OPTIMAL TRANSPORT, $1/d^{\alpha}$-COSTS, AND MEDIAL AXIS TRANSFORMS

J.H. MARTEL

1. Medial Axis Transforms and Optimal Transport

The purpose of this section is to compare some familiar properties of the medial axis transform $A \mapsto M(A)$ (introduced by [Blu67]) with the singularity structures formalized in our Kantorovich contravariant functor $Z : 2^{\partial A} \to 2^{A}$ (introduced in [Mar]). To compare the functors Z with medial axis transform requires we interpret the inclusion $M(A) \hookrightarrow A$ in the category of mass transportation.

Let A be a bounded open subset of $\mathbb{R}^N$. The medial axis $M(A)$ introduced by Blum consists of all $x \in A$ for which $dist(x, \partial A)$ is attained by at least two distinct points,

$$M(A) := \{x \in A \mid \#argmin_{y \in \partial A}\{d(x,y)\} \geq 2\}. \tag{1}$$

A long-known "folk theorem" states that the inclusion $M(A) \hookrightarrow A$ is a homotopy-isomorphism, and even a strong deformation retract. This implies $M(A)$ contains all the topology of A, and a connected subset whenever A is. A formal proof is established [Lie04]. We do not know if $M(A)$ is a strong retract for more general Riemannian spaces (X, d), although results of Alexander–Bishop [AB98], [AB00] prove sufficiently thin Riemannian manifolds deform onto the cut-locus. Our recent thesis [Mar] contains some results, namely "Theorem B", identify conditions for which inclusions denoted $Z_2 \hookrightarrow A$ are homotopy isomorphisms, even strong deformation retracts. This subvariety Z_2 is derived from a contravariant functor $Z = Z(\mu, \nu, c)$ defined by mass transport data (μ, ν, c). The medial axis $M(A)$ and Z_2 will rarely coincide set-theoretically, but this present note demonstrates they are frequently topologically isomorphic.

The medial axis transform corresponds to a "degenerate" transport problem in the following sense: if $A \hookrightarrow \mathbb{R}^N$ is bounded open subset, then we nominate

$$\mu := \frac{1}{\mathscr{H}_A[A]}\mathscr{H}_A \tag{2}$$

as the canonical probability measure on the source A. Consider the probability measures π on $A \times \partial A$ for which $proj_A\#\pi = \mu$ and with unconstrained second

Date: March 31, 2019.

marginal $proj_{\partial A}\#\pi$. Here $proj_A$, $proj_{\partial A}$ are the canonical projections $A \times \partial A \to A, \partial A$. The set-mapping

$$T : x \mapsto argmin_{y \in \partial A}\{d(x,y)\}, \quad \text{for } x \in A, \tag{3}$$

defines a measurable set-valued map $T : A \to \partial A$. The pushforward

$$\nu := T\#\mu \tag{4}$$

is a probability measure on ∂A with $spt(\nu) = \partial A$. With respect to, say, quadratic cost $c = d^2/2$ or distance cost $c = d$, the map $x \mapsto T(x)$ defines a c-optimal transport from μ to ν, with c-optimal coupling $\pi = (Id \times T)\#\mu$ on $A \times \partial A$.

Finally $M(A)$ coincides with the locus-of-discontinuity of $T : A \to \partial A$, or more specifically the singularity Z_2 defined by Kantorovich's contravariant functor $Z = Z(\mu, \nu, d) : 2^{\partial A} \to 2^A$. Thus we arrive at an instance where $M(A) = Z_2$ for the specific coupling program defined by μ, ν, c. This identification suggests the following generalization of medial axis transform: for general probability measures $\nu \in \Delta(\partial A)$ on the boundary of A, we may study the c-optimal couplings π from μ to ν, and obtain a Singularity functor $Z(\mu, \nu, c)$. The generalized medial axis in this setting is Z_2, i.e. the "locus-of-discontinuity" of the c-optimal transport π from μ to ν.

2.

Our thesis developed a Reduction-to-Singularity principle, and identifies conditions for which, say, the inclusion $Z_2 \hookrightarrow Z_1$ is a homotopy-isomorphism. In the above setting with $Z = Z(\mu, \nu, c)$, we find $A = Z_1$, $M(A) = Z_2$. Naturally we inquire whether the hypotheses of our topological theorems are satisfied for any particular costs c. If we fix $c = d^2/2$, then Theorem B takes the following form. For $x \in A = Z_1$, let $y_0 := T(x)$. Then define

$$\eta(x,y) := |c(x,y) - c(x,y_0)|^{-1-\beta} \cdot \nabla_x (c(x,y) - c(x,y_0)),$$

for $y \in \partial A - \{y_0\}$. Observe that $c(x,y) - c(x,y_0) > 0$ is nonvanishing throughout $A - M(A)$. The hypotheses of Theorem B require the following conditions (6), (7) be satisfied. For $x \in A - Z_2$, define $y_0 =: T(x)$, and abbreviate

$$\bar{\nu}(y) := (1 - e^{-d(y,y_0)^2}).\nu(y).$$

Then we define the averaged Bochner integral

$$\eta(x, avg) := \bar{\nu}[\partial A]^{-1} \cdot \int_{\partial A} \eta(x,y) d\bar{\nu}(y). \tag{5}$$

We require that

$$\eta(x, avg) \text{ is nonzero finite tangent vector,} \tag{6}$$

and there exists a constant $C > 0$ such that

$$||\eta(x, avg)|| \geq C > 0 \tag{7}$$

for $x \in A - M(A)$, uniformly with x, for some exponent $\beta > 0$. Typically $\beta = 1$ is sufficient.

The verification of hypotheses (6)–(7) can be difficult to verify. Evidently (7) implies (6). We find $\eta(x, avg)$ is an averaged gradient, even the gradient of the averaged potential $f_{avg}(x)$ defined as follows. Let $f_y(x) := (c(x, y) - c(x, y_0))^{-\beta}$, and consider the average of $f_y(x)$ with respect to the Borel measure $\bar{\nu}(y)$ on ∂A, namely

$$f_{avg}(x) = \bar{\nu}[\partial A]^{-1} \cdot \int_{\partial A} f_y(x) d\bar{\nu}(y).$$

Then

$$\eta(x, avg) = \nabla_x \int_{\partial A} f_{avg}(x) d\bar{\nu}(y).$$

The hypothesis (7) require $f_{avg}(x)$ be critical-point free over the open subset $A - Z_2$. As $x \in A - Z_2$ converges to Z_2, we find the potential $f_{avg} > 0$ and the gradient $\nabla_x f_{avg}$ diverge to infinity.

3. Hubbard's $1/d^\alpha$-distance

We need remark on a complication arising from the nonconvexity of A. What is the natural distance function d on $A \subset \mathbb{R}^N$, and the physical "transport cost" of a unit mass at $x \in A$ to target mass $y \in \partial A$? There are at least two popular possibilities. First we may restrict the ambient euclidean distance $d_{\mathbb{R}^N}(x, y) = ||x - y||$ to $A \times \partial A \subset \mathbb{R}^N \times \mathbb{R}^N$. But this restriction does not represent a path length distance in the sense of Gromov [Gro+01, 1.A-B]. In otherwords the restriction does not represent geodesic transport in A, and there is no variational description of the metric in terms of shortest-length curves.

A second approach defines $d = d_A$ as the induced length distance defined by

$$d_A(x, y) = \inf_\gamma Length(\gamma),$$

where the infimum is over all curves $\gamma : [0, 1] \to A$ contained in A with $\gamma(0) = x$, $\gamma(1) = y$. The reader will observe that both possibilities define coincident medial axes $M(A)$ according to (1), since euclidean balls are geodesically convex. The induced length distance $d = d_A$ is possibly most preferred by metric geometers, yet is difficult to numerically evaluate. Moreover geodesics with respect to the induced path distance $c = d_A$ can oftentimes be branching. The possible branching of geodesics implies gradients $y \mapsto \nabla_x d(x, y)$ are noninjective maps $\partial A \to T_x A$ for $x \in A$. This possible noninjectivity violates an important transport condition called (Twist), and is obstruction to hypothesis (6). Thus neither the restricted distance $c = d|A \times \partial A$ nor the induced distance $c = d_A$ are especially convenient costs.

This article explores a third possibility, namely a variant of Hubbard's so-called $1/d$-metric (see [HH06, Ch. 2.2, pp.33]). Let $A \subset \mathbb{R}^N$ be open subset. Then for every real parameter α we define the Riemannian metric

$$g_\alpha := (dist(x, \mathbb{R}^N - A))^{-\alpha}.ds^2, \tag{8}$$

where

$$ds^2 = dx_1^2 + dx_2^2 + \cdots + dx_N^2$$

is the standard Euclidean metric on $\mathbb{R}^N$. The choice $\alpha = 0$ yields $g_0 = ds^2$. Let $\kappa = \kappa(g)$ denote the sectional curvature of the metric g.

Lemma 1. *For every parameter $\alpha \geq 0$, the Riemannian metric g_α has nonpositive sectional curvature $\kappa \leq 0$ throughout A.*

Proof. We follow Hubbard's proof [HH06, Thm. 2.2.9, pp.36], where the key observation is that: for every $y \in \mathbb{R}^N - A$, the function $f_y(x) := -\log ||x - y||^\alpha$ is subharmonic for $x \in A$ (in fact, the function is harmonic). Therefore the supremum

$$f(x) := \sup_{y \in \partial A} f_y(x) = \sup_{y \in \mathbb{R}^N - A} f_y(x)$$

is subharmonic. But the metric g_α is conformal to the standard Euclidean metric ds^2, and the formula for the sectional curvature of conformal metrics is well-known, namely $\kappa = -\Delta \log f \cdot ds^2$, which is ≤ 0 by above subharmonicity. $\square$

For variable α the metric g_α, and the corresponding path length distance $d_\alpha(x, y)$ is possibly incomplete on A. Geodesics in the g_α metric steer so as to keep as far away as possible from the boundary ∂A. However incompleteness only occurs at the boundary ∂A of A as subset of $\mathbb{R}^N$. For A an open subset there may exist sequences $\{x_k\}_{k=1,2,\ldots}$ in A which have no limit point in A relative to the distance d_α. Despite the metric g_α diverging as $x \to \partial A$, we prefer the lengths of geodesics $\gamma : [0, 1] \to A$ converging to ∂A to have finite length, and seek parameters α for which

$$Length_\alpha(\gamma) = \int_0^1 \sqrt{g_\alpha(\gamma'(t), \gamma'(t))}.dt < +\infty.$$

Example 2. Hubbard's definition of $1/d$-metric corresponds to $\alpha = 2$ in equation (8). Amazingly the $1/d$-metric on the upper halfspace $H := \{x_1 > 0\}$ in $\mathbb{R}^N$ in $x_1, \ldots, x_N$ coordinates is the complete constant-curvature hyperbolic metric on H. For $0 < \alpha < 2$ the metric is incomplete. The curve $\gamma(t) = (1 - t, 0, 0, \cdots)$ for $0 \leq t \leq 1$ is a curve in H. The g_α-length of γ evaluates to $\int_0^1 (1 - t)^{-\alpha/2} dt$, which is improper integral converging to $1 < (1 - \alpha/2)^{-1} < +\infty$ when $0 < \alpha < 2$. While $0 < \alpha < 2$ we can uniquely extend d_α to a complete metric pairing $\tilde{d_\alpha} : \overline{H} \times \overline{H} \to \mathbb{R}$, where $\overline{H} = \{x_1 \geq 0\}$. Note that $\overline{H}$ is not homeomorphic to adjoining a sphere at-infinity S^2_∞ to H.

Example 3. The $1/d$-metric ($\alpha = 2$) on the once-punctured plane $A = \mathbb{R}^2 - \{0\}$ is isometric to a straight cylinder of circumference 2π ([HH06, Ex.2.2.6]). The same computations as previous example show for $0 < \alpha < 2$, the metric d_α is incomplete with completion $\tilde{d}_\alpha$ equal to an infinite cone with angle depending on α at the origin vertex.

Example 4. The Weil-Petersson metric d_{WP} on the Teichmueller space $\mathscr{T}_g$ of a closed genus g hyperbolic surface is asymptotically equivalent to Hubbard's metric with exponent $\alpha = 3/2$, [Wol75].

The above examples have A unbounded open subset. But our applications to medial axes concern bounded open subsets.

Example 5. We modify example 3 by restricting to the punctured disk, say, $D^\times := \{0 < ||x|| < R\}$ for a constant $R > 0$. Then the medial axis $M(D^\times) = \{||x|| = R/2\}$ is a circle in $D^\times$. Now we propose that sufficient (UHS) conditions, namely (6)–(7), are satisfied throughout $D^\times$ and the inclusion $Z_2 \hookrightarrow D^\times$ is homotopy-isomorphism (by Theorem B) for $Z = Z(\mu, \nu, c_\alpha)$ for $0 < \alpha < 2$. Moreover we propose Z_2 is also a circle, diffeomorphic to $M(A)$, but not identical.

Example 6. Let A be convex subset and $0 < \alpha < 2$. Then ${}_\alpha(x, y)$ is proportional to the rescaled Euclidean distance $|x - y|^{1-\alpha/2}$.

Proposition 7. *Let A be open subset of $\mathbb{R}^N$, with topological closure $\overline{A}$. For parameters $0 < \alpha < 2$ the metric g_α is incomplete Riemannian metric on A. Then:*

(i) curves γ in A converging to the boundary ∂A have uniquely defined finite length with respect to the metric g_α; and

(ii) the path length distance $\tilde{d}_\alpha : \overline{A} \times \overline{A} \to \mathbb{R}_{\geq 0}$ defines a complete metric distance throughout $\overline{A}$.

Proof. First we need prove that curves converging to the boundary have finite length. The length is an improper integral which converges for $0 < \alpha < 2$. [INCOMPLETE] □

4.

Now we propose a more interesting mass transport interpretation of medial axis transforms. Let A be bounded open subset of $\mathbb{R}^N$, with boundary ∂A, and probability measures μ, ν as previously defined in (2), (4). Then we choose cost $c = \tilde{d}_\alpha : A \times \partial A \to \mathbb{R}$ defined by restricting the completion to $A \times \partial A \subset \overline{A} \times \overline{A}$. The subvarieties Z_2 and $M(A)$ do not coincide set-theoretically, but we conjecture that they do coincide topologically:

Theorem 8 (Work-In-Progress). *Let A be bounded open subset of $\mathbb{R}^N$. Let $c = \tilde{d_\alpha}$ be the metric completion of d_α to $\overline{A}$ (Prop. 7), and let $Z = Z(\mu, \nu, c) : 2^{\partial A} \to 2^A$ be the Singularity functor with respect to (μ, ν, c) as defined in (2), (4).*

Then sufficient (UHS) Conditions are satisfied to apply Theorem B [Mar, Thm.3.4.3], and the inclusion $Z_2 \hookrightarrow A$ is a homotopy isomorphism and even a strong deformation retract.

We require some preliminary lemmas. Lemma [ref] proves the metric g_α is nonpositively curved, and therefore distance-minimizing geodesics exist between any pair of points x, y in A. However the geodesics are possibly nonunique.

Lemma 9. *For every $\alpha \geq 0$, the restricted cost $c = \tilde{d}^2_\alpha/2 : A \times \partial A \to \mathbb{R}$ satisfies the following (Twist) condition: for every $x \in A$, the gradient mapping*

$$\partial A \to T_x A, \ y \mapsto \nabla_x c(x, y) \textit{ is } \nu\textit{-a.e. injective.}$$

Proof. We take advantage of fact that c is a Lagrangian cost defined by an action principle. According to [Vil09, Prop.10.15, pp.235], the gradient $\nabla_x c(x, y)$ is equal to

$$\frac{-1}{2}.\rho(x).\gamma'(0),$$

where ρ is the conformal factor $\rho(x) = dist(x, \mathbb{R}^N - A)^{-\alpha}$, and where $\gamma'(0)$ is the initial tangent vector of any action-minimizing curve γ in A with $\gamma(0) = x$, $\gamma(1) = y$. The nonpositive curvature of g_α implies action-minimizing curves exist. These geodesics are possibly nonunique, however for ν-a.e. $y \in \partial A$, the geodesic joining x to y will be unique. Since the conformal factor ρ is nonvanishing, and since geodesics in Riemannian manifolds are determined by their initial point and initial tangent vector, we conclude $y \mapsto \nabla_x c(x, y)$ is ν-a.e. injective, as desired. □

That c satisfies the above (Twist) condition implies the uniqueness of c-optimal semicouplings from μ to ν. [ref].

The above Lemma 9 and the identity

$$\nabla_x c(x, y) = \frac{-\rho_x}{2} \cdot \gamma'(0)$$

implies the gradient of the cross-difference c_Δ is readily computed

$$\nabla_x c_\Delta(x, y_0, y_1) = \frac{\rho_x}{2}.[\gamma_1'(0) - \gamma_0'(0)],$$

where γ_0, γ_1 are the g_α-geodesics satisfying $\gamma_0(0) = \gamma_1(0) = x, \gamma_0(1) = y_0, \gamma_1(1) = y_1$.

Following the previous definitions of $\eta(x, avg)$ from (6), we have

$$\eta(x, avg) = \frac{\rho_x}{2} \left\langle |\psi(y) - \psi(y_0) + c_\Delta(x, y_0, y)|^{-1-\beta} \cdot [\gamma_0'(0) - \gamma_y'(0)] \right\rangle. \tag{9}$$

Here $\langle v \rangle$ denotes the average Bochner integral of a vector-valued function v with respect to the Radon measure $\bar{\nu}$ on ∂A, where

$$d\bar{\nu}(y) = (1 - e^{d(y,y_0)^2})d\nu(y).$$

Thus

$$\langle v \rangle := (\bar{\nu}[Y])^{-1}. \int_Y v(y) d\bar{\nu}(y).$$

We remark that the average η_{avg} is weighted by the regions where the potential $\psi(y) - \psi(y_0) + c_\Delta(x, y_0, y) \geq 0$ vanishes $(= 0)$. The weights blow-up to $+\infty$ when $y \to Z_2 - A$, and without the exponential factor $(1 - e^{d(y,y_0)^2})$ the weights would blow-up as $y \to y_0$.

Remark 10. If the region A is convex, then the nonvanishing $\eta(x, avg)$ of (9) can be verified, since the tangent vectors $\{\gamma_y'(0) - \gamma_0'(0) \mid y \in \partial A\}$ lie in a common halfspace of $T_x A$.

The factor $1 - e^{d(y,y_0)^2}$ defines a "scale" on the boundary ∂A. If $x \in A$ is sufficiently close to the boundary ∂A with respect to the above scale, then the gradients $\nabla_x c(x, y)$, for $y \in \partial A$ satisfying $d(y, y_0) \geq 1$, are readily seen to satisfy (UHS) conditions, since the gradients make large angle with respect to $\nabla_x c(x, y_0)$, i.e. the angle is greater than $\pi/2$. Compare Figure 4.

If A is sufficiently "thin", such that all points x are sufficiently close to ∂A, then we see (UHS) conditions are satisfied throughout $A - Z_2$. This is reminiscent of theorems of Alexander-Bishop [ref]: Let (M, g) be a Riemannian manifold with boundary ∂M, and let C be the "cut locus of the boundary". If M is sufficiently "thin" (i.e. has sufficiently small inradius relative to its curvature), then the cut locus C is a strong deformation retract of M and the inclusion $C \hookrightarrow M$ is a homotopy isomorphism. See [ref Thm 2.1, AB2].[Definition: cut locus of boundary]

5. Upper Semicontinuity of Z and $M(A)$.

The completion of Hubbard's $1/d_\alpha$ distance and the cost $c = \tilde{d}_\alpha^2$ yields an alternative to the medial axis $M(A)$ in the subvariety Z_2 defined by c-optimal couplings. We propose this construction of Z_2 is a useful improvement over the conventional definition of $M(A)$ per (1). For instance the medial axis is defined on the category of open subsets A of $\mathbb{R}^N$, whereas the functors Z are more generally defined for measure spaces.

We remark on the differences between C^0, $C^{1,1}$, and C^2 regularity of boundaries ∂A. For C^2 boundary, the medial axis $M(A)$ will be disjoint from A. However for $C^0, C^{1,1}$ regularity, the medial axis $M(A)$ will extend into the boundary ∂A. The $C^{1,1}$ regularity frequently occurs, e.g. whenever A is convex polyhedra. If A is $C^{1.1}$ and $y \in \partial A$ is not uniquely differentiable boundary point, then $y \in M(A)$. I.e. the

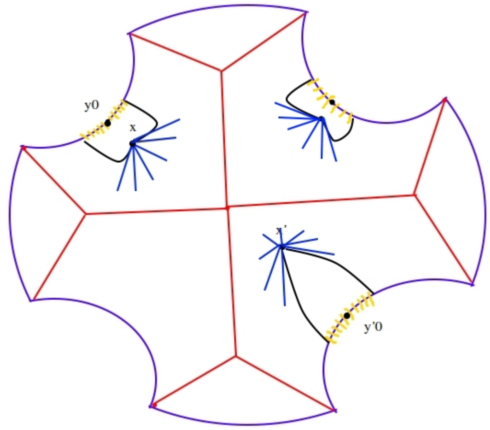

FIGURE 1. When $x \in A$ is sufficiently close to ∂A, and $y_0 := T(x)$, then the gradients $\nabla_x c(x, y)$, for $d(y, y_0) \geq 1$, are readily seen to satisfy HalfSpace Conditions. But for points $x' \in A$ at larger distance from ∂A, the gradients $\nabla_x c(x', y)$ are not contained in a proper halfspace for $d(y, T(x')) \geq 1$.

medial axis $M(A)$ extends into the boundary at y. Therefore "sharp" corners cause the medial axis to "split" and extend into ∂A. That sharp corners can appear under Gromov-Hausdorff variations of the subset $A \subset \mathbb{R}^N$ implies the medial axis is rather unstable. Small perturbations of the open subset A (e.g. background noise) can lead to large changes in the medial axis $M(A)$. Many authors have suggested modified medial axes (c.f. [FLM03], [TH03] and references therein) which "filter out" possible noise. The present article takes another approach to "regularizing" $M(A)$. It's well-known that optimal transportation enjoys strong continuity properties with respect to variations in the datum (μ, ν, c). More precisely we quote the following result from [Vil09, Thm. 28.9, pp.780–790].

Proposition 11. *Let A_k, $k = 1, 2, \ldots$, be bounded open subsets of $\mathbb{R}^N$ with canonical probability measures μ_k, and converging in Gromov-Hausdorff topology*

$$\lim_{k \to +\infty} (A_k, \mu_k) = (A_0, \mu_0).$$

Let $c_0, c_1, c_2, \ldots$ be the costs $c_k = \tilde{d}^2_{\alpha, A_k}/2$. Let $T_k : A_k \to \partial A_k$ be the maps defined in (3)*. Then:*

(i) The sequence of probability measures $\nu_k := T_k \# \mu_k$ converges in the narrow topology to a probability measure

$$\lim_{k \to +\infty} \nu_k := \overline{\nu_0}.$$

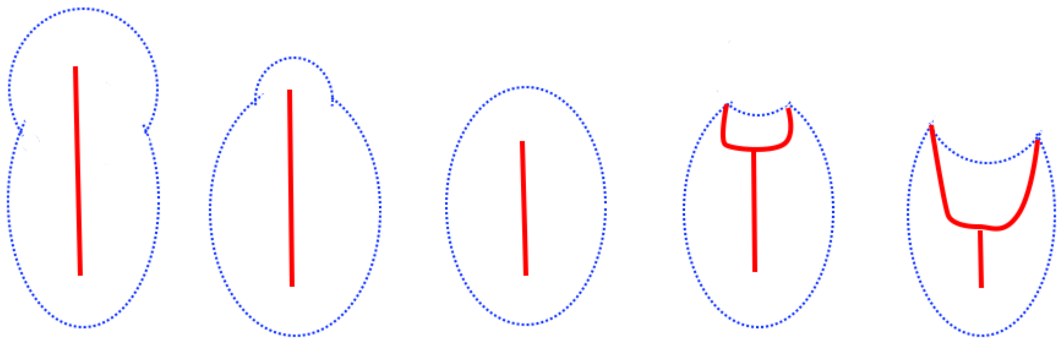

FIGURE 2. The medial axis $A \mapsto M(A)$ varies upper semicontinuously when sharp corners appear, like the two figures on the right.

(ii) The correspondance

$$(\mu_k, \nu_k, c_k) \mapsto Z(\mu_k, \nu_k, c_k)$$

varies upper semicontinuously, and there exists an injective natural transformation between the functors

$$Z(\mu_0, \overline{\nu_0}, c_0) \hookrightarrow \lim_{k \to +\infty} Z(\mu_k, \nu_k, c_k).$$

(iii) In particular there exists natural topological embedding

$$Z_{2,0} \hookrightarrow \lim_{k \to +\infty} Z_{2,k}.$$

We refer the reader to any standard textbook on category theory for the definitions of functors and natural transformations, e.g. [**Lang**]. The point of Proposition 11 is that the singularities of the limit (μ_0, ν_0, c_0) are no more complicated than the approximant singularities of (μ_k, ν_k, c_k). In fact the singularity often simplifies in various limits. The upper semicontinuity of the medial axis $A \mapsto M(A)$ is apparently well-known [ABE09, §5]. The Proposition 11 is familiar property of (lower semicontinuous) convex functions: if $\phi_k : X \to \mathbb{R}$ is a sequence of lsc convex functions which converge pointwise to a limit $\lim_{k \to +\infty} \phi_k = \phi_0$, then for every $x \in X$, the subdifferential $\partial \phi_0(x)$ is a subset of the Gromov-Hausdorff limit $\lim_{k \to +\infty} \partial \phi_k(x)$.

6. Conclusion

In conclusion, Blum identified the medial axis transform as convenient mode of describing objects, and heuristics showed the inclusions $M(A) \hookrightarrow A$ were always homotopy isomorphisms. However Blum's medial axis is only a particular instance of a more useful topological object, namely Z_2 of the contravariant functors $Z(\mu, \nu, c) : 2^{\partial A} \to 2^A$. This Z_2 is stable, and defined with respect to a Hubbard type Riemannian metric g_α, and the inclusions $Z_2 \hookrightarrow A$ are identified as homotopy isomorphisms when the (UHS) Conditions (6), (7) hold throughout the open complement $A - Z_2$. Thus we propose a further "folk-theorem" regarding this mass transport extension of so-called medial axis transforms, and Theorem B from [Mar].

References

[AB98] Stephanie B Alexander and Richard L Bishop. "Thin Riemannian manifolds with boundary". In: *Mathematische Annalen* 311.1 (1998), pp. 55–70.

[AB00] Stephanie B Alexander and Richard L Bishop. "Spines and homology of thin Riemannian manifolds with boundary". In: *Advances in Mathematics* 155.1 (2000), pp. 23–48.

[ABE09] Dominique Attali, Jean-Daniel Boissonnat, and Herbert Edelsbrunner. "Stability and computation of medial axes-a state-of-the-art report". In: *Mathematical foundations of scientific visualization, computer graphics, and massive data exploration*. Springer, 2009, pp. 109–125.

[Blu67] Harry Blum. "A Transformation for Extracting New Descriptors of Shape". In: *Models for the Perception of Speech and Visual Form*. Ed. by Weiant Wathen-Dunn. Cambridge: MIT Press, 1967, pp. 362–380.

[FLM03] Mark Foskey, Ming C. Lin, and Dinesh Manocha. "Efficient Computation of a Simplified Medial Axis". In: *Proceedings of the Eighth ACM Symposium on Solid Modeling and Applications*. SM '03. Seattle, Washington, USA, 2003, pp. 96–107. URL: http://doi.acm.org/10.1145/781606.781623.

[Gro+01] M. Gromov et al. *Metric Structures for Riemannian and Non-Riemannian Spaces*. Progress in Mathematics. Birkhäuser Boston, 2001.

[HH06] JH Hubbard and JH Hubbard. "Teichmüller Theory and Applications to Geometry, Topology and Dynamics, Volume I: Teichmüller Theory". In: (2006).

[Lie04] Andre Lieutier. "Any open bounded subset of Rn has the same homotopy type as its medial axis". In: *Computer-Aided Design* 36.11 (2004), pp. 1029–1046. DOI: https://doi.org/10.1016/j.cad.2004.01.011. URL: http://www.sciencedirect.com/science/article/pii/S0010448504000065.

[Mar] J.H. Martel. "Applications of Optimal Transport to Algebraic Topology: How to Build Spines from Singularity". PhD thesis. University of Toronto. URL: https://github.com/marvinMLKUltra/thesis/blob/master/ut-thesis.pdf.

[TH03] R. Tam and W. Heidrich. "Shape simplification based on the medial axis transform". In: *IEEE Visualization, 2003. VIS 2003.* 2003, pp. 481–488.

[Vil09] C. Villani. *Optimal transport: old and new.* Grundlehren der mathematischen wissenschaften Vol. 338. Springer-Verlag, 2009.

[Wol75] Scott Wolpert. "Noncompleteness of the Weil-Petersson metric for Teichmuller space." In: *Pacific J. Math.* 61.2 (1975), pp. 573–577. URL: https://projecteuclid.org:443/euclid.pjm/1102868050.

Email address: marvinlessknown2714@protonmail.com