_{1,i+1}} - \frac{\beta_{1,i}}{1-\beta_{1,i}} \right| \frac{1}{2} \left( \|\nabla f(z_i)\|^2 + \|\alpha_i m_i / \sqrt{\widehat{v}_i}\|^2 \right) \right] \\
&\leq E \left[ \sum_{i=1}^t \left| \frac{\beta_{1,i+1}}{1-\beta_{1,i+1}} - \frac{\beta_{1,i}}{1-\beta_{1,i}} \right| \frac{1}{2} (H^2 + G^2) \right] \\
&= \sum_{i=1}^t \left( \frac{\beta_{1,i}}{1-\beta_{1,i}} - \frac{\beta_{1,i+1}}{1-\beta_{1,i+1}} \right) \frac{1}{2} (H^2 + G^2) \\
&\leq \left( \frac{\beta_1}{1-\beta_1} - \frac{\beta_{1,t+1}}{1-\beta_{1,t+1}} \right) (H^2 + G^2)
\end{aligned}$$

where the first inequality is due to  $\langle a, b \rangle \leq \frac{1}{2}(\|a\|^2 + \|b\|^2)$ , the second inequality is using due to upper bound on  $\|\nabla f(x_t)\| \leq H$  and  $\|\alpha_i m_i / \sqrt{\widehat{v}_i}\| \leq G$  given by the assumptions in Theorem 3.1, the third equality is because  $\beta_{1,t} \leq \beta_1$  and  $\beta_{1,t}$  is non-increasing, the last inequality is due to telescope sum.

This completes the proof. **Q.E.D.**

**Lemma 7.5.** *Suppose the assumptions in Theorem 3.1 hold. For  $T_4$  in (28), we have*

$$\begin{aligned}
\frac{2}{3L} T_4 &= E \left[ \sum_{i=1}^t \left\| \left( \frac{\beta_{1,t+1}}{1-\beta_{1,t+1}} - \frac{\beta_{1,t}}{1-\beta_{1,t}} \right) \alpha_t m_t / \sqrt{v_t} \right\|^2 \right] \\
&\leq \left( \frac{\beta_1}{1-\beta_1} - \frac{\beta_{1,t+1}}{1-\beta_{1,t+1}} \right)^2 G^2
\end{aligned}$$

**Proof.** [Proof of Lemma 7.5] The proof is similar to the previous lemma.

$$\begin{aligned}
\frac{2}{3L}T_4 &= E \left[ \sum_{i=1}^t \left( \frac{\beta_{1,t+1}}{1-\beta_{1,t+1}} - \frac{\beta_{1,t}}{1-\beta_{1,t}} \right)^2 \|\alpha_t m_t / \sqrt{v_t}\|^2 \right] \\
&\leq E \left[ \sum_{i=1}^t \left( \frac{\beta_{1,t}}{1-\beta_{1,t}} - \frac{\beta_{1,t+1}}{1-\beta_{1,t+1}} \right)^2 G^2 \right] \\
&\leq \left( \frac{\beta_1}{1-\beta_1} - \frac{\beta_{1,t+1}}{1-\beta_{1,t+1}} \right) \sum_{i=1}^t \left( \frac{\beta_{1,t}}{1-\beta_{1,t}} - \frac{\beta_{1,t+1}}{1-\beta_{1,t+1}} \right) G^2 \\
&\leq \left( \frac{\beta_1}{1-\beta_1} - \frac{\beta_{1,t+1}}{1-\beta_{1,t+1}} \right)^2 G^2
\end{aligned}$$

where the first inequality is due to  $\|\alpha_t m_t / \sqrt{v_t}\| \leq G$  by our assumptions, the second inequality is due to non-decreasing property of  $\beta_{1,t}$  and  $\beta_1 \geq \beta_{1,t}$ , the last inequality is due to telescoping sum.

This completes the proof.

**Q.E.D.**

**Lemma 7.6.** Suppose the assumptions in Theorem 3.1 hold. For  $T_5$  in (29), we have

$$\begin{aligned}
\frac{2}{3L}T_5 &= E \left[ \sum_{i=1}^t \left\| \frac{\beta_{1,i}}{1-\beta_{1,i}} \left( \frac{\alpha_t}{\sqrt{\widehat{v}_i}} - \frac{\alpha_{i-1}}{\sqrt{\widehat{v}_{i-1}}} \right) \odot m_{i-1} \right\|^2 \right] \\
&\leq \left( \frac{\beta_1}{1-\beta_1} \right)^2 H^2 E \left[ \sum_{i=2}^t \sum_{j=1}^d \left( \frac{\alpha_t}{\sqrt{\widehat{v}_i}} - \frac{\alpha_{i-1}}{\sqrt{\widehat{v}_{i-1}}} \right)_j^2 \right]
\end{aligned}$$

**Proof.** [Proof of Lemma 7.6]

$$\begin{aligned}
\frac{2}{3L}T_5 &\leq E \left[ \sum_{i=2}^t \left( \frac{\beta_1}{1-\beta_1} \right)^2 \sum_{j=1}^d \left( \left( \frac{\alpha_t}{\sqrt{\widehat{v}_i}} - \frac{\alpha_{i-1}}{\sqrt{\widehat{v}_{i-1}}} \right)_j (m_{i-1})_j^2 \right) \right] \\
&\leq \left( \frac{\beta_1}{1-\beta_1} \right)^2 H^2 E \left[ \sum_{i=2}^t \sum_{j=1}^d \left( \frac{\alpha_t}{\sqrt{\widehat{v}_i}} - \frac{\alpha_{i-1}}{\sqrt{\widehat{v}_{i-1}}} \right)_j^2 \right]
\end{aligned}$$

where the first inequality is due to  $\beta_1 \geq \beta_{1,t}$  and (23), the second inequality is due to  $\|m_i\| < H$ .

This completes the proof.

**Q.E.D.**

**Lemma 7.7.** Suppose the assumptions in Theorem 3.1 hold. For  $T_2$  in (26), we have [Check expectation signs in what follows? I feel that you missed it at several places.]

$$\begin{aligned}
T_2 &= -E \left[ \sum_{i=1}^t \alpha_i \langle \nabla f(z_i), g_i / \sqrt{\widehat{v}_i} \rangle \right] \\
&\leq \sum_{i=2}^t \frac{1}{2} \|\alpha_i g_i / \sqrt{\widehat{v}_i}\|^2 + L^2 \left( \frac{\beta_1}{1-\beta_1} \right)^2 \left( \frac{1}{1-\beta_1} \right)^2 E \left[ \sum_{i=1}^{t-1} \|\alpha_i g_i / \sqrt{\widehat{v}_i}\|^2 \right] \\
&\quad + L^2 H^2 \left( \frac{1}{1-\beta_1} \right)^2 \left( \frac{\beta_1}{1-\beta_1} \right)^4 E \left[ \sum_{j=1}^d \sum_{i=2}^{t-1} \left| \frac{\alpha_i}{\sqrt{\widehat{v}_i}} - \frac{\alpha_{i-1}}{\sqrt{\widehat{v}_{i-1}}} \right|_j^2 \right] \\
&\quad + 2H^2 E \left[ \sum_{i=2}^t \sum_{j=1}^d \left| \left( \frac{\alpha_i}{\sqrt{\widehat{v}_i}} - \frac{\alpha_{i-1}}{\sqrt{\widehat{v}_{i-1}}} \right)_j \right| \right] \\
&\quad + 2H^2 E \left[ \sum_{j=1}^d (\alpha_1 / \sqrt{\widehat{v}_1})_j \right] - E \left[ \sum_{i=1}^t \alpha_i \langle \nabla f(x_i), \nabla f(x_t) / \sqrt{\widehat{v}_i} \rangle \right]. \tag{35}
\end{aligned}$$

**Proof.** [Proof of Lemma 7.7] Recall from the definition (20), we have

$$z_i - x_i = \frac{\beta_{1,i}}{1 - \beta_{1,i}}(x_i - x_{i-1}) = -\frac{\beta_{1,i}}{1 - \beta_{1,i}}\alpha_{i-1}m_{i-1}/\sqrt{\widehat{v}_{i-1}} \quad (36)$$

Further we have  $z_1 = x_1$  by definition of  $z_1$ . We have

$$\begin{aligned} T_2 &= -E \left[ \sum_{i=1}^t \alpha_i \langle \nabla f(z_i), g_i / \sqrt{\widehat{v}_i} \rangle \right] \\ &= -E \left[ \sum_{i=1}^t \alpha_i \langle \nabla f(x_i), g_i / \sqrt{\widehat{v}_i} \rangle \right] - E \left[ \sum_{i=1}^t \alpha_i \langle \nabla f(z_i) - \nabla f(x_i), g_i / \sqrt{\widehat{v}_i} \rangle \right]. \end{aligned} \quad (37)$$

The second term of (37) can be bounded as

$$\begin{aligned} &-E \left[ \sum_{i=1}^t \alpha_i \langle \nabla f(z_i) - \nabla f(x_i), g_i / \sqrt{\widehat{v}_i} \rangle \right] \\ &\leq E \left[ \sum_{i=2}^t \frac{1}{2} \|\nabla f(z_i) - \nabla f(x_i)\|^2 + \frac{1}{2} \|\alpha_i g_i / \sqrt{\widehat{v}_i}\|^2 \right] \\ &\leq \frac{L^2}{2} T_7 + \frac{1}{2} E \left[ \sum_{i=2}^t \|\alpha_i g_i / \sqrt{\widehat{v}_i}\|^2 \right], \end{aligned} \quad (38)$$

where the first inequality is because  $\langle a, b \rangle \leq \frac{1}{2} (\|a\|^2 + \|b\|^2)$  and the fact that  $z_1 = x_1$ , the second inequality is because

$$\|\nabla f(z_i) - \nabla f(x_i)\| \leq L \|z_i - x_i\| = L \left\| \frac{\beta_{1,i}}{1 - \beta_{1,i}} \alpha_{i-1} m_{i-1} / \sqrt{\widehat{v}_{i-1}} \right\|,$$

and  $T_7$  is defined as

$$T_7 = E \left[ \sum_{i=2}^t \left\| \frac{\beta_{1,i}}{1 - \beta_{1,i}} \alpha_{i-1} m_{i-1} / \sqrt{\widehat{v}_{i-1}} \right\|^2 \right]. \quad (39)$$

We next bound the  $T_7$  in (39), by update rule  $m_i = \beta_{1,i} m_{i-1} + (1 - \beta_{1,i}) g_i$ , we have  $m_i = \sum_{k=1}^i [(\prod_{l=k+1}^i \beta_{1,l}) (1 - \beta_{1,k}) g_k]$ . Based on that, we obtain

$$\begin{aligned} T_7 &\leq \left( \frac{\beta_1}{1 - \beta_1} \right)^2 E \left[ \sum_{i=2}^t \sum_{j=1}^d \left( \frac{\alpha_{i-1} m_{i-1}}{\sqrt{\widehat{v}_{i-1}}} \right)_j^2 \right] \\ &= \left( \frac{\beta_1}{1 - \beta_1} \right)^2 E \left[ \sum_{i=2}^t \sum_{j=1}^d \left( \sum_{k=1}^{i-1} \frac{\alpha_{i-1} \left( \prod_{l=k+1}^{i-1} \beta_{1,l} \right) (1 - \beta_{1,k}) g_k}{\sqrt{\widehat{v}_{i-1}}} \right)_j^2 \right] \\ &\leq 2 \left( \frac{\beta_1}{1 - \beta_1} \right)^2 E \left[ \underbrace{\sum_{i=2}^t \sum_{j=1}^d \left( \sum_{k=1}^{i-1} \frac{\alpha_k \left( \prod_{l=k+1}^{i-1} \beta_{1,l} \right) (1 - \beta_{1,k}) g_k}{\sqrt{\widehat{v}_k}} \right)_j^2}_{T_8} \right. \\ &\quad \left. + 2 \left( \frac{\beta_1}{1 - \beta_1} \right)^2 E \left[ \underbrace{\sum_{i=2}^t \sum_{j=1}^d \left( \sum_{k=1}^{i-1} \left( \prod_{l=k+1}^{i-1} \beta_{1,l} \right) (1 - \beta_{1,k}) (g_k)_j \left( \frac{\alpha_{i-1}}{\sqrt{\widehat{v}_{i-1}}} - \frac{\alpha_k}{\sqrt{\widehat{v}_k}} \right) \right)_j^2}_{T_9} \right] \right] \end{aligned} \quad (40)$$

where the first inequality is due to  $\beta_{1,i} \leq \beta_1$ , the second equality is by substituting expression of  $m_t$ , the last inequality is because  $(a + b)^2 \leq 2(\|a\|^2 + \|b\|^2)$ , and we have introduced  $T_8$  and  $T_9$  for ease of notation.

In (40), we first bound  $T_8$  as below

$$\begin{aligned}
T_8 &= E \left[ \sum_{i=2}^t \sum_{j=1}^d \sum_{k=1}^{i-1} \sum_{p=1}^{i-1} \left( \frac{\alpha_k g_k}{\sqrt{\widehat{v}_k}} \right)_j \left( \prod_{l=k+1}^{i-1} \beta_{1,k} \right) (1 - \beta_{1,k}) \left( \frac{\alpha_p g_p}{\sqrt{\widehat{v}_p}} \right)_j \left( \prod_{q=p+1}^{i-1} \beta_{1,p} \right) (1 - \beta_{1,p}) \right] \\
&\stackrel{(i)}{\leq} E \left[ \sum_{i=2}^t \sum_{j=1}^d \sum_{k=1}^{i-1} \sum_{p=1}^{i-1} (\beta_1^{i-1-k}) (\beta_1^{i-1-p}) \frac{1}{2} \left( \left( \frac{\alpha_k g_k}{\sqrt{\widehat{v}_k}} \right)_j^2 + \left( \frac{\alpha_p g_p}{\sqrt{\widehat{v}_p}} \right)_j^2 \right) \right] \\
&\stackrel{(ii)}{=} E \left[ \sum_{i=2}^t \sum_{j=1}^d \sum_{k=1}^{i-1} (\beta_1^{i-1-k}) \left( \frac{\alpha_k g_k}{\sqrt{\widehat{v}_k}} \right)_j^2 \sum_{p=1}^{i-1} (\beta_1^{i-1-p}) \right] \\
&\stackrel{(iii)}{\leq} \frac{1}{1 - \beta_1} E \left[ \sum_{i=2}^t \sum_{j=1}^d \sum_{k=1}^{i-1} (\beta_1^{i-1-k}) \left( \frac{\alpha_k g_k}{\sqrt{\widehat{v}_k}} \right)_j^2 \right] \\
&\stackrel{(iv)}{=} \frac{1}{1 - \beta_1} E \left[ \sum_{k=1}^{t-1} \sum_{j=1}^d \sum_{i=k+1}^t (\beta_1^{i-1-k}) \left( \frac{\alpha_k g_k}{\sqrt{\widehat{v}_k}} \right)_j^2 \right] \\
&\leq \left( \frac{1}{1 - \beta_1} \right)^2 E \left[ \sum_{k=1}^{t-1} \sum_{j=1}^d \left( \frac{\alpha_k g_k}{\sqrt{\widehat{v}_k}} \right)_j^2 \right] = \left( \frac{1}{1 - \beta_1} \right)^2 E \left[ \sum_{i=1}^{t-1} \|\alpha_i g_i / \sqrt{\widehat{v}_i}\|^2 \right] \tag{41}
\end{aligned}$$

where (i) is due to  $ab < \frac{1}{2}(a^2 + b^2)$  and follows from  $\beta_{1,t} \leq \beta_1$  and  $\beta_{1,t} \in [0, 1)$ , (ii) is due to symmetry of  $p$  and  $k$  in the summation, (iii) is because of  $\sum_{p=1}^{i-1} (\beta_1^{i-1-p}) \leq \frac{1}{1-\beta_1}$ , (iv) is exchanging order of summation, and the second-last inequality is due to the similar reason as (iii).

For the  $T_9$  in (40), we have

$$\begin{aligned}
T_9 &= E \left[ \sum_{i=2}^t \sum_{j=1}^d \left( \sum_{k=1}^{i-1} \left( \prod_{l=k+1}^{i-1} \beta_{1,k} \right) (1 - \beta_{1,k}) (g_k)_j \left( \frac{\alpha_{i-1}}{\sqrt{\widehat{v}_{i-1}}} - \frac{\alpha_k}{\sqrt{\widehat{v}_k}} \right)_j \right)^2 \right] \\
&\leq H^2 E \left[ \sum_{i=2}^t \sum_{j=1}^d \left( \sum_{k=1}^{i-1} \left( \prod_{l=k+1}^{i-1} \beta_{1,k} \right) \left| \frac{\alpha_{i-1}}{\sqrt{\widehat{v}_{i-1}}} - \frac{\alpha_k}{\sqrt{\widehat{v}_k}} \right|_j \right)^2 \right] \\
&\leq H^2 E \left[ \sum_{i=1}^{t-1} \sum_{j=1}^d \left( \sum_{k=1}^i \beta_1^{i-k} \left| \frac{\alpha_i}{\sqrt{\widehat{v}_i}} - \frac{\alpha_k}{\sqrt{\widehat{v}_k}} \right|_j \right)^2 \right] \\
&\leq H^2 E \left[ \sum_{i=1}^{t-1} \sum_{j=1}^d \left( \sum_{k=1}^i \beta_1^{i-k} \sum_{l=k+1}^i \left| \frac{\alpha_l}{\sqrt{\widehat{v}_l}} - \frac{\alpha_{l-1}}{\sqrt{\widehat{v}_{l-1}}} \right|_j \right)^2 \right] \tag{42}
\end{aligned}$$

where the first inequality holds due to  $\beta_{1,k} < 1$  and  $|(g_k)_j| \leq H$ , the second inequality holds due to  $\beta_{1,k} \leq \beta_1$ , and the last inequality applied the triangle inequality. For RHS of (42), using Lemma 7.8

(that will be proved later) with  $a_i = \left| \frac{\alpha_i}{\sqrt{\widehat{v}_i}} - \frac{\alpha_{i-1}}{\sqrt{\widehat{v}_{i-1}}} \right|_j$ , we further have

$$\begin{aligned}
T_9 &\leq H^2 E \left[ \sum_{i=1}^{t-1} \sum_{j=1}^d \left( \sum_{k=1}^i \beta_1^{i-k} \sum_{l=k+1}^i \left| \frac{\alpha_l}{\sqrt{\widehat{v}_l}} - \frac{\alpha_{l-1}}{\sqrt{\widehat{v}_{l-1}}} \right|_j \right)^2 \right] \\
&\leq H^2 \left( \frac{1}{1 - \beta_1} \right)^2 \left( \frac{\beta_1}{1 - \beta_1} \right)^2 E \left[ \sum_{j=1}^d \sum_{i=2}^t \left| \frac{\alpha_l}{\sqrt{\widehat{v}_l}} - \frac{\alpha_{l-1}}{\sqrt{\widehat{v}_{l-1}}} \right|_j^2 \right] \tag{43}
\end{aligned}$$



Based on (38), (40), (41) and (43), we can then bound the second term of (37) as

$$\begin{aligned}
& -E \left[ \sum_{i=1}^t \alpha_i \langle \nabla f(z_i) - \nabla f(x_i), g_i / \sqrt{\widehat{v}_i} \rangle \right] \\
& \leq L^2 \left( \frac{\beta_1}{1-\beta_1} \right)^2 \left( \frac{1}{1-\beta_1} \right)^2 E \left[ \sum_{i=1}^{t-1} \|\alpha_i g_i / \sqrt{\widehat{v}_i}\|^2 \right] \\
& \quad + L^2 H^2 \left( \frac{1}{1-\beta_1} \right)^2 \left( \frac{\beta_1}{1-\beta_1} \right)^4 E \left[ \sum_{j=1}^d \sum_{i=2}^{t-1} \left| \frac{\alpha_i}{\sqrt{\widehat{v}_i}} - \frac{\alpha_{i-1}}{\sqrt{\widehat{v}_{i-1}}} \right|_j^2 \right] \\
& \quad + \frac{1}{2} E \left[ \sum_{i=2}^t \|\alpha_i g_i / \sqrt{\widehat{v}_i}\|^2 \right]. \tag{44}
\end{aligned}$$

Let us turn to the first term in (37). Reparameterize  $g_t$  as  $g_t = \nabla f(x_t) + \delta_t$  with  $E[\delta_t] = 0$ , we have

$$\begin{aligned}
& E \left[ \sum_{i=1}^t \alpha_i \langle \nabla f(x_i), g_i / \sqrt{\widehat{v}_i} \rangle \right] \\
& = E \left[ \sum_{i=1}^t \alpha_i \langle \nabla f(x_i), (\nabla f(x_i) + \delta_i) / \sqrt{\widehat{v}_i} \rangle \right] \\
& = E \left[ \sum_{i=1}^t \alpha_i \langle \nabla f(x_i), \nabla f(x_i) / \sqrt{\widehat{v}_i} \rangle \right] + E \left[ \sum_{i=1}^t \alpha_i \langle \nabla f(x_i), \delta_i / \sqrt{\widehat{v}_i} \rangle \right]. \tag{45}
\end{aligned}$$

It can be seen that the first term in RHS of (45) is the desired descent quantity, the second term is a bias term to be bounded. For the second term in RHS of (45), we have

$$\begin{aligned}
& E \left[ \sum_{i=1}^t \alpha_i \langle \nabla f(x_i), \delta_i / \sqrt{\widehat{v}_i} \rangle \right] \\
& = E \left[ \sum_{i=2}^t \langle \nabla f(x_i), \delta_i \odot (\alpha_i / \sqrt{\widehat{v}_i} - \alpha_{i-1} / \sqrt{\widehat{v}_{i-1}}) \rangle \right] + E \left[ \sum_{i=2}^t \alpha_{i-1} \langle \nabla f(x_i), \delta_i \odot (1 / \sqrt{\widehat{v}_{i-1}}) \rangle \right] \\
& \quad + E \left[ \alpha_1 \langle \nabla f(x_1), \delta_1 / \sqrt{\widehat{v}_1} \rangle \right] \\
& \geq E \left[ \sum_{i=2}^t \langle \nabla f(x_i), \delta_i \odot (\alpha_i / \sqrt{\widehat{v}_i} - \alpha_{i-1} / \sqrt{\widehat{v}_{i-1}}) \rangle \right] - 2H^2 E \left[ \sum_{j=1}^d (\alpha_1 / \sqrt{\widehat{v}_1})_j \right] \tag{46}
\end{aligned}$$

where the last equation is because given  $x_i, \widehat{v}_{i-1}$ ,  $E[\delta_i \odot (1 / \sqrt{\widehat{v}_{i-1}}) | x_i, \widehat{v}_{i-1}] = 0$  and  $\|\delta_i\| \leq 2H$  due to  $\|g_i\| \leq H$  and  $\|\nabla f(x_i)\| \leq H$  based on Assumptions A2 and A3. Further, we have

$$\begin{aligned}
& E \left[ \sum_{i=2}^t \langle \nabla f(x_i), \delta_i \odot (\alpha_i / \sqrt{\widehat{v}_i} - \alpha_{i-1} / \sqrt{\widehat{v}_{i-1}}) \rangle \right] \\
& = E \left[ \sum_{i=2}^t \sum_{j=1}^d (\nabla f(x_i))_j (\delta_i)_j (\alpha_i / (\sqrt{\widehat{v}_i})_j - \alpha_{i-1} / (\sqrt{\widehat{v}_{i-1}})_j) \right] \\
& \geq -E \left[ \sum_{i=2}^t \sum_{j=1}^d |(\nabla f(x_i))_j| |(\delta_i)_j| \left| (\alpha_i / (\sqrt{\widehat{v}_i})_j - \alpha_{i-1} / (\sqrt{\widehat{v}_{i-1}})_j) \right| \right] \\
& \geq -2H^2 E \left[ \sum_{i=2}^t \sum_{j=1}^d \left| (\alpha_i / (\sqrt{\widehat{v}_i})_j - \alpha_{i-1} / (\sqrt{\widehat{v}_{i-1}})_j) \right| \right] \tag{47}
\end{aligned}$$

Substituting (46) and (47) into (45), we then bound the first term of (37) as

$$\begin{aligned}
& - E \left[ \sum_{i=1}^t \alpha_i \langle \nabla f(x_i), g_i / \sqrt{\widehat{v}_i} \rangle \right] \\
& \leq 2H^2 E \left[ \sum_{i=2}^t \sum_{j=1}^d \left| (\alpha_i / (\sqrt{\widehat{v}_i})_j - \alpha_{i-1} / (\sqrt{\widehat{v}_{i-1}})_j) \right| \right] + 2H^2 E \left[ \sum_{j=1}^d (\alpha_1 / \sqrt{\widehat{v}_1})_j \right] \\
& - E \left[ \sum_{i=1}^t \alpha_i \langle \nabla f(x_i), \nabla f(x_i) / \sqrt{\widehat{v}_i} \rangle \right]
\end{aligned} \tag{48}$$

We finally apply (48) and (44) to obtain (35). The proof is now complete. Q.E.D.

**Lemma 7.8.** For  $a_i \geq 0$ ,  $\beta \in [0, 1)$ , and  $b_i = \sum_{k=1}^i \beta^{i-k} \sum_{l=k+1}^i a_l$ , we have

$$\sum_{i=1}^t b_i^2 \leq \left( \frac{1}{1-\beta} \right)^2 \left( \frac{\beta}{1-\beta} \right)^2 \sum_{i=2}^t a_i^2$$

**Proof.** [Proof of Lemma 7.8] The result is proved by following

$$\begin{aligned}
\sum_{i=1}^t b_i^2 &= \sum_{i=1}^t \left( \sum_{k=1}^i \beta^{i-k} \sum_{l=k+1}^i a_l \right)^2 \\
&\stackrel{(i)}{=} \sum_{i=1}^t \left( \sum_{l=2}^i \sum_{k=1}^{l-1} \beta^{i-k} a_l \right)^2 = \sum_{i=1}^t \left( \sum_{l=2}^i \beta^{i-l+1} a_l \sum_{k=1}^{l-1} \beta^{l-1-k} \right)^2 \\
&\stackrel{(ii)}{\leq} \left( \frac{1}{1-\beta} \right)^2 \sum_{i=1}^t \left( \sum_{l=2}^i \beta^{i-l+1} a_l \right)^2 = \left( \frac{1}{1-\beta} \right)^2 \sum_{i=1}^t \left( \sum_{l=2}^i \sum_{m=2}^i \beta^{i-l+1} a_l \beta^{i-m+1} a_m \right) \\
&\stackrel{(iii)}{\leq} \left( \frac{1}{1-\beta} \right)^2 \sum_{i=1}^t \sum_{l=2}^i \sum_{m=2}^i \beta^{i-l+1} \beta^{i-m+1} \frac{1}{2} (a_l^2 + a_m^2) \\
&\stackrel{(iv)}{=} \left( \frac{1}{1-\beta} \right)^2 \sum_{i=1}^t \sum_{l=2}^i \sum_{m=2}^i \beta^{i-l+1} \beta^{i-m+1} a_l^2 \stackrel{(v)}{\leq} \left( \frac{1}{1-\beta} \right)^2 \frac{\beta}{1-\beta} \sum_{l=2}^t \sum_{i=l}^t \beta^{i-l+1} a_l^2 \\
&\leq \left( \frac{1}{1-\beta} \right)^2 \left( \frac{\beta}{1-\beta} \right)^2 \sum_{l=2}^t a_l^2
\end{aligned}$$

where (i) is by changing order of summation, (ii) is due to  $\sum_{k=1}^{l-1} \beta^{l-1-k} \leq \frac{1}{1-\beta}$ , (iii) is by the fact that  $ab \leq \frac{1}{2}(a^2 + b^2)$ , (iv) is due to symmetry of  $a_l$  and  $a_m$  in the summation, (v) is because  $\sum_{m=2}^i \beta^{i-m+1} \leq \frac{\beta}{1-\beta}$  and the last inequality is for similar reason.

This completes the proof. Q.E.D.

### 7.2.2 Proof of Theorem 3.1

**Proof.** [Proof of Theorem 3.1] We combine Lemma 7.2, Lemma 7.3, Lemma 7.4, Lemma 7.5, Lemma 7.6, and Lemma 7.7 to bound the overall expected descent of the objective. We have

$$\begin{aligned}
& E[f(z_{t+1}) - f(z_1)] \leq \sum_{i=1}^6 T_i \\
& = -E \left[ \sum_{i=1}^t \langle \nabla f(z_i), \frac{\beta_{1,i}}{1-\beta_{1,i}} \left( \frac{\alpha_i}{\sqrt{\widehat{v}_i}} - \frac{\alpha_{i-1}}{\sqrt{\widehat{v}_{i-1}}} \right) \odot m_{i-1} \rangle \right] - E \left[ \sum_{i=1}^t \alpha_i \langle \nabla f(z_i), g_i / \sqrt{\widehat{v}_i} \rangle \right] \\
& \quad - E \left[ \sum_{i=1}^t \langle \nabla f(z_i), \left( \frac{\beta_{1,i+1}}{1-\beta_{1,i+1}} - \frac{\beta_{1,i}}{1-\beta_{1,i}} \right) \alpha_i m_i / \sqrt{\widehat{v}_i} \rangle \right] \\
& \quad + E \left[ \sum_{i=1}^t \frac{3}{2} L \left\| \left( \frac{\beta_{1,t+1}}{1-\beta_{1,t+1}} - \frac{\beta_{1,t}}{1-\beta_{1,t}} \right) \alpha_t m_t / \sqrt{v_t} \right\|^2 \right] \\
& \quad + E \left[ \sum_{i=1}^t \frac{3}{2} L \left\| \frac{\beta_{1,i}}{1-\beta_{1,i}} \left( \frac{\alpha_t}{\sqrt{\widehat{v}_i}} - \frac{\alpha_{i-1}}{\sqrt{\widehat{v}_{i-1}}} \right) \odot m_{i-1} \right\|^2 \right] + E \left[ \sum_{i=1}^t \frac{3}{2} L \left\| \alpha_i g_i / \sqrt{\widehat{v}_i} \right\|^2 \right] \\
& \leq H^2 \frac{\beta_1}{1-\beta_1} E \left[ \sum_{i=2}^t \sum_{j=1}^d \left| \left( \frac{\alpha_i}{\sqrt{\widehat{v}_i}} - \frac{\alpha_{i-1}}{\sqrt{\widehat{v}_{i-1}}} \right)_j \right| \right] \\
& \quad + \left( \frac{\beta_1}{1-\beta_1} - \frac{\beta_{1,t+1}}{1-\beta_{1,t+1}} \right) (H^2 + G^2) + \left( \frac{\beta_1}{1-\beta_1} - \frac{\beta_{1,t+1}}{1-\beta_{1,t+1}} \right)^2 G^2 \\
& \quad + \left( \frac{\beta_1}{1-\beta_1} \right)^2 H^2 E \left[ \sum_{i=2}^t \sum_{j=1}^d \left( \frac{\alpha_i}{\sqrt{\widehat{v}_i}} - \frac{\alpha_{i-1}}{\sqrt{\widehat{v}_{i-1}}} \right)_j^2 \right] + E \left[ \sum_{i=1}^t \frac{3}{2} L \left\| \alpha_i g_i / \sqrt{\widehat{v}_i} \right\|^2 \right] \\
& \quad + E \left[ \sum_{i=2}^t \frac{1}{2} \left\| \alpha_i g_i / \sqrt{\widehat{v}_i} \right\|^2 \right] + L^2 \frac{\beta_1}{1-\beta_1} \left( \frac{1}{1-\beta_1} \right)^2 E \left[ \sum_{k=1}^{t-1} \sum_{j=1}^d \left( \frac{\alpha_k g_k}{\sqrt{\widehat{v}_k}} \right)_j^2 \right] \\
& \quad + L^2 H^2 \left( \frac{1}{1-\beta_1} \right)^2 \left( \frac{\beta_1}{1-\beta_1} \right)^4 E \left[ \sum_{j=1}^d \sum_{i=2}^{t-1} \left( \frac{\alpha_i}{\sqrt{\widehat{v}_i}} - \frac{\alpha_{i-1}}{\sqrt{\widehat{v}_{i-1}}} \right)_j^2 \right] \\
& \quad + 2H^2 E \left[ \sum_{i=2}^t \sum_{j=1}^d \left| \left( \frac{\alpha_i}{\sqrt{\widehat{v}_i}} - \frac{\alpha_{i-1}}{\sqrt{\widehat{v}_{i-1}}} \right)_j \right| \right] \\
& \quad + 2H^2 E \left[ \sum_{j=1}^d (\alpha_1 / \sqrt{\widehat{v}_1})_j \right] - E \left[ \sum_{i=1}^t \alpha_i \langle \nabla f(x_i), \nabla f(x_i) / \sqrt{\widehat{v}_i} \rangle \right]
\end{aligned}$$

where the first inequality is due to Lemma 7.2, the second inequality is due to Lemma 7.3, Lemma 7.4, Lemma 7.5, Lemma 7.6, Lemma 7.7.

By merging similar terms in above inequality, we further have

$$\begin{aligned}
& E[f(z_{t+1}) - f(z_1)] \\
& \leq \left( H^2 \frac{\beta_1}{1-\beta_1} + 2H^2 \right) E \left[ \sum_{i=2}^t \sum_{j=1}^d \left| \left( \frac{\alpha_i}{\sqrt{\widehat{v}_i}} - \frac{\alpha_{i-1}}{\sqrt{\widehat{v}_{i-1}}} \right)_j \right| \right] \\
& \quad + \left( 1 + L^2 \left( \frac{1}{1-\beta_1} \right)^2 \left( \frac{\beta_1}{1-\beta_1} \right)^2 \right) H^2 \left( \frac{\beta_1}{1-\beta_1} \right)^2 E \left[ \sum_{j=1}^d \sum_{i=2}^{t-1} \left( \frac{\alpha_i}{\sqrt{\widehat{v}_i}} - \frac{\alpha_{i-1}}{\sqrt{\widehat{v}_{i-1}}} \right)_j^2 \right] \\
& \quad + \left( \frac{3}{2}L + \frac{1}{2} + L^2 \frac{\beta_1}{1-\beta_1} \left( \frac{1}{1-\beta_1} \right)^2 \right) E \left[ \sum_{i=1}^t \left\| \alpha_i g_i / \sqrt{\widehat{v}_i} \right\|^2 \right] \\
& \quad + \left( \frac{\beta_1}{1-\beta_1} - \frac{\beta_{1,t+1}}{1-\beta_{1,t+1}} \right) (H^2 + G^2) + \left( \frac{\beta_1}{1-\beta_1} - \frac{\beta_{1,t+1}}{1-\beta_{1,t+1}} \right)^2 G^2 \\
& \quad + 2H^2 E \left[ \sum_{j=1}^d (\alpha_1 / \sqrt{\widehat{v}_1})_j \right] - E \left[ \sum_{i=1}^t \alpha_i \langle \nabla f(x_i), \nabla f(x_i) / \sqrt{\widehat{v}_i} \rangle \right] \tag{49}
\end{aligned}$$

Rearranging (49), we have

$$\begin{aligned}
& E \left[ \sum_{i=1}^t \alpha_i \langle \nabla f(x_i), \nabla f(x_i) / \sqrt{\widehat{v}_i} \rangle \right] \\
& \leq \left( H^2 \frac{\beta_1}{1-\beta_1} + 2H^2 \right) E \left[ \sum_{i=2}^t \sum_{j=1}^d \left| \left( \frac{\alpha_i}{\sqrt{\widehat{v}_i}} - \frac{\alpha_{i-1}}{\sqrt{\widehat{v}_{i-1}}} \right)_j \right| \right] + \\
& \quad + \left( 1 + L^2 \left( \frac{1}{1-\beta_1} \right)^2 \left( \frac{\beta_1}{1-\beta_1} \right)^2 \right) H^2 \left( \frac{\beta_1}{1-\beta_1} \right)^2 E \left[ \sum_{j=1}^d \sum_{i=2}^{t-1} \left( \frac{\alpha_i}{\sqrt{\widehat{v}_i}} - \frac{\alpha_{i-1}}{\sqrt{\widehat{v}_{i-1}}} \right)_j^2 \right] \\
& \quad + \left( \frac{3}{2}L + \frac{1}{2} + L^2 \frac{\beta_1}{1-\beta_1} \left( \frac{1}{1-\beta_1} \right)^2 \right) E \left[ \sum_{i=1}^t \left\| \alpha_i g_i / \sqrt{\widehat{v}_i} \right\|^2 \right] \\
& \quad + \left( \frac{\beta_1}{1-\beta_1} - \frac{\beta_{1,t+1}}{1-\beta_{1,t+1}} \right) (H^2 + G^2) + \left( \frac{\beta_1}{1-\beta_1} - \frac{\beta_{1,t+1}}{1-\beta_{1,t+1}} \right)^2 G^2 \\
& \quad + 2H^2 E \left[ \sum_{j=1}^d (\alpha_1 / \sqrt{\widehat{v}_1})_j \right] + E[f(z_1) - f(z_{t+1})] \\
& \leq E \left[ C_1 \sum_{i=1}^t \left\| \alpha_i g_i / \sqrt{\widehat{v}_i} \right\|^2 + C_2 \sum_{i=2}^t \left\| \frac{\alpha_i}{\sqrt{\widehat{v}_i}} - \frac{\alpha_{i-1}}{\sqrt{\widehat{v}_{i-1}}} \right\|_1 + C_3 \sum_{i=2}^{t-1} \left\| \frac{\alpha_i}{\sqrt{\widehat{v}_i}} - \frac{\alpha_{i-1}}{\sqrt{\widehat{v}_{i-1}}} \right\|^2 \right] + C_4
\end{aligned}$$

where

$$\begin{aligned}
C_1 & \triangleq \left( \frac{3}{2}L + \frac{1}{2} + L^2 \frac{\beta_1}{1-\beta_1} \left( \frac{1}{1-\beta_1} \right)^2 \right) \\
C_2 & \triangleq \left( H^2 \frac{\beta_1}{1-\beta_1} + 2H^2 \right) \\
C_3 & \triangleq \left( 1 + L^2 \left( \frac{1}{1-\beta_1} \right)^2 \left( \frac{\beta_1}{1-\beta_1} \right)^2 \right) H^2 \left( \frac{\beta_1}{1-\beta_1} \right)^2 \\
C_4 & \triangleq \left( \frac{\beta_1}{1-\beta_1} \right) (H^2 + G^2) + \left( \frac{\beta_1}{1-\beta_1} \right)^2 G^2 \\
& \quad + 2H^2 E \left[ \|\alpha_1 / \sqrt{\widehat{v}_1}\|_1 \right] + E[f(z_1) - f(z^*)]
\end{aligned}$$

and  $z^*$  is an optimal of  $f$ , i.e.  $z^* \in \arg \min_z f(z)$ .

This completes the proof. **Q.E.D.**

### 7.2.3 Proof of Corollary 3.2

**Proof.** [Proof of Corollary 3.2]

We first bound non-constant terms in RHS of (6), which is given by

$$E \left[ C_1 \sum_{t=1}^T \left\| \alpha_t g_t / \sqrt{\widehat{v}_t} \right\|^2 + C_2 \sum_{t=2}^T \left\| \frac{\alpha_t}{\sqrt{\widehat{v}_t}} - \frac{\alpha_{t-1}}{\sqrt{\widehat{v}_{t-1}}} \right\|_1 + C_3 \sum_{t=2}^{T-1} \left\| \frac{\alpha_t}{\sqrt{\widehat{v}_t}} - \frac{\alpha_{t-1}}{\sqrt{\widehat{v}_{t-1}}} \right\|^2 \right] + C_4.$$

For the term with  $C_1$ , assume  $\min_{j \in [d]} (\sqrt{\widehat{v}_1})_j \geq c > 0$  (this is natural since if it is 0, division by 0 error will happen), we have

$$\begin{aligned} & E \left[ \sum_{t=1}^T \left\| \alpha_t g_t / \sqrt{\widehat{v}_t} \right\|^2 \right] \\ & \leq E \left[ \sum_{t=1}^T \left\| \alpha_t g_t / c \right\|^2 \right] = E \left[ \sum_{t=1}^T \left\| \frac{1}{\sqrt{t}} g_t / c \right\|^2 \right] = E \left[ \sum_{t=1}^T \left( \frac{1}{c\sqrt{t}} \right)^2 \|g_t\|^2 \right] \\ & \leq H^2 / c^2 \sum_{t=1}^T \frac{1}{t} \leq H^2 / c^2 (1 + \log T) \end{aligned}$$

where the first inequality is due to  $(\widehat{v}_t)_j \geq (\widehat{v}_{t-1})_j$ , and the last inequality is due to  $\sum_{t=1}^T 1/t \leq 1 + \log T$ .

For the term with  $C_2$ , we have

$$\begin{aligned} & E \left[ \sum_{t=2}^T \left\| \frac{\alpha_t}{\sqrt{\widehat{v}_t}} - \frac{\alpha_{t-1}}{\sqrt{\widehat{v}_{t-1}}} \right\|_1 \right] = E \left[ \sum_{j=1}^d \sum_{t=2}^T \left( \frac{\alpha_{t-1}}{(\sqrt{\widehat{v}_{t-1}})_j} - \frac{\alpha_t}{(\sqrt{\widehat{v}_t})_j} \right) \right] \\ & = E \left[ \sum_{j=1}^d \left( \frac{\alpha_1}{(\sqrt{\widehat{v}_1})_j} - \frac{\alpha_T}{(\sqrt{\widehat{v}_T})_j} \right) \right] \leq E \left[ \sum_{j=1}^d \frac{\alpha_1}{(\sqrt{\widehat{v}_1})_j} \right] \leq d/c \end{aligned} \quad (50)$$

where the first equality is due to  $(\widehat{v}_t)_j \geq (\widehat{v}_{t-1})_j$  and  $\alpha_t \leq \alpha_{t-1}$ , and the second equality is due to telescope sum.

For the term with  $C_3$ , we have

$$\begin{aligned} & E \left[ \sum_{t=2}^{T-1} \left\| \frac{\alpha_t}{\sqrt{\widehat{v}_t}} - \frac{\alpha_{t-1}}{\sqrt{\widehat{v}_{t-1}}} \right\|^2 \right] \\ & \leq E \left[ \max_{t \in [T], t > 1} \left\| \frac{\alpha_t}{\sqrt{\widehat{v}_t}} - \frac{\alpha_{t-1}}{\sqrt{\widehat{v}_{t-1}}} \right\| \sum_{t=2}^{T-1} \left\| \frac{\alpha_t}{\sqrt{\widehat{v}_t}} - \frac{\alpha_{t-1}}{\sqrt{\widehat{v}_{t-1}}} \right\| \right] \\ & \leq (d/c) E \left[ \sum_{t=2}^{T-1} \left\| \frac{\alpha_t}{\sqrt{\widehat{v}_t}} - \frac{\alpha_{t-1}}{\sqrt{\widehat{v}_{t-1}}} \right\| \right] \\ & \leq (d/c)^2 \end{aligned}$$

where the second and third inequalities have used (50) and the fact  $\|\cdot\|_2 \leq \|\cdot\|_1$ .

Then we have for AMSGRAD,

$$\begin{aligned} & E \left[ C_1 \sum_{t=1}^T \left\| \alpha_t g_t / \sqrt{\widehat{v}_t} \right\|^2 + C_2 \sum_{t=2}^T \left\| \frac{\alpha_t}{\sqrt{\widehat{v}_t}} - \frac{\alpha_{t-1}}{\sqrt{\widehat{v}_{t-1}}} \right\|_1 + C_3 \sum_{t=2}^{T-1} \left\| \frac{\alpha_t}{\sqrt{\widehat{v}_t}} - \frac{\alpha_{t-1}}{\sqrt{\widehat{v}_{t-1}}} \right\|^2 \right] + C_4 \\ & \leq C_1 H^2 / c^2 (1 + \log T) + C_2 d / c + C_3 (d/c)^2 + C_4 \end{aligned} \quad (51)$$

Now we lower bound the effective stepsizes, since  $\widehat{v}_t$  is exponential moving average of  $g_t^2$  and  $\|g_t\| \leq H$ , we have  $(\widehat{v}_t)_j \leq H^2$ , we have

$$\alpha/(\sqrt{\widehat{v}_t})_j \geq \frac{1}{H\sqrt{t}}$$

And thus

$$E \left[ \sum_{t=1}^T \alpha_t \langle \nabla f(x_t), \nabla f(x_t) / \sqrt{\widehat{v}_t} \rangle \right] \geq E \left[ \sum_{t=1}^T \frac{1}{H\sqrt{t}} \|\nabla f(x_t)\|^2 \right] \geq \frac{\sqrt{T}}{H} \min_{t \in [T]} E [\|\nabla f(x_t)\|^2] \quad (52)$$

Then by (6), (51) and (52), we have

$$\frac{1}{H} \sqrt{T} \min_{t \in [T]} E [\|\nabla f(x_t)\|^2] \leq C_1 H^2 / c^2 (1 + \log T) + C_2 d / c + C_3 (d/c)^2 + C_4$$

which is equivalent to

$$\begin{aligned} & \min_{t \in [T]} E [\|\nabla f(x_t)\|^2] \\ & \leq \frac{H}{\sqrt{T}} (C_1 H^2 / c^2 (1 + \log T) + C_2 d / c + C_3 (d/c)^2 + C_4) \\ & = \frac{1}{\sqrt{T}} (Q_1 + Q_2 \log T) \end{aligned}$$

One more thing is to verify the assumption  $\|\alpha_t m_t / \sqrt{\widehat{v}_t}\| \leq G$  in Theorem 3.1, since  $\alpha_{t+1} / (\sqrt{\widehat{v}_{t+1}})_j \leq \alpha_t / (\sqrt{\widehat{v}_t})_j$  and  $\alpha_1 / (\sqrt{\widehat{v}_1})_j \leq 1/c$  in the algorithm, we have  $\|\alpha_t m_t / \sqrt{\widehat{v}_t}\| \leq \|m_t\| / c \leq H/c$ .

This completes the proof. **Q.E.D.**

### 7.2.4 Proof of Corollary 3.3

**Proof.** [Proof of Corollary 3.3]

#### Algorithm 5. AdaGrad with momentum

(S0). Define  $m_0 = 0, \widehat{v}_0 = 0$ ;  
 For  $t = 1, \dots, T$ , do  
   (S1).  $m_t = \beta_{1,t} m_{t-1} + (1 - \beta_{1,t}) g_t$   
   (S2).  $\widehat{v}_t = (1 - 1/t) \widehat{v}_{t-1} + (1/t) g_t^2$   
   (S3).  $x_{t+1} = x_t - \alpha_t m_t / \sqrt{\widehat{v}_t}$   
 End

The proof is similar to proof for Corollary 3.2, first let's bound RHS of (6) which is

$$E \left[ C_1 \sum_{t=1}^T \left\| \alpha_t g_t / \sqrt{\widehat{v}_t} \right\|^2 + C_2 \sum_{t=2}^T \left\| \frac{\alpha_t}{\sqrt{\widehat{v}_t}} - \frac{\alpha_{t-1}}{\sqrt{\widehat{v}_{t-1}}} \right\|_1 + C_3 \sum_{t=2}^{T-1} \left\| \frac{\alpha_t}{\sqrt{\widehat{v}_t}} - \frac{\alpha_{t-1}}{\sqrt{\widehat{v}_{t-1}}} \right\|^2 \right] + C_4$$

We recall from Table 1 that in AdaGrad,  $\widehat{v}_t = \frac{1}{t} \sum_{i=1}^t g_i^2$ . Thus, when  $\alpha_t = 1/\sqrt{t}$ , we obtain  $\alpha_t / \sqrt{\widehat{v}_t} = 1 / \sum_{i=1}^t g_i^2$ . We assume  $\min_{j \in [d]} |(g_1)_j| \geq c > 0$ , which is equivalent to  $\min_{j \in [d]} (\sqrt{\widehat{v}_1})_j \geq c > 0$  (a requirement of the AdaGrad). For  $C_1$  term we have

$$\begin{aligned} E \left[ \sum_{t=1}^T \left\| \alpha_t g_t / \sqrt{\widehat{v}_t} \right\|^2 \right] &= E \left[ \sum_{t=1}^T \left\| \frac{g_t}{\sqrt{\sum_{i=1}^t g_i^2}} \right\|^2 \right] = E \left[ \sum_{j=1}^d \sum_{t=1}^T \frac{(g_t)_j^2}{\sum_{i=1}^t (g_i)_j^2} \right] \\ &\leq E \left[ \sum_{j=1}^d \left( 1 - \log((g_1)_j^2) + \log \sum_{t=1}^T (g_t)_j^2 \right) \right] \leq d(1 - \log(c^2) + 2 \log H + \log T) \end{aligned}$$

where the third inequality used Lemma 7.9 and the last inequality used  $\|g_t\| \leq H$  and  $\min_{j \in [d]} |(g_1)_j| \geq c > 0$ .

For  $C_2$  term we have

$$\begin{aligned} E \left[ \sum_{t=2}^T \left\| \frac{\alpha_t}{\sqrt{\widehat{v}_t}} - \frac{\alpha_{t-1}}{\sqrt{\widehat{v}_{t-1}}} \right\|_1 \right] &= E \left[ \sum_{j=1}^d \sum_{t=2}^T \left( \frac{1}{\sqrt{\sum_{i=1}^{t-1} (g_i)_j^2}} - \frac{1}{\sqrt{\sum_{i=1}^t (g_i)_j^2}} \right) \right] \\ &= E \left[ \sum_{j=1}^d \left( \frac{1}{\sqrt{(g_1)_j^2}} - \frac{1}{\sqrt{\sum_{i=1}^T (g_i)_j^2}} \right) \right] \leq d/c \end{aligned}$$

For  $C_3$  term we have

$$\begin{aligned} &E \left[ \sum_{t=2}^{T-1} \left\| \frac{\alpha_t}{\sqrt{\widehat{v}_t}} - \frac{\alpha_{t-1}}{\sqrt{\widehat{v}_{t-1}}} \right\|^2 \right] \\ &\leq E \left[ \max_{t \in [T], t > 1} \left\| \frac{\alpha_t}{\sqrt{\widehat{v}_t}} - \frac{\alpha_{t-1}}{\sqrt{\widehat{v}_{t-1}}} \right\| \sum_{t=2}^{T-1} \left\| \frac{\alpha_t}{\sqrt{\widehat{v}_t}} - \frac{\alpha_{t-1}}{\sqrt{\widehat{v}_{t-1}}} \right\| \right] \\ &\leq (d/c) E \left[ \sum_{t=2}^{T-1} \left\| \frac{\alpha_t}{\sqrt{\widehat{v}_t}} - \frac{\alpha_{t-1}}{\sqrt{\widehat{v}_{t-1}}} \right\| \right] \leq (d/c)^2, \end{aligned}$$

where we have used the fact that  $\|\cdot\|_2 \leq \|\cdot\|_1$  and the upper bound on the  $C_2$  term.

Now we lower bound the effective stepsizes  $\alpha_t/(\sqrt{\widehat{v}_t})_j$ ,

$$\frac{\alpha_t}{(\sqrt{\widehat{v}_t})_j} = \frac{1}{\sqrt{\sum_{i=1}^t (g_i)_j^2}} \geq \frac{1}{H\sqrt{t}},$$

where we recall that  $\alpha_t = 1/\sqrt{t}$  and  $\|g_t\| \leq H$ . Following the same argument in the proof of Corollary 3.2 and the previously derived upper bounds, we have

$$\frac{\sqrt{T}}{H} \min_{t \in [T]} E [\|\nabla f(x_t)\|^2] \leq C_1 d(1 - \log(c^2) + 2 \log H + \log T) + C_2 d/c + C_3 (d/c)^2 + C_4$$

which yields

$$\begin{aligned} &\min_{t \in [T]} E [\|\nabla f(x_t)\|^2] \\ &\leq \frac{H}{\sqrt{T}} (C_1 d(1 - \log(c^2) + 2 \log H + \log T) + C_2 d/c + C_3 (d/c)^2 + C_4) \\ &= \frac{1}{\sqrt{T}} (Q'_1 + Q'_2 \log T) \end{aligned}$$

The last thing is to verify the assumption  $\|\alpha_t m_t / \sqrt{\widehat{v}_t}\| \leq G$  in Theorem 3.1, since  $\alpha_{t+1}/(\sqrt{\widehat{v}_{t+1}})_j \leq \alpha_t/(\sqrt{\widehat{v}_t})_j$  and  $\alpha_1/(\sqrt{\widehat{v}_1})_j \leq 1/c$  in the algorithm, we have  $\|\alpha_t m_t / \sqrt{\widehat{v}_t}\| \leq \|m_t\|/c \leq H/c$ .

This completes the proof. **Q.E.D.**

**Lemma 7.9.** For  $a_t \geq 0$  and  $\sum_{i=1}^t a_i \neq 0$ , we have

$$\sum_{t=1}^T \frac{a_t}{\sum_{i=1}^t a_i} \leq 1 - \log a_1 + \log \sum_{i=1}^T a_i.$$

**Proof.** [Proof of Lemma 7.9] We will prove it by induction. Suppose

$$\sum_{t=1}^{T-1} \frac{a_t}{\sum_{i=1}^t a_i} \leq 1 - \log a_1 + \log \sum_{i=1}^{T-1} a_i,$$

we have

$$\sum_{t=1}^T \frac{a_t}{\sum_{i=1}^t a_i} = \frac{a_T}{\sum_{i=1}^T a_i} + \sum_{t=1}^{T-1} \frac{a_t}{\sum_{i=1}^t a_i} \leq \frac{a_T}{\sum_{i=1}^T a_i} + 1 - \log a_1 + \log \sum_{i=1}^{T-1} a_i.$$

Applying the definition of concavity to  $\log(x)$ , with  $f(z) \triangleq \log(z)$ , we have  $f(z) \leq f(z_0) + f'(z_0)(z - z_0)$ , then substitute  $z = x - b$ ,  $z_0 = x$ , we have  $f(x - b) \leq f(x) + f'(x)(-b)$  which is equivalent to  $\log(x) \geq \log(x - b) + b/x$  for  $b < x$ , using  $x = \sum_{i=1}^T a_i$ ,  $b = a_T$ , we have

$$\log \sum_{i=1}^T a_i \geq \log \sum_{i=1}^{T-1} a_i + \frac{a_T}{\sum_{i=1}^T a_i}$$

and then

$$\sum_{t=1}^T \frac{a_t}{\sum_{i=1}^t a_i} \leq \frac{a_T}{\sum_{i=1}^T a_i} + 1 - \log a_1 + \log \sum_{i=1}^{T-1} a_i \leq 1 - \log a_1 + \log \sum_{i=1}^T a_i.$$

Now it remains to check first iteration. We have

$$\frac{a_1}{a_1} = 1 \leq 1 - \log(a_1) + \log(a_1) = 1$$

This completes the proof. **Q.E.D.**

### 7.3 Convergence analysis for Incremental Generalized Adam (Algorithm 2)

In this section, we provide the main convergence analysis for Algorithm 2. We will first give several lemmas prior to proving Theorem 4.1.

#### 7.3.1 Proof of Auxiliary Lemmas

**Lemma 7.10.** *For Algorithm 2, we have*

$$\sum_{t=1}^T f(x_{t+1}) - f(x_t) \leq T_1 + T_2 + T_3$$

where

$$T_1 = \sum_{t=1}^T - \left\langle \nabla f(x_t), \frac{\alpha_t^1}{\sqrt{\hat{v}_t^1}} \odot \sum_{i=1}^n m_t^i \right\rangle \quad (53)$$

$$T_2 = nH^2 \sum_{t=1}^T \sum_{i=1}^n \left\| \frac{\alpha_t^{i+1}}{\sqrt{\hat{v}_t^{i+1}}} - \frac{\alpha_t^i}{\sqrt{\hat{v}_t^i}} \right\|_1 \quad (54)$$

$$T_3 = \frac{L}{2} n \sum_{t=1}^T \sum_{i=1}^n \|\alpha_t^i m_t^i / \sqrt{\hat{v}_t^i}\|^2. \quad (55)$$

**Proof.** [Proof of Lemma 7.10] Recall that  $\beta_{1,t} = \beta_1$  for all  $t$ . First by the smoothness of  $f$ , we obtain

$$f(x_{t+1}) \leq f(x_t) + \langle \nabla f(x_t), d_t \rangle + \frac{L}{2} \|d_t\|^2 \quad (56)$$

where

$$d_t := x_{t+1} - x_t = - \sum_{i=1}^n \frac{\alpha_t^i}{\sqrt{\hat{v}_t^i}} \odot m_t^i + \frac{\alpha_t^i}{\sqrt{\hat{v}_t^i}}. \quad (57)$$



By definition of  $d_t$ , we have

$$\begin{aligned} d_t &= - \sum_{i=1}^n \alpha_t^i m_t^i / \sqrt{\hat{v}_t^i} \\ &= - \sum_{i=1}^n \frac{\alpha_t^1}{\sqrt{\hat{v}_t^1}} \odot m_t^i - \sum_{i=1}^n \left( \frac{\alpha_t^i}{\sqrt{\hat{v}_t^i}} - \frac{\alpha_t^1}{\sqrt{\hat{v}_t^1}} \right) \odot m_t^i \end{aligned}$$

and

$$\begin{aligned} & \sum_{t=1}^T (f(x_{t+1}) - f(x_t)) \\ & \leq \sum_{t=1}^T \left( \langle \nabla f(x_t), d_t \rangle + \frac{L}{2} \|d_t\|^2 \right) \\ & = \sum_{t=1}^T - \left\langle \nabla f(x_t), \frac{\alpha_t^1}{\sqrt{\hat{v}_t^1}} \odot \sum_{i=1}^n m_t^i \right\rangle - \sum_{t=1}^T \left\langle \nabla f(x_t), \sum_{i=1}^n \left( \frac{\alpha_t^i}{\sqrt{\hat{v}_t^i}} - \frac{\alpha_t^1}{\sqrt{\hat{v}_t^1}} \right) \odot m_t^i \right\rangle + \sum_{t=1}^T \frac{L}{2} \|d_t\|^2 \\ & \leq \sum_{t=1}^T - \left\langle \nabla f(x_t), \frac{\alpha_t^1}{\sqrt{\hat{v}_t^1}} \odot \sum_{i=1}^n m_t^i \right\rangle + \sum_{t=1}^T H^2 \sum_{i=1}^n \left\| \frac{\alpha_t^i}{\sqrt{\hat{v}_t^i}} - \frac{\alpha_t^1}{\sqrt{\hat{v}_t^1}} \right\|_1 + \sum_{t=1}^T \frac{L}{2} \|d_t\|^2 \\ & \leq \sum_{t=1}^T - \left\langle \nabla f(x_t), \frac{\alpha_t^1}{\sqrt{\hat{v}_t^1}} \odot \sum_{i=1}^n m_t^i \right\rangle + \sum_{t=1}^T n H^2 \sum_{i=1}^n \left\| \frac{\alpha_t^{i+1}}{\sqrt{\hat{v}_t^{i+1}}} - \frac{\alpha_t^i}{\sqrt{\hat{v}_t^i}} \right\|_1 + \sum_{t=1}^T \frac{L}{2} n \sum_{i=1}^n \|\alpha_t^i m_t^i / \sqrt{\hat{v}_t^i}\|^2 \\ & = T_1 + T_2 + T_3. \end{aligned}$$

where the second inequality holds due to  $\|\nabla f(x_t)\| \leq H$  and  $\|m_t^i\| \leq H$  that follows the same argument in (34), and the last inequality is due to  $\sum_{i=1}^n |a_t^i - a_t^1| \leq \sum_{i=1}^n \sum_{j=1}^{i-1} |a_t^{j+1} - a_t^j| \leq \sum_{i=1}^n \sum_{j=1}^n |a_t^{j+1} - a_t^j| \leq n \sum_{j=1}^n |a_t^{j+1} - a_t^j|$  for any sequence of  $\{a_t^i\}_{i=1}^n$ .

This completes the proof. **Q.E.D.**

We now give a useful lemma that will be used to bound both  $T_1$  and  $T_3$ .

**Lemma 7.11.** *For Algorithm 2, we have*

$$\begin{aligned} \sum_{t=1}^T \sum_{j=1}^n \left\| \alpha_t^j m_t^j / \sqrt{\hat{v}_t^j} \right\|^2 & \leq 2 \sum_{t=1}^T \sum_{i=1}^n \left\| \frac{\alpha_t^i}{\sqrt{\hat{v}_t^i}} \odot \nabla f_i(x_t^i) \right\|^2 \\ & \quad + 2H^2 \left( \frac{\beta_1}{1 - \beta_1} \right)^2 \sum_{t=1}^T \sum_{i=1}^n \left\| \frac{\alpha_t^{i+1}}{\sqrt{\hat{v}_t^{i+1}}} - \frac{\alpha_t^i}{\sqrt{\hat{v}_t^i}} \right\|^2. \end{aligned} \quad (58)$$

**Proof.** [Proof of Lemma 7.11] For  $1 \leq i \leq n$ , define  $g_{n(t-1)+i} = \nabla f_i(x_t^i)$ ,  $\alpha_{n(t-1)+i} = \alpha_t^i$ ,  $v_{n(t-1)+i} = v_t^i$ ,  $m_{n(t-1)+i} = m_t^i$ , it is easy to verify that

$$\begin{aligned} m_{n(t-1)+i} &= m_t^i = (1 - \beta_1) \left( \sum_{j=1}^i \beta_1^{i-j} \nabla f_j(x_t^j) + \sum_{u=1}^{t-1} \sum_{j=1}^n \beta_1^{n(t-u-1)+n-j+i} \nabla f_j(x_u^j) \right) \\ &= (1 - \beta_1) \sum_{k=1}^{n(t-1)+i} \beta_1^{n(t-1)+i-k} g_k, \end{aligned}$$

and thus

$$\begin{aligned}
& \sum_{t=1}^T \sum_{j=1}^n \left\| \alpha_t^j m_t^j / \sqrt{\widehat{v}_t^j} \right\|^2 \\
&= \sum_{q=1}^{nT} \left\| \frac{\alpha_q}{\sqrt{\widehat{v}_q}} \odot (1 - \beta_1) \sum_{k=1}^q \beta_1^{q-k} g_k \right\|^2 \\
&= (1 - \beta_1)^2 \sum_{q=1}^{nT} \left\| \sum_{k=1}^q \frac{\alpha_k}{\sqrt{\widehat{v}_k}} \odot \beta_1^{q-k} g_k + \sum_{k=1}^q \left( \frac{\alpha_q}{\sqrt{\widehat{v}_q}} - \frac{\alpha_k}{\sqrt{\widehat{v}_k}} \right) \odot \beta_1^{q-k} g_k \right\|^2 \\
&\leq 2(1 - \beta_1)^2 \left( \underbrace{\sum_{q=1}^{nT} \left\| \sum_{k=1}^q \frac{\alpha_k}{\sqrt{\widehat{v}_k}} \odot \beta_1^{q-k} g_k \right\|^2}_{D_1} + \underbrace{\sum_{q=1}^{nT} \left\| \sum_{k=1}^q \left( \frac{\alpha_q}{\sqrt{\widehat{v}_q}} - \frac{\alpha_k}{\sqrt{\widehat{v}_k}} \right) \odot \beta_1^{q-k} g_k \right\|^2}_{D_2} \right). \quad (59)
\end{aligned}$$

We next bound  $D_1$  and  $D_2$  in (59). For  $D_1$  we have

$$\begin{aligned}
D_1 &= \sum_{q=1}^{nT} \left\| \sum_{k=1}^q \frac{\alpha_k}{\sqrt{\widehat{v}_k}} \odot \beta_1^{q-k} g_k \right\|^2 = \sum_{q=1}^{nT} \sum_{k=1}^q \sum_{l=1}^q \left( \beta_1^{q-k} \frac{\alpha_k}{\sqrt{\widehat{v}_k}} \odot g_k \right)^T \beta_1^{q-l} \frac{\alpha_l}{\sqrt{\widehat{v}_l}} \odot g_l \\
&\stackrel{(a)}{\leq} \sum_{q=1}^{nT} \sum_{k=1}^q \sum_{l=1}^q \beta_1^{q-k} \beta_1^{q-l} \frac{1}{2} \left( \left\| \frac{\alpha_k}{\sqrt{\widehat{v}_k}} \odot g_k \right\|^2 + \left\| \frac{\alpha_l}{\sqrt{\widehat{v}_l}} \odot g_l \right\|^2 \right) \\
&\stackrel{(b)}{=} \sum_{q=1}^{nT} \sum_{k=1}^q \sum_{l=1}^q \beta_1^{q-k} \beta_1^{q-l} \left\| \frac{\alpha_k}{\sqrt{\widehat{v}_k}} \odot g_k \right\|^2 \leq \sum_{q=1}^{nT} \frac{1}{1 - \beta_1} \sum_{k=1}^q \beta_1^{q-k} \left\| \frac{\alpha_k}{\sqrt{\widehat{v}_k}} \odot g_k \right\|^2 \\
&\stackrel{(c)}{=} \frac{1}{1 - \beta_1} \sum_{k=1}^{nT} \sum_{q=k}^{nT} \beta_1^{q-k} \left\| \frac{\alpha_k}{\sqrt{\widehat{v}_k}} \odot g_k \right\|^2 \leq \left( \frac{1}{1 - \beta_1} \right)^2 \sum_{k=1}^{nT} \left\| \frac{\alpha_k}{\sqrt{\widehat{v}_k}} \odot g_k \right\|^2 \quad (60)
\end{aligned}$$

where (a) is due to  $\langle a, b \rangle \leq \frac{1}{2}(\|a\|^2 + \|b\|^2)$ , (b) is due to symmetry of  $k$  and  $l$  in summation, (c) is by changing order of summation. For  $D_2$  we have

$$\begin{aligned}
D_2 &= \sum_{q=1}^{nT} \left\| \sum_{k=1}^q \beta_1^{q-k} \left( \frac{\alpha_q}{\sqrt{\widehat{v}_q}} - \frac{\alpha_k}{\sqrt{\widehat{v}_k}} \right) \odot g_k \right\|^2 \\
&= \sum_{q=1}^{nT} \left\| \sum_{k=1}^q \beta_1^{q-k} \sum_{r=k+1}^q \left( \frac{\alpha_r}{\sqrt{\widehat{v}_r}} - \frac{\alpha_{r-1}}{\sqrt{\widehat{v}_{r-1}}} \right) \odot g_k \right\|^2 \\
&\leq \sum_{q=1}^{nT} \sum_{k=1}^q \sum_{r=k+1}^q \sum_{l=1}^q \sum_{s=l+1}^q \beta_1^{q-k} \beta_1^{q-l} \left( \left( \frac{\alpha_r}{\sqrt{\widehat{v}_r}} - \frac{\alpha_{r-1}}{\sqrt{\widehat{v}_{r-1}}} \right) \odot g_k \right)^T \left( \frac{\alpha_s}{\sqrt{\widehat{v}_s}} - \frac{\alpha_{s-1}}{\sqrt{\widehat{v}_{s-1}}} \right) \odot g_l \\
&\leq \sum_{q=1}^{nT} \sum_{k=1}^q \sum_{r=k+1}^q \sum_{l=1}^q \sum_{s=l+1}^q \beta_1^{q-k} \beta_1^{q-l} \frac{1}{2} \left( \left\| \left( \frac{\alpha_r}{\sqrt{\widehat{v}_r}} - \frac{\alpha_{r-1}}{\sqrt{\widehat{v}_{r-1}}} \right) \odot g_k \right\|^2 + \left\| \left( \frac{\alpha_s}{\sqrt{\widehat{v}_s}} - \frac{\alpha_{s-1}}{\sqrt{\widehat{v}_{s-1}}} \right) \odot g_l \right\|^2 \right) \\
&= \sum_{q=1}^{nT} \sum_{k=1}^q \sum_{r=k+1}^q \sum_{l=1}^q \sum_{s=l+1}^q \beta_1^{q-k} \beta_1^{q-l} \left\| \left( \frac{\alpha_r}{\sqrt{\widehat{v}_r}} - \frac{\alpha_{r-1}}{\sqrt{\widehat{v}_{r-1}}} \right) \odot g_k \right\|^2 \quad (61)
\end{aligned}$$

where the last inequality is due to  $\langle a, b \rangle \leq \frac{1}{2}(\|a\|^2 + \|b\|^2)$ , the last equality is due to symmetry of  $k, r$  and  $l, s$  in summation. Further, we have

$$\begin{aligned}
D_2 &\leq \sum_{q=1}^{nT} \sum_{k=1}^q \sum_{r=k+1}^q \beta_1^{q-k} H^2 \left\| \left( \frac{\alpha_r}{\sqrt{\widehat{v}_r}} - \frac{\alpha_{r-1}}{\sqrt{\widehat{v}_{r-1}}} \right) \right\|^2 \sum_{l=1}^q \sum_{s=l+1}^q \beta_1^{q-l} \\
&\leq \frac{\beta_1}{(1-\beta_1)^2} H^2 \sum_{q=1}^{nT} \sum_{k=1}^q \sum_{r=k+1}^q \beta_1^{q-k} \left\| \left( \frac{\alpha_r}{\sqrt{\widehat{v}_r}} - \frac{\alpha_{r-1}}{\sqrt{\widehat{v}_{r-1}}} \right) \right\|^2 \\
&= \frac{\beta_1}{(1-\beta_1)^2} H^2 \sum_{q=1}^{nT} \sum_{r=2}^q \sum_{k=1}^{r-1} \beta_1^{q-k} \left\| \left( \frac{\alpha_r}{\sqrt{\widehat{v}_r}} - \frac{\alpha_{r-1}}{\sqrt{\widehat{v}_{r-1}}} \right) \right\|^2 \\
&= \frac{\beta_1}{(1-\beta_1)^2} H^2 \sum_{r=2}^{nT} \sum_{q=r}^{nT} \sum_{k=1}^{r-1} \beta_1^{q-k} \left\| \left( \frac{\alpha_r}{\sqrt{\widehat{v}_r}} - \frac{\alpha_{r-1}}{\sqrt{\widehat{v}_{r-1}}} \right) \right\|^2 \\
&\leq H^2 \left( \frac{1}{1-\beta_1} \right)^2 \left( \frac{\beta_1}{1-\beta_1} \right)^2 \sum_{r=2}^{nT} \left\| \frac{\alpha_r}{\sqrt{\widehat{v}_r}} - \frac{\alpha_{r-1}}{\sqrt{\widehat{v}_{r-1}}} \right\|^2
\end{aligned} \tag{62}$$

where the second inequality is due to  $\sum_{l=1}^q \sum_{s=l+1}^q \beta_1^{q-l} \leq \frac{\beta_1}{(1-\beta_1)^2}$  and the last inequality is due to  $\sum_{q=r}^{nT} \sum_{k=1}^{r-1} \beta_1^{q-k} \leq \frac{\beta_1}{(1-\beta_1)^2}$ . Substituting (60) and (61) into (59), we then obtain (58)

This completes the proof. **Q.E.D.**

In the following we will bound  $T_1$ .

**Lemma 7.12.** *For  $T_1$ , we have*

$$\begin{aligned}
T_1 &\leq G_1 \sum_{t=1}^T \sum_{i=1}^n \left\| \frac{\alpha_t^i}{\sqrt{\widehat{v}_t^i}} \odot \nabla f_i(x_t^i) \right\|^2 + G_2 \sum_{t=1}^T \sum_{i=1}^n \left\| \frac{\alpha_t^{i+1}}{\sqrt{\widehat{v}_t^{i+1}}} - \frac{\alpha_t^i}{\sqrt{\widehat{v}_t^i}} \right\|^2 + G_3 \\
&\quad - (1-\beta_1) \sum_{t=1}^T \left\langle \nabla f(x_t), \frac{\alpha_t^1}{\sqrt{\widehat{v}_t^1}} \odot \nabla f(x_t) \right\rangle
\end{aligned}$$

with

$$\begin{aligned}
G_1 &= 14n^3 L^2 \left( \frac{1}{1-\beta_1} \right)^5 + 2(n^2 M^2 L^2 + n) \\
G_2 &= 14n^3 L^2 H^2 \left( \frac{1}{1-\beta_1} \right)^5 + 2 \left( n^2 M^2 L^2 \left( \frac{\beta_1}{1-\beta_1} \right)^2 + n^2 \right) H^2 \\
G_3 &= M^2 H^2 + \frac{1}{1-\beta_1^n} n M H^2
\end{aligned}$$

**Proof.** [Proof of Lemma 7.12]

By expanding recursive definition of  $m_t^i$ , it is easy to verify that

$$\begin{aligned}
& \sum_{i=1}^n m_t^i - \beta^n \sum_{i=1}^n m_{t-1}^i \\
&= \sum_{i=1}^n (1 - \beta_1) \left( \sum_{j=1}^i \beta_1^{i-j} \nabla f_j(x_t^j) + \sum_{j=i+1}^n \beta_1^{i+n-j} \nabla f_j(x_{t-1}^j) \right) \\
&= \sum_{i=1}^n (1 - \beta_1) \left( \sum_{k=0}^{i-1} \beta_1^k \nabla f_{i-k}(x_t^{i-k}) + \sum_{k=i-n}^{-1} \beta_1^{n+k} \nabla f_{i-k}(x_{t-1}^{i-k}) \right) \\
&= \sum_{i=1}^n (1 - \beta_1) \left( \sum_{k=0}^{i-1} \beta_1^k \nabla f_{i-k}(x_t^{i-k}) + \sum_{l=i}^{n-1} \beta_1^l \nabla f_{i+n-l}(x_{t-1}^{i+n-l}) \right) \\
&= (1 - \beta_1) \left( \sum_{k=0}^{n-1} \sum_{i=k+1}^n \beta_1^k \nabla f_{i-k}(x_t^{i-k}) + \sum_{l=1}^{n-1} \sum_{i=1}^l \beta_1^l \nabla f_{i+n-l}(x_{t-1}^{i+n-l}) \right) \\
&= (1 - \beta_1) \left( \sum_{k=0}^{n-1} \beta_1^k \sum_{i=k+1}^n (\nabla f_{i-k}(x_t^{i-k}) - \nabla f_{i-k}(x_t)) \right) \\
&\quad + (1 - \beta_1) \left( \sum_{k=1}^{n-1} \beta_1^k \sum_{i=1}^k (\nabla f_{i+n-k}(x_{t-1}^{i+n-k}) - \nabla f_{i+n-k}(x_t)) \right) \\
&\quad + (1 - \beta_1) \sum_{k=0}^{n-1} \beta_1^k \sum_{i=1}^n \nabla f_i(x_t)
\end{aligned} \tag{63}$$

where the first equality is by rolling out  $m_t^i$  for  $n$  iterations, the second equality is due to  $k = i - j$  and the third equality is due to  $l = k + n$ , the forth equality is by changing order of summation, the fifth equality holds by letting  $l = k$ .

By recursively applying the above equality, and using the fact that  $\nabla f(x_t) = \sum_{i=1}^n \nabla f_i(x_t)$ , we can then obtain that

$$\begin{aligned}
\sum_{i=1}^n m_t^i &= (1 - \beta_1) \sum_{u=0}^{t-2} \beta_1^{nu} \left( \sum_{k=0}^{n-1} \beta_1^k \sum_{i=k+1}^n (\nabla f_{i-k}(x_{t-u}^{i-k}) - \nabla f_{i-k}(x_t)) \right) \\
&\quad + (1 - \beta_1) \sum_{u=0}^{t-2} \beta_1^{nu} \left( \sum_{k=1}^{n-1} \beta_1^k \sum_{i=1}^k (\nabla f_{i+n-k}(x_{t-1-u}^{i+n-k}) - \nabla f_{i+n-k}(x_t)) \right) \\
&\quad + (1 - \beta_1) \left( \sum_{u=0}^{t-2} \beta_1^{nu} \sum_{k=0}^{n-1} \beta_1^k \nabla f(x_t) \right) + \beta_1^{n(t-1)} \sum_{i=1}^n m_1^i
\end{aligned} \tag{64}$$

Substituting (64) into (53), we obtain that

$$T_1 = \sum_{t=1}^T - \left\langle \nabla f(x_t), \frac{\alpha_t^1}{\sqrt{\hat{v}_t^1}} \odot \sum_{i=1}^n m_t^i \right\rangle = D_1 + D_2 + D_3 + D_4, \tag{65}$$

where

$$\begin{aligned}
D_1 &= - \sum_{t=1}^T \left\langle \nabla f(x_t), \frac{\alpha_t^1}{\sqrt{\hat{v}_t^1}} \odot (1 - \beta_1) \sum_{u=0}^{t-2} \beta_1^{nu} \left( \sum_{k=0}^{n-1} \beta_1^k \sum_{i=k+1}^n (\nabla f_{i-k}(x_{t-u}^{i-k}) - \nabla f_{i-k}(x_t)) \right) \right\rangle, \\
D_2 &= - \sum_{t=1}^T \left\langle \nabla f(x_t), \frac{\alpha_t^1}{\sqrt{\hat{v}_t^1}} \odot (1 - \beta_1) \sum_{u=0}^{t-2} \beta_1^{nu} \left( \sum_{k=1}^{n-1} \beta_1^k \sum_{i=1}^k (\nabla f_{i+n-k}(x_{t-1-u}^{i+n-k}) - \nabla f_{i+n-k}(x_t)) \right) \right\rangle, \\
D_3 &= - \sum_{t=1}^T \left\langle \nabla f(x_t), \frac{\alpha_t^1}{\sqrt{\hat{v}_t^1}} \odot (1 - \beta_1) \left( \sum_{u=0}^{t-2} \beta_1^{nu} \sum_{k=0}^{n-1} \nabla f(x_t) \right) \right\rangle,
\end{aligned}$$

$$D_4 = - \sum_{t=1}^T \left\langle \nabla f(x_t), \frac{\alpha_t^1}{\sqrt{\widehat{v}_t^1}} \odot \beta_1^{n(t-1)} \sum_{i=1}^n m_1^i \right\rangle.$$

In (65), we further bound  $\{D_i\}_{i=1}^4$ . Applying the inequality  $\langle a, b \rangle \leq \frac{1}{2}(\|a\|^2 + \|b\|^2)$  to  $D_1 + D_2$ , we obtain that

$$\begin{aligned} D_1 + D_2 \leq & \frac{1}{2}(1 - \beta_1) \sum_{t=1}^T \left\| \alpha_t^1 \nabla f(x_t) / \sqrt{\widehat{v}_t^1} \right\|^2 \\ & + \frac{1}{2}(1 - \beta_1) \sum_{t=1}^T \left\| \sum_{u=0}^{t-2} \beta_1^{nu} \left( \sum_{k=0}^{n-1} \beta_1^k \sum_{i=k+1}^n (\nabla f_{i-k}(x_{t-u}^{i-k}) - \nabla f_{i-k}(x_t)) \right. \right. \\ & \left. \left. + \sum_{k=1}^{n-1} \beta_1^k \sum_{i=1}^k (\nabla f_{i+n-k}(x_{t-1-u}^{i+n-k}) - \nabla f_{i+n-k}(x_t)) \right) \right\|^2. \end{aligned} \quad (66)$$

Since  $\sum_{u=0}^{t-2} \beta_1^{nu} = 1 + \sum_{u=1}^{t-2} \beta_1^{nu} \geq 1$  (given  $t \geq 2$  without loss of generality), we bound  $D_3$  as below

$$D_3 \leq -(1 - \beta_1) \sum_{t=1}^T \left\langle \nabla f(x_t), \frac{\alpha_t^1}{\sqrt{\widehat{v}_t^1}} \odot \nabla f(x_t) \right\rangle. \quad (67)$$

Next, given the conditions that  $\|\nabla f(x_t)\| \leq H$ ,  $\alpha_t^1 / \sqrt{(\widehat{v}_t^1)_j} \leq M$ ,  $\|m_1^i\| \leq H$  and  $\sum_{t=1}^T \beta_1^{n(t-1)} \leq \frac{1}{1-\beta_1^n}$ , we have

$$D_4 = - \sum_{t=1}^T \beta_1^{n(t-1)} \left\langle \nabla f(x_t), \frac{\alpha_t^1}{\sqrt{\widehat{v}_t^1}} \odot \sum_{i=1}^n m_1^i \right\rangle \leq \sum_{t=1}^T \beta_1^{n(t-1)} n M H^2 \leq \frac{1}{1-\beta_1^n} n M H^2 \quad (68)$$

Substituting (66)-(68) into (65), we obtain that

$$\begin{aligned} T_1 = & D_1 + D_2 + D_3 + D_4 \\ \leq & \frac{1}{2}(1 - \beta_1)(T_4 + T_5) - (1 - \beta_1) \sum_{t=1}^T \left\langle \nabla f(x_t), \frac{\alpha_t^1}{\sqrt{\widehat{v}_t^1}} \odot \nabla f(x_t) \right\rangle + \frac{1}{1-\beta_1^n} n M H^2 \end{aligned} \quad (69)$$

where we define  $T_4$  and  $T_5$  as below

$$T_4 = \sum_{t=1}^T \left\| \alpha_t^1 \nabla f(x_t) / \sqrt{\widehat{v}_t^1} \right\|^2 \quad (70)$$

$$\begin{aligned} T_5 = & \sum_{t=1}^T \left\| \sum_{u=0}^{t-2} \beta_1^{nu} \left( \sum_{k=0}^{n-1} \beta_1^k \sum_{i=k+1}^n (\nabla f_{i-k}(x_{t-u}^{i-k}) - \nabla f_{i-k}(x_t)) \right. \right. \\ & \left. \left. + \sum_{k=1}^{n-1} \beta_1^k \sum_{i=1}^k (\nabla f_{i+n-k}(x_{t-1-u}^{i+n-k}) - \nabla f_{i+n-k}(x_t)) \right) \right\|^2. \end{aligned} \quad (71)$$

We will bound  $T_4$  and  $T_5$  in the next. Let us first study  $T_5$ . Upon defining

$$\begin{aligned} e(t, u) = & \sum_{k=0}^{n-1} \beta_1^k \sum_{i=k+1}^n (\nabla f_{i-k}(x_{t-u}^{i-k}) - \nabla f_{i-k}(x_t)) \\ & + \sum_{k=1}^{n-1} \beta_1^k \sum_{i=1}^k (\nabla f_{i+n-k}(x_{t-1-u}^{i+n-k}) - \nabla f_{i+n-k}(x_t)), \end{aligned}$$

then we have

$$\begin{aligned}
T_5 &= \sum_{t=1}^T \left\| \sum_{u=0}^{t-2} \beta_1^{nu} e(t, u) \right\|^2 = \sum_{t=1}^T \sum_{u=0}^{t-2} \sum_{w=0}^{t-2} \beta_1^{nu} \beta_1^{nw} e(t, u)^T e(t, w) \\
&\leq \sum_{t=1}^T \sum_{u=0}^{t-2} \sum_{w=0}^{t-2} \beta_1^{nu} \beta_1^{nw} \frac{1}{2} (\|e(t, u)\|^2 + \|e(t, w)\|^2) \\
&= \sum_{t=1}^T \sum_{u=0}^{t-2} \sum_{w=0}^{t-2} \beta_1^{nu} \beta_1^{nw} (\|e(t, u)\|^2) \leq \frac{1}{1 - \beta_1^n} \sum_{t=1}^T \sum_{u=0}^{t-2} \beta_1^{nu} \|e(t, u)\|^2
\end{aligned} \tag{72}$$

In (72), we further bound  $\|e(t, u)\|^2$

$$\begin{aligned}
&\|e(t, u)\|^2 \\
&\leq 2 \left\| \sum_{k=0}^{n-1} \beta_1^k \sum_{i=k+1}^n (\nabla f_{i-k}(x_{t-u}^{i-k}) - \nabla f_{i-k}(x_t)) \right\|^2 \\
&\quad + 2 \left\| \sum_{k=1}^{n-1} \beta_1^k \sum_{i=1}^k (\nabla f_{i+n-k}(x_{t-1-u}^{i+n-k}) - \nabla f_{i+n-k}(x_t)) \right\|^2 \\
&\leq 2 \frac{1}{1 - \beta_1} \sum_{k=0}^{n-1} \beta_1^k \left\| \sum_{i=k+1}^n (\nabla f_{i-k}(x_{t-u}^{i-k}) - \nabla f_{i-k}(x_t)) \right\|^2 \\
&\quad + 2 \frac{1}{1 - \beta_1} \sum_{k=1}^{n-1} \beta_1^k \left\| \sum_{i=1}^k (\nabla f_{i+n-k}(x_{t-1-u}^{i+n-k}) - \nabla f_{i+n-k}(x_t)) \right\|^2 \\
&\leq 2n \frac{1}{1 - \beta_1} \sum_{k=0}^{n-1} \beta_1^k \sum_{i=k+1}^n \|\nabla f_{i-k}(x_{t-u}^{i-k}) - \nabla f_{i-k}(x_t)\|^2 \\
&\quad + 2n \frac{1}{1 - \beta_1} \sum_{k=1}^{n-1} \beta_1^k \sum_{i=1}^k \|\nabla f_{i+n-k}(x_{t-1-u}^{i+n-k}) - \nabla f_{i+n-k}(x_t)\|^2,
\end{aligned} \tag{73}$$

where the second inequality used the facts that  $\sum_k \beta_1^k \leq \frac{1}{1 - \beta_1}$ , and  $\|\sum_k \beta_1^k s_k\|^2 \leq \frac{1}{1 - \beta_1} \|\sum_k \beta_1^k s_k\|$ .

Suppose  $p < t$ , we have

$$\begin{aligned}
&\|\nabla f_i(x_p^i) - \nabla f_i(x_t)\|^2 \leq L^2 \|x_p^i - x_t\|^2 = L^2 \|x_t - x_{p+1} + x_{p+1} - x_p^i\|^2 \\
&\stackrel{(a)}{=} L^2 \left\| \sum_{l=p+1}^{t-1} (x_{l+1} - x_l) + \sum_{j=i}^n (x_p^{j+1} - x_p^j) \right\|^2 \\
&\stackrel{(b)}{=} L^2 \left\| \sum_{l=p+1}^{t-1} (x_l^{n+1} - x_l^1) + \sum_{j=i}^n (x_p^{j+1} - x_p^j) \right\|^2 = L^2 \left\| \sum_{l=p+1}^{t-1} \sum_{j=1}^n (x_l^{j+1} - x_l^j) + \sum_{j=i}^n (x_p^{j+1} - x_p^j) \right\|^2 \\
&\stackrel{(c)}{=} L^2 \left\| \sum_{l=p+1}^{t-1} \sum_{j=1}^n \alpha_l^j m_l^j / \sqrt{\hat{v}_l^j} + \sum_{j=i}^n \alpha_p^j m_p^j / \sqrt{\hat{v}_p^j} \right\|^2 \\
&\stackrel{(d)}{=} L^2 \left\| \sum_{l=p+1}^{t-1} \sum_{j=1}^n \alpha_l^j m_l^j / \sqrt{\hat{v}_l^j} + \sum_{j=1}^n \tau_j \alpha_p^j m_p^j / \sqrt{\hat{v}_p^j} \right\|^2 \\
&\stackrel{(e)}{\leq} n(t-p) L^2 \sum_{l=p}^{t-1} \sum_{j=1}^n \left\| \alpha_l^j m_l^j / \sqrt{\hat{v}_l^j} \right\|^2 \leq n(t-p+1) L^2 \sum_{l=p}^t \sum_{j=1}^n \left\| \alpha_l^j m_l^j / \sqrt{\hat{v}_l^j} \right\|^2
\end{aligned} \tag{74}$$

where the equality (a) holds due to  $x_p^{n+1} = x_{p+1}$  from (S2) of Algorithm 2, the equality (b) holds due to  $x_{l+1} = x_l^{n+1}$  and  $x_l = x_l^1$  from (S1) and (S2) of Algorithm 2, the equality (c) holds due to (S1-3) of Algorithm 2, the equality (d) holds since we introduce a new variable  $\tau_j$  which is equal to 0 if  $j < i$ , and 1 otherwise, and the inequality (e) used the facts that  $\|\sum_{i=1}^n a_i\|^2 \leq n \sum_{i=1}^n \|a_i\|^2$  and  $\tau_j \leq 1$ .

Although we start from  $p < t$  in (74), it also holds for  $p = t$  since  $\|x_t^i - x_t\|^2 = \|\sum_{j=1}^{i-1} \alpha_t^j m_t^j / \sqrt{\widehat{v}_t^j}\|^2 \leq n \sum_{j=1}^n \|\alpha_t^j m_t^j / \sqrt{\widehat{v}_t^j}\|^2$ . Substituting (74) into (73), we have

$$\begin{aligned}
& \|e(t, u)\|^2 \\
& \leq 2n \frac{1}{1 - \beta_1} \sum_{k=0}^{n-1} \beta_1^k \sum_{i=k+1}^n \|\nabla f_{i-k}(x_{t-u}^{i-k}) - \nabla f_{i-k}(x_t)\|^2 \\
& \quad + 2n \frac{1}{1 - \beta_1} \sum_{k=1}^{n-1} \beta_1^k \sum_{i=1}^k \|\nabla f_{i+n-k}(x_{t-1-u}^{i+n-k}) - \nabla f_{i+n-k}(x_t)\|^2 \\
& \leq 2n \frac{1}{1 - \beta_1} \sum_{k=0}^{n-1} \beta_1^k \sum_{i=k+1}^n n(u+1)L^2 \sum_{l=t-u}^t \sum_{j=1}^n \left\| \alpha_l^j m_l^j / \sqrt{\widehat{v}_l^j} \right\|^2 \\
& \quad + 2n \frac{1}{1 - \beta_1} \sum_{k=1}^{n-1} \beta_1^k \sum_{i=1}^k n(u+2)L^2 \sum_{l=t-1-u}^t \sum_{j=1}^n \left\| \alpha_l^j m_l^j / \sqrt{\widehat{v}_l^j} \right\|^2 \\
& \leq 2n \frac{1}{1 - \beta_1} \sum_{k=0}^{n-1} \beta_1^k \sum_{i=1}^n n(u+2)L^2 \sum_{l=t-1-u}^t \sum_{j=1}^n \left\| \alpha_l^j m_l^j / \sqrt{\widehat{v}_l^j} \right\|^2 \\
& \leq 2n^3 L^2 \left( \frac{1}{1 - \beta_1} \right)^2 (u+2) \sum_{l=t-1-u}^t \sum_{j=1}^n \left\| \alpha_l^j m_l^j / \sqrt{\widehat{v}_l^j} \right\|^2 \tag{75}
\end{aligned}$$

Substitute (75) into (72), we further have

$$\begin{aligned}
T_5 & \leq 2n^3 L^2 \frac{1}{1 - \beta_1^n} \left( \frac{1}{1 - \beta_1} \right)^2 \sum_{t=1}^T \sum_{u=0}^{t-2} \beta_1^{nu} (u+2) \sum_{l=t-1-u}^t \sum_{j=1}^n \left\| \alpha_l^j m_l^j / \sqrt{\widehat{v}_l^j} \right\|^2 \\
& = 2n^3 L^2 \frac{1}{1 - \beta_1^n} \left( \frac{1}{1 - \beta_1} \right)^2 \sum_{t=1}^T \sum_{l=1}^t \sum_{u=\max(t-1-l, 0)}^{t-2} \sum_{j=1}^n \beta_1^{nu} (u+2) \left\| \alpha_l^j m_l^j / \sqrt{\widehat{v}_l^j} \right\|^2 \\
& = 2n^3 L^2 \frac{1}{1 - \beta_1^n} \left( \frac{1}{1 - \beta_1} \right)^2 \sum_{l=1}^T \sum_{t=l}^T \sum_{u=\max(t-1-l, 0)}^{t-2} \sum_{j=1}^n \beta_1^{nu} (u+2) \left\| \alpha_l^j m_l^j / \sqrt{\widehat{v}_l^j} \right\|^2 \\
& = 2n^3 L^2 \frac{1}{1 - \beta_1^n} \left( \frac{1}{1 - \beta_1} \right)^2 \sum_{l=1}^T \sum_{j=1}^n \left\| \alpha_l^j m_l^j / \sqrt{\widehat{v}_l^j} \right\|^2 \underbrace{\left( \sum_{t=l}^T \sum_{u=\max(t-1-l, 0)}^{t-2} \beta_1^{nu} (u+2) \right)}_{D_5} \tag{76}
\end{aligned}$$

where the first equality holds since in summation of indices  $u$  and  $l$ , the region formed by  $0 \leq u \leq t-2$  and  $t-1-u \leq l \leq t$  is equivalent to the region formed by  $\min_{0 \leq u \leq t-2} (t-1-u) \leq l \leq t$  and  $\max(0, t-1-l) \leq u \leq \min(t-2, t)$ , and the second equality holds since in summation of indices  $t$  and  $l$ , the region formed by  $1 \leq t \leq T$  and  $1 \leq l \leq t$  is equivalent to the region formed by  $1 \leq l \leq T$  and  $l \leq t \leq T$ .

In (76), we further bound  $D_5$  as below.

$$\begin{aligned}
D_5 &= \sum_{t=l}^T \sum_{u=\max(t-1-l,0)}^{t-2} \beta_1^{nu} (u+2) \\
&= \sum_{u=0}^{l-2} \beta_1^{nu} (u+2) + \sum_{t=l+1}^T \sum_{u=t-1-l}^{t-2} \beta_1^{nu} (u+2) \\
&\stackrel{(a)}{\leq} \left( \frac{1}{1-\beta_1^n} \right)^2 + 2 \frac{1}{1-\beta_1^n} + \sum_{t=l+1}^T \sum_{u=t-1-l}^{t-2} \beta_1^{nu} (u+2) \\
&= \left( \frac{1}{1-\beta_1^n} \right)^2 + 2 \frac{1}{1-\beta_1^n} + \sum_{t=l+1}^T \beta_1^{n(t-1-l)} \sum_{u=t-1-l}^{t-2} \beta_1^{n(u-(t-1-l))} (u+2) \\
&= \left( \frac{1}{1-\beta_1^n} \right)^2 + 2 \frac{1}{1-\beta_1^n} + \sum_{t=l+1}^T \beta_1^{n(t-1-l)} \sum_{q=0}^{l-1} \beta_1^{nq} (q+t-l+1) \\
&\stackrel{(b)}{\leq} \left( \frac{1}{1-\beta_1^n} \right)^2 + 2 \frac{1}{1-\beta_1^n} + \sum_{t=l+1}^T \beta_1^{n(t-1-l)} \left( \left( \frac{1}{1-\beta_1^n} \right)^2 + (t-l+1) \frac{1}{1-\beta_1^n} \right) \\
&\stackrel{(c)}{\leq} \left( \frac{1}{1-\beta_1^n} \right)^2 + 2 \frac{1}{1-\beta_1^n} + 2 \left( \frac{1}{1-\beta_1^n} \right)^3 + 2 \left( \frac{1}{1-\beta_1^n} \right)^2 \leq 7 \left( \frac{1}{1-\beta_1^n} \right)^3,
\end{aligned}$$

where the inequality (a) holds since  $\sum_{u=0}^{l-2} \beta_1^{nu} u = \sum_{u=0}^{l-2} \sum_{k=1}^u \beta_1^{nu} = \sum_{k=1}^{l-2} \sum_{u=k}^{l-2} \beta_1^{nu} \leq \sum_{k=1}^{l-2} \beta_1^k \frac{1}{1-\beta_1^n} \leq \left( \frac{1}{1-\beta_1^n} \right)^2$ , inequality (b) holds because  $\sum_{q=0}^{l-1} \beta_1^{nq} q \leq \left( \frac{1}{1-\beta_1^n} \right)^2$ , inequality (c) follows from  $\sum_{t=l+1}^T \beta_1^{n(t-1-l)} (t-l+1) = \sum_{k=1}^{T-l-1} \beta_1^{nk} k \leq \left( \frac{1}{1-\beta_1^n} \right)^2$ .

By (76) and the above upper bound on  $D_5$ , we have

$$\begin{aligned}
T_5 &= 2n^3 L^2 \frac{1}{1-\beta_1^n} \left( \frac{1}{1-\beta_1} \right)^2 \sum_{l=1}^T \sum_{j=1}^n \left\| \alpha_l^j m_l^j / \sqrt{\widehat{v}_l^j} \right\|^2 D_5 \\
&\leq 14n^3 L^2 \left( \frac{1}{1-\beta_1^n} \right)^4 \left( \frac{1}{1-\beta_1} \right)^2 \sum_{l=1}^T \sum_{j=1}^n \left\| \alpha_l^j m_l^j / \sqrt{\widehat{v}_l^j} \right\|^2.
\end{aligned} \tag{77}$$

Now let's turn to  $T_4$  in (70). Recall that  $\nabla f(x_t) = \sum_{i=1}^n \nabla f_i(x_t)$ , from (70) we have

$$\begin{aligned}
T_4 &= \sum_{t=1}^T \left\| \alpha_t^1 \nabla f(x_t) / \sqrt{\widehat{v}_t^1} \right\|^2 \leq \left\| \alpha_1^1 \nabla f(x_1) / \sqrt{\widehat{v}_1^1} \right\|^2 + \sum_{t=2}^T n \sum_{i=1}^n \left\| \alpha_t^1 \nabla f_i(x_t) / \sqrt{\widehat{v}_t^1} \right\|^2 \\
&\stackrel{(a)}{\leq} M^2 H^2 + \sum_{t=2}^T n \sum_{i=1}^n \left( 2 \left\| \alpha_t^1 \nabla f_i(x_{t-1}^i) / \sqrt{\widehat{v}_t^1} \right\|^2 + 2 \left\| \alpha_t^1 (\nabla f_i(x_t) - \nabla f_i(x_{t-1}^i)) / \sqrt{\widehat{v}_t^1} \right\|^2 \right) \\
&= M^2 H^2 + \sum_{t=2}^T n \sum_{i=1}^n \left( 2 \left\| \nabla f_i(x_{t-1}^i) \odot \left( \frac{\alpha_{t-1}^i}{\sqrt{\widehat{v}_{t-1}^i}} + \frac{\alpha_t^1}{\sqrt{\widehat{v}_t^1}} - \frac{\alpha_{t-1}^i}{\sqrt{\widehat{v}_{t-1}^i}} \right) \right\|^2 \right) \\
&\quad + \sum_{t=2}^T n \sum_{i=1}^n \left( 2 \left\| \alpha_t^1 (\nabla f_i(x_t) - \nabla f_i(x_{t-1}^i)) / \sqrt{\widehat{v}_t^1} \right\|^2 \right) \\
&\stackrel{(b)}{\leq} M^2 H^2 + 4n \sum_{t=2}^T \sum_{i=1}^n \left\| \alpha_{t-1}^i \nabla f_i(x_{t-1}^i) / \sqrt{\widehat{v}_{t-1}^i} \right\|^2 + 4n D_6 + 2n M^2 L^2 D_7,
\end{aligned} \tag{78}$$



where (a) holds since  $\alpha_t^i/(\sqrt{\hat{v}_t^i})_j \leq M$  and  $\nabla\|f(x_1)\| \leq H$ , (b) holds because  $\alpha_t^1/(\sqrt{\hat{v}_t^1})_j \leq M$  and  $\|\nabla f_i(x_t) - \nabla f_i(x_{t-1}^i)\| \leq L\|x_t - x_{t-1}^i\|$ , and  $D_6$  and  $D_7$  are defined by

$$D_6 = \sum_{t=2}^T \sum_{i=1}^n \left\| \left( \frac{\alpha_t^1}{\sqrt{\hat{v}_t^1}} - \frac{\alpha_{t-1}^i}{\sqrt{\hat{v}_{t-1}^i}} \right) \odot \nabla f_i(x_{t-1}^i) \right\|^2, \quad D_7 = \sum_{t=1}^T \sum_{i=1}^n \|x_t - x_{t-1}^i\|^2.$$

In (78), we next bound  $D_6$ . Let us denote  $\alpha_{t-1}^{n+1}/\sqrt{\hat{v}_{t-1}^{n+1}} \triangleq \alpha_t^1/\sqrt{\hat{v}_t^1}$ . We then have

$$\begin{aligned} D_6 &\leq \sum_{t=2}^T \sum_{i=1}^n H^2 \left\| \left( \frac{\alpha_t^1}{\sqrt{\hat{v}_t^1}} - \frac{\alpha_{t-1}^i}{\sqrt{\hat{v}_{t-1}^i}} \right) \right\|^2 \leq \sum_{t=2}^T \sum_{i=1}^n H^2 \left\| \sum_{j=i}^n \left( \frac{\alpha_{t-1}^{j+1}}{\sqrt{\hat{v}_{t-1}^{j+1}}} - \frac{\alpha_{t-1}^j}{\sqrt{\hat{v}_{t-1}^j}} \right) \right\|^2 \\ &\leq n H^2 \sum_{t=2}^T \sum_{i=1}^n \sum_{j=1}^n \left\| \left( \frac{\alpha_{t-1}^{j+1}}{\sqrt{\hat{v}_{t-1}^{j+1}}} - \frac{\alpha_{t-1}^j}{\sqrt{\hat{v}_{t-1}^j}} \right) \right\|^2 = n^2 H^2 \sum_{t=2}^T \sum_{j=1}^n \left\| \frac{\alpha_{t-1}^{j+1}}{\sqrt{\hat{v}_{t-1}^{j+1}}} - \frac{\alpha_{t-1}^j}{\sqrt{\hat{v}_{t-1}^j}} \right\|^2. \end{aligned} \quad (79)$$

Moreover, we bound  $D_7$  in (78) by

$$D_7 = \sum_{t=1}^T \sum_{i=1}^n \left\| \sum_{j=i+1}^n \alpha_t^j m_t^j / \sqrt{\hat{v}_t^j} \right\|^2 \leq n^2 \sum_{t=1}^T \sum_{j=1}^n \left\| \alpha_t^j m_t^j / \sqrt{\hat{v}_t^j} \right\|^2, \quad (80)$$

where the first equality holds due to (S1-3) of Algorithm 2.

Substituting (79), (80) and (58) into (78), we then obtain that

$$\begin{aligned} T_4 &\leq M^2 H^2 + 4n \sum_{t=2}^T \sum_{i=1}^n \left\| \alpha_{t-1}^i \nabla f_i(x_{t-1}^i) / \sqrt{\hat{v}_{t-1}^i} \right\|^2 + 4n^3 H^2 \sum_{t=1}^T \sum_{i=1}^n \left\| \frac{\alpha_t^{i+1}}{\sqrt{\hat{v}_t^{i+1}}} - \frac{\alpha_t^i}{\sqrt{\hat{v}_t^i}} \right\|^2 \\ &\quad + 4n^3 M^2 L^2 \left( \sum_{t=1}^T \sum_{i=1}^n \left\| \frac{\alpha_t^i}{\sqrt{\hat{v}_t^i}} \odot \nabla f_i(x_t^i) \right\|^2 + H^2 \left( \frac{\beta_1}{1-\beta_1} \right)^2 \sum_{t=1}^T \sum_{i=1}^n \left\| \frac{\alpha_t^{i+1}}{\sqrt{\hat{v}_t^{i+1}}} - \frac{\alpha_t^i}{\sqrt{\hat{v}_t^i}} \right\|^2 \right) \\ &\leq M^2 H^2 + 4(n^3 M^2 L^2 + n) \sum_{t=1}^T \sum_{i=1}^n \left\| \frac{\alpha_t^i}{\sqrt{\hat{v}_t^i}} \odot \nabla f_i(x_t^i) \right\|^2 \\ &\quad + 4 \left( n^3 M^2 L^2 \left( \frac{\beta_1}{1-\beta_1} \right)^2 + n^3 \right) H^2 \sum_{t=1}^T \sum_{i=1}^n \left\| \frac{\alpha_t^{i+1}}{\sqrt{\hat{v}_t^{i+1}}} - \frac{\alpha_t^i}{\sqrt{\hat{v}_t^i}} \right\|^2. \end{aligned} \quad (81)$$

Substituting (58) into (77), we have

$$\begin{aligned} T_5 &\leq 28n^3 L^2 \left( \frac{1}{1-\beta_1^n} \right)^4 \left( \frac{1}{1-\beta_1} \right)^2 \sum_{t=1}^T \sum_{i=1}^n \left\| \frac{\alpha_t^i}{\sqrt{\hat{v}_t^i}} \odot \nabla f_i(x_t^i) \right\|^2 \\ &\quad + 28n^3 L^2 H^2 \left( \frac{1}{1-\beta_1^n} \right)^4 \left( \frac{1}{1-\beta_1} \right)^2 \left( \frac{\beta_1}{1-\beta_1} \right)^2 \sum_{t=1}^T \sum_{i=1}^n \left\| \frac{\alpha_t^{i+1}}{\sqrt{\hat{v}_t^{i+1}}} - \frac{\alpha_t^i}{\sqrt{\hat{v}_t^i}} \right\|^2 \\ &\leq 28n^3 L^2 \left( \frac{1}{1-\beta_1} \right)^6 \sum_{t=1}^T \sum_{i=1}^n \left\| \frac{\alpha_t^i}{\sqrt{\hat{v}_t^i}} \odot \nabla f_i(x_t^i) \right\|^2 \\ &\quad + 28n^3 L^2 H^2 \left( \frac{1}{1-\beta_1} \right)^6 \sum_{t=1}^T \sum_{i=1}^n \left\| \frac{\alpha_t^{i+1}}{\sqrt{\hat{v}_t^{i+1}}} - \frac{\alpha_t^i}{\sqrt{\hat{v}_t^i}} \right\|^2 \end{aligned} \quad (82)$$

Finally, substituting (81) and (82) into (69), we obtain

$$\begin{aligned}
T_1 &\leq \frac{1}{2}(1 - \beta_1)(T_4 + T_5) - (1 - \beta_1) \sum_{t=1}^T \left\langle \nabla f(x_t), \frac{\alpha_t^1}{\sqrt{\hat{v}_t^1}} \odot \nabla f(x_t) \right\rangle + \frac{1}{1 - \beta_1^n} M^2 H^2 \\
&\leq G_1 \sum_{t=1}^T \sum_{i=1}^n \left\| \frac{\alpha_t^i}{\sqrt{\hat{v}_t^i}} \odot \nabla f_i(x_t^i) \right\|^2 + G_2 \sum_{t=1}^T \sum_{i=1}^n \left\| \frac{\alpha_t^{i+1}}{\sqrt{\hat{v}_t^{i+1}}} - \frac{\alpha_t^i}{\sqrt{\hat{v}_t^i}} \right\|^2 + G_3 \\
&\quad - (1 - \beta_1) \sum_{t=1}^T \left\langle \nabla f(x_t), \frac{\alpha_t^1}{\sqrt{\hat{v}_t^1}} \odot \nabla f(x_t) \right\rangle
\end{aligned}$$

where

$$\begin{aligned}
G_1 &= 14n^3 L^2 \left( \frac{1}{1 - \beta_1} \right)^5 + 2(n^3 M^2 L^2 + n) \\
G_2 &= 14n^3 L^2 H^2 \left( \frac{1}{1 - \beta_1} \right)^5 + 2 \left( n^3 M^2 L^2 \left( \frac{\beta_1}{1 - \beta_1} \right)^2 + n^3 \right) H^2 \\
G_3 &= M^2 H^2 + \frac{1}{1 - \beta_1^n} n M H^2.
\end{aligned}$$

This completes the proof. **Q.E.D.**

### 7.3.2 Proof of Theorem 4.1

**Proof.** [Proof of Theorem 4.1] By Lemma 7.10, we have

$$\sum_{t=1}^T (f(x_{t+1}) - f(x_t)) \leq T_1 + T_2 + T_3$$

where

$$\begin{aligned}
T_1 &= \sum_{t=1}^T - \left\langle \nabla f(x_t), \frac{\alpha_t^1}{\sqrt{\hat{v}_t^1}} \odot \sum_{i=1}^n m_t^i \right\rangle \\
T_2 &= n H^2 \sum_{t=1}^T \sum_{i=1}^n \left\| \frac{\alpha_t^{i+1}}{\sqrt{\hat{v}_t^{i+1}}} - \frac{\alpha_t^i}{\sqrt{\hat{v}_t^i}} \right\|_1 \\
T_3 &= \frac{L}{2} n \sum_{t=1}^T \sum_{i=1}^n \|\alpha_t^i m_t^i / \sqrt{\hat{v}_t^i}\|^2
\end{aligned}$$

And by Lemma 7.11 and Lemma 7.12, we have

$$\begin{aligned}
\sum_{t=1}^T (f(x_{t+1}) - f(x_t)) &\leq F_1 \sum_{t=1}^T \sum_{i=1}^n \left\| \frac{\alpha_t^i}{\sqrt{\hat{v}_t^i}} \odot \nabla f_i(x_t^i) \right\|^2 + F_2 \sum_{t=1}^T \sum_{i=1}^n \left\| \frac{\alpha_t^{i+1}}{\sqrt{\hat{v}_t^{i+1}}} - \frac{\alpha_t^i}{\sqrt{\hat{v}_t^i}} \right\|^2 + F_3 \\
&\quad + n H^2 \sum_{t=1}^T \sum_{i=1}^n \left\| \frac{\alpha_t^{i+1}}{\sqrt{\hat{v}_t^{i+1}}} - \frac{\alpha_t^i}{\sqrt{\hat{v}_t^i}} \right\|_1 - (1 - \beta_1) \sum_{t=1}^T \left\langle \nabla f(x_t), \frac{\alpha_t^1}{\sqrt{\hat{v}_t^1}} \odot \nabla f(x_t) \right\rangle
\end{aligned}$$

with

$$\begin{aligned}
F_1 &= 14n^3 L^2 + nL \left( \frac{1}{1 - \beta_1} \right)^5 + 2(n^3 M^2 L^2 + n) + nL \\
F_2 &= 14n^3 L^2 H^2 n H^2 \left( \frac{1}{1 - \beta_1} \right)^5 + 2 \left( n^3 M^2 L^2 \left( \frac{\beta_1}{1 - \beta_1} \right)^2 + n^3 \right) H^2 + nL H^2 \left( \frac{\beta_1}{1 - \beta_1} \right)^2 \\
F_3 &= M^2 H^2 + \frac{1}{1 - \beta_1^n} n M H^2
\end{aligned}$$

After rearranging, we have

$$\begin{aligned}
& \sum_{t=1}^T \left\langle \nabla f(x_t), \frac{\alpha_t^1}{\sqrt{\widehat{v}_t^1}} \odot \nabla f(x_t) \right\rangle \\
& \leq \frac{1}{1-\beta_1} F_1 \sum_{t=1}^T \sum_{i=1}^n \left\| \frac{\alpha_t^i}{\sqrt{\widehat{v}_t^i}} \odot \nabla f_i(x_t^i) \right\|^2 + \frac{1}{1-\beta_1} F_2 \sum_{t=1}^T \sum_{i=1}^n \left\| \frac{\alpha_t^{i+1}}{\sqrt{\widehat{v}_t^{i+1}}} - \frac{\alpha_t^i}{\sqrt{\widehat{v}_t^i}} \right\|^2 \\
& \quad + \frac{1}{1-\beta_1} n H^2 \sum_{t=1}^T \sum_{i=1}^n \left\| \frac{\alpha_t^{i+1}}{\sqrt{\widehat{v}_t^{i+1}}} - \frac{\alpha_t^i}{\sqrt{\widehat{v}_t^i}} \right\|_1 + \frac{1}{1-\beta_1} (F_3 + f(x_1) - f(x^*))
\end{aligned}$$

where  $x^* \in \arg \min_x f(x)$ .

This completes the proof. Q.E.D.

### 7.3.3 Proof of Corollary 4.1

**Proof.** [Proof of Corollary 4.1] The proof is the same as proof of Corollary 3.2. We first show that the assumption  $\alpha_t^i / \sqrt{(\widehat{v}_t^i)_j} \leq M$  in Theorem 4.1 is satisfied. Because  $(v_t^i)_j$  is non-decreasing (by the max in update rule of  $\widehat{v}_t^i$ ) and for some constant  $c$ ,  $(v_1^1)_j \geq c^2 > 0$  (because  $\nabla f(x_1^1)$  is required to be element-wise non-zero), we have  $\alpha_t^i / \sqrt{(\widehat{v}_t^i)_j} \leq \alpha_1^1 / \sqrt{(\widehat{v}_1^1)_j} \leq 1/c = M$ . Next, let's turn to convergence rate.

By Theorem 4.1, we have

$$\begin{aligned}
& \sum_{t=1}^T \alpha_t^1 \langle \nabla f(x_t), \nabla f(x_t) / \sqrt{\widehat{v}_t^1} \rangle \\
& \leq \sum_{t=1}^T \sum_{i=1}^n \left( C'_1 \left\| \alpha_t^i \nabla f(x_t^i) / \sqrt{\widehat{v}_t^i} \right\|^2 + C'_2 \left\| \frac{\alpha_t^{i+1}}{\sqrt{\widehat{v}_t^{i+1}}} - \frac{\alpha_t^i}{\sqrt{\widehat{v}_t^i}} \right\|_1 + C'_3 \left\| \frac{\alpha_t^{i+1}}{\sqrt{\widehat{v}_t^{i+1}}} - \frac{\alpha_t^i}{\sqrt{\widehat{v}_t^i}} \right\|^2 \right) + C'_4.
\end{aligned} \tag{83}$$

We need to upper bound the terms weighted by  $C'_1$ ,  $C'_2$ ,  $C'_3$  and lower bound  $\sum_{t=1}^T \sum_{i=1}^n \min_{j \in [d]} \alpha_t^i / \sqrt{(\widehat{v}_t^i)_j}$  and to get convergence rate.

Since  $(v_t^i)_j$  is non-decreasing, for some constant  $c$ ,  $(v_t^i)_j \geq c^2 > 0$  and  $\alpha_t^i = 1 / \sqrt{n(t-1) + i}$ , we have

$$\sum_{t=1}^T \sum_{i=1}^n \left\| \alpha_t^i \nabla f(x_t^i) / \sqrt{\widehat{v}_t^i} \right\|^2 \leq \sum_{k=1}^{nT} \frac{H^2}{c} \frac{1}{k} \leq \frac{H^2}{c} \log nT$$

Further, because  $(v_t^i)_j$  is non-decreasing and  $\alpha_t^i$  is non-increasing, we have

$$\begin{aligned}
\sum_{t=1}^T \sum_{i=1}^n \left\| \frac{\alpha_t^{i+1}}{\sqrt{\widehat{v}_t^{i+1}}} - \frac{\alpha_t^i}{\sqrt{\widehat{v}_t^i}} \right\|_1 &= \sum_{t=1}^T \sum_{i=1}^n \sum_{j=1}^d \left( \frac{\alpha_t^i}{\sqrt{(\widehat{v}_t^i)_j}} - \frac{\alpha_t^{i+1}}{\sqrt{(\widehat{v}_t^{i+1})_j}} \right) \\
&= \sum_{j=1}^d \left( \frac{\alpha_1^1}{\sqrt{(\widehat{v}_1^1)_j}} - \frac{\alpha_T^{n+1}}{\sqrt{(\widehat{v}_T^{n+1})_j}} \right) \leq \sum_{j=1}^d \left( \frac{\alpha_1^1}{\sqrt{\widehat{v}_t^1}} \right) \leq \frac{d}{c}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{t=1}^T \sum_{i=1}^n \left\| \frac{\alpha_t^{i+1}}{\sqrt{\widehat{v}_t^{i+1}}} - \frac{\alpha_t^i}{\sqrt{\widehat{v}_t^i}} \right\|^2 &\leq \left\| \frac{\alpha_1^1}{\sqrt{\widehat{v}_1^1}} \right\|_1 \sum_{t=1}^T \sum_{i=1}^n \left\| \frac{\alpha_t^{i+1}}{\sqrt{\widehat{v}_t^{i+1}}} - \frac{\alpha_t^i}{\sqrt{\widehat{v}_t^i}} \right\| \\
&\leq \left\| \frac{\alpha_1^1}{\sqrt{\widehat{v}_1^1}} \right\|_1 \sum_{t=1}^T \sum_{i=1}^n \left\| \frac{\alpha_t^{i+1}}{\sqrt{\widehat{v}_t^{i+1}}} - \frac{\alpha_t^i}{\sqrt{\widehat{v}_t^i}} \right\|_1 \leq \left\| \frac{\alpha_1^1}{\sqrt{\widehat{v}_1^1}} \right\|_1^2 \leq \frac{d^2}{c^2}
\end{aligned}$$

Since  $\|\widehat{v}_t\| \leq H$  by update rule, the sum of minimum step size is lower bounded by

$$\sum_{t=1}^T \sum_{i=1}^n \min_{j \in [d]} \alpha_t^i / \sqrt{(\widehat{v}_t^i)_j} \geq \sum_{k=1}^{nT} \frac{1}{H\sqrt{k}} \geq \sqrt{nT}/H$$

Then by above inequalities we have

$$\min_{t \in [T]} \|\nabla f(x_t)\|^2 \leq \frac{1}{\sqrt{T}} (R_1 + R_2 \log T)$$

for some constant  $R_1$  and  $R_2$  independent of  $T$ .

This completes the proof. **Q.E.D.**

### 7.3.4 Proof of Corollary 4.2

**Proof.** [Proof of Corollary 4.2]

#### Algorithm 6. Incremental AdaGrad with momentum

**(S0).** Initialize  $x_1, m_0^n = 0, \widehat{v}_0^n = 0$ ;  
For  $t = 1, \dots, T$ , do  
**(S1).** Let  $m_t^0 = m_{t-1}^n, x_t^1 = x_t, \widehat{v}_t^0 = \widehat{v}_{t-1}^n$   
For  $i = 1, \dots, n$ , do  
**(S1-1).**  $m_t^i = \beta_{1,t} m_{t-1}^{i-1} + (1 - \beta_{1,t}) \nabla f_i(x_t^i)$   
**(S1-2).**  $\widehat{v}_t^i = \frac{n(t-1)+i-1}{n(t-1)+i} \widehat{v}_{t-1}^{i-1} + \frac{1}{n(t-1)+i} \nabla f_i(x_t^i)^2$   
**(S1-3).**  $x_t^{i+1} = x_t^i - \alpha_t^i m_t^i / \sqrt{\widehat{v}_t^i}$   
End  
**(S2).**  $x_{t+1} = x_t^{n+1}$   
End

The proof is the similar to proof of Corollary 3.3. We first show that the assumption  $\alpha_t^i / \sqrt{(\widehat{v}_t^i)_j} \leq M$  in Theorem 4.1 is satisfied. Because  $\alpha_t^i / \sqrt{(\widehat{v}_t^i)_j}$  is non-decreasing with the choice  $\alpha_t^i = \frac{1}{\sqrt{n(t-1)+i}}$  (which will be clear later) and for some constant  $c$ ,  $(v_1^1)_j \geq c^2 > 0$  (because  $\nabla f(x_1^1)$  must be element-wise non-zero), we have  $\alpha_t^i / \sqrt{(\widehat{v}_t^i)_j} \leq \alpha_1^1 / \sqrt{(\widehat{v}_1^1)_j} \leq 1/c = M$ . Next, let's turn to convergence rate.

By Theorem 4.1, we have

$$\begin{aligned}
&\sum_{t=1}^T \alpha_t^1 \langle \nabla f(x_t), \nabla f(x_t) / \sqrt{\widehat{v}_t^1} \rangle \\
&\leq \sum_{t=1}^T \sum_{i=1}^n \left( C'_1 \left\| \alpha_t^i \nabla f(x_t^i) / \sqrt{\widehat{v}_t^i} \right\|^2 + C'_2 \left\| \frac{\alpha_t^{i+1}}{\sqrt{\widehat{v}_t^{i+1}}} - \frac{\alpha_t^i}{\sqrt{\widehat{v}_t^i}} \right\| + C'_3 \left\| \frac{\alpha_t^{i+1}}{\sqrt{\widehat{v}_t^{i+1}}} - \frac{\alpha_t^i}{\sqrt{\widehat{v}_t^i}} \right\|^2 \right) + C'_4
\end{aligned} \tag{84}$$

We need to upper bound the terms weighted by  $C'_1$ ,  $C'_2$ ,  $C'_3$  and lower bound  $\sum_{t=1}^T \sum_{i=1}^n \min_{j \in [d]} \alpha_t^i / \sqrt{(\widehat{v}_t^i)_j}$  and to get convergence rate.

Since  $(v_t^i)_j$  is average of past gradient and  $\alpha_t^i = 1/\sqrt{n(t-1)+i}$ , denote  $g_{n(t-1)+i} = \nabla f_i(x_t^i)$ ,  $\alpha_{n(t-1)+i} = \alpha_t^i$ ,  $\widehat{v}_{n(t-1)+i} = \widehat{v}_t^i$ , we have  $\widehat{v}_k = 1/k \sum_{l=1}^k g_l^2$  and

$$\begin{aligned} \sum_{t=1}^T \sum_{i=1}^n \left\| \alpha_t^i \nabla f(x_t^i) / \sqrt{\widehat{v}_t^i} \right\|^2 &= \sum_{k=1}^{nT} \left\| \alpha_k g_k / \sqrt{\widehat{v}_k} \right\|^2 = \sum_{k=1}^{nT} \left\| g_k / \sqrt{\sum_{l=1}^k g_l^2} \right\|^2 \\ &= \sum_{k=1}^{nT} \sum_{j=1}^d (g_k)_j^2 / \sum_{l=1}^k (g_l)_j^2 \\ &\leq \sum_{j=1}^d \left( 1 - \log((g_1^2)_j) + \log \left( \sum_{k=1}^{nT} (g_k)_j^2 \right) \right) \\ &\leq d(1 - \log(c^2) + \log(nTH^2)) \end{aligned}$$

where the first inequality is due to Lemma 7.9 and  $|(g_1)_j| \geq c$

Further, because  $\alpha_t^i / \sqrt{(\widehat{v}_t^i)_j}$  is non-increasing and  $\alpha_1^1 / \sqrt{(\widehat{v}_1^1)_j} = 1/\sqrt{(g_1)_j^2} \leq 1/c$  for some constant  $c > 0$ , we have

$$\begin{aligned} \sum_{t=1}^T \sum_{i=1}^n \left\| \frac{\alpha_t^{i+1}}{\sqrt{\widehat{v}_t^{i+1}}} - \frac{\alpha_t^i}{\sqrt{\widehat{v}_t^i}} \right\|_1 &= \sum_{t=1}^T \sum_{i=1}^n \sum_{j=1}^d \left( \frac{\alpha_t^i}{\sqrt{(\widehat{v}_t^i)_j}} - \frac{\alpha_t^{i+1}}{\sqrt{(\widehat{v}_t^{i+1})_j}} \right) \\ &= \sum_{j=1}^d \left( \frac{\alpha_1^1}{\sqrt{(\widehat{v}_1^1)_j}} - \frac{\alpha_{T+1}^{n+1}}{\sqrt{(\widehat{v}_{T+1}^{n+1})_j}} \right) \leq \sum_{j=1}^d \left( \frac{\alpha_1^1}{\sqrt{\widehat{v}_1^1}} \right) \leq \frac{d}{c} \end{aligned}$$

and

$$\begin{aligned} \sum_{t=1}^T \sum_{i=1}^n \left\| \frac{\alpha_t^{i+1}}{\sqrt{\widehat{v}_t^{i+1}}} - \frac{\alpha_t^i}{\sqrt{\widehat{v}_t^i}} \right\|^2 &\leq \left\| \frac{\alpha_1^1}{\sqrt{\widehat{v}_1^1}} \right\|_1 \sum_{t=1}^T \sum_{i=1}^n \left\| \frac{\alpha_t^{i+1}}{\sqrt{\widehat{v}_t^{i+1}}} - \frac{\alpha_t^i}{\sqrt{\widehat{v}_t^i}} \right\| \\ &\leq \left\| \frac{\alpha_1^1}{\sqrt{\widehat{v}_1^1}} \right\|_1 \sum_{t=1}^T \sum_{i=1}^n \left\| \frac{\alpha_t^{i+1}}{\sqrt{\widehat{v}_t^{i+1}}} - \frac{\alpha_t^i}{\sqrt{\widehat{v}_t^i}} \right\|_1 \leq \left\| \frac{\alpha_1^1}{\sqrt{\widehat{v}_1^1}} \right\|_1^2 \leq d^2/c^2 \end{aligned}$$

Since  $\|\widehat{v}_t\| \leq H$  by update rule, the sum of minimum step size is lower bounded by

$$\sum_{t=1}^T \sum_{i=1}^n \min_{j \in [d]} \alpha_t^i / \sqrt{(\widehat{v}_t^i)_j} \geq \sum_{k=1}^{nT} \frac{1}{H\sqrt{k}} \geq \sqrt{nT}/H$$

Then by above inequalities we have

$$\min_{t \in [T]} \|\nabla f(x_t)\|^2 \leq \frac{1}{\sqrt{T}} (R'_1 + R'_2 \log T)$$

for some constant  $R'_1$  and  $R'_2$  independent of T.

This completes the proof. **Q.E.D.**