ALEXANDROV SPACES, KANTOROVICH SINGULARITY, SOULS AND SPLITTING THEOREMS

J. H. MARTEL

1.

Throughout this article we let (X, d) designate a noncompact, complete, connected finite-dimensional Alexandrov space. So X is a length space satisfying $\kappa \geq 0$ sectional curvature conditions. In otherwords every quadruple (a, b, c, d) in X satisfies Toponogov's comparison condition, c.f. [Vil09, Ch.26, pp.738], [MT07, pp.53–55]. For a basepoint $x_0 \in X$, let $M = M(x_0)$ be the set of all geodesics λ in X satisfying:

(i) the geodesic λ passes through x_0;

(ii) the geodesic λ is distance minimizing over every compact subinterval;

(iii) the geodesic is maximally nonextendible.

For every $x_0 \in X$, we abbreviate $M^*(x_0) \subset M(x_0)$ as the subset of noncompact geodesics.

Lemma 1. *If X is connected complete noncompact Alexandrov, then $M^*(x_0)$ is nonempty for every $x_0 \in X$.*

Proof. [MT07, Lemma 2.1] □

In otherwords there exists distance minimizing asymptotic geodesic rays. The set M of geodesics contains evidently three types: the geodesics λ are either

(a) compact; or

(b) noncompact and doubly-ended; or

(c) noncompact and singly-ended.

It is necessary to emphasize that $M^*(x_0)$ varies lower semicontinuously with respect to the choice of x_0. Lemma [ref] says Alexandrov spaces X are "ruled" by maximal minimizing geodesics in $M^*(x_0)$, $x_0 \in X$. For instance if x_0 is a regular point on an infinite flat cone, then $M^*(x_0)$ is a singleton, whereas if $x_0 = v$ is the cone vertex, then $M^*(v)$ is infinite and parameterized by an $N-1$-sphere on N-dimension cones.

Our purpose is to demonstrate how methods of Kantorovich Singularity and optimal semicouplings from [Mar] establishes two basic theorems of Alexandrov geometry

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nearly simultaneously. In case M^* contains a doubly-ended geodesic, then our arguments below will establish Gromoll-Cheeger's Splitting theorem [MT07, Thm.2.11, pp.58]; otherwise we use the geodesics of M^* to establish the Cheeger-Gromoll-Perelman's Soul theorem [MT07, Thm.2.7, pp.56] for singular Alexandrov spaces.

For any $x_0 \in X$, $\lambda \in M^*(x_0)$, let $h_\lambda : X \to \mathbb{R}$ be the unique horofunction satisfying $h_\lambda(x_0) = 0$, and defined by the usual formula

$$h_{\lambda,x_0}(x) := \lim_{t \to +\infty} d(\lambda(t), x) - t.$$

We observe $h_{\lambda,x_0}(x) \geq -d(x, x_0)$, and h_{λ,x_0} diverges to $-\infty$ along the geodesic λ. Our curvature hypothesis $\kappa \geq 0$ implies h_λ is geodesically concave function and superlevel sets $\{h_{\lambda,x_0} \geq T\}$ are totally convex subsets of X for all $T \in \mathbb{R}$. (The same definition implies h_λ is convex in nonpositive curvature $\kappa \leq 0$). If the geodesic λ is doubly-ended, then h_λ will be symmetric with respect to x_0 and approaches values $\pm\infty$ as arc-parameter $\lambda(s)$ diverges to $s \to \mp\infty$.

For $\lambda \in M^*$, we choose real numbers $t = t(\lambda) \in \mathbb{R}$ for which $0 < |t|$ is numerically small, say $t = .00001$.

Lemma 2. *For every $x_0 \in X$, if the parameter $t : M^*(x_0) \to \mathbb{R}$ is sufficiently small ($t \approx 0^+$), then the excision $X_0 := X[t]$ is a nontrivial compact totally-convex subset of X.*

Proof. The horofunctions h_λ are concave, therefore the excision $X - H_{\lambda,t}$ is a totally convex subset of X. The intersection $\cap_{\lambda \in M^*} X - H_{\lambda,t}$ is a totally convex subset. Moreover the completeness of X implies all the minimizing geodesics in X_0 are compact and Lemma [ref] implies X_0 is a compact subset. □

The excision X_0 is a compact convex boundary ∂X_0. The boundary ∂X_0 is "cellulated" by the boundaries $\partial H_{\lambda,t}$ of the excised horoballs. Moreover the following claim may be established:

Lemma 3. *The inclusion $X_0 \hookrightarrow X$ is a homotopy-isomorphism, and there exists a continuous strong deformation retract $X \rightsquigarrow X_0$.*

Proof. □

The above constructions lead us to our semicoupling program. The excision X_0 has a canonical Hausdorff measure $\sigma := \mathscr{H}_{X_0} = \mathscr{H}_X 1_{X_0}$, and the excision boundary ∂X_0 has canonical Hausdorff measure $\tau := \mathscr{H}_{\partial X_0}$. The measures σ, τ are designated our source and target measures, respectively.

We need determine cost. For pairs $x, y \in X$, we may compare $d(x, y)$ to the signed distances between horospheres $h_\lambda(x) - h_\lambda(y)$, which we observe is independant of the basepoint x_0 defining $h_\lambda = h_{\lambda,x_0}$. Concavity of h_λ implies the function

$$b(x, y) := \inf_\lambda \{h_\lambda(x) - h_\lambda(y)\} \tag{1}$$

is concave in the x-variable, for every choice of $y \in X$. If $(x, y) \in X_0 \times \partial X_0$, then $b(x, y) \geq 0$ with equality if and only if $x \in X_0$ and occupies the same horosphere component as y. Compactness of X_0 implies the superlevels of $\{x \in X_0 \mid b(x, y) \geq T\}$, for fixed $y \in \partial X_0$, $T \geq 0$, are compact convex subsets of X_0.

The triangle inequality implies

$$0 \leq b(x, y) \leq h_\lambda(x) - h_\lambda(y) \leq d(x, y), \text{ for } (x, y) \in X_0 \partial X_0 \tag{2}$$

with equality $h_\lambda(x) - h_\lambda(y) = d(x, y)$ if and only if x, y lie on minimizing ray λ. Observe $0 < b(x, y)$ whenever $x \in X_0 - \partial X_0$.

The pairing $b : X \times X \to \mathbb{R}_{\geq 0}$ defined by equation (1) is a (possibly degenerate) distance function. Throughout $X_0 \times X_0$ we find $b \geq 0$ satisfies a triangle inequality and is symmetric $b(x, y) = b(y, x)$ if we add absolute-values to (1). However caution needs be exercised since b is possibly degenerate, having $b(x, y) = 0$ for $x \neq y$. For instance if M^* is a singleton, then $b(x, y) = 0$ if x, y both occupy the same horosphere centred at λ.

The basic idea of this article is to treat $b(x, y)$ as a type of distance on X, and restrict b to the subset $X_0 \times \partial X_0$ defined earlier in Lemma 2.

Having nominated a distance $c = b$, we next turn to distance maximizing transports. Indeed distance minimization appears less useful for our purposes given the nonnegativity (2), where equality is if and only if $x \in \partial X_0$.

Definition 4. Fix x_0, M^*, definition of b (1). Let $X_0, \partial X_0$ be excisions with small parameter t. Let μ, ν be the measures on $X_0, \partial X_0$. Then let $\Pi' = \Pi'_{x_0}$ be the set of maximizers of the following maximization program:

$$\max_{\pi \in SC(\mu,\nu)} \int_{X_0 \times \partial X_0} b(x, y) d\pi(x, y). \tag{3}$$

The regularity properties of maximizing measures $\pi \in \Pi_{x_0}$ will have strong dependance on the basepoint x_0. Indeed when b is degenerate, then Π' is not a singleton, and we need choice of "canonical" coupling π'. This canonical choice is achieved by a second variational problem, say as described by [ref], [**Kim-Pass**].

Choose an auxiliary $\alpha : [0, +\infty) \to [0, +\infty)$ which is strictly convex nondecreasing function such that

$$\int_{X_0 \times \partial X_0} \alpha(d(x, y)) d\mu(x) \otimes d\nu(y) < +\infty.$$

E.g. $\alpha(t) = t^2/2$ or $\alpha(t) = \sqrt{1 + t^2}$. A canonical choice of π^* in Π', with respect to auxiliary functions α, is the coupling which minimizes the total $\alpha \circ d$ cost:

$$\min_{\pi \in \Pi'} \int_{X_0 \times \partial X_0} \alpha(d(x, y)) d\pi(x, y). \tag{4}$$

We let $\Pi'' \subset \Pi'$ be the set of b-maximal couplings which have minimal $\alpha \cdot d$ cost. Here d refers to the distance of the initial Alexandrov space X.

The existence and structure of minimizers π^* of (4) follows from the arguments of [ref]. Here is the key property. Define a secondary cost $\tilde{c}$ by the rule

$$\tilde{c}(x,y) := \begin{cases} \alpha(d(x,y)), & \text{if } -\psi(y)+\phi(x) \leq b(x,y), \\ +\infty, & \text{else .} \end{cases} \tag{5}$$

If $\pi \in \Pi'$ is any maximizer, then $b(x,y) = -\psi(y) + \phi(x)$ for π-a.e. throughout $X_0 \times \partial X_0$. With the above definition of $\tilde{c}$, we find $\pi \in \Pi'$ if and only if $\tilde{c} < +\infty$ π-a.e., and then $\tilde{c} = \alpha(d)$ π-a.e.. It follows that we can rephrase the secondary variational problem (4) as

$$\inf_{\pi \in \Pi} \int_{X_0 \times \partial X_0} \tilde{c}(x,y) d\pi(x,y). \tag{6}$$

[Thm 3.1][ref] implies the existence of a $\tilde{c}$-cyclically monotone set Γ on which $\tilde{c}$ is finite and $\pi \in \Pi''$ is concentrated.

We sketch a second construction of canonical measures $\pi \in \Pi''$ minimizing (4). For $\epsilon > 0$, let

$$b_\epsilon(x,y) := b(x,y) - \epsilon\alpha(d(x,y))$$

be defined for $(x,y) \in X_0 \times \partial X_0$. One readily deduces the existence of b_ϵ-maximal semicouplings π_ϵ. As $\epsilon \to 0^+$, we may extract a convergent subsequence π_k, for $k = 1, 2, \ldots$, with weak-* limit π_0. Then one finds the limit coupling π_0 is minimizer of (4).

Proposition 5. *For every basepoint $x_0 \in X$, there exists a unique b-maximizing coupling $\pi'' \in \Pi'_{x_0}$ which has minimal $d^2/2$-transport cost. I.e. $\Pi''_{x_0} = \{\pi''\}$ is a nonempty singleton.*

Proof. □

[Background: Kantorovich potentials]

If $\psi : \partial X_0 \to \mathbb{R} \cup \{+\infty\}$ is a potential, then the b-transform of ψ is defined

$$\psi^b(x) := \sup_{y \in \partial X_0} \{\psi(y) + b(x,y)\}.$$

If $\phi : X_0 \to \mathbb{R} \cup \{-\infty\}$ is a potential on X_0, the b-transform of ϕ is defined

$$\phi^b(y) := \inf_{x \in X_0} \{\phi(x) - b(x,y)\}.$$

A potential ψ is b-convex, and satisfying $(\psi^b)^b = \psi$ if

$$\psi(\overline{y}) = \inf_{x \in X_0} \sup_{y \in \partial X_0} \{\psi(y) + b(x,y) - b(x,\overline{y})\}$$

for all $\overline{y} \in \partial X_0$.

The reader will observe that $\psi(y) \leq \psi^{bb}(y)$ for arbitrary, possibly nonconvex functions ψ. The key observation is the inequality

$$\psi^b(x) - \psi(y) \geq b(x, y),$$

with equality if and only if $x \in \partial^b \psi(y)$ for b-convex potentials $\psi^{bb} = \psi$. The following technical lemma is important to our results.

Lemma 6. *Let ψ be a b-convex potential on ∂X_0, $\psi^{bb} = \psi$. For all $y \in spt(\psi) \subset \partial X_0$, the b-subdifferential $\partial^b \psi(y)$ is a totally convex subset of X_0.*

Proof. □

Consequently if $Z = Z(\mu, \nu, b)$ is the Kantorovich functor $Z : 2^{\partial X_0} \to 2^X$, then for all closed subsets Y_I of ∂X_I, the cell $Z(Y_I) := \cap_{y \in Y_I} \partial^b \psi(y)$ is a totally convex subset of X_0.

Remark. Our thesis constructs deformation retracts via gradient flow towards the poles of a vector field denoted $\eta(x, avg)$. To apply the methods of [Mar] to singular Alexandrov spaces requires the definition of nonsmooth gradients and gradient projections. Petrunin-Perelman prove the existence of well-defined gradient flows in singular Alexandrov spaces [PP94].

2. Splitting

Theorem 7. *Let μ be source measure on X_0, and ν a target measure on ∂X_0, and with cost b as defined in (1) with respect to a basepoint x_0.*

If $\int_{X_0} d\mu / \int_{\partial X_0} d\nu \approx 1^+$, then the active domain A of the "canonical" b-maximal semicoupling π^* defined in (4) is a strong deformation retract of X_0.

Moreover if M^* contains a doubly-ended minimizing ray, then the active domain $A = Z_1$ splits isometrically $A \simeq [-T, +T] \times Z_2$, where Z_2 consists of all source points $x \in X_0$ such that $\partial^b \psi^b(x) \geq 2$.

Incomplete. □

The cross-difference $||\nabla_x b_\Delta|| \neq 0$ is nonvanishing throughout X, and this implies $b_\Delta : X \to \mathbb{R}$ is a submersion in the topological sense, i.e. there exists a topological splitting $X \simeq X' \times \mathbb{R}$. However the usual Splitting Theorem of Cheeger-Gromoll requires the construction of Riemannian submersion. Recall a Riemannian submersion $f : X \to \mathbb{R}$ is a smooth function such that $||\nabla_x f|| = 1$ throughout X. The existence of a Riemannian submersion f on X implies the existence of isometric splitting $X \simeq f^{-1}(pt) \times X'$, where $f^{-1}(pt)$ is a generic fibre and X' is a closed subvariety of X.

3. Nonnegative Ricci Curvature

Let (X, g) be Riemannian manifold with nonnegative Ricci curvature. Then for every ray λ in X the horofunction $h_\lambda : X \to \mathbb{R}$ is superharmonic. This means the divergence of the gradient flow $div(\nabla_x h_\lambda) \leq 0$ is nonpositive throughout X, and for every subdomain D of X, the restriction $1_D \cdot h_\lambda$ achieves its absolute minimum along the boundary ∂D.

The important Splitting theorem of Cheeger-Gromoll [ref] is the following:

Theorem 8. *Let (X, g) be a complete Riemannian manifold with nonnegative Ricci curvature. Suppose $M^*(x_0)$ contains a doubly-ended minimizing ray for some basepoint $x_0 \in X$. Then there exists a totally convex subset Y of X and an isometric splitting $X \simeq Y \times \mathbb{R}$.*

It's well-known that Toponogov proved the existence of isometric splittings when (X, g) is a smooth Alexandrov manifold [ref]. Our goal is to establish a splitting theorem for singular Alexandrov spaces using the Kantorovich Singularity functor. [Include McCann's interpretation using convex functions: proves Toponogov for singular Alexandrov?]

References

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