Winter 2022 MTH 261 Mini Test 2

Marvin Lin

Feburary 2022

Question 1

Example 0.0.1

Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix}$, $\mathbf{a}_2 \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}$, $\mathbf{a}_3 = \begin{bmatrix} -7 \\ -2 \\ \alpha \end{bmatrix}$.

Determine what value(s) of α will make $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ a linearly dependent set. Show all work that supports your conclusion, including all row-reduction steps.

$$\begin{bmatrix} 1 & -2 & -7 & 0 \\ 2 & 5 & -2 & 0 \\ -7 & 3 & \alpha & 0 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & -2 & -7 & 0 \\ 0 & 9 & 12 & 0 \\ -7 & 3 & \alpha & 0 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & -2 & -7 & 0 \\ 0 & 9 & 12 & 0 \\ 0 & -11 & \alpha - 49 & 0 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & -2 & -7 & 0 \\ 0 & 1 & \frac{12}{9} & 0 \\ 0 & -11 & \alpha - 49 & 0 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & -2 & -7 & 0 \\ 0 & 1 & \frac{12}{9} & 0 \\ 0 & 0 & \alpha - 34\frac{1}{3} & 0 \end{bmatrix}$$

For the following set of vectors to be linearly dependent, the $\mathbf{x_3}$ vector must be free. In order for this to occur, then $\alpha - 34\frac{1}{3}$ must be 0.

$$0 = \alpha - 34\tfrac{1}{3}$$
$$\alpha = 34\tfrac{1}{3}$$

Therefore, when $\alpha = 34\frac{1}{3}$, $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ is a linearly dependent set.

Question 2

Example 0.0.2

Find the standard matrix T for the linear transform $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that T first reflects points through the horizontal x_1-axis and then reflects points through the line $x_2 = x_1$.

Starting with the identity matrix, I, we can map each vector with the appropriate transformation. Let's start with the reflection through the horizontal x_1 axis.

$$\mathbf{e_1} \mapsto \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$\mathbf{e_2} \mapsto \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

On the next transformation, we reflect through $x_2 = x_1$.

$$\mathbf{e_1} \mapsto \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$\mathbf{e_2} \mapsto \begin{bmatrix} 0 \\ -1 \end{bmatrix} \mapsto \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Therefore, we can determine the standard matrix is $T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Question 3

Example 0.0.3

An *affine transformation* $T : \mathbb{R}^n \to \mathbb{R}^m$ has the form $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$, where A is $m \times n$ and $\mathbf{b} \in \mathbb{R}^m$. Why is T not linear when $\mathbf{b} \neq \mathbf{0}$? Explain in as much detail as possible. Show any computations that support your conclusion.

Fundamentally, a linear transformation means that vector addition and scalar multiplication is preserved. In the case where the vector $\mathbf{b} \neq 0$, this means that vector addition and scalar multiplication is not preserved

Furthermore, by definition, when a linear transformation occurs, then $T(\mathbf{0}) = \mathbf{0}$. Let's evaluate the given form of $T(\mathbf{x})$ to see the outcome:

$$T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$$
$$T(\mathbf{0}) = A\mathbf{0} + \mathbf{b}$$
$$T(\mathbf{0}) = \mathbf{b}$$

Given $\mathbf{b} \neq \mathbf{0}$, we can see that $T(\mathbf{0}) = \mathbf{0}$ is not true, and therefore the affine transformation is not linear.

Question 4

Example 0.0.4

Let $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$. Determine if A is invertible or singular. If A is invertible, find A^{-1}. If A is singular, state as specifically as possible why this is so.

$$\begin{bmatrix} 1 & 1 & -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -2 & 2 & -1 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -2 & 2 & -1 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 1 \\ 0 & 2 & 0 & 1 & 0 & 1 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 & 1 & 1 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 1 & 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Since A can be reduced down to RREF, we know that A is invertible.

$$\text{Furthermore, } A^{-1} = \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Question 5

Example 0.0.5

Let $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 4 & -1 \end{bmatrix}$. Determine if AB is defined or not. If it is defined, find the resulting matrix. If it is undefined, state as specifically as possible why this is so.

We can tell if AB is defined be analyzing if the size of the matrix matches up.

$$3 \times 3 \text{ and } 3 \times 2$$

Since the number of columns of A match with the number of rows for B, we know that AB is defined.

$$A = \begin{bmatrix} (1)(1) + (1)(2) + (-1)(4) & (1)(2) + (1)(3) + (-1)(-1) \\ (1)(1) + (-1)(2) + (1)(4) & (1)(2) + (-1)(3) + (1)(-1) \\ (-1)(1) + (1)(2) + (1)(4) & (-1)(2) + (1)(3) + (1)(-1) \end{bmatrix}$$

$$A = \begin{bmatrix} 1 + 2 - 4 & 2 + 3 + 1 \\ 1 + (-2) + 4 & 2 + (-3) + (-1) \\ (-1) + 2 + 4 & (-2) + 3 + (-1) \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & 6 \\ 3 & -2 \\ 5 & 0 \end{bmatrix}$$

$$\text{Therefore, } \begin{bmatrix} -1 & 6 \\ 3 & -2 \\ 5 & 0 \end{bmatrix} \text{ is the resulting matrix.}$$