MTH 261 Lecture Notes

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Contents

Chapter 1

Introduction to Linear Algebra

1.1 System of Linear Equations

Linear Algebra is the area of math concerning liner equations/functions that are represented in vector space and through matrices. If Calculus is the foundational language of mathematics, then Linear Algebra is the foundational language of STEM.

> **Remark.** Keep in mind that while Linear Algebra utilizes matrices, it is just a tool to solve problems with. The study is **NOT** of matrices.

Definition 1.1 (Linear Equation)

An equation where $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$, where $a_1, a_2, a_n, b \in$ as constants of $\mathbb{C}$ and in $\mathbb{R}$

Definition 1.2 (Linear System)

A collection of linear equations with the same variable

Definition 1.3 (Solution & Solution Set)

A solution satisfies all equations in a system simultaneously, while a solution set is all possible solutions

Definition 1.4 (Equivalent Solutions)

Two systems with the same identical solution set

Definition 1.5 (Types of Solutions)

There are two major classification with solution, which are broken into three major solutions:

- Inconsistent (no solutions)
- Consistent (at least one solution)
  - Unique Solution
  - Infinite Solutions

Since the systems we are exploring and purely linear, there will not be two, three, or more solutions.

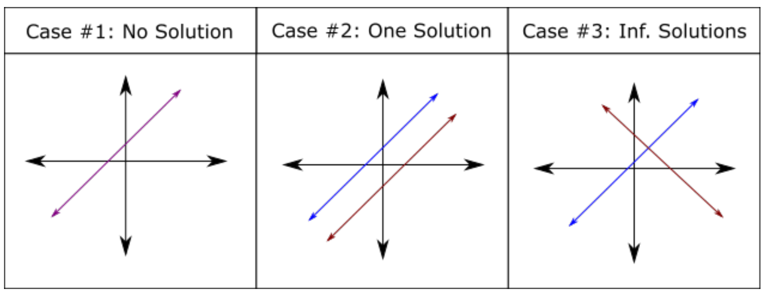

Figure 1.1: Types of solutions in a linear system

> **Remark.** This is true for all linear systems in all space

Definition 1.6 (Matrix)

A rectangular array of numbers, often used to compress systems. Let's say the following system of equations is given:

$$\begin{aligned} x - 7y \qquad + 6t &= 5 \\ z - 2t &= -3 \\ -x + 7y - 4z + 2t &= 7 \end{aligned}$$

It can be reduced into the following matrix: $\begin{bmatrix} 1 & -7 & 0 & 6 \\ 0 & 0 & 1 & -2 \\ -1 & 7 & -4 & 2 \end{bmatrix}$

Definition 1.7 (Augmented Matrix)

A standard matrix which also includes the **b** coefficient in the matrix. The following is the augmented matrix of the system of equations from Definition 1.6: $\begin{bmatrix} 1 & -7 & 0 & 6 \\ 0 & 0 & 1 & -2 \\ -1 & 7 & -4 & 2 \end{bmatrix}$ OR $\left[\begin{array}{ccc|c} 1 & -7 & 0 & 6 \\ 0 & 0 & 1 & -2 \\ -1 & 7 & -4 & 2 \end{array}\right]$

Definition 1.8 (Elementary Row Operations)

Elementary row operations are used to change matrices in a revertible manner.

1. **Replacement.** Sum of itself and a multiple of another row.
2. **Interchange.** Swap/Interchange two rows.
3. **Scaling.** Multiply all entries in a row by a nonzero constant.

These operations can be used with any matrix, and are typically used to create variations of the matrix that is beneficial to finding specific information.

Definition 1.9 (Row Equivilance)

Matrices are referred to as row equivilant if the matrices can be transformed into each other through elementary row operations.

Existence and Uniqueness

In analyzing a linear system, a few key questions are being asked:

1. Is the system consistent or inconsistent? (Solution or no solution)
2. Is the solution unique (one solution or many)?

> **Remark.** Keep in mind, since we are looking into linear systems, there may only be one solution or infinite solutions, none in between.

1.2 Row Reduction and Echelon Forms

There are specific forms in which a matrix can be in to provide valuable information about the matrices.

Definition 1.10 (Leading Entry)

The leftmost nonzero entry in a nonzero row.

In the case of the matrix below, x is the leading entry:

$$\begin{bmatrix} \underline{x} & 2 & 0 & 5 \end{bmatrix}$$

Definition 1.11 (Echelon Form (REF))

One of the primary forms, refereed to as row echelon form (often just called echelon form) has the following properties:

- All nonzero rows are above any rows of zeros
- Each leading entry is in a column to the right of the leading entry above it
- All entries in a column below a leading entry are zeros

Below is an example of a matrix in Echelon Form (REF):

$$\begin{bmatrix} \underline{4} & 0 & 4 & 3 \\ 0 & \underline{3} & 6 & 1 \\ 0 & 0 & \underline{1} & 8 \end{bmatrix}$$

The underlined numbers indicate the leading entries.

Definition 1.12 (Reduced Echelon Form (RREF))

Similar to REF, the reduced echelon form (RREF) takes the definition of the REF a few steps further

- The leading entries are 1
- Te leading entry (1) is the only nonzero entry in its column

Below is an example of a matrix in Echelon Form (REF):

$$\begin{bmatrix} \underline{1} & 0 & 0 & 3 \\ 0 & \underline{1} & 0 & 1 \\ 0 & 0 & \underline{1} & 8 \end{bmatrix}$$

Theorem 1.1 (Uniqueness of RREF)

Each matrix is only row equivalent to a single RREF matrix.

This is ONLY true for RREF. REF can be adjusted with row operations to have multiple matrices.

Pivots

Definition 1.13 (Pivot Positions and Columns)

Leading entries in REF and the leading 1's in RREF are referred to a pivot position. Similarly, a column containing a pivot position is referred to as a pivot column.

These pivots can be identified in the original matrix, and are not terms exclusively for REF and RREF (unlike leading entries).

Gauss-Jordan Elimination

Step 1. The leftmost non-zero column must be a pivot column, with the first row in the column being the pivot position.

$$A = \begin{bmatrix} 0 & 4 & 4 & 3 \\ 4 & 4 & 6 & 1 \\ 2 & 0 & 4 & 8 \end{bmatrix}$$

Step 2. Any nonzero entry in the pivot column can be a pivot. Use the interchange ERO to make it the first column.

$$\sim \begin{bmatrix} 2 & 0 & 4 & 8 \\ 4 & 4 & 6 & 1 \\ 0 & 4 & 4 & 3 \end{bmatrix}$$

Step 3. Use EROs to make all positions below pivot into zeros.

$$\sim \begin{bmatrix} 2 & 0 & 4 & 8 \\ 0 & 4 & -2 & -15 \\ 0 & 4 & 4 & 3 \end{bmatrix}$$

Step 4. Ignore the pivot row and repeat for all remaining rows (ad terminum; until completion/termination).

$$\sim \begin{bmatrix} 2 & 0 & 4 & 8 \\ 0 & 4 & -2 & -15 \\ 0 & 0 & 6 & 18 \end{bmatrix} = REF(A)$$

This will create a $A \to REF(A)$, and is common referred to as **Gaussian Elimination**. The following 2 steps will create $A \to RREF(A)$ and is referred to as **Gauss-Jordan Elimination**

Step 5. Make all pivots 1 through the scaling ERO.

$$\sim \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -\frac{1}{2} & -3\frac{3}{4} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Step 6. Begin with the rightmost pivot and use row operations to work upwards, making all entries above pivot positions to be zeros.

$$\sim \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -2\frac{1}{4} \\ 0 & 0 & 1 & 3 \end{bmatrix} = RREF(A)$$

Gauss-Jordan Elimination in Linear Systems

This method also helps solves linear systems when reduced down to RREF. Assume column 1 is x_1, column 2 is x_2, column 3 is x_3, and column 4 is b.

$$\begin{gathered} 4x_2 + 2x_3 = 3 \\ 4x_1 + 4x_2 + 6x_3 = 1 \\ 2x_1 + 4x_3 = 8 \end{gathered}$$

And therefore after applying Gauss-Jordan Elimination is:

$$\begin{cases} x_1 = -2 \\ x_2 = -2\frac{1}{4} \\ x_3 = 3 \end{cases}$$

In the case of the following where one row is only zeros,

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -2\frac{1}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix} = RREF(A)$$

we get:

$$\begin{cases} x_1 = -2 \\ x_2 = -2\frac{1}{4} \\ x_3 = x_3 \to x_3 \text{ is free} \end{cases}$$

Definition 1.14

Variables corresponding with a pivot column is called **basic variable**, while variables corresponding to a non-pivot column is referred to as a **free variable**.

There are three possible definitions of the solution set:

- **No solution.** We write ϕ (phi).
- **Unique Solution.** We write the single solution in a set (ie. $(1, 0, -1)$).
- **Infinite solutions.** Form a general solution by expressing basic variables in terms of the free variable, and notating free variables as free. Also called parametric solution, where free variables are the parameters.

Theorem 1.2

The following can be determined by analyzing the augmented matrix:

1. **A linear system is consistent** if and only if the rightmost column if not a pivot column, in addition to no false statements (ie. $0/neq1$). A false statement is a clear indicator of an inconsistent system.
2. **A linear system is consistent and unique** if there are no free variables.
3. **A linear system is consistent and has infinite solutions** if there are free variables.

1.3 Vector Equations

Definition 1.15

Matrix with one column is called column vector, or just a vector.

Definition 1.16 (Equality, Sum, and Scalar)

A vector is only equal if corresponding entries are equal:

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Vectors can only be added if they are in the same space (ie. $\mathbb{R}^2 vs \mathbb{R}^3$):

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \text{undefined}$$

A vector is scaled by multiplying each entry by the scalar, c:

$$c\mathbf{u} = 4 \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4(2) \\ 4(3) \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \end{bmatrix}$$

Graphing Vectors in $\mathbb{R}^2$

Fundamentally, vectors can be graphed with the definition:

$$\begin{bmatrix} a \\ b \end{bmatrix} = (a, b)$$

Then, vectors are graphed with a point and and arrow staring from the origin.

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$\mathbb{R}^2$ is the set of all points (vectors) on the plane.

Definition 1.17 (Parallelogram Rule for Addition)

If $\mathbf{u}$ and $\mathbf{v}\epsilon\mathbb{R}^2$, then $\mathbf{u}+\mathbf{v}$ is the 4th vertex of the parallelogram whose other vertices are $\mathbf{0}, \mathbf{u}, \mathbf{v}$.

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Vectors in $\mathbb{R}^3$ & $\mathbb{R}^n$

In $\mathbb{R}^3$, nearly all properties are identical to $\mathbb{R}^2$. Equality, addition, and scalar multiplication are identical. $\mathbb{R}^3$ is represented by:

$$\mathbb{R}^3 = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} | a, b, c\epsilon\mathbb{R} \right\}$$

Vectors in $\mathbb{R}^n$

Definition 1.18 ($\mathbb{R}^n$)

$\mathbb{R}^n$ is defined as the collection of n-tuples (or all lists) of real numbers, usually written as:

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

Moreover, the zero vector $\mathbf{0}$ is written as all entries are zero. The same definitions of equality, summation, and scaling apply as $\mathbb{R}^2$

Theorem 1.3 (Properties of $\mathbb{R}^n$)

Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ ϵ $\mathbb{R}^n$, c & d ϵ $\mathbb{R}$:

1. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ (Commutativity of Addition.)
2. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{v}+(\mathbf{u}+\mathbf{w})$ (Associativity of $+$.)
3. $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$ ($\mathbf{0}$ is the Additive Identity.)
4. $\mathbf{u}-\mathbf{u}=\mathbf{0}+(-1)\mathbf{u}=\mathbf{0}$) ($-\mathbf{u}$ is the Additive Inverse of $\mathbf{u}$.)
5. $c(\mathbf{u}+\mathbf{v})=c\mathbf{v}+c\mathbf{u}$ (Distributive I.)
6. $(c+d)\mathbf{u}=c\mathbf{u}+c\mathbf{u}$ (Distributive II.)
7. $c(d\mathbf{u})=(cd)\mathbf{u}$ (Associative Property of Scalar Multiplication.)
8. $1\mathbf{u}=\mathbf{u}$ (1 is the scalar Multiplicative Identity.)

Definition 1.19

The linear combination of $\mathbf{v}_1$,... ,$\mathbf{v}_p \epsilon \mathbb{R}^n$ with corresponding weights c_1,... ,$c_p \epsilon \mathbb{R}^n$ is the $\mathbb{R}^n$ vector.

$$c_1\mathbf{v}_1, c_2\mathbf{v}_2, c_3\mathbf{v}_3, \ldots, c_p\mathbf{v}_p$$

Example 1.3.1

Suppose $\mathbf{v}_1, \mathbf{v}_2 \epsilon \mathbb{R}^n$. Identify the linear combinations of $\mathbf{v}_1$ & $\mathbf{v}_2$.

$$\begin{gathered}
\mathbf{v}_1 + \sqrt{3}\mathbf{v}_2 \ \checkmark \\
\mathbf{v}_1 - \mathbf{v}_2 \ \checkmark \\
\mathbf{0} \ \checkmark \\
\mathbf{v}_1 \ \checkmark \\
\mathbf{v}_1 + \sqrt{3\mathbf{v}_2} \ \times \\
\mathbf{v}_1^2 + \mathbf{v}_2 \ \times \\
-\mathbf{v}_1 + 4\mathbf{v}_2 \ \checkmark \\
\pi\mathbf{v}_2 \ \checkmark
\end{gathered}$$

Example 1.3.2

We can see that a vector equation $x_1\mathbf{v}_1, x_2\mathbf{v}_2, \ldots, x_p\mathbf{v}_p = \mathbf{b}$ has the same solution set as the linear system whose augmented matrix is:

$$A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_p & \mathbf{b} \end{bmatrix}$$

So $\mathbf{b}$ can be generated by a linear combination of $\mathbf{v}_1$, ..., $\mathbf{v}_p$ iff there exists a solution to the linear system whose augmented matrix is A.

Definition 1.20

If $\mathbf{v}_1$, ..., $\mathbf{v}_p \epsilon \mathbb{R}^3$, we define the span of $\mathbf{v}_1$, ..., $\mathbf{v}_p$ to be the set of all linear combinations of $\mathbf{v}_1$, ..., $\mathbf{v}_p$ denoted:

$$span = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$$

INCLUDE EXAMPLE

Theorem 1.4

If $\mathbf{v}_1$, ..., $\mathbf{v}_p \epsilon \mathbb{R}^3$, then $\mathbf{0} \ \epsilon \ span\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$

Visualizing *span*

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1.4 The Matrix Equation $A\mathbf{x} = \mathbf{b}$

Definition 1.21

If A is an $m \times n$ matrix with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$ and $\mathbf{x} \epsilon \mathbb{R}^n$, the the product $A\mathbf{x}$ is the linear combination of the column of A with the corresponding weights in $\curvearrowleft$.

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n$$

INSERT EXAMPLE HERE

Theorem 1.5

If A is an $m \times n$ matrix with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and $\mathbf{b} \epsilon \mathbb{R}^m$, then:

$$A\mathbf{x} = \mathbf{b} \text{ (matrix equation)}$$

has the same solution set as

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{b} \text{ (vector equation)}$$

which has the same solution set as the linear system whose augmented matrix is:

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}$$

Proposition 1.1

The matrix equation $A\mathbf{x} = \mathbf{b}$ has a solution (ie. is consistent) iff $\mathbf{b}$ is a linear combination of the columns of A iff $\mathbf{b}$ is in the span of the columns of A.

Is $A\mathbf{x} = \mathbf{b}$ always consistent? No!

Example 1.4.1

INSERT EXAMPLE HERE

Theorem 1.6

Let A be an $m \times n$ matrix. TFAE (the following are equivalent; all true or all false):

1. $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b} \epsilon \mathbb{R}^m$
2. A has a pivot in every row
3. Each $\mathbf{b} \epsilon \mathbb{R}^m$ is a linear combination of the columns of A.
4. The columns of A $span \mathbb{R}^m$

Example 1.4.2

INSERT EXAMPLE HERE

Definition 1.22

A square matrix with 1's on the main diagonal and 0's elsewhere is called an identity matrixm denoted I. Typically, I_n is the $n \times n$ identity matrix.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ldots$$

Proposition 1.2

If $\mathbf{x} \epsilon \mathbb{R}^n$, then $I\mathbf{x} = \mathbf{x}$.

Theorem 1.7

If A is an $m \times n$ matrix, $\mathbf{u}, \mathbf{v} \epsilon \mathbb{R}^n$, $c \epsilon \mathbb{R}$m then:

- $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$, and
- $A(c\mathbf{u}) = c(A\mathbf{u})$

1.5 Solution Sets of Linear Systems

Definition 1.23

A linear system is called homogenous if it can be written as $A\mathbf{x} = \mathbf{0}$, where A is an $m \times n$ matrix and $\mathbf{0}$ is the zero vector.

Is $A\mathbf{x} = \mathbf{0}$ always consistent? Yes, since if $\mathbf{x} = \mathbf{0}$, and this is always true: $\mathbf{0} = \mathbf{0}$.

Definition 1.24

Note $A\mathbf{x} = \mathbf{0}$ is always consistent with solution $\mathbf{x} = \mathbf{0}$–we call this the trivial solution.

Example 1.5.1

INSERT EXAMPLE HERE

Generally, the solution of a homogenous system $A\mathbf{x} = \mathbf{0}$ is $span\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ for some $\mathbf{v}_1, \ldots, \mathbf{v}_p$.

- 0 free variables $\to$ Solution set is a point ($\mathbf{0}$)
- 1 free variables $\to$ Solution set is a line (through $\mathbf{0}$)
- 2 free variables $\to$ Solution set is a plane (through $\mathbf{0}$)
- etc.

Definition 1.25

A parametric vector equation is an equation of the form $\mathbf{x} = c_1\mathbf{v}_1, c_2\mathbf{v}_2, c_3\mathbf{v}_3, \ldots, c_p\mathbf{v}_p$. A solution presented in this form (eg. $\mathbf{x} = x_2\mathbf{u}, x_3\mathbf{v}, x_4\mathbf{w}$) is said to be in paraetric vector form.

Algorithm for Solving $A\mathbf{x} = \mathbf{b}$

1. Compute $RREF(\begin{bmatrix} A & \mathbf{b} \end{bmatrix})$.
2. Express basic variables in terms of free variables.
3. Write $\mathbf{x}$ in parametric vector form. Decompose $\mathbf{x}$ in a linear combination of vectors with weights as the free variables.

Example 1.5.2

INSERT EXAMPLE HERE

1.6 Linear Independence

Definition 1.26

An indexed set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ where $\mathbf{v}_i \epsilon \mathbb{R}^n$, is called linearly independant if the vector equation (homogeneous).

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. If the trivial solution is not unique, the set is linearly dependent.
If a set of indexed vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$, where $\mathbf{v}_i \epsilon \mathbb{R}^n$, is linearly dependent, then these exist weights $c_1, \ldots, c_p \epsilon \mathbb{R}^n$ not all zero such that

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_p\mathbf{v}_p = \mathbf{0}$$

This equation is called a linear dependence relation.

Example 1.6.1

INSERT EXAMPLE HERE

Example 1.6.2

INSERT EXAMPLE HERE

Example 1.6.3

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Theorem 1.8 (The Characterization of Lienarly Dependent Sets)

An indexed set

Example 1.6.4

INSERT EXAMPLE HERE

1.7 Introduction to Linear Transformations

Definition 1.27

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Definition 1.28

A transformation T is called linear (linear transformation) if for all $\mathbf{u}$, $\mathbf{v}$ in the domain of T and c $\in \mathbb{R}$,

- $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u} + T(\mathbf{v})$
- $T(c\mathbf{u}) = cT(\mathbf{u}$

Therefore, we can determine that every matrix transformation $(T(\mathbf{x} = A\mathbf{x})$ is a linear transformation

To show that a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear:

1. Introduce $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n andc \in \mathbb{R}$. These must be arbitrary
2. Show that $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u} + T(\mathbf{v}$
3. Show that $T(c\mathbf{u}) = cT(\mathbf{u}$

Turns out, we could show that IF WE ALREADY KNOW that T is linear, then:

- $T(\mathbf{0}) = \mathbf{0}$ and
- $T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v}$

To show that T is NOT linear, either:

- Show $T(\mathbf{0} \neq \mathbf{0})$, or
- $T(\mathbf{u}+\mathbf{v}) \neq T(\mathbf{u}) + T(\mathbf{v})$ for specific vectors u & v, or

1.8 The Matrix of a Linear Transformation

Definition 1.29

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Example 1.8.1

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ that first dilates vectors by a size of 2 then reflects across the line $x_2 = -x_1$. Assuming this transformation is linear, find the standard matrix for T

Solution

1. Dilates (expands) by 2 $\to$ Doubles in size.
2. Reflects across $x_1 = -X_1(y = -x)$.

Chapter 2

Matrix Algebra

2.1 Matrix Operations

There are several ways of representing a matrix:

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}_i j \end{bmatrix} =$$

Definition 2.1

The <u>diagonal entries</u> of the $m \times n$ matrix i

Definition 2.2

The main diagonal is the collection diagonal entries starting from the top left. **NOTE:** This may not include the bottom right due to the size of the matrix

Definition 2.3

Diagonal matrix has the form:

$$\begin{bmatrix} \square & 0 & 0 & 0 \\ 0 & \square & 0 & 0 \\ 0 & 0 & \square & 0 \end{bmatrix}$$

Definition 2.4

The zero matrix is any matrix who's entries are all zero.

Basic operations are the same between matrices and vectors (addition, subtraction, multiplication, and equality). Keep in mind, matrices must be the same size for addition/subtraction operations.

Theorem 2.1 (Properties of Matrix Arithmitic)

Let A, B, C, be matrices of the same size, $r, s, \in \mathbb{R}$.

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$
3. $A + O = A$
4. $r(A + B) = rA + rB$
5. $(r + s)A = rA + sA$
6. $r(sA) = (rs)A$

NOTE: O is used a 0 in mathematics.

Matrix Multiplication

When B multiplies a vector $\mathbf{x}$ it starts from $\mathbf{x} \mapsto B\mathbf{x}$

Definition 2.5

If A is $m \times n$ and B is $n \times p$, then the product AB is the $m \times p$ matrix.

$$AB = [A\mathbf{b}_1, A\mathbf{b}_2 \ldots A\mathbf{b}_p]$$

Remark. The idea of non-communicative multiplication is uncommon, however does occur on occasion.

Theorem 2.2

Let A, B, C be matrices so that these are defined, $r \in \mathbb{R}$

1. $A(BC) = (AB)C$
2. $A(B + C) = AB + AC$
3. $(B + C)A = BA + CA$
4. $r(AB) = (rA)B = A(rB)$
5. If A is $m \times n$, then $I_m A = A = AI_n$

A few common pitfalls. In general:

- $AB \neq BA$
- Cancellation does not hold: $AB = AC \nRightarrow B = C$
- The Zero Product Principle does not hold: $AB = O \nRightarrow A = O$ or $B = O$

Theorem 2.3 (Transpose Properties)
Let A, B, be matrices so that these are defined, $r \in (R)$.

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(rA)^T = rA^T$
4. $(AB)^T = B^T A^T$

> **Remark.** This idea of transposition will come up frequently in Linear Algebra.

2.2 The Inverse of a Matrix

Definition 2.6 (Multiplicative Inverse)
If $c \in \mathbb{R}$ with $c \neq O$,then:

$$c \cdot c^{-1} = 1 \text{ and } c^{-1} \cdot c = 1$$

In other words:

$$A \cdot A^{-1} = I \text{ and } A^{-1} \cdot A = I$$

Definition 2.7
An $n \times n$ matrix A is invertible if there is another $n \times n$ matrix C such that $CA = I$ and $AC = I$. We call C the inverse of A and write $C = A^{-1}$.

Proposition 2.1
The inverse of an invertible matrix is unique.

Definition 2.8

An $n \times n$ matrix that is not invertible is called singular.

$$\begin{aligned} &\text{invertible} = \text{nonsingular} \\ &\text{singular} = \text{noninvertible} \end{aligned}$$

Theorem 2.4

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq O$, then A is invertible, and $A^{-1} = \frac{1}{ad=bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. If $ad - bc = O$, A is singular.

Definition 2.9

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the <u>determinant</u> of A, written $detA$ or $|A|$, is the number $ad - bc$.

Example 2.2.1

Let $A = \begin{bmatrix} 2 & 6 \\ -1 & 3 \end{bmatrix}$. Find $\det A$ & A^{-1}.

Solution: $\det A = (2)(3) - (6)(-1) = 12$

$A^{-1} = \frac{1}{detA} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$= \frac{1}{12} \begin{bmatrix} 3 & -6 \\ 1 & 2 \end{bmatrix}$

$=$

Theorem 2.5

If A is nonsingular, then for each $\mathbf{b} \in \mathbb{R}^n$, $A\mathbf{x} = \mathbf{b}$ is consistent with the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

$A\mathbf{x} = \mathbf{b}$

Since A is singular, A^{-1} exists, so:

$A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}$

$(A^{-1}A)\mathbf{x} = A^{-1}\mathbf{b}$

$I\mathbf{x} = A^{-1}\mathbf{b}$

$\mathbf{x} = A^{-1}\mathbf{b}$

Example 2.2.2

Let A, B, C, D be invertible $n \times m$ matrices. Solve for C if $A^{-1}B^{-1}ADCD^{-1}B = D$

$$
\begin{gathered}
(A^{-1}A)B^{-1}ADCD^{-1}B = AD \\
B^{-1}ADCD^{-1}B = AD \\
BB^{-1}ADCD^{-1}B = BAD \\
ADCD^{-1}B = BAD \\
A^{-1}ADCD^{-1}B = A^{-1}BAD \\
DCD^{-1}B = A^{-1}BAD \\
D^{-1}DCD^{-1}B = D^{-1}A^{-1}BAD \\
CD^{-1}B = D^{-1}A^{-1}BAD \\
CD^{-1}BB^{-1} = D^{-1}A^{-1}BADB^{-1} \\
CD^{-1} = D^{-1}A^{-1}BADB^{-1} \\
CD^{-1}D = D^{-1}A^{-1}BADB^{-1}D \\
C = D^{-1}A^{-1}BADB^{-1}D
\end{gathered}
$$

This tells us that the order in which you multiply makes a difference. Do you multiply on the left or the right?

Theorem 2.6 (Properties of the Inverse)

If A, B are $n \times n$ invertible matrices, then:

1. A^{-1} is invertible, and $(A^{-1})^{-1} = A$
2. AB is invertible, and $(AB)^{-1} = A^{-1}B^{-1}$
3. A^T is invertible, and $(A^T)^{-1} = (A^{-1})^T$

Theorem 2.7

Building on number 2 of the previous theorem, the product of $n \times n$ invertible matrices is invertible, and the inverse is the product of the inverses in reverse order.
eg. $(E_5E_4E_3E_2E_1)^{-1} = E_5^{-1}E_4^{-1}E_3^{-1}E_2^{-1}E_1^{-1}$

This is very similar to the socks and show rule: you put on your socks, then put on your shoes. To take them off, you take off your shoe then take off your socks.

Definition 2.10

An <u>elemntary matrix</u> is a matrix obtained by performing one Elementary Row Operation on an identity matrix.

Example 2.2.3

$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$, Compute $E_1 \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

$= \begin{bmatrix} a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} + g \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & b \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + e \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} + h \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & c \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + f \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} + i \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix}$

$= \begin{bmatrix} a & b & c \\ d & e & f \\ -2d + g & -2e + h & -2f + i \end{bmatrix}$

$$E_1 A \text{ replaces } R_3 \text{ with } -2R_2 + R_3$$

Elementary matrices are simply a method of completing elementary row operations through matrices. They are very predictable.

Example 2.2.4

Find $E_2 A$ where
$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ & $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

$$I \xrightarrow{-3R_2} E_2$$
$$A \xrightarrow{-3R_2} E_2 A$$

$E_2 A = \begin{bmatrix} 1 & 2 & 3 \\ -12 & -15 & -18 \\ 7 & 8 & 9 \end{bmatrix}$

Elementary matrices can tell you what to do without making any calculations, similar to the idea of how the vertex form of quadratics tell you where the vertex is.

Proposition 2.2

If an ERO is performed on an $m \times n$ matrix A, the resulting matrix can be written EA, where E is the $m \times m$ elementary matrix found by preforming the same ERO on I_m.

Proposition 2.3

If E is elementary, then E is invertible. Idea for the following theorem:
Suppose A is $n \times n$. We want to invert A. Therefore, row reduce $A \to I$.
$E_p \ldots E_4 E_3 E_2 E_1 A = I \leftarrow$ Identity Matrix
Where E_p are elementary matrices

$$(E_p \ldots E_4 E_3 E_2 E_1 A) = I$$

Thus, $A^{-1} = E_p \ldots E_4 E_3 E_2 E_1$

Theorem 2.8

An $n \times n$ matrix A is invertible iff A is row equivalent to A. In this case, any sequence of EROs that reduced $A \to I$ would also transform $I \to A^{-1}$

Algorithm for Finding A^{-1}

Start by augmenting $[A\ I]$. To find A^{-1} if A is invertible, RREF$[A\ I] = [I\ A^{-1}]$

There are other algorithms that are typically computationally longer. The adjoint method (one of these algorithms) is far more work.

Example 2.2.5

Determine if $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & -2 & 3 \end{bmatrix}$ is invertible or singular. If A is invertible, find A^{-1}.

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 1 & -2 & 3 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 0 & -3 & 2 & -1 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -2 & 1 & 0 \\ 0 & -3 & 2 & -1 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & -3 & 2 & -1 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 5 & -3 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 5/2 & -3/2 & 1/2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -3/2 & 3/2 & -1/2 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 5/2 & -3/2 & 1/2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -7/2 & 5/2 & -1/2 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 5/2 & -3/2 & 1/2 \end{bmatrix}$$

Therefore, $A^{-1} = \begin{bmatrix} 1 & 0 & 0 & -7/2 & 5/2 & -1/2 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 5/2 & -3/2 & 1/2 \end{bmatrix}$

Another note: If you cannot row reduce to RREF, then you know that it is not invertible.

2.3 Characterizations of Invertible Matrices

Theorem 2.9 (The (small) Invertible Matrix Theorem)

Suppose A is $n \times n$. The following are equivalent (TFAE):

1. A is invertible
2. A is row equivalent to I.
3. A has n pivot positions.
4. $A\mathbf{x} = \mathbf{O}$ has only the trivial solution
5. The columns of A are linearly independent
6. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is $1-1$.
7. $A\mathbf{x} = \mathbf{b}$ is consistent with the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.
8. The columns of A span $\mathbb{R}^n$.
9. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto $\mathbb{R}^n$
10. There exists an $n \times n$ matrix C such that $CA = I$
11. There exists an $n \times n$ matrix D suc that $AD = I$
12. A^T is invertible

There will be additional statements in the future to supplement these initial equivalencies. This theorem above will allow us to **change our goal** to something that is equivalent, which will make solving the problem easier.

Definition 2.11

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is invertible if there exists a function $S : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$S(T(\mathbf{x})) \text{ for all } \mathbf{x} \in \mathbb{R}^{\mathbf{2}}, \text{ and}$$
$$T(S(\mathbf{x})) \text{ for all } \mathbf{x} \in \mathbb{R}^{\mathbf{2}}$$

Theorem 2.10

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be linear with standard matrix A. Then T is invertible iff A is invertible. In this case, the linear transformation S given by $S(\mathbf{x}) = A^{-1}\mathbf{x}$ is the unique function satisfying the invertible definition for T, and $S = T^{-1}$.

Example 2.3.1

Suppose $T : \mathbb{R}^7 \to \mathbb{R}^6$ is linear and $1-1$. Show that T is onto (the range = codomain).

We know T is linear, so it has a standard matrix A, and it is a 7×7 matrix. Since it is square, the IMT can be used.

Since T defined by $T(\mathbf{x}) = A\mathbf{x}$ is $1-1$, we can use the IMT.

By the IMT, T is onto $\mathbb{R}^7$

Chapter 3

Determinants

3.1 Introduction to Determinants

Definition 3.1

For the *uninteresting* 1×1 case, we define $detA = det[a_11] = a_11$

> **Remark.** While this is not used in a practical context, its used to build on this with a recursive definition.

Definition 3.2

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we defined $detA = ad - bc$.

Consider $A = [a_ij]$ is 3×3 with $a_11/neqO$. Then,

$$REF \begin{bmatrix} a_11 & a_12 & a_13 \\ a_21 & a_22 & a_23 \\ a_31 & a_32 & a_33 \end{bmatrix} = \begin{bmatrix} a_11 & a_12 & a_13 \\ 0 & a_11a_22 - a_12a_21 & a_11a_23 - a_13a_21 \\ 0 & 0 & a_11\Delta \end{bmatrix}$$

Where $\Delta = a_11a_22a_33 + a_12a_23a_31 + a_13a_21a_32 - a_11a_23a_32 - a_12a_21a_33 - a_13a_22a_31$

> **Remark.** This is A METHOD, but not a good method. Do not need to memorize this, it is just a step towards a larger definition.

Definition 3.3

For $A = a_ij]$ as a 3×3 matrix, we define $detA = \Delta$

Determinants help you determine invertability, as if the determinant is 0, the we know it is not invertible. A determinant determines if A is invertible or singular

Definition 3.4

The ij minor matrix A_ij, where A_ij is the matrix obtained from A by deleting its i^th row nd j^th column.

Example 3.1.1

Let $A = \begin{bmatrix} 1 & 0 & -7 & 6 & 1 \\ 2 & 3 & 5 & -1 & 1 \\ 0 & 1 & 1 & 2 & 1 \\ \pi & e & 6 & 4 & 1 \end{bmatrix}$. Find A_{32}.

$A_{32} = \begin{bmatrix} 1 & -7 & 6 & 1 \\ 2 & 5 & -1 & 1 \\ \pi & 6 & 4 & 1 \end{bmatrix}$

3.2 Properties of Determinants

Theorem 3.1 (Row Operations & Determinants)

Let A be a square matrix. Let B be obtained from A by a single ERO. Then:

ERO	Effect on det	$detB$
Scale by k	Scale by k	$detA = kdetA$
Interchange	Negates	$detA = kdetA$
Replacement	Invariant	$detA = detA$

Remark. Scaling in determinants is like factoring!

Example 3.2.1

Find $\begin{vmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix}$

$= -\begin{vmatrix} 1 & -4 & 0 & 6 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 2 & -8 & 6 & 8 \end{vmatrix}$

$= -\begin{vmatrix} 1 & -4 & 0 & 6 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 2 & -8 & 6 & 8 \end{vmatrix}$

Row reduce rows bellow 1 to be 0 in under the first pivot.

$= -\begin{vmatrix} 1 & -4 & 0 & 6 \\ 0 & 3 & 5 & -8 \\ 0 & -12 & 1 & 16 \\ 0 & 0 & 6 & -4 \end{vmatrix}$

Reduce row 3 to continue making triangle matrix.

$= -\begin{vmatrix} 1 & -4 & 0 & 6 \\ 0 & 3 & 5 & -8 \\ 0 & 0 & 21 & -16 \\ 0 & 0 & 6 & -4 \end{vmatrix}$

Take $\frac{7}{2}$ of row 3.

$= -\frac{7}{2}\begin{vmatrix} 1 & -4 & 0 & 6 \\ 0 & 3 & 5 & -8 \\ 0 & 0 & 6 & \frac{-32}{7} \\ 0 & 0 & 6 & -4 \end{vmatrix}$

Eliminate position 3 in row 4 to complete triangle matrix.

$= -\frac{7}{2}\begin{vmatrix} 1 & -4 & 0 & 6 \\ 0 & 3 & 5 & -8 \\ 0 & 0 & 6 & \frac{-32}{7} \\ 0 & 0 & 0 & \frac{4}{7} \end{vmatrix}$

$= \frac{7}{2}(1)(3)(6)(\frac{4}{7})$

$= -36$

Suppose A is an $n \times n$ matrix. Let $U = REF(A)$. So:

$$U = \begin{bmatrix} \blacksquare & \star & \star & \dots & \star \\ 0 & \blacksquare & \star & \dots & \star \\ 0 & 0 & \blacksquare & \dots & \star \\ 0 & 0 & 0 & \dots & \blacksquare \end{bmatrix}$$

is obtained through interchange & replacement. Then det $A = \pm detU = \pm$ (product of main diagonal entries of U)

By IMT, A is invertible if it has n pivots. Also, A is singular if it has less than n pivots. If A has less than n pivots, then at least one $\blacksquare = 0$. If this is the case, then $detU = 0$. Since $detA = \pm detU = \pm 0$

Theorem 3.2 (Addition to the IMT)

A square matrix A is invertible iff $detA \neq 0$

Theorem 3.3

If A is square, $detA = det(A^T)$

Therefore, the transposed matrix does not make a difference in the determinant.

Theorem 3.4 (Multiplicative Property of Determinants)

If A, B are $n \times n$ matrices, then:

$$det(AB) = det(A)det(b)$$

Keep in mind that this is only tru for multiplication, not addition.

Chapter 4

Vector Spaces

4.1 Introduction to Vector Spaces

NOTE: This is one of the most important units of Linear Algebra.

Definition 4.1

A vector space is a nonempty set V of objects (called vectors) on which two operations are defined–vector addition and scalar multiplication–via ten axioms that mus be true for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \epsilon V$ and $c, d \epsilon \mathbb{R}$.

1. $\mathbf{u} + \mathbf{v} \epsilon V$ (Closed Under $+$)
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ ($+$ is Commutative)
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{v} + (\mathbf{u} + \mathbf{w})$ ($+$ is Associative)
4. There exists a vector $\mathbf{0} \epsilon v$ such that $\mathbf{u} + \mathbf{0} = \mathbf{0}$ ($\mathbf{0}$ is the $+$ identity)
5. For all $\mathbf{u} \epsilon V$, there exists $(-\mathbf{u}) \epsilon V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ ($-\mathbf{u}$ is the $+$ inverse)
6. $c\mathbf{u} \epsilon V$ (Closed under scalar Multiplication)
7. $(c + d)\mathbf{u} = c\mathbf{u} + c\mathbf{b}$ (Vector Distribution)
8. $c(d\mathbf{u}) = (cd)\mathbf{u}$ (Scalar Associativity)
9. $1\mathbf{u} = \mathbf{u}$ (1 is the Scalar Multiplicative Identity)

Proposition 4.1

Let V be a v.s. (vector space) with $\mathbf{u} \epsilon V$ and $c \epsilon \mathbb{R}$:

- The additive inverse is unique
- Additive inverses are unique
- $\mathbf{0u} = \mathbf{0}$
- $c\mathbf{0} = \mathbf{0}$
- $-\mathbf{u} = (-1)\mathbf{u}$

Consider $\mathbb{R}^n$. We looked at (ii-v, vii-x). Suppose $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \dots \\ u_n \end{bmatrix} \epsilon \mathbb{R}^n$ & $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix} \epsilon \mathbb{R}^n$. Let $c \epsilon \mathbb{R}$. Then,

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 \\ u_2 \\ \dots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \dots \\ u_n + v_n \end{bmatrix}$$

Since $u_n \epsilon \mathbb{R}$ and $v_n \epsilon \mathbb{R}$, $u_n + v_n \epsilon \mathbb{R}$.
Therefore R^n is a v.s.

Definition 4.2

Definition 4.3

Let D be a subset of $\mathbb{R}$. Define $\mathbb{F}_D$ to be the set of all functions whose domain is D and whose codomain is $\mathbb{R}$. Let $\mathbf{f}, \mathbf{g} \epsilon \mathbb{F}_D$, $c, t \epsilon \mathbb{R}$. Define $\mathbf{f} + \mathbf{g}$ to be the function whose output at t is $\mathbf{f}(t) + \mathbf{g}(t)$. Define $c\mathbf{f}$ to be the function whose output at t is $(c\mathbf{f})(t) = c(\mathbf{f}(t))$.

Subspace Test

Definition 4.4

Let V be a v.s. A set H is a subspace of V if H is a v.s. and all elements of H are also in V

Theorem 4.1 (Subspace Test)

Let V be a v.s. Then H is a subspace of V if:

- H is nonempty
- H is closed under addition
- H is closed under scalar multiplication

Subspaces adopt the features of the greater space!

> **Remark.** This is a MAJOR time saver in linear algebra.

Definition 4.5

The set $\mathbf{P}$ is the set of all polynomials with real coefficients.

Theorem 4.2

$\mathbb{P}$ is a subspace of $\mathbb{F}_{\mathbb{R}}$.

Definition 4.6

The set $\mathbb{P}_n$ is the set of all polynomials with real coefficients of degree at most n.

We know that $\mathbb{R}^n$, $M_{m\times n}$, $\mathbb{F}_D$, $\mathbb{P}$, $\mathbb{P}_n$ are vector spaces.

Example 4.1.1

Show $\mathbb{P}_n$ is a v.s.s for $n = 1, 2, 3, \ldots$
Let's use the subspace test. Notice $\mathbb{P}_n$ is a subspace of $\mathbb{F}_{\mathbb{R}}$, since ever polynomial in $\mathbb{P}_n$ is a function whose domain is $\mathbb{R}$.
We want to show that $\mathbb{P}_n$ is nonempt. Notice $t^n \epsilon \mathbb{P}_n$, so $\mathbb{P}_n$ is nonempty.
Let $\mathbf{a}(t) = a_0 + a_1 t + \ldots + a_n t^n$
$\mathbf{b}(t) = b_0 + b_1 t + \ldots + b_n t^n$
Who both $\epsilon \mathbb{P}_n$. We want to show that $\mathbf{a}(t) + \mathbf{b}(t) \epsilon \mathbb{P}_n$. Then:

Chapter 5

hi

Chapter 6

hi

6.1 hi

6.2 hi

6.3 hi

6.4 hi

The Gram-Schmidt Process is used to find an orthogonal basis that can be used in lieu of a given basis. Given a basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ for a subspace W of $\mathbb{R}^n$, let:

$$\begin{gathered}\mathbf{v}_1 = \mathbf{x}_1 \\ \mathbf{v}_2 = \mathbf{x}_2 - proj_{\mathbf{v}_1}\mathbf{x}_2 \\ \mathbf{v}_3 = \mathbf{x}_3 - proj_{\mathbf{v}_1}\mathbf{x}_3 - proj_{\mathbf{v}_2}\mathbf{x}_3 \\ \vdots \\ \mathbf{v}_p = \mathbf{x}_p - proj_{\mathbf{v}_1}\mathbf{x}_p - proj_{\mathbf{v}_2}\mathbf{x}_p - \ldots - proj_{\mathbf{v}_{p-1}}\mathbf{x}_p\end{gathered}$$

Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthogonal basis for W.

Example 6.4.1

If $\mathbf{x}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$, and $X = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$, then X is a basis for the subspace W of $\mathbb{R}^4$ Construct an orthonormal basis for W.

$$\text{Let } A = \begin{bmatrix}1&0&0\\1&1&0\\1&1&1\\1&1&1\end{bmatrix} \sim \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{bmatrix}$$

Since $W = \text{Col}A$, we can write the vectors as:

$$\mathbf{y} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3$$

Applying the Gram-Schmidt Process, we can find that:

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

$$\mathbf{v}_2 = \mathbf{x}_2 - proj_{\mathbf{v}_1}\mathbf{x}_2 = \begin{bmatrix}-\frac{3}{4}\\\frac{1}{4}\\\frac{1}{4}\\\frac{1}{4}\end{bmatrix}$$

$$\mathbf{v}_3 = \mathbf{x}_3 - proj_{\mathbf{v}_1}\mathbf{x}_3 - proj_{\mathbf{v}_2}\mathbf{x}_3 = \begin{bmatrix}0\\-\frac{2}{3}\\\frac{1}{3}\\\frac{1}{3}\end{bmatrix}$$

Then, we can find the orthogonal basis for W to be $\{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}-\frac{3}{4}\\\frac{1}{4}\\\frac{1}{4}\\\frac{1}{4}\end{bmatrix}, \begin{bmatrix}0\\-\frac{2}{3}\\\frac{1}{3}\\\frac{1}{3}\end{bmatrix}\}$

We can then simply normalize each to find the orthonormal basis for W