OPTIMAL TRANSPORT, $1/d^\alpha$-COSTS, AND MEDIAL AXIS TRANSFORMS

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1. Medial Axis Transforms and Optimal Transport

The purpose of this section is to compare some familiar properties of the medial axis transform $A \mapsto M(A)$ (introduced by [Blu67]) with the singularity structures formalized in our Kantorovich contravariant functor $Z : 2^{\partial A} \to 2^A$ (introduced in [Mar]). To compare the functors Z with medial axis transform requires we interpret the inclusion $M(A) \hookrightarrow A$ in the category of mass transportation.

Let A be a bounded open subset of $\mathbb{R}^N$. The medial axis $M(A)$ introduced by Blum consists of all $x \in A$ for which $dist(x, \partial A)$ is attained by at least two distinct points,

$$M(A) := \{x \in A \mid \#argmin_{y \in \partial A}\{d(x,y)\} \geq 2\}. \tag{1}$$

A long-known "folk theorem" states that the inclusion $M(A) \hookrightarrow A$ is a homotopy-isomorphism, and even a strong deformation retract. This implies $M(A)$ contains all the topology of A, and a connected subset whenever A is. A formal proof is established [Lie04]. We do not know if $M(A)$ is a strong retract for more general Riemannian spaces (X, d), and a search through the literature does not address the question. But our recent thesis [Mar] contains some results, namely "Theorem B", identify conditions for which inclusions denoted $Z_2 \hookrightarrow A$ are homotopy isomorphisms, even strong deformation retracts. This subvariety Z_2 is derived from a contravariant functor $Z = Z(\mu, \nu, c)$ defined by mass transport data (μ, ν, c). The medial axis $M(A)$ and Z_2 will rarely coincide set-theoretically, but this present note demonstrates they are frequently topologically isomorphic.

The medial axis transform corresponds to a "degenerate" transport problem in the following sense: if $A \hookrightarrow \mathbb{R}^N$ is bounded open subset, then we nominate

$$\mu := \frac{1}{\mathscr{H}_A[A]} \mathscr{H}_A \tag{2}$$

as the canonical probability measure on the source A. Consider the probability measures π on $A \times \partial A$ for which $proj_A \# \pi = \mu$ and with unconstrained second

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marginal $proj_{\partial A}\#\pi$. Here $proj_A$, $proj_{\partial A}$ are the canonical projections $A \times \partial A \to A, \partial A$. The set-mapping

$$T : x \mapsto argmin_{y \in \partial A}\{d(x,y)\}, \quad \text{for } x \in A, \tag{3}$$

defines a measurable set-valued map $T : A \to \partial A$. The pushforward

$$\nu := T\#\mu \tag{4}$$

is a probability measure on ∂A with $spt(\nu) = \partial A$. With respect to, say, quadratic cost $c = d^2/2$ or distance cost $c = d$, the map $x \mapsto T(x)$ defines a c-optimal transport from μ to ν, with c-optimal coupling $\pi = (Id \times T)\#\mu$ on $A \times \partial A$.

Finally $M(A)$ coincides with the locus-of-discontinuity of $T : A \to \partial A$, or more specifically the singularity Z_2 defined by Kantorovich's contravariant functor $Z = Z(\mu, \nu, d) : 2^{\partial A} \to 2^A$. Thus we arrive at an instance where $M(A) = Z_2$ for the specific coupling program defined by μ, ν, c. This identification suggests the following generalization of medial axis transform: for general probability measures $\nu \in \Delta(\partial A)$ on the boundary of A, we may study the c-optimal couplings π from μ to ν, and obtain a Singularity functor $Z(\mu, \nu, c)$. The generalized medial axis in this setting is Z_2, i.e. the "locus-of-discontinuity" of the c-optimal transport π from μ to ν.

2.

Our thesis developed a Reduction-to-Singularity principle, and identifies conditions for which, say, the inclusion $Z_2 \hookleftarrow Z_1$ is a homotopy-isomorphism. In the above setting with $Z = Z(\mu, \nu, c)$, we have $A = Z_1$, $M(A) = Z_2$, and naturally we inquire whether the hypotheses of our topological theorems are satisfied for any particular costs c.

If we fix $c = d^2/2$, then our Theorem B takes the following form. For $x \in A = Z_1$, let $y_0 := T(x)$. Then define

$$\eta(x,y) := |c(x,y) - c(x,y_0)|^{-1/2} \cdot \nabla_x(c(x,y) - c(x,y_0)), \text{ for } y \in \partial A - \{y_0\}.$$

Observe that $c(x,y) - c(x,y_0) > 0$ is nonvanishing throughout $A - M(A)$ in the above notations. Our Theorem B requires the following hypotheses (6), (7) be satisfied for $x \in A - M(A) = Z_1 - Z_2$: the averaged Bochner integral defined as

$$\eta(x, avg) := (\nu[\partial A - \{y_0\}])^{-1} \cdot \int_{\partial A - \{y_0\}} \eta(x,y) d\nu(y), \tag{5}$$

and we require that

$$\eta(x, avg) \text{ is nonzero finite tangent vector,} \tag{6}$$

and there exists a constant $C > 0$ such that

$$||\eta(x, avg)|| \geq C > 0 \tag{7}$$

for $x \in A - M(A)$, uniformly with x. The verification of hypotheses (6)–(7) can be difficult to verify. Evidently (7) implies (6). Equivalently, we find $\eta(x, avg)$ is an averaged gradient and therefore the gradient of the averaged potential

$$f_{avg}(x) := \int_{\partial A} \nabla_x \sqrt{c(x,y) - c(x,y_0)}.$$

The hypothesis (7) is simply the claim that $f_{avg}(x)$ is critical-point free over the open subset A.

We need also remark on a complication arising from the nonconvexity of A. What is the natural distance function d on $A \subset \mathbb{R}^N$, and the physical "transport cost" of a unit mass at $x \in A$ to target mass $y \in \partial A$? There are at least two popular possibilities. First we may restrict the ambient euclidean distance $d_{\mathbb{R}^N}(x,y) = ||x-y||$ to $A \times \partial A \subset \mathbb{R}^N \times \mathbb{R}^N$. But this restriction does not represent a path length distance in the sense of Gromov [Gro+01, 1.A-B]. In otherwords the restriction does not represent geodesic transport in A, and there is no variational description of the metric in terms of shortest-length curves.

A second approach defines $d = d_A$ as the induced length distance defined by

$$d_A(x,y) = \inf_\gamma \int_\gamma Length(\gamma),$$

where the infimum is over all curves $\gamma : [0,1] \to A$ contained in A with $\gamma(0) = x$, $\gamma(1) = y$. The reader will observe that both possibilities define coincident medial axes $M(A)$ according to (1), since euclidean balls are geodesically convex. The induced length distance $d = d_A$ is possibly most preferred by metric geometers, yet is difficult to numerically evaluate. Moreover geodesics with respect to the induced path distance $c = d_A$ can oftentimes be branching. The possible branching of geodesics implies gradients $y \mapsto \nabla_x d(x,y)$ are noninjective maps $\partial A \to T_x A$ for $x \in A$. This possible noninjectivity violates an important transport condition called (Twist), and is obstruction to hypothesis (6). Thus neither the restricted distance $c = d|A \times \partial A$ nor the induced distance $c = d_A$ are especially convenient costs.

3. Hubbard's $1/d^\alpha$-distance

This article explores a third possibility: namely a variant of Hubbard's so-called $1/d$-metric (see [HH06, Ch. 2.2, pp.33]). Let $A \subset \mathbb{R}^N$ be open subset. Then for every real parameter α we define the Riemannian metric

$$g_\alpha := (dist(x, \mathbb{R}^N - A))^{-\alpha}.ds^2, \tag{8}$$

where

$$ds^2 = dx_1^2 + dx_2^2 + \cdots + dx_N^2$$

is the standard Euclidean metric on $\mathbb{R}^N$. The choice $\alpha = 0$ yields $g_0 = ds^2$. Let $\kappa = \kappa(g)$ denotes the sectional curvature of the metric g.

Lemma 1. *For every parameter $\alpha \geq 0$, the Riemannian metric g_α has nonpositive sectional curvature $\kappa \leq 0$ throughout A.*

Proof. We follow Hubbard's proof [HH06, Thm. 2.2.9, pp.36], where the key observation is that: for every $y \in \mathbb{R}^N - A$, the function $f_y(x) := -\log ||x-y||^\alpha$ is subharmonic for $x \in A$ (in fact, the function is harmonic). Therefore the supremum

$$f(x) := \sup_{y \in \partial A} f_y(x) = \sup_{y \in \mathbb{R}^N - A} f_y(x)$$

is subharmonic. But the metric g_α is conformal to the standard Euclidean metric ds^2, and the formula for the sectional curvature of conformal metrics is well-known, namely $\kappa = -\Delta \log f \cdot ds^2$, which is ≤ 0 by above subharmonicity. □

For variable α the metric g_α, and the corresponding path length distance $d_\alpha(x, y)$ is possibly incomplete on A. However incompleteness only occurs at the boundary ∂A of A as subset of $\mathbb{R}^N$. For A an open subset there may exist sequences (relative to the distance d_α) $\{x_k\}_{k \in \mathbb{N}}$ in A which have no limit point in A. Despite the metric g_α diverging as $x \to \partial A$, we prefer the lengths of geodesics $\gamma : [0, 1] \to A$ converging to ∂A to have finite length, and seek parameters α for which

$$Length_\alpha(\gamma) = \int_0^1 \sqrt{g_\alpha(\gamma'(t), \gamma'(t))}.dt < +\infty.$$

Example 2. Hubbard's definition of $1/d$-metric corresponds to $\alpha = 2$ in equation (8). Amazingly the $1/d$-metric on the upper halfspace $H := \{x_1 > 0\}$ in $\mathbb{R}^N$ in $x_1, \ldots, x_N$ coordinates is the complete constant-curvature hyperbolic metric on H!

Yet for $0 < \alpha < 2$ the metric is incomplete. The curve $\gamma(t) = (1-t, 0, 0, \cdots)$ for $0 \leq t \leq 1$ is a curve in H. The g_α-length of γ evaluates to $\int_0^1 (1-t)^{-\alpha/2} dt$, which is improper integral converging to

$$1 < (1 - \alpha/2)^{-1} < +\infty$$

when $0 < \alpha < 2$. While $0 < \alpha < 2$ we can uniquely extend d_α to a complete metric pairing

$$\tilde{d_\alpha} : \overline{H} \times \overline{H} \to \mathbb{R},$$

where $\overline{H} = \{x_1 \geq 0\}$. Note that $\overline{H}$ is not homeomorphic to adjoining a sphere at-infinity S^2_∞ to H.

Example 3. The $1/d$-metric ($\alpha = 2$) on the once-punctured plane $A = \mathbb{R}^2 - \{0\}$ is isometric to a straight cylinder of circumference 2π ([HH06, Ex.2.2.6]). The same computations as previous example show for $0 < \alpha < 2$, the metric d_α is incomplete with completion $\tilde{d_\alpha}$ equal to an infinite cone with angle [FORMULA] at the origin vertex.

Example 4. The Weil-Petersson metric d_{WP} on the Teichmueller space $\mathscr{T}_g$ of a closed genus g hyperbolic surface is asymptotically equivalent to Hubbard's metric with exponent $\alpha = 3/2$, [**Wolpert1975**].

The above examples have A unbounded open subset. But our applications to medial axes concern bounded open subsets.

Example 5. We modify example 3 by restricting to the punctured disk, say, $D^\times := \{0 < ||x|| < R\}$ for a constant $R > 0$. Then the medial axis $M(D^\times) = \{||x|| = R/2\}$ is a circle in $D^\times$. Now we propose that sufficient (UHS) conditions, namely (6)–(7), are satisfied throughout $D^\times$ and the inclusion $Z_2 \hookrightarrow D^\times$ is homotopy-isomorphism (by Theorem B) for $Z = Z(\mu, \nu, c_\alpha)$ for $0 < \alpha < 2$. Moreover we propose Z_2 is also a circle, diffeomorphic to $M(A)$, but not identical.

Proposition 6. *[Work-In-Progress] Let A be open subset of $\mathbb{R}^N$. For parameters $0 < \alpha < 2$ the metric g_α is incomplete Riemannian metric on A, and geodesics in A converging to the boundary ∂A have uniquely defined finite length with respect to the metric g_α. Consequently the path length metric*

$$\tilde{d_\alpha} : \overline{A} \times \overline{A} \to \mathbb{R}^{\geq 0}$$

is well-defined throughout the closure $\overline{A}$.

Proof. □

We remark on the differences between C^0, $C^{1,1}$, and C^2 regularity of boundaries ∂A. For C^2 boundary, the medial axis $M(A)$ will be disjoint from A. However for $C^0, C^{1,1}$ regularity, the medial axis $M(A)$ will extend into the boundary ∂A. For simplicity we prefer C^2-regularity, although $C^{1,1}$ regularity frequently occurs (i.e. when A is convex polyhedra).

4.

Now we propose a more interesting mass transport interpretation of medial axis transforms. Let A be bounded open subset of $\mathbb{R}^N$, with boundary ∂A, and probability measures μ, ν as previously defined in (2), (4). Then we choose cost $c = \tilde{d_\alpha} : A \times \partial A \to \mathbb{R}$ defined by restricting the completion to $A \times \partial A \subset \overline{A} \times \overline{A}$. The subvarieties Z_2 and $M(A)$ do not coincide set-theoretically, but we conjecture that they do coincide topologically:

Theorem 7 (Work-In-Progress)**.** *Let A be bounded open subset of $\mathbb{R}^N$. Let $c = \tilde{d_\alpha}$ be the metric completion of d_α to $\overline{A}$ (Prop. 6), and let $Z = Z(\mu, \nu, c) : 2^{\partial A} \to$*

2^A *be the Singularity functor with respect to* (μ, ν, c) *as defined in* (2), (4). *Then sufficient (UHS) Conditions are satisfied to apply Theorem B [Mar, Thm.3.4.3.], and the inclusion* $Z_2 \hookrightarrow A$ *is a homotopy isomorphism and even a strong deformation retract.*

Lemma 8. *For every* $\alpha \geq 0$, *the restricted cost* $c = \tilde{d_\alpha}^2/2 : A \times \partial A \to \mathbb{R}$ *satisfies the following (Twist) condition: for every* $x \in A$, *the gradient mapping*

$$\partial A \to T_x A, \ y \mapsto \nabla_x c(x, y)$$

is injective.

Proof. We take advantage of fact that c is a Lagrangian cost defined by an action principle. According to [Vil09, Prop.10.15, pp.235], the gradient $\nabla_x c(x, y)$ is equal to $\frac{-1}{2}\rho(x).\gamma'(0)$, where ρ is the conformal factor $\rho(x) = dist(x, \mathbb{R}^N - A)^{-\alpha}$, and where $\gamma'(0)$ is the initial tangent vector of an action-minimizing curve γ in A with $\gamma(0) = x$, $\gamma(1) = y$. The nonpositive curvature of g_α implies action-minimizing curves exist. Since the conformal factor ρ is nonvanishing, and since geodesics in Riemannian manifolds are determined by their initial point and initial tangent vector, we conclude $y \mapsto \nabla_x c(x, y)$ is injective, as desired. □

That c satisfies the above (Twist) condition implies the uniqueness of c-optimal semicouplings from μ to ν. [ref]. The above Lemma [ref], and the identity

$$\nabla_x c(x, y) = \frac{-1}{2}\rho(x)\gamma'(0)$$

implies the gradient of the cross-difference c_Δ is readily computed

$$\nabla_x c_\Delta(x, y_0, y_1) = \frac{\rho(x)}{2}.[\gamma_1'(0) - \gamma_0'(0)],$$

where γ_0, γ_1 are the g_α-geodesics satisfying

$$\gamma_0(0) = \gamma_1(0) = x, \ \gamma_0(1) = y_0, \ \gamma_1(1) = y_1.$$

[Incomplete:]

5.

The completion of Hubbard's $1/d$-distance and the cost $c = \tilde{d_\alpha}$ yields an alternative to the medial axis $M(A)$ in the subvariety Z_2 defined by c-optimal couplings. We propose this construction of Z_2 yields a useful improvement over the conventional definition of $M(A)$ per (1). For instance the medial axis is defined on the category of open subsets A of $\mathbb{R}^N$, whereas the functors Z are more generally defined for measure spaces.

An oftentimes frustrating property of $M(A)$ is it's notorious instability (c.f. [Sun+13, §1]). Small perturbations of the open subset A often leads to large changes in the medial axis $M(A)$. Therefore $M(A)$ suffers large variability when A has background noise. Many authors have suggested modified media axes (c.f. [FLM03], [TH03] and references therein) which "filter out" possible noise. Moreover, when A_k, $k = 1, 2, \ldots$, is a sequence of $C^{1,1}$ open subsets converging in Gromov-Hausdorff topology to a C^2 subset A_∞, then the sequence of medial axes $M(A_k)$ will not converge to $M(A_\infty)$, i.e. we find $\lim_{k\to+\infty} M(A_k) \neq M(A_\infty)$. Therefore the medial axis transform is not continuous with respect to Gromov-Hausdorff convergence.

On the other hand, it's well-known that optimal transportation enjoys strong continuity properties, and c-optimal semicouplings vary continuously (in appropriate narrow topology) with perturbations of μ, ν (c.f. [Vil09, Thm. 28.9, pp.780]). More precisely:

6. Conclusion

In conclusion, Blum identified the medial axis transform as convenient/efficient mode of describing objects, and heuristics showed the inclusions $M(A) \hookrightarrow A$ were always homotopy isomorphisms. However Blum's medial axis is but a particular instance of a more useful topological object, namely Z_2 the contravariant functors $Z(\mu, \nu, c) : 2^{\partial A} \to 2^A$. This Z_2 is stable topological object, and the inclusions $Z_2 \hookrightarrow A$ are identified as homotopy isomorphisms when the (UHS) Conditions (6), (7) hold throughout the open complement $A - Z_2$. Thus we propose more stable topological "folk-theorems" regarding a mass transport extension of so-called medial axis transforms, and Theorem B from [Mar].

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