ALEXANDROV SPACES, KANTOROVICH SINGULARITY, SOULS AND SPLITTING THEOREMS

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1. Ruling by Maximal Minimizing Rays

Throughout this article (X, d) designates a noncompact, complete, connected finite-dimensional Alexandrov space. So X is a length space satisfying $\kappa \geq 0$ sectional curvature conditions. In otherwords every quadruple (a, b, c, d) in X satisfies Toponogov's comparison condition, c.f. [Vil09, Ch.26, pp.738], [MT07, pp.53–55]. For a basepoint $x_0 \in X$, let $M = M(x_0)$ be the set of all geodesics λ in X satisfying:

(i) the geodesic λ passes through x_0;
(ii) the geodesic λ is distance minimizing over every compact subinterval;
(iii) the geodesic is maximally nonextendible.

For every $x_0 \in X$, we abbreviate $M^*(x_0) \subset M(x_0)$ as the subset of noncompact geodesics.

Lemma 1. *If X is connected complete noncompact Alexandrov, then $M^*(x_0)$ is nonempty for every $x_0 \in X$.*

Proof. [MT07, Lemma 2.1] □

In otherwords there exists distance minimizing asymptotic geodesic rays. The set M of geodesics contains evidently three types: the geodesics λ are either

(a) compact; or
(b) noncompact and doubly-ended; or
(c) noncompact and singly-ended.

It is necessary to emphasize that $M^*(x_0)$ varies lower semicontinuously with respect to the choice of x_0. Lemma 1 says Alexandrov spaces X are "ruled" by maximal minimizing geodesics in $M^*(x_0)$, $x_0 \in X$. For instance if x_0 is a regular point on an infinite flat cone, then $M^*(x_0)$ is a singleton, whereas if $x_0 = v$ is the cone vertex, then $M^*(v)$ is infinite and parameterized by an $N-1$-sphere on N-dimension cones.

Our purpose is to demonstrate how methods of Kantorovich Singularity and optimal semicouplings from [Mar] establishes two basic theorems of Alexandrov geometry nearly simultaneously. In case M^* contains a doubly-ended geodesic, then our arguments below will establish Gromoll-Cheeger's Splitting theorem [MT07, Thm.2.11,

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pp.58]; otherwise we use the geodesics of M^* to establish the Cheeger-Gromoll-Perelman's Soul theorem [MT07, Thm.2.7, pp.56] for singular Alexandrov spaces.

For any $x_0 \in X$, $\lambda \in M^*(x_0)$, let $h_\lambda : X \to \mathbb{R}$ be the unique horofunction satisfying $h_\lambda(x_0) = 0$, and defined by the usual formula

$$h_{\lambda,x_0}(x) := \lim_{t\to+\infty} d(\lambda(t), x) - t.$$

We observe $h_{\lambda,x_0}(x) \geq -d(x, x_0)$, and h_{λ,x_0} diverges to $-\infty$ along the geodesic λ. Our curvature hypothesis $\kappa \geq 0$ implies h_λ is geodesically concave function and superlevel sets $\{h_{\lambda,x_0} \geq T\}$ are totally convex subsets of X for all $T \in \mathbb{R}$. (We remark that the same definition implies h_λ is convex in nonpositive curvature $\kappa \leq 0$). If the geodesic λ is doubly-ended, then h_λ will be symmetric with respect to x_0 and approaches values $\pm\infty$ as arc-parameter $\lambda(s)$ diverges to $s \to \mp\infty$.

For $\lambda \in M^*$, we choose real numbers $t = t(\lambda) \in \mathbb{R}$ for which $0 < |t|$ is numerically small, say $t = .00001$.

Lemma 2. *For every $x_0 \in X$, if the parameter $t : M^*(x_0) \to \mathbb{R}$ is sufficiently small ($t \approx 0^+$), then the excision*

$$X_0 := X[t] = X - \cup_{\lambda\in M^*(x_0)}\{h_\lambda \geq t(\lambda)\}$$

is a nontrivial compact totally convex subset of X.

Proof. We abbreviate $H_{\lambda,t} = \{h_\lambda \geq t(\lambda)\}$. The horofunctions h_λ are concave, therefore the excision $X - H_{\lambda,t}$ is a totally convex subset of X. Therefore the intersection $\cap_{\lambda\in M^*} X - H_{\lambda,t}$ is a totally convex subset. Moreover the completeness of X implies all the minimizing geodesics in X_0 are compact and Lemma 1 implies X_0 is a compact subset. □

The excision X_0 is a compact convex boundary ∂X_0. The boundary ∂X_0 is "cellulated" by the boundaries $\partial H_{\lambda,t}$ of the excised horoballs. Moreover one easily establishes that the homotopy types of X_0, X coincide.

Lemma 3. *If the excision parameter t is sufficiently small, then the inclusion $X_0 \hookrightarrow X$ is a homotopy-isomorphism, and there exists a continuous strong deformation retract $X \rightsquigarrow X_0$.*

Proof. □

The above constructions lead us to our semicoupling program. The excision X_0 has a canonical Hausdorff measure $\sigma := \mathscr{H}_{X_0} = \mathscr{H}_X 1_{X_0}$, and the excision boundary ∂X_0 has canonical Hausdorff measure $\tau := \mathscr{H}_{\partial X_0}$. The measures σ, τ are designated the source and target measures, respectively.

We need determine cost. For pairs $x, y \in X$, we may compare $d(x,y)$ to the signed distances between horospheres $h_\lambda(x) - h_\lambda(y)$, which we observe is independant of the basepoint x_0 defining $h_\lambda = h_{\lambda, x_0}$. Concavity of h_λ implies the function

$$b(x,y) := \inf_\lambda \{h_\lambda(x) - h_\lambda(y)\} \tag{1}$$

is concave in the x-variable, for every choice of $y \in X$. If $(x,y) \in X_0 \times \partial X_0$, then $b(x,y) \geq 0$ with equality if and only if $x \in X_0$ and occupies the same horosphere component as y. Compactness of X_0 implies the superlevels of $\{x \in X_0 \mid b(x,y) \geq T\}$, for fixed $y \in \partial X_0$, $T \geq 0$, are compact convex subsets of X_0.

Furthermore the triangle inequality implies

$$0 \leq b(x,y) \leq h_\lambda(x) - h_\lambda(y) \leq d(x,y), \text{ for } (x,y) \in X_0 \partial X_0 \tag{2}$$

with equality $h_\lambda(x) - h_\lambda(y) = d(x,y)$ if and only if x, y lie on minimizing ray λ. Observe $0 < b(x,y)$ whenever $x \in X_0 - \partial X_0$.

The pairing $b : X \times X \to \mathbb{R}_{\geq 0}$ defined by equation (1) is a possibly degenerate distance function. Throughout $X_0 \times X_0$ we find $b \geq 0$ satisfies a triangle inequality and is symmetric $b(x,y) = b(y,x)$ if we add absolute-values to (1). However caution again needs be exercised since b is possibly degenerate, having $b(x,y) = 0$ for $x \neq y$. For instance if M^* is a singleton, then $b(x,y) = 0$ if x, y both occupy the same horosphere centred at λ.

The basic idea of this article is to treat $b(x,y)$ as a type of distance on X, and restrict b to the subset $X_0 \times \partial X_0$ defined earlier in Lemma 2. Having nominated a distance $c = b$, we next turn to distance maximizing transports. Indeed distance minimization appears less useful for our purposes given the nonnegativity (2), where equality is if and only if $x \in \partial X_0$.

Definition 4. Fix x_0, M^*, definition of b (1). Let $X_0, \partial X_0$ be excisions with small parameter t. Let μ, ν be the measures on $X_0, \partial X_0$. Then let $\Pi' = \Pi'_{x_0}$ be the set of maximizers of the following maximization program:

$$\max_{\pi \in \Pi(\mu,\nu)} \int_{X_0 \times \partial X_0} b(x,y) d\pi(x,y). \tag{3}$$

Here $\Pi(\mu, \nu)$ designates the set of couplings π from μ to ν.

The regularity properties of maximizing measures $\pi \in \Pi_{x_0}$ will have strong dependance on the basepoint x_0. Indeed when b is degenerate, then Π' is not a singleton, and we need choice of "canonical" coupling π'. This canonical choice is achieved by a secondary variational problem, where we follow the ideas of [AP03], [AKP+04], [KP18]. Briefly, we select a canonical coupling π' by finding couplings $\pi \in \Pi'$ which have minimal $d^2/2$ transport cost. Our goal in this section is to prove the existence and uniqueness of this minimal coupling.

We begin by defining a family of auxiliary costs. For $\epsilon > 0$, let

$$c_\epsilon(x,y) := \epsilon^2 d^2(x,y)/2 - b(x,y) \tag{4}$$

be defined for $(x,y) \in X_0 \times \partial X_0$. Consider the minimization program:

$$\min_{\pi \in \Pi(\mu,\nu)} \int_{X_0 \times \partial X_0} c_\epsilon(x,y) d\pi(x,y). \tag{5}$$

The idea is that if π_ϵ is a c_ϵ-optimal transport, then taking $\epsilon \to 0^+$ we obtain limit transports π_0. These limit transports are $(-b)$-optimal transports, i.e. $\pi_0 \in \Pi'$, and in fact have minimal $d^2/2$ cost among couplings in Π'. More formally, we first prove the program (5) has unique minimizers for every $\epsilon > 0$.

Lemma 5. *Let $X_0, \partial X_0$ be the excisions constructed in [ref] with a sufficiently small parameter t.*

(i) For $y \in \partial X_0$, the function $x \mapsto d^2(x,y)/2$ is continuously differentiable function of $x \in X_0$.

(ii) For $\epsilon > 0$, the cost $c_\epsilon : X_0 \times \partial X_0 \to \mathbb{R}$ satisfies (Twist) with respect to source variable $x \in X_0$. So for every $y \in \partial X_0$, the rule $x \mapsto \nabla_x c_\epsilon(x,y)$ defines an injective map $\nabla_x c_\epsilon(\cdot, y) : \partial X_0 \to T_x X_0$.

If μ, ν are probability measures on $X, \partial X$, as above, then there exists unique c_ϵ-optimal couplings from μ to ν for $\epsilon > 0$. Let π_e be the unique c_e-optimal couplings. Compactness of $X, \partial X$ implies the family π_ϵ is weak-* compact, and there exists convergent subsequenecs of π_e. According to the (Twist) condition, we deduce the existence of Monge maps $T_\epsilon : X \to \partial X$ satisfying $T_e \# \mu = \nu$ and

$$\int c_e(x, T_e(x)) d\mu(x) = \min_{\pi \in \Pi(\mu,\nu)} \int c_e d\pi.$$

Lemma 6. *Let π_e be c_e-optimal couplings. Every accumulation point π_0 of the family π_e, $e > 0$, is a c-optimal coupling, where $c = c_0 = -b$ according to equation (4).*

Proof. First we have $c_e \geq c$ for all $e > 0$, with pointwise monotone convergence

$$\lim_{e \to 0^+} c_e = c.$$

The Monotone Convergence theorem implies

$$\lim_{e \to 0^+} \int c_e d\pi = \int c d\pi$$

for all couplings π. If $T_e : X \to \partial X$ is the Monge map describing the c_e-optimal transport from μ to ν, then every convergent subsequence of couplings π_e yields the μ-a.e. pointwise convergence of the Monge maps T_e to a map

$$\lim_{e \to 0^+} T_e = T_0.$$

The pointwise convergence and the Dominated Convergence theorem implies

$$\lim_{e\to 0^+}\int c_e(x,T_e(x))d\mu(x)=\int c(x,T_0(x))d\mu(x)=\int cd\pi_0.$$

If π is an arbitrary coupling, then

$$\int c_e d\pi \geq \int c_e(x,T_e(x))d\mu = \int c_e d\pi_e$$

for all $e>0$. Therefore

$$\liminf_{e\to 0^+}\int c_e d\pi \geq \liminf_{e\to 0^+}\int c_e d\pi_e = \int cd\pi_0.$$

The final equation implies the limit π_0 is c-optimal. □

Proposition 7. *For every basepoint $x_0\in X$, there exists a unique b-maximizing coupling $\pi''\in\Pi'_{x_0}$ which has minimal $d^2/2$-transport cost. I.e. $\Pi''_{x_0}=\{\pi''\}$ is a nonempty singleton.*

INCOMPLETE. □

2.

[Background: Kantorovich potentials]

If $\psi:\partial X_0\to\mathbb{R}\cup\{+\infty\}$ is a potential, then the b-transform of ψ is defined

$$\psi^b(x):=\sup_{y\in\partial X_0}\{\psi(y)+b(x,y)\}.$$

If $\phi:X_0\to\mathbb{R}\cup\{-\infty\}$ is a potential on X_0, the b-transform of ϕ is defined

$$\phi^b(y):=\inf_{x\in X_0}\{\phi(x)-b(x,y)\}.$$

A potential ψ is b-convex, and satisfying $(\psi^b)^b=\psi$ if

$$\psi(\overline{y})=\inf_{x\in X_0}\sup_{y\in\partial X_0}\{\psi(y)+b(x,y)-b(x,\overline{y})\}$$

for all $\overline{y}\in\partial X_0$.

The reader will observe that $\psi(y)\leq\psi^{bb}(y)$ for arbitrary, possibly nonconvex functions ψ. The key observation is the inequality

$$\psi^b(x)-\psi(y)\geq b(x,y),$$

with equality if and only if $x\in\partial^b\psi(y)$ for b-convex potentials $\psi^{bb}=\psi$. The following technical lemma is important to our results.

Lemma 8. *Let ψ be a b-convex potential on ∂X_0, $\psi^{bb}=\psi$. For all $y\in spt(\psi)\subset\partial X_0$, the b-subdifferential $\partial^b\psi(y)$ is a totally convex subset of X_0.*

Proof. □

Consequently if $Z = Z(\mu, \nu, b)$ is the Kantorovich functor $Z : 2^{\partial X_0} \to 2^X$, then for all closed subsets Y_I of ∂X_I, the cell $Z(Y_I) := \cap_{y \in Y_I} \partial^b \psi(y)$ is a totally convex subset of X_0.

Remark. Our thesis constructs deformation retracts via gradient flow towards the poles of a vector field denoted $\eta(x, avg)$. To apply the methods of [Mar] to singular Alexandrov spaces requires the definition of nonsmooth gradients and gradient projections. Petrunin-Perelman prove the existence of well-defined gradient flows in singular Alexandrov spaces [PP94].

3. Splitting

Theorem 9. *Let μ be source measure on X_0, and ν a target measure on ∂X_0, and with cost b as defined in (1) with respect to a basepoint x_0.*

If $\int_{X_0} d\mu / \int_{\partial X_0} d\nu \approx 1^+$, then the active domain A of the "canonical" b-maximal semicoupling π^* defined in (5) is a strong deformation retract of X_0.

Moreover if M^* contains a doubly-ended minimizing ray, then the active domain $A = Z_1$ splits isometrically $A \simeq [-T, +T] \times Z_2$, where Z_2 consists of all source points $x \in X_0$ such that $\partial^b \psi^b(x) \geq 2$.

Incomplete. □

The cross-difference $||\nabla_x b_\Delta|| \neq 0$ is nonvanishing throughout X, and this implies $b_\Delta : X \to \mathbb{R}$ is a submersion in the topological sense, i.e. there exists a topological splitting $X \simeq f^{-1}(pt) \times \mathbb{R}$, where $f^{-1}(pt)$ is a generic fibre. However the usual Splitting Theorem of Cheeger-Gromoll requires the construction of Riemannian submersion, where we recall a Riemannian submersion $f : X \to \mathbb{R}$ is a smooth function such that $||\nabla_x f|| = 1$ throughout X. The existence of a Riemannian submersion f on X implies the existence of isometric splitting $X \simeq f^{-1}(pt) \times \mathbb{R}$.

4. Nonnegative Ricci Curvature

Let (X, g) be Riemannian manifold with nonnegative Ricci curvature. Then for every ray λ in X the horofunction $h_\lambda : X \to \mathbb{R}$ is superharmonic. This means the divergence of the gradient flow $div(\nabla_x h_\lambda) \leq 0$ is nonpositive throughout X, and for every subdomain D of X, the restriction $1_D \cdot h_\lambda$ achieves its absolute minimum along the boundary ∂D.

The important Splitting theorem of Cheeger-Gromoll [ref] is the following:

Theorem 10. *Let (X, g) be a complete Riemannian manifold with nonnegative Ricci curvature. Suppose $M^*(x_0)$ contains a doubly-ended minimizing ray for some basepoint $x_0 \in X$. Then there exists a totally convex subset Y of X and an isometric splitting $X \simeq Y \times \mathbb{R}$.*

It's well-known that Toponogov proved the existence of isometric splittings when (X, g) is a smooth Alexandrov manifold [ref]. Our goal is to establish a splitting theorem for singular Alexandrov spaces using the Kantorovich Singularity functor. [Include McCann's interpretation using convex functions: proves Toponogov for singular Alexandrov?]

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