MTH 261 Lecture Notes

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Contents

Chapter 1

Introduction to Linear Algebra

1.1 System of Linear Equations

Linear Algebra is the area of math concerning liner equations/functions that are represented in vector space and through matrices. If Calculus is the foundational language of mathematics, then Linear Algebra is the foundational language of STEM.

> ***Remark.*** Keep in mind that while Linear Algebra utilizes matrices, it is just a tool to solve problems with. The study is **NOT** of matrices.

Definition 1.1 (Linear Equation)

An equation where $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$, where $a_1, a_2, a_n, b \in$ as constants of $\mathbb{C}$ and in $\mathbb{R}$

Definition 1.2 (Linear System)

A collection of linear equations with the same variable

Definition 1.3 (Solution & Solution Set)

A solution satisfies all equations in a system simultaneously, while a solution set is all possible solutions

Definition 1.4 (Equivalent Solutions)

Two systems with the same identical solution set

Definition 1.5 (Types of Solutions)

There are two major classification with solution, which are broken into three major solutions:

- Inconsistent (no solutions)
- Consistent (at least one solution)
  - Unique Solution
  - Infinite Solutions

Since the systems we are exploring and purely linear, there will not be two, three, or more solutions.

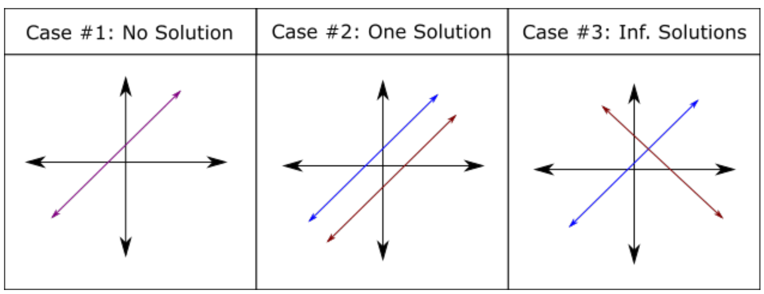

Figure 1.1: Types of solutions in a linear system

Remark. This is true for all linear systems in all space

Definition 1.6 (Matrix)

A rectangular array of numbers, often used to compress systems. Let's say the following system of equations is given:

$$\begin{aligned} x - 7y \qquad\quad + 6t &= 5 \\ z - 2t &= -3 \\ -x + 7y - 4z + 2t &= 7 \end{aligned}$$

It can be reduced into the following matrix:

$$\begin{bmatrix} 1 & -7 & 0 & 6 \\ 0 & 0 & 1 & -2 \\ -1 & 7 & -4 & 2 \end{bmatrix}$$

Definition 1.7 (Augmented Matrix)

A standard matrix which also includes the $\mathbf{b}$ coefficient in the matrix. The following is the augmented matrix of the system of equations from Definition 1.6:

$$\begin{bmatrix} 1 & -7 & 0 & 6 \\ 0 & 0 & 1 & -2 \\ -1 & 7 & -4 & 2 \end{bmatrix} OR \left[\begin{array}{ccc|c} 1 & -7 & 0 & 6 \\ 0 & 0 & 1 & -2 \\ -1 & 7 & -4 & 2 \end{array}\right]$$

1.2 Row Reduction and Echelon Forms

1.3 Vector Equations

1.4 The Matrix Equation $A\mathbf{x} = \mathbf{b}$

1.5 Solution Sets of Linear Systems

1.6 Linear Independence

1.7 Introduction to Linear Transformations

Definition 1.8

A transformation T is called linear (linear transformation) if for all $\mathbf{u}$, $\mathbf{v}$ in the domain of T and c $\in \mathbb{R}$,

- $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u} + T(\mathbf{v})$
- $T(c\mathbf{u}) = cT(\mathbf{u}$

Therefore, we can determine that every matrix transformation ($T(\mathbf{x} = A\mathbf{x}$) is a linear transformation

To show that a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear:

1. Introduce $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n andc \in \mathbb{R}$. These must be arbitrary
2. Show that $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u} + T(\mathbf{v}$
3. Show that $T(c\mathbf{u}) = cT(\mathbf{u}$

Turns out, we could show that IF WE ALREADY KNOW that T is linear, then:

- $T(\mathbf{0}) = \mathbf{0}$ and
- $T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v}$

To show that T is NOT linear, either:

- Show $T(\mathbf{0} \neq \mathbf{0})$, or
- $T(\mathbf{u} + \mathbf{v}) \neq T(\mathbf{u}) + T(\mathbf{v})$ for specific vectors u & v, or

1.8 The Matrix of a Linear Transformation

Example 1.8.1

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ that first dilates vectors by a size of 2 then reflects across the line $x_2 = -x_1$. Assuming this transformation is linear, find the standard matrix for T

Solution

1. Dilates (expands) by $2 \to$ Doubles in size.
2. Reflects across $x_1 = -X_1(y = -x)$.

Chapter 2

Matrix Algebra

2.1 Matrix Operations

There are several ways of representing a matrix:

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}_i j \end{bmatrix} =$$

Definition 2.1

The diagonal entries of the $m \times n$ matrix i

Definition 2.2

The main diagonal is the collection diagonal entrias starting from the top left. **NOTE:** This may not include the bottom right due to the size of the matrix

Definition 2.3

Diagonal matrix has the form:

$$\begin{bmatrix} \square & 0 & 0 & 0 \\ 0 & \square & 0 & 0 \\ 0 & 0 & \square & 0 \end{bmatrix}$$

Definition 2.4

The zero matrix is any matrix whos entries are all zero.

Basic operations are the same between matrices and vectors (addition, subtraction, multiplication, and equality). Keep in mind, matrices must be the same size for addition/subtration operations.

Example 2.1.1 (Properties of Matrix Arithmitic)

Let A, B, C, be matrices of the same size, $r, s, \in \mathbb{R}$.

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$
3. $A + O = A$
4. $r(A + B) = rA + rB$
5. $(r + s)A = rA + sA$
6. $r(sA) = (rs)A$

NOTE: O is used a 0 in mathematics.

Matrix Multiplication

When B multiplies a vector $\mathbf{x}$ it starts from $\mathbf{x} \mapsto B\mathbf{x}$

Definition 2.5

If A is $m \times n$ and B is $n \times p$, then the product AB is the $m \times p$ matrix.

$$AB = [A\mathbf{b}_1, A\mathbf{b}_2 \ldots A\mathbf{b}_p]$$

Remark. The idea of non-communative multiplication is uncommon, however does occur on ocassion.

Example 2.1.2

Let A, B, C be matricies so that these are defined, $r \in \mathbb{R}$

1. $A(BC) = (AB)C$
2. $A(B + C) = AB + AC$
3. $(B + C)A = BA + CA$
4. $r(AB) = (rA)B = A(rB)$
5. If A is $m \times n$, then $I_m A = A = AI_n$

A few common pitfalls. In general:

- $AB \neq BA$
- Cancellation does not hold: $AB = AC \not\Rightarrow B = C$
- The Zero Product Principle does not hold: $AB = O \not\Rightarrow A = O$ or $B = O$

Example 2.1.3 (Transpose Properties)
Let A, B, be matricies so that these are defined, $r \in (R)$.

1. $(A^T)^T = A$
2. $(A+B)^T = A^T + B^T$
3. $(rA)^T = rA^T$
4. $(AB)^T = B^T A^T$

Remark. This idea of ranposition will come up frequently in Linear Algebra.

2.2 The Inverse of a Matrix

Definition 2.6 (Multiplicative Inverse)
If $c \in \mathbb{R}$ with $c \neq O$,then:

$$c \cdot c^{-1} = 1 \text{ and } c^{-1} \cdot c = 1$$

In other words:

$$A \cdot A^{-1} = I \text{ and } A^{-1} \cdot A = I$$

Example 2.2.1
Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq O$, then A is invertible, and $A^{-1} = \frac{1}{ad = bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. If $ad - bc = O$, A is singular.

Definition 2.7
If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the <u>determinant</u> of A, written $detA$ or $|A|$, is the number $ad - bc$.

Example 2.2.2

Let $A = \begin{bmatrix} 2 & 6 \\ -1 & 3 \end{bmatrix}$. Find $\det A$ & A^{-1}.

Solution: $\det A = (2)(3) - (6)(-1) = 12$

$A^{-1} = \frac{1}{detA} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$= \frac{1}{12} \begin{bmatrix} 3 & -6 \\ 1 & 2 \end{bmatrix}$

$=$

2.3 Characterizations of Invertible Matrices