# OPTIMAL TRANSPORT, $1/d^{\alpha}$-COSTS, AND MEDIAL AXIS TRANSFORMS

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## 1. Medial Axis Transforms and Optimal Transport

The purpose of this section is to compare some familiar properties of the medial axis transform $A \mapsto M(A)$ (introduced by [Blu67]) with the singularity structures formalized in our Kantorovich contravariant functor $Z : 2^{\partial A} \to 2^{A}$ (introduced in [Mar]). We compare the functor $Z$ with medial axis transform by interpreting the inclusion $M(A) \hookrightarrow A$ in terms of mass transportation.

Let $A$ be a bounded open subset of $\mathbb{R}^N$. The medial axis $M(A)$ defined by Blum consists of all $x \in A$ for which $dist(x, \partial A)$ is attained by at least two distinct points,

$$M(A) := \{x \in A \mid \#argmin_{y \in \partial A}\{d(x,y)\} \geq 2\}. \tag{1}$$

Caution needs be exercised since $M(A)$ is possibly not a closed subset of $A$. The closure $\overline{M(A)}$ coincides with the so-called "cut locus of the boundary $\partial A$".

A long-known "folk theorem" states that the inclusion $M(A) \hookrightarrow A$ is a homotopy-isomorphism, and even a strong deformation retract. (Strictly speaking, the closure $\overline{M(A)}$ is a strong retract). This implies $M(A)$ contains all the topology of $A$, and a connected subset whenever $A$ is. A formal proof is established [Lie04]. We do not know if $M(A)$ is a strong retract for more general Riemannian spaces $(X, d)$, although results of Alexander–Bishop [AB98], [AB00] prove sufficiently thin Riemannian manifolds deform onto the cut-locus. Our recent thesis [Mar] contains some results, namely "Theorem B", identify conditions for which inclusions denoted $Z_2 \hookrightarrow A$ are homotopy isomorphisms, even strong deformation retracts. This subvariety $Z_2$ is derived from a contravariant functor $Z = Z(\mu, \nu, c)$ defined by mass transport data $(\mu, \nu, c)$. The medial axis $M(A)$ and $Z_2$ will rarely coincide set-theoretically, but this present note demonstrates they are frequently topologically isomorphic.

The medial axis transform corresponds to a "degenerate" transport problem in the following sense: if $A \hookrightarrow \mathbb{R}^N$ is bounded open subset, then we nominate

$$\mu := \frac{1}{\mathscr{H}_A[A]} \mathscr{H}_A \tag{2}$$

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as the canonical probability measure on the source $A$. Consider the probability measures $\pi$ on $A \times \partial A$ for which $proj_A \# \pi = \mu$ and with unconstrained second marginal $proj_{\partial A} \# \pi$. Here $proj_A$, $proj_{\partial A}$ are the canonical projections $A \times \partial A \to A, \partial A$. The set-mapping

$$T : x \mapsto argmin_{y \in \partial A} \{d(x,y)\}, \quad \text{for } x \in A, \tag{3}$$

defines a measurable set-valued map $T : A \to \partial A$. The pushforward

$$\nu := T \# \mu \tag{4}$$

is a probability measure on $\partial A$ with $spt(\nu) = \partial A$. With respect to, say, quadratic cost $c = d^2/2$ or distance cost $c = d$, the map $x \mapsto T(x)$ defines a $c$-optimal transport from $\mu$ to $\nu$, with $c$-optimal coupling $\pi = (Id \times T)\#\mu$ on $A \times \partial A$.

Finally $M(A)$ coincides with the locus-of-discontinuity of $T : A \to \partial A$, or more specifically the singularity $Z_2$ defined by Kantorovich's contravariant functor $Z = Z(\mu, \nu, d) : 2^{\partial A} \to 2^A$. Thus we arrive at an instance where $M(A) = Z_2$ for the specific coupling program defined by $\mu, \nu, c$. This identification suggests the following generalization of medial axis transform: for general probability measures $\nu \in \Delta(\partial A)$ on the boundary of $A$, we may study the $c$-optimal couplings $\pi$ from $\mu$ to $\nu$, and obtain a Singularity functor $Z(\mu, \nu, c)$. The generalized medial axis in this setting is $Z_2$, i.e. the "locus-of-discontinuity" of the $c$-optimal transport $\pi$ from $\mu$ to $\nu$.

## 2. Reduction-to-Singularity

Our thesis developed a Reduction-to-Singularity principle, and identifies conditions for which, say, the inclusion $Z_2 \hookrightarrow Z_1$ is a homotopy-isomorphism. In the above setting with $Z = Z(\mu, \nu, c)$, we find $A = Z_1$, $M(A) = Z_2$. Naturally we inquire whether the hypotheses of our topological theorems are satisfied for any particular costs $c$. If we fix $c = d^2/2$, then Theorem B takes the following form. For $x \in A = Z_1$, let $y_0 := T(x)$. Then define

$$\eta(x,y) := |c(x,y) - c(x,y_0)|^{-1-\beta} \cdot \nabla_x (c(x,y) - c(x,y_0)),$$

for $y \in \partial A - \{y_0\}$. Observe that $c(x,y) - c(x,y_0) > 0$ is nonvanishing throughout $A - M(A)$. The hypotheses of Theorem B require the following conditions (6), (7) be satisfied. For $x \in A - Z_2$, define $y_0 =: T(x)$, and abbreviate

$$\bar{\nu}(y) := (1 - e^{-d(y,y_0)^2}).\nu(y).$$

Then we define the averaged Bochner integral

$$\eta(x, avg) := \bar{\nu}[\partial A]^{-1} \cdot \int_{\partial A} \eta(x,y) d\bar{\nu}(y). \tag{5}$$

We require that

$$\eta(x, avg) \text{ is nonzero finite tangent vector,} \tag{6}$$

and there exists a constant $C > 0$ such that

$$||\eta(x, avg)|| \geq C > 0 \tag{7}$$

for $x \in A - M(A)$, uniformly with $x$, for some exponent $\beta > 0$. Typically $\beta = 1$ is sufficient.

The verification of hypotheses (6)–(7) can be difficult to verify. Evidently (7) implies (6). We find $\eta(x, avg)$ is an averaged gradient, even the gradient of the averaged potential $f_{avg}(x)$ defined as follows. Let $f_y(x) := (c(x,y) - c(x, y_0))^{-\beta}$, and consider the average of $f_y(x)$ with respect to the Borel measure $\bar{\nu}(y)$ on $\partial A$, namely

$$f_{avg}(x) = \bar{\nu}[\partial A]^{-1} \cdot \int_{\partial A} f_y(x) d\bar{\nu}(y).$$

Then

$$\eta(x, avg) = \nabla_x \int_{\partial A} f_{avg}(x) d\bar{\nu}(y).$$

The hypothesis (7) requires $f_{avg}(x)$ be critical-point free over the open subset $A - Z_2$. As $x \in A - Z_2$ converges to $Z_2$, we find the potential $f_{avg} > 0$ and the gradient $\nabla_x f_{avg}$ diverge to infinity.

## 3. Hubbard's $1/d^{\alpha}$-distance

We need remark on a complication arising from the nonconvexity of $A$. What is the natural distance function $d$ on $A \subset \mathbb{R}^N$, and the physical "transport cost" of a unit mass at $x \in A$ to target mass $y \in \partial A$? There are at least two popular possibilities. First we may restrict the ambient euclidean distance $d_{\mathbb{R}^N}(x,y) = ||x-y||$ to $A \times \partial A \subset \mathbb{R}^N \times \mathbb{R}^N$. But this restriction does not represent a path length distance in the sense of Gromov [Gro+01, 1.A-B]. In otherwords the restriction does not represent geodesic transport in $A$, and there is no variational description of the metric in terms of shortest-length curves.

A second approach defines $d = d_A$ as the induced length distance defined by

$$d_A(x,y) = \inf_{\gamma} Length(\gamma),$$

where the infimum is over all curves $\gamma : [0,1] \to A$ contained in $A$ with $\gamma(0) = x$, $\gamma(1) = y$. The reader will observe that both possibilities define coincident medial axes $M(A)$ according to (1), since euclidean balls are geodesically convex. The induced length distance $d = d_A$ is possibly most preferred by metric geometers, yet is difficult to numerically evaluate. Moreover geodesics with respect to the induced path distance $c = d_A$ can oftentimes be branching. The possible branching of geodesics implies gradients $y \mapsto \nabla_x d(x,y)$ are noninjective maps $\partial A \to T_x A$ for $x \in A$. This possible noninjectivity violates an important transport condition called (Twist), and is obstruction to hypothesis (6). Thus neither the restricted distance $c = d|A \times \partial A$ nor the induced distance $c = d_A$ are especially convenient costs.

This article explores a third possibility, namely a variant of Hubbard's so-called $1/d$-metric (see [HH06, Ch. 2.2, pp.33]). Let $A \subset \mathbb{R}^N$ be open subset. Then for every real parameter $\alpha$ we define the Riemannian metric

$$g_\alpha := (dist(x, \mathbb{R}^N - A))^{-\alpha}.ds^2, \tag{8}$$

where

$$ds^2 = dx_1^2 + dx_2^2 + \cdots + dx_N^2$$

is the standard Euclidean metric on $\mathbb{R}^N$. The choice $\alpha = 0$ yields $g_0 = ds^2$. Let $\kappa = \kappa(g)$ denote the sectional curvature of the metric $g$.

**Lemma 1.** *For every parameter $\alpha \geq 0$, the Riemannian metric $g_\alpha$ has nonpositive sectional curvature $\kappa \leq 0$ throughout $A$.*

*Proof.* We follow Hubbard's proof [HH06, Thm. 2.2.9, pp.36], where the key observation is that: for every $y \in \mathbb{R}^N - A$, the function $f_y(x) := -\log ||x-y||^\alpha$ is subharmonic for $x \in A$ (in fact, the function is harmonic). Therefore the supremum

$$f(x) := \sup_{y \in \partial A} f_y(x) = \sup_{y \in \mathbb{R}^N - A} f_y(x)$$

is subharmonic. But the metric $g_\alpha$ is conformal to the standard Euclidean metric $ds^2$, and the formula for the sectional curvature of conformal metrics is well-known, namely $\kappa = -\Delta \log f \cdot ds^2$, which is $\leq 0$ by above subharmonicity. □

For variable $\alpha$ the metric $g_\alpha$, and the corresponding path length distance $d_\alpha(x,y)$ is possibly incomplete on $A$. Geodesics in the $g_\alpha$ metric steer so as to keep as far away as possible from the boundary $\partial A$. However incompleteness only occurs at the boundary $\partial A$ of $A$ as subset of $\mathbb{R}^N$. For $A$ an open subset there may exist sequences $\{x_k\}_{k=1,2,\ldots}$ in $A$ which have no limit point in $A$ relative to the distance $d_\alpha$. Despite the metric $g_\alpha$ diverging as $x \to \partial A$, we prefer the lengths of geodesics $\gamma : [0,1] \to A$ converging to $\partial A$ to have finite length, and seek parameters $\alpha$ for which

$$Length_\alpha(\gamma) = \int_0^1 \sqrt{g_\alpha(\gamma'(t), \gamma'(t))}.dt < +\infty.$$

**Example 2.** Hubbard's definition of $1/d$-metric corresponds to $\alpha = 2$ in equation (8). Amazingly the $1/d$-metric on the upper halfspace $H := \{x_1 > 0\}$ in $\mathbb{R}^N$ in $x_1, \ldots, x_N$ coordinates is the complete constant-curvature hyperbolic metric on $H$. For $0 < \alpha < 2$ the metric is incomplete. The curve $\gamma(t) = (1-t, 0, 0, \cdots)$ for $0 \leq t \leq 1$ is a curve in $H$. The $g_\alpha$-length of $\gamma$ evaluates to $\int_0^1 (1-t)^{-\alpha/2} dt$, which is improper integral converging to $1 < (1-\alpha/2)^{-1} < +\infty$ when $0 < \alpha < 2$. While $0 < \alpha < 2$ we can uniquely extend $d_\alpha$ to a complete metric pairing $\tilde{d_\alpha} : \overline{H} \times \overline{H} \to \mathbb{R}$, where $\overline{H} = \{x_1 \geq 0\}$. Note that $\overline{H}$ is not homeomorphic to adjoining a sphere at-infinity $S^2_\infty$ to $H$.

**Example 3.** The $1/d$-metric ($\alpha = 2$) on the once-punctured plane $A = \mathbb{R}^2 - \{0\}$ is isometric to a straight cylinder of circumference $2\pi$ ([HH06, Ex.2.2.6]). The same computations as previous example show for $0 < \alpha < 2$, the metric $d_\alpha$ is incomplete with completion $\tilde{d_\alpha}$ equal to an infinite cone with angle depending on $\alpha$ at the origin vertex.

**Example 4.** The Weil-Petersson metric $d_{WP}$ on the Teichmueller space $\mathscr{T}_g$ of a closed genus $g$ hyperbolic surface is asymptotically equivalent to Hubbard's metric with exponent $\alpha = 3/2$, [Wol75].

The above examples have $A$ unbounded open subset. But our applications to medial axes concern bounded open subsets.

**Example 5.** We modify example 3 by restricting to the punctured disk, say, $D^\times := \{0 < ||x|| < R\}$ for a constant $R > 0$. Then the medial axis $M(D^\times) = \{||x|| = R/2\}$ is a circle in $D^\times$. Now we propose that sufficient (UHS) conditions, namely (6)–(7), are satisfied throughout $D^\times$ and the inclusion $Z_2 \hookrightarrow D^\times$ is homotopy-isomorphism (by Theorem B) for $Z = Z(\mu, \nu, c_\alpha)$ for $0 < \alpha < 2$. Moreover we propose $Z_2$ is also a circle, diffeomorphic to $M(A)$, but not identical.

**Example 6.** Let $A$ be convex subset and $0 < \alpha < 2$. Then ${}_\alpha(x, y)$ is proportional to the rescaled Euclidean distance $|x - y|^{1-\alpha/2}$.

**Proposition 7.** *Let $A$ be open subset of $\mathbb{R}^N$, with topological closure $\overline{A}$. For parameters $0 < \alpha < 2$ the metric $g_\alpha$ is incomplete Riemannian metric on $A$. Then:*

*(i) curves $\gamma$ in $A$ converging to the boundary $\partial A$ have uniquely defined finite length with respect to the metric $g_\alpha$; and*

*(ii) the path length distance $\tilde{d_\alpha} : \overline{A} \times \overline{A} \to \mathbb{R}_{\geq 0}$ defines a complete metric distance throughout $\overline{A}$.*

*Proof.* □

## 4. $M(A)$ VERSUS $Z_2$

Now we propose a more interesting mass transport interpretation of medial axis transforms. Let $A$ be bounded open subset of $\mathbb{R}^N$, with boundary $\partial A$, and probability measures $\mu, \nu$ as previously defined in (2), (4). Then we choose cost $c = \tilde{d_\alpha}$ : $A \times \partial A \to \mathbb{R}$ defined by restricting the completion to $A \times \partial A \subset \overline{A} \times \overline{A}$. The subvarieties $Z_2$ and $M(A)$ do not coincide set-theoretically, but we conjecture that they do coincide topologically:

**Theorem 8.** *Let $A$ be bounded open subset of $\mathbb{R}^N$. Let $c = \tilde{d_\alpha}$ be the metric completion of $d_\alpha$ to $\overline{A}$ (Prop. 7), and let $Z = Z(\mu, \nu, c) : 2^{\partial A} \to 2^A$ be the Singularity functor with respect to $(\mu, \nu, c)$ as defined in (2), (4).*

*Then sufficient (UHS) Conditions are satisfied to apply Theorem B [Mar, Thm.3.4.3], and the inclusion $Z_2 \hookrightarrow A$ is a homotopy isomorphism and even a strong deformation retract.*

We require some preliminary lemmas. Lemma [ref] proves the metric $g_\alpha$ is nonpositively curved, and therefore distance-minimizing geodesics exist between any pair of points $x, y$ in $A$. However the geodesics are possibly nonunique.

**Lemma 9.** *For every $\alpha \geq 0$, the restricted cost $c = \tilde{d}_\alpha^2/2 : A \times \partial A \to \mathbb{R}$ satisfies the following (Twist) condition: for every $x \in A$, the gradient mapping*

$$\partial A \to T_x A, \;\; y \mapsto \nabla_x c(x, y) \text{ is } \nu\text{-a.e. injective.}$$

*Proof.* We take advantage of fact that $c$ is a Lagrangian cost defined by an action principle. According to [Vil09, Prop.10.15, pp.235], the gradient $\nabla_x c(x, y)$ is equal to

$$\frac{-1}{2}.\rho(x).\gamma'(0),$$

where $\rho$ is the conformal factor $\rho(x) = dist(x, \mathbb{R}^N - A)^{-\alpha}$, and where $\gamma'(0)$ is the initial tangent vector of any action-minimizing curve $\gamma$ in $A$ with $\gamma(0) = x$, $\gamma(1) = y$. The nonpositive curvature of $g_\alpha$ implies action-minimizing curves exist. These geodesics are possibly nonunique, however for $\nu$-a.e. $y \in \partial A$, the geodesic joining $x$ to $y$ will be unique. Since the conformal factor $\rho$ is nonvanishing, and since geodesics in Riemannian manifolds are determined by their initial point and initial tangent vector, we conclude $y \mapsto \nabla_x c(x, y)$ is $\nu$-a.e. injective, as desired. □

That $c$ satisfies the above (Twist) condition implies the uniqueness of $c$-optimal semicouplings from $\mu$ to $\nu$. [ref].

The above Lemma 9 and the identity

$$\nabla_x c(x, y) = \frac{-\rho_x}{2} \cdot \gamma'(0)$$

implies the gradient of the cross-difference $c_\Delta$ is readily computed

$$\nabla_x c_\Delta(x, y_0, y_1) = \frac{\rho_x}{2}.[\gamma_1'(0) - \gamma_0'(0)],$$

where $\gamma_0, \gamma_1$ are the $g_\alpha$-geodesics satisfying $\gamma_0(0) = \gamma_1(0) = x, \gamma_0(1) = y_0, \gamma_1(1) = y_1$.

Following the previous definitions of $\eta(x, avg)$ from (6), we have

$$\eta(x, avg) = \frac{\rho_x}{2} \left\langle |\psi(y) - \psi(y_0) + c_\Delta(x, y_0, y)|^{-1-\beta} \cdot [\gamma_0'(0) - \gamma_y'(0)] \right\rangle. \tag{9}$$

Here $\langle v \rangle$ denotes the average Bochner integral of a vector-valued function $v$ with respect to the Radon measure $\bar{\nu}$ on $\partial A$, where

$$d\bar{\nu}(y) = (1 - e^{d(y,y_0)^2}) d\nu(y).$$

That is

$$\langle v \rangle := (\bar{\nu}[Y])^{-1} . \int_Y v(y) d\bar{\nu}(y).$$

We remark that the average $\eta_{avg}$ is weighted by the regions where the potential $\psi(y) - \psi(y_0) + c_\Delta(x, y_0, y) \geq 0$ vanishes $(= 0)$. The weights blow-up to $+\infty$ when $y \to Z_2 - A$, and without the exponential factor $(1 - e^{d(y,y_0)^2})$ the weights would blow-up as $y \to y_0$.

*Remark* 10. If the region $A$ is convex, then the nonvanishing $\eta(x, avg)$ of (9) can be verified, since the tangent vectors $\{\gamma_y'(0) - \gamma_0'(0) \mid y \in \partial A\}$ lie in a common halfspace of $T_x A$.

The factor $1 - e^{d(y,y_0)^2}$ defines a "scale" on the boundary $\partial A$. If $x \in A$ is sufficiently close to the boundary $\partial A$ with respect to the above scale, then the gradients $\nabla_x c(x, y)$, for $y \in \partial A$ satisfying $d(y, y_0) \geq 1$, are readily seen to satisfy (UHS) conditions, since the gradients make large angle with respect to $\nabla_x c(x, y_0)$, i.e. the angle is greater than $\pi/2$. Compare Figure 4.

If $A$ is sufficiently "thin", such that all points $x$ are sufficiently close to $\partial A$, then we see (UHS) conditions are satisfied throughout $A - Z_2$. This is reminiscent of theorems of Alexander-Bishop [AB00, Thm 2.1]:

> Let $(M, g)$ be a Riemannian manifold with boundary $\partial M$, and let $C$ be the "cut locus of the boundary". If $M$ is sufficiently "thin" and has sufficiently small inradius relative to its curvature, then the cut locus $C$ is a strong deformation retract of $M$ and the inclusion $C \hookrightarrow M$ is a homotopy isomorphism.

There is alternative method for obtaining continuous deformation retracts. [insert]

## 5.

There is difficulty in applying our methods whenever $Z_2$ is not closed. Here is a necessary criterion for the successful application of the above deformation retracts. Criterion: for every $y_0 \in spt(Z)$, there needs exist at least one $y_1 \neq y_0$ with $Z(y_0, y_1) \neq 0$.

## 6. Upper Semicontinuity of $Z$ and $M(A)$.

The completion of Hubbard's $1/d_\alpha$ distance and the cost $c = \tilde{d}_\alpha^2$ yields an alternative to the medial axis $M(A)$ in the subvariety $Z_2$ defined by $c$-optimal couplings.

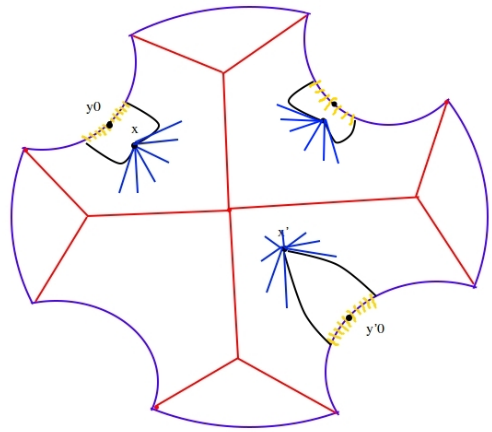


Figure 1. When $x \in A$ is sufficiently close to $\partial A$, and $y_0 := T(x)$, then the gradients $\nabla_x c(x, y)$, for $d(y, y_0) \geq 1$, are readily seen to satisfy HalfSpace Conditions. But for points $x' \in A$ at larger distance from $\partial A$, the gradients $\nabla_x c(x', y)$ are not contained in a proper halfspace for $d(y, T(x')) \geq 1$.

We propose this construction of $Z_2$ is a useful improvement over the conventional definition of $M(A)$ per (1). For instance the medial axis is defined on the category of open subsets $A$ of $\mathbb{R}^N$, whereas the functors $Z$ are more generally defined for measure spaces.

We remark on the differences between $C^0$, $C^{1,1}$, and $C^2$ regularity of boundaries $\partial A$. For $C^2$ boundary, the medial axis $M(A)$ will be disjoint from $A$. However for $C^0, C^{1,1}$ regularity, the medial axis $M(A)$ will extend into the boundary $\partial A$. The $C^{1,1}$ regularity frequently occurs, e.g. whenever $A$ is convex polyhedra. If $A$ is $C^{1.1}$ and $y \in \partial A$ is not uniquely differentiable boundary point, then $y \in M(A)$. I.e. the medial axis $M(A)$ extends into the boundary at $y$. Therefore "sharp" corners cause the medial axis to "split" and extend into $\partial A$. That sharp corners can appear under Gromov-Hausdorff variations of the subset $A \subset \mathbb{R}^N$ implies the medial axis is rather unstable. Small perturbations of the open subset $A$ (e.g. background noise) can lead to large changes in the medial axis $M(A)$. Many authors have suggested modified medial axes (c.f. [FLM03], [TH03] and references therein) which "filter out" possible noise. The present article takes another approach to "regularizing" $M(A)$. It's well-known that optimal transportation enjoys strong continuity properties with respect to variations in the datum $(\mu, \nu, c)$. More precisely we quote the following result from [Vil09, Thm. 28.9, pp.780–790].

**Proposition 11.** *Let $A_k$, $k = 1, 2, \ldots$, be bounded open subsets of $\mathbb{R}^N$ with canonical probability measures $\mu_k$, and converging in Gromov-Hausdorff topology*

$$\lim_{k\to+\infty}(A_k, \mu_k) = (A_0, \mu_0).$$

*Let $c_0, c_1, c_2, \ldots$ be the costs $c_k = \tilde{d}^2_{\alpha,A_k}/2$. Let $T_k : A_k \to \partial A_k$ be the maps defined in (3). Then:*

*(i) The sequence of probability measures $\nu_k := T_k\#\mu_k$ converges in the narrow topology to a probability measure*

$$\lim_{k\to+\infty}\nu_k := \overline{\nu_0}.$$

*(ii) The correspondance*

$$(\mu_k, \nu_k, c_k) \mapsto Z(\mu_k, \nu_k, c_k)$$

*varies upper semicontinuously, and there exists an injective natural transformation between the functors*

$$Z(\mu_0, \overline{\nu_0}, c_0) \hookrightarrow \lim_{k\to+\infty} Z(\mu_k, \nu_k, c_k).$$

*(iii) In particular there exists natural topological embedding*

$$Z_{2,0} \hookrightarrow \lim_{k\to+\infty} Z_{2,k}.$$

We refer the reader to any standard textbook on category theory for the definitions of functors and natural transformations, e.g. [Lan05, Ch.1]. The point of Proposition 11 is that the singularities of the limit $(\mu_0, \nu_0, c_0)$ are no more complicated than the approximant singularities of $(\mu_k, \nu_k, c_k)$. In fact the singularity often simplifies in various limits. The upper semicontinuity of the medial axis $A \mapsto M(A)$ is apparently well-known [ABE09, §5]. The Proposition 11 is familiar property of (lower semicontinuous) convex functions: if $\phi_k : X \to \mathbb{R}$ is a sequence of lsc convex functions which converge pointwise to a limit $\lim_{k\to+\infty}\phi_k = \phi_0$, then for every $x \in X$, the subdifferential $\partial\phi_0(x)$ is a subset of the Gromov-Hausdorff limit $\lim_{k\to+\infty}\partial\phi_k(x)$.

## 7. Conclusion

In conclusion, Blum identified the medial axis transform as convenient mode of describing objects, and heuristics showed the inclusions $M(A) \hookrightarrow A$ were always homotopy isomorphisms. However Blum's medial axis is only a particular instance of a more useful topological object, namely $Z_2$ of the contravariant functors $Z(\mu, \nu, c) : 2^{\partial A} \to 2^A$. This $Z_2$ is stable, and defined with respect to a Hubbard type Riemannian metric $g_\alpha$, and the inclusions $Z_2 \hookrightarrow A$ are identified as homotopy isomorphisms when the (UHS) Conditions (6), (7) hold throughout the open complement $A - Z_2$. Thus we

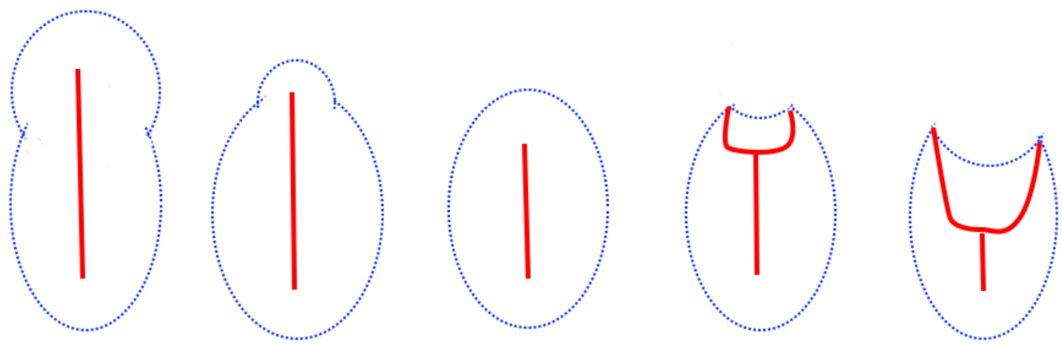

FIGURE 2. The medial axis $A \mapsto M(A)$ varies upper semicontinuously when sharp corners appear, like the two figures on the right.

propose a further "folk-theorem" regarding this mass transport extension of so-called medial axis transforms, and Theorem B from [Mar].

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