The Value of Waiting to Invest

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THE VALUE OF WAITING TO INVEST*

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This paper studies the optimal timing of investment in an irreversible project where the benefits from the project and the investment cost follow continuous-time stochastic processes. The optimal investment rule and an explicit formula for the value of the option to invest are derived, assuming that the option is valued by risk-averse investors who are well diversified. The same analysis is applied to the scrapping decision. Simulations show that this option value can be significant, and that for reasonable parameter values it is optimal to wait until benefits are twice the investment costs.

I. INTRODUCTION

Suppose that a firm is considering building a synthetic fuel plant. What is the appropriate way to decide whether or not to build? Clearly, one calculates the present values of profits and the direct costs of construction. It would be incorrect, however, simply to compare these present values and build the plant if the present value of profits exceeds that of the direct costs.

The decision to build the plant is irreversible; the plant cannot be used for any other purpose. The decision to defer building, however, is reversible. This asymmetry, when properly taken into account, leads to a rule that says build the plant only if benefits exceed costs by a certain positive amount. The correct calculation involves comparing the value of investing today with the (present) value of investing at all possible times in the future. This is a comparison of mutually exclusive alternatives.

In this paper we explore the practical importance of the value of waiting to invest, assuming that investment timing decisions are made by risk-averse investors who hold well-diversified portfolios. We derive explicit formulas for the value of the option to invest in an irreversible project and the rule for when to invest when both the value of the project and the cost of investing are stochastic. The formulas enable us to compute exactly the optimal investment timing rule, as well as the dollar value lost by a firm that takes a project at a suboptimal time.

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In Section II we present a general model of investment with the option to wait, and apply the model to real investment problems. In Section III we solve the valuation problem assuming that both the present value of benefits and the investment cost follow geometric Brownian motion, and also assuming that the present value of benefits usually follows Brownian motion but also may jump discretely to zero. In every case we assume that the option is infinitely lived.[1] Risk aversion is introduced by assuming that investment options are owned by well-diversified investors. In Section IV we compute examples of the option value and investment rule for a wide variety of parameters. The general conclusion is that timing considerations are quantitatively important. For reasonable parameter values it is optimal to defer investing until the present value of the benefits from a project is *double* the investment cost. The rule, "invest if the net present value of investing exceeds zero" is only valid if the variance of the present value of future benefits and costs is zero or if the expected rate of growth of the present value is minus infinity; the value lost by following this suboptimal investment policy can be substantial. Section V presents an example in which a firm with a Cobb-Douglas production function faces a random demand curve. We compute the option value and investment rule as a function of exogenous parameters. Section VI concludes.

A number of related papers have studied investment timing. Baldwin [1982]; Baldwin and Meyer [1979]; Brock, Rothschild, and Stiglitz [1983]; and Venezia and Brenner [1979] have all analyzed the investment timing problem under risk neutrality and have obtained many of the same comparative static results as we do. Bernanke [1983] and Venezia [1983] have added Bayesian learning, so that investors learn not only about the value of the project by waiting, but also about the underlying stochastic

1. Many real-life investment opportunities are not infinitely lived, but expire or become valueless at some point. We deal with this by allowing the present value of the benefit from undertaking the project to have an average downward drift and by allowing the present value of the project's cash flows to jump to zero. In the latter case the option eventually becomes valueless, but at an unknown date. The important omission in our model is the case where the option to invest expires at a known date in the future. A finitely lived patent, for example, would give the holder an option to invest with a known expiration date, and would be worth less than an infinitely lived patent. It is typically not possible to solve analytically for the option value in this case; however, numerical solutions may be obtained.

process.[2] This paper's main contributions are providing a tractable and realistic means of incorporating risk aversion considerations into the timing problem and presenting examples that show timing considerations to be important.

II. The Investment Problem

We study the investment decision of a firm that is considering the following investment opportunity: at any time t (up to a possible expiration date T), the firm can pay F_t to install an investment project, for which expected future net cash flows conditional on undertaking the project have a present value V_t. We emphasize that V_t is a present value; it represents the appropriately discounted *expected* cash flows, given the information available at time t. For the firm, V_t represents the market value of a claim on the stream of net cash flows that arise from installing the investment project at time t. Typically, V_t and F_t are stochastic. The installation of capacity is irreversible, in that the capacity can be used only for this specific project.

We assume that V_t follows geometric Brownian motion of the form,

$$\frac{dV}{V} = \alpha_v dt + \sigma_v dz_v, \tag{1a}$$

where z_v is a standard Wiener process. Thus, the firm knows the present value of future net cash flows if it installs the project today. However, the present value may be different if the capacity is installed in the future. (We shall also consider the possibility that at some [random] time in the future, the present value of net cash flows drops at once to zero.) Similarly, we assume that F_t follows:

$$\frac{dF}{F} = \alpha_f dt + \sigma_f dz_f. \tag{1b}$$

2. Bernanke also provides a useful discussion of previous papers dealing with irreversibility and their relation to financial option models and search theory models. See also Krutilla [1967]; Henry [1974]; Cukierman [1980]; Greenley, Walsh, and Young [1981]; and Myers and Majd [1983]. Roberts and Weitzman [1981] study the related problem of optimal stopping of R and D expenditures. We do not address issues of the relationship between first-mover advantages and investment timing as in Spence [1981].

For both V and F, the geometric Brownian motion assumption is crucial for the derivation of the formulas below. This assumption is reasonable for the project value V, but may be less so for the investment cost F. The project value V in many applications is the market value of an asset; if the project were undertaken and a company owned only this asset, V is the price for which the company's stock would sell, which is to say that V is the price of a financial asset. The rate of growth of V will equal the rate of return on the stock, less cash flow that is earned on the project and paid out. Thus, as long as the payout rate is relatively constant, the assumption of geometric Brownian motion for V is as reasonable as assuming that a stock price obeys geometric Brownian motion (a standard assumption in the finance literature), although the rate of appreciation of V will typically be less than the total rate of return on a comparable stock.[3] Nevertheless, the example in Section V will show that the assumption of geometric Brownian motion can lead to unrealistic conclusions. If V is not a present value, our analysis is still valid, but (1a) may be a less reasonable specification. The investment cost F is typically the price of a physical asset and not a present value.[4] We discuss scenarios, however, in which F is also a present value.

Examples of the Investment Problem

The essential feature of the problem is that the firm is faced with the mutually exclusive choice of taking an irreversible project today or in the future. Uncertainty about the project's value and the cost of the project is being continuously resolved. There are a number of situations embodying these assumptions. A monopolist, for example, may have an investment opportunity such that once he installs his capacity, he is protected from competition. Alternatively, one can think of a firm in a competitive industry exhibiting temporary rents. The investment opportunity consists of a project whose future net cash flows have a present value exceeding the investment cost now, but which tend on average

3. For an installed and producing project, equilibrium requires that capital gains plus cash flow less depreciation equals the required rate of return on the project. The uninstalled project is not depreciating, so its expected price appreciation is less than the required rate of return by the cash flow net of depreciation which the project would have earned if it were installed. The earnings-to-price ratio therefore measures the extent to which α_v is less than the required rate of return.

4. Brock, Rothschild, and Stiglitz [1983] also draw this distinction, and note that the price of a financial asset should grow geometrically, unlike other prices.

toward the investment cost because of lagged entry. In equation (1a) this is represented by $\alpha_v < 0$. Many examples of this kind of project occur in high technology industries. When a firm is considering introducing a new product, it realizes that others may introduce similar products. As the others enter, profits disappear. These industries also provide examples of how V_t might at some point drop to zero. While the firm is waiting to introduce its product, a new, more sophisticated or cheaper version might be introduced by another firm, rendering the former's product useless.

More generally, we shall be describing a situation in which the investor can swap one risky asset (F) for another (V). This can be thought of either as a straightforward investment problem or as an asset replacement problem. The two problems are the same. For example, suppose that the government is considering building a canal through Everglades National Park. V_t represents the present value of the benefits from building the canal, while F_t represents both the direct costs and the present value of forgone benefits of the park as a recreational area.

Optimal Scrapping of a Project

By reinterpreting variables, the model can also be used to study the problem of optimal scrapping. Interpret F as the value of the project in place and V as the value of the project if it were to be sold. The model then provides the value of the option to scrap a project. In general, it will be optimal to scrap only when the selling price exceeds the project value by a positive amount.

III. Investment Timing and the Value of Waiting

This section studies the problem of the optimal timing of the installation of an irreversible investment project. We derive an optimal decision rule and the value of the investment opportunity. We begin with the case of V and F following (1a) and (1b) and then show how the correct discount rate is obtained when investors are risk averse. We then consider the possibility that V_t may suddenly fall to zero.

V and F Stochastic

To introduce the problem, suppose initially that V_t follows (1a), but that F is constant. The firm receives $V_t - F$ when it invests. The investment timing problem consists of finding a number C_t^*, for every time t, such that if $V_t/F \geqslant C_t^*$, the investment is undertaken, and deferred otherwise. This investment decision schedule $\{C_t^*\}$ is chosen so as to maximize the time zero expected present value of the payoff $V_t - F$.

For example, let the investment opportunity expire at T. It is obvious that if we reach T and have not already undertaken investment, then it will be optimal to do so provided that $V_T \geqslant F$. Thus, $C_T^* = 1$ constitutes a boundary at T, at which the investment opportunity is undertaken. In a similar way, working backwards, for any t it is possible to derive a C_t^* such that, if the investment opportunity is still unexercised at t, then undertaking it will be optimal if $V_t/F \geqslant C_t^*$, and not otherwise. When $V_t/F = C_t^*$, the firm invests, and the net present value of the project is then $F[C_t^* - 1]$.

For an arbitrary boundary $\{C_t'\}_0^T$, the value of the investment opportunity is the expected present value of the payoff:

$$X(T) = E_0\{e^{-\mu t'}[V_{t'} - F]\}, \tag{2}$$

where t' is the date at which V/F first reaches the boundary $\{C_t'\}$ and $X(T)$ is the time zero value of an investment opportunity that expires at T. The expectation is taken over the first passage times t', and μ is the appropriate discount rate, which we take as given for the moment.

In the special case where the investment opportunity is infinitely lived, it is possible to solve for the maximized value of (2) explicitly. When $T = \infty$, it is possible to remove calendar time from the problem. Hence C^* cannot depend on t, so $C_t^* = C^*$ for all t. Because $V_{t'} - F = F(C^* - 1)$ is constant, maximizing (2) reduces to the problem,

$$\max_{C'} F[C' - 1]E_0\{e^{-\mu t'}\}.$$

The solution to this problem is a special case of the problem we solve below.

Now consider the same problem, except that F_t is also random and follows the stochastic process (1b). The problem is formulated as a first passage problem as before, but the characterization of the boundary at which investment should occur is not as obvious. A simple argument establishes that investment should occur when the ratio V/F reaches a boundary. The problem we have been considering involves choosing a boundary B to maximize

$$E_0\,[(V_t - F_t)e^{-\mu t}],$$

subject to (1a) and (1b). Let $V' = kV$ and $F' = kF$, where k is an arbitrary positive number, and consider the problem of choosing a boundary B' to maximize

$$E_0\,[(V'_t - F'_t)e^{-\mu t}],$$

subject to

$$\frac{dV'}{V'} = \alpha_v dt + \sigma_v dz_v; \qquad \frac{dF'}{F'} = \alpha_f dt + \sigma_f dz_f.$$

The two problems are formally identical, so the boundaries B and B' must be the same and hence independent of k. Since the boundary is independent of k, it is homogeneous of degree zero in V and F. Also, as before, the boundary is independent of calendar time when $T = \infty$. Thus, the correct rule is to invest when the ratio V/F reaches a fixed boundary. There is an additional question as to whether there are multiple boundaries. The Appendix proves that the solution is to invest if V/F exceeds C^*, and wait otherwise.

Because the optimal rule is to invest when V_t/F_t reaches a barrier C^*, the expected present value of the payoff is

$$E_0\{F_{t'}\,[C^* - 1]e^{-\mu t'}\} = [C^* - 1]E_0\,\{F_{t'}e^{-\mu t'}\}, \tag{3}$$

where the expectation is taken over the joint density of F_t and the first-passage times for $V_t/F_{t'}$. Fortunately, it is not necessary to derive the joint density for $F_{t'}$ and t' in order to evaluate (3). The evaluation of (3) is involved, however, so it is relegated to the Appendix. From the Appendix the value of the opportunity is

$$X = (C^* - 1)F_0\left(\frac{V_0/F_0}{C^*}\right)^{\varepsilon}, \tag{4}$$

where

$$\varepsilon = \sqrt{\left(\frac{\alpha_v - \alpha_f}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2(\mu - \alpha_f)}{\sigma^2}} + \left(\frac{1}{2} - \frac{\alpha_v - \alpha_f}{\sigma^2}\right),$$

$$C^* = \varepsilon/(\varepsilon - 1), \tag{5}$$

$\sigma^2 = \sigma_v^2 + \sigma_f^2 - 2\rho_{vf}\sigma_v\sigma_f$, and ρ_{vf} is the instantaneous correlation between the rates of increase of V and F.

The condition $\alpha_v < \mu$ ensures that $\varepsilon > 1$, so that the solution is well defined. If $\alpha_v \geqslant \mu$, then the growth rate of the value of the project is expected to exceed the discount rate. Consequently, the value of the investment opportunity would be infinite, and it will never pay to invest. When F is nonstochastic, (4) is Samuelson's [1970] formula for the price of an infinitely lived American call option on a dividend-paying stock. The case where T is finite has not been solved analytically [Samuelson, 1970]. The general solution procedure in such cases involves using a discrete approximation to the continuous-time problem and applying a dynamic programming argument to obtain numerical approximations to the solution (cf. Ingersoll [1977]). Numerical procedures can also be used when the parameters (α, μ, and σ) are functions of V, F, and t.

Both the value of the investment option and the level of V/F at which investment should occur are increasing functions of the variance of V/F, σ^2.[5] The reason for this is well-known: an increase in variance increases the spread of possible future values for V/F, and hence the maximum possible gain, while leaving unchanged the maximum possible loss. Note that only the variance of the proportional change in V/F, σ^2, enters the formula; this occurs since the investment rule depends only on V/F. Therefore, an increase in either σ_v^2 or σ_f^2, or a decrease in the correlation ρ_{vf}, will increase the value of the investment option.

The value of the investment option is also an increasing function of α_v and a decreasing function of α_f and μ. As we shall see in the next section, however, comparative statics for the drifts alone are uninteresting.

Optimal Scrapping.[6] As noted earlier, a simple reinterpretation of the solution with V and F random (equation (4)) provides the value of the option to scrap a project. Interpret V as the random scrap value of the project. F is the random value to the firm of the project in place. The payoff from scrapping is therefore $V - F$. An argument like that above establishes that the optimal policy is to sell the asset when the ratio F/V reaches a boundary.

5. It should be noted that this result is a consequence of the assumption that V follows geometric Brownian motion with constant parameters. Brock, Rothschild, and Stiglitz [1983] show that when the stochastic process for V has a lower absorbing barrier sufficiently close to the current value of V, then an increase in variance can lower the value of the option. With processes like (1), zero is a natural absorbing barrier, but one that is never reached in finite time.

6. Myers and Majd [1983] use numerical option-pricing techniques to value the abandonment option for a project when the option is finitely lived.

Let c^* represent the level to which F/V must fall before it is optimal to scrap. Equation (4) then becomes

$$(6)\quad X = (C^* - 1)F_0\left(\frac{V_0/F_0}{C^*}\right)^{\varepsilon} = (1 - c^*)V_0\left(\frac{F_0/V_0}{c^*}\right)^{1-\varepsilon},$$

where $c^* = (\varepsilon - 1)/\varepsilon$ and ε is given by (5).[7] Note that c^* is less than 1. Thus, in general the firm waits to scrap until the value of the project is less than its scrap value by some positive amount. The intuition here is the same as before: by waiting, the firm can benefit from increases in $V - F$, but is protected against decreases.

Computing the Correct Discount Rate

We have taken as given μ, the rate at which future payoffs are discounted. We now show that μ—which is the equilibrium expected rate of return on the investment opportunity—must be a weighted average of the equilibrium expected rates of return on assets with the same risk as V and F.

Risk aversion by investors is here introduced by supposing that options to invest are owned by well-diversified investors, who need only be compensated for the systematic component of the risk of projects and options to invest. This is in contrast to Venezia and Brenner [1979], for example, who assume that the entire project is owned by a single risk-averse investor. Assuming that investors are well diversified describes publicly owned corporations in the United States and simplifies the computation of the option value.

The actual rate of return on the investment opportunity is computed by taking an Ito derivative[8] of the option value, equation (4):

$$\begin{aligned}\frac{dX}{X} &= \frac{\varepsilon dV}{V} + (1 - \varepsilon)\frac{dF}{F} + \varepsilon(\varepsilon - 1)\left(\tfrac{1}{2}\sigma_v^2 + \tfrac{1}{2}\sigma_f^2 - \rho_{vf}\sigma_v\sigma_f\right)dt \\ (8)\qquad &= [\varepsilon\alpha_v + (1 - \varepsilon)\alpha_f + \varepsilon(\varepsilon - 1)\tfrac{1}{2}\sigma^2]dt \\ &\qquad + \varepsilon\sigma_v dz_v + (1 - \varepsilon)\sigma_f dz_f.\end{aligned}$$

In (8) the unanticipated component of the return on X is

7. For the special case when V is constant, $\alpha_f = \mu$, and μ is the risk-free rate, (6) reduces to the formula obtained in Merton [1973] for the value of a perpetual put option on a nondividend paying stock.

8. If $X = g(V,F,t)$, with V and F given by (1), Ito's lemma states that

$$dX = g_t dt + g_v dV + g_f dF + \tfrac{1}{2}[\sigma_v^2 V^2 g_{vv} + \sigma_f^2 F^2 g_{ff} + 2\sigma_{vf} VF g_{vf}]dt.$$

$\varepsilon\sigma_v dz_v + (1 - \varepsilon)\sigma_f dz_f$, which is a weighted average of the unanticipated components in the rates of change of V and F. With standard asset pricing models, the risk premium earned on an asset is proportional to the riskiness of the asset. For example, in the Capital Asset Pricing Model,

$$\hat{\alpha}_i - r = \phi\rho_{im}\sigma_i, \tag{9}$$

where $\hat{\alpha}_i$ is the required rate of return on asset i, r is the risk-free rate, ϕ is the market price of risk, and ρ_{im} is the correlation between the rate of return on the asset and that on the market portfolio. If $\hat{\alpha}_j$ is the required rate of return for an asset with unexpected rate of return $\sigma_j dz_j$, then from (9) an asset with unexpected rate of return $\nu\sigma_j dz_j$ will have required rate of return $r + \nu(\hat{\alpha}_j - r)$. Therefore, μ, which is the discount rate for future payoffs and hence the equilibrium expected rate of return on the investment opportunity, will be given by

$$\mu = r + \varepsilon(\hat{\alpha}_v - r) + (1 - \varepsilon)(\hat{\alpha}_f - r) = \varepsilon\hat{\alpha}_v + (1 - \varepsilon)\hat{\alpha}_f, \tag{10}$$

where $\hat{\alpha}_v$ and $\hat{\alpha}_f$ are determined by (9), and are the expected rates of return required by investors for assets that are perfectly correlated with, and have the same standard deviations as V and F.

Equating the required expected rate of return (10) with the actual expected rate of return on X in (8) yields a quadratic equation in ε:

$$\mu = \varepsilon\hat{\alpha}_v + (1 - \varepsilon)\hat{\alpha}_f = \varepsilon\alpha_v + (1 - \varepsilon)\alpha_f + \tfrac{1}{2}\varepsilon(\varepsilon - 1)\sigma^2. \tag{11}$$

Equation (11) has the solution,

$$\varepsilon = \sqrt{\left(\frac{\delta_f - \delta_v}{\sigma^2} - \frac{1}{2}\right)^2 + 2\,\frac{\delta_f}{\sigma^2}} + \left(\frac{1}{2} - \frac{\delta_f - \delta_v}{\sigma^2}\right), \tag{12}$$

where $\delta_v = \hat{\alpha}_v - \alpha_v$ and $\delta_f = \hat{\alpha}_f - \alpha_f$. $\delta_v > 0$ insures that $\varepsilon > 1$. Equation (12) is the same as (5), since (12) can be obtained from (5) by substituting $\varepsilon\hat{\alpha}_v + (1 - \varepsilon)\hat{\alpha}_f$ for μ in equation (5) and then solving the resulting quadratic equation for ε.

Equation (12) shows that the drift terms *per se* are unimportant; the value of the option and the investment rule are affected only by the difference between the actual drifts and the required rates of return on assets with the same risk as V and F. A project with α_v of -0.10 and $\hat{\alpha}_v$ of -0.05 (a project with negative systematic risk) will less likely pay off than a project with α_v of 0.10 and $\hat{\alpha}_v$ of 0.15 (a project with positive systematic risk). The two projects will be valued the same, however, since those infrequent states in which the first project pays off are states in which the market as a whole performs poorly, and thus are valued more highly than states in which the second project pays off.

The parameter δ_v represents the portion of the required return on V that is forgone by merely receiving the price increases in V. The greater is δ_v, the greater is the cost to holding the option, which amounts to holding V indirectly. Thus, if δ_v rises, investment will optimally occur at a lower V/F, and the option is worth less. Holding α_v fixed, we see that increases in the systematic risk of V will raise δ_v and lower the option value. Similarly, δ_f represents the portion of the return on F that is forgone by obtaining only the price increases in F. An increase in δ_f has the opposite effects from an increase in δ_v, since F is a cost that is deferred by waiting. The larger is δ_f, the greater is the gain from deferral. An increase in the systematic risk of F raises δ_f and the value of the option.

More insight into the roles of δ_v and δ_f can be gained by considering the formula when $\sigma_v^2 = \sigma_f^2 = 0$. In that case, it can be shown that

$$\hat{X} = (\hat{C} - 1)\, F_0 \left(\frac{V_0/F_0}{\hat{C}}\right)^{\delta_f/(\delta_f - \delta_v)} \tag{13}$$
$$\hat{C} = \delta_f/\delta_v.$$

Expression (13) implies that in the deterministic case, investing is optimal only if

$$V_0\delta_v \geqslant F_0\delta_f. \tag{14}$$

This condition says that it is optimal to invest when the opportunity cost from not installing the project, $V_0\delta_v$, equals or exceeds the opportunity cost saved by deferring installation, $F_0\delta_f$. The formula is defined only for $\delta_v < \delta_f$. Otherwise it is optimal to invest immediately, or (if $V < F$) never.

When $\sigma^2 > 0$, the investment condition is

$$V_0\delta_v \geqslant F_0\delta_f + h(V_0,F_0); \qquad h > 0. \tag{14'}$$

When $\delta_v = 0$, there is no loss from waiting, and it is never optimal to invest; from (4) and (12) it can be verified that in this case $X = V$ and $C^* = \infty$.[9] The condition $\delta_f = 0$ is not of comparable

9. Venezia [1983] also obtains a condition when waiting is optimal at any current price. This is due, however, to the value of waiting to acquire information about uncertain parameters in a Bayesian setting. Our result holds with known parameters and is driven by time-value considerations.

importance, since with uncertainty the opportunity saving on F is not the only gain from waiting—there is also value from waiting (represented by h), since the option provides the ability to capture gains in V and avoid losses.

Jumps in V_t

Assume that there is a positive probability that the present value of net future cash flows, V_t, can take a discrete jump to zero. If this happens, the investment opportunity becomes worthless. Thus the stochastic process for V_t is a mixed Poisson-Wiener process of the form,

$$\frac{dV}{V} = \alpha_v dt + \sigma_v dz_v + dq, \tag{15}$$

where

$$dq = \begin{cases} -1 & \text{with probability } \lambda dt \\ 0 & \text{with probability } 1 - \lambda dt. \end{cases}$$

The occurrence of the Poisson event induces the process to stop, since zero is a natural absorbing barrier for a geometric Brownian motion process.

Notice that when the Poisson event occurs, it is as if the investment opportunity expires, since its value becomes zero. Thus, calculating the value of the investment opportunity when V_t can jump to zero is like calculating the value of an investment opportunity with an uncertain expiration date. The value in this case is easily calculated thanks to a result in Merton [1971].

The distribution of first occurrence times for a Poisson event with parameter λ is exponential. Suppose that the Poisson event is uncorrelated with the market portfolio and V, and that F is constant. Then the expected present value of the payoff from the investment opportunity with uncertain expiration date is

$$X^* = \int_0^\infty \lambda e^{-\lambda T} X(T) dT. \tag{16}$$

This may be integrated by parts [Merton, 1971] to give

$$X^* = \max_{\{C'_t\}} E_0(e^{-(\mu+\lambda)t'} F[C'_t - 1]). \tag{17}$$

This is exactly the problem we solved in subsection A above with no Poisson jump, except that the discount rate μ has been replaced by $\mu + \lambda$.[10] The formula is therefore the same as (4) with the discount rate adjusted in (5).

To compute the discount rate in this case, we can use Ito's lemma for Poisson processes [Merton, 1971] to calculate the expected rate of return on the option to invest. Assume for simplicity that F is fixed. The Ito derivative of (4) is then

$$\frac{dX^*}{X^*} = \varepsilon(\alpha_v dt + \sigma_v dz_v) + \varepsilon(\varepsilon - 1)\tfrac{1}{2}\sigma_v^2 dt - \lambda dt. \tag{18}$$

The only risk in (18) is due to the term $\varepsilon\sigma_v dz_v$, so the required rate of return on X^* is $r + \varepsilon(\hat{\alpha}_v - r)$. Equating this to the expected rate of return gives

$$\begin{aligned}[r + \varepsilon(\hat{\alpha}_v - r)]dt &= E\left(\frac{dX^*}{X^*}\right) \\ &= [\varepsilon\alpha_v - \lambda + \varepsilon(\varepsilon - 1)\tfrac{1}{2}\sigma_v^2]dt.\end{aligned} \tag{19}$$

Solving for ε yields

$$\varepsilon = \sqrt{\left(\frac{r - \delta_v}{\sigma_v^2} - \frac{1}{2}\right)^2 + \frac{2(r + \lambda)}{\sigma_v^2}} + \left(\frac{1}{2} - \frac{r - \delta_v}{\sigma_v^2}\right). \tag{20}$$

For the case when F is random, every occurrence of r in (20) is replaced by δ_f.

As would be expected from the analogy between an increase in the jump probability and an increase in the discount rate, an increase in λ reduces the value of the option and lowers C^*.[11] For a given C^*, if the payoff occurs, it occurs at the same time as with no jump, but has a lower present value.

IV. Numerical Examples

Tables I and II display the value of the investment option and the investment rule for a wide range of parameters. For the

10. Merton [1976] first obtained this result, when he showed that the formula for a call option written on a stock for which there is a possibility of complete ruin, is obtained by replacing r with $r + \lambda$ in the Black-Scholes formula.

11. Merton [1976] shows that the possibility of complete ruin for the stock makes a call option *more* valuable. In our case the possibility of complete ruin makes the option to invest *less* valuable. Our experiment is equivalent to having both δ_v and δ_f rise with λ. Because of the assumption that the installed project earns a fair rate of return, the uncaptured cash flow from the project (as a percentage of V) is implicitly assumed to increase as λ is increased. This differs from the assumption made in Merton [1976], in which the uncaptured cash flow (a stock dividend) is held fixed. In effect, δ_f is increased, and δ_v is unchanged. Consequently, Merton obtains the opposite comparative static results.

TABLE I

VALUE OF INVESTMENT OPPORTUNITY WHEN $V = F = 1$

δ_v		0.05			0.10			0.25	
ρ_{vf}	−0.5	0.0	0.5	−0.5	0.0	0.5	−0.5	0.0	0.5
σ_v^2,σ_f^2									
0.01	0.33	0.30	0.28	0.14	0.12	0.08	0.03	0.02	0.01
0.02	0.38	0.34	0.30	0.20	0.16	0.12	0.06	0.04	0.02
0.04	0.45	0.40	0.34	0.27	0.23	0.16	0.11	0.08	0.04
0.10	0.57	0.51	0.43	0.40	0.34	0.25	0.21	0.16	0.09
0.20	0.67	0.61	0.51	0.52	0.45	0.34	0.32	0.25	0.16
0.30	0.73	0.67	0.57	0.60	0.52	0.40	0.39	0.32	0.21
λ									
0.00	0.45	0.40	0.34	0.27	0.23	0.16	0.11	0.08	0.04
0.05	0.33	0.29	0.24	0.23	0.19	0.13	0.10	0.07	0.04
0.10	0.27	0.23	0.19	0.20	0.16	0.12	0.10	0.07	0.04
0.25	0.19	0.16	0.12	0.15	0.12	0.09	0.09	0.06	0.04
δ_f									
0.01	0.30	0.23	0.14	0.18	0.13	0.07	0.08	0.06	0.03
0.05	0.37	0.31	0.23	0.22	0.17	0.10	0.09	0.06	0.03
0.10	0.45	0.40	0.34	0.27	0.23	0.16	0.11	0.08	0.04
0.25	0.60	0.58	0.56	0.42	0.39	0.36	0.18	0.15	0.10

Note. Entries are calculated using (4) and (12) in the text. Base case parameters are $\sigma_v^2 = \sigma_f^2 = 0.04$; $\delta_v = \delta_f = 0.10$; $\lambda = 0.00$.

TABLE II

VALUE OF BENEFITS RELATIVE TO INVESTMENT COST (V/F) AT WHICH INVESTMENT IS OPTIMAL

δ_v		0.05			0.10			0.25	
ρ_{vf}	−0.5	0.0	0.5	−0.5	0.0	0.5	−0.5	0.0	0.5
σ_v^2,σ_f^2									
0.01	2.50	2.35	2.18	1.47	1.37	1.25	1.09	1.06	1.03
0.02	2.91	2.64	2.35	1.72	1.56	1.37	1.18	1.12	1.06
0.04	3.65	3.17	2.64	2.13	1.86	1.56	1.34	1.24	1.12
0.10	5.65	4.56	3.41	3.19	2.62	2.00	1.77	1.54	1.29
0.20	8.77	6.70	4.56	4.79	3.73	2.62	2.44	2.00	1.54
0.30	11.83	8.77	5.65	6.34	4.79	3.19	3.07	2.44	1.77
λ									
0.00	3.65	3.17	2.64	2.13	1.86	1.56	1.34	1.24	1.12
0.05	2.50	2.23	1.92	1.86	1.67	1.44	1.32	1.23	1.12
0.10	2.10	1.90	1.67	1.72	1.56	1.37	1.30	1.22	1.12
0.25	1.67	1.54	1.40	1.51	1.40	1.27	1.27	1.19	1.11
δ_f									
0.01	2.31	1.89	1.46	1.64	1.43	1.22	1.25	1.17	1.08
0.05	2.85	2.38	1.86	1.83	1.58	1.32	1.28	1.19	1.10
0.10	3.65	3.17	2.64	2.13	1.86	1.56	1.34	1.24	1.12
0.25	6.42	5.96	5.49	3.35	3.09	2.81	1.62	1.49	1.33

Note. Entries are calculated using equations (4) and (12) in the text. Base case parameters are $\delta_v = \delta_f = 0.10$; $\sigma_v^2 = \sigma_f^2 = 0.04$; $\lambda = 0.00$.

base case we set $\sigma_v^2 = \sigma_f^2 = 0.04$ and $\delta_v = \delta_f = 0.10$. If V is interpreted as a present value, a reasonable estimate for σ_v is the average standard deviation for unlevered equity in the United States, which is about 0.20.[12] δ_v measures the extent to which expected price increases in V alone fail to compensate investors for the risk of price changes in V. We set this value at 0.10.[13] Appropriate choices for F are less clear. If the investment cost is nonstochastic, δ_f should be the risk-free rate. If F is systematically risky (which is likely), but $\alpha_f = 0$, then it would be greater.

With no investment timing option, max[0, $V_0 - F_0$] would be the value of the investment opportunity because the opportunity could only be exercised now. The value of the investment timing option is the difference between the infinitely lived investment opportunity and max[0, $V_0 - F_0$]. Table I shows that the cost of following a suboptimal investment rule can be substantial. The entries in Table I represent the loss per dollar of V if the project were undertaken at $V/F = 1$, rather than waiting until the optimal time. The value of the option to invest can never exceed V, so 1.00 is an upper bound for the entries in Table I. For example, if $\sigma_v^2 = \sigma_f^2 = 0.02$, $\rho_{vf} = 0.00$, and $\delta_v = \delta_f = 0.10$, then the investment option is worth 16 percent of V. If the other parameters stayed the same but the drift in V were lowered by 15 percent annually, then $\delta_v = 0.25$, and the option would be worth 4 percent of V.

Table II shows that investment in the first case above would be optimal if V/F reached 1.56, and in the second case at 1.12. It is clear from Table II that the level of V/F at which investment is optimal is typically much greater than 1.00. (From equation (5), a firm will be willing to invest at $V = F$ only when $\sigma^2 = 0.00$ or $\delta_v = \infty$.) For reasonable parameters, it is optimal to wait to invest until V is more than twice F.

Tables I and II demonstrate that both the option value and investment rule are sensitive to changes in variance. The option value and investment rule are also sensitive to a change in δ_v from 0.05 to 0.10. This sensitivity is to be expected in the vicinity of $\delta_v = 0.0$, since at that point we obtain the limiting values of $X = 1.00$ and $C^* = \infty$. X and C^* are less sensitive to changes in δ_f. For options that already have little value (for example, in the

12. From Table I in Stoll and Whaley [1983] the average standard deviation of stocks on the New York Stock Exchange is approximately 0.30. Thus, the average unlevered standard deviation (assuming a debt to value ratio of 1/3) is in the vicinity of 0.20.

13. As noted in footnote 3 above, the earnings-price ratio of an installed project, with earnings measured net of depreciation, would be an estimate of δ_v.

last column of Table I), even a dramatic increase in λ from 0.00 to 0.25 has little effect on the option value. This occurs because the option value in those cases is already primarily due to variance, because of the large δ_v.

One can compare the value of the timing option with the value the option would have under certainty. Table III presents the percentage of the option value which is due to uncertainty, computed by comparing equation (4) with equation (13). To concentrate on the "pure" uncertainty component, we assume that increases in σ^2 are not accompanied by changes in δ_v or δ_f.[14] As the table shows, increases in both δ_v and σ^2, holding δ_f constant, increase the percentage of the value attributable to uncertainty. For an investment opportunity with $\delta_v \geqslant \delta_f$, all of the value is due to uncertainty, since otherwise waiting would be suboptimal.

V. Example: The Investment Decision of a Monopolist

In this section we present an example in which V and the option value are derived in terms of production and demand parameters. Consider the example of a project that produces a commodity, using a Cobb-Douglas production function,

$$Q_t = \overline{K}^{\alpha} L_t^{\beta}, \tag{21}$$

TABLE III

Percentage of Value of Investment Option which is Due to Uncertainty

	$\sigma^2 = 0.02$	$\sigma^2 = 0.04$	$\sigma^2 = 0.10$	$\sigma^2 = 0.30$
$\delta_v = 0.02$	5.3	9.4	17.2	28.4
$\delta_v = 0.04$	12.5	20.1	32.6	47.1
$\delta_v = 0.06$	25.6	36.4	50.8	64.6
$\delta_v = 0.08$	51.4	61.9	73.0	82.1
$\delta_v = 0.10$	100.0	100.0	100.0	100.0

Note. Entries are calculated using equations (4), (12), and (13) in the text. All of these calculations assume that $V = F = 1$ and $\delta_f = 0.10$.

14. The tables ignore the possibility that changes in σ_v and σ_f affect the required rates of return $\hat{\alpha}_v$ and $\hat{\alpha}_f$. This assumption would be valid if the uncertainty is uncorrelated with the market portfolio or if investors are risk neutral. If the risk is systematic, then changes in σ^2 might be accompanied by changes in δ_v or δ_f; it is then possible to use an asset pricing relationship such as (9) to specify the effect on required rates of return of changes in variance (cf. McDonald and Siegel [1985]). This can lead to ambiguity in the comparative static results. Venezia and Brenner [1979] make the same point, although without distinguishing between systematic and total risk.

where $\overline{K}$ is the fixed level of capital, and Q_t and L_t are quantity produced and labor employed at time t. The firm faces an inverse demand curve given by

$$P_t = \theta_t Q_t^{-1/\eta}, \tag{22}$$

where P_t is the price of the commodity at time t, η is the price elasticity of demand, and θ_t is a demand shift parameter following the stochastic process,

$$\frac{d\theta}{\theta} = \alpha_\theta dt + \sigma_\theta dz_\theta. \tag{23}$$

Let $\rho_{\theta m}$ be the correlation of θ with the market portfolio.

After the project is installed, instantaneous profits are given by $\pi_t = P_t Q_t - \overline{w} L_t$, where $\overline{w}$ is the wage. At each t, the firm chooses labor usage to maximize instantaneous profits. Maximized profits at t are then $B\theta_t^\gamma$, where $\gamma = [1 - \beta(1 - 1/\eta)]^{-1} > 1$ and B is a constant. When the project lives forever, the present value of expected maximized profits is

$$V(\theta_0) = B\theta_0^\gamma/(\hat{\alpha}_v - \alpha_v), \tag{24}$$

where $\alpha_v = \gamma\alpha_\theta + \frac{1}{2}\gamma(\gamma - 1)\sigma_\theta^2$ and

$$\hat{\alpha}_v = r + \phi\rho_{\theta m}\gamma\sigma_\theta, \tag{25}$$

which follows from (9) together with the fact that $\sigma_v = \gamma\sigma_\theta$ and $\rho_{\theta m} = \rho_{vm}$. Equation (24) can be rewritten as

$$\hat{\alpha}_v = \alpha_v + B\theta_0^\gamma/V_0 = \alpha_v + \delta_v, \tag{26}$$

so that δ_v represents the payout ratio of the installed project.

Table IV displays the option values and investment rules corresponding to different values of the underlying exogenous parameters. It is important to realize that the comparative static experiment in this table is different from that in Tables I and II. In particular, a change in σ_θ not only changes the option value directly, but also changes $\hat{\alpha}_v$ and α_v. It is possible for an increase in σ_θ to lower the value of the option (see footnote 14), although this does not occur in Table IV. For example, if $\eta = 1.0001$, $\alpha_\theta = -0.03$, and $\rho_{\theta m} = 0.95$, then an increase in σ_θ^2 from 0.0025 to 0.0050 (σ_v^2 increases by the same amount) will lower the value of the option from 0.0293 to 0.0286. The table shows sizable option values and optimal investment levels for reasonable parameters. Several of the entries are asterisked, which signifies that it is always optimal to defer investment. It is implausible, however,

TABLE IV

OPTION VALUES AND INVESTMENT RULES

α_θ		−0.03			0.00			0.01	
$\rho_{\theta m}$	0.00	0.20	0.50	0.00	0.20	0.50	0.00	0.20	0.50
σ_θ^2									
0.01	0.0740	0.0597	0.0459	0.1890	0.1305	0.0837	0.2902	0.1890	0.1110
	(1.22)	(1.18)	(1.13)	(1.68)	(1.43)	(1.26)	(2.25)	(1.68)	(1.35)
0.04	0.2227	0.1657	0.1177	0.4127	0.2659	0.1657	0.5493	0.3257	0.1905
	(1.85)	(1.58)	(1.38)	(3.28)	(2.09)	(1.58)	(5.28)	(2.50)	(1.69)
0.10	0.4399	0.3044	0.2056	0.7545	0.4302	0.2592	*	0.4973	0.2835
	(3.58)	(2.35)	(1.76)	(13.7)	(3.47)	(2.05)	*	(4.36)	(2.20)
0.20	0.7647	0.4736	0.3053	*	0.6318	0.3631	*	0.7151	0.3874
	(14.6)	(4.02)	(2.35)	*	(7.37)	(2.80)	*	(11.0)	(3.02)

Note. Values are option values computed using (4) and (12) in the text. Values in parentheses are optimal investment levels. Technological parameters are chosen such that $V = 1$. F is nonstochastic and equal to 1. Other parameter values are $\eta = 2.0$; $r = 0.05$; $\beta = 2/3$; and $\phi = 0.50$.

*For these values $\delta_v < 0$, so the option value is undefined.

to expect that one would never invest in a real asset, however large V becomes. This demonstrates a weakness of the assumption that θ follows geometric Brownian motion. The parameter θ represents a real shock; as such, it is unreasonable to expect uncertainty about future values of the shock to grow linearly with time. A mean-reverting process for θ (for which it is not possible to obtain an analytic solution) would capture the notion that in the long run, demand can be expected to be at a "normal" level.

One might expect the variance of total returns (changes in V *plus* payouts) from an investment to grow linearly with time. The variance of V, however, would not grow linearly with time if δ_v, the payout rate, increased with V.[15] This would place limits on V and induce a finite C^*.

VI. CONCLUSION

This paper has studied the investment timing problem and shown for quite reasonable parameters that investment timing considerations are important. In particular, the value lost by sub-

15. If the value of IBM's assets followed geometric Brownian motion, there would be a chance that IBM would become indefinitely large relative to the economy as a whole. It is more reasonable to suppose that there are limits to IBM's potential growth, although the total returns from investing in IBM have no such limits.

optimally adopting a project with zero net present value can easily range from 10 to 20 percent or more of a project's value.

The analysis has several limitations. First, as noted, the assumption of geometric Brownian motion for V and F is most plausible when these values represent present values. Even then, however, the example in Section V demonstrated that undesirable results are possible. A more realistic specification would have payout rates and variances be functions of prices, and the result would almost certainly be lower option values. For prices which are not present values, a mean-reverting process would capture the notion that prices tend toward equilibrium levels. For these more general processes, numerical methods would likely be needed for valuation of the timing option.

The analysis also imposes implicit restrictions on the investment process. The investment is assumed to be lumpy, the importance of which can be seen by supposing that investment is not lumpy and that the marginal reward from investing is large for small investments. Some investment will then always be optimal, at least initially, and the waiting problem is less important. The analysis also ignores the possibility that the investment may be partially reversed or scrapped after the project is adopted. One could interpret V as including the value of the scrapping option, but this begs the question of how such an option affects the timing problem. Reversibility is implicitly included in the analysis to the extent that a project is deemed more reversible the more quickly it depreciates. A high depreciation rate implies a high payout rate and hence a high δ_v. The timing option is worth less for such projects.

APPENDIX: DERIVATION OF $E_0\{F_{t'}e^{-\mu t'}\}$

The purpose of this Appendix is to derive the expectation on the right-hand side of (3). We discuss three results: first, we present the partial differential equation that governs the behavior of the option value; second, we derive the option value; and third, we verify that the stopping region is $[C^*,\infty)$. Let

$$L(V_3,F_3,0) = E_0\{F_{t'}e^{-\mu t'}\}. \tag{A.1}$$

Using Theorem 7.5 from Malliaris and Brock [1982] (pp. 100–01), together with the fact that $L_3 = \mu L$, we note that L must satisfy the partial differential equation:

$$\mu L = \tfrac{1}{2}[L_{vv}V^2\sigma_v^2 + L_{ff}F^2\sigma_f^2 + 2L_{vf}VF\sigma_{vf}] + L_v\alpha_v V + L_f\alpha_f F. \tag{A.2}$$

The solution to (A.2) must satisfy certain boundary conditions: (i) $L = F$ when $C = V/F = C^*$; and (ii) $L \to 0$ as $V/F \to 0$. Assume for the moment that the stopping region is $[C^*,\infty)$. Guess that the form of L is

$$L = kF^aC^b, \tag{A.3}$$

with k a constant. This guess satisfies (A.2). Boundary condition (i) then requires that $k = C^{*-b}$ and that $a = 1$. With these constraints (A.2) can be written as

$$\mu = \tfrac{1}{2}b(b - 1)\sigma^2 + b\alpha_v + (1 - b)\alpha_f, \tag{A.4}$$

where $\sigma^2 = \sigma_v^2 + \sigma_f^2 - 2\sigma_{vf}$. As long as $\alpha_f < \mu$, (A.4) will have both a positive and a negative root. Boundary condition (ii) requires that $b > 0$, so the positive solution is the correct one. This is the solution given by equation (5) in the text, with $b = \varepsilon$. Therefore, (A.3), with $a = 1$, $k = C^{*-1}$, and $b = \varepsilon$ given by (10), solves the partial differential equation (A.2). This solution works for arbitrary C^*. Choosing the optimal C^*, which is done in the text, amounts to imposing an additional boundary condition, variously known as "high contact" or "smooth pasting."

The optimal scrapping problem is formally identical to the optimal investment problem, and it is again appropriate to take the positive root. This formal similarity of options to buy and options to sell is implied by Merton [1973] and Margrabe [1978].

Finally, we argue that the stopping region is $[C^*,\infty)$. Suppose, contrary to this, that it was optimal to exercise the option with $C_0 = V_0/F_0$, but that it was not optimal to exercise the option with $C_1 = V_1/F_1 > C_0$. This would imply that

$$V_1 - F_1 < X(V_1,F_1) \quad \text{and} \quad V_0 - F_0 = X(V_0,F_0). \tag{A.5}$$

It is straightforward to show by change of variable in (A.2) and the boundary conditions that X is homogeneous of degree one in V and F. (This is done by showing that (A.2) and the boundary conditions depend only upon $V/F = C$. See also Merton [1973].) Using this homogeneity property, we see that (A.5) becomes

$$X(C_1,1) > C_1 - 1 \quad \text{and} \quad X(C_0,1) = C_0 - 1. \tag{A.6}$$

Multiplying the left-hand side of (A.6) by C_0/C_1 and again using homogeneity yields

$$X(C_0,C_0/C_1) > C_0 - C_0/C_1 \quad \text{and} \quad X(C_0,1) = C_0 - 1. \tag{A.7}$$

This implies that if there are two identical investment opportunities with different investment costs, it could be optimal to invest in the opportunity with the higher investment cost while leaving unexercised the opportunity with the lower investment cost. This is never optimal for call options, however, no matter how C is distributed. (See Cox and Rubinstein [1985], p. 140.) Thus, (A.6) must be false, and the stopping region must be $[C^*,\infty)$.

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