

Algebraic Topology and Optimal Transportation:

...How to build Spines from Singularity...

J.H.Martel

Our favourite problem of algebraic topology:

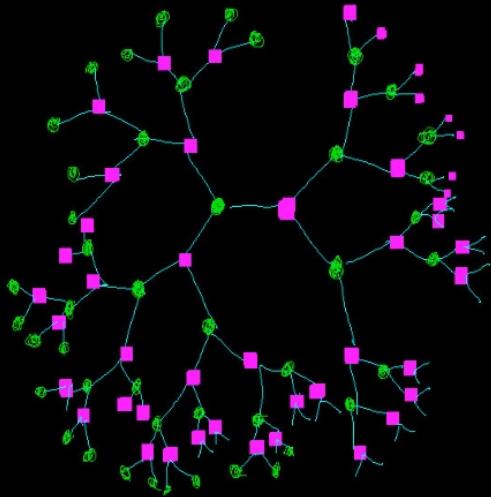
If (X, d, vol_X) is a finite-dimensional, complete, noncompact, finite-volume length space,

then:

construct a *minimal* subvariety Z of X ,
s.t. the inclusion $Z \rightarrow X$ is a homotopy-isomorphism,
And strong-deformation retract.

Def: maximal-codimension retracts Z are called *Spines* / *Souls* of (X, d, vol_X) .
 $(\kappa \leq 0)$ / $(\kappa \geq 0)$

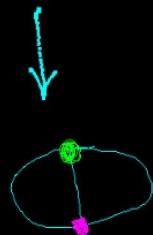
“subvariety” means satisfying a finite number of locally-lipschitz equations (“local DC”)
[definition from Poincaré’s ‘`Analysis Situs’’]



$X \quad E\Gamma$ -model

=Universal
Cover
of space M

$$\pi_1(M) = \Gamma$$



X / Γ quotient
with $\pi_1 = \Gamma$



Poincaré ~1895 constructs Algebraic-Topology:

← ← *Goal: construct Spines from $E\Gamma$ models ($\kappa \leq 0$)*

*Fundamental group, Universal
Covering Spaces, orbit quotients.*

Our thesis introduces and develops a general method
for explicitly constructing **SMALL**-dimensional **$E\Gamma$** classifying spaces.

Hypothesis: require Γ is infinite, discrete, Bieri-Eckmann duality group

with finite cohomological dimension $cd(\Gamma)=v < +\infty$ and dualizing module **D**.

Our method presumes a user has explicit geometric $E\Gamma$ model X:

i.e. (X,d) is finite-dimensional Cartan-Hadamard space (NPC contractible)

where group action $X \times \Gamma \rightarrow X$ is isometric, proper discontinuous, free,

and quotient X/Γ finite volume w.r.t. volume measure vol_X .

Our arguments apply simultaneously to families of numerous groups Γ :

Ex: $\Gamma = \mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}^3, \dots$ torsion-free abelian groups

- = $GL(\mathbb{Z}^2), GL(\mathbb{Z}^3), Sp(\mathbb{Z}^4), Sp(\mathbb{Z}^6), \dots$ arithmetic subgroups $G(\mathbb{Z})$ of Q -reductive groups,
- = $MCG(\Sigma_g)$... mapping class groups of closed orientable surfaces,
- = knot groups,
- = braid groups, etc.

..... all models not created equal.....

Ex: - $MCG(\Sigma_g)$ acts isometrically on $Teich(\Sigma_g)$ with *Weil-Petersson* geometry.

- $G(\mathbb{Z})$ acts isometrically on spaces of quadratic (or hermitian) forms.
- $GL(\mathbb{Z}^2)$ acts isometrically on Voronoi's cone, projectivizes to H^2
(Poincare disk)

If X is geometric $E\Gamma$ model, then typically $\dim(X) \gg \dots \gg cd(\Gamma)$.

Bieri-Eckmann formula:

$$cd(\Gamma) = \dim(X) - (q + 1),$$

(homological duality)

q =spherical-dimension(**D**).

Eilenberg-Ganea: $cd(\Gamma) == \text{minimal dimension of } E\Gamma\text{-models} \text{ ** if } cd(\Gamma) > 2$

Ex: Teichmueller space T_g

$$\dim(T_g) = 6g-6 , \quad vcd(MCG(\Sigma_g)) = 4g-5 ,$$

(Harer/Ivanov)

- Open Problem: where is explicit $(4g-5)$ -dimensional model of $EMCG(\Sigma_g)$??
- Our “final solution” is presently obstructed by a problem we call
“Closing the Steinberg symbol”

Let (X, d, vol_X) be a geometric $E\Gamma$ model, where Γ is Bieri-Eckmann duality group.

Our thesis constructs Γ -invariant closed subsets Z of X , with $\dim(Z) \approx \text{cd}(\Gamma)$,

for which **the inclusion $Z \rightarrow X$ is homotopy-isomorphism**,

and explicit Γ -equivariant continuous deformation retracts X onto Z ;

and moreover we describe a technique

(based on solutions of a problem called "Closing the Steinberg symbol")

for **achieving Z with MAX codimension $\dim(Z) == \text{cd}(\Gamma)$.**

Spines and Souls : Tradition in Geometric-Homology:

Klein, Minkowski, Poincare, Steenrod, Thom, Lefschetz, Wall, Eilenberg, Ganea, Borel, Serre, Thurston, Gromov, Neeman, Mumford, Gromoll-Cheeger-Perelman, Soule, Ash, McConnell,

...how to construct NEW models of “old” spaces, and as explicit as possible?

“Textbook” constructions of $E\Gamma$ are abstract/external/dislocated

- Requires perfect knowledge of Γ , i.e. generators and relations.
- Milnor: $E\Gamma = \text{joins}(\Gamma, \Gamma, \Gamma, \dots)$.
- Wall: inductive wedges of spherical-complexes and attaching maps.
- Postnikov towers, Cayley graphs, Rips complex,

We presume limited knowledge of Γ , but require explicit geometric $E\Gamma$ -model (X, d, vol_X).

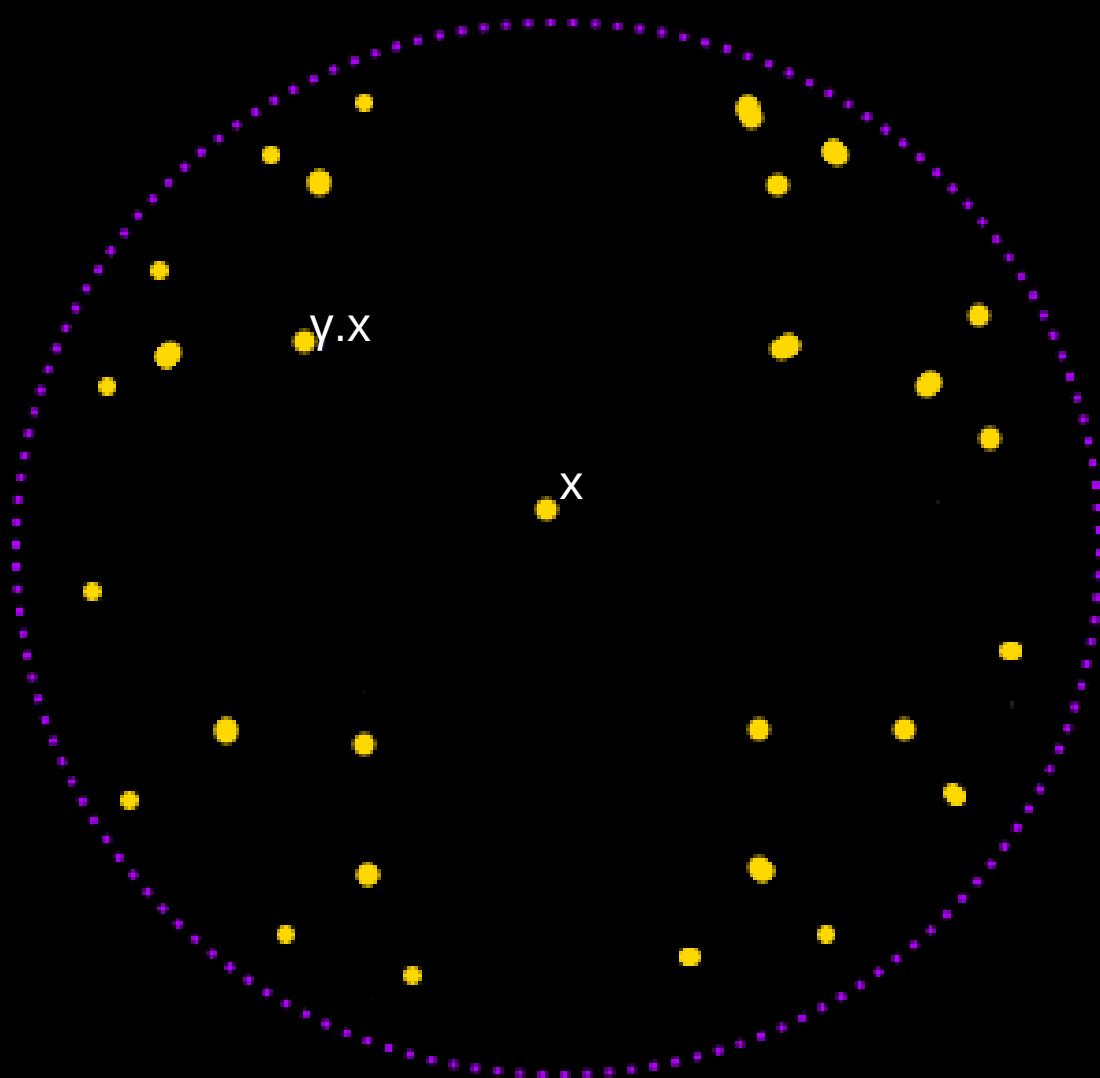
Our thesis develops a new program for In Situ Reduction-to-Spine,

- *exhibits Spines as explicit subsets of initial model X*
- *nonlinear extension of Soule-Ash’s Well Rounded Retract
[1980s, explicit Spines for $EGL(Z^N)$, $N \geq 1$]*

...illustrating our approach:

Begin with initial geometric
 $E\Gamma$ model (X, d, vol_X).

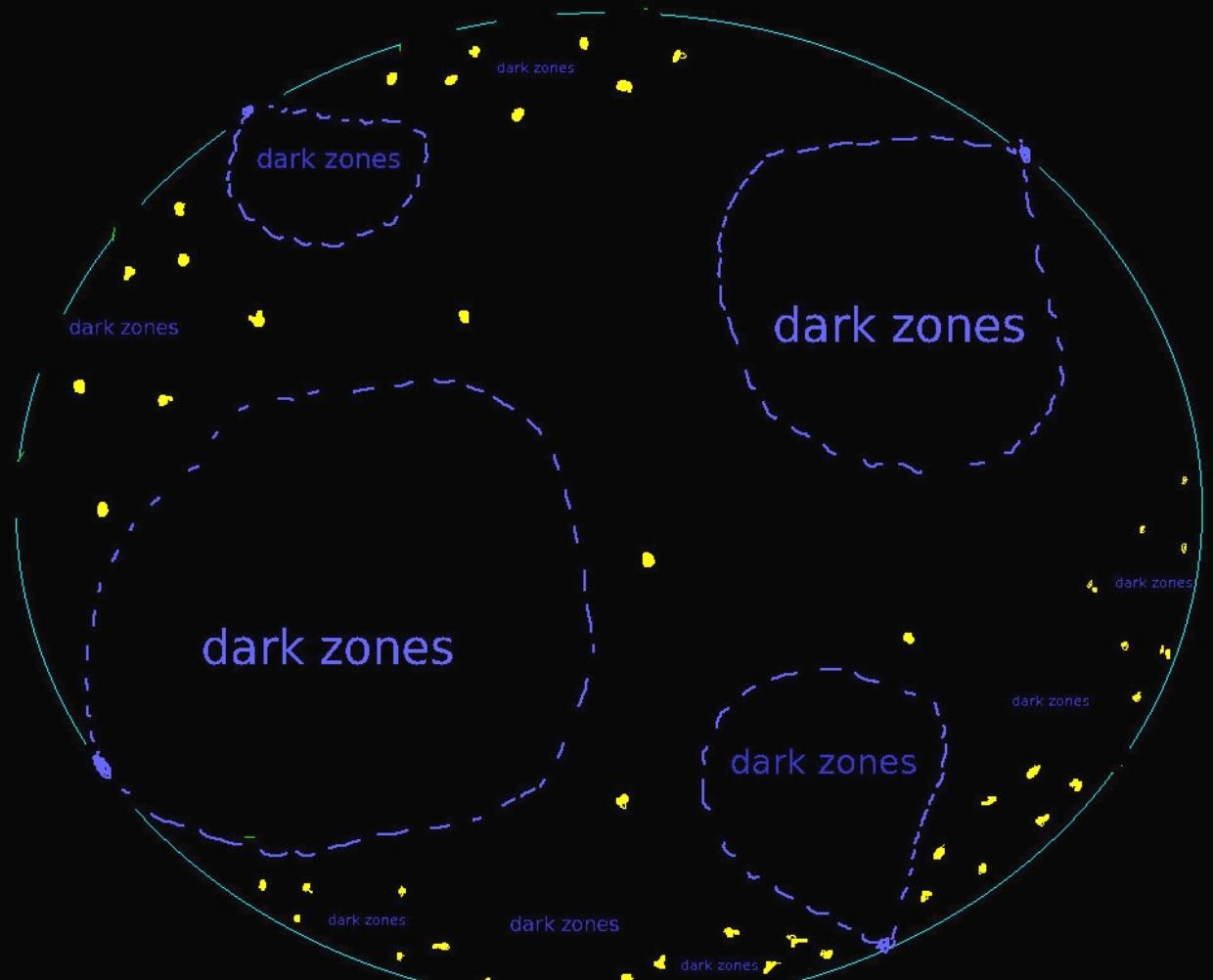
Yellow dots = Γ orbit of pt. x



Observe:

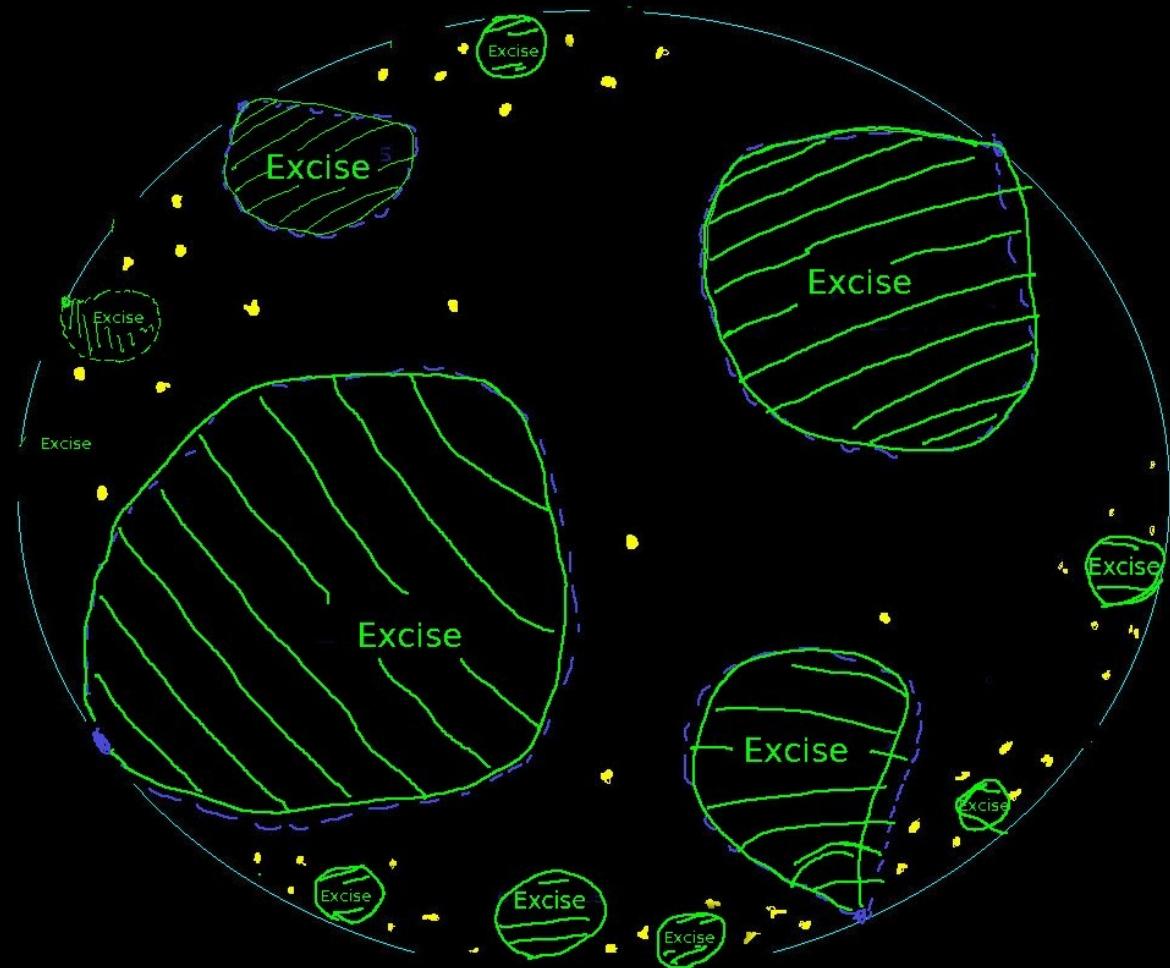
Γ -orbit avoid dark zones.

*Dark zones =
countable family of
 Γ -rational horoballs $V[t]$*



Orbit avoids Dark zones

*Excision $X[t]$ obtained by
scooping out / excising
the dark zones from X .*



Orbit avoids Dark zones

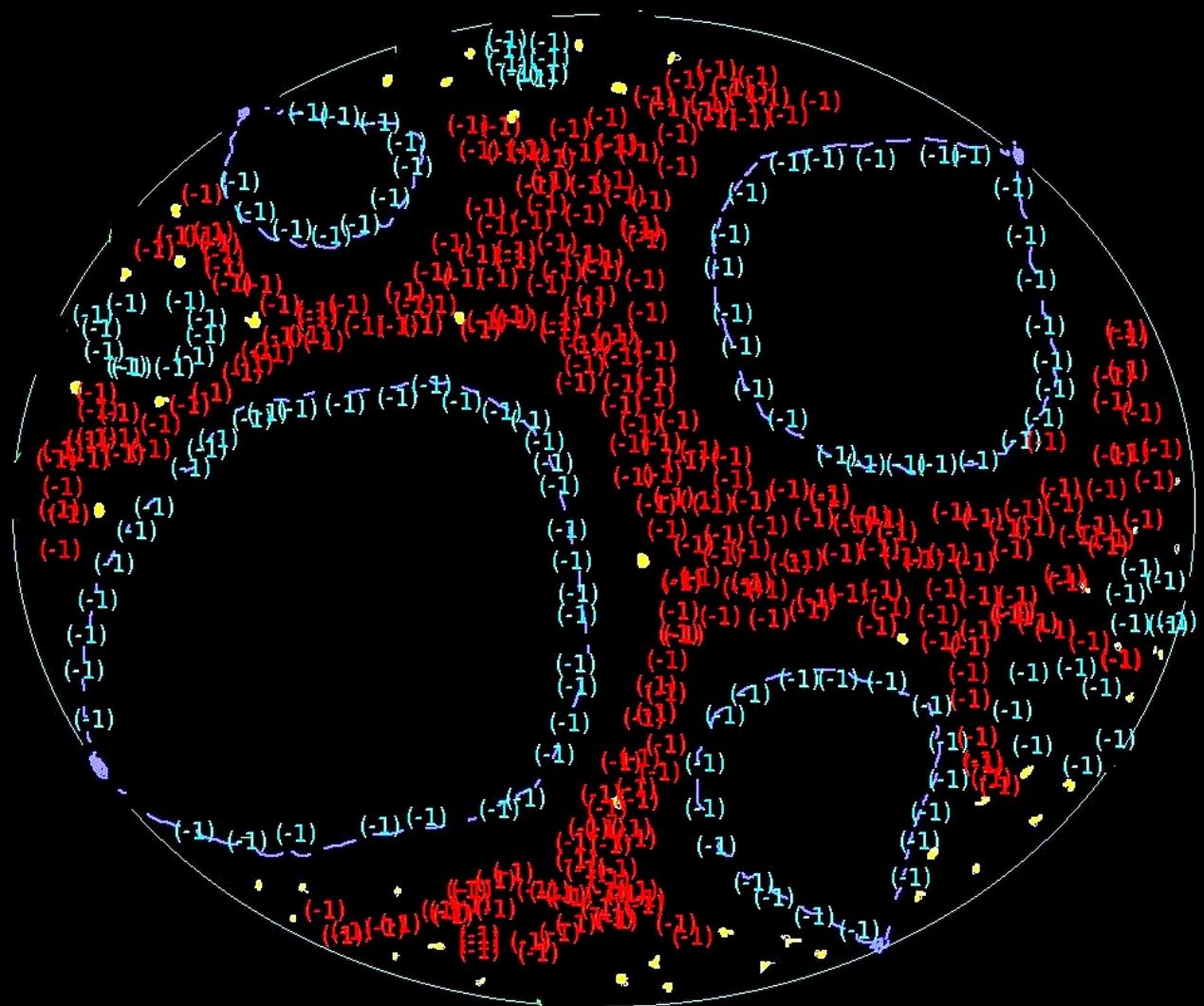
Scoop/Excise convex deep dark zones at-infinity.

*Excision ==>
manifold-with-corners
 $X[t] \times \delta X[t]$.*

Define:

(-1) source
measure σ
on $X[t]$

(-1) target
measure τ
on $\delta X[t]$

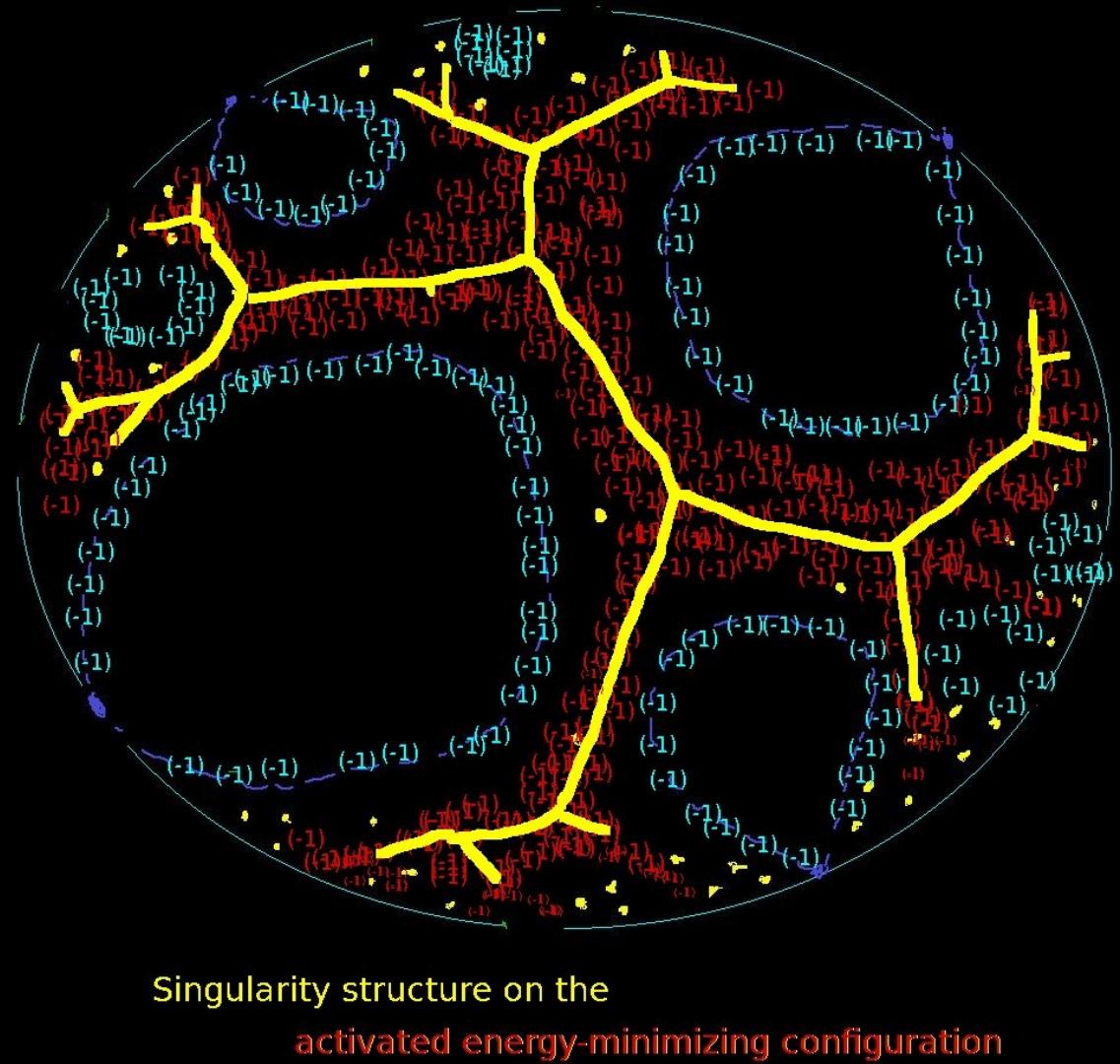


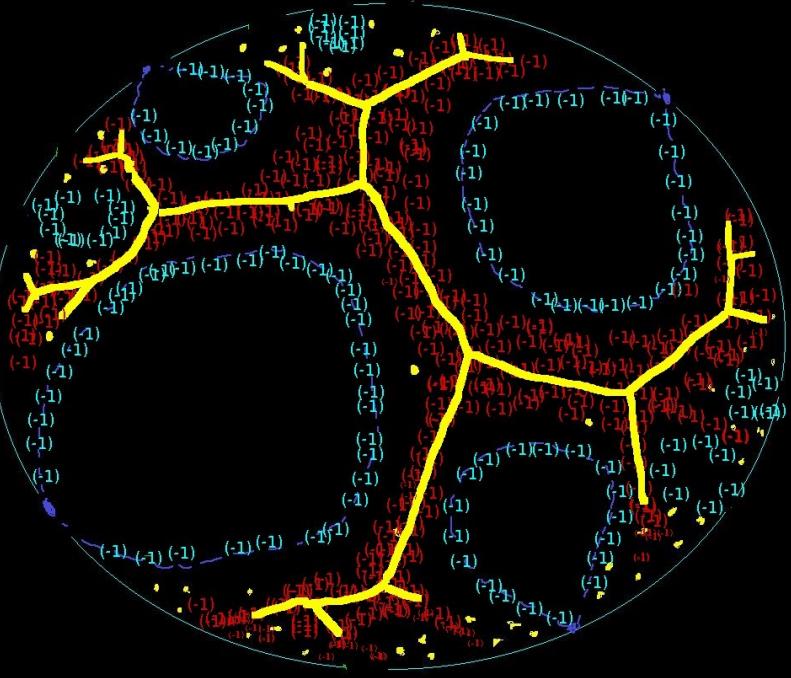
With measures σ , τ , we next study the *Singularity structure* of “Energy-minimizing” semicouplings from source to target.

$$Z: \begin{matrix} \delta X[t] \\ 2 \end{matrix} \rightarrow \begin{matrix} X[t] \\ 2 \end{matrix}$$

We propose:

Spines are readily displayed in the locus-of-discontinuity (Singularities) of optimal semicouplings from $X[t]$ to $\delta X[t]$





*Remarks: this **one-dimensional tree T** is long-known, and many retractions of X onto T exist.*

Except our construction (via Singularity structure) applies verbatim to higher dimensions...

...Spines are not hidden...

Spines are readily displayed
in *locus-of-discontinuity* Z
of “deformation retracts $r : X[t] \rightarrow \delta X[t]$ ”

“*the idea*”

...Spines are not hidden...

“the formalization”

Spines readily displayed in
Kantorovich's Contravariant Singularity Functor $Z(\sigma, \tau, v)$
of v -optimal semicouplings π from $(X[t], \sigma)$ to $(\delta X[t], \tau)$

... where $v: X[t] \times \delta X[t]$ is the ``visible repulsion'' cost on a Γ -rational excision $X[t]$

Terms to define:

Singularity functor $Z(\sigma, \tau, v)$, $Z : 2^{\delta X[t]} \rightarrow 2^{X[t]}$

of v -optimal semicouplings π

from source $(X[t], \sigma)$ to target $(\delta X[t], \tau)$

of Γ -rational excision $X[t]$

Terms to define:

Topology:

Source excision models $(X[t], \sigma)$

Target $(\partial X[t], \tau)$.

Steinberg modules $D := \tilde{H}_q(\partial X[t]; \mathbb{Z})$.

Steinberg symbols $B \in H_q(\partial X[t]; \mathbb{Z})$

and $\text{FILL}[B] = H_{q+1}(X[t], \partial X[t]; \mathbb{Z})$.

Chain sums $\underline{F} = \sum_{\gamma \in \Gamma} F \cdot \gamma$

with well-separated gates $\{G\} = \{\text{FILL}[B].\gamma \quad | \quad \gamma \in \Gamma\}$.

Terms to define:

Mass transport:

Costs $c : X[t] \times \partial X[t] \rightarrow \mathbb{R}$.

Two-pointed repulsion and visibility costs c^*, v

c -optimal semicouplings π .

c -concave potentials $\psi^{cc} = \psi$.

c -subdifferentials $\partial^c \psi(y) \subset X[t]$ for $y \in \partial X[t]$.

Monge-Kantorovich duality: c -optimal semicouplings π supported on graph of $\partial^c \psi$.

Kantorovich Singularity functor $Z : 2^{\partial X[t]} \rightarrow 2^{X[t]}$.

Filtrations $Z_0 \hookleftarrow Z_1 \hookleftarrow Z_2 \hookleftarrow \dots$.

Kantorovich's Contravariant Singularity Functor

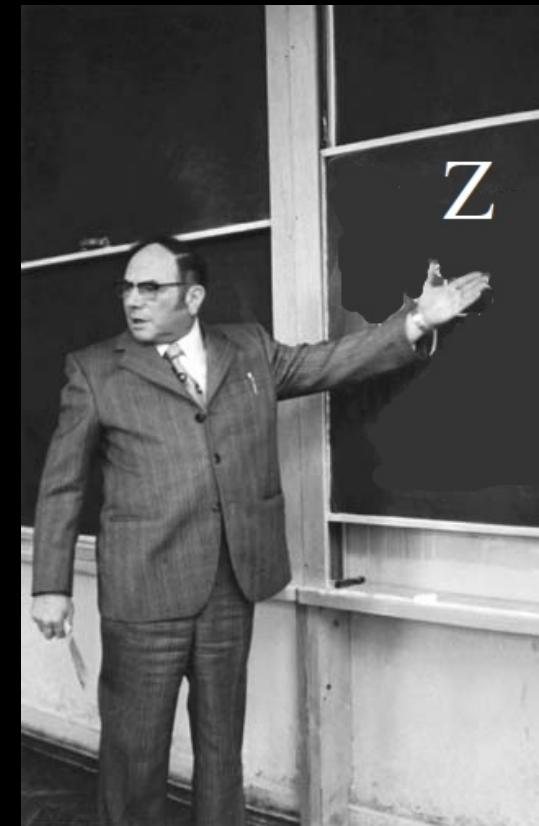
$$Z : 2^{\partial X} \rightarrow 2^X, \quad Z(Y_I) = \cap_{y \in Y_I} \partial^c \psi(y) = " \cap_{y \in Y_I} r^{-1}(y)" .$$

... but everything summarized in: Kantorovich's Contravariant Singularity Functor

$$Z : 2^{\partial X} \rightarrow 2^X,$$

$$Z(Y_I) = \cap_{y \in Y_I} \partial^c \psi(y)$$

$$= " \cap_{y \in Y_I} r^{-1}(y)" .$$



$\psi^{CC} = \psi$ is c-concave potential, $\psi: \delta X[t] \rightarrow \mathbb{R}$, and $Y_I \subset \delta X[t]$ closed subset.

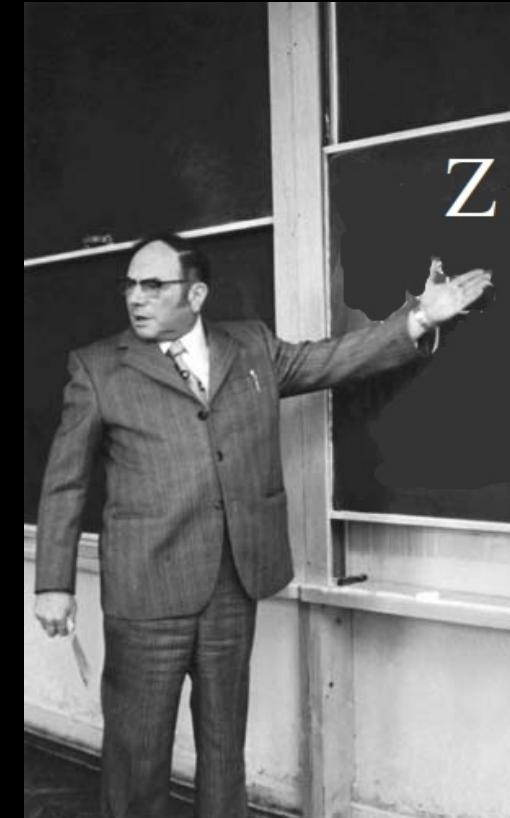
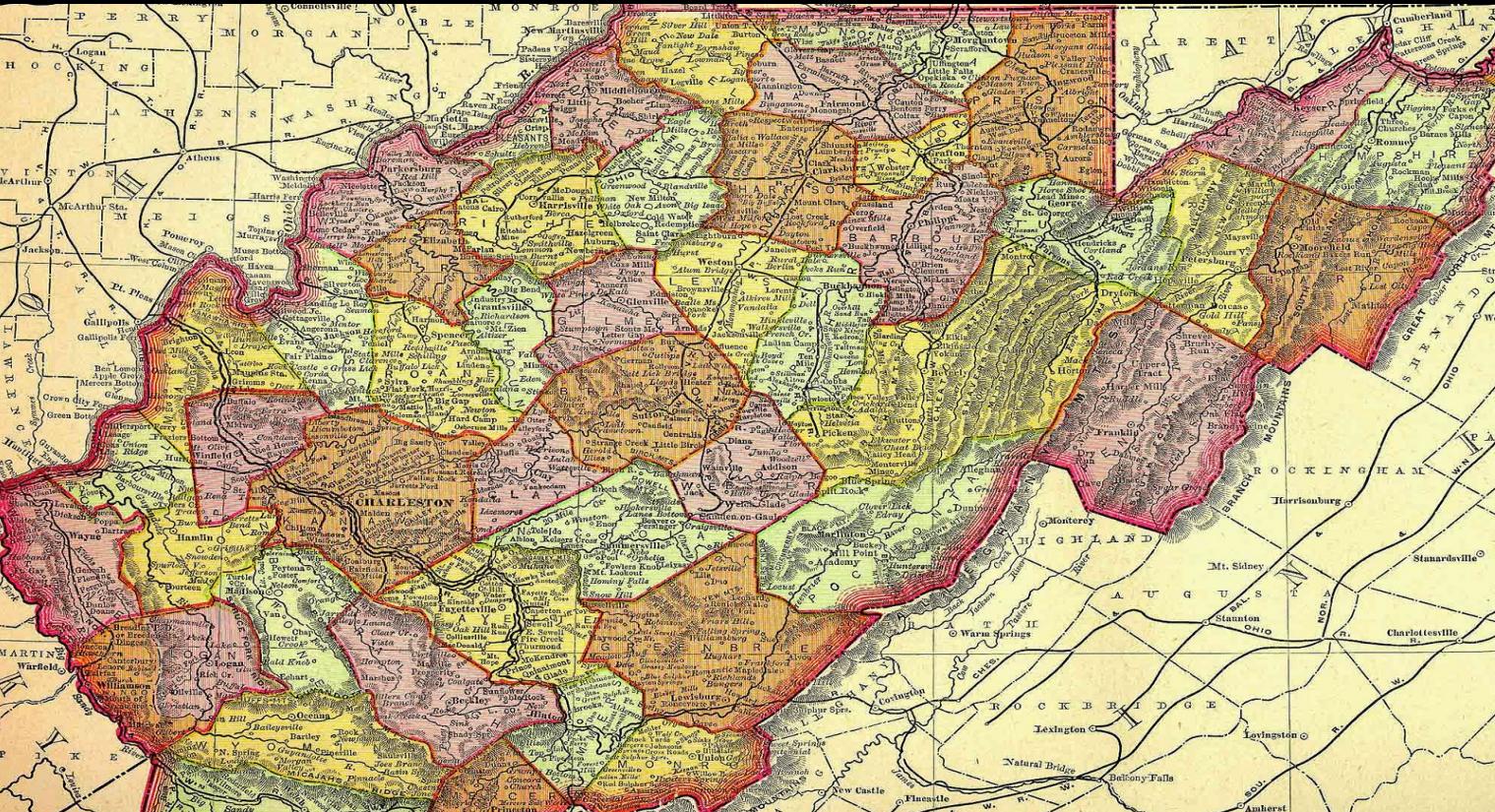
Singularity is overburdened term.

$$Z : 2^{\partial X} \rightarrow 2^X, \quad Z(Y_I) = \cap_{y \in Y_I} \partial^c \psi(y)$$

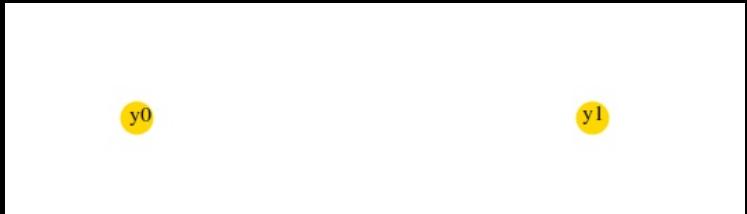
Economic Definition:

Singularity arises wherever there is competition for limited common resources.

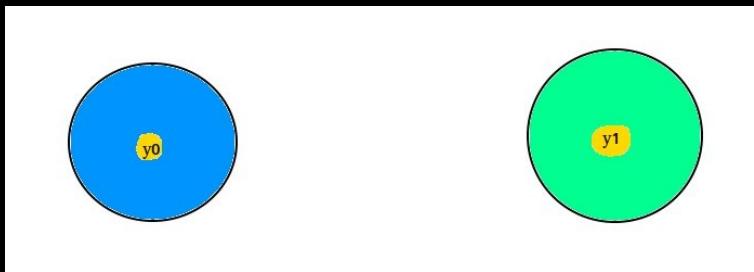
Singularity is Why countries exist with borders.



Singularity: wherever competition for limited resources:

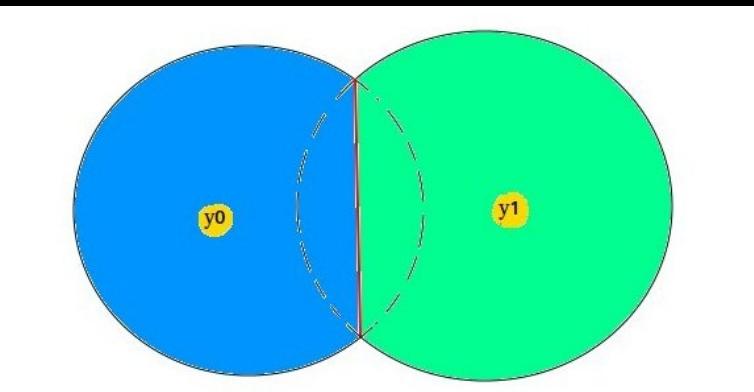


$c=d^2/2$ quadratic cost
(+) → ← (−) attraction



- y_0, y_1 no competition
(no interact)

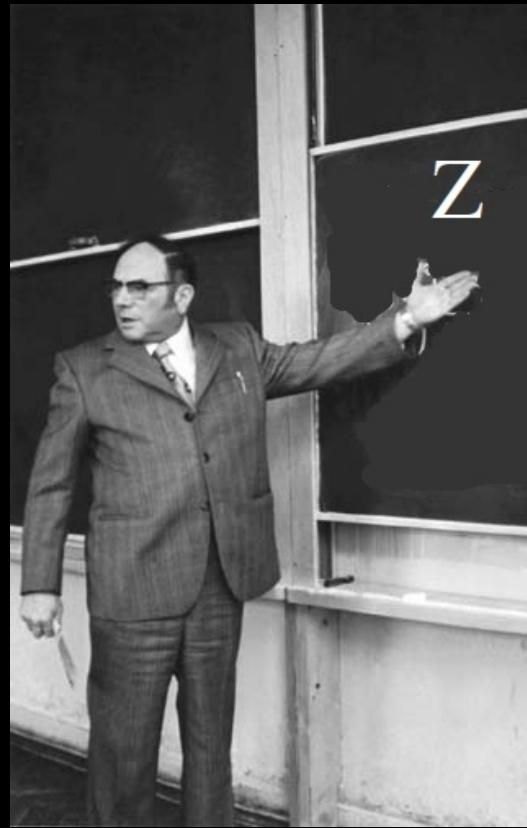
- *Singularity* = Empty



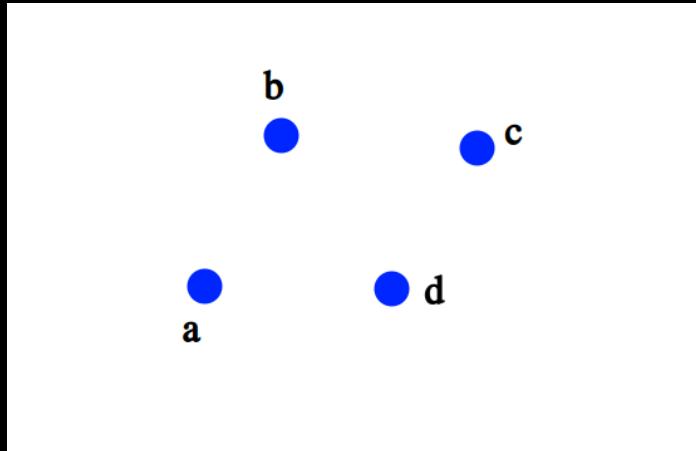
- y_0, y_1 compete/interact

- *Singularity* nonempty
& stable/persistent
w.r.t. target measure τ

$$Z : 2^{\partial X} \rightarrow 2^X,$$

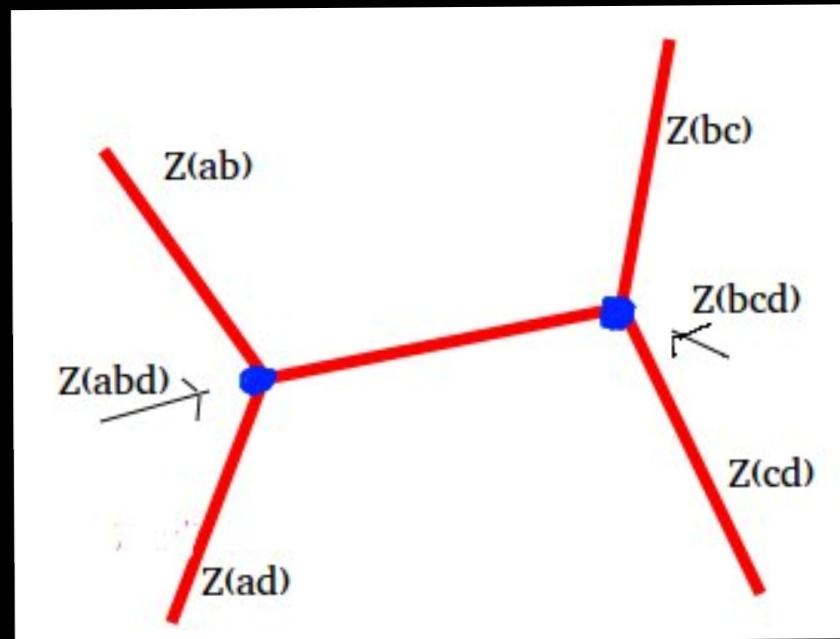
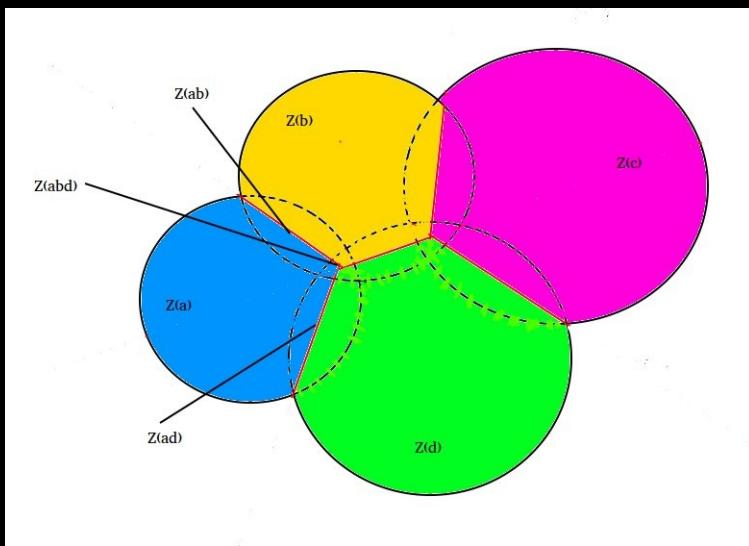
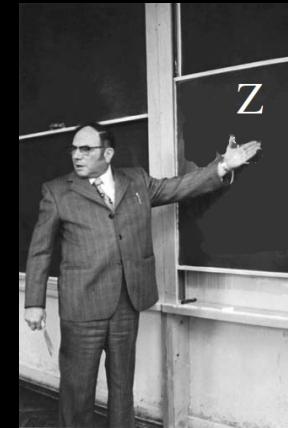


$$Z(Y_I) = \cap_{y \in Y_I} \partial^c \psi(y)$$



Source = 2-dim background field

Target = $+a+b+c+d$ (4 point masses)



$$c = d^{**} 2/2$$

quadratic cost

$(+)$ \rightarrow \leftarrow $(-)$
attraction

...for every cost $c: X \times \delta X \rightarrow \mathbf{R}$, there is c -Legendre-Fenchel transform

Key definitions: - c -concave potentials $\psi^{cc} = \psi$.

- c -subdifferentials $\partial^c \psi(y)$ are subset of X for every $y \in \delta X$



Kantorovich says `` c -optimal semicouplings π are supported on the graphs of c -subdifferential of c -concave potentials $\psi^{cc} = \psi$ ''

Monge-Kantorovich Duality:

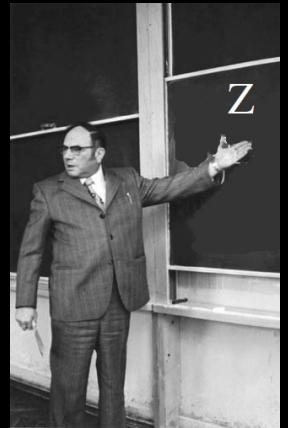
$$\max_{\psi \text{ c-concave}} \left[\int_X -\psi^c(x) d\sigma(x) + \int_{\partial X} \psi(y) d\tau(y) \right] = \inf_{\pi \in SC(\sigma, \tau)} \int_{X \times \partial X} c(x, y) d\pi(x, y)$$

$-\psi^c(x) + \psi(y) \leq c(x, y)$

Kantorovich's Contravariant Singularity Functor IS EXPLICIT.

- c -concavity $\psi^{cc} = \psi$ of a potential $\psi : \partial X[t] \rightarrow \mathbb{R}$ represents a pointwise inequality

$$-\psi^c(x) + \psi(y) \leq c(x, y)$$



for all $(x, y) \in X[t] \times \partial X[t]$,

with equality $\psi(y) - \psi^c(x) = c(x, y)$

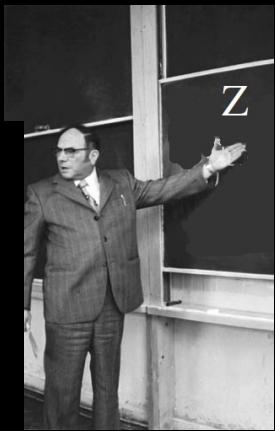
if and only if $y \in \partial^c \psi^c(x)$ iff $x \in \partial^c \psi(y)$

iff $y \in \text{argmax}[\{\psi(y_*) - c(x, y_*) + c(x, y) | y_* \in Y\}]$.

Variational defn. of
 c -subdifferential

Define $Z(\{y\}) := \partial^c \psi(y)$

Kantorovich's Contravariant Singularity Functor IS EXPLICIT.



$Z(Y_I)$ consists of $x \in X$ for which

*Variational defn. of
c-subdifferential*

$\text{argmax}[\{\psi(y_*) - c(x, y_*) + c(x, y) | y_* \in Y\}]$ contains Y_I ,

where $y_0 \in Y_I$ is reference point.

Abbreviate $c_\Delta(x; y, y') := c(x, y) - c(x, y')$ two-pointed cross difference.

Implies equations

$$Z(Y_I) = \{0 = \psi(y_0) - \psi(y) - c_\Delta(x; y_0, y) \mid y, y_0 \in Y_I, y \neq y_0\}$$

Reduces to $\#(Y_I) - 1$ equations. Symmetry y_0, y_1 .

Applications to Algebraic Topology:

Contravariance says $Z(Y_I)$'s are local cells in $X[t]$, parameterized contravariantly by subsets Y_I of $Y = \delta X[t]$

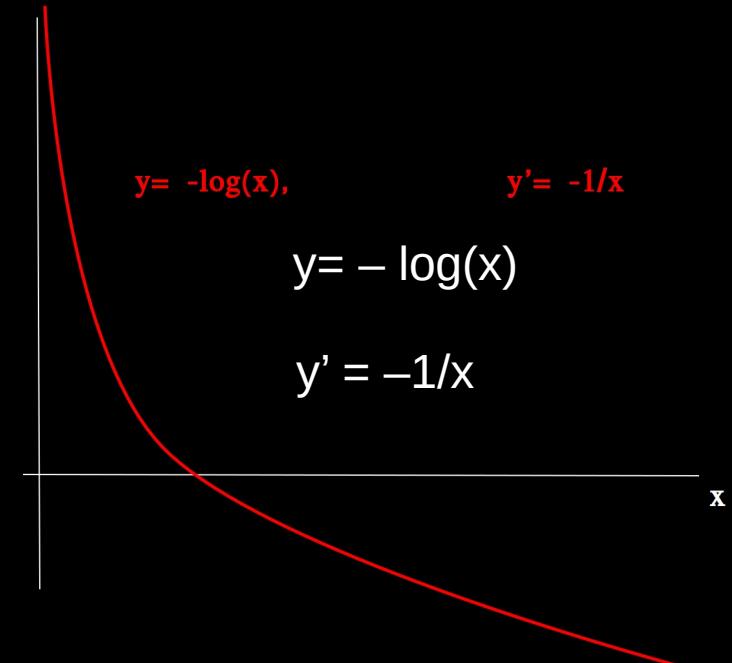
$$\boxed{\text{if } Y_I \hookrightarrow Y_J. \quad \text{then} \quad Z(Y_I) \hookleftarrow Z(Y_J)}$$

Theorem (Local Reduction)

Let $Z = Z(\sigma, \tau, c)$ be Kantorovich functor for cost c , source σ , target τ , with $\sigma[X] > \tau[\delta X[t]]$.

Then we find local criterion (UHS conditions) which ensures $Z(Y_I) \hookleftarrow Z(Y_J)$ is homotopy-isomorphism, and construct explicit continuous deformation retracts wherever (UHS) satisfied.

- Proof:
- Variational definition of c -subdifferentials, and gradient flow toward positive poles (not zeros!).
 - Model: gradient flow to $x=0$ of $f(x) = -\log(x)$, $x>0$.
 - flow accelerates into the cusp.



Applications to Algebraic Topology:

If we “skewer the cube diagonally” and filtrate according to dimension, we obtain descending chain of closed subsets

$$X[t] \leftarrow Z\{1\} \leftarrow Z\{2\} \leftarrow Z\{3\} \leftarrow \dots, \dots \quad \text{where codim } Z\{k\} = k-1$$

-Contravariance implies $Z\{1\}, Z\{2\}, Z\{3\}, \dots$ are homology-cycles in X , $\delta Z\{k\} = 0$ (consequence of adjunction formula)

Theorem (Global Reduction):

Let $Z = Z(\sigma, \tau, c)$ be Kantorovich functor for cost c , source σ , target τ , with $\sigma[X] > \tau[\delta X[t]]$.

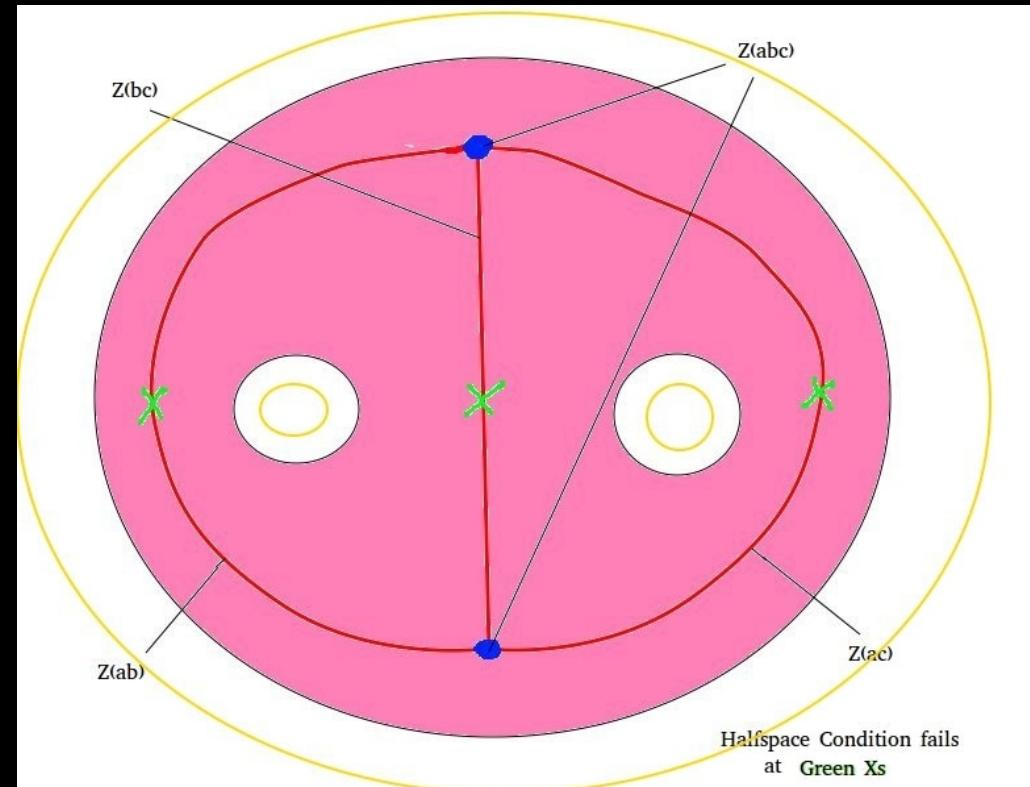
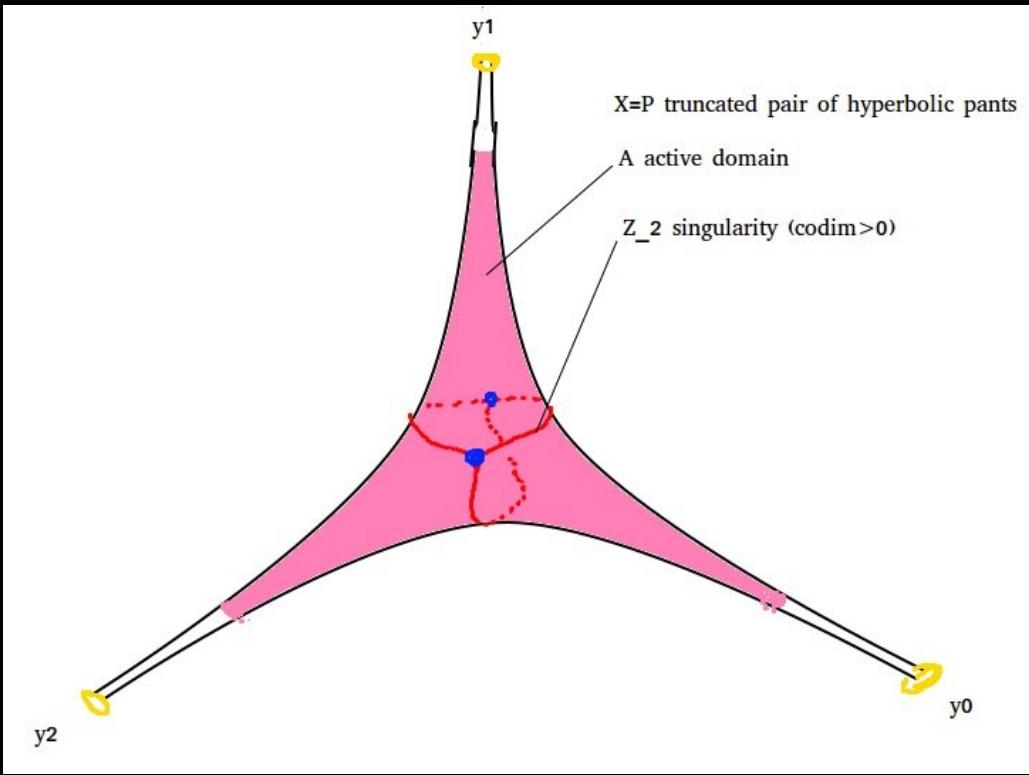
We identify index $J \geq 0$ such that local cells $\{Z(Y_I) \mid Y_I \rightarrow Y\}$ and their local homotopy reductions assemble into global continuous reductions

$$X[t] \rightarrow Z\{1\} \rightarrow Z\{2\} \rightarrow \dots \rightarrow Z\{J+1\}$$

and such that $Z\{J+1\}$ is a codimension- J closed subvariety of X , with inclusion $Z\{J+1\} \rightarrow X$ a continuous homotopy-isomorphism.

- index J defined by max. codimension of cells $Z(Y_I)$ where (UHS) satisfied.

Global Reduction Theorem ==>
 Singularity of Repulsion cost between excised source pant $P[t]$
 and target boundary $\delta P[t]$ produces the familiar Θ -graph



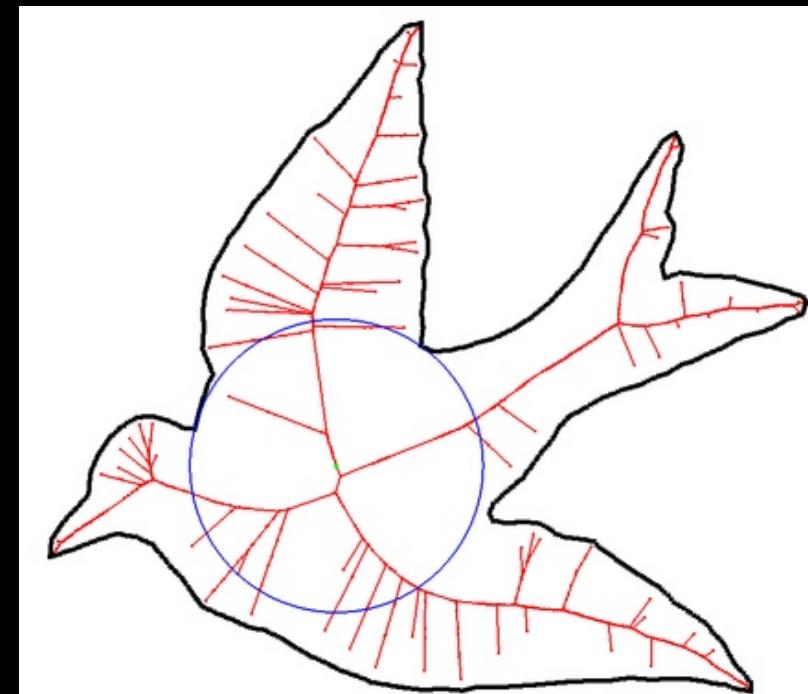
Local & Global Reduction Theorems are “generalizations” of following “folk-theorem”:

Thm [Leutier, 2004]

If A is open subset of \mathbb{R}^N ,
then the “medial axis” $M(A)$ includes into A as homotopy-isomorphism
(and is actually strong def. retract).

Recall $M(A) := \{ x \in A \mid \text{dist}(x, \delta A) \text{ attained}$
by at least two points $y_1, y_2 \in \delta A \}$

- $M(A)$ unstable w.r.t. perturbations of A
- $M(A)$ generally codimension 1 hypersurface



Our Local and Global Reduction theorems are general.

- Valid for every choice of cost $c: X[t] \times \delta X[t] \rightarrow \mathbb{R}$

Applications seek index J be large as possible.

Many obstructions exist!

*Ex. Quadratic cost $c=d^{**}2/2$*

- *inclusion $Z\{1\} \rightarrow (X=Z\{0\})$ is generally **NOT** homotopy-isomorphism*

Best results obtained with anti-quadratic ``repulsion'' costs (defined below)

Example: [Attractive (-1)+ (+1) Quadratic Cost]

Consider closed unit interval $X = [0,1]$ with boundary $\delta X = \{0, 1\}$.

σ is uniform distribution of (-1) charges. mass(σ) = 15(-)

τ is uniform distribution of (+1) charges. Mass(τ) = 6(+)

mass(σ) > mass(τ)

{+1}
{+1}
{+1}

{+1}
{+1}
{+1}

(-1) (-1) (-1) (-1) (-1) (-1) (-1) (-1) (-1) (-1) (-1) (-1) (-1) (-1)

()
()
()
()()()
y=0

Z(01)={empty}

()()()
Z(1)
()()()
y=1

Ground state

Ground state

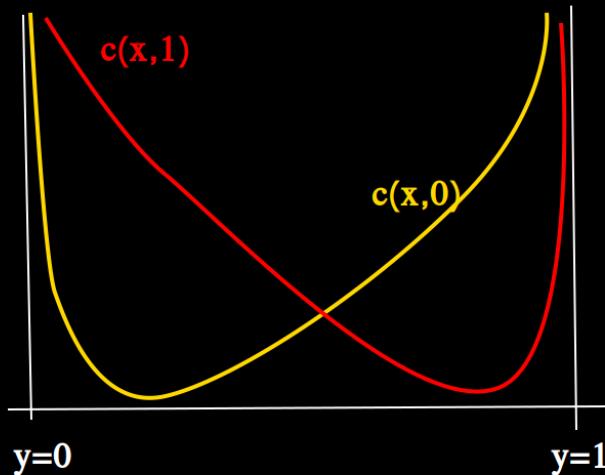
One-dimensional Repulsion cost c^* :

Unit interval $X = [0,1]$, boundary $\delta X = \{0,1\}$

$$c^*(x,0) = |x|^{-2} + 2|1-x|^{-2}$$

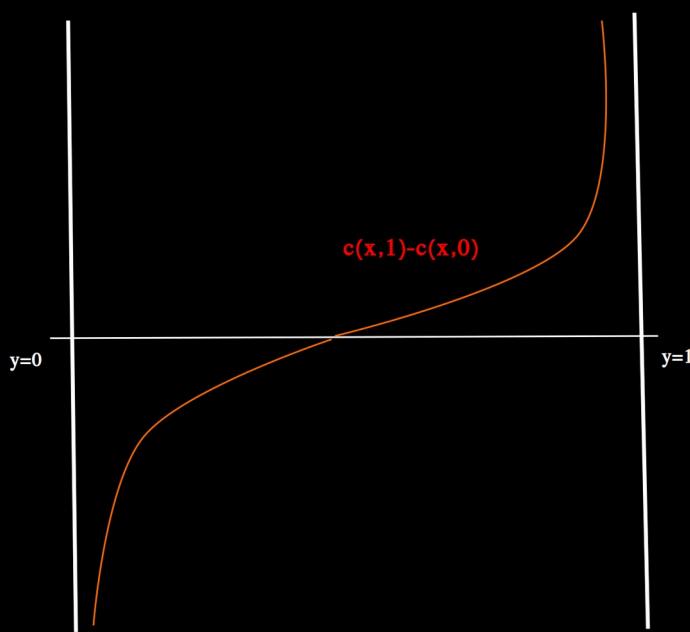
$$c^*(x,1) = 2|x|^{-2} + |1-x|^{-2}$$

$$c^*(x,1) - c^*(x,0) = |1-x|^{-2} - |x|^{-2}$$



- cross-difference is critical-point free!
(critical points at poles $x=0,1$)

- every fibre is connected.



Consider closed unit interval $X = [0,1]$ with boundary $\delta X = \{0, 1\}$.

σ is uniform distribution of (-1) charges. $\text{mass}(\sigma) = 15(-)$
 τ is uniform distribution of (-1) charges. $\text{Mass}(\tau) = 4(-)$

$\boxed{\text{mass}(\sigma) >> \dots >> \text{mass}(\tau)}$

(-1)
(-1)
(-1) (-1) (-1) (-1) (-1) (-1) (-1) (-1) (-1) (-1) (-1) (-1) (-1) (-1)



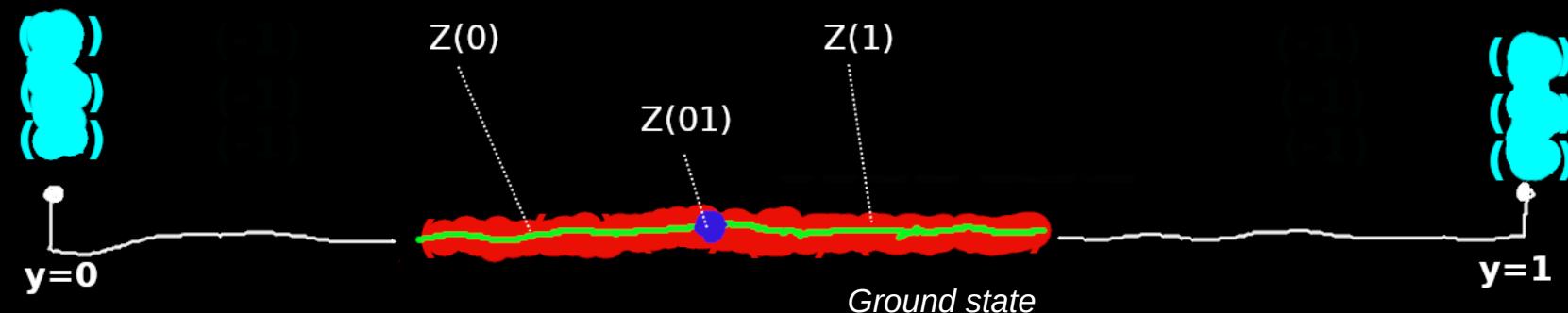
Ground state

Consider closed unit interval $X = [0,1]$ with boundary $\delta X = \{0, 1\}$.

σ is uniform distribution of (-1) charges. $\text{mass}(\sigma) = 15(-)$
 τ is uniform distribution of (-1) charges. $\text{Mass}(\tau) = 6(-)$

mass(σ) >> mass(τ)

(-1)
(-1)
(-1)
(-1) (-1) (-1) (-1) (-1) (-1) (-1) (-1) (-1) (-1) (-1) (-1) (-1) (-1)



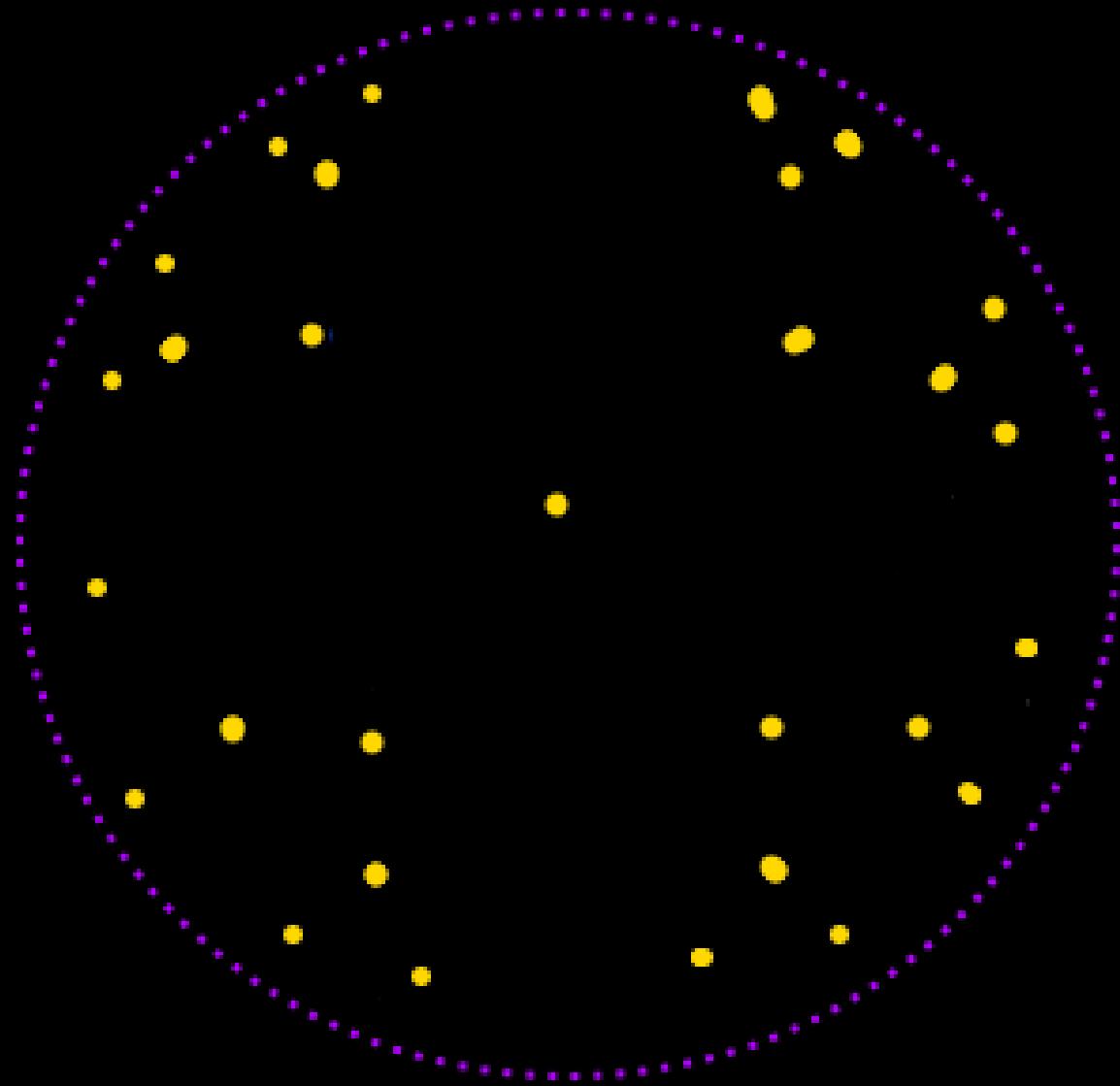
- Kantorovich functor $Z=Z(\sigma,\tau,c)$ defined for general costs c ,
- we have defined the one-dimensional repulsion cost $c=c^*$

Now we describe an applications to a topological Extension Problem.

- requires “Closing the Steinberg symbol” and visibility cost v^*

Recall our excision $X[t]$, $\delta X[t]$ and the hypothesis
that Γ satisfies Bieri-Eckmann homological duality:

Initial geometric $E\Gamma$ -model X



Γ -rational Excision $X[t] \times \delta X[t]$

Given initial geometric $E\Gamma$ model (X, d, vol_X) :

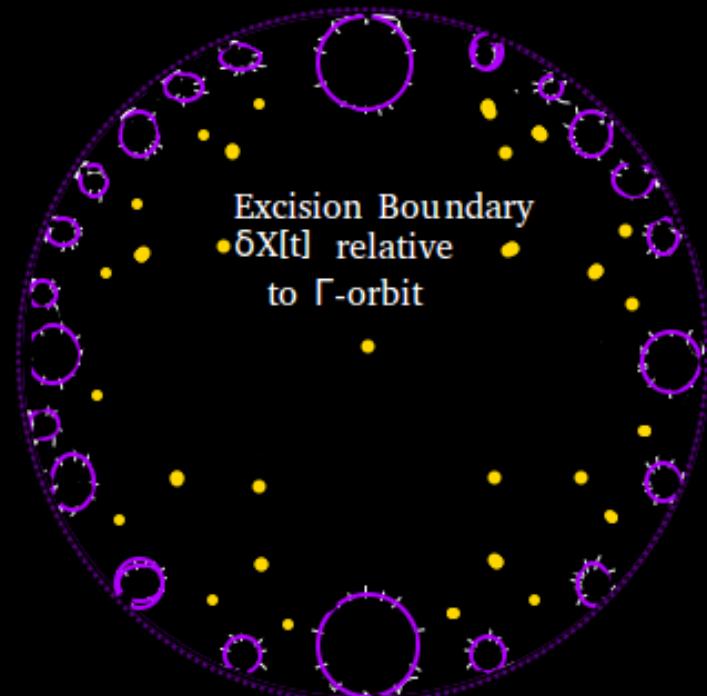
Construct Γ -rational excision $X[t]$ with proper-discontinuous boundary $\delta X[t]$,

Obtain: Source ($X[t], \sigma$) / Target ($\delta X[t], \tau$)

Excision $X[t] := X - U V[t]$

obtained by scooping-out/excising from X a countable family of Γ -rational horoballs $V[t]$ which are “nearly” at-infinity.

- Γ -rationality implies $X[t]$ and boundary $\delta X[t]$ are Γ -invariant subsets, and inherit proper-discontinuous Γ -actions.



Theorem [Curtis-Solomon-Tits]:

The excision boundary $\delta X[t]$ of maximal Γ -rational excision has the homotopy-type of a countable wedge of q-spheres.

Theorem [Bieri-Eckmann]:

The reduced singular homology group, with natural $Z\Gamma$ -module structure

$$D = \sim H_*(\delta X[t])$$

is homological-dualizing module (a.k.a. “Steinberg module”).

[Solomon-Tits]==>

Steinberg module D is infinite cyclic $\mathbb{Z}\Gamma$ -module

- generated by boundary spheres B "at-infinity"

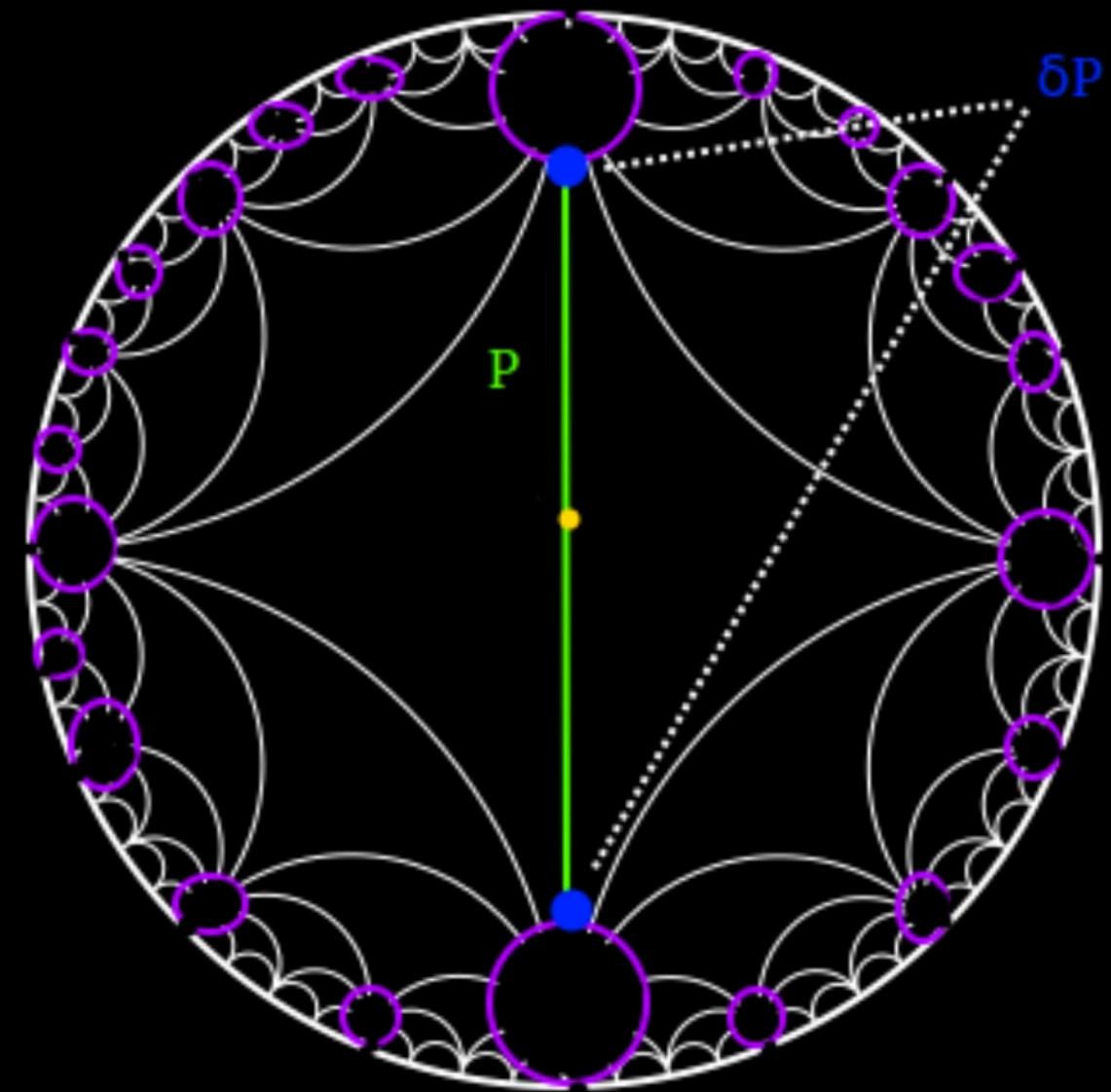
$\{X[t] \text{ Cartan-Hadamard}\} + \{\text{LES relative homology}\} ==>$

The boundary map ("connecting homomorphism") is isomorphism,
with canonical inverse δ^{-1} defined by "Flat Filling" on singular chains

$$\partial : H_{q+1}(X[t], \partial X[t]) \xrightarrow{\sim} \tilde{H}_q(\partial X[t])$$

$$P = \text{FILL}[B] \quad B$$

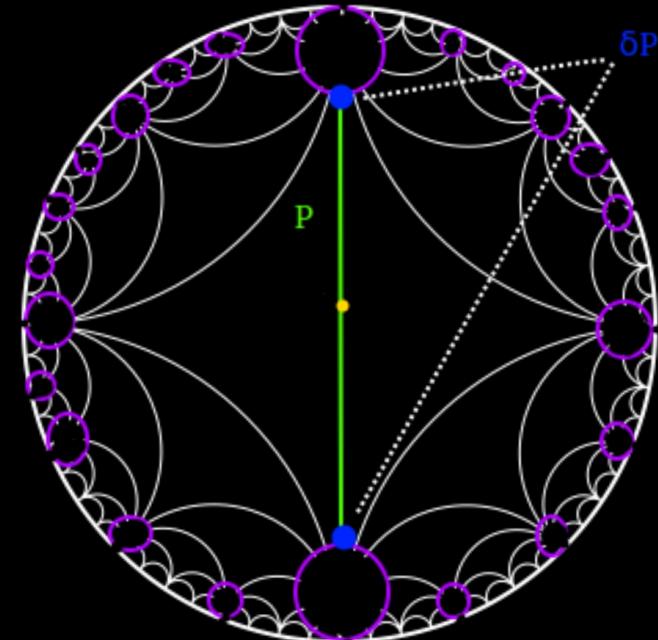
Steinberg symbol P is
relative 1-cycle; with boundary
 δP a boundary 0-sphere.



Homological duality:

- Steinberg symbols are relative cycles $\text{FILL}[B] \in H_{q+1}(X[t], \partial X[t])$
- Bieri-Eckmann duality implies $\text{FILL}[B]$ is dual cycle to minimal spines.
- $\dim(\text{FILL}[B])$ is max codimension of minimal spines (homological duality formula)
- Observe $\text{FILL}[B]$ deformation retracts to $\{\text{pt}\}$
 $\text{FILL}[B] \rightarrow \{\text{pt}\}$.

$$\boxed{\text{Duality} \implies \text{cd}(\Gamma) + \dim(\text{FILL}[B]) = \dim(X)}$$

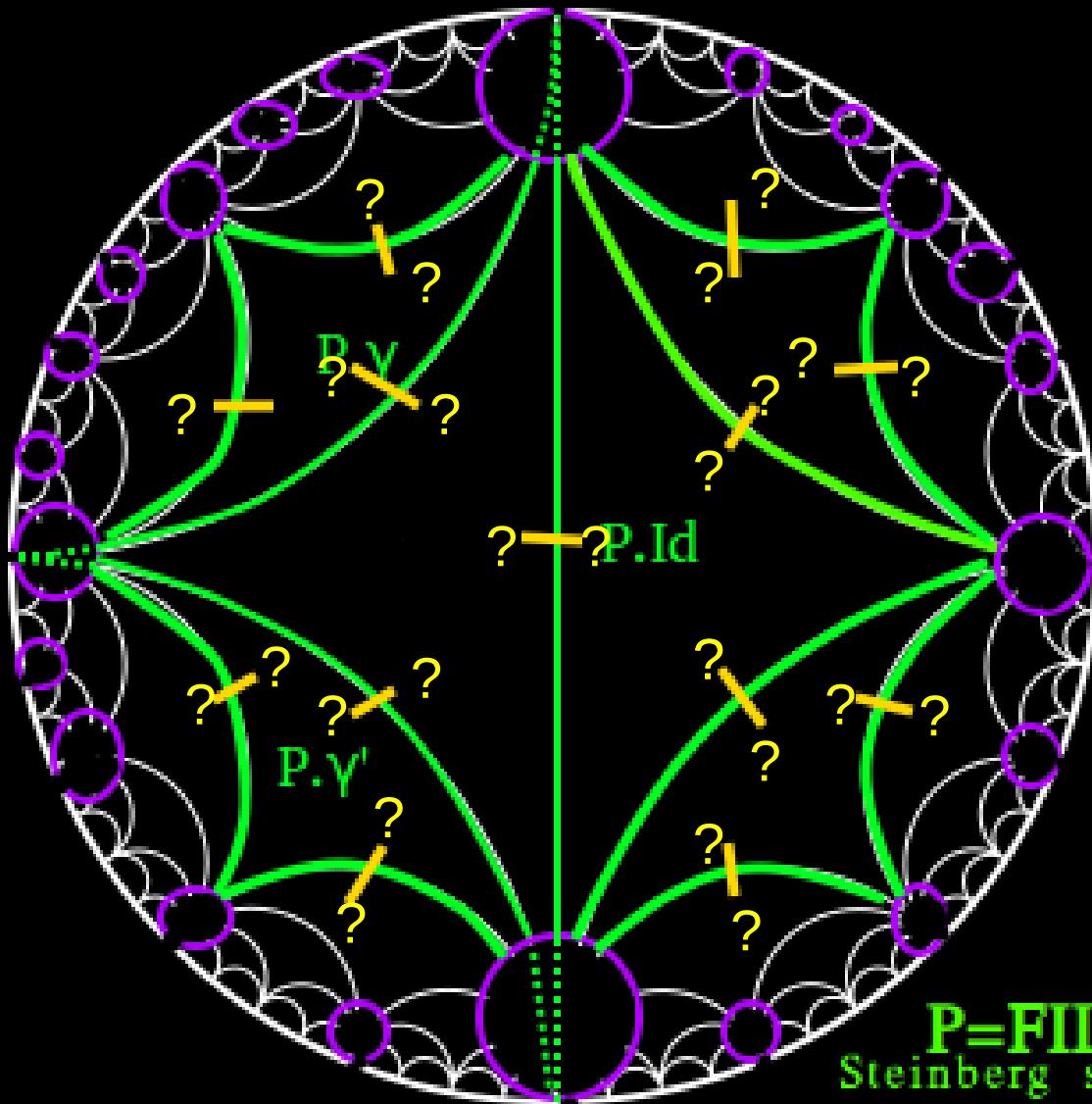


Extension Problem:

*Find a
continuous extension
of the local reductions*

$$\{ P.y \rightarrow \{pt\}, y \in \Gamma \}$$

*and obtain
a global continuous
reduction of $X[t]$
onto Spine Z*



Today we cannot announce a positive general solution to this Extension Problem.

We only have numerical evidence confirming a Conjecture contingent on two hypotheses:

- *that we have a finite subset I of Γ which successfully “Closes Steinberg”, and*
- *that the definition of visible-repulsion cost v satisfies
a (Twist) condition familiar from optimal transport theory*

Conjecture / (Work-In-Progress):

Let (X, d, vol_X) be a geometric $E\Gamma$ model, where Γ satisfies Bieri-Eckmann duality.

IF:

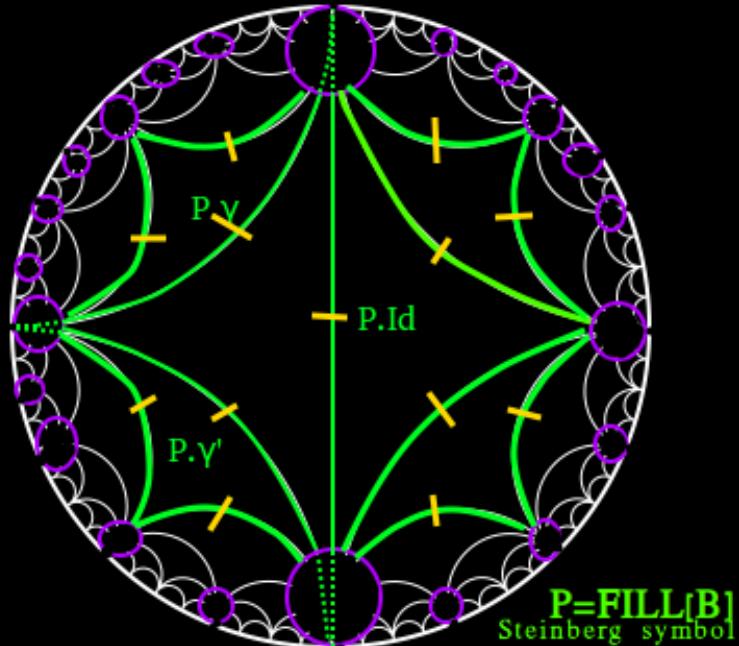
- I is finite subset of Γ which successfully “Closes Steinberg symbol”,
- and if v is the visible repulsion cost defined on the corresponding chain sum E replacing the Γ -rational excision $X[t]$,

THEN:

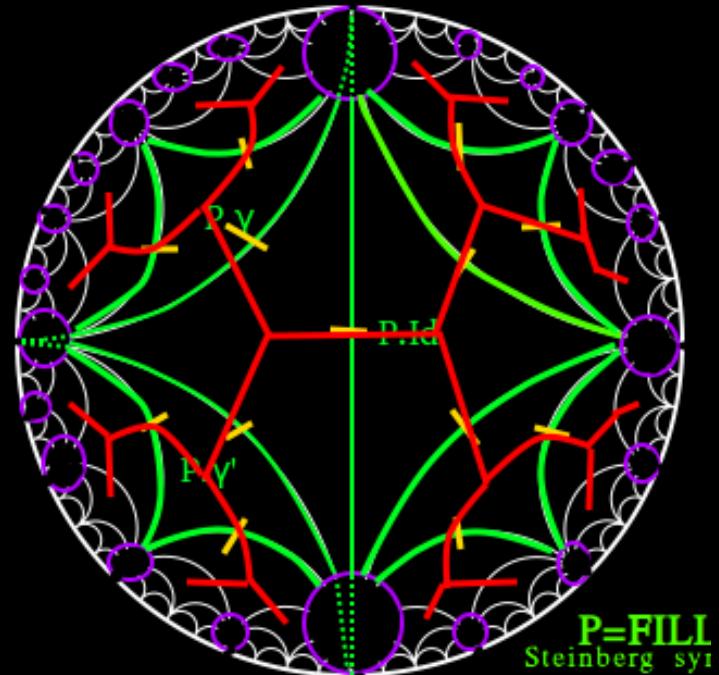
- the Global Reduction Theorem applied to functor $Z(\sigma, \tau, v)$ gives positive solution to Extension Problem.
- (UHS) conditions are satisfied up to index $J := q = \dim(D)$. (= spherical dimension of Steinberg module)
- the closed subvariety $Z\{q+1\}$ is a maximal codimension retract of X .

We propose:

$\{Closing\ Steinberg\} + \{Visible\ Repulsion\ Cost\ v\} + \{Global\ Reduction\ Thm.\} ==> \{Spine\}$



→ → → → → →

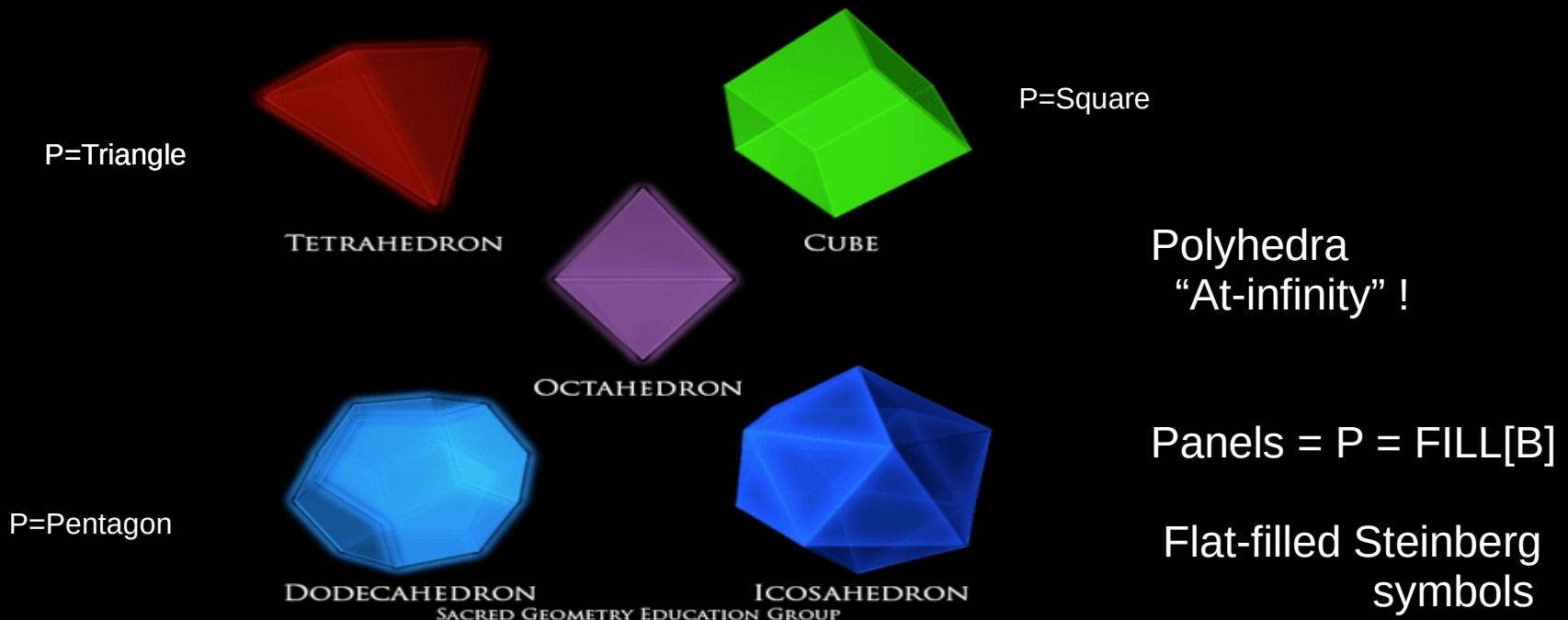


Closing Steinberg (CS):

In low dimensions (CS) is the problem of stitching a football from collection of panels $\{P\}$.

In applications: the panels $\{P\} = \{ \text{FILL}[B].y \mid y \in \Gamma \}$ are flat-filled Steinberg symbols.

Ex: Regular platonic solids solve (CS) with panels P = triangle, square, pentagon.



Closing Steinberg (Definition):

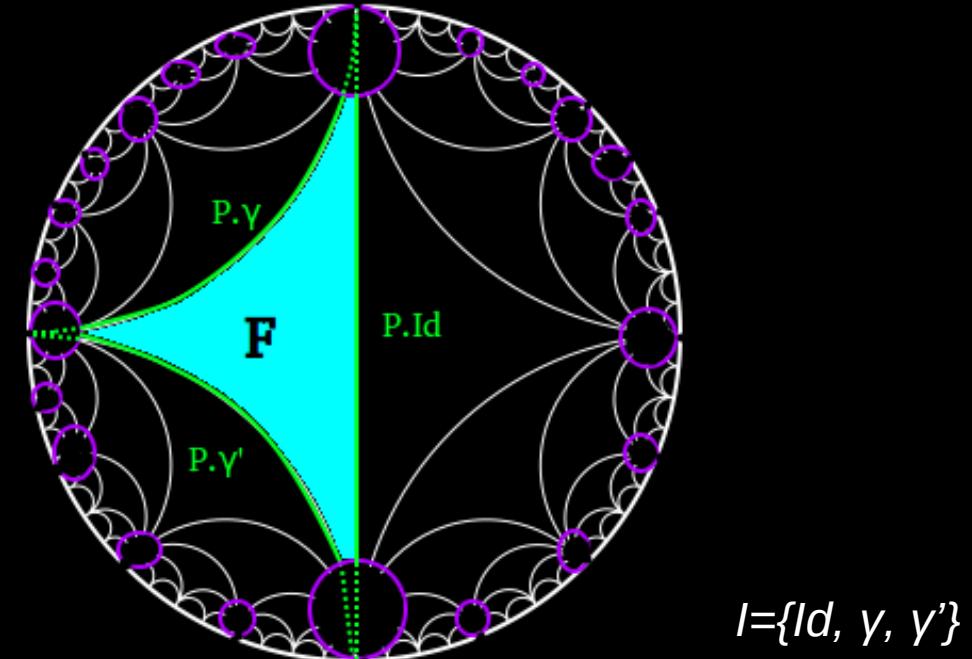
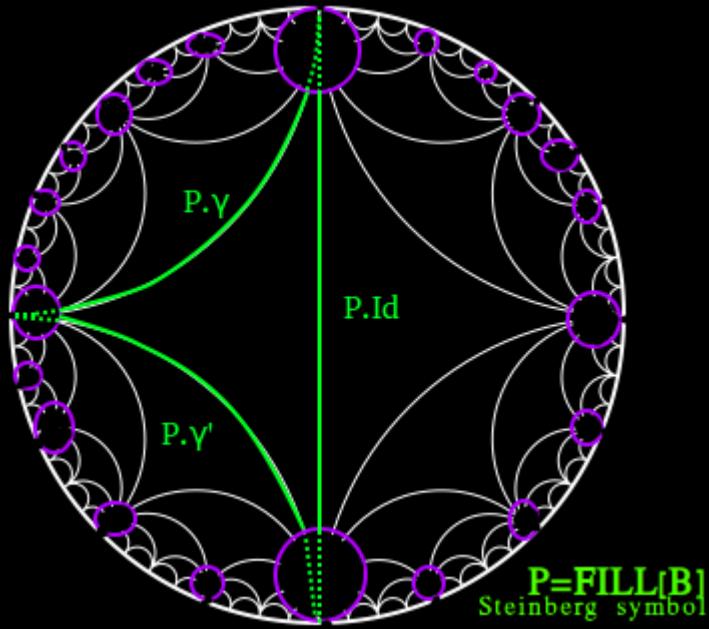
- A finite subset I of Γ successfully “Closes Steinberg” if :

- (CS1) the chain sum $\sum_{\gamma \in I} P.\gamma \neq 0 \bmod 2$ (nontrivial over $\mathbb{Z}/2$ coefficients).
- (CS2) the chain sum $\sum_{\gamma \in I} \partial P.\gamma = 0 \bmod 2$ (vanishing boundary over $\mathbb{Z}/2$ coefficients).
- (CS3) there exists $x \in X[t]$ which is simultaneously visible from $P.\gamma, \gamma \in I$, in $X[t]$ (well-defined closed convex hull).
- (CS4) if we define $F := \overline{\text{conv}}\{P.\gamma \mid \gamma \in I\}$, then the convex chain sum $\underline{F} = \sum_{\gamma \in \Gamma} F.\gamma$ has well-separated gates structure with gates $\{G\} = \{P.\gamma \mid \gamma \in \Gamma\}$. (well-separated gates)

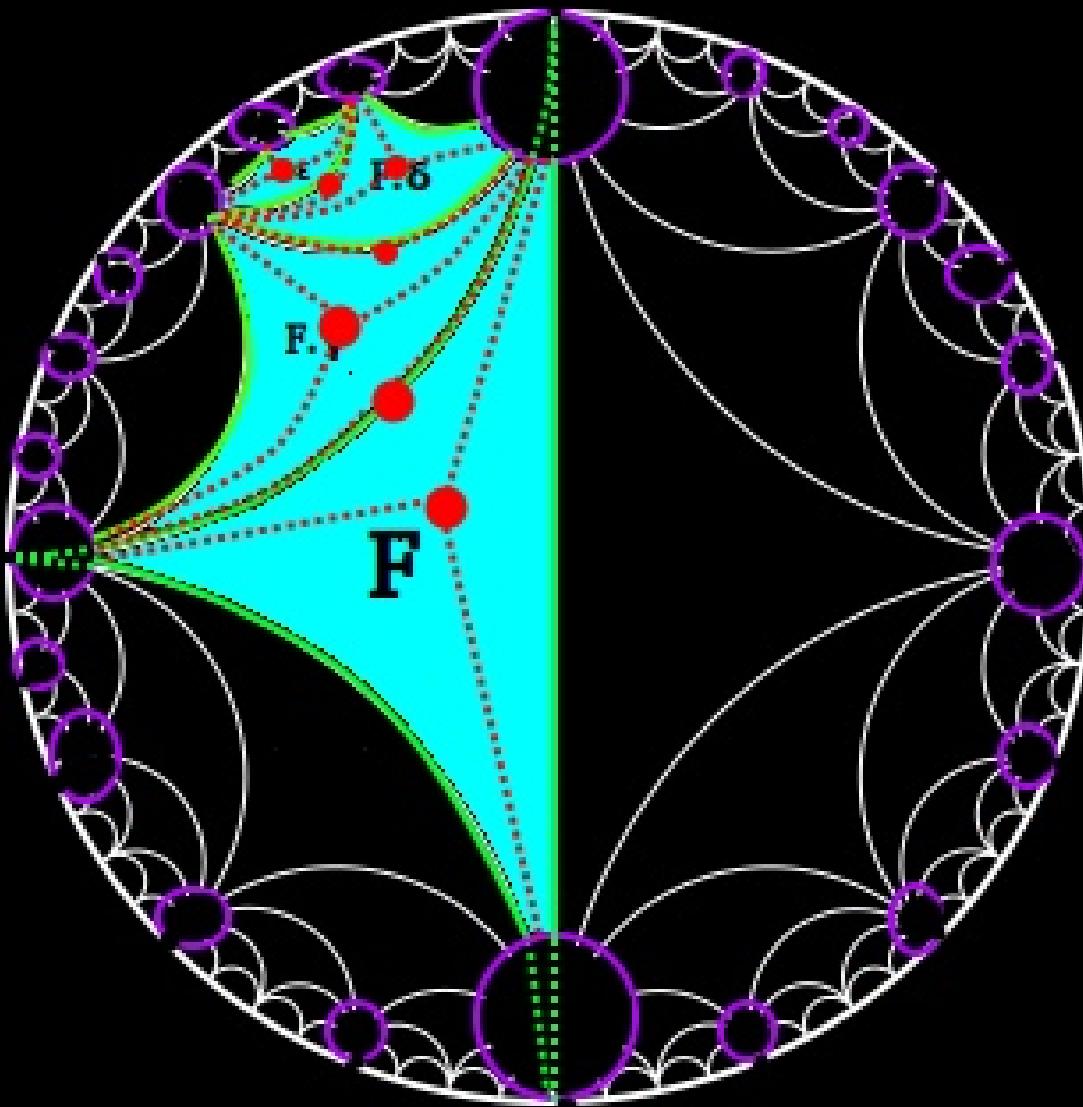
Closing Steinberg:

CS1, CS2 ==> constructing nonzero $\xi \in H_0(\Gamma, \mathbb{Z}_2\Gamma \times \mathbf{D}) \neq 0$
 - equivalent to a syzygy in projective homological resolution of \mathbf{D} .

CS4 ==> the translates F, Γ define chain sum $E = \sum F.y$
 - Γ acts on chain summands of E like “shift operator” equivalent to right-action $\Gamma \times \Gamma \rightarrow \Gamma$



Chain sum F with well-separated gates:



Well-separated gates ==>
summands F, F' have
either trivial intersection
or $F \cap F' = P$

CS replaces $X[t]$ with a chain sum E ,
- ``partition of unity'' of a
proper subset of $X[t]$.

- the translates $F.y, y \in \Gamma$,
do not necessarily fill $X[t]$!

Ex: $\Gamma = PGL(\mathbb{Z}^3)$

The following symbol ξ' is a solution to (CS):

$$\xi' := \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

$I := \{3 \times 3 \text{ minors of } \xi'\}$ is the finite subset $PGL(\mathbb{Z}^3)$ which solves (CS).

Solutions to (CS) replace X , or $X[t]$, with chain sums:

Proposition 37. *The convex hull of the rank-one states spanned by the columns of ξ' , i.e.*

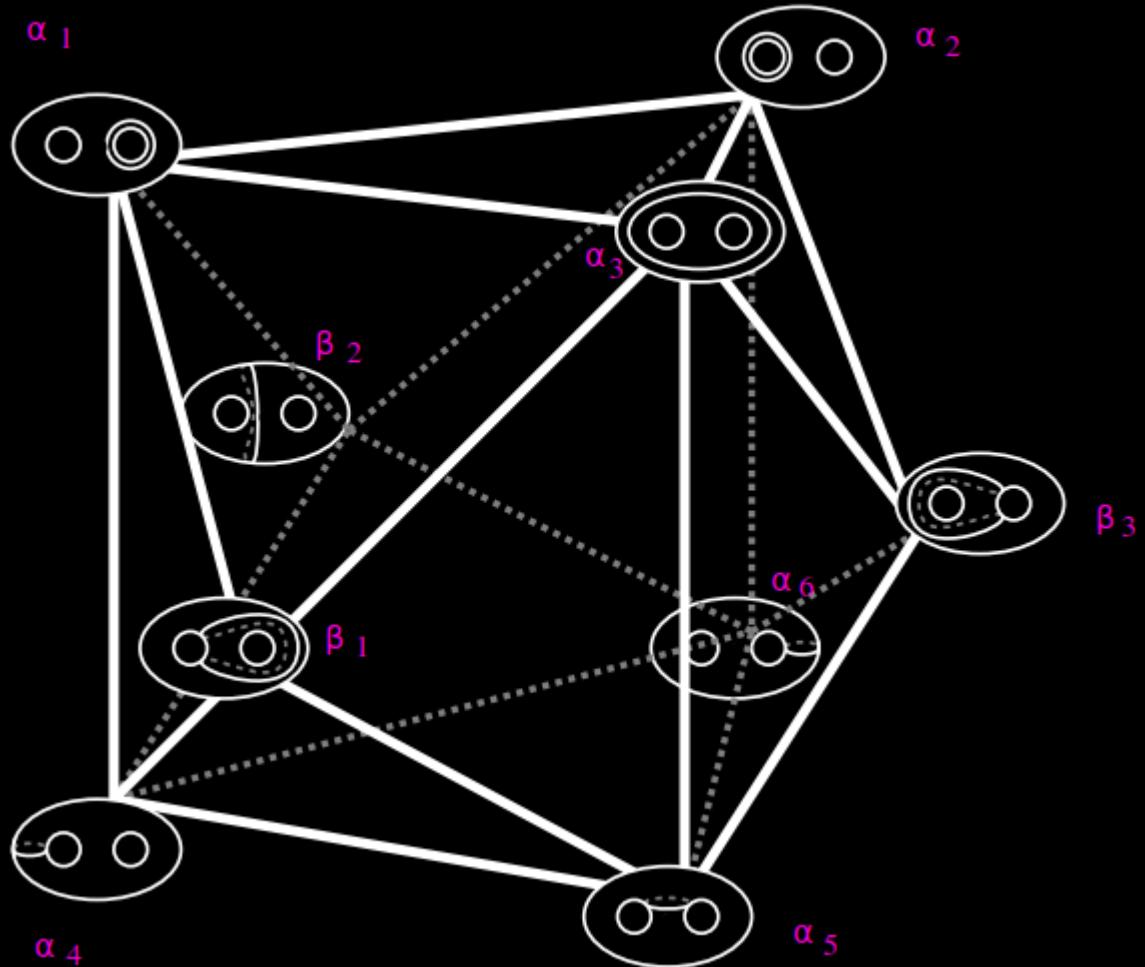
$$F := \text{conv}[\{x^2, y^2, z^2, (x+z)^2, (y+z)^2, (x+y+z)^2\}] \subset Q$$

forms a convex set F , whose translates $F.GL(\mathbb{Z}^3)$ tessellate Voronoi's cone Q of three-dimensional positive-semidefinite real states.

(CS) for $\Gamma = MCG(\Sigma_2)$: Open Problem

N. Broaddus [2012] identifies the panel P=3-ball in genus 2 Teichmueller space.

(CS) is primary obstruction to building explicit spine of Teich_2

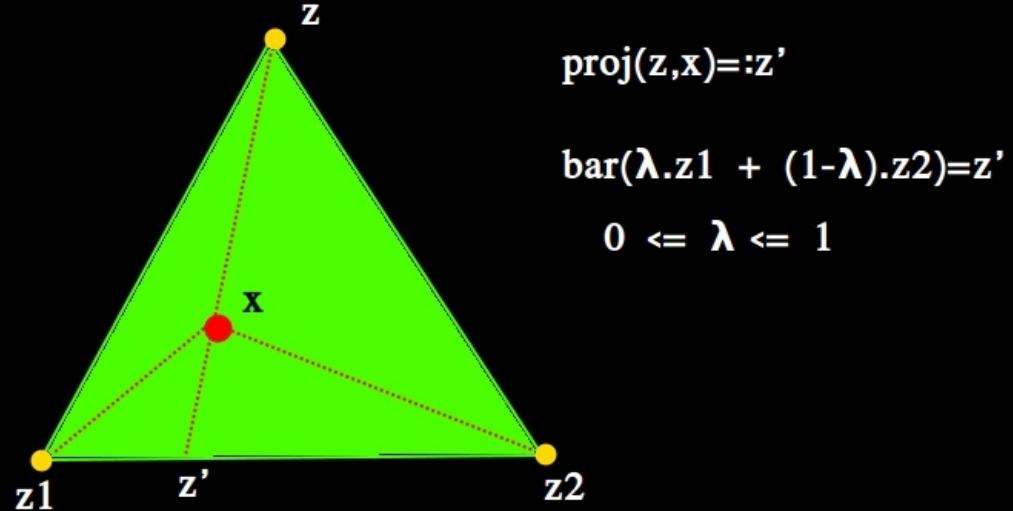


Next: the visible repulsion costs v , v^ ...*

For brevity we describe the “visible repulsion” cost v in the simplest case of a finite compact 2-simplex Δ .

Source = F

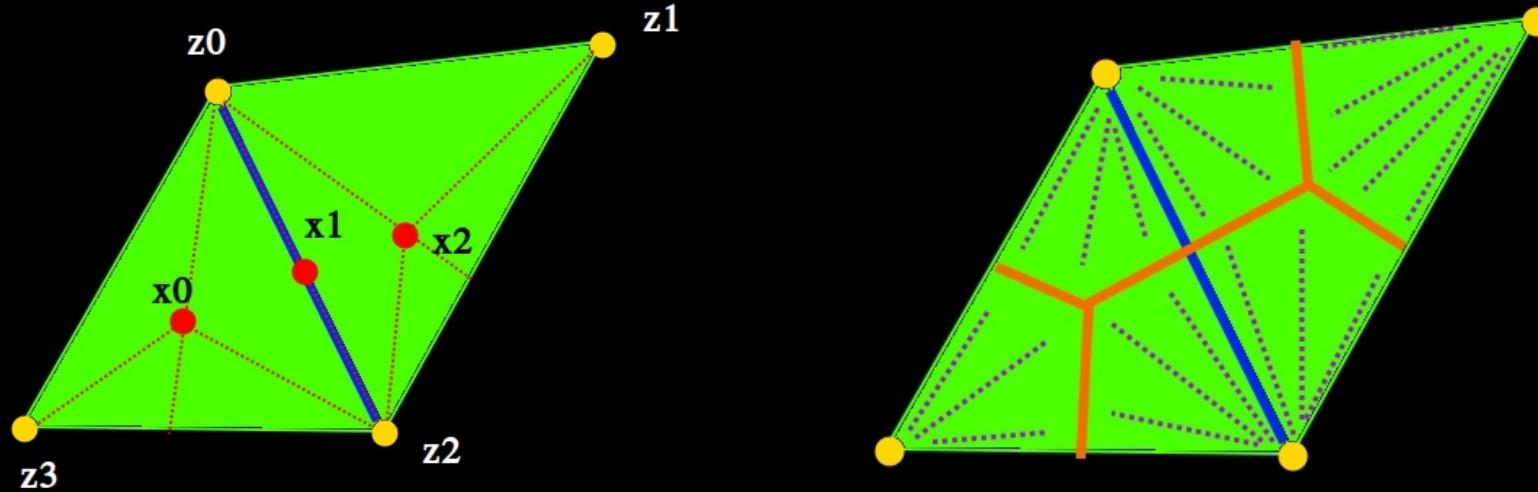
Target = Extreme points $E[F]$
 $= \{z, z_1, z_2\}$



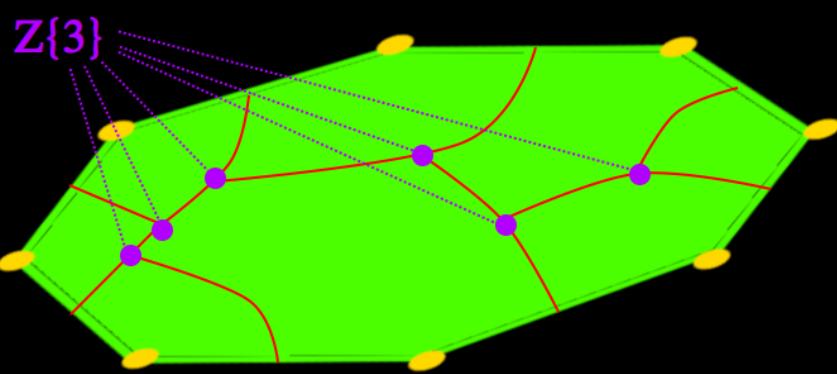
$$v(x,z) = 1/2 \cdot \|x-z\|^{-2} + \lambda \cdot \|x-z_1\|^{-2} + (1-\lambda) \cdot \|x-z_2\|^{-2}$$

$v: F \times E[F] \rightarrow \mathbb{R}$ visibility cost

- the definition of v extends to repulsion cost v^* on chain sum E with well-separated gates $\{G\}$
- gates G geodesically convex imply v^* is continuous extension of the restricted repulsion costs $v^*|G=v$
- Singularity structures $Z(\sigma, \tau, v^*)$ are continuous interpolations of restricted singularity structures $Z(\sigma, \tau, v^*|G)$ over gates $\{G\}$ of E

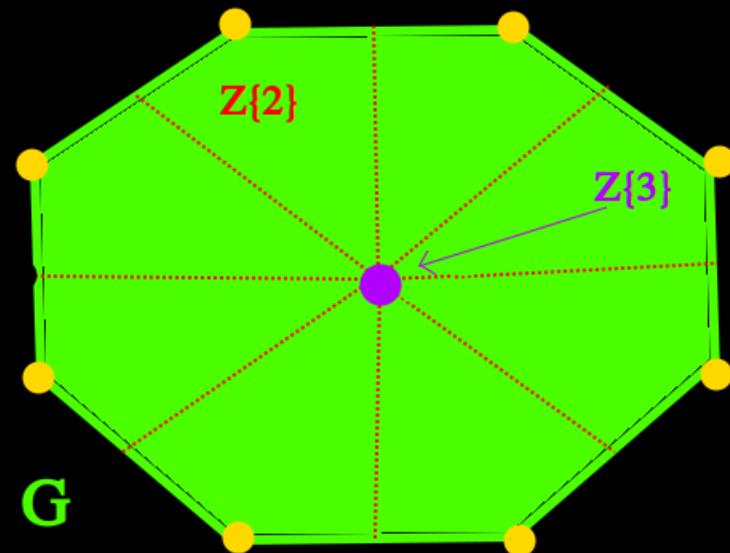


Symmetry of Gates ==> (UHS) Conditions satisfied:

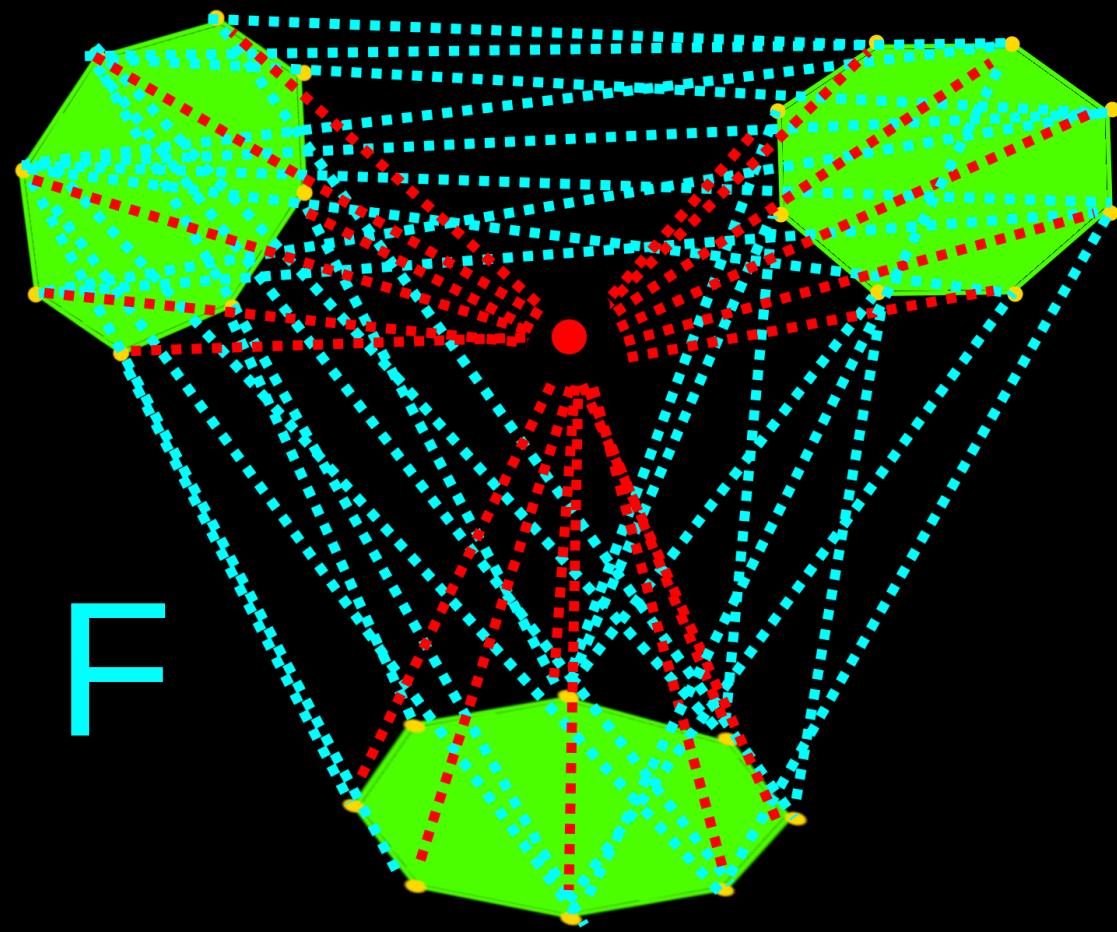


G deformation retracts to Z{2}

- $Z\{3\}$ is disconnected
for generic convex panels G



- Symmetry implies
 G homotopy-reduces to $Z\{3\} = \{\text{pt}\}$

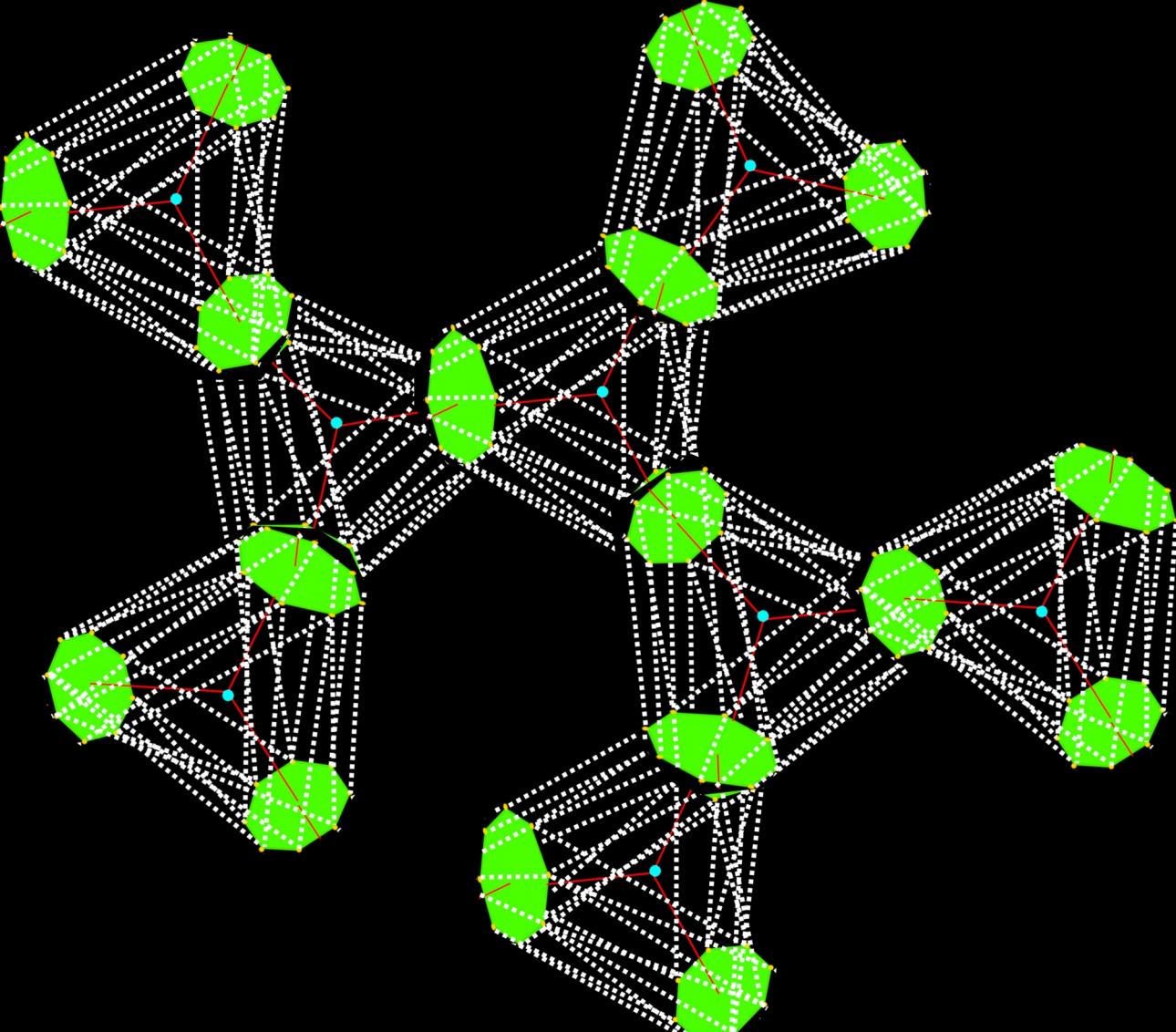


*Successfully Closing Steinberg symbol
replaces X with a chain sum*

$$E = \sum F.y$$

where $F := \text{conv}[P, P.y, P.y', \dots]$

- (CS) Hypotheses imply:
 - the convex hull F is well-defined,
 - and
- translates $F.y, y \in \Gamma$, have well-separated gates $\{G\}$ coincident with Γ -orbit of Steinberg symbols
 $G=P=FILE[B]$



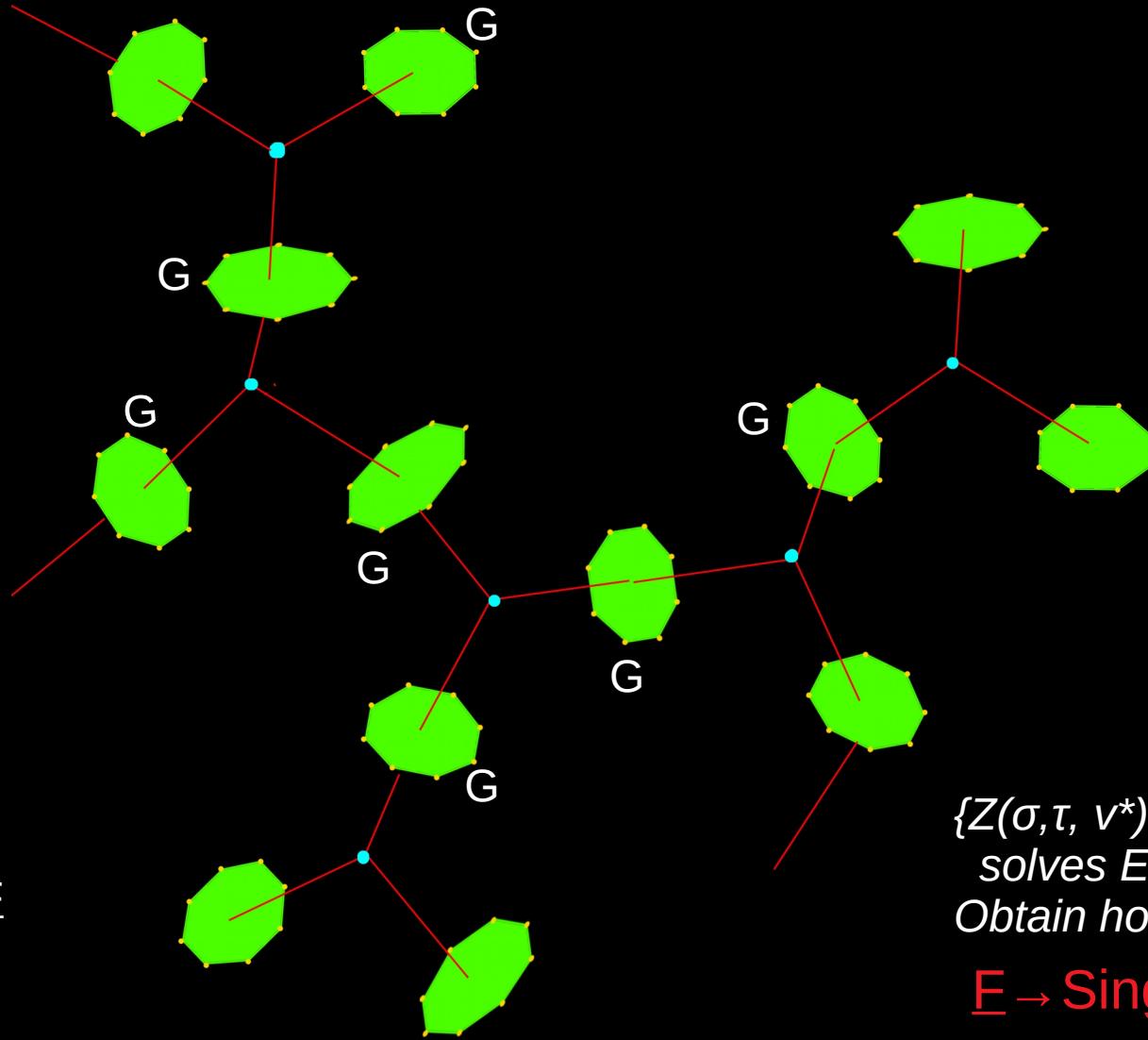
*(CS) replaces X
with a chain sum*

$$\underline{E} = \sum F_i \gamma$$

where

$$F := \text{conv}[P, P\gamma, P\gamma', \dots]$$

v^* -optimal
semicouplings
from source $X = E$
to target $Y = E[E]$



$\{Z(\sigma, \tau, v^*) + \text{Global Red. Thm}\}$
solves Extension Problem
Obtain homotopy-reduction:
 $E \rightarrow \text{Singularity } Z\{2\}$

The End.



Thank you.