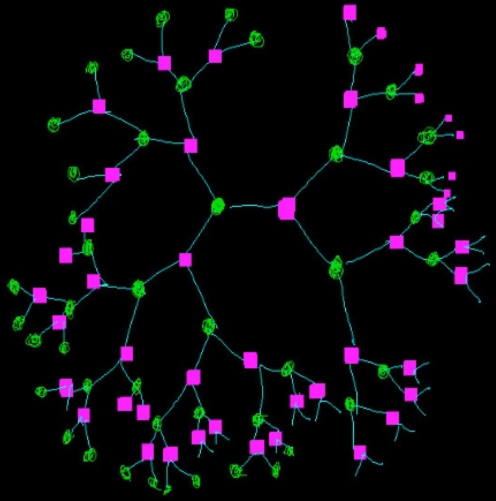
Algebraic Topology
and Optimal Transportation:

...How to build Spines from Singularity...

J.H.Martel



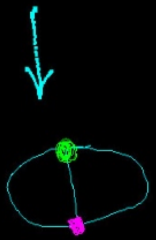
X EΓ-model

=Universal Cover of space M

$\pi_1(M) = \Gamma$

group Γ = Space Symmetries of universal cover X

Group-cohomology of Γ = *Topological Symmetries of* EΓ *action*



X / Γ quotient with $\pi_1 = \Gamma$

Poincare ~1895 constructs Algebraic-Topology:

Fundamental group, Universal Covering Spaces, orbit quotients.

Our thesis introduces and develops a general technique
for explicitly constructing **SMALL**-dimensional $E\Gamma$ classifying spaces.

Hypothesis: when Γ is infinite, discrete, Bieri-Eckmann duality group

with finite cohomological dimension $cd(\Gamma)=\nu < +\infty$ and dualizing module **D**.

Ex: $\Gamma= \mathbf{Z}, \mathbf{Z}^2, \mathbf{Z}^3, \ldots$ torsion-free abelian groups

= GL(**Z**^2), GL(**Z**^3), Sp(**Z**^4), Sp(**Z**^6), … arithmetic groups **G**(**Z**) of **Q**-reductive groups,

= MCG(Σ_g) … mapping class groups of closed surfaces,

= …, knotgroups,

= …., etc.

Our method presumes a user has initial explicit geometric $E\Gamma$ model X

i.e. (X,d) is finite-dimensional Cartan-Hadamard space (NPC contractible)

where group action $X \times \Gamma \to X$ isometric, proper discontinuous, free,

and quotient X/Γ finite volume.

……. all models not created equal…..

Ex: - MCG(Σ_g) acts isometrically on Teich(Σ_g) with Weil-Petersson geometry.

- G(Z) acts isometrically on spaces of quadratic (or hermitian) forms.

- GL(Z^2) acts isometrically on Voronoi's cone, projectivizes to H^2
(Poincare disk)

If X is geometric EΓ model, then typically dim(X)>> …. >> cd(Γ).

Bieri-Eckmann formula: $\boxed{\mathrm{cd}(\Gamma) = \dim(X) - (q + 1),}$ q=spherical-dimension(D).

(homological duality)

Our thesis constructs **Γ-invariant closed subsets Z of X, with dim(Z) ≈ cd(Γ)**,

for which **the inclusion Z → X is homotopy-isomorphism**, and

explicit **Γ-equivariant continuous deformation retracts X onto Z,** and

we describe technique for **achieving Z with MAX codimension dim(Z) != cd(Γ)**.

Definition: such maximal-codimension retracts Z are called *minimal spines/souls.*

Spines and Souls : Tradition in Geometric-Homology.:

Klein, Minkowski, Poincare, Steenrod, Thom, Lefschetz, Thurston, Gromov, Neeman, Mumford, Gromoll-Cheeger-Perelman, Soule, Ash, McConnell, ….

...how to construct NEW models of old spaces, and as explicit as possible?

"Textbook" constructions of EΓ are abstract/external/dislocated

- requires perfect knowledge of Γ, i.e. generators and relations.

- Milnor: EΓ=joins(Γ,Γ,Γ,...).
- Wall: inductive wedges of spherical-complexes and attaching maps.
- Postnikov towers, Cayley graphs, Rips complex, … .

We presume limited knowledge of Γ,
but *require explicit geometric* EΓ-model (X,d,vol_X).

Our thesis develops a new program for In Situ reduction-to-spine,

- ! construct spines as explicit subsets of initial model X !

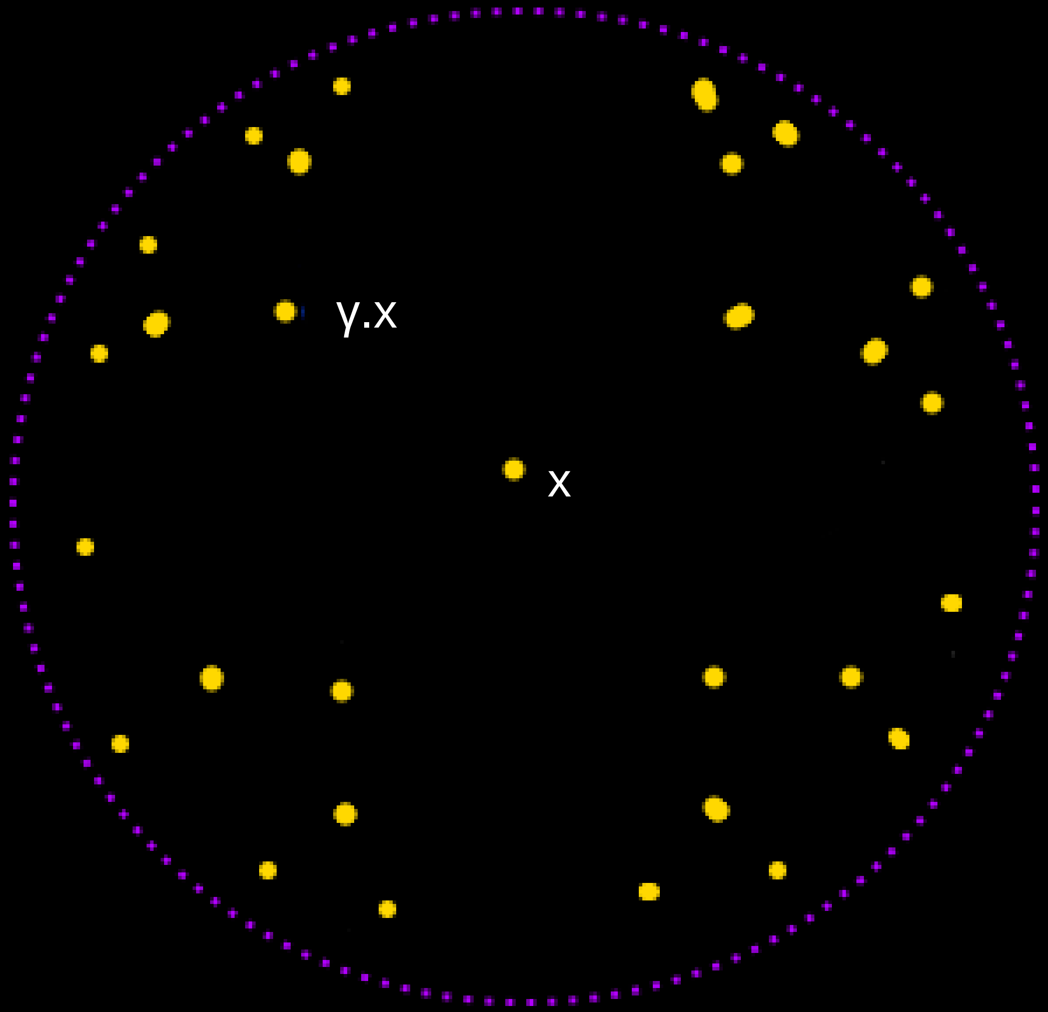
- ready-made for implementation on WOLFRAM.

- Nonlinear extension of Soule-Ash's Well Rounded Retract !

...illustrating our approach:

Initial EΓ model (X,d,volX).

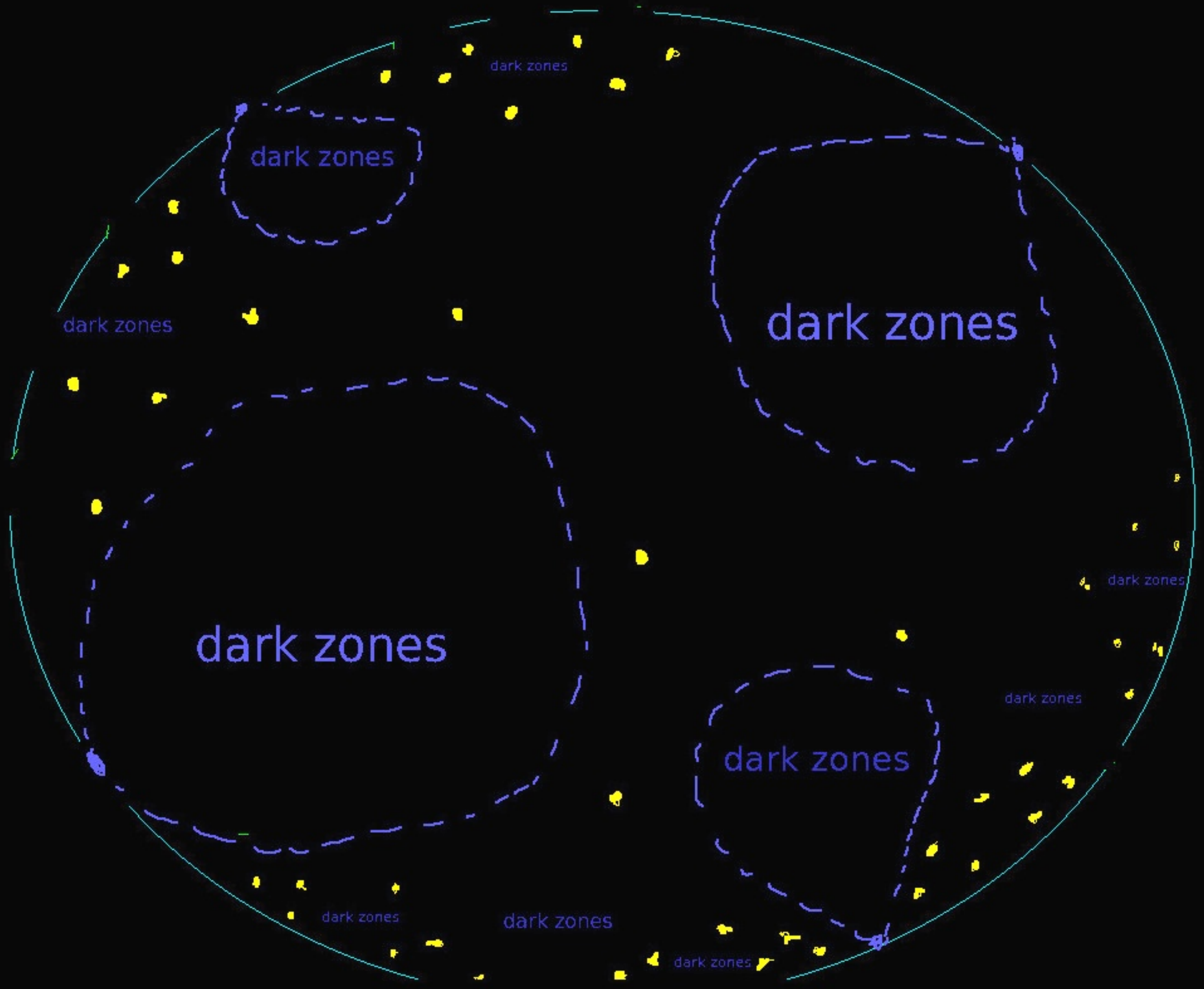
Yellow dots = Γ orbit of pt. x



- Γ orbit avoid dark zones.

Dark zones
=
Γ-rational horoballs
$V[t]$

Excision $X[t]$ obtained by scooping out / excising the dark zones from X.



Orbit avoids Dark zones

- Γ orbit avoid dark zones.

Dark zones
=
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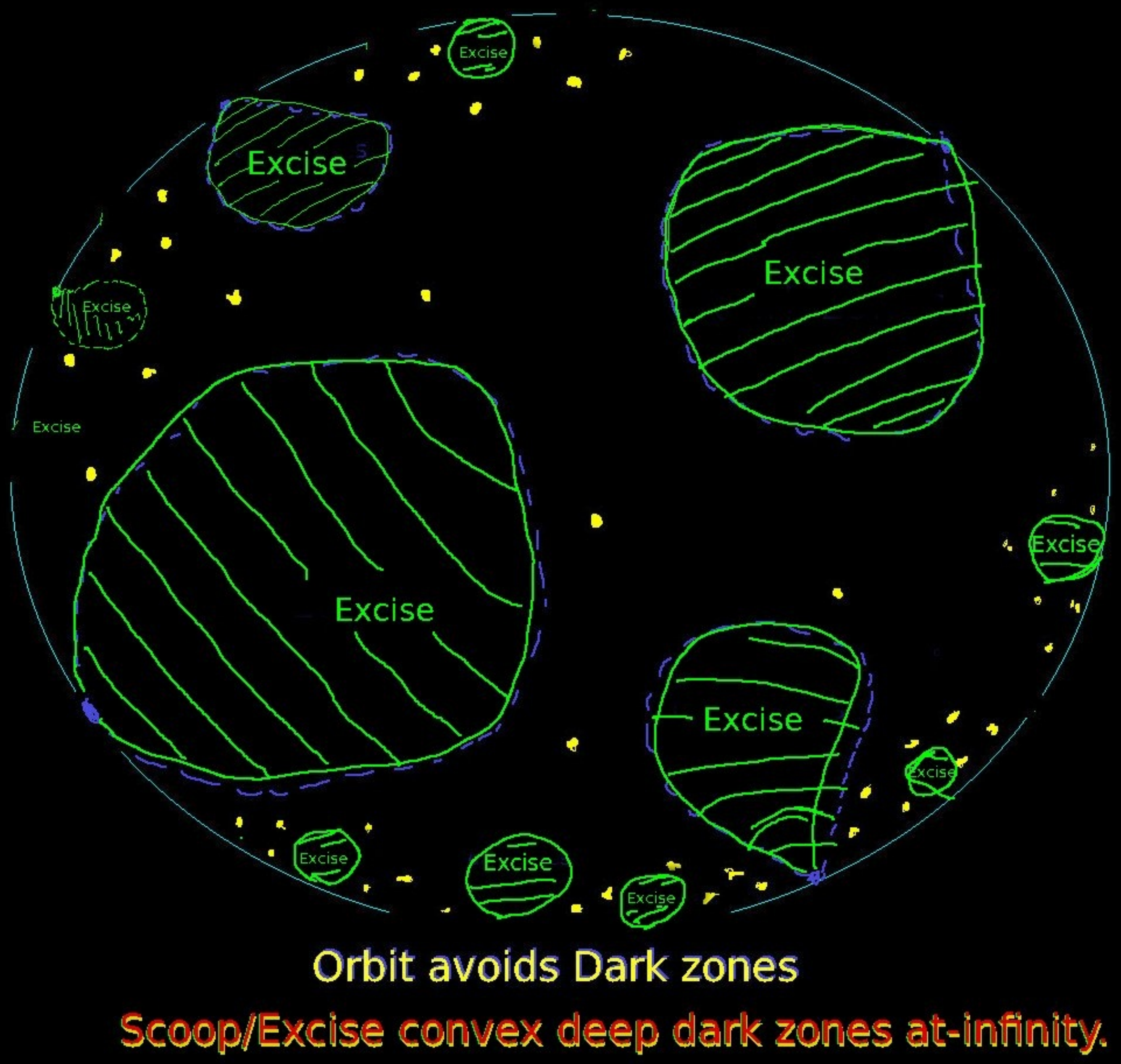
Excision $X[t]$ obtained by scooping out / excising the dark zones from X.

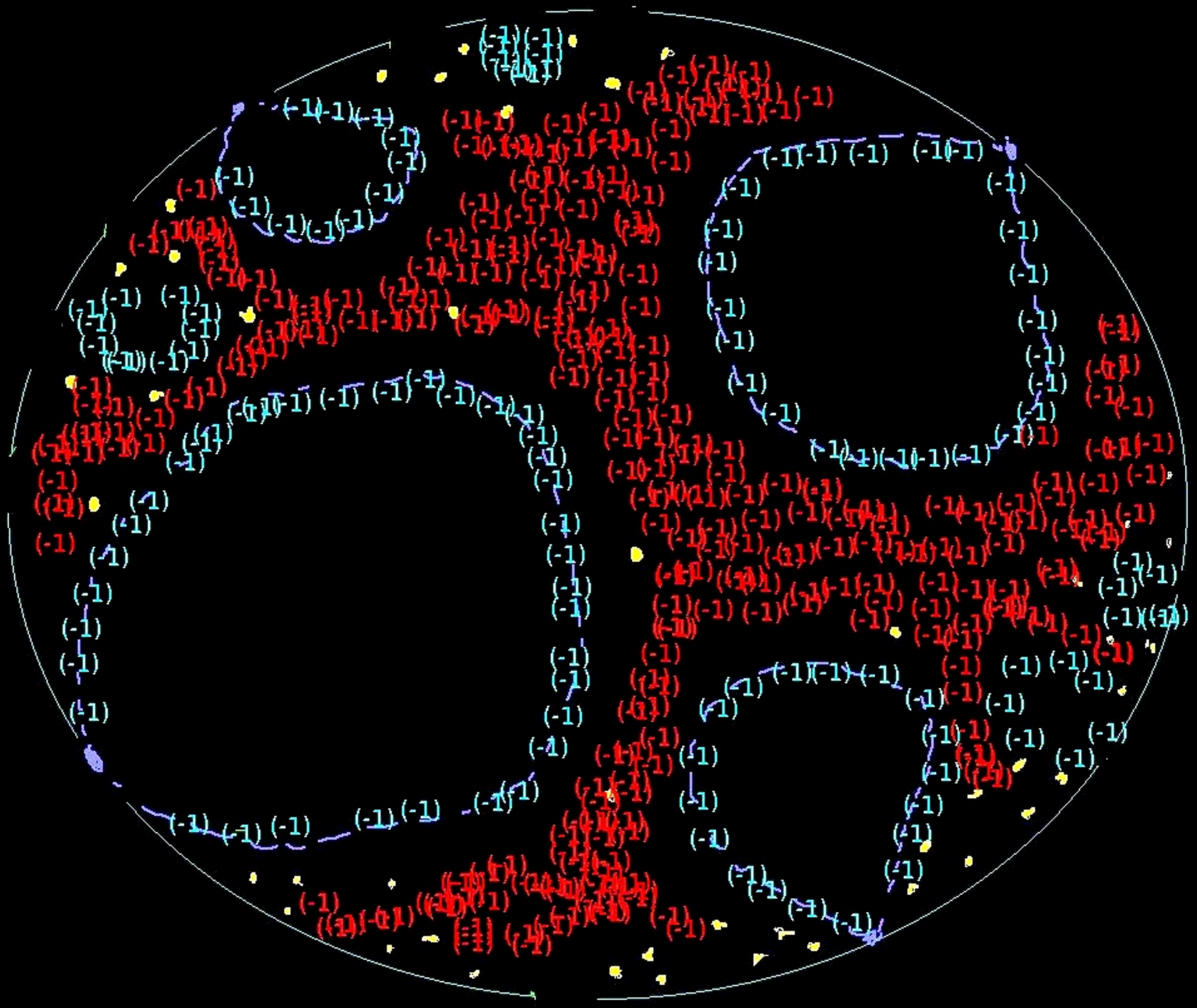


Excision model is
manifold-with-corners
X[t] x δ X[t] .

Next: define
source measures σ
and target measures τ

(-1) source
measure
on X[t]

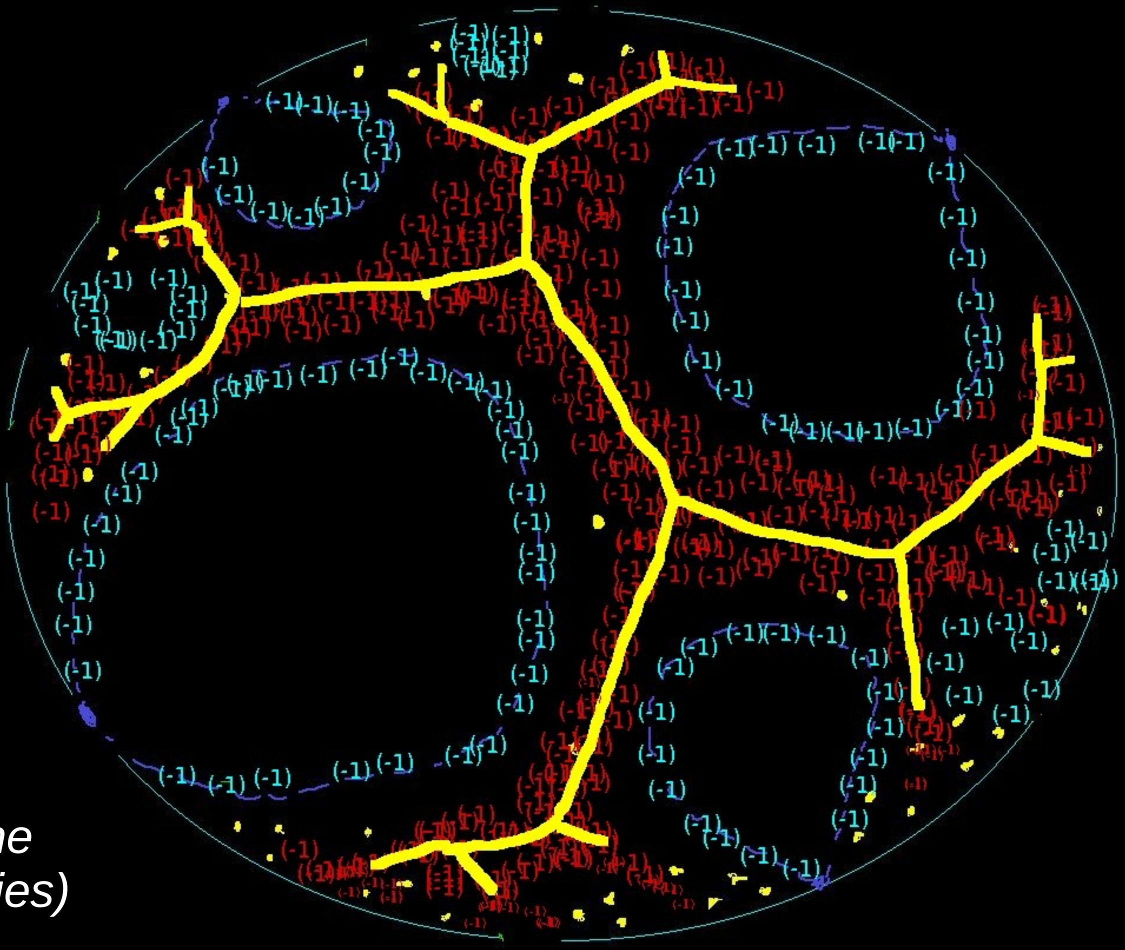
(-1) target measure
on δX[t]



With measures σ, τ, we next study the singularities of c-optimal semicouplings from source to target.

$$Z: 2^{\delta X[t]} \rightarrow 2^{X[t]}$$

We propose:
Spines are readily displayed in the locus-of-discontinuity (Singularities) of optimal semicouplings.



Singularity structure on the activated energy-minimizing configuration

...Spines are not hidden…

...Spines are not hidden…

Spines are readily displayed
in *locus-of-discontinuity Z*
of deformation retracts $r : X[t] \to \delta X[t]$

...Spines are not hidden…

Spines readily displayed in
Kantorovich’s Contravariant Singularity Functor Z(σ,τ, c*)
of c*-optimal semicouplings π from (X[t], σ) to (δX[t], τ)

source target

… *where* c*: X[t] x δX[t] *is our two-pointed repulsion cost (defined below).*

Terms to define:

Singularity functor $Z(\sigma, \tau, c^*)$

of c*-optimal semicouplings π

from source $(X[t], \sigma)$ to target $(\delta X[t], \tau)$

Terms to define:

<u>Topology:</u>

Source excision models $(X[t], \sigma)$

Target $(\partial X[t], \tau)$.

Steinberg modules $D := \tilde{H}_q(\partial X[t]; \mathbb{Z})$.

Steinberg symbols $B \in H_q(\partial X[t]; \mathbb{Z})$

and $FILL[B] = H_{q+1}(X[t], \partial X[t]; \mathbb{Z})$.

Chain sums $\underline{F} = \sum_{\gamma \in \Gamma} F.\gamma$

with well-separated gates $\{G\} = \{$FILL[B].γ $\mid \gamma \in \Gamma\}$.

Terms to define:

Mass transport:

Costs $c : X[t] \times \partial X[t] \to \mathbb{R}$.

Two-pointed repulsion and visibility costs c* , v

c-optimal semicouplings π.

c-concave potentials $\psi^{cc} = \psi$.

c-subdifferentials $\partial^c \psi(y) \subset X[t]$ for $y \in \partial X[t]$.

Monge-Kantorovich duality: c-optimal semicouplings π supported on graph of $\partial^c \psi$.

Kantorovich Singularity functor $Z : 2^{\partial X[t]} \to 2^{X[t]}$.

Filtrations $Z_0 \hookleftarrow Z_1 \hookleftarrow Z_2 \hookleftarrow \cdots$.

Kantorovich's Contravariant Singularity Functor

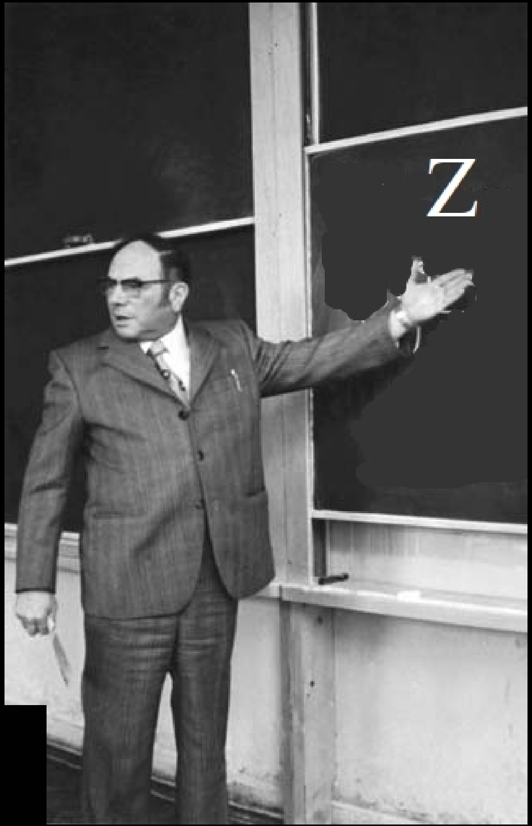
$$Z : 2^{\partial X} \to 2^X, \quad Z(Y_I) = \cap_{y \in Y_I} \partial^c \psi(y) = \text{“} \cap_{y \in Y_I} r^{-1}(y) \text{”}.$$

… but everything summarized in: Kantorovich's Contravariant Singularity Functor

$$Z : 2^{\partial X} \to 2^{X},$$

$$Z(Y_I) = \cap_{y \in Y_I} \partial^c \psi(y)$$

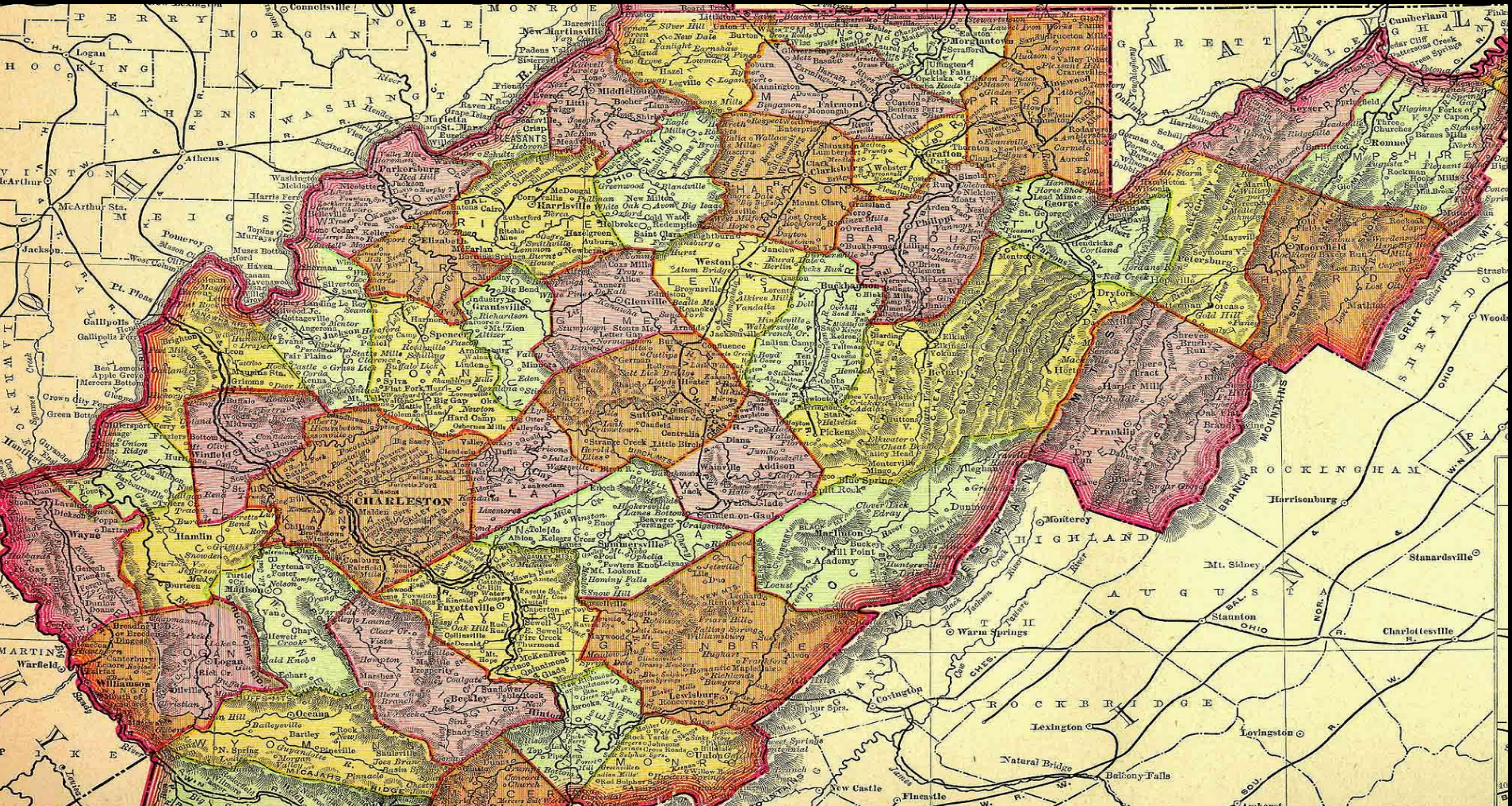
$$= \text{``} \cap_{y \in Y_I} r^{-1}(y) \text{''}.$$

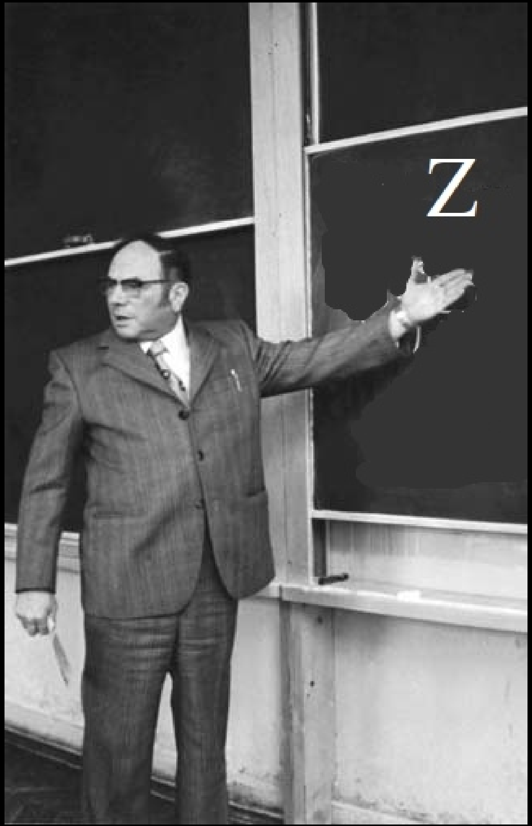


$$Z : 2^{\partial X} \to 2^{X}, \qquad Z(\mathrm{Y}_I) = \cap_{y \in \mathrm{Y}_I} \partial^c \psi(y)$$

...**Singularity** is overburdened term.

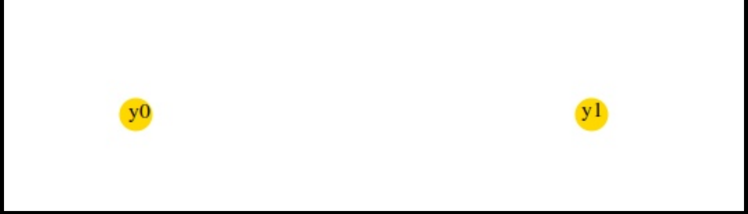
Economic Definition:
Singularity arises wherever there is competition for limited common resources.
Singularity is Why countries exist with borders.





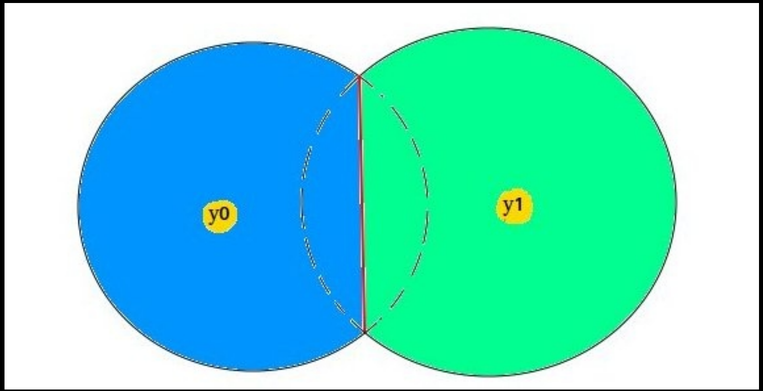
Singularity: *wherever competition for limited resources:*

$$Z : 2^{\partial X} \to 2^{X},$$

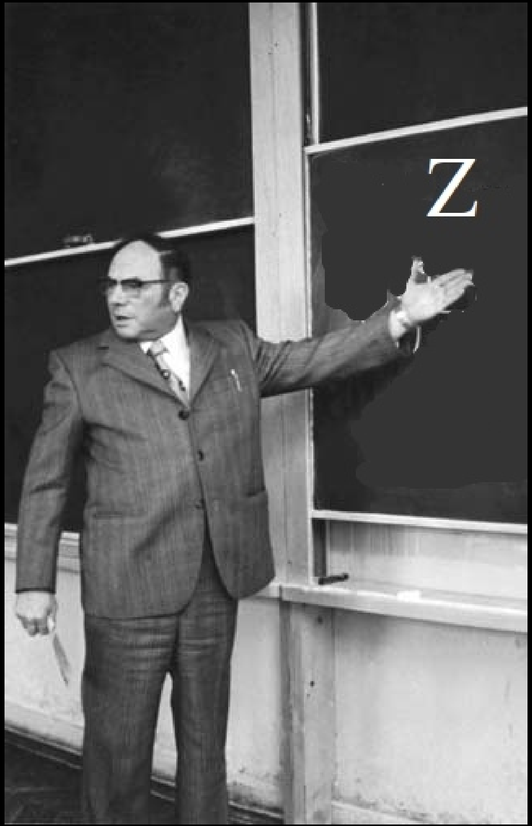


c=d**2/2 quadratic cost
(+)→ ← (–) attraction

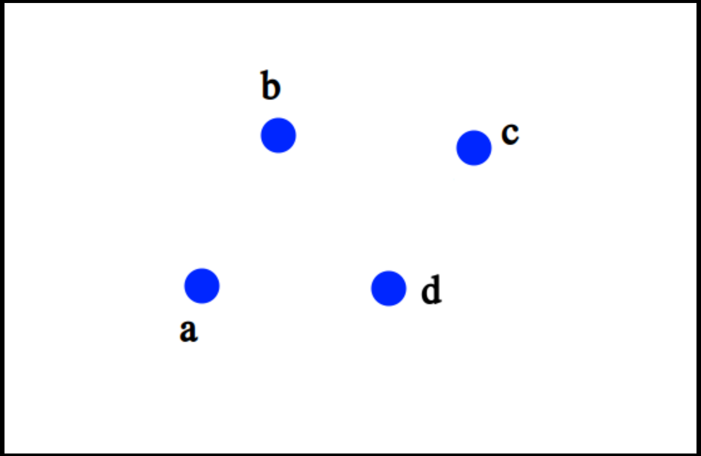
- y0,y1 no competition (no interact)
- Singularity=Empty



- y0,y1 compete/interact
- *Singularity* nonempty and *stable/persistent* w.r.t. target mass

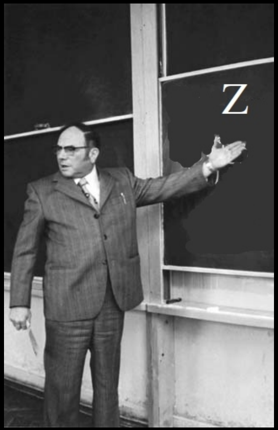


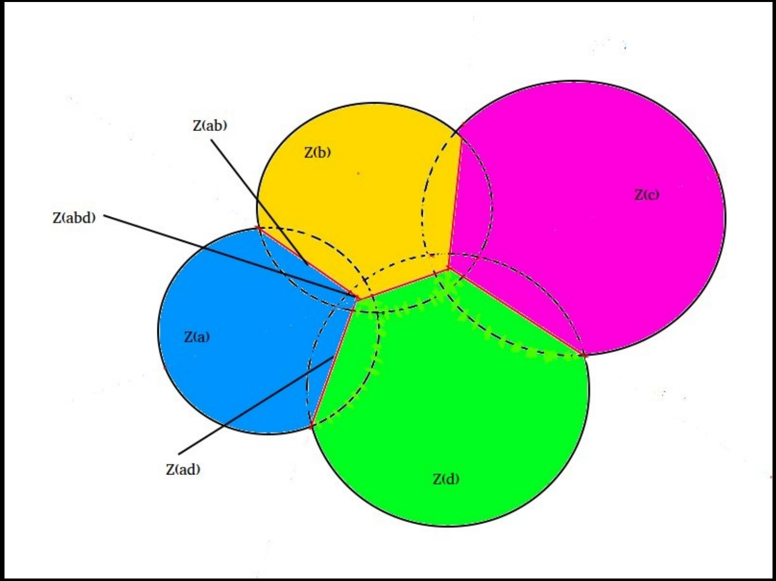
$$Z(Y_I) = \cap_{y \in Y_I} \partial^c \psi(y)$$

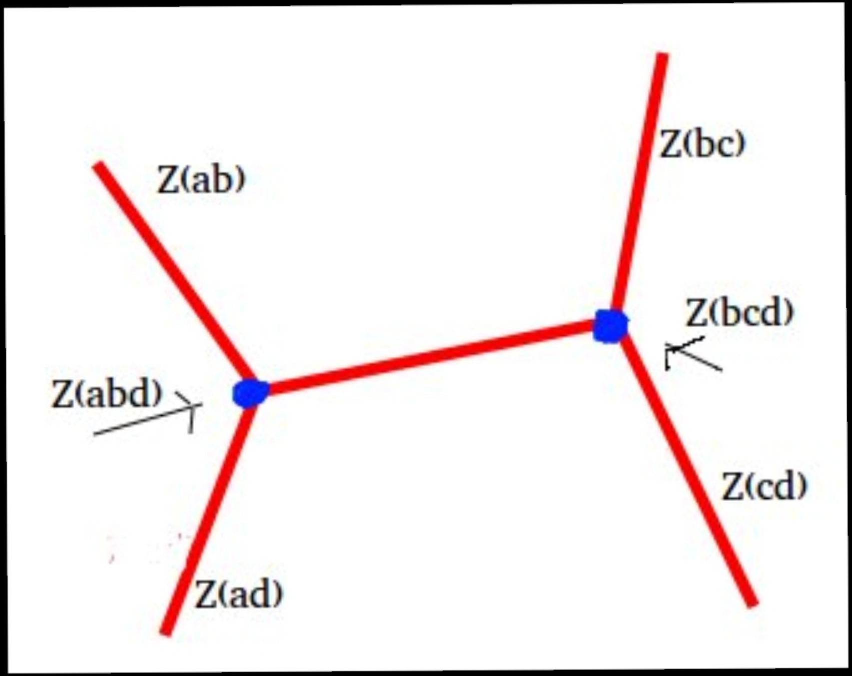


Source = 2-dim background field

Target = +a+b+c+d (4 point masses)



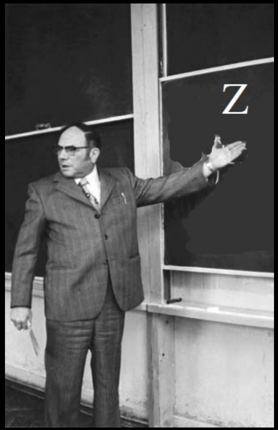




c=d**2/2
quadratic
cost

(+) → ← (–)
attraction

Kantorovich Functor is EXPLICIT.



Key definitions: - c-concave potentials $\psi^{cc} = \psi$.

- c-subdifferentials $\partial^c \psi(y)$

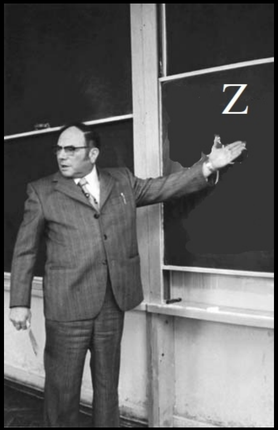
Kantorovich says ``*c-optimal semicouplings π are supported on graph of c-subdifferential of c-concave potentials potentials $\psi = \psi^{cc}$* "

Monge-Kantorovich Duality:

$$\max_{\psi\ c\text{-concave}} \left[\int_X -\psi^c(x)\,d\sigma(x) + \int_{\partial X} \psi(y)\,d\tau(y) \right] = \inf_{\pi \in SC(\sigma,\tau)} \int_{X \times \partial X} c(x,y)\,d\pi(x,y).$$

$-\psi^c(x) + \psi(y) =< c(x,y)$

<u>*Kantorovich's Contravariant Singularity Functor*</u> IS EXPLICIT.



- c-concavity $\psi^{cc} = \psi$ of a potential $\psi : \partial X[t] \to \mathbb{R}$ represents a pointwise inequality

$$-\psi^c(x) + \psi(y) \leq c(x, y)$$

for all $(x, y) \in X[t] \times \partial X[t]$,

with equality $\psi(y) - \psi^c(x) = c(x, y)$

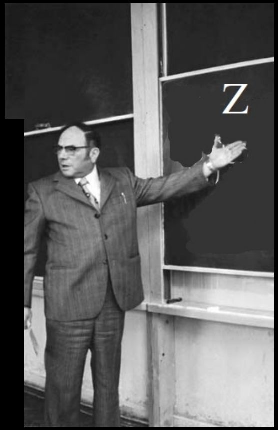
if and only if $y \in \partial^c \psi^c(x)$ iff $x \in \partial^c \psi(y)$

iff $y \in argmax[\{\psi(y_*) - c(x, y_*) + c(x, y) | y_* \in Y\}]$.

Variational defn. of c-subdifferential

Define Z({y}) := $\partial^c \psi(y)$

Kantorovich's Contravariant Singularity Functor IS EXPLICIT.



$Z(Y_I)$ consists of $x \in X$ for which

Variational defn. of c-subdifferential

$$argmax[\{\psi(y_*) - c(x, y_*) + c(x, y) | y_* \in Y\}] \text{ contains } Y_I \ ,$$

where $y_0 \in Y_I$ is reference point.

Abbreviate $c_\Delta(x; y, y') := c(x, y) - c(x, y')$ ....two-pointed cross difference.

Implies equations

$$Z(Y_I) = \{0 = \psi(y_0) - \psi(y) - c_\Delta(x; y_0, y) \mid \ y, y_0 \in Y_I, y \neq y_0\}$$

Reduces to $\#(Y_I) - 1$ equations. Symmetry y_0, y_1.

Applications to Algebraic Topology:

Contravariance says Z(Y_I)'s are local cells in X[t], parameterized contravariantly by subsets Y_I of Y=δX[t]

if $Y_I \hookrightarrow Y_J$. then $Z(Y_I) \hookleftarrow Z(Y_J)$

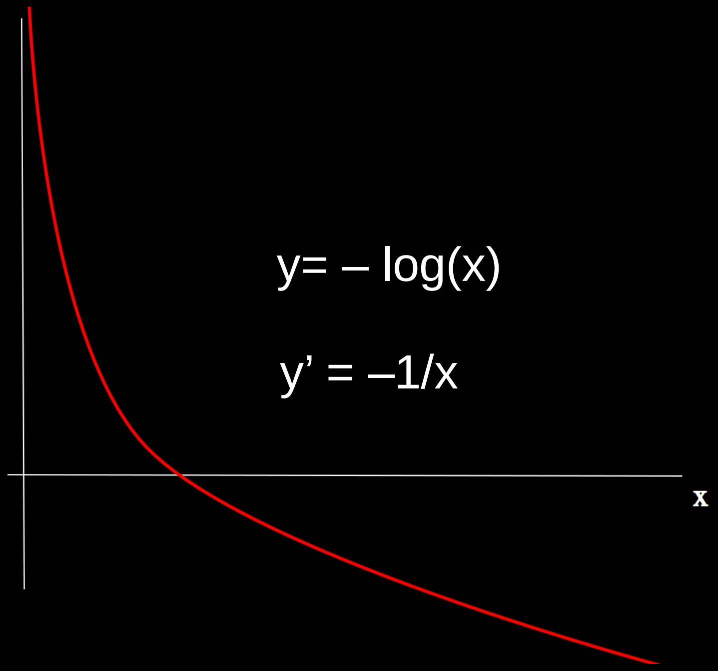
Theorem (Local Reduction)

We find local criterion (UHS conditions) which ensures Z(Y_I) ← Z(Y_J) is homotopy-isomorphism, and construct explicit continuous deformation retracts wherever (UHS) satisfied.

Proof: - Variational definition of c-subdifferentials,
and gradient flow toward positive poles (not zeros!).

- Modelled on gradient flow to +∞ of f(x)= – log(x)

- flow accelerates into the + ∞ cusp !



Applications to Algebraic Topology:

If we “skewer the cube diagonally” and filtrate according to dimension, we obtain descending chain of closed subsets

$X[t] \leftarrow Z\{1\} \leftarrow Z\{2\} \leftarrow Z\{3\} \leftarrow \ldots,$ ………………… where codim $Z\{k\}=k-1$

-Contravariance implies $Z\{1\}$, $Z\{2\}$, $Z\{3\}$,… are homology-cycles in X, $\delta Z\{k\}=0$ ….. consequence of adjunction formula

Theorem (Global Reduction):

We identify index J >= 0 such that local cells $\{Z(Y\_I) \mid Y\_I \to Y\}$ and their local homotopy reductions assemble into global continuous reductions

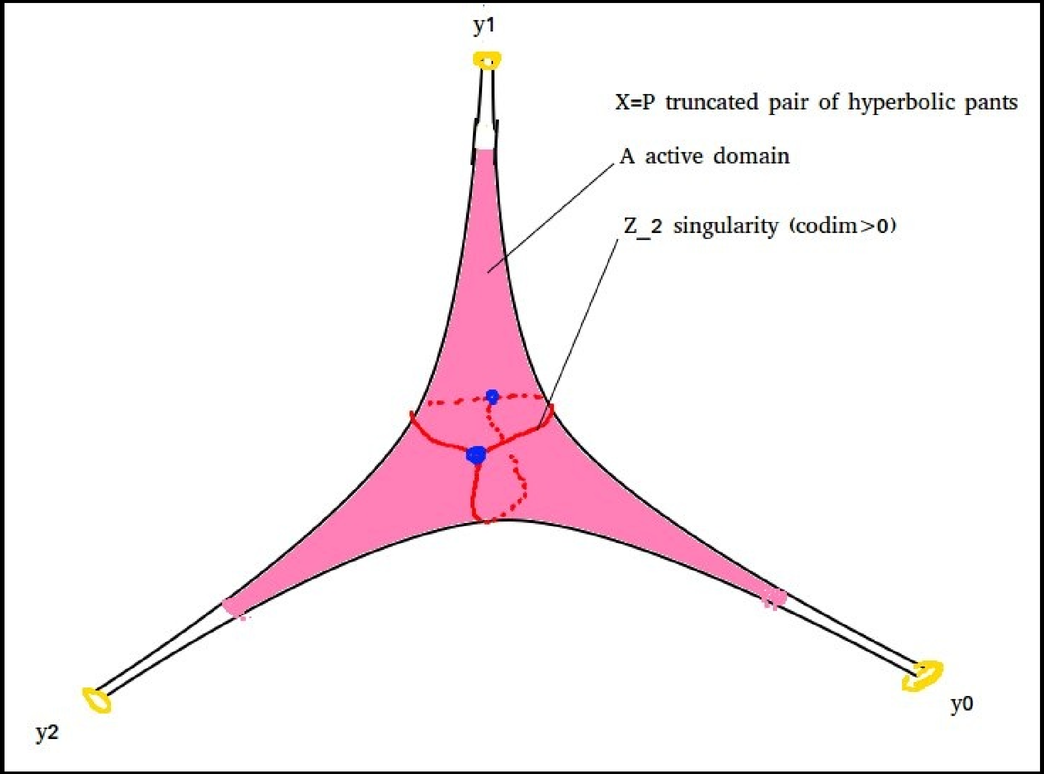
$X \to X[t] \to Z1 \to Z2 \to \ldots \to Z\{J+1\}$

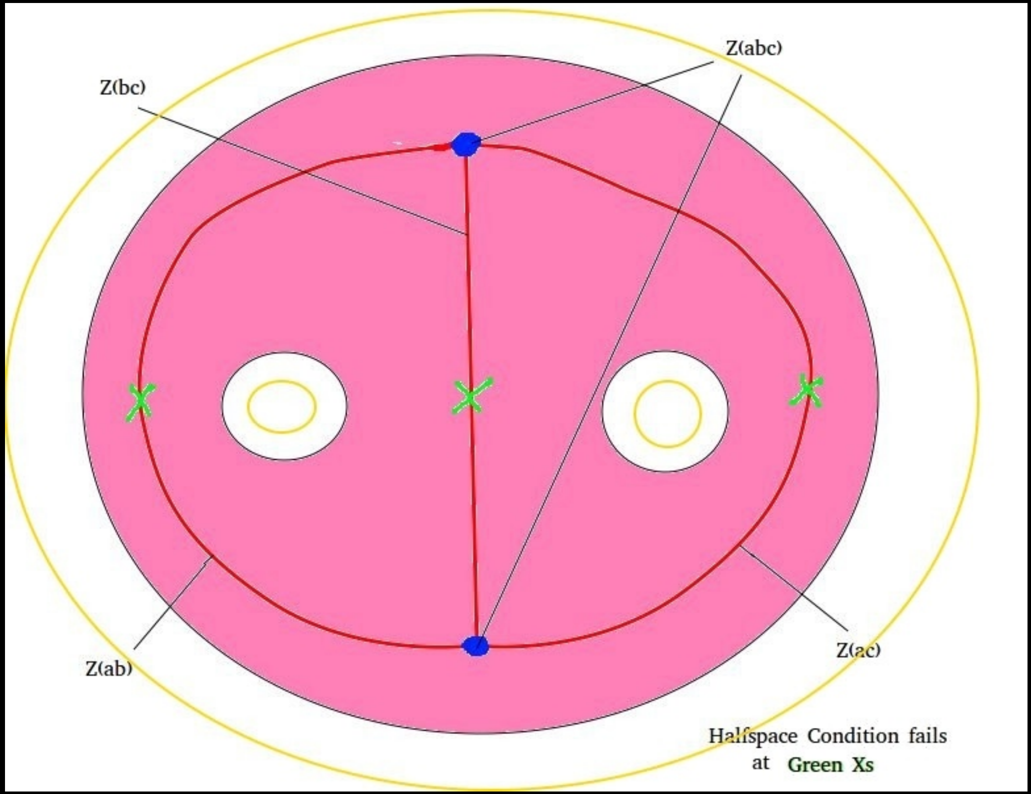
and such that $Z\{J+1\}$ is a codimension-J closed subvariety of X, with inclusion $Z\{J+1\} \to X$ a continuous homotopy-isomorphism.

Applications need index J be large as possible.
J defined by max. codimension of cell Z(Y_I) where (UHS) satisfied.

- Best results obtained with anti-quadratic repulsion (visibility) costs.

Θ-graph Singularity of excised pant P[t] $\to \delta$P[t]





...back to Application of Kantorovich Singularity:

...Spines are not hidden…

~~*Spines are readily displayed*~~
~~*in locus-of-discontinuity Z*~~
~~*of def. retracts r : X[t] → δX[t]*~~

Spines readily displayed in
Kantorovich's Contravariant Singularity Functor $Z(\sigma,\tau, c^*)$
of c*-optimal semicouplings π from (X[t], σ) to (δX[t], τ)

source target

Γ-rational Excision X[t] x δX[t]

Given initial geometric EΓ model (X,d, vol) :

- require Γ-invariant boundary
... but the familiar Gromov visual boundary X(∞) is not useful
since Γ acts ergodically on X(∞) with respect to natural Lebesgue measure.

- implies Γ-invariant Radon measures on X(∞)
do NOT descend to Radon measures on X(∞) / Γ.

Step (0): Construct Γ-rational excision X[t] with proper-discontinuous boundary δX[t] .

Obtain: Source (X[t], σ) ------- >> Target (δX[t], τ)

Excision: X[t]=X – U V[t] obtained by scooping-out/excising
a countable {t}-family of Γ-rational horoballs V[t] in X.

Γ-rationality implies X[t] and boundary δX[t] are Γ-invariant subsets
- inherit proper-discontinuous Γ-actions.

Excision: Bieri-Eckmann duality:

Key Property: $\delta X[t]$ has homotopy-type of countable wedge of q-spheres, and

$$\boxed{D = \tilde{H}_*(\delta X[t])}$$

is Bieri-Eckmann dualizing module ("Steinberg module")

- *Steinberg module D is infinite cyclic ZΓ-module generated by spheres at-infinity B (called "Steinberg symbol")*

X[t] contractible + LES relative homology + NPC ==>>

$$\partial : \underbrace{H_{q+1}(X[t], \partial X[t])}_{FILL[B]} \xrightarrow{\simeq} \underbrace{\tilde{H}_q(\partial X[t])}_{B}$$

isomorphism with inverse flat-filling FILL=δ^{-1}

Homological duality:

-Steinberg symbols are relative cycles FILL[B] ε $H_{q+1}(X[t], \partial X[t])$

-Bieri-Eckmann duality implies FILL[B] is dual cycle to minimal spines fundamental class.

- dim(FILL[B]) is max codimension of minimal spine (homological duality formula)

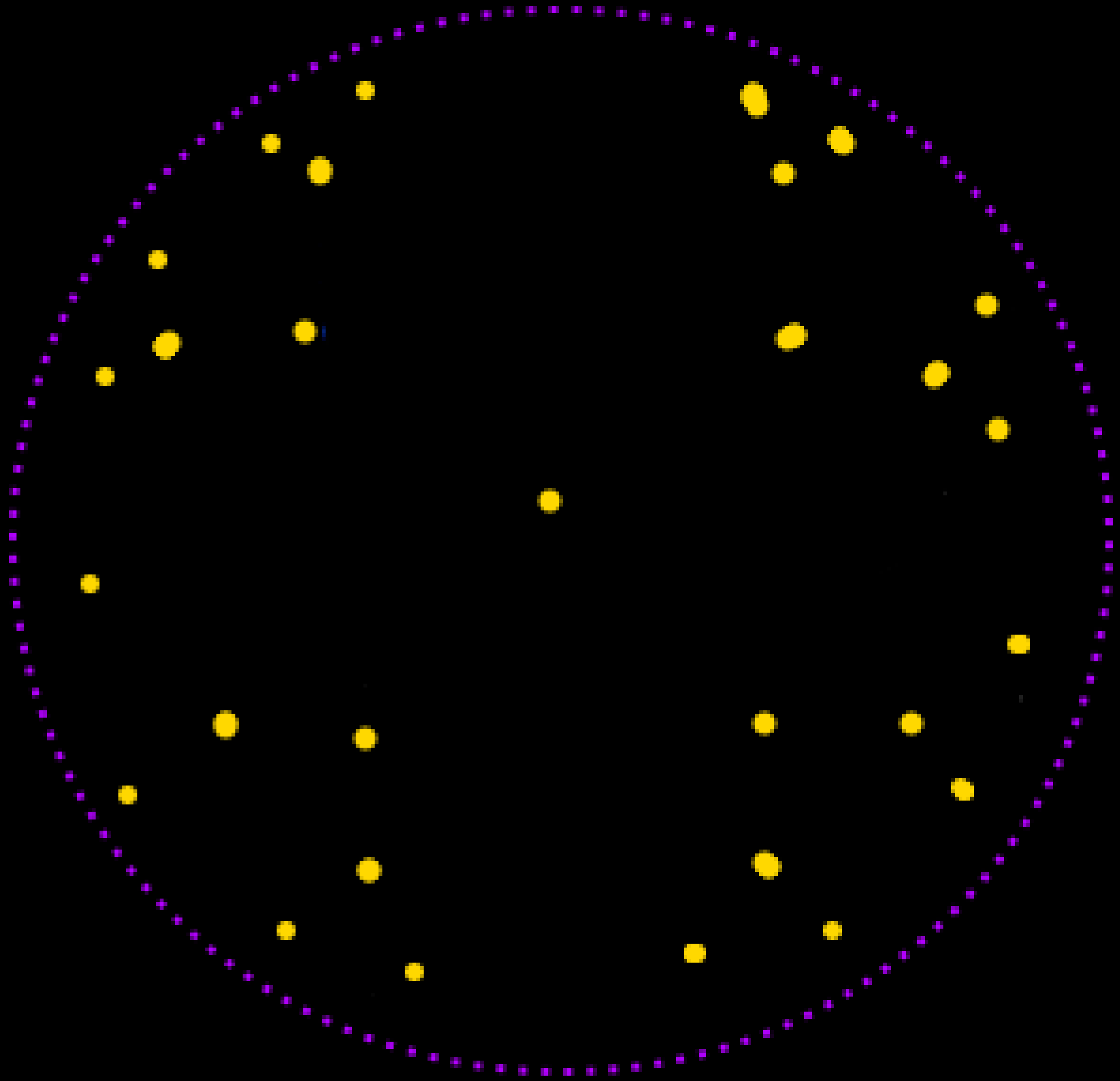
- Observe FILL[B] deformation retracts to {pt} …………. FILL[B]→{pt}.

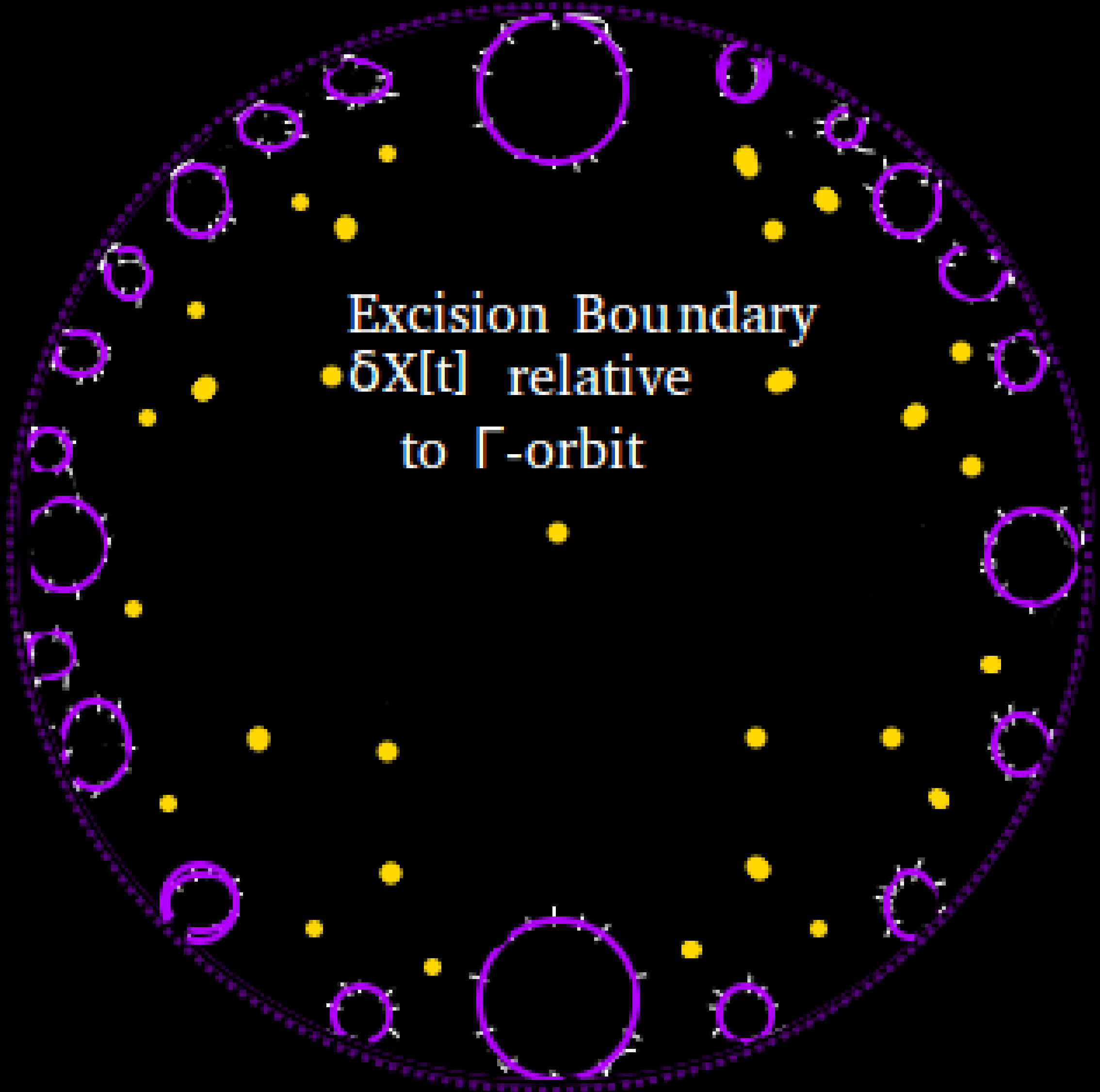
Our goal: continuously interpolate/extend the local retractions $FILL[B].\gamma \rightsquigarrow \{pt\}$, $\gamma \in \Gamma$ throughout $X[t]$ such that the singularity-to-reduction $X[t] \rightsquigarrow \mathscr{X}$ has exact minimal dimension $cd\Gamma$.

Interpolation Problem solved
with Two Pointed-(Visibility)-Repulsion cost c=v,
and Kantorovich Functor Z=Z(σ, τ, v) on
Γ-rational excision X[t], δ X[t]

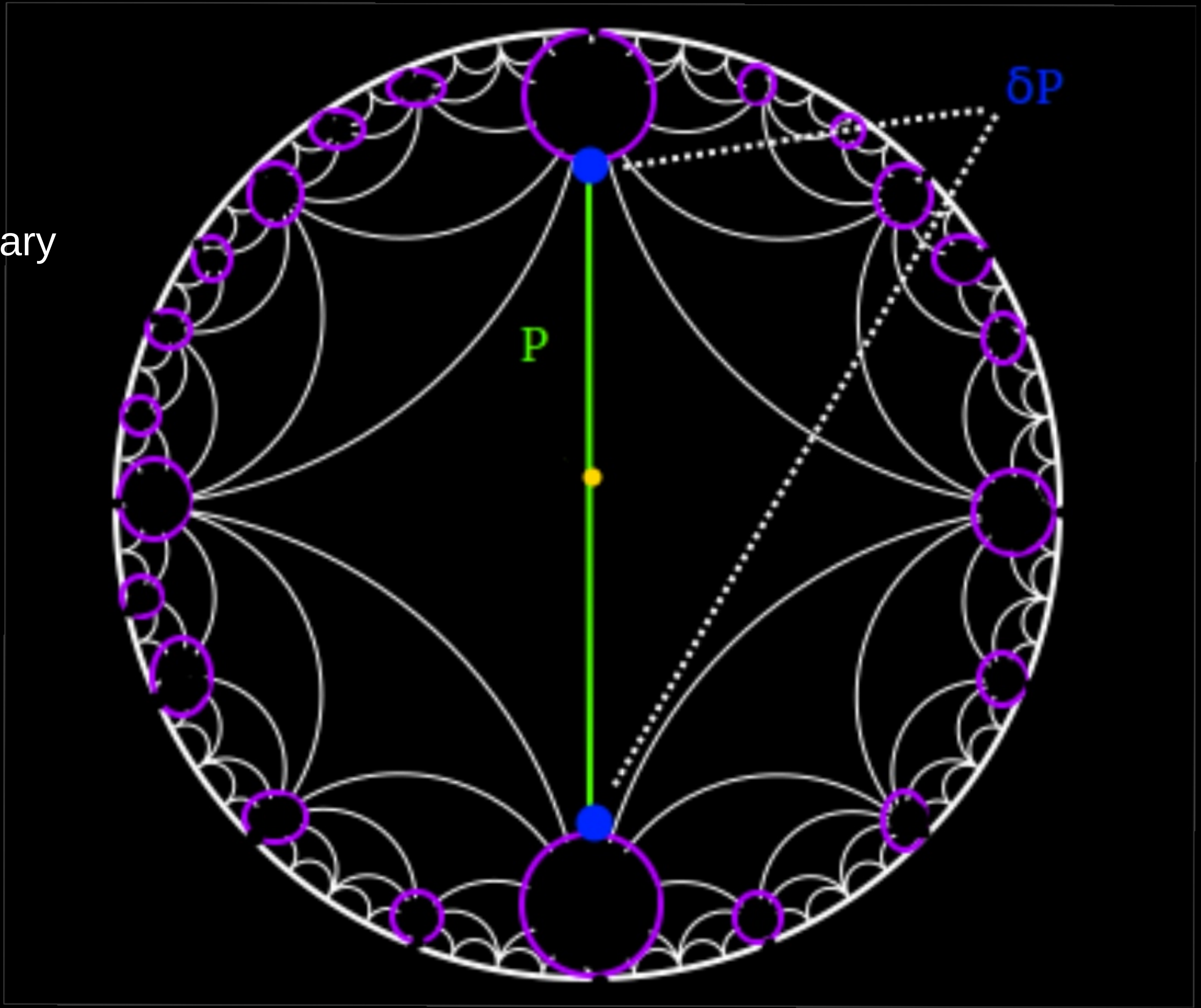
...*to illustrate the Interpolation Problem:*

Initial geometric EΓ-model X

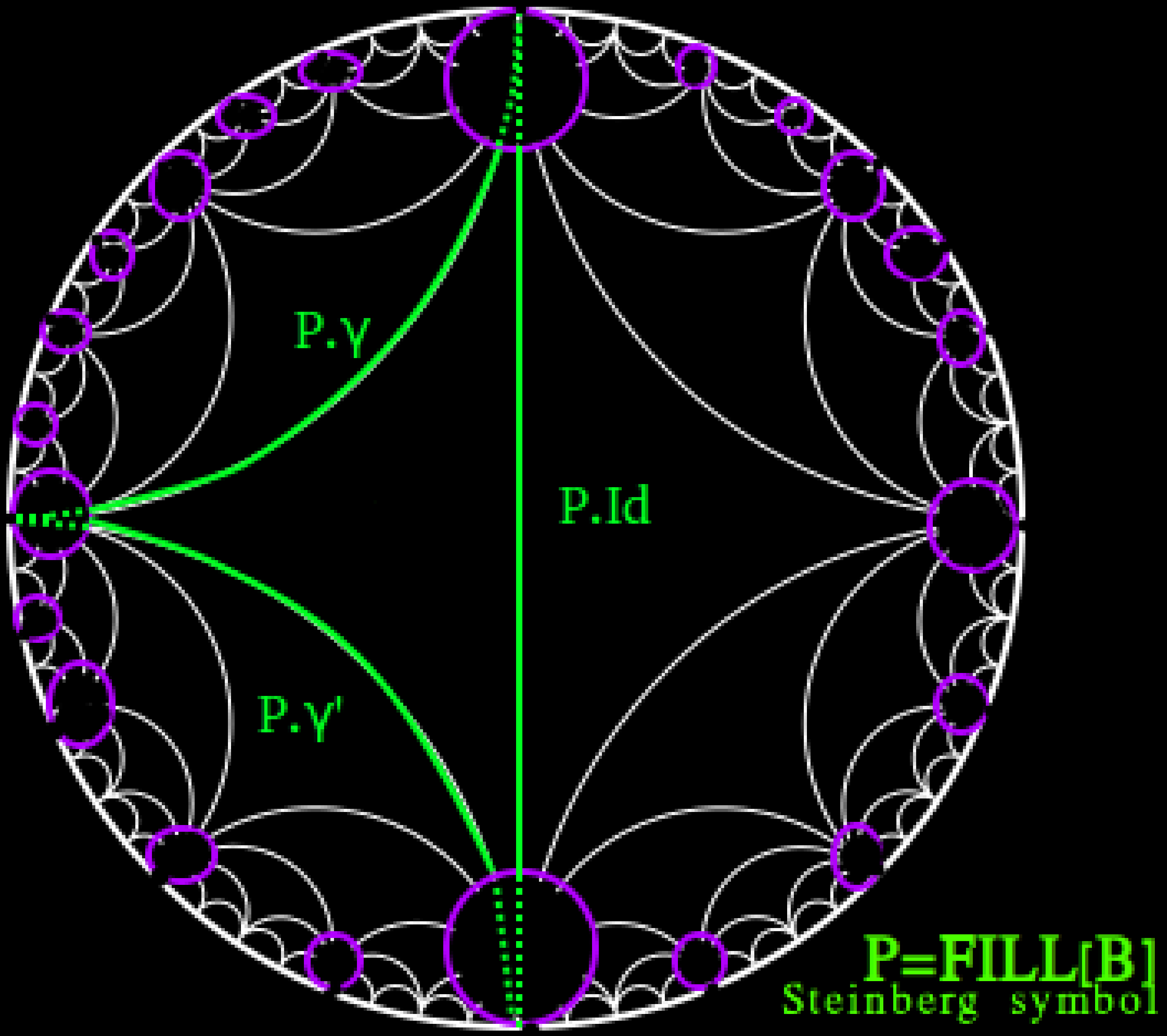



Excision Boundary
δX[t] relative
to Γ-orbit

Steinberg symbol P is
relative 1-cycle; with boundary
δP a boundary 0-sphere.



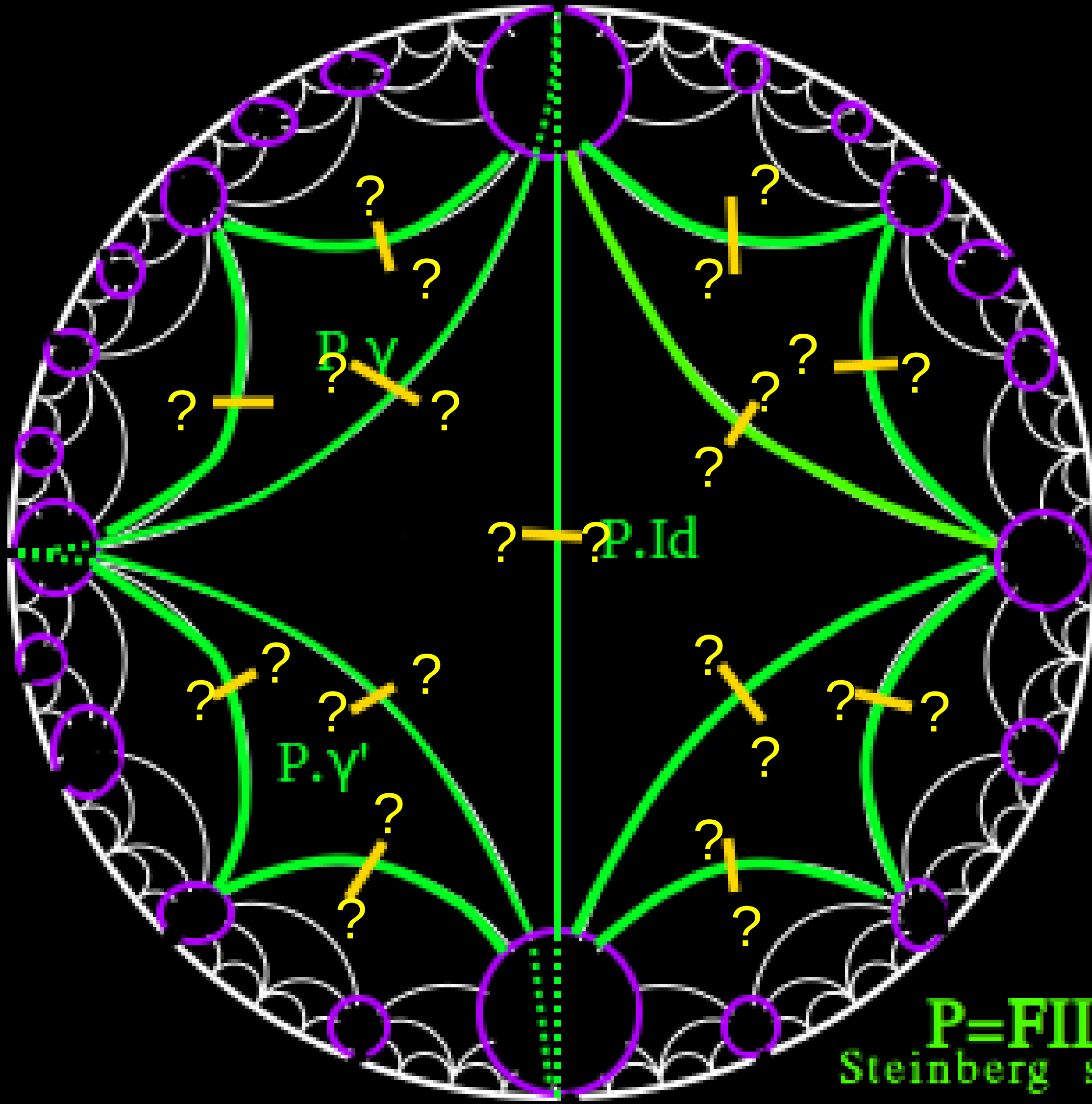
The subset {Id, γ, γ'} successfully Closes the Steinberg symbol.



Homological duality ==>>
Minimal spine is dual to
Steinberg symbols P, P.γ, γ ε Γ.

¿¿ How to assemble/interpolate the local reductions
{ P.γ → {pt}, γ ε Γ }

to obtain a Spine throughout X[t] ??



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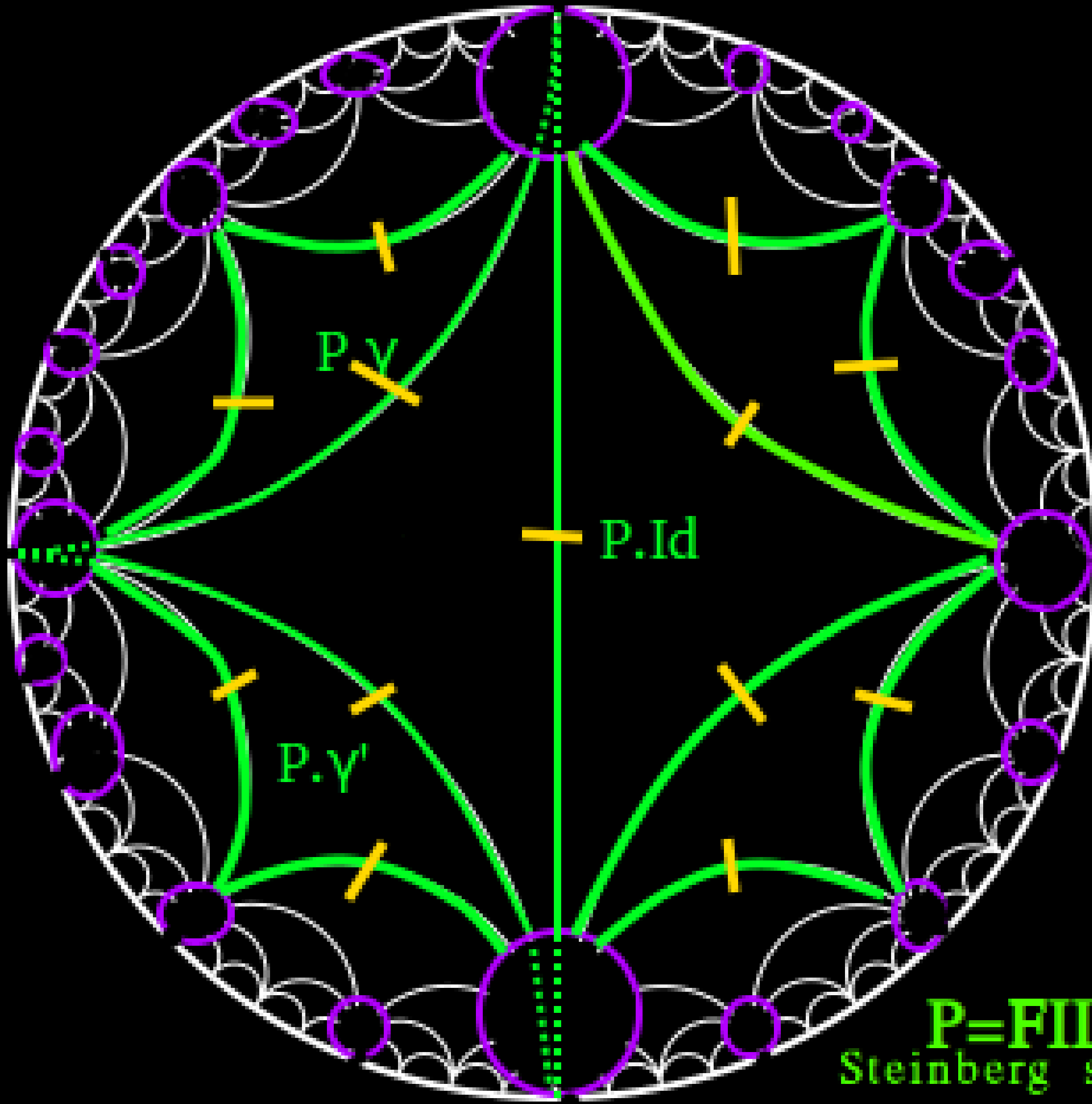
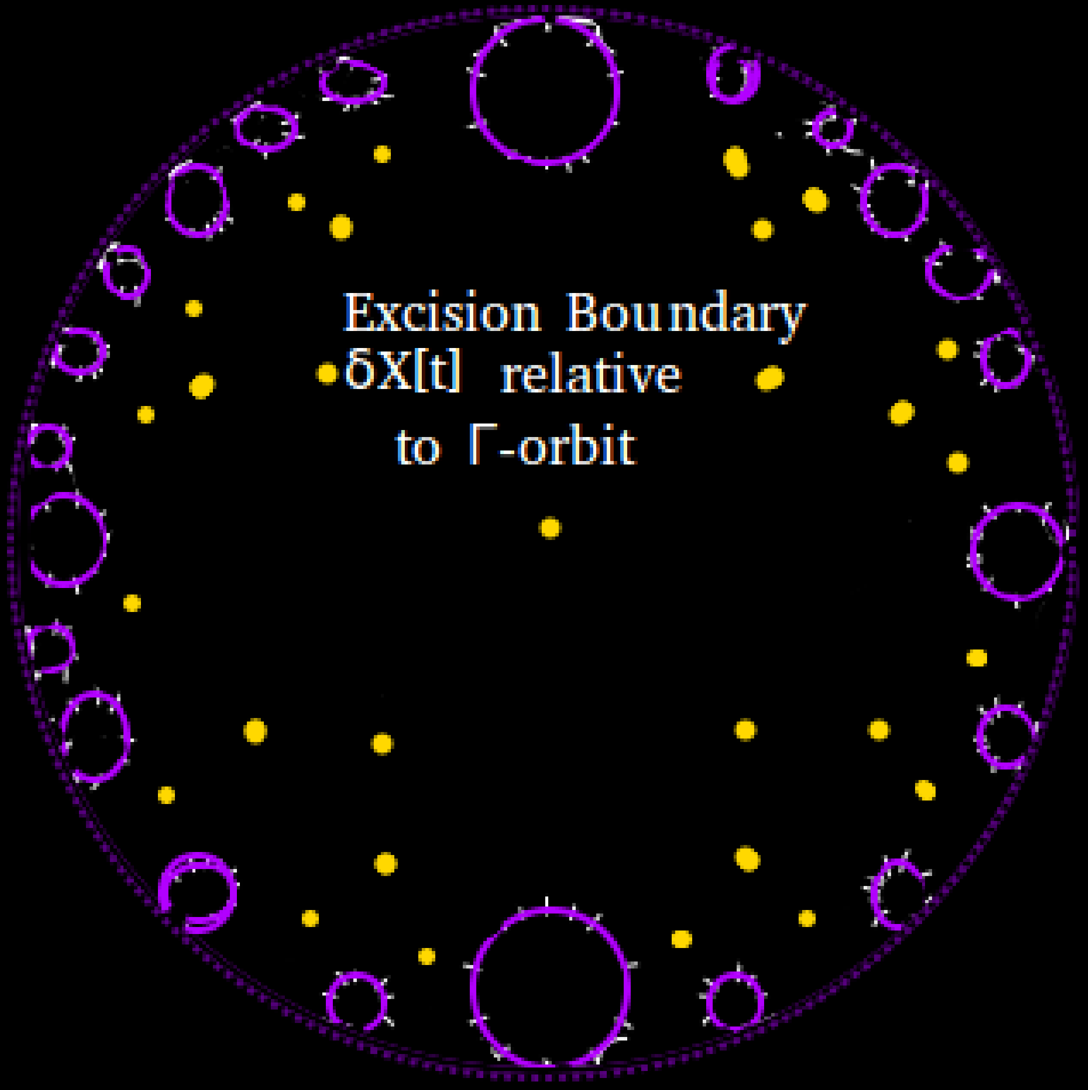
*¿¿ How to assemble/interpolate
the local reductions
{ P.γ → {pt}, γ ε Γ }*

*to obtain a Spine
 throughout X[t] ??*

*Answer:
 Kantorovich Singularity functor
 Z(σ,τ,v) of two-pointed
 visible repulsion cost
 v: X[t] x δX[t] → R*




Excision Boundary
δX[t] relative
to Γ-orbit


Activated Domain
of v-optimal
Semicoupling


Z{2}

One-dimensional repulsion cost:

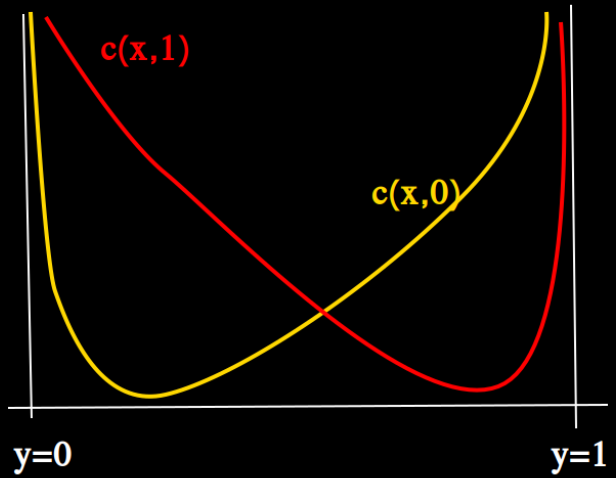
Unit interval I=[0,1] , boundary δI={0,1}

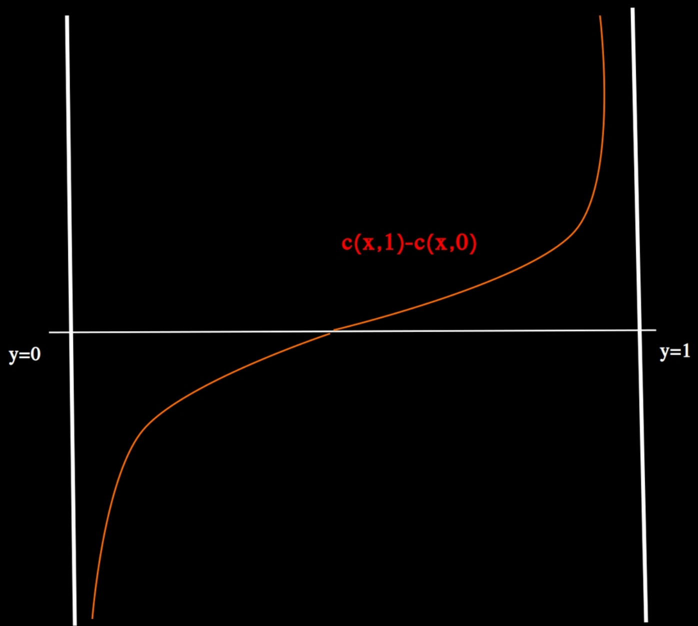
$$c(x,0) = |x|^{-2} + 2|1-x|^{-2}$$

$$c(x,1) = 2|x|^{-2} + |1-x|^{-2}$$

$$c(x,1) - c(x,0) = |1-x|^{-2} - |x|^{-2}$$

- cross-difference is critical-point free!
  (critical points at poles x=0,1)
  every fibre is connected.

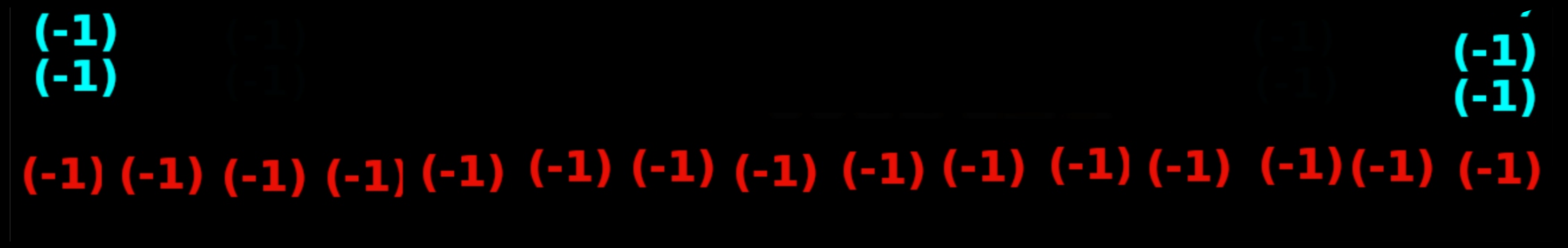


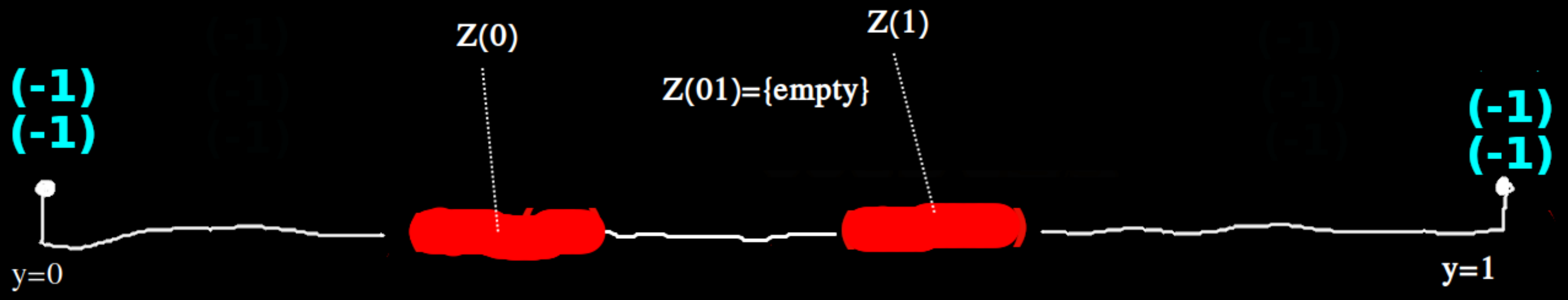


Consider closed unit interval X = [0,1] with boundary δX={0, 1}.

σ is uniform distribution of (-1) charges. mass(σ) =15(-)
τ is uniform distribution of (-1) charges. Mass(τ) = 4(-)

mass(σ) >>...>> mass(τ)





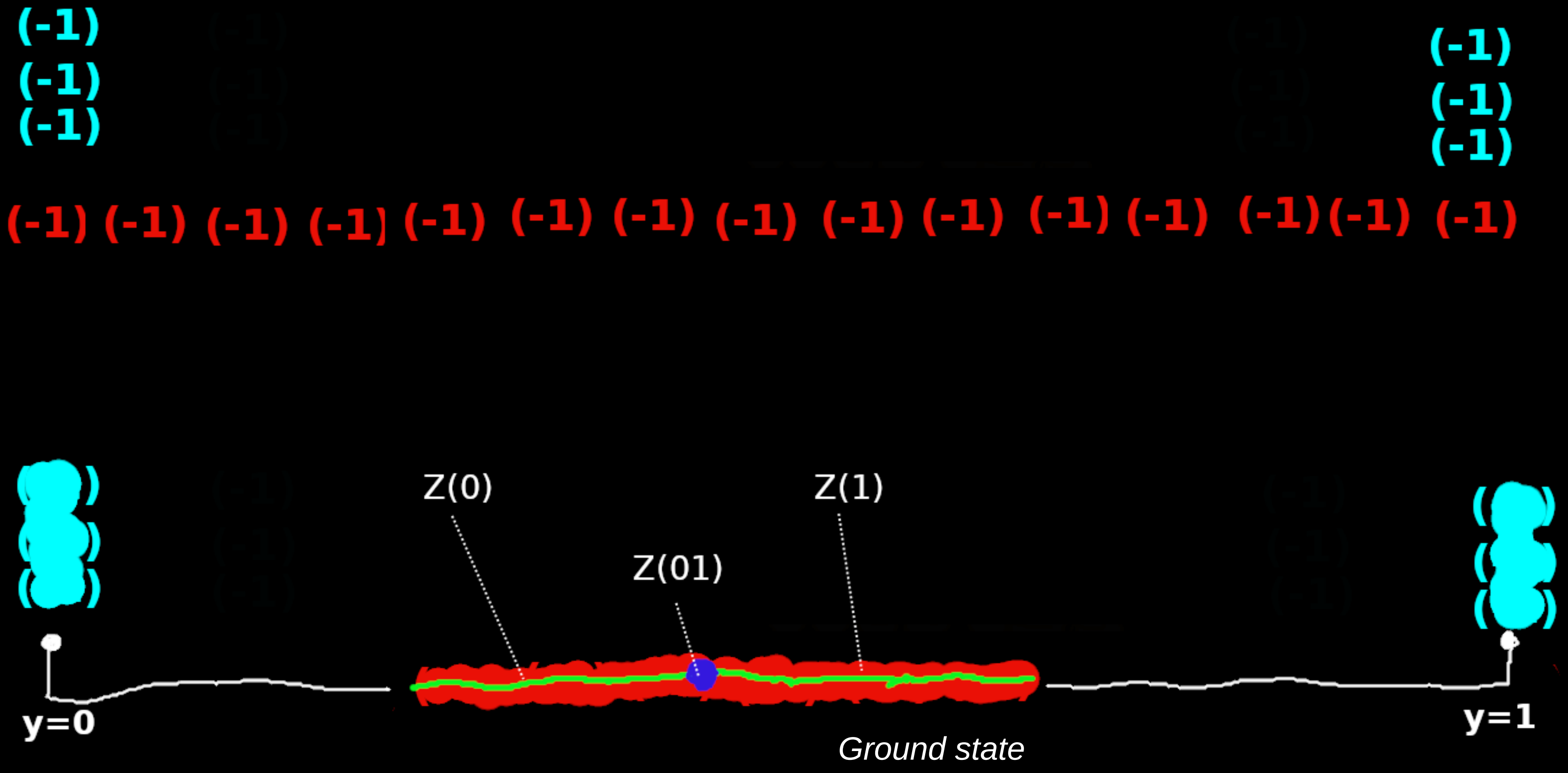
Ground state

Consider closed unit interval X = [0,1] with boundary δX={0, 1}.

σ is uniform distribution of (-1) charges. mass(σ) =15(-)

τ is uniform distribution of (-1) charges. Mass(τ) = 6(-)

mass(σ) >>> mass(τ)



Consider closed unit interval X = [0,1] with boundary δX={0, 1}.

σ is uniform distribution of (-1) charges. mass(σ) =15(-)
τ is uniform distribution of (+1) charges. Mass(τ) = 6(+)

mass(σ) > mass(τ)

Ground state *Ground state*

Closing Steinberg (CS):

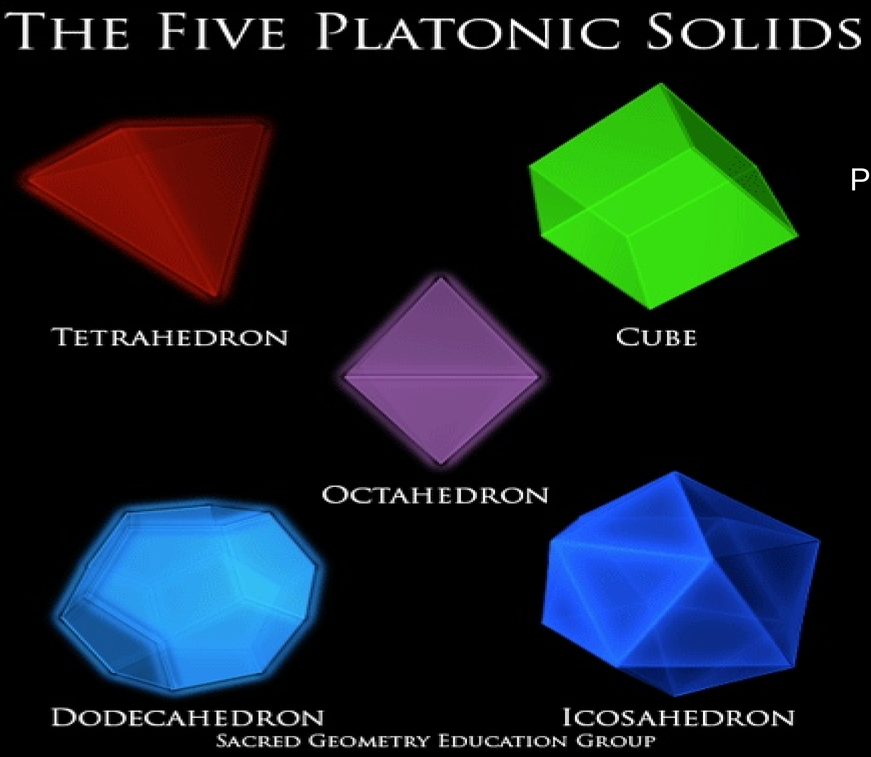
In low dimensions (CS) is the problem of stitching a football from collection of panels {P}.

Ex: Regular platonic solids solve (CS) with panels P = triangle, square, pentagon.

P=Triangle

P=Square

P=Pentagon



Polyhedra
At-infinity !

(CS) implies
$F=\mathrm{conv}(P,P',\ldots)$
is relative cycle
$F \,\varepsilon\, H_*(X[t], \delta X[t])$

Closing Steinberg:

In our applications the panels {P} = { FILL[B].γ | γ ε Γ } are flat-filled Steinberg symbols.
- Seek finite subset I of Γ such that panel translates { P.γ | γ ε I } assemble to "closed football"

Formal definition:

(CS1) the chain sum $\sum_{\gamma\in I} P.\gamma \neq 0 \bmod 2$ (nontrivial over $\mathbb{Z}/2$ coefficients).

(CS2) the chain sum $\sum_{\gamma\in I} \partial P.\gamma = 0 \bmod 2$ (vanishing boundary over $\mathbb{Z}/2$ coefficients).

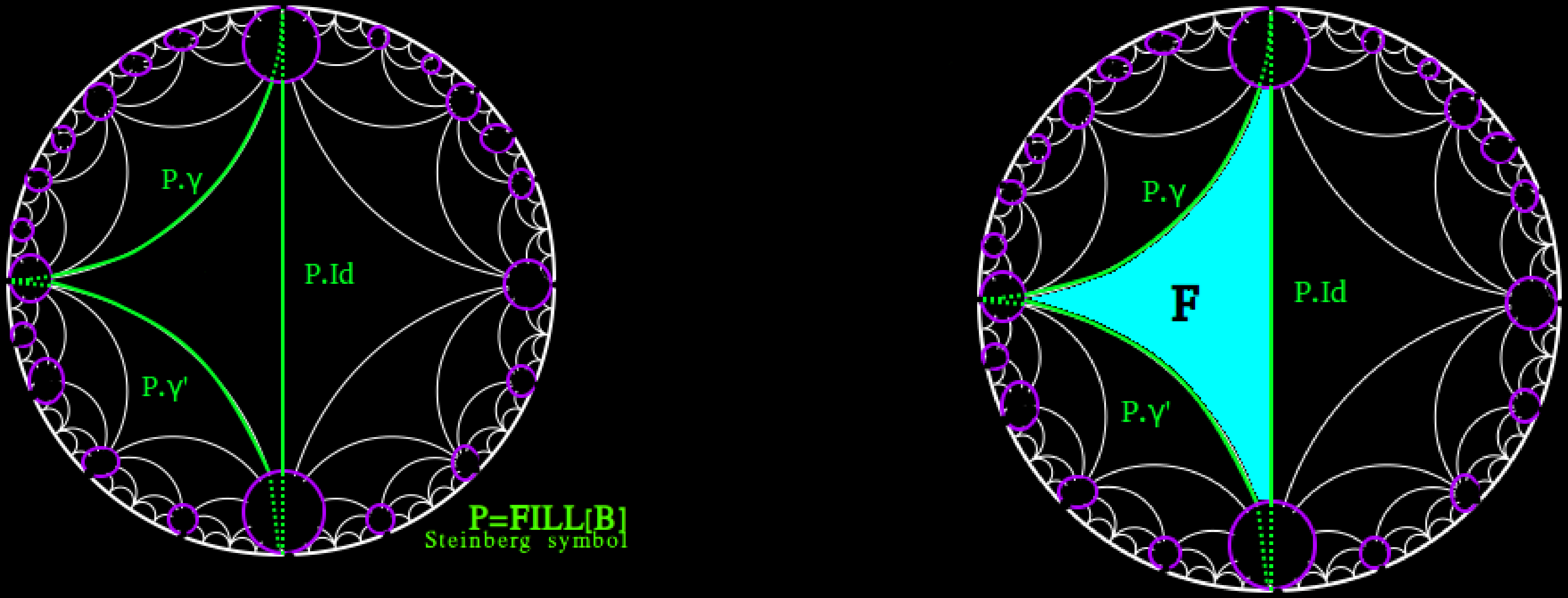
(CS3) there exists $x \in X[t]$ which is simultaneously visible from $P.\gamma$, $\gamma \in I$, in $X[t]$ (well-defined closed convex hull).

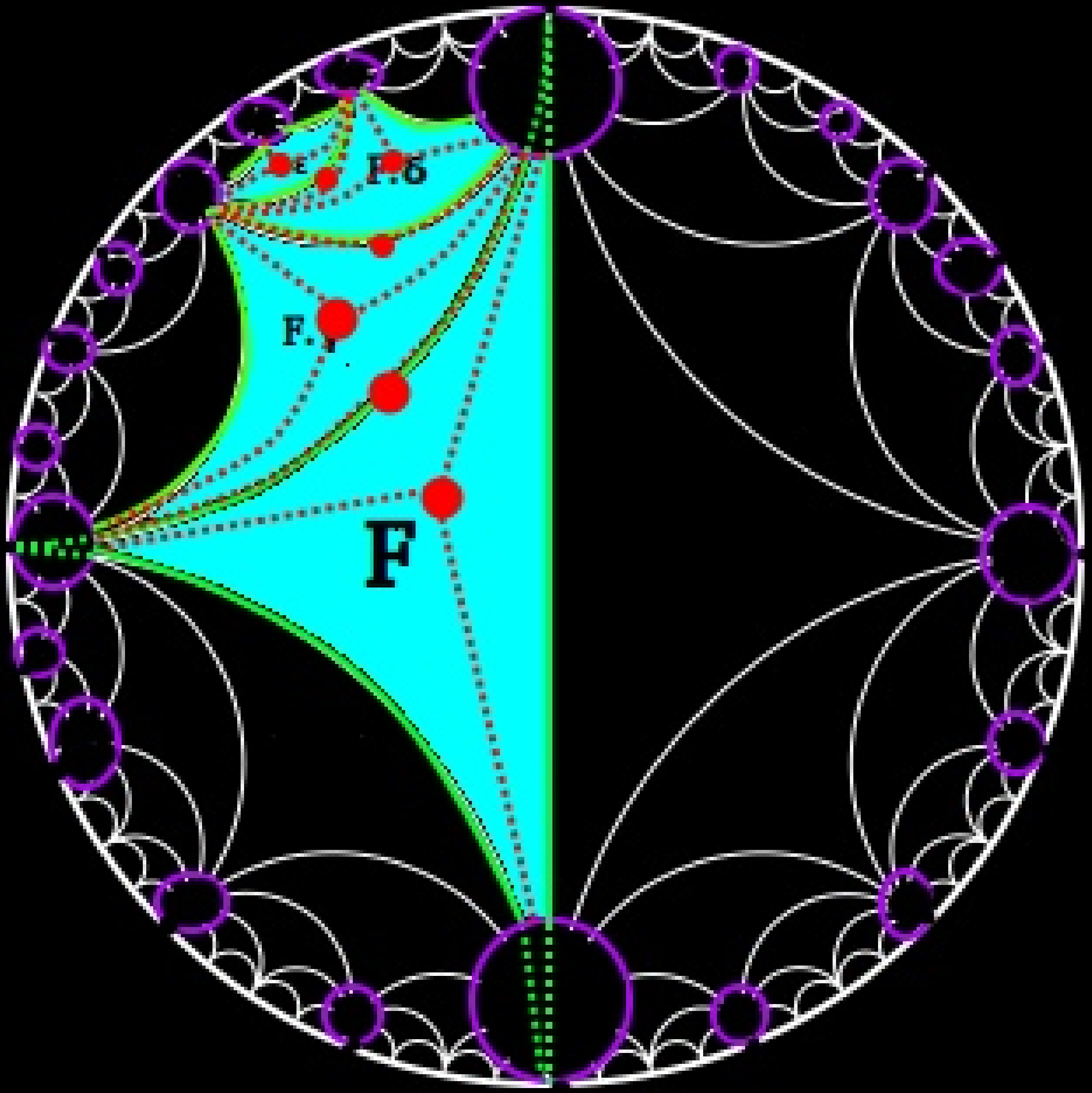
(CS4) if we define $F := \overline{conv}\{P.\gamma |\ \gamma \in I\}$, then the convex chain sum $\underline{F} = \sum_{\gamma\in\Gamma} F.\gamma$ has well-seperated gates structure with gates $\{G\} = \{P.\gamma|\ \gamma \in \Gamma\}$. (well-separated gates)

Closing Steinberg:

CS1, CS2 ==> constructing nonzero ξ ε H_0 (Γ, Z_2 Γ x D).
 - equivalent to a syzsygy in projective homological resolution of D.

CS4 ==> the translates F. Γ define chain sum F = Σ F.γ
 - Γ acts on chain summands of F like “shift operator” equivalent to right-action Γ x Γ → Γ





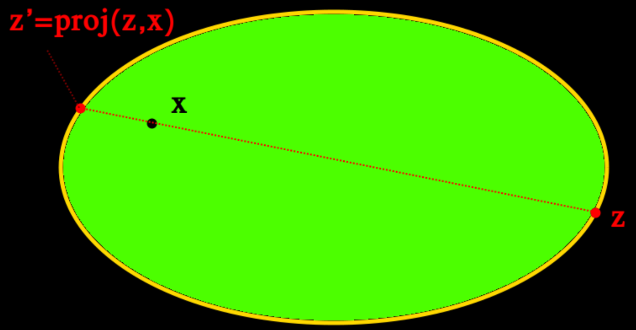
Chain sum F with well-separated gates:

CS replaces X[t] with a chain sum F, a type of partition of unity.

Well-separated gates ==>> summands F,F' have either trivial intersection or F∩F'=P

Two-pointed Repulsion cost c :*

c*(x,z) := ½. dist(x,z)^{-2} + dist(x, proj(z,x))^{-2}

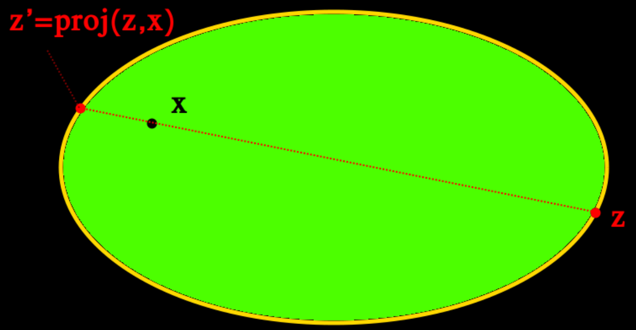


- *the 2-pointed repulsion cost c* leads to singularity functor Z(σ, τ, c*)*

- *Our proposal is finally that (UHS) conditions are satisfied up to index* ***J=dim(D)*** *,
and Kantorovich singularity equivariantly homotopy-reduces the source X
to minimal spine Z .*

<u>*Two-pointed Repulsion cost c* :*</u>

$c^*(x,z) := \frac{1}{2}.\, \mathrm{dist}(x,z)^{-2} + \mathrm{dist}(x, \mathrm{proj}(z,x))^{-2}$



- c extends to repulsion cost on chain sum <u>F</u> with well-separated gates {G}*

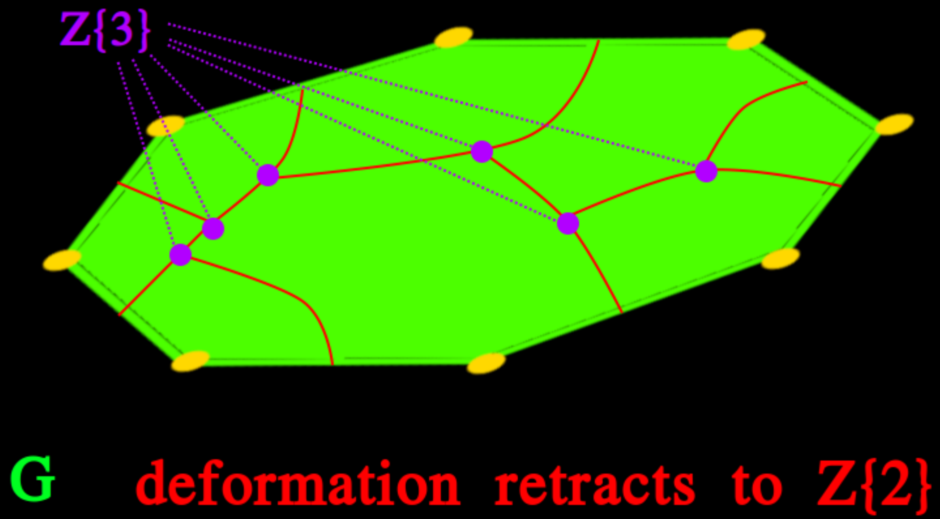
- gates G geodesically convex implies c is continuous interpolation of the restricted 2-pointed repulsion costs c*|G*

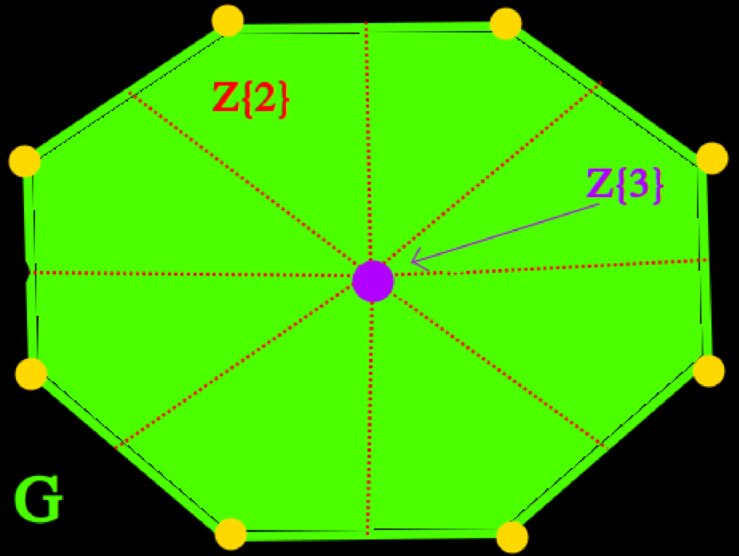
- Singularity structures Z(σ,τ, c) are continuous interpolation of restricted singularity structures Z(σ, τ, c*|G) , over gates {G} of <u>F</u>*

... Closing Steinberg (CS) and Interpolating Local Reductions:

- in our applications the gates {G} of the chain sum $\underline{F}$ will coincide with the Γ-orbit of Steinberg symbols P.γ = FILL[B].γ , γ ε Γ .

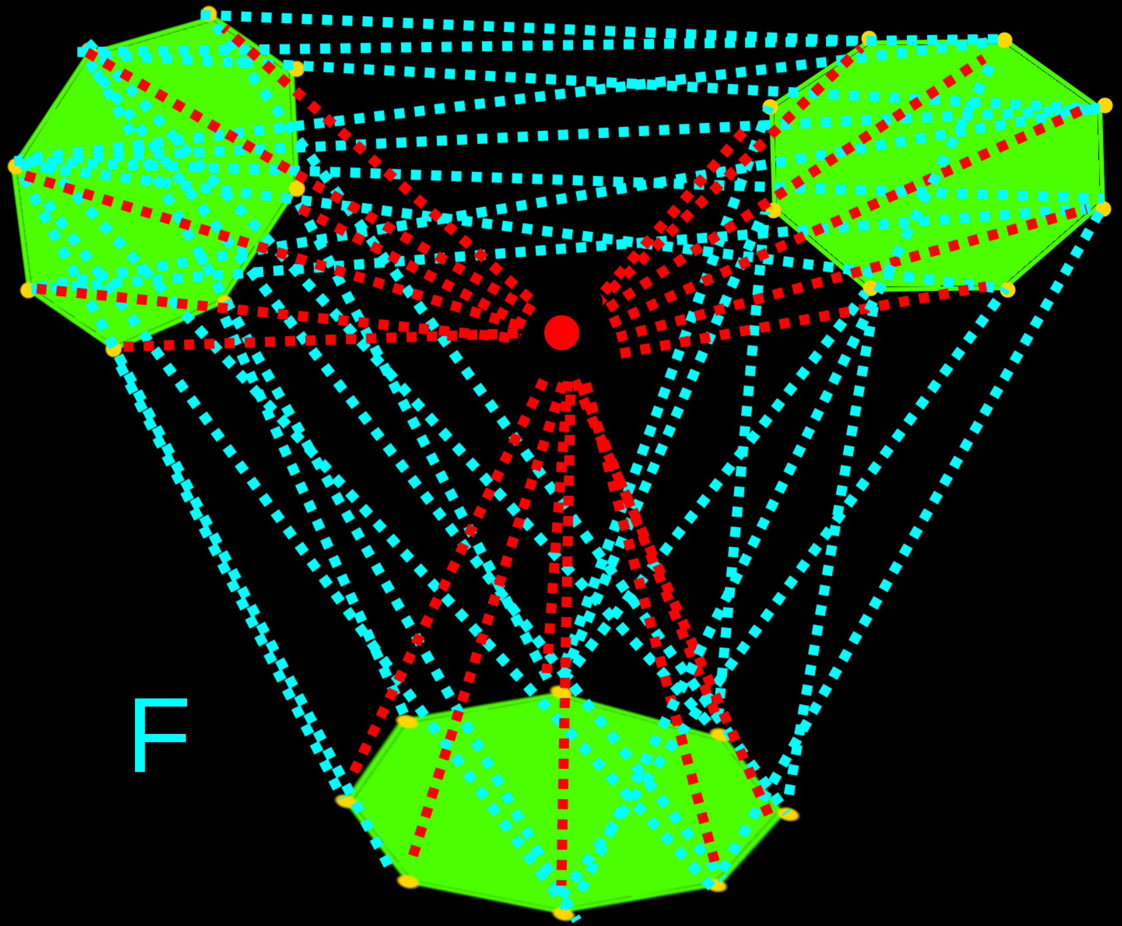
- the singularity structures of the two-pointed repulsion costs c and visibility cost produce homotopy-reductions of the gates G=P to points, G → {pt} .





- generic convex panels G deformation retract to Z{2}, and Z{3} is disconnected.

- gate G with large symmetry implies G reduces to Z{3}={pt}



Successfully Closing Steinberg symbol replaces X with a convex chain sum

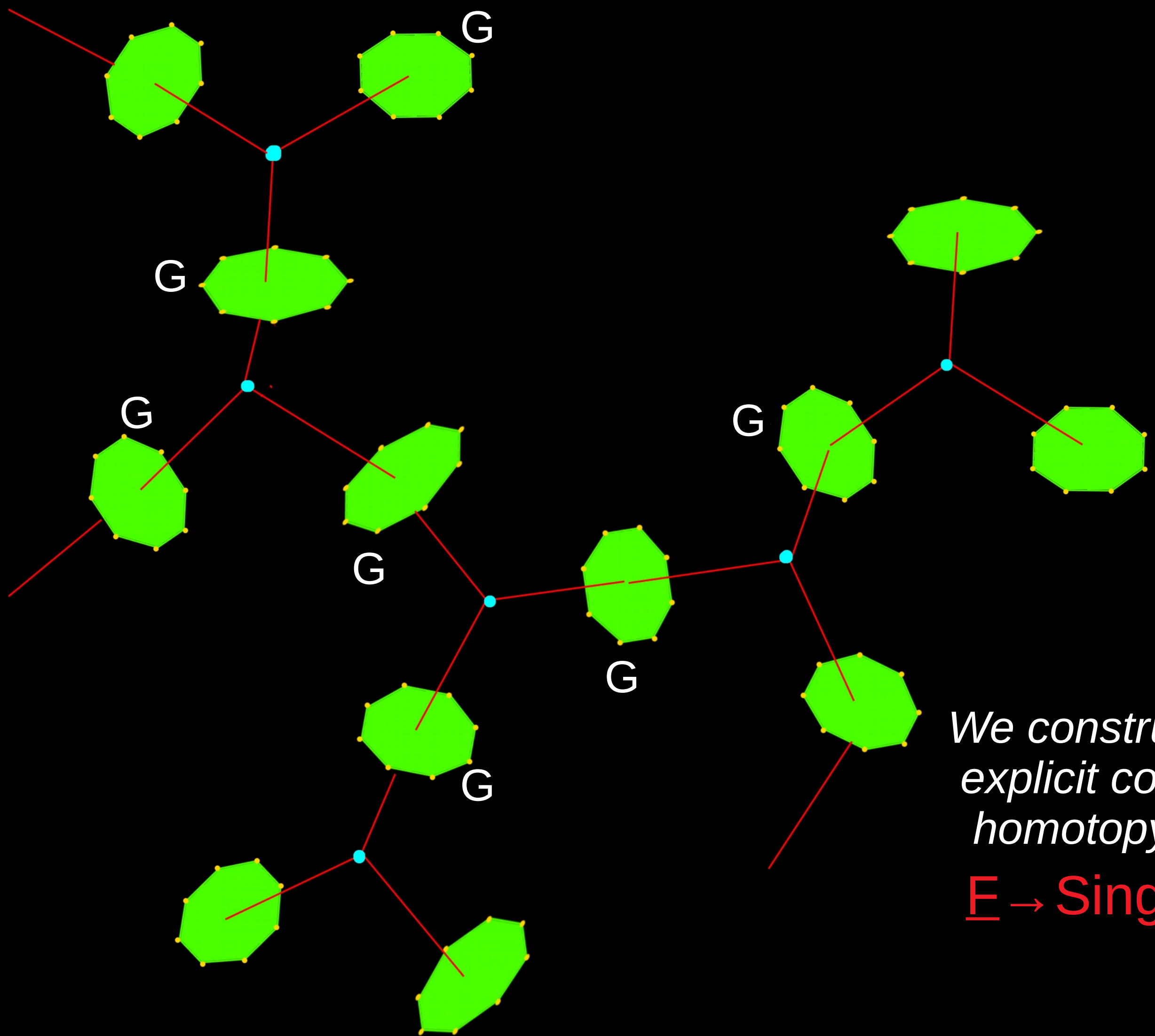
$$\underline{F} = \Sigma\ F.\gamma$$

where F := conv[P, P.γ, P.γ', ...]

- *Hypotheses of (CS) implies: the convex hull F is well-defined,*

and

- *translates F.γ, γ ε Γ , have well-separated gates {G} coincident with Γ-orbit of Steinberg symbols G=FILL[B]*



c*-optimal semicouplings from source $X=\underline{F}$ to target $Y=E[\underline{F}]$

We construct explicit continuous homotopy-reductions

$\underline{F} \rightarrow$ Singularity Z{2}

Given initial geometric EΓ model X,

- we construct excision X[t] with boundary δX[t]
- we Close Steinberg and replace X[t] with chain sum $\underline{F}$
- we install two-pointed visibile repulsion cost v: X[t] x δX[t] $\to$ R

Then Kantorovich's Singularity functor Z=Z(σ,τ,v) constructs explicit homotopy-reductions

$$X \to X[t] \to Z\{1\} \to Z\{2\} \to \ldots \to Z\{J+1\},$$ where index J>1 depends on (UHS) conditions.

We conjecture (and prove in special cases) that costs c* , v satisfy sufficient (UHS) conditions:

e.g. *If X[t] is rational excision model of initial geometric EΓ-model X, then Z(σ,τ,v) satisfies sufficient (UHS) conditions up to J=dim(D) such that Z{J+1} is explicit minimal spine.*

The End.

T

Thank you.