OPTIMAL TRANSPORT, $1/d^{\alpha}$-COSTS, AND MEDIAL AXIS TRANSFORMS

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1. Medial Axis Transforms and Optimal Transport

The purpose of this section is to compare some familiar properties of the medial axis transform $A \mapsto M(A)$ (introduced by [Blu67]) with the singularity structures formalized in our Kantorovich contravariant functor $Z : 2^{\partial A} \to 2^{A}$ (introduced in [Mar]). To compare the functors Z with medial axis transform requires we interpret the inclusion $M(A) \hookrightarrow A$ in the category of mass transportation.

Let A be a bounded open subset of $\mathbb{R}^N$. The medial axis $M(A)$ introduced by Blum consists of all $x \in A$ for which $dist(x, \partial A)$ is attained by at least two distinct points,

$$M(A) := \{x \in A \mid \#argmin_{y \in \partial A}\{d(x,y)\} \geq 2\}. \tag{1}$$

A long-known "folk theorem" states that the inclusion $M(A) \hookrightarrow A$ is a homotopy-isomorphism, and even a strong deformation retract. This implies $M(A)$ contains all the topology of A, and a connected subset whenever A is. A formal proof is established [Lie04]. We do not know if $M(A)$ is a strong retract for more general Riemannian spaces (X, d), and a search through the literature does not address the question. But our recent thesis [Mar] contains some results, namely "Theorem B", identify conditions for which inclusions denoted $Z_2 \hookrightarrow A$ are homotopy isomorphisms, even strong deformation retracts. This subvariety Z_2 is derived from a contravariant functor $Z = Z(\mu, \nu, c)$ defined by mass transport data (μ, ν, c). The medial axis $M(A)$ and Z_2 will rarely coincide set-theoretically, but this present note demonstrates they are frequently topologically isomorphic.

The medial axis transform corresponds to a "degenerate" transport problem in the following sense: if $A \hookrightarrow \mathbb{R}^N$ is bounded open subset, then we nominate

$$\mu := \frac{1}{\mathscr{H}_A[A]} \mathscr{H}_A \tag{2}$$

as the canonical probability measure on the source A. Consider the probability measures π on $A \times \partial A$ for which $proj_A \# \pi = \mu$ and with unconstrained second

Date: March 6, 2019.

marginal $proj_{\partial A}\#\pi$. Here $proj_A$, $proj_{\partial A}$ are the canonical projections $A \times \partial A \to A, \partial A$. The set-mapping

$$T : x \mapsto argmin_{y \in \partial A}\{d(x,y)\}, \quad \text{for } x \in A, \tag{3}$$

defines a measurable set-valued map $T : A \to \partial A$. The pushforward

$$\nu := T\#\mu \tag{4}$$

is a probability measure on ∂A with $spt(\nu) = \partial A$. With respect to, say, quadratic cost $c = d^2/2$ or distance cost $c = d$, the map $x \mapsto T(x)$ defines a c-optimal transport from μ to ν, with c-optimal coupling $\pi = (Id \times T)\#\mu$ on $A \times \partial A$.

Finally $M(A)$ coincides with the locus-of-discontinuity of $T : A \to \partial A$, or more specifically the singularity Z_2 defined by Kantorovich's contravariant functor $Z = Z(\mu, \nu, d) : 2^{\partial A} \to 2^A$. Thus we arrive at an instance where $M(A) = Z_2$ for the specific coupling program defined by μ, ν, c. This identification suggests the following generalization of medial axis transform: for general probability measures $\nu \in \Delta(\partial A)$ on the boundary of A, we may study the c-optimal couplings π from μ to ν, and obtain a Singularity functor $Z(\mu, \nu, c)$. The generalized medial axis in this setting is Z_2, i.e. the "locus-of-discontinuity" of the c-optimal transport π from μ to ν.

2.

Our thesis developed a Reduction-to-Singularity principle, and identifies conditions for which, say, the inclusion $Z_2 \hookleftarrow Z_1$ is a homotopy-isomorphism. In the above setting with $Z = Z(\mu, \nu, c)$, we have $A = Z_1$, $M(A) = Z_2$, and naturally we inquire whether the hypotheses of our topological theorems are satisfied for any particular costs c.

If we fix $c = d^2/2$, then our Theorem B takes the following form. For $x \in A = Z_1$, let $y_0 := T(x)$. Then define

$$\eta(x,y) := (1 - e^{-d(y,y_0)^2}) \cdot |c(x,y) - c(x,y_0)|^{-1/2} \cdot \nabla_x(c(x,y) - c(x,y_0)), \text{ for } y \in \partial A - \{y_0\}.$$

Observe that $c(x,y) - c(x,y_0) > 0$ is nonvanishing throughout $A - M(A)$ in the above notations. The factor $0 \le 1 - e^{-d(y,y_0)^2} \le 1$ is approximately equal to 1 for $y \neq y_0$. In this setting our Theorem B requires the following hypotheses (6), (7) be satisfied for $x \in A - M(A) = Z_1 - Z_2$: the averaged Bochner integral defined as

$$\eta(x, avg) := (\nu[\partial A - \{y_0\}])^{-1} \cdot \int_{\partial A - \{y_0\}} \eta(x,y) d\nu(y), \tag{5}$$

and we require that

$$\eta(x, avg) \text{ is nonzero finite tangent vector,} \tag{6}$$

and there exists a constant $C > 0$ such that

$$||\eta(x, avg)|| \geq C > 0 \tag{7}$$

for $x \in A - M(A)$, uniformly with x. The verification of hypotheses (6)–(7) can be difficult to verify. Evidently (7) implies (6). Equivalently, we find $\eta(x, avg)$ is an averaged gradient and therefore the gradient of the averaged potential

$$f_{avg}(x) := \int_{\partial A} \nabla_x \sqrt{c(x, y) - c(x, y_0)}.$$

The hypothesis (7) is simply the claim that $f_{avg}(x)$ is critical-point free over the open subset $A - Z_2$.

We need also remark on a complication arising from the nonconvexity of A. What is the natural distance function d on $A \subset \mathbb{R}^N$, and the physical "transport cost" of a unit mass at $x \in A$ to target mass $y \in \partial A$? There are at least two popular possibilities. First we may restrict the ambient euclidean distance $d_{\mathbb{R}^N}(x, y) = ||x-y||$ to $A \times \partial A \subset \mathbb{R}^N \times \mathbb{R}^N$. But this restriction does not represent a path length distance in the sense of Gromov [Gro+01, 1.A-B]. In otherwords the restriction does not represent geodesic transport in A, and there is no variational description of the metric in terms of shortest-length curves.

A second approach defines $d = d_A$ as the induced length distance defined by

$$d_A(x, y) = \inf_{\gamma} \int_{\gamma} Length(\gamma),$$

where the infimum is over all curves $\gamma : [0, 1] \to A$ contained in A with $\gamma(0) = x$, $\gamma(1) = y$. The reader will observe that both possibilities define coincident medial axes $M(A)$ according to (1), since euclidean balls are geodesically convex. The induced length distance $d = d_A$ is possibly most preferred by metric geometers, yet is difficult to numerically evaluate. Moreover geodesics with respect to the induced path distance $c = d_A$ can oftentimes be branching. The possible branching of geodesics implies gradients $y \mapsto \nabla_x d(x, y)$ are noninjective maps $\partial A \to T_x A$ for $x \in A$. This possible noninjectivity violates an important transport condition called (Twist), and is obstruction to hypothesis (6). Thus neither the restricted distance $c = d|A \times \partial A$ nor the induced distance $c = d_A$ are especially convenient costs.

3. Hubbard's $1/d^{\alpha}$-distance

This article explores a third possibility: namely a variant of Hubbard's so-called $1/d$-metric (see [HH06, Ch. 2.2, pp.33]). Let $A \subset \mathbb{R}^N$ be open subset. Then for every real parameter α we define the Riemannian metric

$$g_{\alpha} := (dist(x, \mathbb{R}^N - A))^{-\alpha}.ds^2, \tag{8}$$

where

$$ds^2 = dx_1^2 + dx_2^2 + \cdots + dx_N^2$$

is the standard Euclidean metric on $\mathbb{R}^N$. The choice $\alpha = 0$ yields $g_0 = ds^2$. Let $\kappa = \kappa(g)$ denotes the sectional curvature of the metric g.

Lemma 1. *For every parameter $\alpha \geq 0$, the Riemannian metric g_α has nonpositive sectional curvature $\kappa \leq 0$ throughout A.*

Proof. We follow Hubbard's proof [HH06, Thm. 2.2.9, pp.36], where the key observation is that: for every $y \in \mathbb{R}^N - A$, the function $f_y(x) := -\log ||x-y||^\alpha$ is subharmonic for $x \in A$ (in fact, the function is harmonic). Therefore the supremum

$$f(x) := \sup_{y \in \partial A} f_y(x) = \sup_{y \in \mathbb{R}^N - A} f_y(x)$$

is subharmonic. But the metric g_α is conformal to the standard Euclidean metric ds^2, and the formula for the sectional curvature of conformal metrics is well-known, namely $\kappa = -\Delta \log f \cdot ds^2$, which is ≤ 0 by above subharmonicity. □

For variable α the metric g_α, and the corresponding path length distance $d_\alpha(x,y)$ is possibly incomplete on A. Geodesics in the g_α metric steer so as to keep as far away as possible from the boundary ∂A. However incompleteness only occurs at the boundary ∂A of A as subset of $\mathbb{R}^N$. For A an open subset there may exist sequences (relative to the distance d_α) $\{x_k\}_{k \in \mathbb{N}}$ in A which have no limit point in A. Despite the metric g_α diverging as $x \to \partial A$, we prefer the lengths of geodesics $\gamma : [0,1] \to A$ converging to ∂A to have finite length, and seek parameters α for which

$$Length_\alpha(\gamma) = \int_0^1 \sqrt{g_\alpha(\gamma'(t), \gamma'(t))}.dt < +\infty.$$

Example 2. Hubbard's definition of $1/d$-metric corresponds to $\alpha = 2$ in equation (8). Amazingly the $1/d$-metric on the upper halfspace $H := \{x_1 > 0\}$ in $\mathbb{R}^N$ in $x_1, \ldots, x_N$ coordinates is the complete constant-curvature hyperbolic metric on H. For $0 < \alpha < 2$ the metric is incomplete. The curve $\gamma(t) = (1-t, 0, 0, \cdots)$ for $0 \leq t \leq 1$ is a curve in H. The g_α-length of γ evaluates to $\int_0^1 (1-t)^{-\alpha/2} dt$, which is improper integral converging to $1 < (1-\alpha/2)^{-1} < +\infty$ when $0 < \alpha < 2$. While $0 < \alpha < 2$ we can uniquely extend d_α to a complete metric pairing $\tilde{d_\alpha} : \overline{H} \times \overline{H} \to \mathbb{R}$, where $\overline{H} = \{x_1 \geq 0\}$. Note that $\overline{H}$ is not homeomorphic to adjoining a sphere at-infinity S^2_∞ to H.

Example 3. The $1/d$-metric ($\alpha = 2$) on the once-punctured plane $A = \mathbb{R}^2 - \{0\}$ is isometric to a straight cylinder of circumference 2π ([HH06, Ex.2.2.6]). The same computations as previous example show for $0 < \alpha < 2$, the metric d_α is incomplete with completion $\tilde{d_\alpha}$ equal to an infinite cone with angle [FORMULA] at the origin vertex.

Example 4. The Weil-Petersson metric d_{WP} on the Teichmueller space $\mathscr{T}_g$ of a closed genus g hyperbolic surface is asymptotically equivalent to Hubbard's metric with exponent $\alpha = 3/2$, [Wol75].

The above examples have A unbounded open subset. But our applications to medial axes concern bounded open subsets.

Example 5. We modify example 3 by restricting to the punctured disk, say, $D^\times := \{0 < ||x|| < R\}$ for a constant $R > 0$. Then the medial axis $M(D^\times) = \{||x|| = R/2\}$ is a circle in $D^\times$. Now we propose that sufficient (UHS) conditions, namely (6)–(7), are satisfied throughout $D^\times$ and the inclusion $Z_2 \hookrightarrow D^\times$ is homotopy-isomorphism (by Theorem B) for $Z = Z(\mu, \nu, c_\alpha)$ for $0 < \alpha < 2$. Moreover we propose Z_2 is also a circle, diffeomorphic to $M(A)$, but not identical.

Example 6. Let A be convex subset and $0 < \alpha < 2$. Then $_\alpha(x, y)$ is proportional to the rescaled Euclidean distance $|x - y|^{1-\alpha/2}$.

Proposition 7. *[Work-In-Progress] Let A be open subset of $\mathbb{R}^N$. For parameters $0 < \alpha < 2$ the metric g_α is incomplete Riemannian metric on A, and geodesics in A converging to the boundary ∂A have uniquely defined finite length with respect to the metric g_α. Consequently the path length metric*

$$\tilde{d_\alpha} : \overline{A} \times \overline{A} \to \mathbb{R}^{\geq 0}$$

is well-defined throughout the closure $\overline{A}$.

Proof. □

We remark on the differences between C^0, $C^{1,1}$, and C^2 regularity of boundaries ∂A. For C^2 boundary, the medial axis $M(A)$ will be disjoint from A. However for $C^0, C^{1,1}$ regularity, the medial axis $M(A)$ will extend into the boundary ∂A. The $C^{1,1}$ regularity frequently occurs, e.g. whenever A is convex polyhedra.

4.

Now we propose a more interesting mass transport interpretation of medial axis transforms. Let A be bounded open subset of $\mathbb{R}^N$, with boundary ∂A, and probability measures μ, ν as previously defined in (2), (4). Then we choose cost $c = \tilde{d_\alpha}$: $A \times \partial A \to \mathbb{R}$ defined by restricting the completion to $A \times \partial A \subset \overline{A} \times \overline{A}$. The subvarieties Z_2 and $M(A)$ do not coincide set-theoretically, but we conjecture that they do coincide topologically:

Theorem 8 (Work-In-Progress)**.** *Let A be bounded open subset of $\mathbb{R}^N$. Let $c = \tilde{d_\alpha}$ be the metric completion of d_α to $\overline{A}$ (Prop. 7), and let $Z = Z(\mu, \nu, c) : 2^{\partial A} \to 2^A$ be the Singularity functor with respect to (μ, ν, c) as defined in (2), (4). Then* *sufficient (UHS) Conditions are satisfied to apply Theorem B [Mar, Thm.3.4.3.], and the inclusion $Z_2 \hookrightarrow A$ is a homotopy isomorphism and even a strong deformation retract.*

Lemma 9. *For every $\alpha \geq 0$, the restricted cost $c = \tilde{d_\alpha}^2/2 : A \times \partial A \to \mathbb{R}$ satisfies the following (Twist) condition: for every $x \in A$, the gradient mapping*

$$\partial A \to T_x A, \ \ y \mapsto \nabla_x c(x,y) \text{ is injective.}$$

Proof. We take advantage of fact that c is a Lagrangian cost defined by an action principle. According to [Vil09, Prop.10.15, pp.235], the gradient $\nabla_x c(x,y)$ is equal to

$$\frac{-1}{2}.(1 - e^{-d(y,y_0)^2}).\rho(x).\gamma'(0),$$

where ρ is the conformal factor $\rho(x) = dist(x, \mathbb{R}^N - A)^{-\alpha}$, and where $\gamma'(0)$ is the initial tangent vector of an action-minimizing curve γ in A with $\gamma(0) = x$, $\gamma(1) = y$. The nonpositive curvature of g_α implies action-minimizing curves exist. Since the conformal factor ρ is nonvanishing, and since geodesics in Riemannian manifolds are determined by their initial point and initial tangent vector, we conclude $y \mapsto \nabla_x c(x,y)$ is injective, as desired. □

That c satisfies the above (Twist) condition implies the uniqueness of c-optimal semicouplings from μ to ν. [ref].

The above Lemma 9 and the identity

$$\nabla_x c(x,y) = -(1 - e^{-d(y,y_0)^2}) \cdot \frac{\rho_x}{2} \cdot \gamma'(0)$$

implies the gradient of the cross-difference c_Δ is readily computed

$$\nabla_x c_\Delta(x, y_0, y_1) = (1 - e^{-d(y_1,y_0)^2}).\frac{\rho_x}{2}.[\gamma_1'(0) - \gamma_0'(0)],$$

where γ_0, γ_1 are the g_α-geodesics satisfying $\gamma_0(0) = \gamma_1(0) = x, \gamma_0(1) = y_0$, $\gamma_1(1) = y_1$. Following the definition of $\eta(x, avg)$ from [ref], we find

$$\eta(x, avg) = \frac{\rho_x}{2} \left\langle (1 - e^{-d(y,y_0)^2}) \cdot |\psi(y) - \psi(y_0) + c_\Delta(x, y_0, y)|^{-1/2} \ . \ [\gamma_0'(0) - \gamma_y'(0)] \right\rangle .$$

Here $\langle v \rangle$ denotes the averaged Bochner integral of a vector-valued function v with respect to the restricted measure $\nu|_{\partial A - \{y_0\}}$ on ∂A, that is

$$\langle v \rangle = (\nu[Y - \{y_0\}])^{-1}. \int_{Y - \{y_0\}} v(y) d\nu(y).$$

The average defining η_{avg} is weighted by the regions where the potential $\psi(y) - \psi(y_0) + c_\Delta(x, y_0, y) \geq 0$ vanishes $(= 0)$. The weights blow-up to $+\infty$ when $y \to Z_2 - A$. Without the exponential factor $(1 - e^{d(y,y_0)^2})$, the weights would blow-up as $y \to y_0$.

Remark. If the region A is convex, then the nonvanishing $\eta(x, avg) \neq 0$ is easily established since the tangent vectors $\{\gamma_y'(0) - \gamma_0'(0) \mid y \in \partial A\}$ lie in a common halfspace of $T_x A$. In this case the Bochner average is nonzero.

5. Discontinuity of $M(A)$ versus Continuity of Z_2.

The completion of Hubbard's $1/d_\alpha$ distance and the cost $c = \tilde{d}_\alpha^2$ yields an alternative to the medial axis $M(A)$ in the subvariety Z_2 defined by c-optimal couplings. We propose this construction of Z_2 yields a useful improvement over the conventional definition of $M(A)$ per (1). For instance the medial axis is defined on the category of open subsets A of $\mathbb{R}^N$, whereas the functors Z are more generally defined for measure spaces.

If A is $C^{1.1}$ and $y \in \partial A$ is not uniquely differentiable boundary point, then $y \in M(A)$. I.e. the medial axis $M(A)$ extends into the boundary at y. Therefore "sharp" corners cause the medial axis to "split" and extend into ∂A. That sharp corners can appear under Gromov-Hausdorff variations of the subset $A \subset \mathbb{R}^N$ implies the medial axis is rather unstable. Small perturbations of the open subset A (e.g. background noise) can lead to large changes in the medial axis $M(A)$. Many authors have suggested modified medial axes (c.f. [FLM03], [TH03] and references therein) which "filter out" possible noise. The present article takes another approach to "regularizing" $M(A)$. It's well-known that optimal transportation enjoys strong continuity properties with respect to variations in the datum (μ, ν, c). More precisely we quote the following result from [Vil09, Thm. 28.9, pp.780–790].

[insert continuity]

Suppose A_k is a sequence of bounded open subsets which converge in GH-topology to A_0. The pointwise densities ρ_{x,A_k} converge pointwise to ρ_{x,A_0}. We also see $\overline{A}_k$ GH-converges to $\overline{A}_0$. The measures μ_k will converge in narrow-topology to μ_0. However the pushforwards ν_k do not generally converge in narrow-topology to ν_0. Indeed the transport maps T_k defined in (3) may not converge pointwise to T_0.

Proposition 10. *Let A_k, $k = 1, 2, \ldots$, be bounded open subsets of $\mathbb{R}^N$, with canonical probability measures μ_k, and converging in Gromov-Hausdorff topology*

$$\lim_{k \to +\infty} (A_k, \mu_k) = (A_0, \mu_0).$$

Let $c_0, c_1, c_2, \ldots$ be the costs $c_k = \tilde{d}_{\alpha, A_k}^2 / 2$.

Then the sequence of probability measures $\nu_k := T_k \# \mu_k$ converges in the narrow topology to a probability measure

$$\lim_{k \to +\infty} \nu_k := \overline{\nu_0},$$

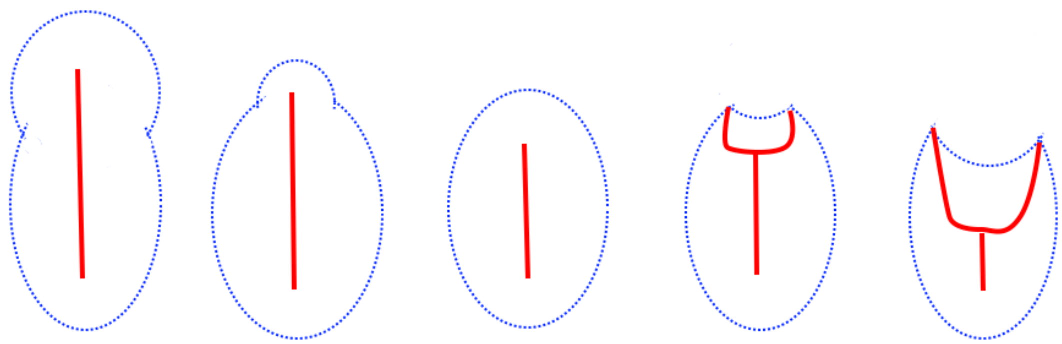

FIGURE 1. The medial axis $A \mapsto M(A)$ is discontinuous when sharp corners appear, like the two figures on the right.

and the contravariant functors $Z(\mu_k, \nu_k, c_k)$ *converge in GH-topology to the contravariant functor* $Z(\mu_0, \overline{\nu_0}, c_0)$*. In particular,*

$$\lim_{k \to +\infty} Z_{2,A_k} = Z_{2,A_0}.$$

So the medial axis has uniquely defined limits in the appropriate topologies, namely with respect to continuous variations in (μ, ν, c). The point of 10 is the identification of the limit medial axis as the singularity of a limit transport problem.

6. Conclusion

In conclusion, Blum identified the medial axis transform as convenient mode of describing objects, and heuristics showed the inclusions $M(A) \hookrightarrow A$ were always homotopy isomorphisms. However Blum's medial axis is but a particular instance of a more useful topological object, namely Z_2 of the contravariant functors $Z(\mu, \nu, c) : 2^{\partial A} \to 2^A$. This Z_2 is stable topological object, and the inclusions $Z_2 \hookrightarrow A$ are identified as homotopy isomorphisms when the (UHS) Conditions (6), (7) hold throughout the open complement $A - Z_2$. Thus we propose more stable topological "folk-theorems" regarding a mass transport extension of so-called medial axis transforms, and Theorem B from [Mar]. That is, we propose replacing the medial axis $M(A)$ with the more stable topological objects Z_2 of optimal transport programs.

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