

1. MEDIAL AXIS TRANSFORM

The purpose of this section is to compare some familiar properties of the medial axis transform $A \mapsto M(A)$ (introduced by [Blu67]) with the singularity structures formalized in our Kantorovich contravariant functor $Z : 2^{\partial A} \rightarrow 2^A$ (introduced in [Mar]). To compare the functors Z with medial axis transform requires we interpret the inclusion $M(A) \hookrightarrow A$ in the category of mass transportation.

Let A be a bounded open subset of \mathbb{R}^N . The medial axis $M(A)$ introduced by Blum consists of all $x \in A$ for which $\text{dist}(x, \partial A)$ is attained by at least two distinct points,

$$(1) \quad M(A) := \{x \in A \mid \#\text{argmin}_{y \in \partial A} \{d(x, y)\} \geq 2\}.$$

A long-known “folk theorem” states that the inclusion $M(A) \hookrightarrow A$ is a homotopy-isomorphism, and even a strong deformation retract. This implies $M(A)$ contains all the topology of A , and a connected subset whenever A is. A formal proof is established [Lie04]. We do not know if $M(A)$ is a strong retract for more general Riemannian spaces (X, d) , and a search through the literature does not address the question. But our recent thesis [Mar] contains some results, namely “Theorem B”, identify conditions for which inclusions denoted $Z_2 \hookrightarrow A$ are homotopy isomorphisms, even strong deformation retracts. This subvariety Z_2 is derived from a contravariant functor $Z = Z(\mu, \nu, c)$ defined by mass transport data (μ, ν, c) . The medial axis $M(A)$ and Z_2 will rarely coincide set-theoretically, but this present note demonstrates they are usually topologically isomorphic.

The medial axis transform corresponds to a “degenerate” transport problem in the following sense: if $A \hookrightarrow \mathbb{R}^N$ is bounded open subset, then we nominate

$$(2) \quad \mu := \frac{1}{\mathcal{H}_A[A]} \mathcal{H}_A$$

as the canonical probability measure on the source A . Consider the probability measures π on $A \times \partial A$ for which $\text{proj}_A \# \pi = \mu$ and with unconstrained second marginal $\text{proj}_{\partial A} \# \pi$. Here $\text{proj}_A, \text{proj}_{\partial A}$ are the canonical projections $A \times \partial A \rightarrow A, \partial A$. The set-mapping

$$(3) \quad T : x \mapsto \text{argmin}_{y \in \partial A} \{d(x, y)\}, \quad \text{for } x \in A,$$

defines a measurable set-valued map $T : A \rightarrow \partial A$. The pushforward

$$(4) \quad \nu := T \# \mu$$

is a probability measure on ∂A with $\text{spt}(\nu) = \partial A$. With respect to, say, quadratic cost $c = d^2/2$ or distance cost $c = d$, the map $x \mapsto T(x)$ defines a c -optimal transport from μ to ν , with c -optimal coupling $\pi = (Id \times T) \# \mu$ on $A \times \partial A$.

Finally $M(A)$ coincides with the locus-of-discontinuity of $T : A \rightarrow \partial A$, or more specifically the singularity Z_2 defined by Kantorovich's contravariant functor $Z = Z(\mu, \nu, d) : 2^{\partial A} \rightarrow 2^A$. Thus we arrive at an instance where $M(A) = Z_2$ for the specific coupling program defined by μ, ν, c . This identification suggests the following generalization of medial axis transform: for general probability measures $\nu \in \Delta(\partial A)$ on the boundary of A , we may study the c -optimal couplings π from μ to ν , and obtain a Singularity functor $Z(\mu, \nu, c)$. The generalized medial axis in this setting is Z_2 , i.e. the “locus-of-discontinuity” of the c -optimal transport π from μ to ν .

Our thesis developed a Reduction-to-Singularity principle, and identifies conditions for which, say, the inclusion $Z_2 \hookleftarrow Z_1$ is a homotopy-isomorphism. In the above setting with $Z = Z(\mu, \nu, c)$, we have $A = Z_1$, $M(A) = Z_2$, and naturally we inquire whether the hypotheses of our topological theorems are satisfied for any particular costs c .

If we fix $c = d^2/2$, then our Theorem B takes the following form. For $x \in A = Z_1$, let $y_0 := T(x)$. Then define

$$\eta(x, y) := |c(x, y) - c(x, y_0)|^{-1/2} \cdot \nabla_x(c(x, y_0) - c(x, y)), \text{ for } y \in \partial A - \{y_0\}.$$

Observe that $c(x, y) - c(x, y_0) > 0$ is nonvanishing throughout $A - M(A)$ in the above notations. Our Theorem B requires the following hypotheses (6), (7) be satisfied for $x \in A - M(A) = Z_1 - Z_2$: the averaged Bochner integral defined as

$$(5) \quad \eta(x, \text{avg}) := (\nu[\partial A - \{y_0\}])^{-1} \cdot \int_{\partial A - \{y_0\}} \eta(x, y) d\nu(y),$$

and we require that

$$(6) \quad \eta(x, \text{avg}) \text{ is nonzero finite tangent vector,}$$

and there exists a constant $C > 0$ such that

$$(7) \quad \|\eta(x, \text{avg})\| \geq C > 0$$

for $x \in A - M(A)$, uniformly with x . The verification of hypotheses (6)–(7) can be difficult to verify. Evidently (7) implies (6).

We need also remark on a complication arising from the nonconvexity of A . What is the natural distance function d on $A \subset \mathbb{R}^N$, and the physical “transport cost” of a unit mass at $x \in A$ to target mass $y \in \partial A$? There are at least two popular possibilities. First we may restrict the ambient euclidean distance $d_{\mathbb{R}^N}(x, y) = \|x - y\|$ to $A \times \partial A \subset \mathbb{R}^N \times \mathbb{R}^N$. But this restriction does not represent a path length distance in the sense of Gromov [Gro+01, 1.A-B]. In otherwords the restriction does not represent geodesic transport in A , and there is no variational description of the metric in terms of shortest-length curves.

A second approach defines $d = d_A$ as the induced length distance defined by

$$d_A(x, y) = \inf_{\gamma} \int_{\gamma} \text{Length}(\gamma),$$

where the infimum is over all curves $\gamma : [0, 1] \rightarrow A$ contained in A with $\gamma(0) = x$, $\gamma(1) = y$. The reader will observe that both possibilities define coincident medial axes $M(A)$ according to (1), since euclidean balls are geodesically convex. The induced length distance $d = d_A$ is possibly most preferred by metric geometers, yet is difficult to numerically evaluate. Moreover geodesics with respect to the induced path distance $c = d_A$ can oftentimes be branching. The possible branching of geodesics implies gradients $y \mapsto \nabla_x d(x, y)$ are noninjective maps $\partial A \rightarrow T_x A$ for $x \in A$. This possible noninjectivity violates an important transport condition called (Twist), and is obstruction to hypothesis (6). Thus neither the restricted distance $c = d|_{A \times \partial A}$ nor the induced distance $c = d_A$ are especially convenient costs.

We article explores a third possibility: namely a variant of Hubbard's so-called $1/d$ -metric (see [HH06, Ch. 2.2, pp.33]). Let $A \subset \mathbb{R}^N$ be open subset. Then for every real parameter α we define the Riemannian metric

$$(8) \quad g_{\alpha} := (\text{dist}(x, \mathbb{R}^N - A))^{-\alpha} \cdot ds^2,$$

where

$$ds^2 = dx_1^2 + dx_2^2 + \dots$$

is the standard Euclidean metric on \mathbb{R}^N . The choice $\alpha = 0$ yields $g_0 = ds^2$. Let $\kappa = \kappa(g)$ denotes the sectional curvature of the metric g .

Lemma 1. *For every parameter $\alpha \geq 0$, the Riemannian metric g_{α} has nonpositive sectional curvature $\kappa \leq 0$ throughout A .*

Proof. We follow Hubbard's proof [HH06, Thm. 2.2.9, pp.36], where the key observation is that: for every $y \in \mathbb{R}^N - A$, the function $f_y(x) := -\log ||x - y||^{\alpha}$ is subharmonic for $x \in A$ (in fact, the function is harmonic). Therefore the supremum

$$f(x) := \sup_{y \in \partial A} f_y(x) = \sup_{y \in \mathbb{R}^N - A} f_y(x)$$

is subharmonic. But the metric g_{α} is conformal to the standard Euclidean metric ds^2 , and the formula for the sectional curvature of conformal metrics is well-known, namely $\kappa = -\Delta \log f \cdot ds^2$, which is ≤ 0 by above subharmonicity. \square

For variable α the metric g_{α} , and the corresponding path length distance $d_{\alpha}(x, y)$ is possibly incomplete on A . However incompleteness only occurs at the boundary ∂A of A as subset of \mathbb{R}^N . For A an open subset there may exist sequences (relative to the distance d_{α}) $\{x_k\}_{k \in \mathbb{N}}$ in A which have no limit point in A . Despite the metric

g_α diverging as $x \rightarrow \partial A$, we prefer the lengths of geodesics $\gamma : [0, 1] \rightarrow A$ converging to ∂A to have finite length, and seek parameters α for which

$$\text{Length}_\alpha(\gamma) = \int_0^1 \sqrt{g_\alpha(\gamma'(t), \gamma'(t))} \cdot dt < +\infty.$$

Example 2. Hubbard's definition of $1/d$ -metric corresponds to $\alpha = 2$ in equation (8). Amazingly the $1/d$ -metric on the upper halfspace $H := \{x_1 > 0\}$ in \mathbb{R}^N in x_1, \dots, x_N coordinates is the complete constant-curvature hyperbolic metric on H !

Yet for $0 < \alpha < 2$ the metric is incomplete. The curve $\gamma(t) = (1 - t, 0, 0, \dots)$ for $0 \leq t \leq 1$ is a curve in H . The g_α -length of γ evaluates to $\int_0^1 (1 - t)^{-\alpha/2} dt$, which is improper integral converging to

$$1 < (1 - \alpha/2)^{-1} < +\infty$$

when $0 < \alpha < 2$. While $0 < \alpha < 2$ we can uniquely extend d_α to a complete metric pairing

$$\tilde{d}_\alpha : \overline{H} \times \overline{H} \rightarrow \mathbb{R},$$

where $\overline{H} = \{x_1 \geq 0\}$. Note that \overline{H} is not homeomorphic to adjoining a sphere at-infinity S_∞^2 to H .

Example 3. The $1/d$ -metric ($\alpha = 2$) on the once-punctured plane $A = \mathbb{R}^2 - \{0\}$ is isometric to a straight cylinder of circumference 2π ([HH06, Ex.2.2.6]). The same computations as previous example show for $0 < \alpha < 2$, the metric d_α is incomplete with completion \tilde{d}_α equal to an infinite cone with angle [FORMULA] at the origin vertex.

The above two examples have A unbounded open subset. But our applications to medial axes concern bounded open subsets.

Example 4. We modify example 3 by restricting to the punctured disk, say, $D^\times := \{0 < \|x\| < R\}$ for a constant $R > 0$. Then the medial axis $M(D^\times) = \{\|x\| = R/2\}$ is a circle in D^\times . Now we propose that sufficient (UHS) conditions, namely (6)–(7), are satisfied throughout D^\times and the inclusion $Z_2 \hookrightarrow D^\times$ is homotopy-isomorphism (by Theorem B) for $Z = Z(\mu, \nu, c_\alpha)$ for $0 < \alpha < 2$. Moreover we propose Z_2 is also a circle, diffeomorphic to $M(A)$, but not identical!

Proposition 5 (Work-In-Progress). *Let A be open subset of \mathbb{R}^N . For parameters $0 < \alpha < 2$ the metric g_α is incomplete metric on A , and geodesics in A converging to the boundary ∂A have uniquely defined finite length with respect to the metric g_α .*

Proof. □

Now we propose a more interesting mass-transport interpretation of medial axis transforms. Let A be bounded open subset of \mathbb{R}^N , with boundary ∂A , and probability

measures μ, ν as previously defined in (2), (4). Then we choose cost $c = \tilde{d}_\alpha : A \times \partial A \rightarrow \mathbb{R}$ defined by restricting the completion to $A \times \partial A \subset \bar{A} \times \bar{A}$. The subvarieties $Z_2, M(A)$ will not coincide set-theoretically, but we conjecture that they do coincide topologically:

Theorem 6 (Work-In-Progress). *Let A be bounded open subset of \mathbb{R}^N . Let $c = \tilde{d}_\alpha$ be the metric completion of d_α to \bar{A} , and let $Z = Z(\mu, \nu, c) : 2^{\partial A} \rightarrow 2^A$ be the Singularity functor with respect to (μ, ν, c) as defined in (2), (4). Then sufficient (UHS) Conditions are satisfied to apply Theorem B [ref][section], and the inclusion $Z_2 \hookrightarrow A$ is a homotopy isomorphism and even a strong deformation retract.*

The completion of Hubbard’s $1/d$ -distance and the cost $c = \tilde{d}_\alpha$ yields an alternative to the medial axis $M(A)$ in the subvariety Z_2 defined by c -optimal couplings. We propose this construction of Z_2 yields a useful improvement over the conventional definition of $M(A)$ per (1). For instance the medial axis is defined on the category of open subsets A of \mathbb{R}^N , whereas the functors Z are more generally defined for measure spaces.

Another objection to $M(A)$ is it’s notorious instability (c.f. [Sun+13, §1]). Small perturbations of the open subset A often leads to large changes in the medial axis $M(A)$. Therefore $M(A)$ suffers large variability when A has background noise. Many authors have suggested modified media axes (c.f. [FLM03], [TH03] and references therein) which “filter out” possible noise. On the other hand, it’s well-known that optimal transportation enjoys strong continuity properties, and c -optimal semicouplings vary continuously (in appropriate narrow topology) with perturbations of μ, ν (c.f. [Vil09, Thm. 28.9, pp.780]).

In conclusion, Blum identified the medial axis transform as convenient/efficient mode of describing objects, and heuristics showed the inclusions $M(A) \hookrightarrow A$ were always homotopy isomorphisms. However Blum’s medial axis is but a particular instance of a more useful topological object, namely Z_2 the contravariant functors $Z(\mu, \nu, c) : 2^{\partial A} \rightarrow 2^A$. This Z_2 is stable topological object, and the inclusions $Z_2 \hookrightarrow A$ are identified as homotopy isomorphisms when the (UHS) Conditions (6), (7) hold throughout the open complement $A - Z_2$. Thus we propose more stable topological “folk-theorems” regarding a mass transport extension of so-called medial axis transforms, and Theorem B from [Mar].

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