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Exercise sheet *Ringvorlesung*

The Helmholtz equation - Scattering from a sound-soft obstacle

We consider the Helmholtz equation. Let $k > 0$ and let $D \subset \mathbb{R}^3$ be a scattering object. For a given $u^i \in H^1_{\text{loc}}(\mathbb{R}^3)$, for which

$$\Delta u^i + k^2 u^i = 0 \quad \text{in } \mathbb{R}^3$$

we want to find $u^s \in H^1_{\text{loc}}(\mathbb{R}^3 \setminus \overline{D})$ such that $u = u^s + u^i$ solves the partial differential equation

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D} \tag{1a}$$

$$u = 0 \quad \text{on } \partial D \tag{1b}$$

$$\lim_{r=|x|\to\infty} r\left(\frac{\partial u^s}{\partial r} - \mathrm{i}ku^s\right) = 0 \quad \text{uniformly.} \tag{1c}$$

One can show: There exists exactly one $u \in H^1_{\text{loc}}(\mathbb{R}^3 \setminus \overline{D})$ that satisfies (1).
In order to obtain a boundary integral equation, we make the (naive, since $k > 0$) ansatz

$$u^s(\mathbf{x}) = \mathcal{S}\varphi(\mathbf{x}) := \int_{\partial D} \Phi(\mathbf{x},\mathbf{y})\varphi(\mathbf{y})\mathrm{d}s(\mathbf{y}) \quad \text{where } \Phi(\mathbf{x},\mathbf{y}) = \frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|}, \quad \text{for } \mathbf{x} \in \mathbb{R}^3 \setminus \overline{D},$$

which implies that

$$-u^i(\mathbf{x}) = \mathrm{V}\varphi(\mathbf{x}) := \gamma\mathcal{S}\varphi(\mathbf{x}) = \int_{\partial D} \Phi(\mathbf{x},\mathbf{y})\varphi(\mathbf{y})\mathrm{d}s(\mathbf{y}), \quad \text{for } \mathbf{x} \in \partial D.$$

Therefore,

$$u^s(\mathbf{x}) = -(\mathcal{S}\mathrm{V}^{-1}u^i)(\mathbf{x}). \tag{2}$$

Exercise 1. Let $D = B_1(0)$ and let $u^i(\mathbf{x}) = \mathrm{e}^{\mathrm{i}k\mathbf{x}\cdot\mathbf{d}}$,where $\mathbf{d} = [1,0,0]^\top \in S^2$.
Implement (2) using the boundary element library BEMPP. Visualize the scattered field u^s, as well as the total field u in the plane $z = 0$.
Home work 1. Choose

$$u^i(\mathbf{x}) = \frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|} \quad \text{where } |\mathbf{y}| < a < 1.$$

Compare the exact solution

$$u^s(\mathbf{x}) = -\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|} \quad \text{where } |\mathbf{x}| > 1$$

to the approximation that is computed by BEMPP using the ansatz (2).

The functions $\delta_{\mathbf{TR}}$ and $\delta_{\mathbf{BDF2}}$

We saw that the functions δ_{TR} and δ_{BDF2}, which will determine the wave numbers later on, are given by

$$\delta_{\mathrm{TR}}(z) \;=\; 2\frac{1-z}{1+z}\,, \quad \text{and} \quad \delta_{\mathrm{BDF2}} \;=\; (1-z)+\frac{1}{2}(1-z)^2\,.$$

Exercise 2. Choose $M = 100$ time steps, $T = 5$ as the final time and $\lambda = (10^{-14})^{1/(2M+1)}$ as the radius of the circle from the Cauchy integral theorem. The temporal step size is given by $\tau = T/M$. Plot the values

$$\frac{1}{\tau}\delta_{\mathrm{S}}\left(\lambda\xi_{M+1}^{-\ell}\right)\,, \quad \ell = 0,1,\dots,M, \quad \xi_{M+1} = \mathrm{e}^{2\pi\mathrm{i}/(M+1)}\,, \quad \mathrm{S}\in\{\mathrm{TR},\mathrm{BDF2}\} \tag{3}$$

and interpolate these to obtain a curve in $\mathbb{C}$. Modify M. How do the curves described by the values in (3) change?
Home work 2.

Do the same as before with the truncated trapezoidal rule (TTR). You can find it on page 120 in the book by L. Banjai and F. Sayas *Integral Equation Methods for Evolutionary PDE*, Springer, Cham, 2022.

The time-dependent scattering problem - Scattering from a sound-soft object

Now, we study the time-dependent scattering problem, which reads as follows.
For a given $g \in \mathbf{TD}(H^{1/2}(\partial D))$, find $u^s \in \mathbf{TD}(H^1_\Delta(\mathbb{R}^3\setminus\overline{D}))$, such that

$$\partial_t^2 u^s - \Delta u^s \;=\; 0 \quad \text{in } \mathbf{TD}(L^2(\mathbb{R}^3\setminus\overline{D}))\,, \tag{4a}$$

$$\gamma u^s \;=\; g \quad \text{on } \mathbf{TD}(H^{1/2}(\partial D))\,. \tag{4b}$$

In what follows, let $g = -\gamma u^i$, where u^i is a solution to

$$\partial_t^2 u^i - \Delta u^i \;=\; 0 \quad \text{in } \mathbb{R}^3\,.$$

As we saw before, the unique solution to (4) is given by $u = (\mathcal{S}\mathrm{V}^{-1})(\partial_t)g$.
Now we want to find a numerical approximation to it.
Exercise 3. Implement the convolution quadrature method to approximate a solution to (4). Let $D = B_1(0)$,

$$u^i(\mathbf{x},t) \;=\; f(A(\mathbf{x}\cdot\mathbf{d}-(t-t_{\mathrm{lag}}))) \quad \text{where} \quad f(z) \;=\; \begin{cases} \mathrm{e}^{-1/(1-z^2)}\,, & |z|<1 \\ 0\,, & |z|\ge 1 \end{cases}$$

and let $A = 2$, $\mathbf{d} = [1,0,0]^\top \in S^2$ and $t_{\mathrm{lag}} = 2$. Choose $M = 100$ time steps, $T = 5$ as the final time and $\tau = T/M$ as the temporal step size. Use the BDF2 method in for implementing the convolution quadrature method.
Plot the total field $u + u^i$ in the planes $z = 0$ and $y = 0$ together with the scattering object D at the times $t_n = n\tau, n = 0,1,\dots,M$ or generate a video.
Home work 3.

(a) Modify $A, \mathbf{d}, t_{\mathrm{lag}}, f, T$ and M. What changes? Does the numerical solution become unstable for small τ? If so, why? Is it possible to resolve this issue? If so, how?

(b) (Requires lots of computational resources) Verify the order of the convolution quadrature method numerically. For this purpose, use

$$u^i(\mathbf{x},t) \;=\; \frac{f(A(t-|\mathbf{x}|))}{4\pi|\mathbf{x}|}$$

and f as above and evaluate the numerical approximation to $u = -(\mathcal{S}\mathrm{V}^{-1})(\partial_t)u^i$ at the point $\mathbf{x} = [2,0,0]$. Compare the approximation to the exact solution given by $-u^i$.