ALEXANDROV SPACES, KANTOROVICH SINGULARITY, AND SOULS AND SPLITTING THEOREMS

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1.

Throughout this article we let (X, d) designate a noncompact, complete, connected finite-dimensional Alexandrov space. So X is a length space satisfying $\kappa \geq 0$ sectional curvature conditions. In otherwords every quadruple (a, b, c, d) in X satisfies Toponogov's comparison condition, c.f. [Vil09, Ch.26, pp.738], [MT07, pp.53–55]. For a basepoint $x_0 \in X$, let $M = M(x_0)$ be the set of all geodesics λ in X satisfying:

(i) the geodesic λ passes through x_0;
(ii) the geodesic λ is distance minimizing over every compact subinterval;
(iii) the geodesic is maximally nonextendible.

For every $x_0 \in X$, we abbreviate $M^*(x_0) \subset M(x_0)$ as the subset of noncompact geodesics.

Lemma 1. *If X is connected complete noncompact Alexandrov, then $M^*(x_0)$ is nonempty for every $x_0 \in X$.*

Proof. [MT07, Lemma 2.1] □

In otherwords there exists distance minimizing asymptotic geodesic rays. The set M of geodesics contains evidently three types: the geodesics λ are either

(a) compact; or
(b) noncompact and doubly-ended; or
(c) noncompact and singly-ended.

It is necessary to emphasize that $M^*(x_0)$ varies lower semicontinuously with respect to the choice of x_0. For instance if x_0 is a regular point on an infinite flat cone, then $M^*(x_0)$ is a singleton, whereas if $x_0 = v$ is the cone vertex, then $M^*(v)$ is infinite and parameterized by an $N-1$-sphere on N-dimension cones.

Our purpose is to demonstrate how methods of Kantorovich Singularity and optimal semicouplings from [Mar] establishes two basic theorems of Alexandrov geometry nearly simultaneously. In case M^* contains a doubly-ended geodesic, then our arguments below will establish Gromoll-Cheeger's Splitting theorem [MT07, Thm.2.11,

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pp.58]; otherwise we use the geodesics of M^* to establish the Cheeger-Gromoll-Perelman's Soul theorem [MT07, Thm.2.7, pp.56] for singular Alexandrov spaces.

For any $x_0 \in X$, $\lambda \in M^*(x_0)$, let $h_\lambda : X \to \mathbb{R}$ be the unique horofunction satisfying $h_\lambda(x_0) = 0$, and defined by the familiar formula

$$h_{\lambda,x_0}(x) := \lim_{t \to +\infty} d(\lambda(t), x) - t.$$

We observe $h_{\lambda,x_0}(x) \geq -d(x, x_0)$, and h_{λ,x_0} diverges to $-\infty$ along the geodesic λ. Our curvature hypothesis $\kappa \geq 0$ implies h_λ is geodesically concave function and superlevel sets $\{h_{\lambda,x_0} \geq T\}$ are totally convex subsets of X for all $T \in \mathbb{R}$. (The same definition implies h_λ is convex in nonpositive curvature $\kappa \leq 0$). If the geodesic λ is doubly-ended, then h_λ will be symmetric with respect to x_0 and approaches values $\pm\infty$ as arc-parameter $\lambda(s)$ diverges to $s \to \mp\infty$.

For $\lambda \in M^*$, we choose real numbers $t = t(\lambda) \in \mathbb{R}$ for which $1/|t|$ is numerically large, say $t < .00001$.

Lemma 2. *For every $x_0 \in X$, if the parameter $t : M^*(x_0) \to \mathbb{R}$ is sufficiently small ($t \approx 0$), then the excision $X_0 := X[t]$ is a nontrivial compact totally-convex subset of X.*

Proof. The horofunctions h_λ are concave, therefore the excision $X - H_{\lambda,t}$ is a totally convex subset of X. The intersection $\cap_{\lambda \in M^*} X - H_{\lambda,t}$ is a totally convex subset. Moreover the completeness of X implies all the minimizing geodesics in X_0 are compact and Lemma [ref] implies X_0 is a compact subset. □

The excision X_0 is a compact convex boundary ∂X_0. The boundary ∂X_0 is "cellulated" by the boundaries $\partial H_{\lambda,t}$ of the excised horoballs. Moreover the following claim may be established:

Lemma 3. *The inclusion $X_0 \hookrightarrow X$ is a homotopy-isomorphism, and there exists a continuous strong deformation retract $X \rightsquigarrow X_0$.*

Proof. □

The above constructions lead us to our semicoupling program. The excision X_0 has a canonical Hausdorff measure $\sigma := \mathscr{H}_{X_0} = \mathscr{H}_X 1_{X_0}$, and the excision boundary ∂X_0 has canonical Hausdorff measure $\tau := \mathscr{H}_{\partial X_0}$. The measures σ, τ are designated our source and target measures, respectively.

We need determine cost. For pairs $x, y \in X$, we may compare $d(x, y)$ to the signed distances between horospheres $h_\lambda(x) - h_\lambda(y)$, which we observe is independant of the basepoint x_0 defining $h_\lambda = h_{\lambda,x_0}$. Concavity of h_λ implies the function

$$b(x, y) := \inf_\lambda \{h_\lambda(x) - h_\lambda(y)\} \tag{1}$$

is concave in the x-variable, for every choice of $y \in X$. If $(x, y) \in X_0 \times \partial X_0$, then $b(x, y) \geq 0$ with equality if and only if $x \in X_0$ and occupies the same horosphere component as y. Compactness of X_0 implies the superlevels of $\{x \in X_0 \mid b(x, y) \geq T\}$, for fixed $y \in \partial X_0$, $T \geq 0$, are compact convex subsets of X_0.

The triangle inequality implies

$$0 \leq b(x, y) \leq h_\lambda(x) - h_\lambda(y) \leq d(x, y), \text{ for } (x, y) \in X_0 \partial X_0 \tag{2}$$

with equality $h_\lambda(x) - h_\lambda(y) = d(x, y)$ if and only if x, y lie on minimizing ray λ.

Observe $0 < b(x, y)$ whenever $x \in X_0 - \partial X_0$. Throughout $X_0 \times X_0$ we find b defines a type of "distance" function. Indeed b is readily found to satisfy a triangle inequality and is symmetric $b(x, y) = b(y, x)$. However caution needs be exercised since b is possibly degenerate, having $b(x, y) = 0$ for $x \neq y$.

Lemma 4. *The pairing $b : X \times X \to \mathbb{R}_{\geq 0}$ defined by equation (1) is a (possibly degenerate) distance function.*

The basic idea of this article is to treat $b(x, y)$ as a type of distance on X, and restrict b to the subset $X_0 \times \partial X_0$ defined earlier in 2. For instance if M^* is a singleton, then $b(x, y) = 0$ if x, y both occupy the same horosphere centred at λ.

Lemma 5. *If M^* contains sufficiently many rays at large angle, then b is a nondegenerate distance function on X_0.*

Having nominated a distance $c = b$, we next turn to distance maximizing transports. Indeed distance minimization appears less useful for our purposes given the nonnegativity (2) and equality occuring exactly when $x \in \partial X_0$. This is analogous to our insistence on anti-quadratic "repulsion" costs in [Mar, c.f. Ch.4].

Definition 6. Fix x_0, M^*, definition of b (1). Let $X_0, \partial X_0$ be excisions with small parameter t. Let μ, ν be the measures on $X_0, \partial X_0$. Then let $\Pi = \Pi_{x_0}$ be the set of maximizers of the following maximization program:

$$\max_{\pi \in SC(\mu,\nu)} \int_{X_0 \times \partial X_0} b(x, y) d\pi(x, y). \tag{3}$$

The regularity properties of maximizing measures $\pi \in \Pi_{x_0}$ will have strong dependance on the basepoint x_0. Indeed when b is degenerate, then Π is not a singleton, and we need choice of "canonical" coupling π'. Here we follow an idea of Kim-Pass []. The canonical choice of π^* in Π is the coupling which minimizes the total $d^2/2$ cost:

$$\{\pi^*\} = argmin_{\pi \in \Pi} \int_{X_0 \times \partial X_0} d(x, y)^2/2 d\pi(x, y). \tag{4}$$

Here $d^2/2$ refers to the original distance d of the Alexandrov space X.

Proposition 7. *For every basepoint $x_0 \in X$, there exists a unique b-maximizing coupling $\pi \in \Pi_{x_0}$ which has minimal $d^2/2$-transport cost.*

Proof. □

If $\psi : \partial X_0 \to \mathbb{R} \cup \{+\infty\}$ is a potential, then the b-transform of ψ is defined

$$\psi^b(x) := \sup_{y \in \partial X_0} \{\psi(y) + b(x, y)\}.$$

If $\phi : X_0 \to \mathbb{R} \cup \{-\infty\}$ is a potential on X_0, the b-transform of ϕ is defined

$$\phi^b(y) := \inf_{x \in X_0} \{\phi(x) - b(x, y)\}.$$

A potential ψ is b-convex, and satisfying $(\psi^b)^b = \psi$ if

$$\psi(\overline{y}) = \inf_{x \in X_0} \sup_{y \in \partial X_0} \{\psi(y) + b(x, y) - b(x, \overline{y})\}$$

for all $\overline{y} \in \partial X_0$.

The reader will observe that $\psi(y) \leq \psi^{bb}(y)$ for arbitrary (possibly nonconvex) functions ψ. The key observation is the inequality

$$\psi^b(x) - \psi(y) \geq b(x, y),$$

with equality if and only if $x \in \partial^b \psi(y)$ for b-convex potentials $\psi^{bb} = \psi$. The following technical lemma is important to our results.

Lemma 8. *Let ψ be a b-convex potential on ∂X_0, $\psi^{bb} = \psi$. For all $y \in spt(\psi) \subset \partial X_0$, the b-subdifferential $\partial^b \psi(y)$ is a totally convex subset of X_0.*

Proof. □

Consequently if $Z = Z(\mu, \nu, b)$ is the Kantorovich functor $Z : 2^{\partial X_0} \to 2^X$, then for all closed subsets Y_I of ∂X_I, the cell $Z(Y_I)$ is a totally convex subset of X_0.

A difficulty in extending our arguments from [Mar] to singular Alexandrov spaces is that gradients and gradient projections are not easy to define on the singular spaces. Indeed our thesis constructs deformation retracts defined via gradient flow towards poles of a vector field denoted $\eta(x, avg)$. Moreover retractions deeper into the singularity structure require gradients ∇^Z over closed subvarieties Z of X. If f is function, then we shall require the gradient of the restriction $f|Z$ of f to Z, having $\nabla^Z f = proj_Z \nabla f$. [Petrunin/Perelman prove existence of well-defined gradient flows in (singular) Alexandrov spaces [PP94]]

2. Splitting

Theorem 9. *Let μ be source measure on X_0, and ν a target measure on ∂X_0, and with cost b as defined in (1) with respect to a basepoint x_0. If $mass(\mu)/mass(\nu) \approx 1^+$, then the active domain A of the "canonical" b-maximal semicoupling π^* defined in (4) is a strong deformation retract of X_0.*

Moreover if M^* contains a doubly-ended minimizing ray, then the active domain $A = Z_1$ splits isometrically $A \simeq [-T, +T] \times Z_2$, where Z_2 consists of all source points $x \in X_0$ such that $\partial^b \psi^b(x) \geq 2$.

Incomplete. □

If λ is a doubly-ended minimizing ray, then we can define the excision $X_0, \partial X_0$ obtained by excising horoballs centred at $\lambda(-\infty)$, $\lambda(+\infty)$ and equal small radius. If y_-, y_+ are points on the "opposite" horospheres, then the cross-difference function $b_\Delta(x, y_-, y_+)$ defines a "submersion-type" mapping $b_\Delta(-, y_-, y_+) : A \to \mathbb{R}$. Strictly speaking, a Riemannian submersion $b_\Delta : A \to \mathbb{R}$ is a mapping such that $||\nabla_x b_\Delta|| = 1$ for all $x \in A$.

[Error: need use $d^2/2$-minimization to specifically define/correct the cross-difference b_Δ?]

So we find two parameters T and $a := \int_{X_0} \mu / \int_{\partial X_0} \nu$. As a decreases 1^+, the active domain $A = A_a$ of the b-maximal semicoupling fills the source X_0. As $T \to +\infty$ the isometric splitting

$$[-T, +T] \times Z_2$$

converges to an isometric splitting

$$(-\infty, +\infty) \times Z_2.$$

This recovers Toponogov's splitting principle for complete finite-dimensional singular Alexandrov spaces.

3. Nonnegative Ricci Curvature

Let (X, g) be Riemannian manifold with nonnegative Ricci curvature. Then for every ray γ in X the horofunction $h_\gamma : X \to \mathbb{R}$ is subharmonic, and the divergence of the gradient flow $div(\nabla_x h_\gamma)$ is nonpositive (≤ 0) throughout X. The important Splitting Theorem of Cheeger-Gromoll is the following:

Theorem 10 (Cheeger-Gromoll [ref)**.** *] Let (X, g) be Riemannian manifold with nonnegative Ricci curvature. Suppose $M^*(x_0)$ contains a doubly-ended minimizing ray for some basepoint $x_0 \in X$. Then there exists totally convex subset $Y \subset X$ and an isometric splitting $X \simeq Y \times \mathbb{R}$.*

It's well-known that Toponogov proved the existence of isometric splittings when (X, g) is a smooth Alexandrov manifold [ref]. Our goal is to establish a splitting theorem for singular Alexandrov spaces using the Kantorovich Singularity functor.

[Include McCann's interpretation using convex functions: proves Toponogov for singular Alexandrov?]

4. Cross-Differences and Connected Fibres

Definition 11 (Cross-Difference b_Δ)**.** For cost $b : X \times Y \to \mathbb{R}$ we define the cross-difference $b_\Delta(x) = b_\Delta(x; y_0, y_1) := b(x, y_0) - b(x, y_1)$ with variables $x \in X$ and $y_0, y_1 \in Y$.

If c satisfies (Twist) condition, then $\nabla_x b_\Delta(x; y_0, y_1)$ is nonvanishing for $y_0 \neq y_1$. The hypothesis that b_Δ is critical point free has many interesting topological consequences. The following section elaborates an elementary fibration argument, which together with nonvanishing gradients, is the basis of our topological applications. The Proposition **??** has the following basic corollary: if the cross-difference b_Δ has a single connected fibre, then all fibres are connected.

If (Twist) condition is satisfied, then a basic form of the (UHS) conditions further states the nonvanishing gradients $\nabla_x b_\Delta(x, y_0, y_1)$, for fixed $y_0 \neq y_1$, are bounded away from 0 uniformly with x.

Remark 12. Whenever C, D are closed connected subsets of $\mathbb{R}^N$, then $C \cup D$ is connected if and only if C, D intersect $C \cap D \neq 0$. This finds useful consequences concerning the connectivity of Kantorovich's contravariant functor $Z : 2^Y \to 2^X$, with respect to c-optimal semicouplings. When the gradients $\nabla_x b_\Delta(x; y_0, y_1)$ are nonvanishing, then fibres $b_\Delta(-; y_0, y_1)^{-1}(t) \subset X$ are topologically-connected subsets, and this implies the cells $Z(Y_I)$ are connected subsets for Y_I supported on $suppZ$.

References

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