A SHORT PROOF THAT MEDIAL AXIS TRANSFORM IS HOMOTOPY-ISOMORPHISM

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The goal of this article is to present a simple proof of a well-known folk theorem concerning Blum's Medial Axis Transform [Blu67]. Let A be a bounded open subset of $\mathbb{R}^N$. The medial axis $M(A)$ defined by Blum consists of all $x \in A$ for which $dist(x, \partial A)$ is attained by at least two distinct points,

$$M(A) := \{x \in A \mid \#argmin_{y\in\partial A}\{d(x,y)\} \geq 2\}. \tag{1}$$

Blum conjectured that the inclusion $M(A) \hookrightarrow A$ is a homotopy-isomorphism. This implies $M(A)$ contains all the topology of A, and is connected whenever A is. A formal proof of the conjecture for bounded open subsets of $\mathbb{R}^N$ is established in [Lie04]. The present article contains a short proof of the above homotopy-isomorphism. The reader should compare the methods of the present article to Lieutier's.

A "ball" in this article designates some Euclidean open ball contained in $\mathbb{R}^N$. The geodesic segment between a pair of points x, y is denoted $[x, y]$. The Riemannian exponential function is denoted exp_x for every x.

For $x \in A$, let $m(x)$ be the centre of the maximal ball containing x and contained in A. The maximal ball is unique for every x. The boundary of the maximal balls is always tangent to the boundary ∂A at two points (or more).

Definition 1 (Max-Centre Map)**.** Let A be open bounded subset of $\mathbb{R}^N$. For every $x \in A$, let $m(x) \in A$ be the centre of the unique maximal ball containing x and contained in A.

The max-centre $x \mapsto m(x)$ defines a self map $m : A \to A$. The self-map $m : A \to A$ is not one-to-one, and the inverse m^{-1} is not everywhere uniquely defined.

Lemma 2. *Let A be a bounded open subset of $\mathbb{R}^N$. The max-centre map $m : A \to A$ is continuously differentiable.*

Definition 3. Let $V = V(A) \subset A$ be the locus of non-differentiability of the inverse m^{-1} of the max-centre map $m : A \to A$.

The Inverse Function Theorem [Spi71][pp.35] says the locus of non-differentiability $V(A)$ coincides with zero locus $Jac(m) = det(Dm) = 0$ and is therefore a closed subset of A.

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The max-centre construction has direct interpretation in terms of shape operator, principal curvatures, and Tube Formula as explained by Gromov [Gro91][pp.16-17], [MS09][Appendix, pp.118]. If $\mathbf{n}_0, \mathbf{n}_1$ are inward pointing unit normals at $y_0, y_1 \in \partial A$ (where inward is relative to A), then inward equidistant deformations

$$exp_{y_0}(\epsilon \mathbf{n}_0) \text{ and } exp_{y_1}(\epsilon \mathbf{n}_1) \tag{2}$$

are disjoint for sufficiently small $\epsilon > 0$. If A is locally concave at y_0, y_1, then inward deformations will diverge and be disjoint for every $\epsilon > 0$. However convexity of A (positive curvature) will refocus the normals $\mathbf{n}_0, \mathbf{n}_1$, and inward deformations will converge, and the exponentials in (2) will coincide. But now it is convenient to recall an important fact from Riemannian geometry, that "focalization is impossible before the cut locus", [Vil09][pp.180-1]. No focalization before the cut locus implies a useful No-Crossing result. Namely, if y_0, y_1 are distinct boundary points, then the geodesic segments $[y_0, m(y_0)]$ and $[y_1, m(y_1)]$ are disjoint except when $m(y_0) = m(y_1)$, in which case

$$[y_0, m(y_0)] \cap [y_1, m(y_1)] = \{m(y_0)\}.$$

No Crossing also implies the union

$$A = \cup_{y \in \partial A} [y, m(y)]$$

is a type of "needle decomposition" of A. No Crossing also implies the existence and continuity of a map $h : A \times [0, 1] \to A$ defined by

$$h(x, s) = [x, m(x)]_{s.dist(x,m(x))}. \tag{3}$$

Here $z = [x, y]_s$ denotes means the unique point z on the geodesic segment joining x, y and with $dist(x, z) = s.dist(x, y)$.

Proposition 4. *The map $h : A \times [0, 1] \to A$ defined by equation (3) is a continuous strong deformation retract of A onto $V(A)$ (Def.3). That is h satisfies:*
(i) $h(x, 0) = x$ for all $x \in A$.
(ii) $h(x, 1) \in V(A)$ for all $x \in A$.
(iii) $h(x, s) = x$ for all $x \in V(A)$, $0 \leq s \leq 1$.

Corollary 5. *The inclusion $V(A) \hookrightarrow A$ is a homotopy-isomorphism.*

Thus we find the *closed* subset $V(A)$ contains all the topology of A. Strictly speaking $V(A)$ contains Blum's $M(A)$. Oftentimes $M(A)$ is not a closed subset of A, and in fact we find $V(A)$ is the topological closure $V(A) = \overline{M(A)}$.

The proof of 4 appears to extend to complete Riemannian manifolds which admit unique geodesics between the necessary points.

References

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