Winter 2022 MTH 261 Mini Test 4

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Question 1

Example 0.0.1

Let $A = \begin{bmatrix} 1 & 3 & -4 & 2 & -1 & 6 \\ 3 & 9 & -12 & 7 & 1 & 15 \\ -3 & -9 & 13 & -10 & 6 & -15 \\ 1 & 3 & -3 & -1 & 6 & 6 \end{bmatrix}$.

1. Find a basis for $ColA$.
2. Find a basis for $NulA$.
3. Find a basis for $RowA$.
4. Find a basis for $LNulA$.
5. Find $dimColA$, $dimNulA$, $dimColA^T$, $dimNulA^T$.

Part 1

Begin with the homogenous equation: $A\mathbf{x} = \mathbf{0}$

$$\begin{bmatrix} 1 & 3 & -4 & 2 & -1 & 6 \\ 3 & 9 & -12 & 7 & 1 & 15 \\ -3 & -9 & 13 & -10 & 6 & -15 \\ 1 & 3 & -3 & -1 & 6 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & -4 & 2 & -1 & 6 & 0 \\ 3 & 9 & -12 & 7 & 1 & 15 & 0 \\ -3 & -9 & 13 & -10 & 6 & -15 & 0 \\ 1 & 3 & -3 & -1 & 6 & 6 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -4 & 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 1 & 4 & -3 & 0 \\ 0 & 0 & 1 & -4 & 3 & 3 & 0 \\ 0 & 0 & 1 & -3 & 7 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -4 & 2 & -1 & 6 & 0 \\ 0 & 0 & 1 & -4 & 3 & 3 & 0 \\ 0 & 0 & 1 & -3 & 7 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 & -3 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -4 & 2 & -1 & 6 & 0 \\ 0 & 0 & 1 & -4 & 3 & 3 & 0 \\ 0 & 0 & 0 & 1 & 4 & -3 & 0 \\ 0 & 0 & 0 & 1 & 4 & -3 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -4 & 2 & -1 & 6 & 0 \\ 0 & 0 & 1 & -4 & 3 & 3 & 0 \\ 0 & 0 & 0 & 1 & 4 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 0 & 2 & 75 & -30 & 0 \\ 0 & 0 & 1 & 0 & 19 & -9 & 0 \\ 0 & 0 & 0 & 1 & 4 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 0 & 0 & 67 & -24 & 0 \\ 0 & 0 & 1 & 0 & 19 & -9 & 0 \\ 0 & 0 & 0 & 1 & 4 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The Column space of A is simply the linear combination of all of the pivot columns of A. Therefore:

$$ColA = \{ \begin{bmatrix} 1 \\ 3 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ -12 \\ 13 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ -10 \\ -1 \end{bmatrix} \}$$

Part 2

We start with outcome of the homogenous equation to get the following:

$$\sim \begin{bmatrix} 1 & 3 & 0 & 0 & 67 & -24 & 0 \\ 0 & 0 & 1 & 0 & 19 & -9 & 0 \\ 0 & 0 & 0 & 1 & 4 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 + 3x_2 + 67x_5 - 24x_6 = 0$$
$$x_3 + 19x_5 - 9x_6 = 0$$
$$x_4 + 4x_5 - 3x_6 = 0$$

$$x_1 = -3x_2 - 67x_5 + 24x_6$$
$$x_3 = -19x_5 + 9x_6$$
$$x_4 = -4x_5 + 3x_6$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -3x_2 \\ 1x_2 \\ 0x_2 \\ 0x_2 \\ 0x_2 \\ 0x_2 \end{bmatrix} + \begin{bmatrix} -67x_5 \\ 0x_5 \\ -19x_5 \\ -4x_5 \\ 1x_5 \\ 0x_5 \end{bmatrix} + \begin{bmatrix} 24x_6 \\ 0x_6 \\ 9x_6 \\ 3x_6 \\ 0x_6 \\ 1x_6 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -67 \\ 0 \\ -19 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 24 \\ 0 \\ 9 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{So } NulA = span\{ \begin{bmatrix} -3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -67 \\ 0 \\ -19 \\ -4 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 24 \\ 0 \\ 9 \\ 3 \\ 0 \\ 0 \end{bmatrix} \}$$

Part 3

Once again, we begin with the homogenous equation: $A^T\mathbf{x} = \mathbf{0}$

$$\begin{bmatrix} 1 & 3 & -3 & 1 \\ 3 & 9 & -9 & 3 \\ -4 & -12 & 13 & -3 \\ 2 & 7 & -10 & -1 \\ -1 & 1 & 6 & 6 \\ 6 & 15 & -15 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & -3 & 1 & 0 \\ 3 & 9 & -9 & 3 & 0 \\ -4 & -12 & 13 & -3 & 0 \\ 2 & 7 & -10 & -1 & 0 \\ -1 & 1 & 6 & 6 & 0 \\ 6 & 15 & -15 & 6 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & -4 & -3 & 0 \\ 0 & 4 & 3 & 7 & 0 \\ 0 & -3 & 3 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -3 & 1 & 0 \\ 0 & 1 & -4 & -3 & 0 \\ 0 & 4 & 3 & 7 & 0 \\ 0 & -3 & 3 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -3 & 1 & 0 \\ 0 & 1 & -4 & -3 & 0 \\ 0 & 0 & 19 & 19 & 0 \\ 0 & 0 & -9 & -9 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -3 & 1 & 0 \\ 0 & 1 & -4 & -3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 19 & 19 & 0 \\ 0 & 0 & -9 & -9 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -3 & 1 & 0 \\ 0 & 1 & -4 & -3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 0 & 4 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The row space is simply the column space of the transposed matrix.
Therefore, the basis of the column space can be identified as:

$$ColA = \{ \begin{bmatrix} 1 \\ 3 \\ -4 \\ 2 \\ -1 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ -12 \\ 7 \\ 1 \\ 15 \end{bmatrix}, \begin{bmatrix} -3 \\ -9 \\ 13 \\ -10 \\ 6 \\ -15 \end{bmatrix} \}$$

Part 4

We can take the result of the RREF of A^T to find the Left Null space, which is the null space of the transposed matrix:

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 + x_4 = 0$$
$$x_2 + x_4 = 0$$
$$x_3 + x_4 = 0$$

$$x_1 = -x_4$$
$$x_2 = -x_4$$
$$x_3 = -x_4$$

$$\text{LNul } A = \text{Nul}A^T = \{ \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \}$$

Part 5

The dimension of the spaces is simply the number of vectors in each space. Therefore:

$$\dim \text{ Col } A = 3$$
$$\dim \text{ Nul } A = 3$$
$$\dim \text{ Row } A = 3$$
$$\dim \text{ LNul } A = 1$$

Question 2

Example 0.0.2

Let $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$, and$\mathbf{w} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$.

1. Show that the characteristic polynomial for A is $-\lambda^3 + \lambda^2 + 4\lambda - 4$.
2. Find the eigenvalues of A.
3. Find a basis for the eigenspace of each eigenvalue.
4. Find $A\mathbf{u}, A\mathbf{v}$, and $A\mathbf{w}$ without using a matrix-vector product.

Part 1

$$det(A - \lambda I) = 0$$

$$det(\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}) = 0$$

$$\begin{vmatrix} 1-\lambda & 1 & -1 \\ 1 & -1-\lambda & 1 \\ -1 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)((-1-\lambda)(1-\lambda)-1)-(1)((1)(1-\lambda)+1)+(-1)((1)(1)-(-1-\lambda)(-1)) = 0$$

$$(1-\lambda)((-1-\lambda)(1-\lambda)-1)-(2+\lambda)-(-\lambda) = 0$$

$$(1-\lambda)((-1-\lambda)(1-\lambda)-1)-2+2\lambda = 0$$

$$(1-\lambda)(-1+\lambda-\lambda+\lambda^2-1)-2+2\lambda = 0$$

$$(1-\lambda)(\lambda^2-2)-2+2\lambda = 0$$

$$\lambda^2-2-\lambda^3+2\lambda-2+2\lambda = 0$$

$$-\lambda^3+\lambda^2+4\lambda-4 = 0$$

Therefore, the characteristic equation is $-\lambda^3 + \lambda^2 + 4\lambda - 4$

Part 2

Simply factor the equation to get:

$$-(\lambda-1)(\lambda+2)(\lambda-2) = 0$$

Therefore, the eignvalues are $\lambda = 1, -2, 2$

Part 3

$$\mathbb{B}_{\lambda=1} = nul \begin{bmatrix} 1-1 & 1 & -1 \\ 1 & -1-1 & 1 \\ -1 & 1 & 1-1 \end{bmatrix} = nul \begin{bmatrix} 0 & 1 & -1 \\ 1 & -2 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 - x_3 = 0$$
$$x_2 - x_3 = 0$$
$$x_3 = x_3$$

$$x_1 = x_3$$
$$x_2 = x_3 \; x_3 = x_3$$

$$\text{Therefore, } \mathbb{B}_{\lambda=1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

$$\mathbb{B}_{\lambda=2} = nul \begin{bmatrix} 1-2 & 1 & -1 \\ 1 & -1-2 & 1 \\ -1 & 1 & 1-2 \end{bmatrix} = nul \begin{bmatrix} -1 & 1 & -1 \\ 1 & -3 & 1 \\ -1 & 1 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ -1 & 1 & -1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3 & 1 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 + x_3 = 0$$
$$x_2 = 0 \; x_3 = x_3$$

$$x_1 = -x_3$$
$$x_2 = 0 \; x_3 = x_3$$

$$\text{Therefore, } \mathbb{B}_{\lambda=2} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

$$\mathbb{B}_{\lambda=-2} = nul \begin{bmatrix} 1 - -2 & 1 & -1 \\ 1 & -1 - -2 & 1 \\ -1 & 1 & 1 - -2 \end{bmatrix} = nul \begin{bmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 3 & 1 & -1 & 0 \\ -1 & 1 & 3 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -2 & -4 & 0 \\ 0 & 2 & 4 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -2 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 - x_3 = 0$$
$$x_2 + 2x_3 = 0 \; x_3 = x_3$$

$$x_1 = -2x_3$$
$$x_2 = 0 \; x_3 = x_3$$

$$\text{Therefore, } \mathbb{B}_{\lambda=-2} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

Part 4

Since we can simply use the linear combinations of the basis of the eigenspace of each eigenvalue (the eigenvectors), we can easily calculate the following:

$$A\mathbf{u} = A \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix}$$

$$A\mathbf{v} = A \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} = 2(A \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}) = 2(2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} -4 \\ 0 \\ 4 \end{bmatrix}$$

$$A\mathbf{w} = A \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} = -1(A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}) = -1(1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$