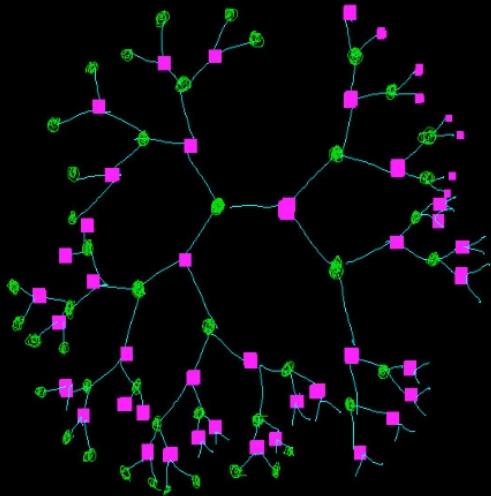


Algebraic Topology and Optimal Transportation:

...How to build Spines from Singularity...

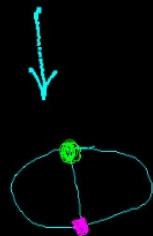
J.H.Martel



$X \rightarrow E\Gamma$ -model

=Universal
Cover
of space M

$$\pi_1(M) = \Gamma$$



X / Γ quotient
with $\pi = \pi_1$



Poincaré ~1895 constructs Algebraic-Topology:

*Group-cohomology
of Γ*

= *Topological
Symmetries
of $E\Gamma$ action*

*Fundamental group, Universal
Covering Spaces, orbit quotients.*

Our thesis introduces and develops a general technique
for explicitly constructing **SMALL**-dimensional **E Γ** classifying spaces.

Hypothesis: ... Γ is infinite, discrete, Bieri-Eckmann duality group
with finite cohomological dimension $cd(\Gamma)=v < +\infty$ and dualizing module **D**.

Our method presumes a user has initial explicit geometric E Γ model X
i.e. (X,d) is finite-dimensional Cartan-Hadamard space (NPC contractible)
where group action $X \times \Gamma \rightarrow X$ isometric, proper discontinuous, free,
and quotient X/Γ finite volume.

Ex: $\Gamma = \mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}^3, \dots$ torsion-free abelian groups

- = $GL(\mathbb{Z}^2), GL(\mathbb{Z}^3), Sp(\mathbb{Z}^4), Sp(\mathbb{Z}^6), \dots$ arithmetic groups **G(Z)**
of **Q**-reductive groups,
- = $MCG(\Sigma_g)$... mapping class groups of closed surfaces,
- = ..., Knot Groups,
- =, etc.

..... all models not created equal.....

Ex: - $MCG(\Sigma_g)$ acts isometrically on $Teich(\Sigma_g)$ with Weil-Petersson geometry.

- **G(Z)** acts isometrically on spaces of quadratic (or hermitian) forms.
- $GL(\mathbb{Z}^2)$ acts isometrically on Voronoi's cone, projectivizes to H^2
(Poincare disk)

If X is geometric $E\Gamma$ model, then typically $\dim(X) \gg \dots \gg cd(\Gamma)$.

Bieri-Eckmann formula:
$$cd(\Gamma) = \dim(X) - (q + 1),$$
 q=spherical-dimension(D).
(homological duality)

If X is geometric $E\Gamma$ model, then typically $\dim(X) \gg \dots \gg cd(\Gamma)$.

Bieri-Eckmann formula: $cd(\Gamma) = \dim(X) - (q + 1)$,
(homological duality) q=spherical-dimension(D).

Our thesis **constructs Γ -invariant closed subsets Z of X , with $\dim(Z) \approx cd(\Gamma)$,**

for which **the inclusion $Z \rightarrow X$ is homotopy-isomorphism**, and

explicit Γ -equivariant continuous deformation retracts X onto Z , and

we describe technique for **achieving Z with MAX codimension $\dim(Z) \neq cd(\Gamma)$** .

Definition: such maximal-codimension retracts Z are called *minimal spines/souls*.

Spines and Souls : Tradition in Geometric-Homology:

Klein, Minkowski, Poincare, Steenrod, Thom, Lefschetz, Borel, Serre, Wall, Thurston, Gromov, Neeman, Mumford, Gromoll-Cheeger-Perelman, Soule, Ash, McConnell,

...how to construct NEW models of old spaces, and as explicit as possible?

“Textbook” constructions of $E\Gamma$ are abstract/external/dislocated

- requires perfect knowledge of Γ , i.e. generators and relations.

- Milnor: $E\Gamma = \text{joins}(\Gamma, \Gamma, \Gamma, \dots)$.
- Wall: inductive wedges of spherical-complexes and attaching maps.
- Postnikov towers, Cayley graphs, Rips complex,

We presume limited knowledge of Γ ,

but require explicit geometric $E\Gamma$ -model (X, d, vol_X).

Our thesis develops a new program for In Situ reduction-to-spine,

- ! construct spines as explicit subsets of initial model X !

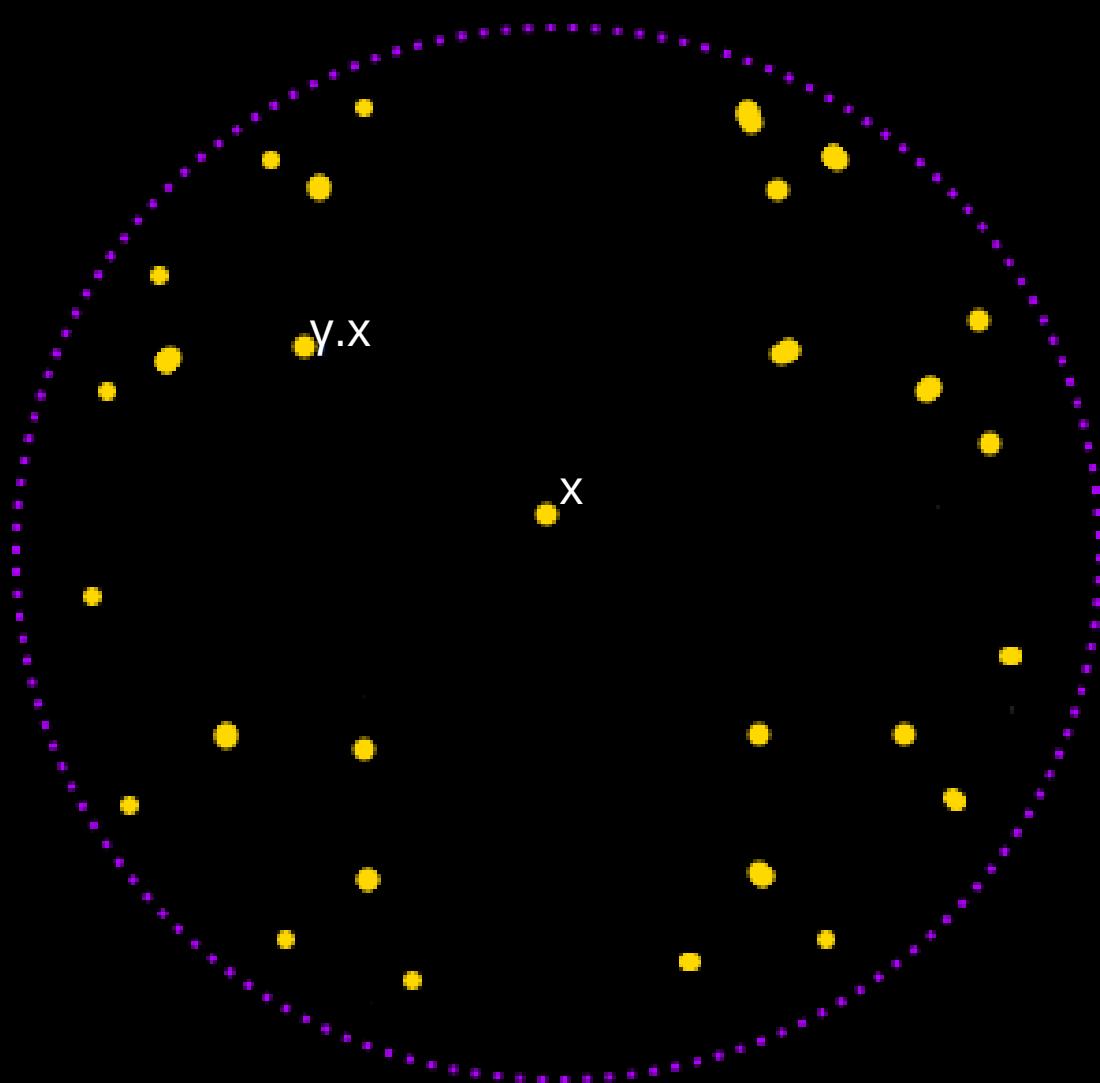
- ready-made for implementation on WOLFRAM.

- Nonlinear extension of Soule-Ash’s Well Rounded Retract !

...illustrating our approach:

Initial EG model (X, d, vol_X).

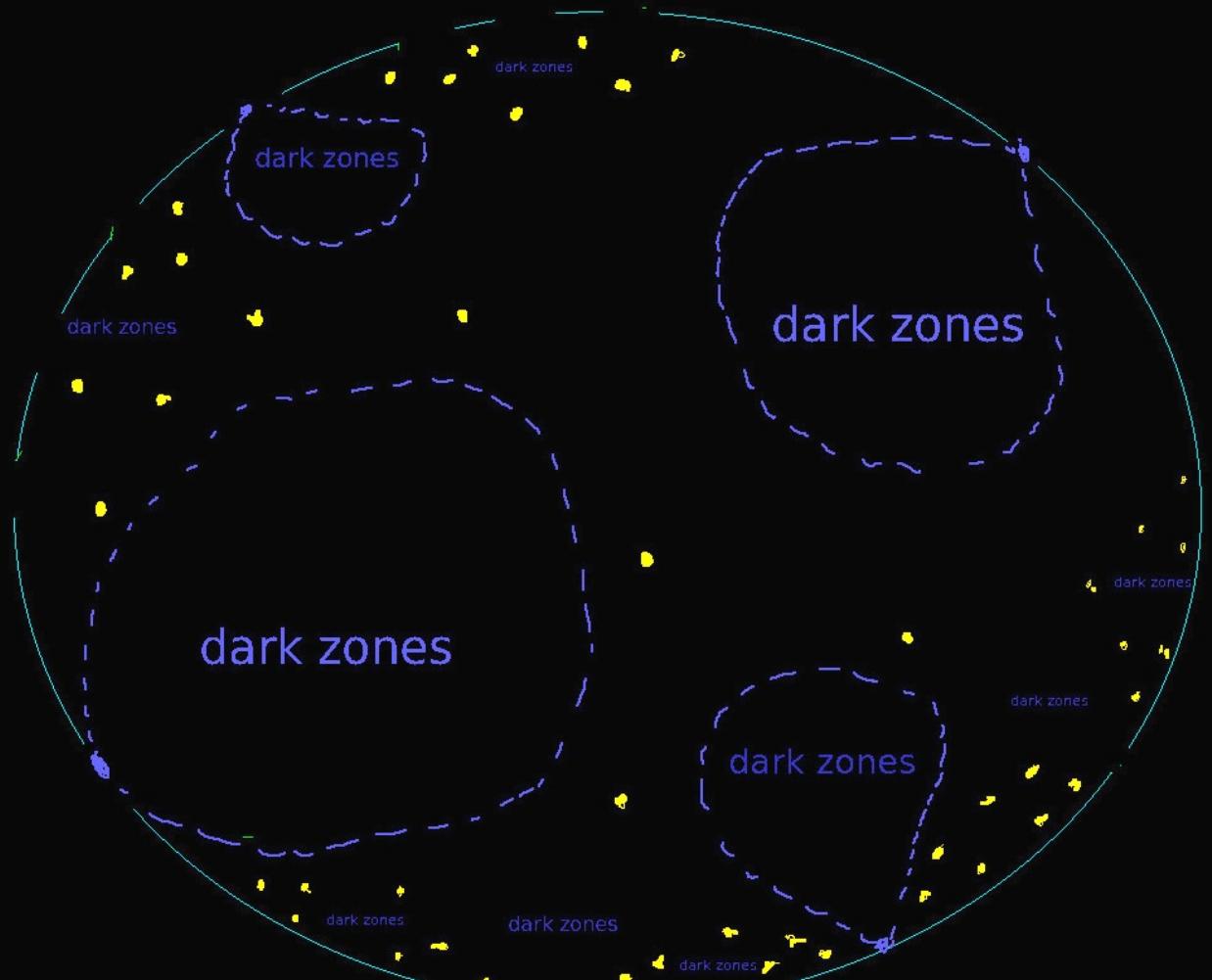
Yellow dots = Γ orbit of pt. x



- Γ orbit avoid dark zones.

Dark zones
=
 Γ -rational horoballs
 $V[t]$

*Excision $X[t]$ obtained by
scooping out / excising
the dark zones from X .*

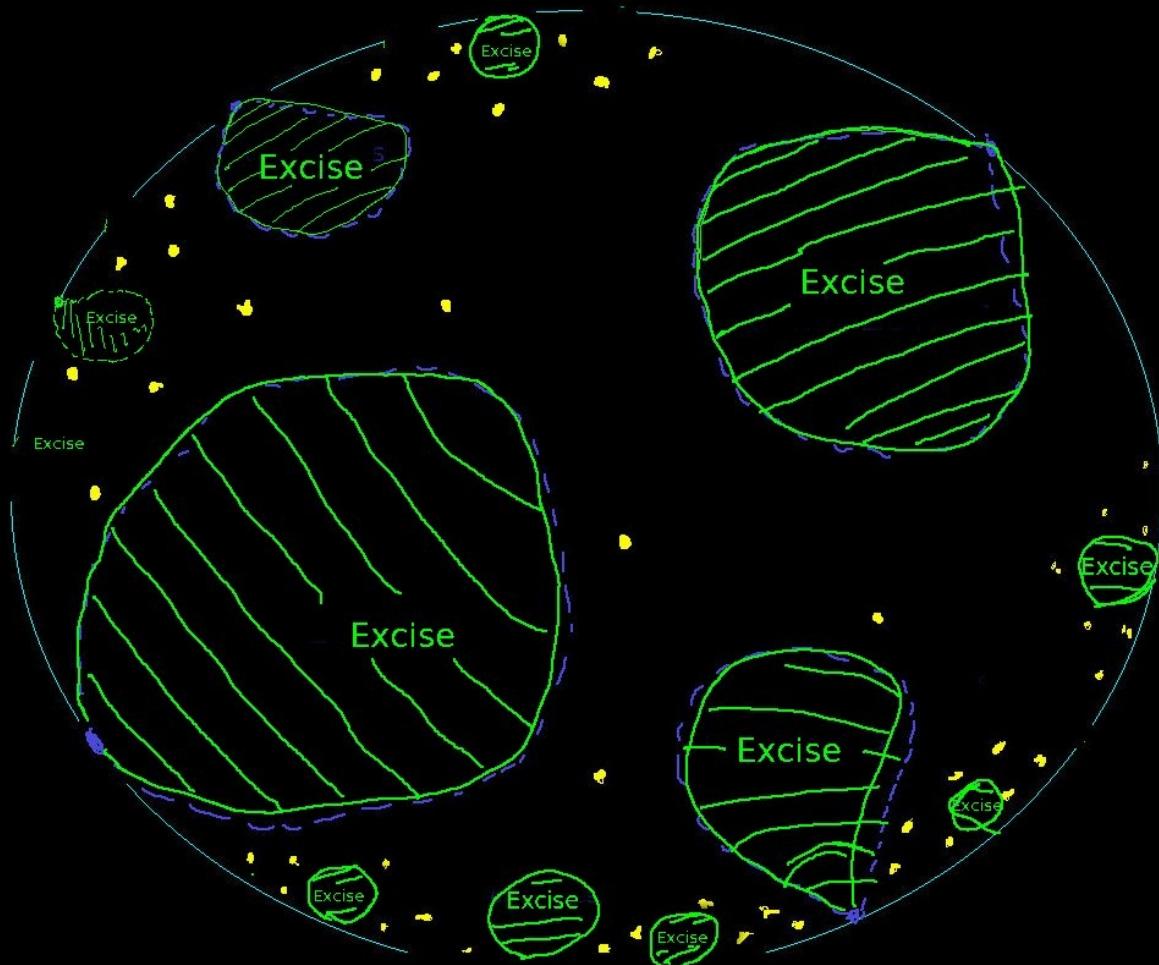


Orbit avoids Dark zones

- Γ orbit avoid dark zones.

Dark zones
=
 Γ -rational horoballs
 $V[t]$

Excision $X[t]$ obtained by scooping out / excising the dark zones from X .



Orbit avoids Dark zones

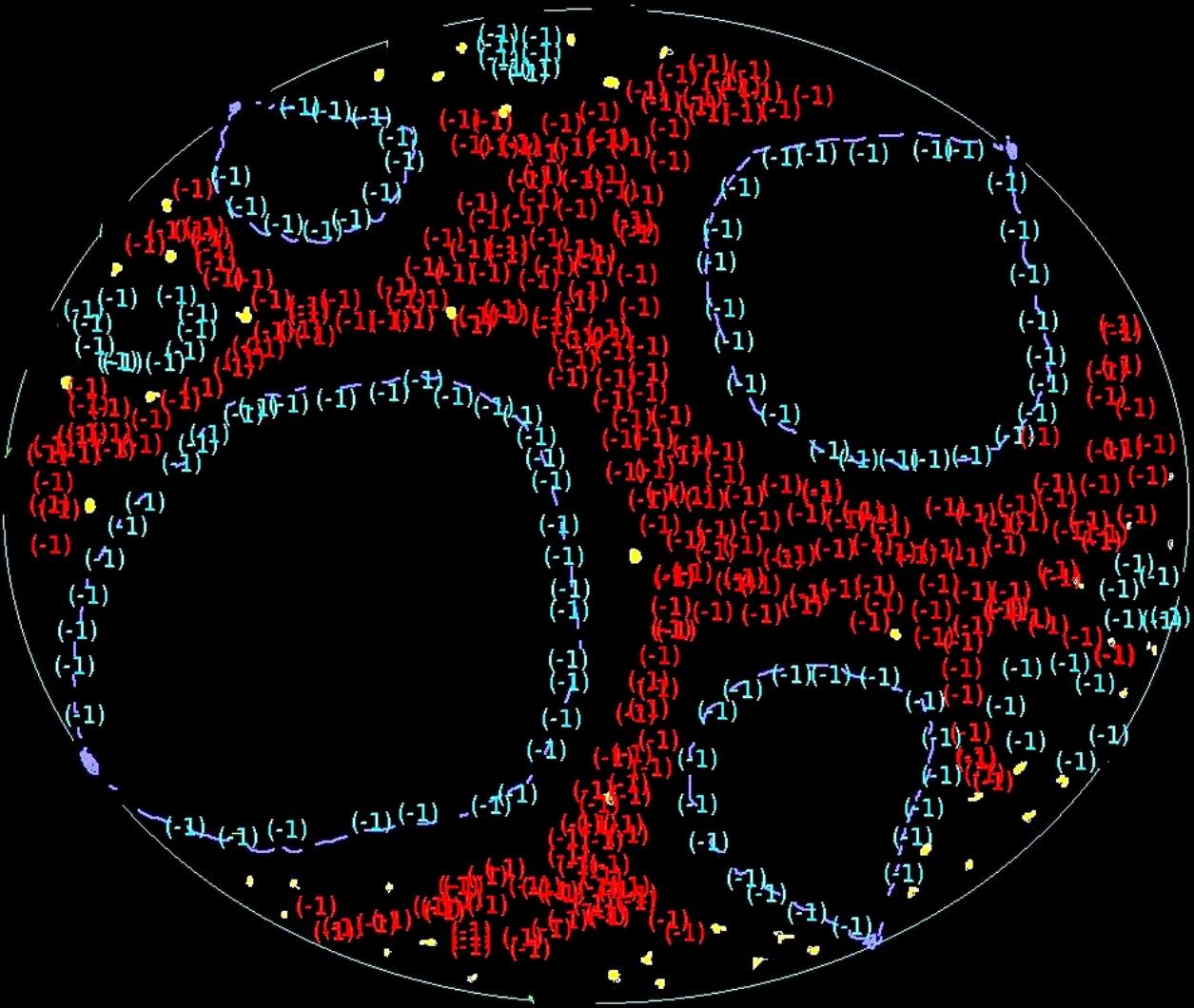
Scoop/Excise convex deep dark zones at-infinity.

*Excision model is
manifold-with-corners
 $X[t] \times \delta X[t]$.*

Next: define
source measures σ
and target measures τ

(-1) source
measure
on $X[t]$

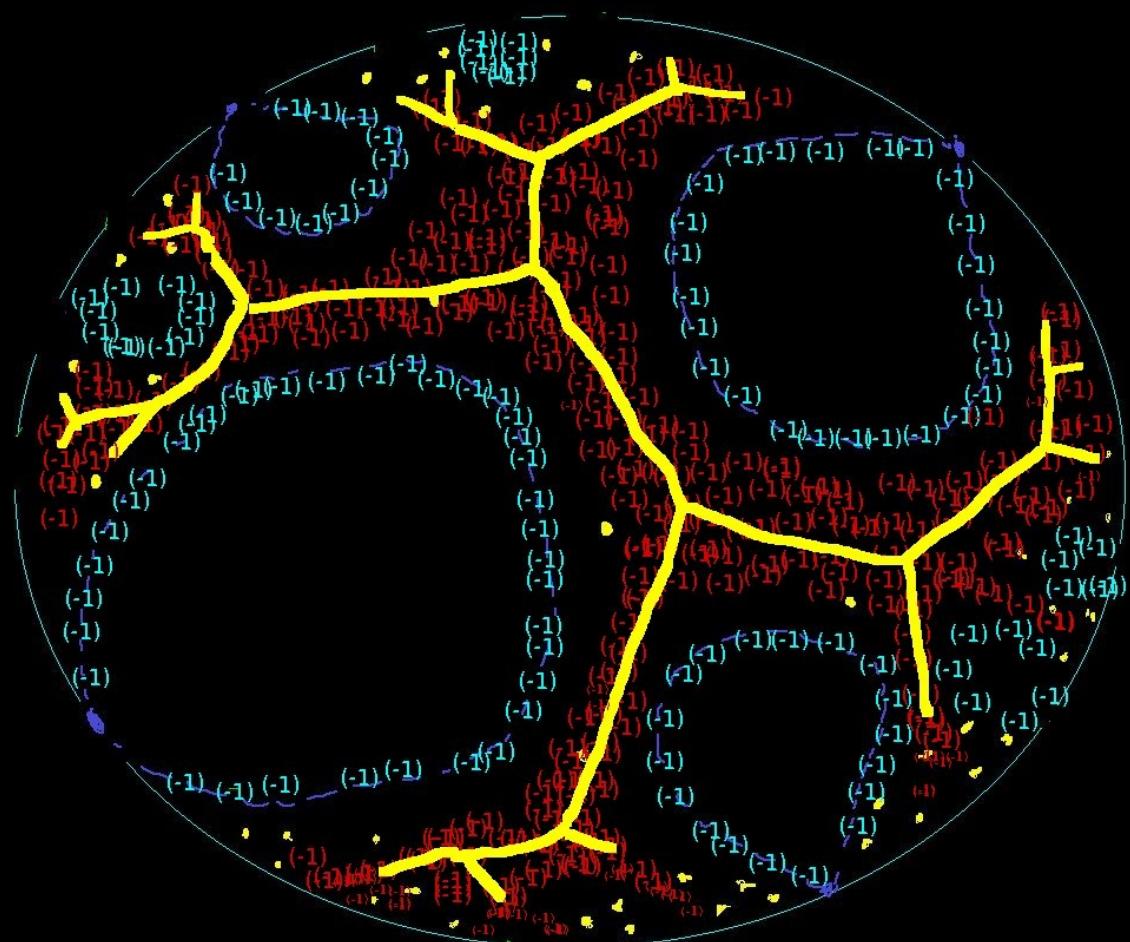
(-1) target measure
on $\delta X[t]$



With measures σ , τ , we next study the singularities of c-optimal semicouplings from source to target.

$$Z: \begin{matrix} \delta X[t] \\ 2 \end{matrix} \rightarrow \begin{matrix} X[t] \\ 2 \end{matrix}$$

We propose:
 Spines are readily displayed in the
 locus-of-discontinuity (Singularities)
 of optimal semicouplings.



Singularity structure on the
 activated energy-minimizing configuration

...Spines are not hidden...

Spines are readily displayed
in *locus-of-discontinuity* Z
of deformation retracts $r : X[t] \rightarrow \delta X[t]$

...Spines are not hidden...

Spines readily displayed in
Kantorovich's Contravariant Singularity Functor $Z(\sigma, \tau, c^*)$
of c^* -optimal semicouplings π from $(X[t], \sigma)$ to $(\delta X[t], \tau)$

source target

... where $c^*: X[t] \times \delta X[t]$ is our two-pointed visible repulsion cost (defined below).

Terms to define:

Singularity functor $Z(\sigma, \tau, c^*)$

of c^* -optimal semicouplings π

from source $(X[t], \sigma)$ to target $(\delta X[t], \tau)$

Terms to define:

Topology:

Source excision models $(X[t], \sigma)$

Target $(\partial X[t], \tau)$.

Steinberg modules $D := \tilde{H}_q(\partial X[t]; \mathbb{Z})$.

Steinberg symbols $B \in H_q(\partial X[t]; \mathbb{Z})$

and $\text{FILL}[B] = H_{q+1}(X[t], \partial X[t]; \mathbb{Z})$.

Chain sums $\underline{F} = \sum_{\gamma \in \Gamma} F \cdot \gamma$

with well-separated gates $\{G\} = \{\text{FILL}[B].\gamma \quad | \quad \gamma \in \Gamma\}$.

Terms to define:

Mass transport:

Costs $c : X[t] \times \partial X[t] \rightarrow \mathbb{R}$.

Two-pointed repulsion and visibility costs c^*, v

c -optimal semicouplings π .

c -concave potentials $\psi^{cc} = \psi$.

c -subdifferentials $\partial^c \psi(y) \subset X[t]$ for $y \in \partial X[t]$.

Monge-Kantorovich duality: c -optimal semicouplings π supported on graph of $\partial^c \psi$.

Kantorovich Singularity functor $Z : 2^{\partial X[t]} \rightarrow 2^{X[t]}$.

Filtrations $Z_0 \hookleftarrow Z_1 \hookleftarrow Z_2 \hookleftarrow \dots$.

Kantorovich's Contravariant Singularity Functor

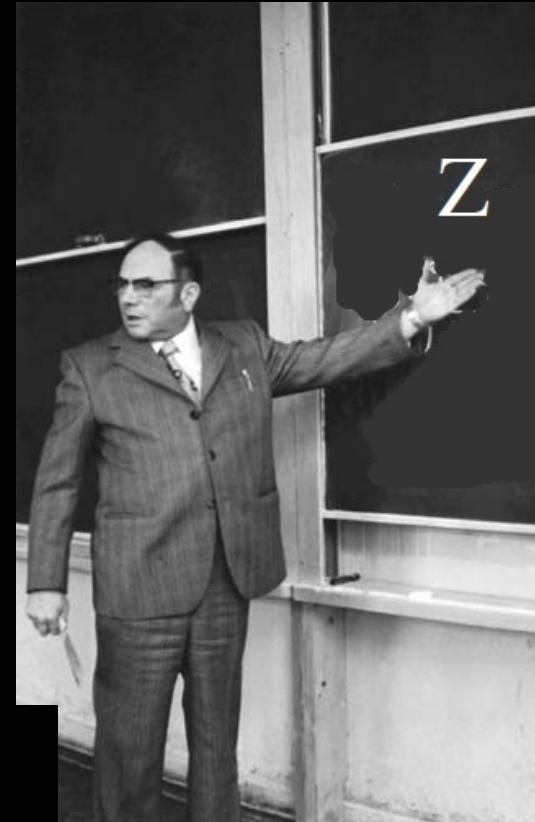
$$Z : 2^{\partial X} \rightarrow 2^X, \quad Z(Y_I) = \cap_{y \in Y_I} \partial^c \psi(y) = " \cap_{y \in Y_I} r^{-1}(y)" .$$

... but everything summarized in: Kantorovich's Contravariant Singularity Functor

$$Z : 2^{\partial X} \rightarrow 2^X,$$

$$Z(Y_I) = \cap_{y \in Y_I} \partial^c \psi(y)$$

$$= " \cap_{y \in Y_I} r^{-1}(y)" .$$



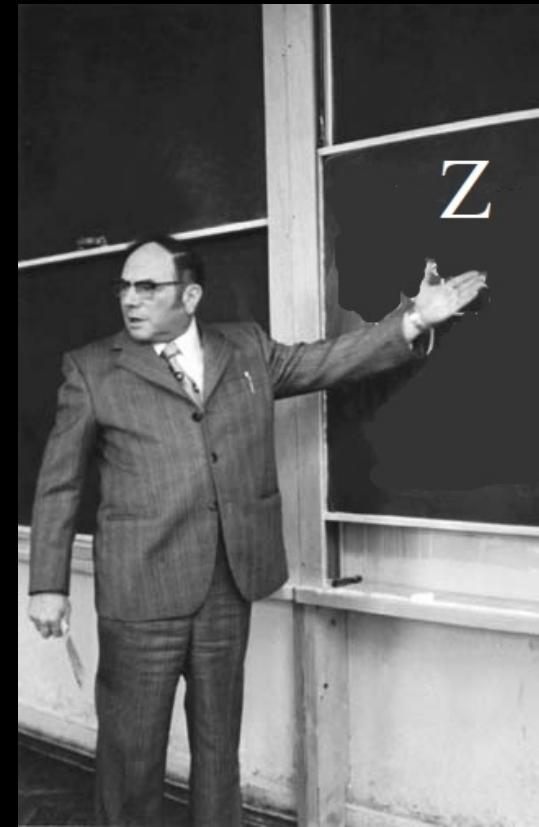
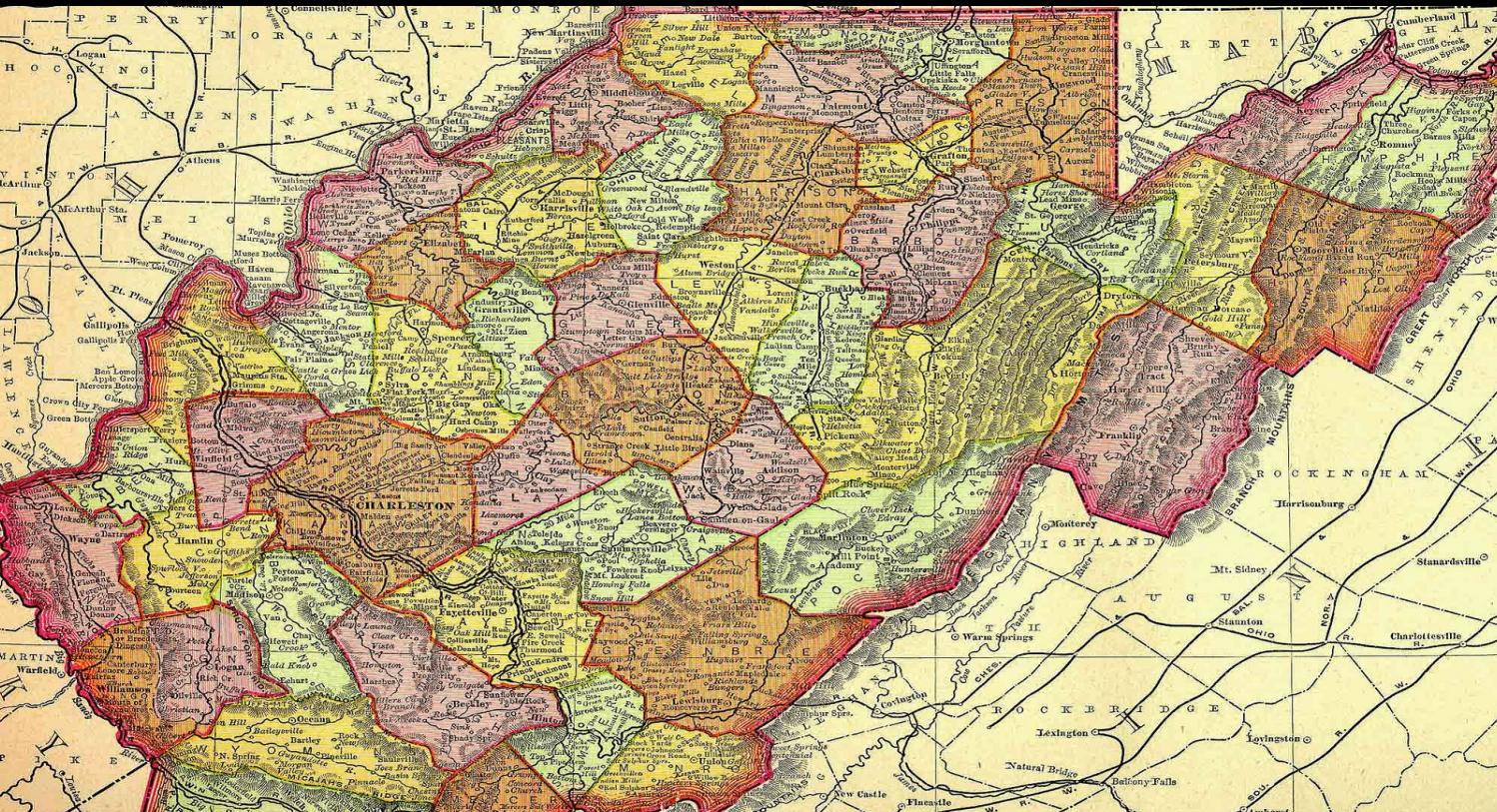
...Singularity is overburdened term.

$$Z : 2^{\partial X} \rightarrow 2^X, \quad Z(Y_I) = \cap_{y \in Y_I} \partial^c \psi(y)$$

Economic Definition:

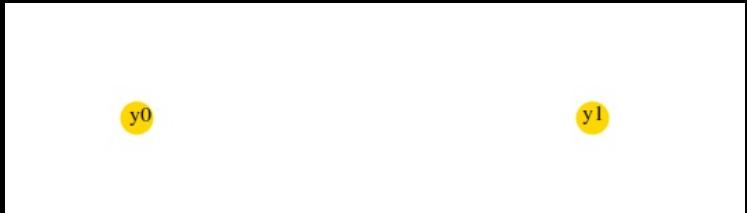
Singularity arises wherever there is competition for limited common resources.

Singularity is Why countries exist with borders.



Singularity: wherever competition for limited resources:

$$Z : 2^{\partial X} \rightarrow 2^X,$$



$c=d^{**2}/2$ quadratic cost
 $(+)\rightarrow \leftarrow (-)$ attraction

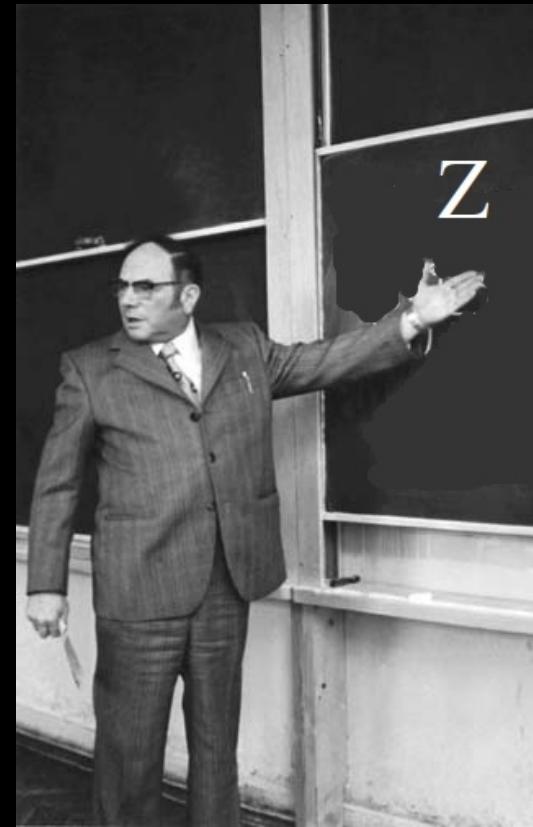
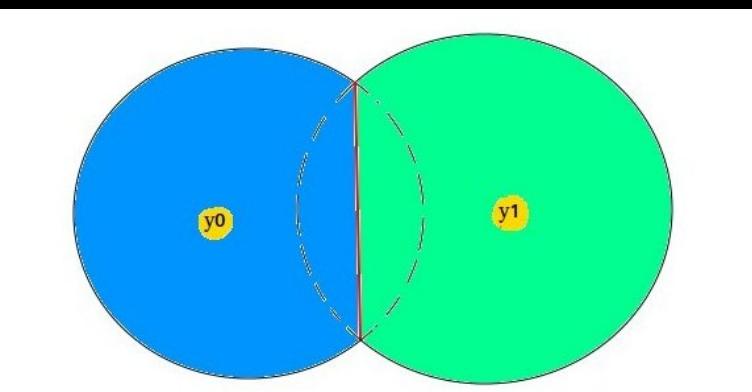


- y_0, y_1 no competition
(no interact)

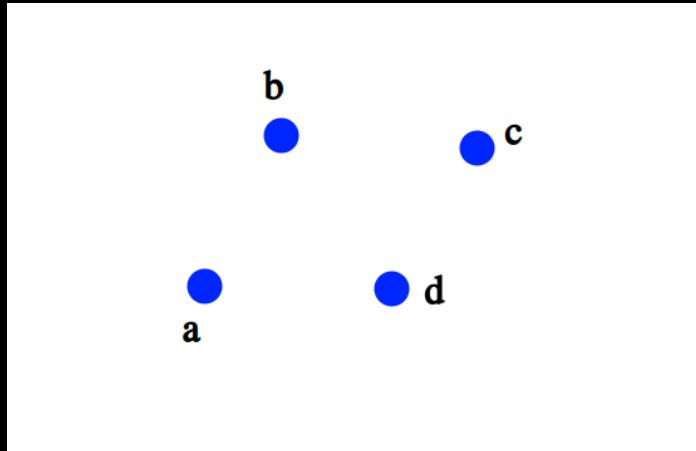
- Singularity=Empty

- y_0, y_1 compete/interact

- *Singularity nonempty*
and stable/persistent
w.r.t. target mass

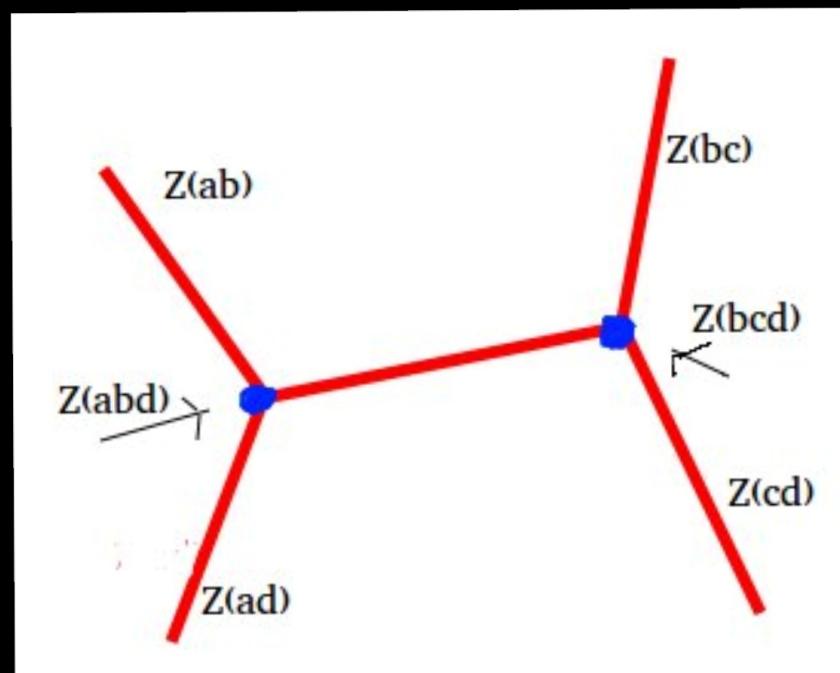
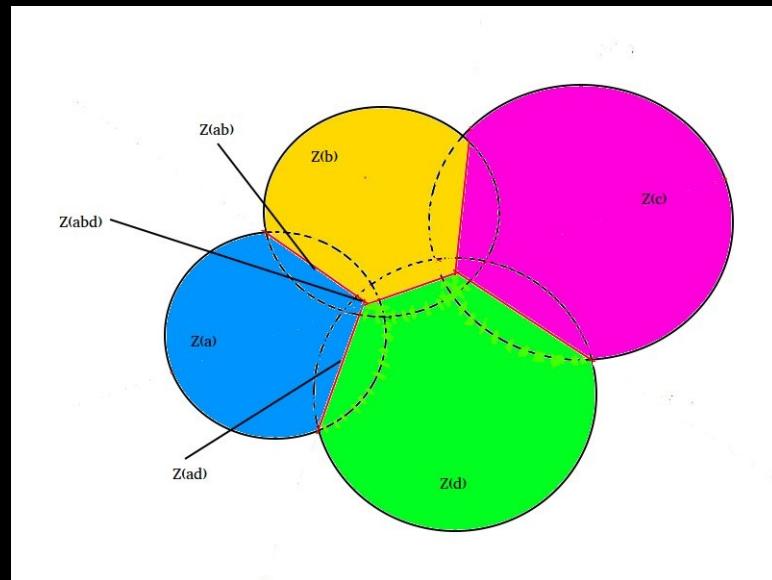
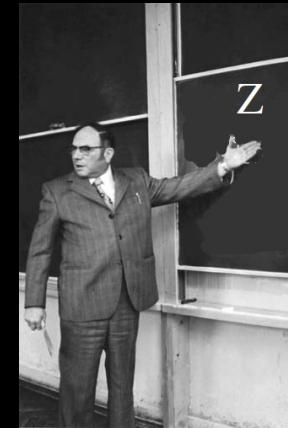


$$Z(Y_I) = \cap_{y \in Y_I} \partial^c \psi(y)$$



Source = 2-dim background field

Target = $+a+b+c+d$ (4 point masses)



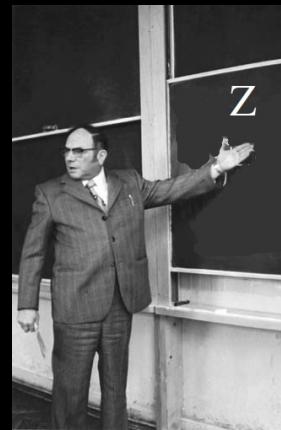
$c=d^{**}2/2$
quadratic
cost

$(+)$ \rightarrow \leftarrow $(-)$
attraction

Kantorovich Functor is EXPLICIT.

Key definitions: - c-concave potentials $\psi^{\text{cc}} = \psi$.

- c-subdifferentials $\partial^c \psi(y)$



Kantorovich says ``*c-optimal semicouplings π are supported on graph of c-subdifferential of c-concave potentials potentials $\psi = \psi^{\text{cc}}$* ''

Monge-Kantorovich Duality:

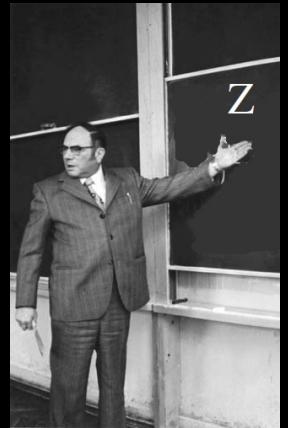
$$\max_{\psi \text{ c-concave}} \left[\int_X -\psi^c(x) d\sigma(x) + \int_{\partial X} \psi(y) d\tau(y) \right] = \inf_{\pi \in SC(\sigma, \tau)} \int_{X \times \partial X} c(x, y) d\pi(x, y)$$

$-\psi^c(x) + \psi(y) \leq c(x, y)$

Kantorovich's Contravariant Singularity Functor IS EXPLICIT.

- c -concavity $\psi^{cc} = \psi$ of a potential $\psi : \partial X[t] \rightarrow \mathbb{R}$ represents a pointwise inequality

$$-\psi^c(x) + \psi(y) \leq c(x, y)$$



for all $(x, y) \in X[t] \times \partial X[t]$,

with equality $\psi(y) - \psi^c(x) = c(x, y)$

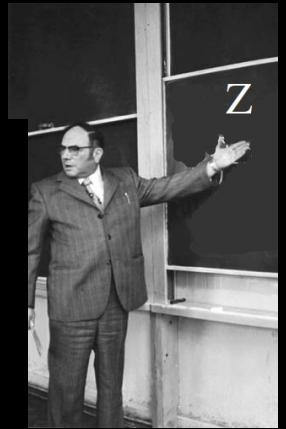
if and only if $y \in \partial^c \psi^c(x)$ iff $x \in \partial^c \psi(y)$

iff $y \in \text{argmax}[\{\psi(y_*) - c(x, y_*) + c(x, y) | y_* \in Y\}]$.

Variational defn. of
 c -subdifferential

Define $Z(\{y\}) := \partial^c \psi(y)$

Kantorovich's Contravariant Singularity Functor IS EXPLICIT.



$Z(Y_I)$ consists of $x \in X$ for which

*Variational defn. of
c-subdifferential*

$\text{argmax}[\{\psi(y_*) - c(x, y_*) + c(x, y) | y_* \in Y\}]$ contains Y_I ,

where $y_0 \in Y_I$ is reference point.

Abbreviate $c_\Delta(x; y, y') := c(x, y) - c(x, y')$ two-pointed cross difference.

Implies equations

$$Z(Y_I) = \{0 = \psi(y_0) - \psi(y) - c_\Delta(x; y_0, y) \mid y, y_0 \in Y_I, y \neq y_0\}$$

Reduces to $\#(Y_I) - 1$ equations. Symmetry y_0, y_1 .

Applications to Algebraic Topology:

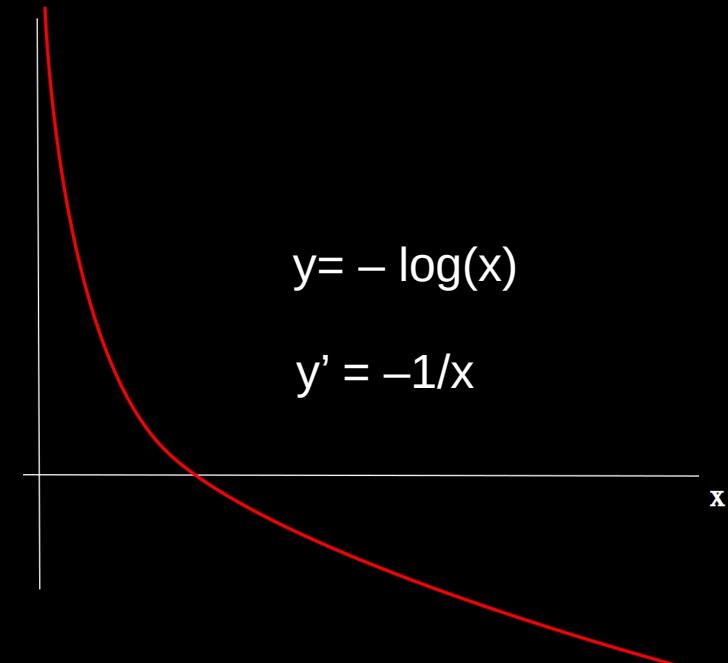
Contravariance says $Z(Y_I)$'s are local cells in $X[t]$, parameterized contravariantly by subsets Y_I of $Y = \delta X[t]$

$$\boxed{\text{if } Y_I \hookrightarrow Y_J. \quad \text{then} \quad Z(Y_I) \hookleftarrow Z(Y_J)}$$

Theorem (Local Reduction)

We find local criterion (UHS conditions) which ensures $Z(Y_I) \hookleftarrow Z(Y_J)$ is homotopy-isomorphism, and construct explicit continuous deformation retracts wherever (UHS) satisfied.

- Proof:
- Variational definition of c-subdifferentials, and gradient flow toward positive poles (not zeros!).
 - Modelled on gradient flow to $+\infty$ of $f(x) = -\log(x)$
 - flow accelerates into the $+\infty$ cusp!



Applications to Algebraic Topology:

If we “skewer the cube diagonally” and filtrate according to dimension, we obtain descending chain of closed subsets

$$X[t] \leftarrow Z\{1\} \leftarrow Z\{2\} \leftarrow Z\{3\} \leftarrow \dots, \dots \quad \text{where } \text{codim } Z\{k\} = k-1$$

-Contravariance implies $Z\{1\}, Z\{2\}, Z\{3\}, \dots$ are homology-cycles in X , $\delta Z\{k\} = 0$ consequence of adjunction formula

Theorem (Global Reduction):

We identify index $J \geq 0$ such that local cells $\{Z(Y_I) \mid Y_I \rightarrow Y\}$ and their local homotopy reductions assemble into global continuous reductions

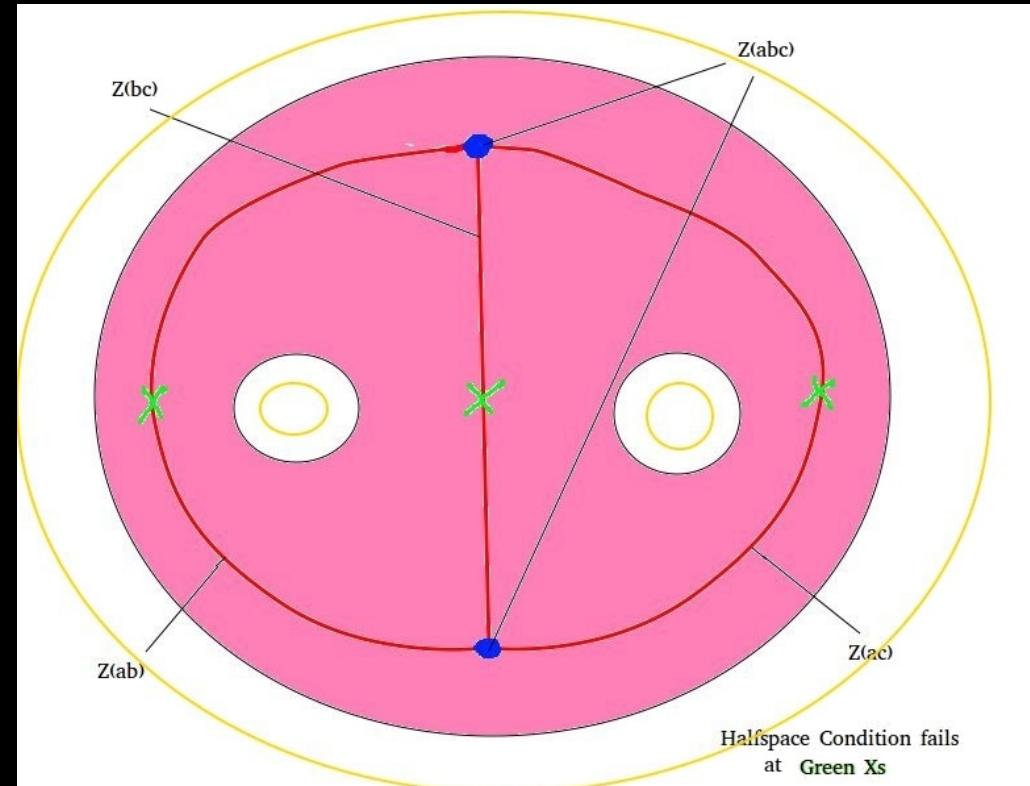
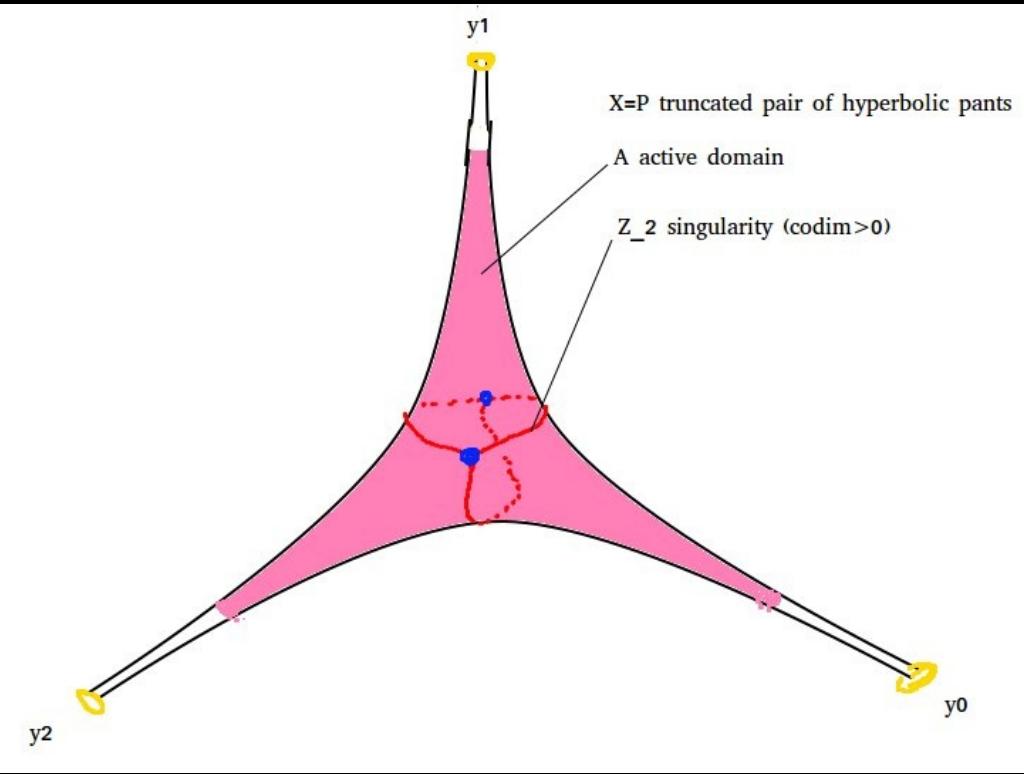
$$X \rightarrow X[t] \rightarrow Z1 \rightarrow Z2 \rightarrow \dots \rightarrow Z\{J+1\}$$

and such that $Z\{J+1\}$ is a codimension- J closed subvariety of X , with inclusion $Z\{J+1\} \rightarrow X$ a continuous homotopy-isomorphism.

Applications need index J be large as possible.
 J defined by max. codimension of cell $Z(Y_I)$ where (UHS) satisfied.

- Best results obtained with anti-quadratic repulsion (visibility) costs.

Θ -graph Singularity of excised pant $P[t] \rightarrow \delta P[t]$



...back to Application of Kantorovich Singularity:

...Spines are not hidden...

~~Spines are readily displayed
in locus of discontinuity Z
of def. retracts $r : X[t] \rightarrow \delta X[t]$~~

Spines readily displayed in
Kantorovich's Contravariant Singularity Functor $Z(\sigma, \tau, c^*)$
of c^* -optimal semicouplings π from $(X[t], \sigma)$ to $(\delta X[t], \tau)$

source target

Given initial geometric Γ -model (X, d, vol) :

Γ -rational Excision $X[t] \times \delta X[t]$

- require Γ -invariant boundary
... but the familiar Gromov visual boundary $X(\infty)$ is not useful
since Γ acts ergodically on $X(\infty)$ with respect to natural Lebesgue measure.
- implies Γ -invariant Radon measures on $X(\infty)$
do NOT descend to Radon measures on $X(\infty) / \Gamma$.

Step (0): Construct Γ -rational excision $X[t]$ with proper-discontinuous boundary $\delta X[t]$.

Obtain: Source ($X[t], \sigma$) ----- >> Target ($\delta X[t], \tau$)

Excision: $X[t] = X - \cup V[t]$ obtained by scooping-out/excising
a countable $\{t\}$ -family of Γ -rational horoballs $V[t]$ in X .

Γ -rationality implies $X[t]$ and boundary $\delta X[t]$ are Γ -invariant subsets
- inherit proper-discontinuous Γ -actions.

Excision: Bieri-Eckmann duality:

Key Property: $\delta X[t]$ has homotopy-type of countable wedge of q-spheres, and

$$D = \tilde{H}_*(\delta X[t])$$
 is Bieri-Eckmann dualizing module (“Steinberg module”)

- Steinberg module D is infinite cyclic $Z\Gamma$ -module generated by spheres at-infinity B (called “Steinberg symbols”)

$X[t]$ contractible + LES relative homology + NPC ==>>

$$\partial : H_{q+1}(X[t], \partial X[t]) \xrightarrow{\sim} \tilde{H}_q(\partial X[t])$$

*isomorphism with inverse
flat-filling*

$FILL[B] \qquad B \qquad FILL = \delta^{\wedge \{-1\}}$

Homological duality:

- Steinberg symbols are relative cycles $\text{FILL}[B] \in H_{q+1}(X[t], \partial X[t])$
- Bieri-Eckmann duality implies $\text{FILL}[B]$ is dual cycle to minimal spines fundamental class.
- $\dim(\text{FILL}[B])$ is max codimension of minimal spine (homological duality formula)
- Observe $\text{FILL}[B]$ deformation retracts to $\{\text{pt}\}$ $\text{FILL}[B] \rightarrow \{\text{pt}\}$.

Our goal: continuously interpolate/extend the local retractions $\text{FILL}[B].\gamma \rightsquigarrow \{\text{pt}\}$, $\gamma \in \Gamma$ throughout $X[t]$ such that the singularity-to-reduction $X[t] \rightsquigarrow \mathcal{Z}$ has exact minimal dimension $cd\Gamma$.

Interpolation Problem solved

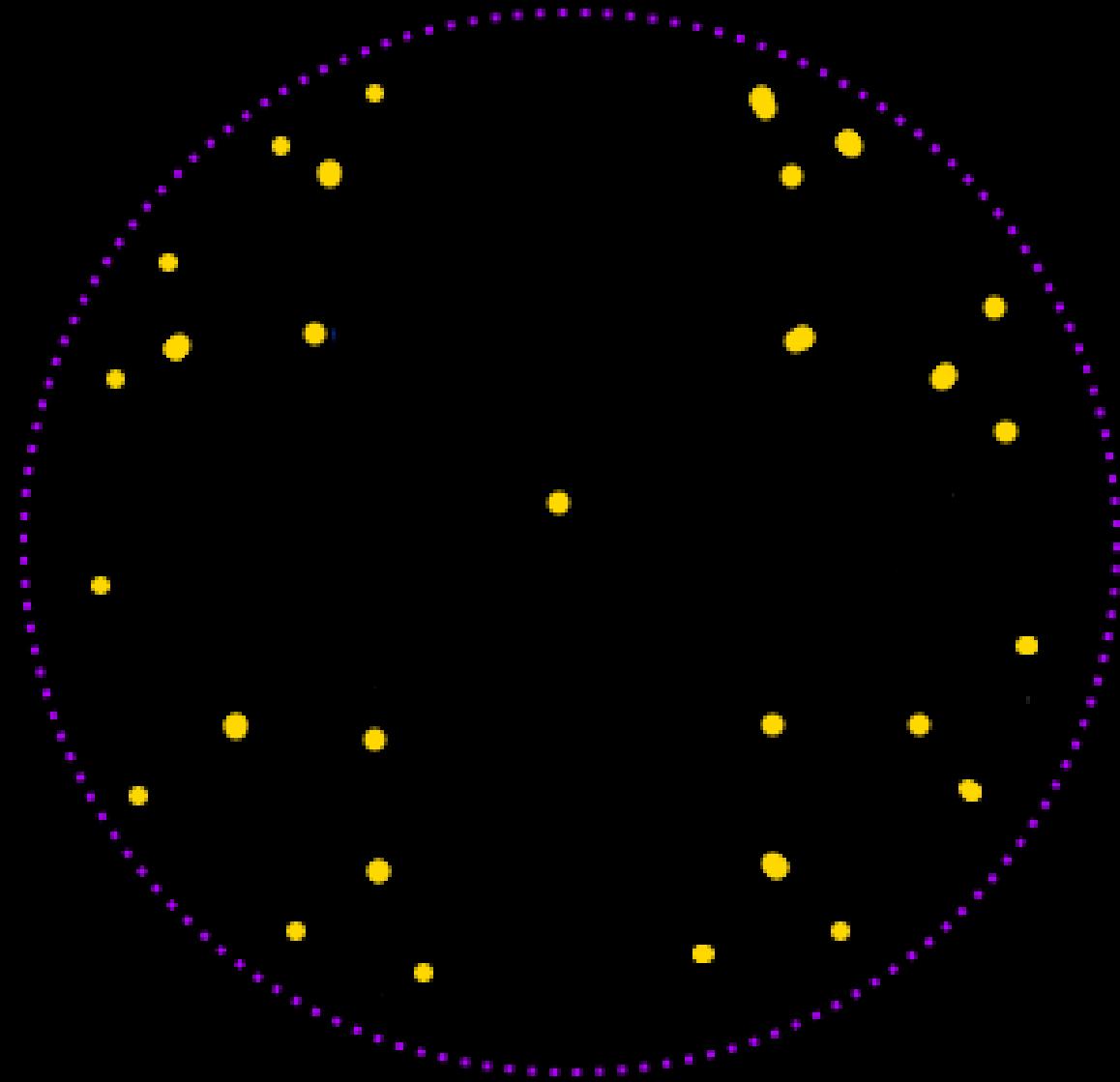
with Two Pointed-(Visibility)-Repulsion cost $c=v$,

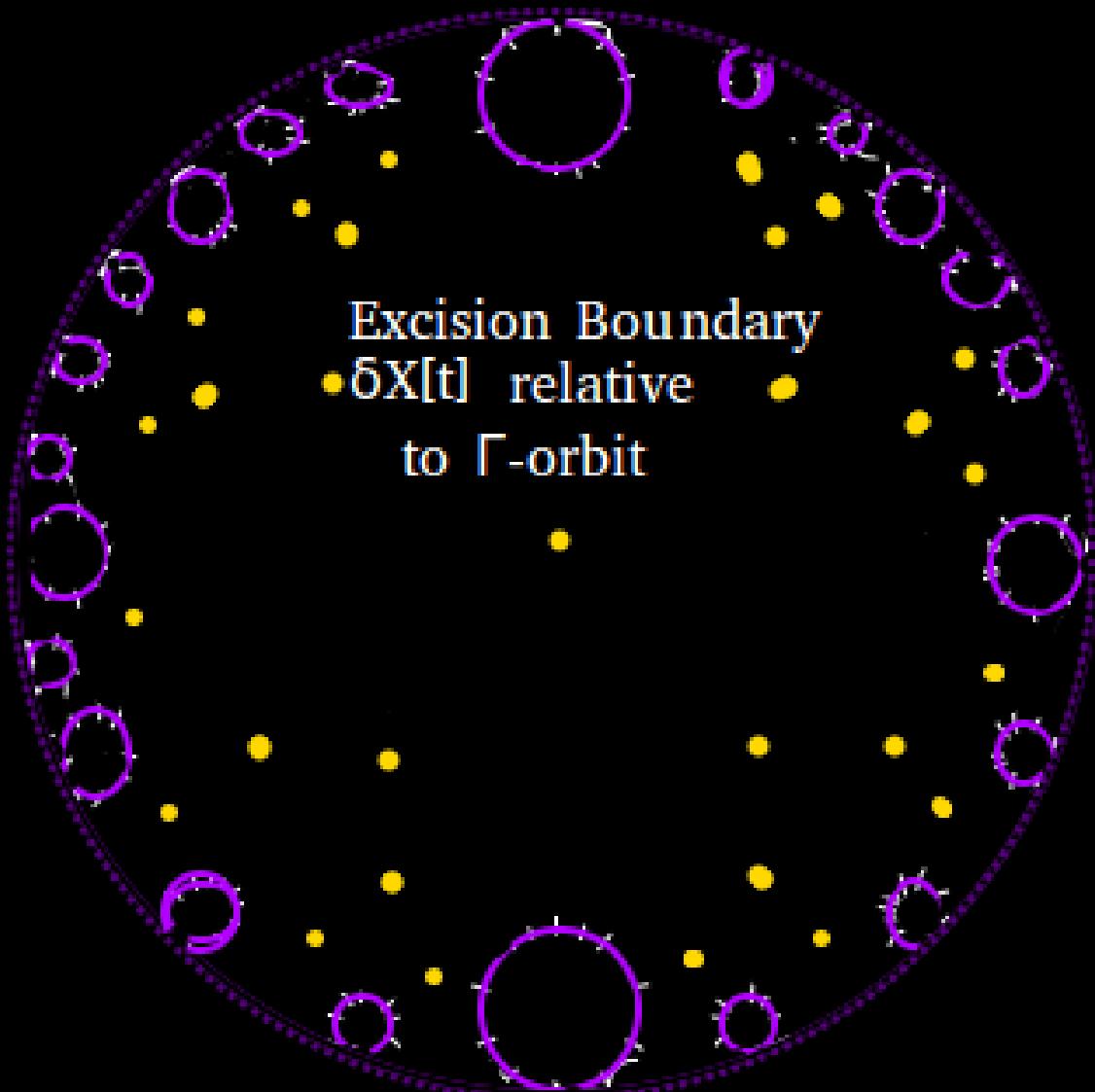
and Kantorovich Functor $Z=Z(\sigma, \tau, v)$ on

Γ -rational excision $X[t], \delta X[t]$

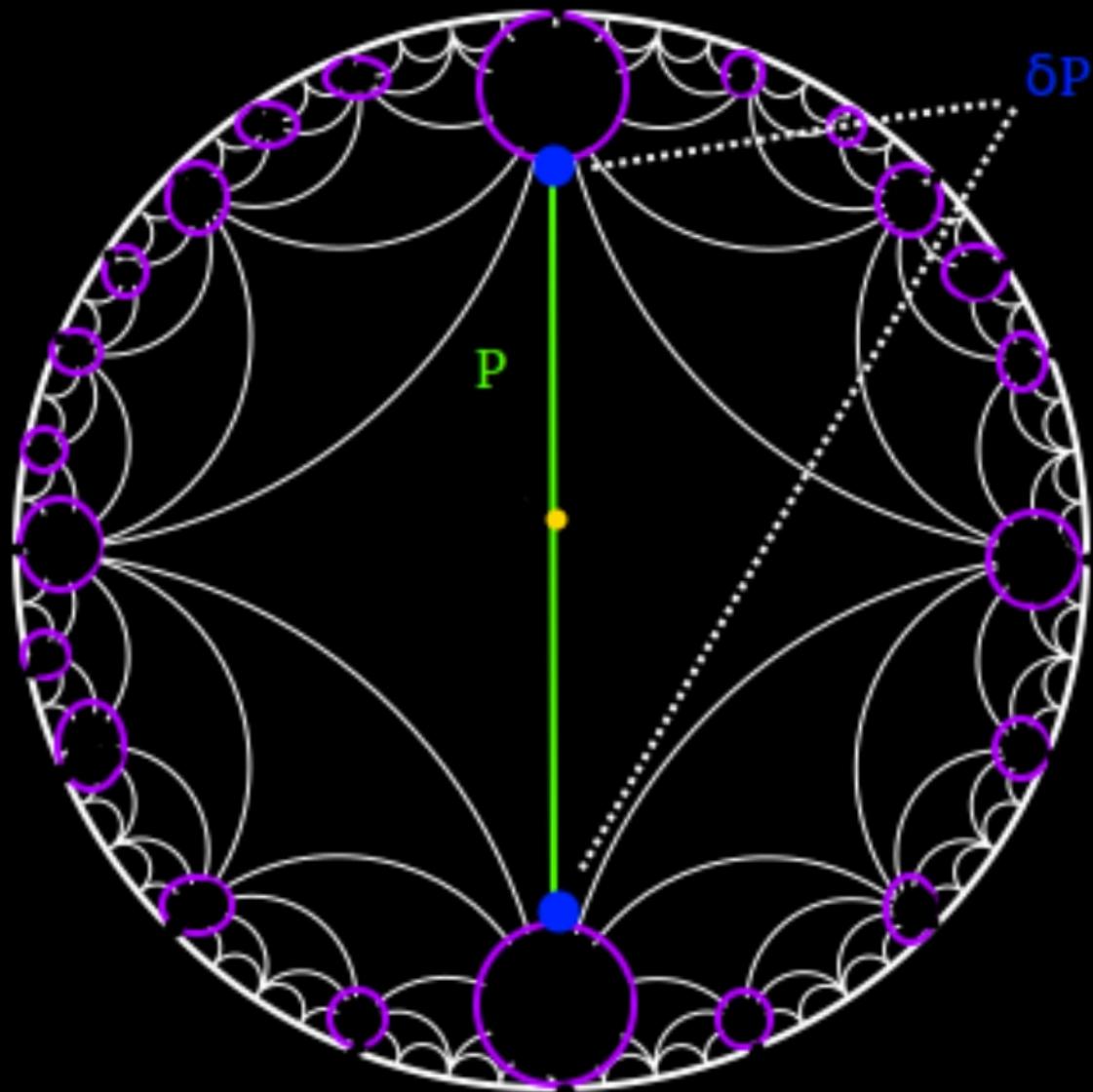
...to illustrate the Interpolation Problem:

Initial geometric $E\Gamma$ -model X

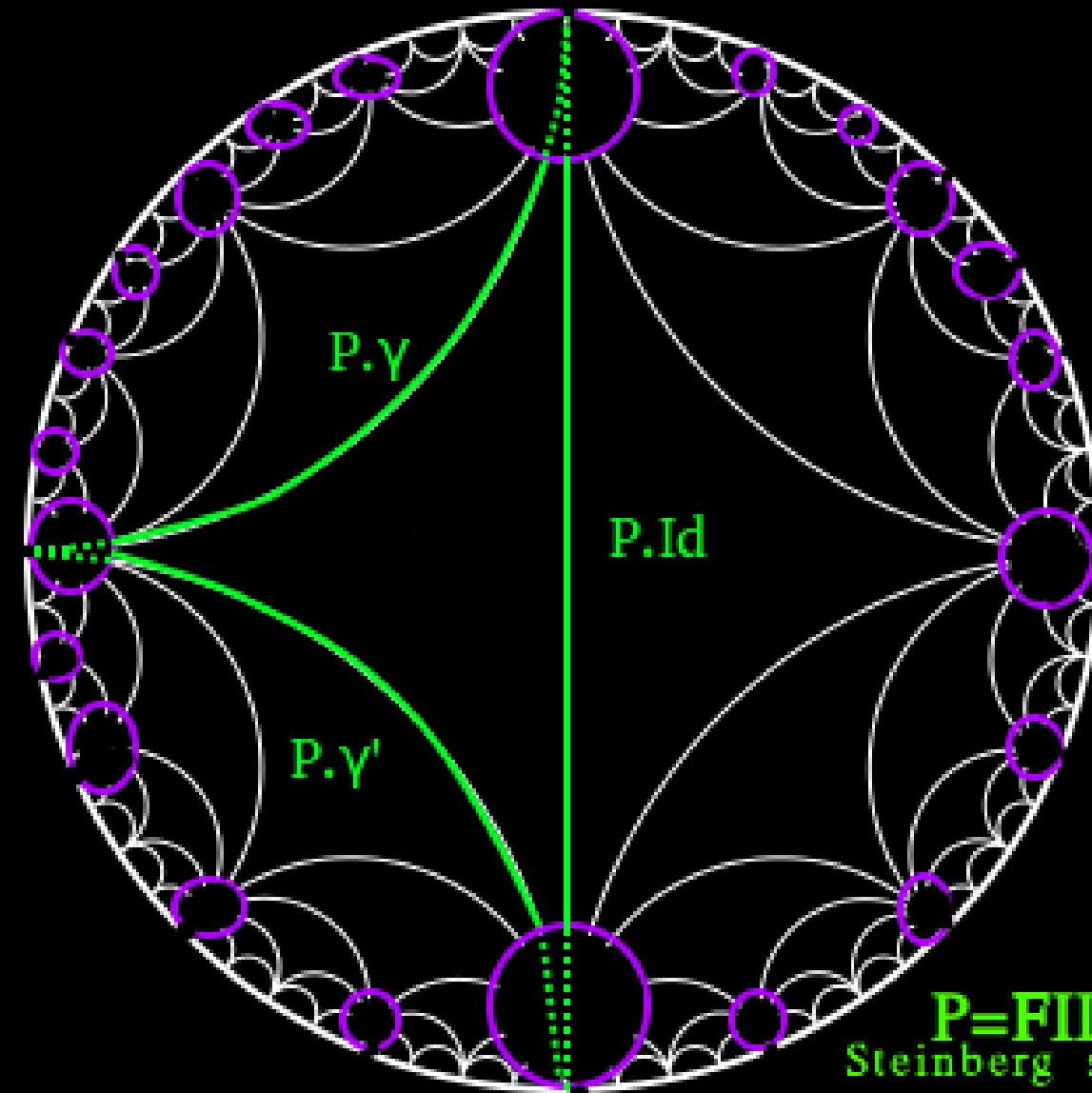




Steinberg symbol P is relative 1-cycle; with boundary δP a boundary 0-sphere.



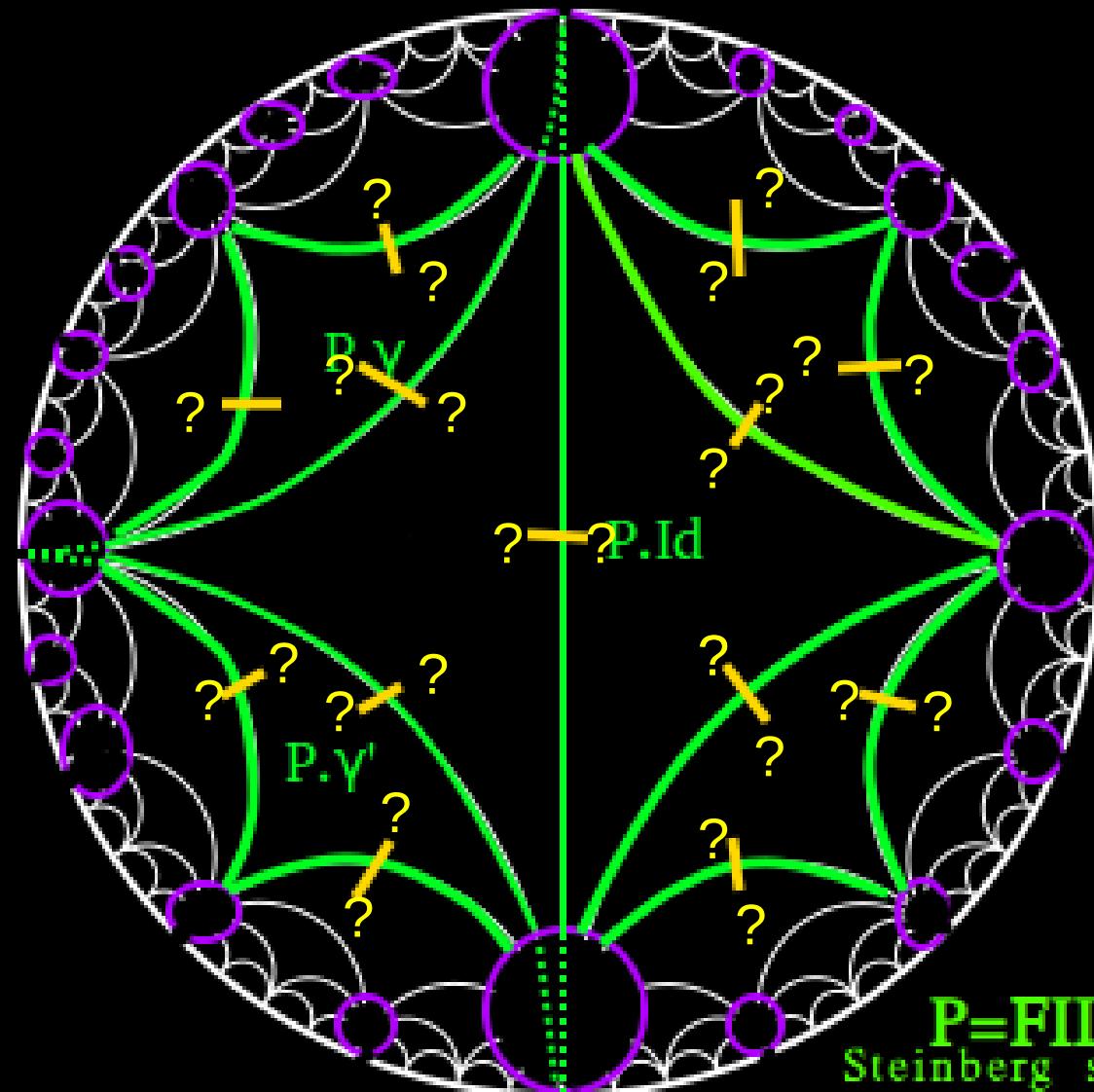
The subset $\{Id, \gamma, \gamma'\}$ successfully
Closes the Steinberg symbol.



Homological duality ==>>
Minimal spine is dual to
Steinberg symbols $P, P.\gamma, \gamma \in \Gamma$.

¿¿ How to assemble/interpolate
the local reductions
 $\{P.\gamma \rightarrow \{pt\}, \gamma \in \Gamma\}$

to obtain a Spine
throughout $X[t] ??$

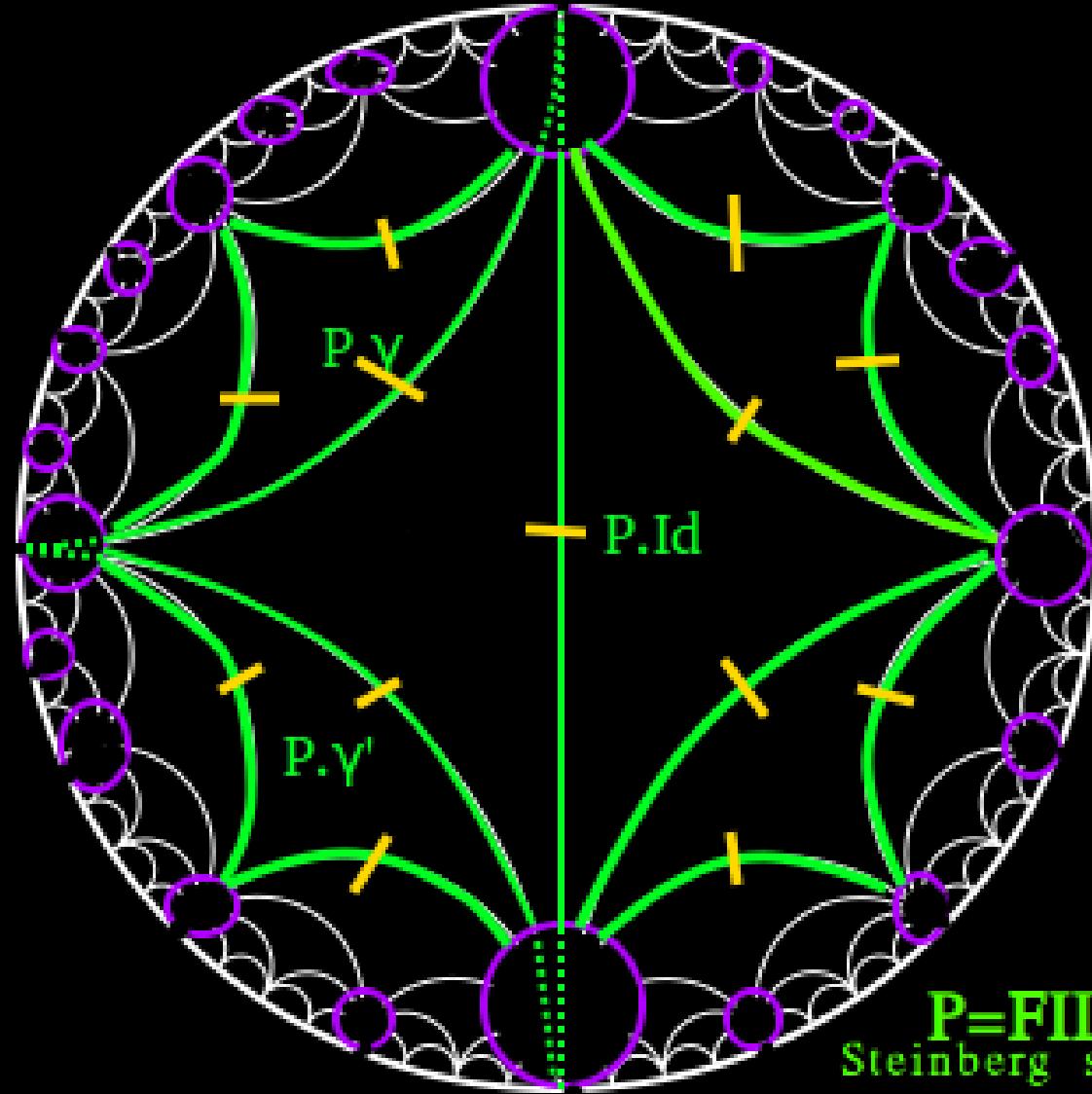


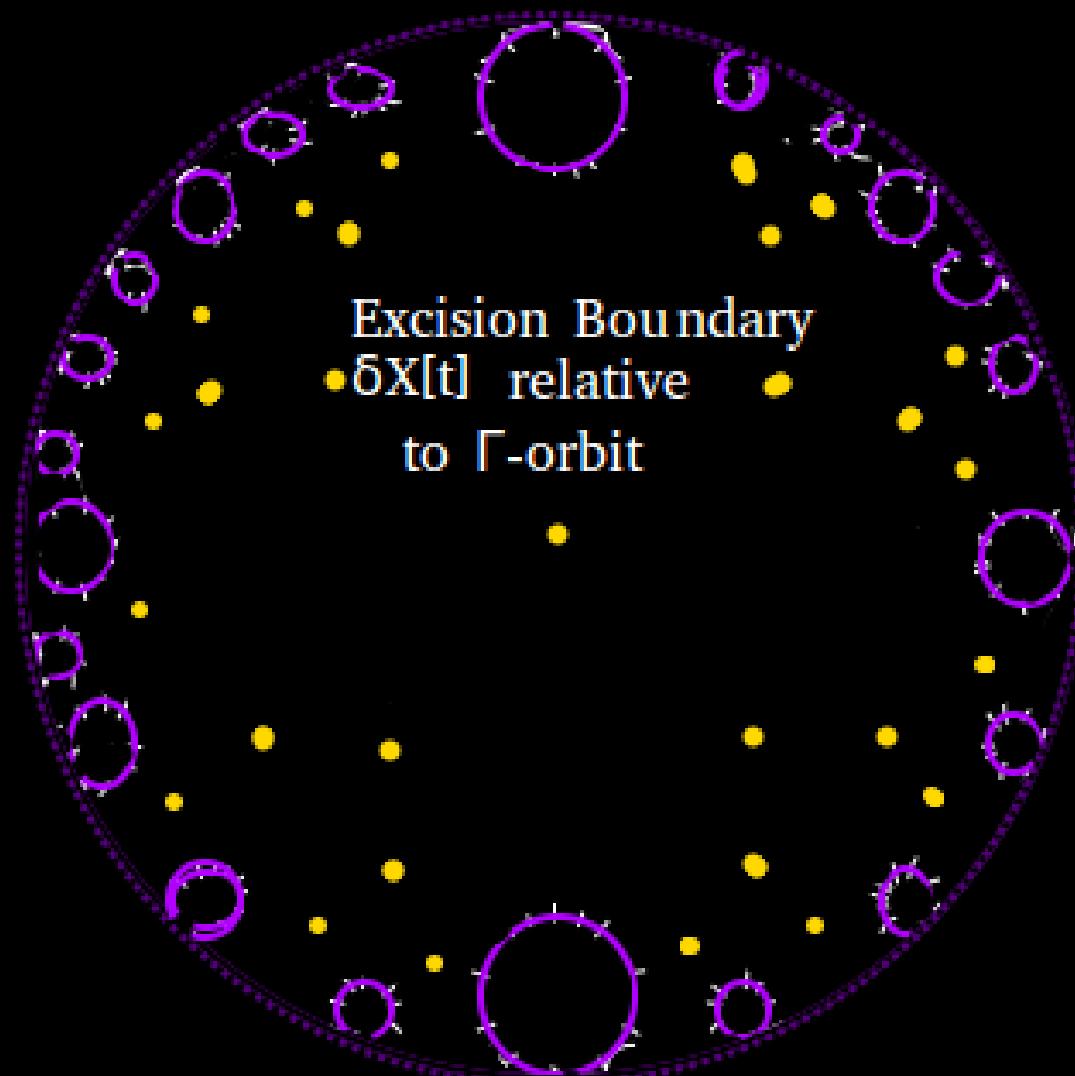
Homological duality ==>
Minimal spine is dual to
Steinberg symbols $P, P.\gamma, \gamma \in \Gamma$.

How to assemble/interpolate
the local reductions
 $\{P.\gamma \rightarrow \{pt\}, \gamma \in \Gamma\}$

to obtain a Spine
throughout $X[t] ??$

Answer:
Kantorovich Singularity functor
 $Z(\sigma, \tau, v)$ of two-pointed
visible repulsion cost
 $v: X[t] \times \delta X[t] \rightarrow R$





A diagram of a brain slice, represented by a black circle with a white grid pattern. A large, irregularly shaped light blue area is centered in the middle. Inside this blue area, the text "Activated Domain
of v-optimal
Semicoupling" is written in red. Several purple circles are scattered across the brain slice, some containing small green dots. Green arrows point from the purple circles towards the center of the blue area. A yellow arrow points from the bottom left towards the center. A yellow "C" is located at the bottom right.

Activated Domain
of v-optimal
Semicoupling



Singularity functor $Z=Z(\sigma, \tau, c)$ depends on choice of cost c : $X[t] \times \delta X[t] \rightarrow R$,

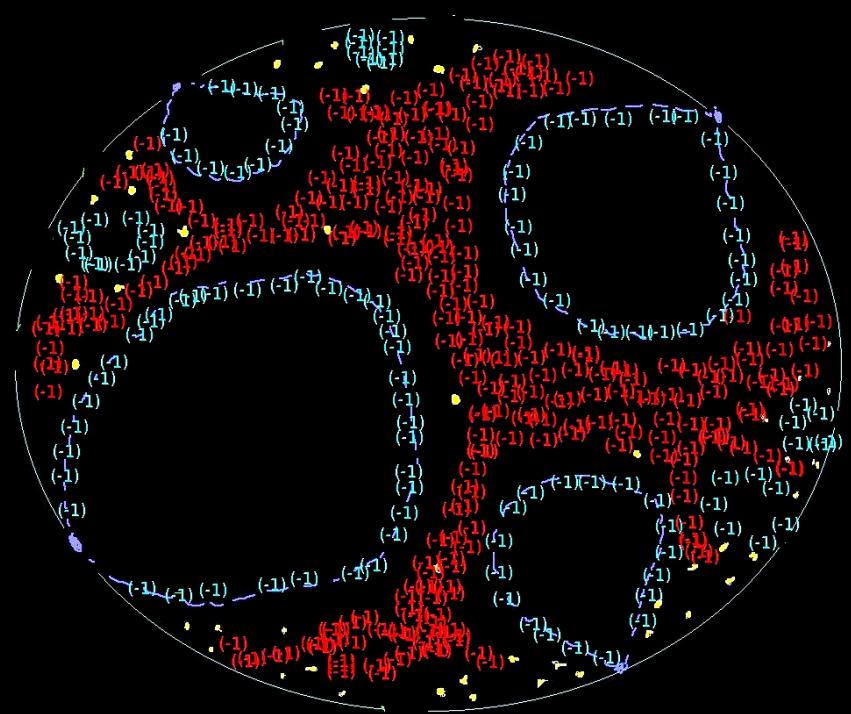
Measures cost of transporting from unit source mass to a unit target mass.

- Best topological results obtained with anti-quadratic repulsion costs.

- Introduce new costs:

$c=c^*$ (two-pointed repulsion cost)

$c=v$ (two-pointed visible repulsion cost)



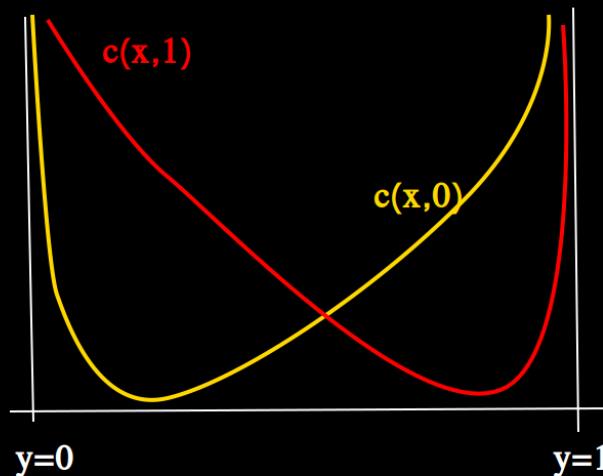
One-dimensional repulsion cost:

Unit interval $I = [0,1]$, boundary $\delta I = \{0,1\}$

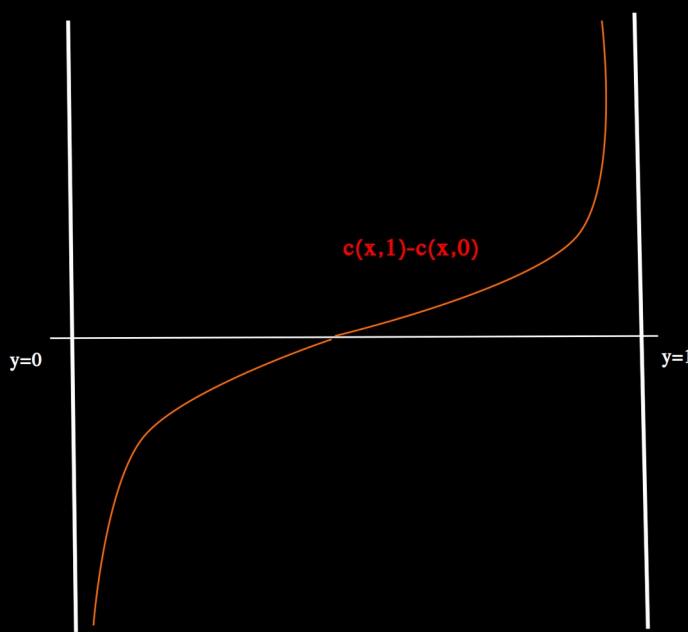
$$c(x,0) = |x|^{-2} + 2|1-x|^{-2}$$

$$c(x,1) = 2|x|^{-2} + |1-x|^{-2}$$

$$c(x,1) - c(x,0) = |1-x|^{-2} - |x|^{-2}$$



- cross-difference is critical-point free!
(critical points at poles $x=0,1$)
- every fibre is connected.



Consider closed unit interval $X = [0,1]$ with boundary $\delta X = \{0, 1\}$.

σ is uniform distribution of (-1) charges. $\text{mass}(\sigma) = 15(-)$
 τ is uniform distribution of (-1) charges. $\text{Mass}(\tau) = 4(-)$

$\boxed{\text{mass}(\sigma) >> \dots >> \text{mass}(\tau)}$

(-1)
(-1)
(-1) (-1) (-1) (-1) (-1) (-1) (-1) (-1) (-1) (-1) (-1) (-1) (-1) (-1)



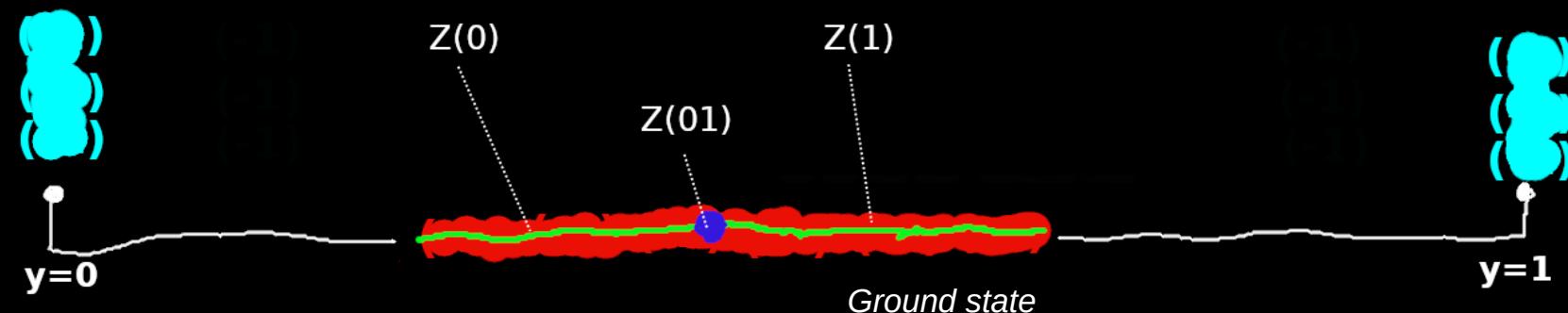
Ground state

Consider closed unit interval $X = [0,1]$ with boundary $\delta X = \{0, 1\}$.

σ is uniform distribution of (-1) charges. $\text{mass}(\sigma) = 15(-)$
 τ is uniform distribution of (-1) charges. $\text{Mass}(\tau) = 6(-)$

mass(σ) >> mass(τ)

(-1)
(-1)
(-1)
(-1) (-1) (-1) (-1) (-1) (-1) (-1) (-1) (-1) (-1) (-1) (-1) (-1) (-1)



Consider closed unit interval $X = [0,1]$ with boundary $\delta X = \{0, 1\}$.

σ is uniform distribution of (-1) charges. mass(σ) = 15(-)

τ is uniform distribution of (+1) charges. Mass(τ) = 6(+)

mass(σ) > mass(τ)

(+1)
(+1)
(+1)

(+1)
(+1)
(+1)

(-1) (-1) (-1) (-1) (-1) (-1) (-1) (-1) (-1) (-1) (-1) (-1) (-1) (-1)

()
()
()

() () ()
y=0

Z(0)

Z(01) = {empty}

Z(1)

() () ()
y=1

Ground state

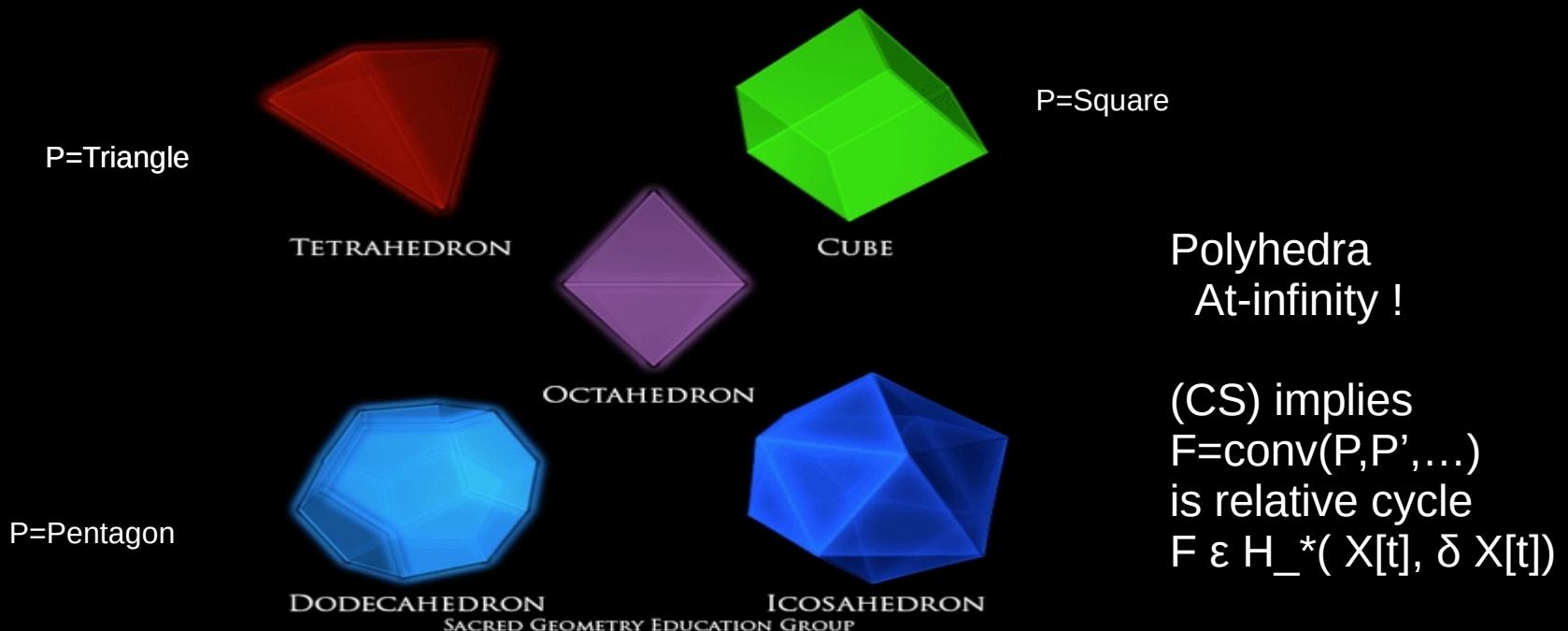
Ground state

Closing Steinberg (CS):

In low dimensions (CS) is the problem of stitching a football from collection of panels $\{P\}$.

Ex: Regular platonic solids solve (CS) with panels $P = \text{triangle, square, pentagon}$.

THE FIVE PLATONIC SOLIDS



Closing Steinberg:

In our applications the panels $\{P\} = \{\text{FILL}[B].y \mid y \in \Gamma\}$ are flat-filled Steinberg symbols.

- Seek finite subset I of Γ such that panel translates $\{P.y \mid y \in I\}$ assemble to “closed football”

Formal definition: (CS1) the chain sum $\sum_{\gamma \in I} P.\gamma \neq 0 \pmod{2}$ (nontrivial over $\mathbb{Z}/2$ coefficients).

(CS2) the chain sum $\sum_{\gamma \in I} \partial P.\gamma = 0 \pmod{2}$ (vanishing boundary over $\mathbb{Z}/2$ coefficients).

(CS3) there exists $x \in X[t]$ which is simultaneously visible from $P.\gamma, \gamma \in I$, in $X[t]$ (well-defined closed convex hull).

(CS4) if we define $F := \overline{\text{conv}}\{P.\gamma \mid \gamma \in I\}$, then the convex chain sum $\underline{F} = \sum_{\gamma \in \Gamma} F.\gamma$ has well-separated gates structure with gates $\{G\} = \{P.\gamma \mid \gamma \in \Gamma\}$. (well-separated gates)

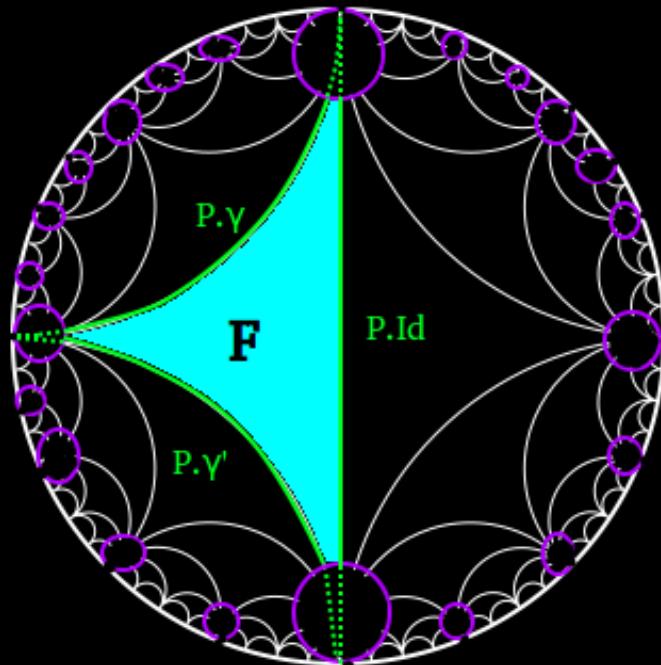
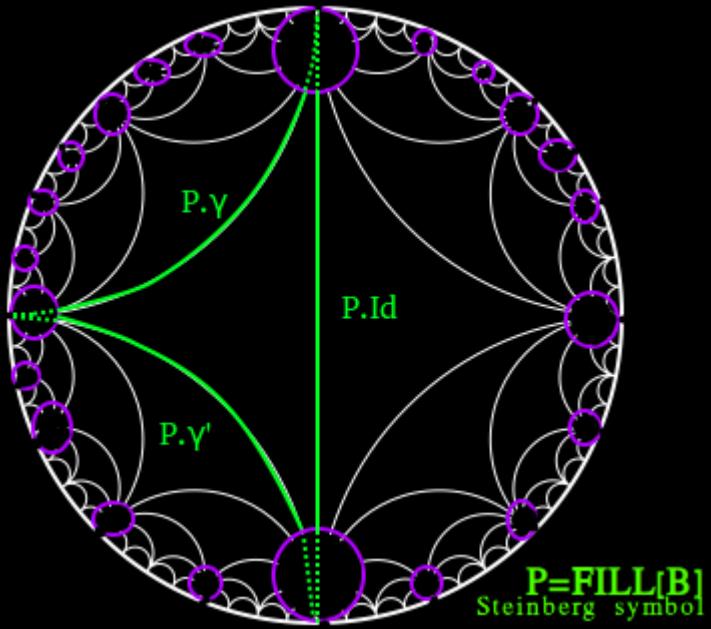
Closing Steinberg:

CS1, CS2 ==> constructing nonzero $\xi \in H_0(\Gamma, \mathbb{Z}_2\Gamma \times D)$.

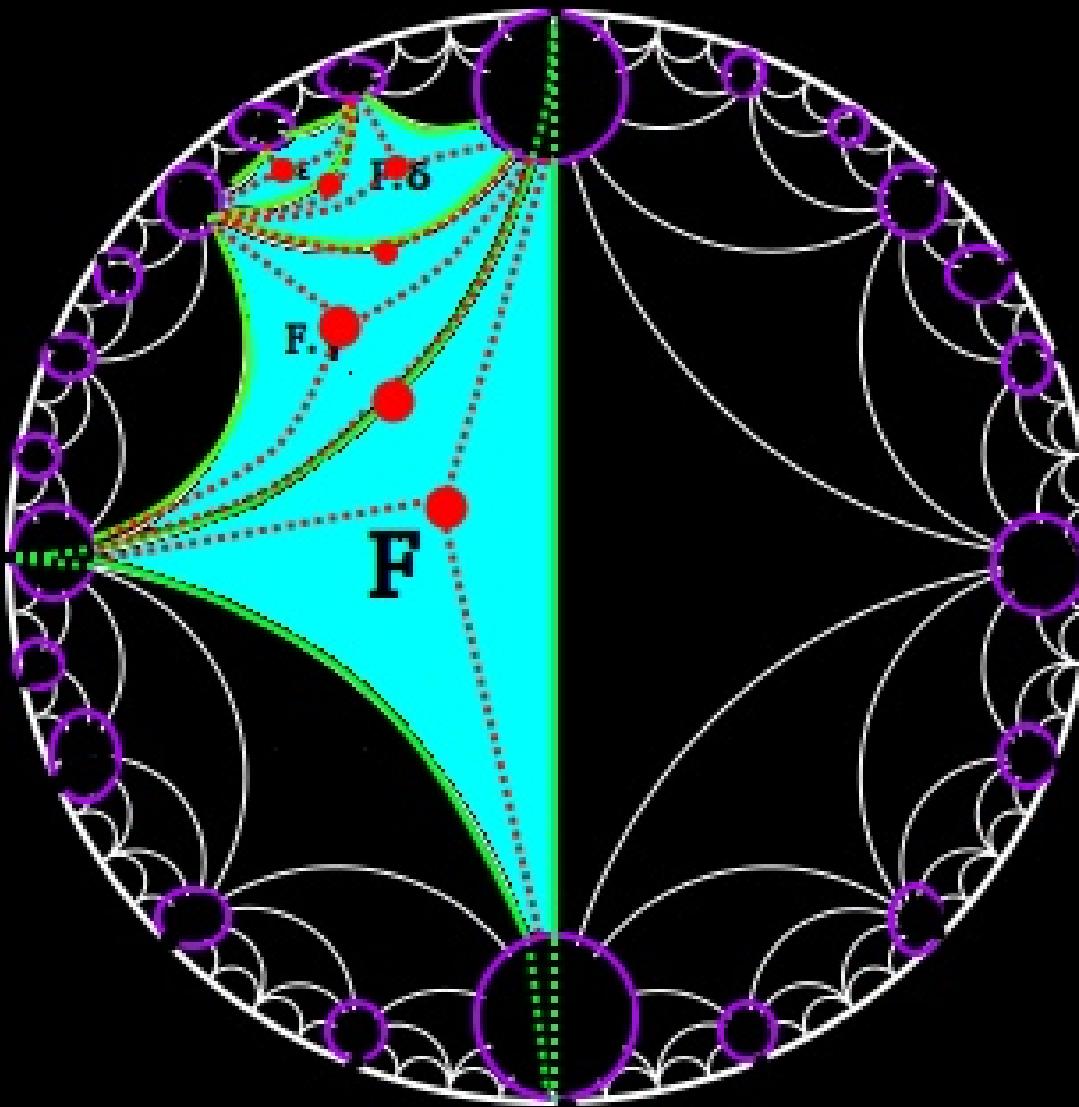
- equivalent to a syzygy in projective homological resolution of D.

CS4 ==> the translates F, Γ define chain sum $E = \sum F.y$

- Γ acts on chain summands of E like “shift operator” equivalent to right-action $\Gamma \times \Gamma \rightarrow \Gamma$



Chain sum F with well-separated gates:

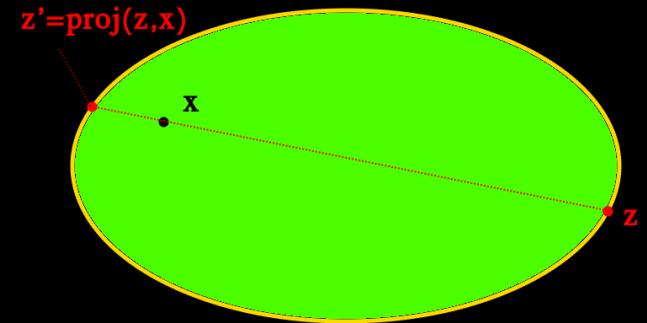


CS replaces $X[t]$ with a chain sum E , a type of partition of unity.

Well-separated gates ==>
summands F, F' have
either trivial intersection
or $F \cap F' = P$

Two-pointed Repulsion cost c^* :

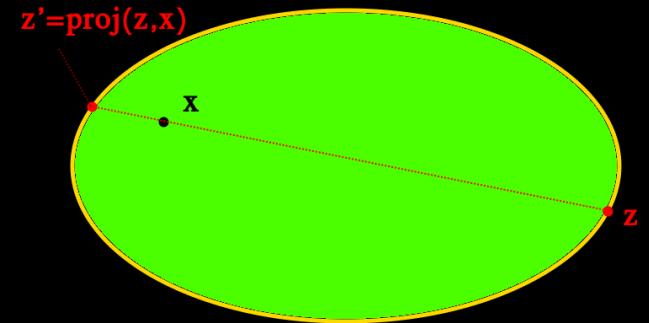
$$c^*(x, z) := \frac{1}{2} \cdot \text{dist}(x, z)^{-2} + \text{dist}(x, \text{proj}(z, x))^{-2}$$



- the 2-pointed repulsion cost c^* leads to singularity functor $Z(\sigma, \tau, c^*)$
- Our proposal is finally that (UHS) conditions are satisfied up to index **J=dim(D)** , and Kantorovich singularity equivariantly homotopy-reduces the source X to minimal spine Z .

Two-pointed Repulsion cost c^* :

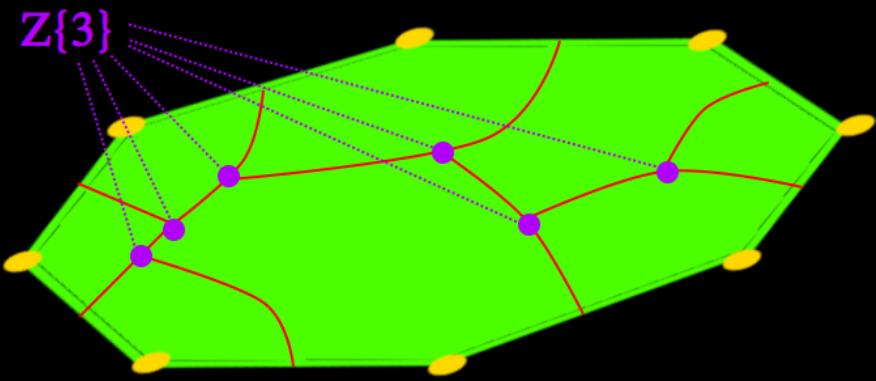
$$c^*(x, z) := \frac{1}{2} \cdot \text{dist}(x, z)^{-2} + \text{dist}(x, \text{proj}(z, x))^{-2}$$



- c^* extends to repulsion cost on chain sum E with well-separated gates $\{G\}$
- gates G geodesically convex implies c^* is continuous interpolation of the restricted 2-pointed repulsion costs $c^*|G$
- Singularity structures $Z(\sigma, \tau, c^*)$ are continuous interpolation of restricted singularity structures $Z(\sigma, \tau, c^*|G)$, over gates $\{G\}$ of E

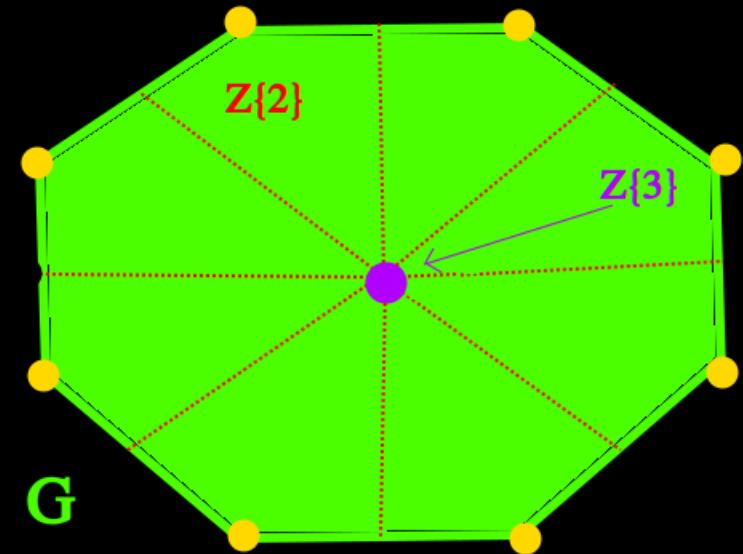
... Closing Steinberg (CS) and Interpolating Local Reductions:

- in our applications the gates $\{G\}$ of the chain sum E will coincide with the Γ -orbit of Steinberg symbols $P.y = \text{FILL}[B].y$, $y \in \Gamma$.
- the singularity structures of the two-pointed repulsion costs c and visibility cost produce homotopy-reductions of the gates $G=P$ to points, $G \rightarrow \{\text{pt}\}$.

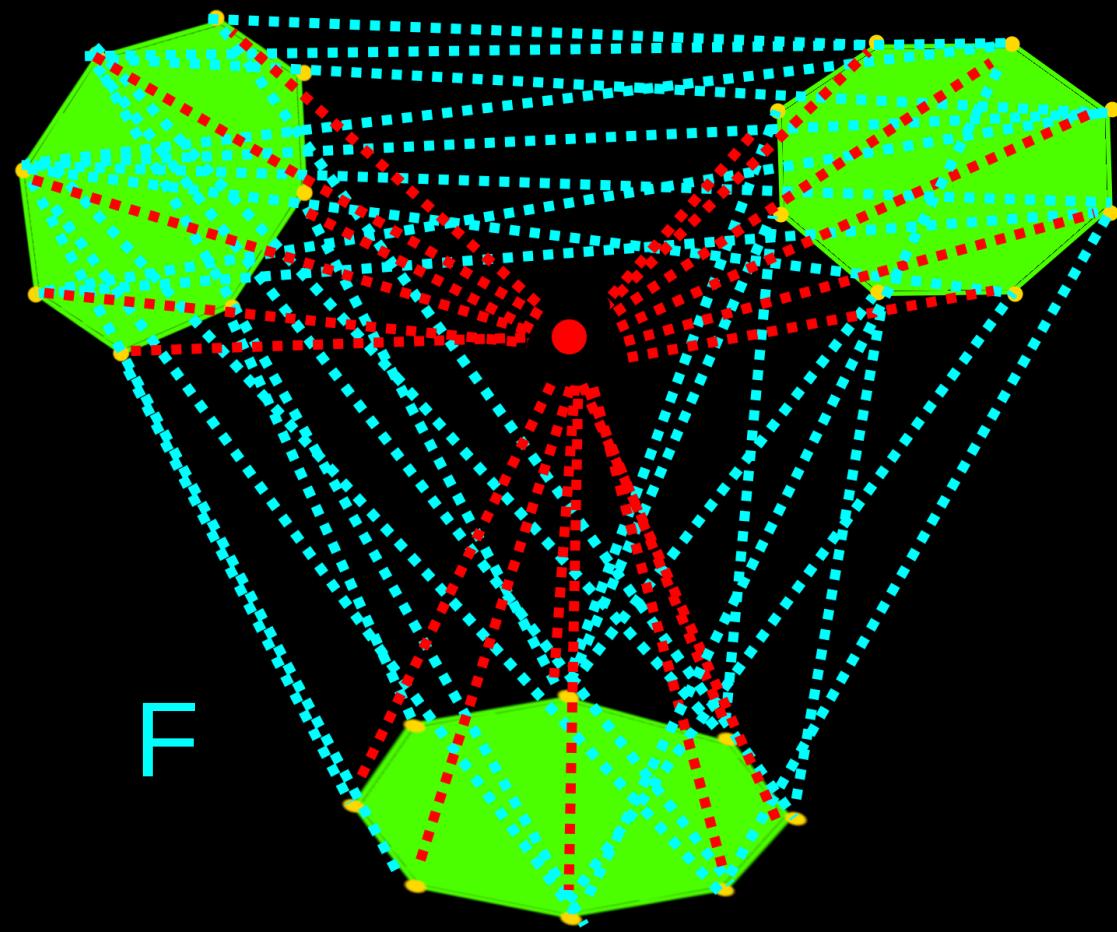


G deformation retracts to Z{2}

- generic convex panels **G** deformation retract to **Z{2}**, and **Z{3}** is disconnected.



- gate **G** with large symmetry implies **G** reduces to **Z{3}={pt}**



*Successfully Closing Steinberg symbol
replaces X with a convex chain sum*

$$E = \sum F.y$$

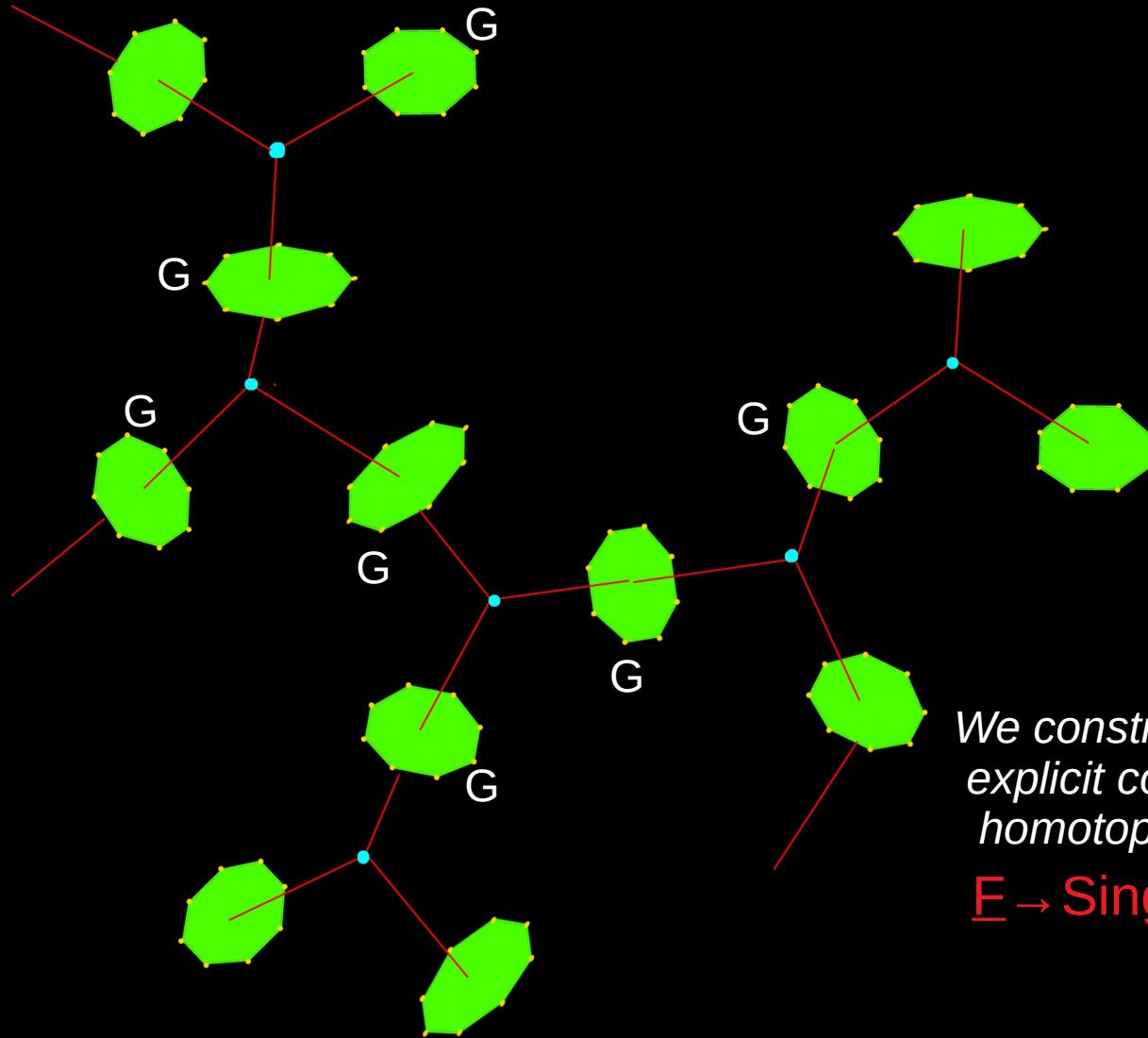
where $F := \text{conv}[P, P.y, P.y', \dots]$

- Hypotheses of (CS) implies:
the convex hull F is well-defined,

and

- translates $F.y, y \in \Gamma$, have well-separated gates $\{G\}$ coincident with Γ -orbit of Steinberg symbols
 $G=FILL[B]$

c^* -optimal
semicouplings
from source $X = \underline{E}$
to target $Y = E[\underline{E}]$



We construct
explicit continuous
homotopy-reductions
 $\underline{E} \rightarrow \text{Singularity } Z\{2\}$

Given initial geometric $E\Gamma$ model X ,

- we construct excision $X[t]$ with boundary $\delta X[t]$
- we Close Steinberg and replace $X[t]$ with chain sum E
- we install two-pointed visible repulsion cost $v: X[t] \times \delta X[t] \rightarrow \mathbb{R}$

Then Kantorovich's Singularity functor $Z = Z(\sigma, \tau, v)$ constructs explicit homotopy-reductions

$X \rightarrow X[t] \rightarrow Z\{1\} \rightarrow Z\{2\} \rightarrow \dots \rightarrow Z\{J+1\}$, where index $J > 1$ depends on (UHS) conditions.

We conjecture (and prove in special cases) that costs c^* , v satisfy sufficient (UHS) conditions:

*If $X[t]$ is rational excision model of initial geometric $E\Gamma$ -model X ,
then $Z(\sigma, \tau, v)$ satisfies sufficient (UHS) conditions up to $J = \dim(D)$
such that $Z\{J+1\}$ is explicit minimal spine.*

The End.



Thank you.