

MTH 261 Lecture Notes

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Winter 2022

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Chapter 1

Introduction to Linear Algebra

1.1 System of Linear Equations

Linear Algebra is the area of math concerning linear equations/functions that are represented in vector space and through matrices. If Calculus is the foundational language of mathematics, then Linear Algebra is the foundational language of STEM.

Remark. Keep in mind that while Linear Algebra utilizes matrices, it is just a tool to solve problems with. The study is **NOT** of matrices.

Definition 1.1 (Linear Equation)

An equation where $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$, where $a_1, a_2, a_n, b \in \mathbb{C}$ as constants of \mathbb{C} and in \mathbb{R}

Definition 1.2 (Linear System)

A collection of linear equations with the same variable

Definition 1.3 (Solution & Solution Set)

A solution satisfies all equations in a system simultaneously, while a solution set is all possible solutions

Definition 1.4 (Equivalent Solutions)

Two systems with the same identical solution set

Definition 1.5 (Types of Solutions)

There are two major classification with solution, which are broken into three major solutions:

- Inconsistent (no solutions)
- Consistent (at least one solution)
 - Unique Solution
 - Infinite Solutions

Since the systems we are exploring are purely linear, there will not be two, three, or more solutions.

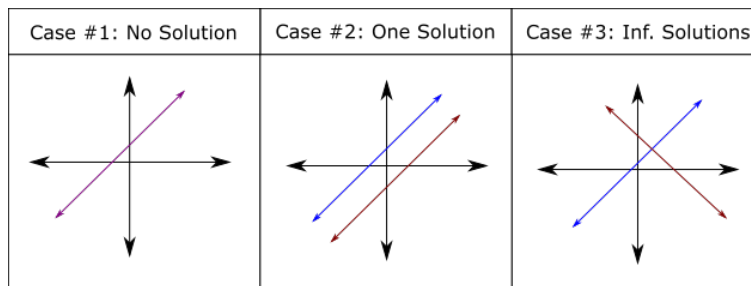


Figure 1.1: Types of solutions in a linear system

Remark. This is true for all linear systems in all space

Definition 1.6 (Matrix)

A rectangular array of numbers, often used to compress systems. Let's say the following system of equations is given:

$$\begin{aligned} x - 7y + 6t &= 5 \\ z - 2t &= -3 \\ -x + 7y - 4z + 2t &= 7 \end{aligned}$$

It can be reduced into the following matrix: $\begin{bmatrix} 1 & -7 & 0 & 6 \\ 0 & 0 & 1 & -2 \\ -1 & 7 & -4 & 2 \end{bmatrix}$

Definition 1.7 (Augmented Matrix)

A standard matrix which also includes the \mathbf{b} coefficient in the matrix. The following is the augmented matrix of the system of equations from Definition 1.6:

$$\begin{bmatrix} 1 & -7 & 0 & 6 \\ 0 & 0 & 1 & -2 \\ -1 & 7 & -4 & 2 \end{bmatrix} \text{ OR } \left[\begin{array}{ccc|c} 1 & -7 & 0 & 6 \\ 0 & 0 & 1 & -2 \\ -1 & 7 & -4 & 2 \end{array} \right]$$

Definition 1.8 (Elementary Row Operations)

Elementary row operations are used to change matrices in a reversible manner.

1. **Replacement.** Sum of itself and a multiple of another row.
2. **Interchange.** Swap/Interchange two rows.
3. **Scaling.** Multiply all entries in a row by a nonzero constant.

These operations can be used with any matrix, and are typically used to create variations of the matrix that is beneficial to finding specific information.

Definition 1.9 (Row Equivalence)

Matrices are referred to as row equivalent if the matrices can be transformed into each other through elementary row operations.

Existence and Uniqueness

In analyzing a linear system, a few key questions are being asked:

1. Is the system consistent or inconsistent? (Solution or no solution)
2. Is the solution unique (one solution or many)?

Remark. Keep in mind, since we are looking into linear systems, there may only be one solution or infinite solutions, none in between.

1.2 Row Reduction and Echelon Forms

There are specific forms in which a matrix can be in to provide valuable information about the matrices.

Definition 1.10 (Leading Entry)

The leftmost nonzero entry in a nonzero row.

In the case of the matrix below, x is the leading entry:

$$\begin{bmatrix} \underline{x} & 2 & 0 & 5 \end{bmatrix}$$

Definition 1.11 (Echelon Form (REF))

One of the primary forms, referred to as row echelon form (often just called echelon form) has the following properties:

- All nonzero rows are above any rows of zeros
- Each leading entry is in a column to the right of the leading entry above it
- All entries in a column below a leading entry are zeros

Below is an example of a matrix in Echelon Form (REF):

$$\begin{bmatrix} \underline{4} & 0 & 4 & 3 \\ 0 & \underline{3} & 6 & 1 \\ 0 & 0 & \underline{1} & 8 \end{bmatrix}$$

The underlined numbers indicate the leading entries.

Definition 1.12 (Reduced Echelon Form (RREF))

Similar to REF, the reduced echelon form (RREF) takes the definition of the REF a few steps further

- The leading entries are 1
- The leading entry (1) is the only nonzero entry in its column

Below is an example of a matrix in Echelon Form (REF):

$$\begin{bmatrix} \underline{1} & 0 & 0 & 3 \\ 0 & \underline{1} & 0 & 1 \\ 0 & 0 & \underline{1} & 8 \end{bmatrix}$$

Theorem 1.1 (Uniqueness of RREF)

Each matrix is only row equivalent to a single RREF matrix.

This is ONLY true for RREF. REF can be adjusted with row operations to have multiple matrices.

Pivots

Definition 1.13 (Pivot Positions and Columns)

Leading entries in REF and the leading 1's in RREF are referred to a pivot position. Similarly, a column containing a pivot position is referred to as a pivot column.

These pivots can be identified in the original matrix, and are not terms exclusively for REF and RREF (unlike leading entries).

Gauss-Jordan Elimination

Step 1. The leftmost non-zero column must be a pivot column, with the first row in the column being the pivot position.

$$A = \begin{bmatrix} 0 & 4 & 4 & 3 \\ 4 & 4 & 6 & 1 \\ 2 & 0 & 4 & 8 \end{bmatrix}$$

Step 2. Any nonzero entry in the pivot column can be a pivot. Use the interchange ERO to make it the first column.

$$\sim \begin{bmatrix} 2 & 0 & 4 & 8 \\ 4 & 4 & 6 & 1 \\ 0 & 4 & 4 & 3 \end{bmatrix}$$

Step 3. Use EROs to make all positions below pivot into zeros.

$$\sim \begin{bmatrix} 2 & 0 & 4 & 8 \\ 0 & 4 & -2 & -15 \\ 0 & 4 & 4 & 3 \end{bmatrix}$$

Step 4. Ignore the pivot row and repeat for all remaining rows (ad terminum; until completion/termination).

$$\sim \begin{bmatrix} 2 & 0 & 4 & 8 \\ 0 & 4 & -2 & -15 \\ 0 & 0 & 6 & 18 \end{bmatrix} = REF(A)$$

This will create a $A \rightarrow REF(A)$, and is common referred to as **Gaussian Elimination**. The following 2 steps will create $A \rightarrow RREF(A)$ and is referred to as **Gauss-Jordan Elimination**

Step 5. Make all pivots 1 through the scaling ERO.

$$\sim \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -\frac{1}{2} & -3\frac{3}{4} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Step 6. Begin with the rightmost pivot and use row operations to work upwards, making all entries above pivot positions to be zeros.

$$\sim \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -2\frac{1}{4} \\ 0 & 0 & 1 & 3 \end{bmatrix} = RREF(A)$$

Gauss-Jordan Elimination in Linear Systems

This method also helps solve linear systems when reduced down to RREF. Assume column 1 is x_1 , column 2 is x_2 , column 3 is x_3 , and column 4 is b .

$$\begin{aligned} 4x_2 + 2x_3 &= 3 \\ 4x_1 + 4x_2 + 6x_3 &= 1 \\ 2x_1 + 4x_3 &= 8 \end{aligned}$$

And therefore after applying Gauss-Jordan Elimination is:

$$\begin{cases} x_1 = -2 \\ x_2 = -2\frac{1}{4} \\ x_3 = 3 \end{cases}$$

In the case of the following where one row is only zeros,

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -2\frac{1}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix} = RREF(A)$$

we get:

$$\begin{cases} x_1 = -2 \\ x_2 = -2\frac{1}{4} \\ x_3 = x_3 \rightarrow x_3 \text{ is free} \end{cases}$$

Definition 1.14

Variables corresponding with a pivot column is called **basic variable**, while variables corresponding to a non-pivot column is referred to as a **free variable**.

There are three possible definitions of the solution set:

- **No solution.** We write ϕ (phi).
- **Unique Solution.** We write the single solution in a set (ie. $(1, 0, -1)$).
- **Infinite solutions.** Form a general solution by expressing basic variables in terms of the free variable, and notating free variables as free. Also called parametric solution, where free variables are the parameters.

Theorem 1.2

The following can be determined by analyzing the augmented matrix:

1. **A linear system is consistent** if and only if the rightmost column is not a pivot column, in addition to no false statements (ie. $0 \neq 1$).
A false statement is a clear indicator of an inconsistent system.
2. **A linear system is consistent and unique** if there are no free variables.
3. **A linear system is consistent and has infinite solutions** if there are free variables.

1.3 Vector Equations

Definition 1.15

Matrix with one column is called column vector, or just a vector.

Definition 1.16 (Equality, Sum, and Scalar)

A vector is only equal if corresponding entries are equal:

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Vectors can only be added if they are in the same space (ie. \mathbb{R}^2 vs \mathbb{R}^3):

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \text{undefined}$$

A vector is scaled by multiplying each entry by the scalar, c :

$$c\mathbf{u} = 4 \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4(2) \\ 4(3) \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \end{bmatrix}$$

Graphing Vectors in \mathbb{R}^2

Fundamentally, vectors can be graphed with the definition:

$$\begin{bmatrix} a \\ b \end{bmatrix} = (a, b)$$

Then, vectors are graphed with a point and arrow starting from the origin.

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\mathbb{R}^2 is the set of all points (vectors) on the plane.

Definition 1.17 (Parallelogram Rule for Addition)

If \mathbf{u} and $\mathbf{v} \in \mathbb{R}^2$, then $\mathbf{u} + \mathbf{v}$ is the 4th vertex of the parallelogram whose other vertices are $\mathbf{0}, \mathbf{u}, \mathbf{v}$.

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Vectors in \mathbb{R}^3 & \mathbb{R}^n

In \mathbb{R}^3 , nearly all properties are identical to \mathbb{R}^2 . Equality, addition, and scalar multiplication are identical. \mathbb{R}^3 is represented by:

$$\mathbb{R}^3 = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\}$$

Vectors in \mathbb{R}^n **Definition 1.18 (\mathbb{R}^n)**

\mathbb{R}^n is defined as the collection of n-tuples (or all lists) of real numbers, usually written as:

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

Moreover, the zero vector $\mathbf{0}$ is written as all entries are zero. The same definitions of equality, summation, and scaling apply as \mathbb{R}^2

Theorem 1.3 (Properties of \mathbb{R}^n)

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, c & $d \in \mathbb{R}$:

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (Commutativity of Addition.)
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{v} + (\mathbf{u} + \mathbf{w})$ (Associativity of $+$.)
3. $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$ ($\mathbf{0}$ is the Additive Identity.)
4. $\mathbf{u} - \mathbf{u} = \mathbf{0} + (-1)\mathbf{u} = \mathbf{0}$ ($-\mathbf{u}$ is the Additive Inverse of \mathbf{u} .)
5. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{v} + c\mathbf{u}$ (Distributive I.)
6. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ (Distributive II.)
7. $c(d\mathbf{u}) = (cd)\mathbf{u}$ (Associative Property of Scalar Multiplication.)
8. $1\mathbf{u} = \mathbf{u}$ (1 is the scalar Multiplicative Identity.)

Definition 1.19

The linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_p \in \mathbb{R}^n$ with corresponding weights $c_1, \dots, c_p \in \mathbb{R}$ is the \mathbb{R}^n vector.

$$c_1 \mathbf{v}_1, c_2 \mathbf{v}_2, c_3 \mathbf{v}_3, \dots, c_p \mathbf{v}_p$$

INCLUDE EXAMPLE

We can see that a vector equation $x_1 \mathbf{v}_1, x_2 \mathbf{v}_2, \dots, x_p \mathbf{v}_p = \mathbf{b}$ has the same solution set as the linear system whose augmented matrix is:

$$A = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \dots \quad \mathbf{v}_p \quad \mathbf{b}]$$

So \mathbf{b} can be generated by a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_p$ iff there exists a solution to the linear system whose augmented matrix is A .

Definition 1.20

If $\mathbf{v}_1, \dots, \mathbf{v}_p \in \mathbb{R}^3$, we define the span of $\mathbf{v}_1, \dots, \mathbf{v}_p$ to be the set of all linear combinations of $\mathbf{v}_1, \dots, \mathbf{v}_p$ denoted:

$$\text{span} = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$$

INCLUDE EXAMPLE

Theorem 1.4

If $\mathbf{v}_1, \dots, \mathbf{v}_p \in \mathbb{R}^3$, then $\mathbf{0} \in \text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$

Visualizing *span*

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1.4 The Matrix Equation $A\mathbf{x} = \mathbf{b}$

Definition 1.21

1.5 Solution Sets of Linear Systems

1.6 Linear Independence

1.7 Introduction to Linear Transformations

Definition 1.22

A transformation T is called linear (linear transformation) if for all \mathbf{u}, \mathbf{v} in the domain of T and $c \in \mathbb{R}$,

- $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
- $T(c\mathbf{u}) = cT(\mathbf{u})$

Therefore, we can determine that every matrix transformation ($T(\mathbf{x}) = A\mathbf{x}$) is a linear transformation

To show that a transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is linear:

1. Introduce $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. These must be arbitrary
2. Show that $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
3. Show that $T(c\mathbf{u}) = cT(\mathbf{u})$

Turns out, we could show that IF WE ALREADY KNOW that T is linear, then:

- $T(\mathbf{0}) = \mathbf{0}$ and
- $T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$

To show that T is NOT linear, either:

- Show $T(\mathbf{0}) \neq \mathbf{0}$, or
- $T(\mathbf{u} + \mathbf{v}) \neq T(\mathbf{u}) + T(\mathbf{v})$ for specific vectors \mathbf{u} & \mathbf{v} , or

1.8 The Matrix of a Linear Transformation

Example 1.8.1

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that first dilates vectors by a size of 2 then reflects across the line $x_2 = -x_1$. Assuming this transformation is linear, find the standard matrix for T

Solution

1. Dilates (expands) by 2 \rightarrow Doubles in size.
2. Reflects across $x_2 = -x_1$.

Chapter 2

Matrix Algebra

2.1 Matrix Operations

There are several ways of representing a matrix:

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n] = [\mathbf{a}_{ij}] =$$

Definition 2.1

The diagonal entries of the $m \times n$ matrix \mathbf{A}

Definition 2.2

The main diagonal is the collection diagonal entries starting from the top left. **NOTE:** This may not include the bottom right due to the size of the matrix

Definition 2.3

Diagonal matrix has the form:

$$\begin{bmatrix} \square & 0 & 0 & 0 \\ 0 & \square & 0 & 0 \\ 0 & 0 & \square & 0 \end{bmatrix}$$

Definition 2.4

The zero matrix is any matrix whose entries are all zero.

Basic operations are the same between matrices and vectors (addition, subtraction, multiplication, and equality). Keep in mind, matrices must be the same size for addition/subtraction operations.

Theorem 2.1 (Properties of Matrix Arithmetic)

Let A, B, C , be matrices of the same size, $r, s, \in \mathbb{R}$.

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$
3. $A + O = A$
4. $r(A + B) = rA + rB$
5. $(r + s)A = rA + sA$
6. $r(sA) = (rs)A$

NOTE: O is used a 0 in mathematics.

Matrix Multiplication

When B multiplies a vector \mathbf{x} it starts from $\mathbf{x} \mapsto B\mathbf{x}$

Definition 2.5

If A is $m \times n$ and B is $n \times p$, then the product AB is the $m \times p$ matrix.

$$AB = [A\mathbf{b}_1, A\mathbf{b}_2 \dots A\mathbf{b}_p]$$

Remark. The idea of non-commutative multiplication is uncommon, however does occur on occasion.

Theorem 2.2

Let A, B, C be matrices so that these are defined, $r \in \mathbb{R}$

1. $A(BC) = (AB)C$
2. $A(B + C) = AB + AC$
3. $(B + C)A = BA + CA$
4. $r(AB) = (rA)B = A(rB)$
5. If A is $m \times n$, then $I_m A = A = A I_n$

A few common pitfalls. In general:

- $AB \neq BA$
- Cancellation does not hold: $AB = AC \nRightarrow B = C$
- The Zero Product Principle does not hold: $AB = O \nRightarrow A = O$ or $B = O$

Theorem 2.3 (Transpose Properties)

Let A, B , be matrices so that these are defined, $r \in (R)$.

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(rA)^T = rA^T$
4. $(AB)^T = B^T A^T$

Remark. This idea of transposition will come up frequently in Linear Algebra.

2.2 The Inverse of a Matrix

Definition 2.6 (Multiplicative Inverse)

If $c \in \mathbb{R}$ with $c \neq 0$, then:

$$c \cdot c^{-1} = 1 \text{ and } c^{-1} \cdot c = 1$$

In other words:

$$A \cdot A^{-1} = I \text{ and } A^{-1} \cdot A = I$$

Definition 2.7

An $n \times n$ matrix A is invertible if there is another $n \times n$ matrix C such that $CA = I$ and $AC = I$. We call C the inverse of A and write $C = A^{-1}$.

Proposition 2.1

The inverse of an invertible matrix is unique.

Definition 2.8

An $n \times n$ matrix that is not invertible is called singular.

invertible = nonsingular

singular = noninvertible

Theorem 2.4

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible, and $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. If $ad - bc = 0$, A is singular.

Definition 2.9

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant of A , written $\det A$ or $|A|$, is the number $ad - bc$.

Example 2.2.1

Let $A = \begin{bmatrix} 2 & 6 \\ -1 & 3 \end{bmatrix}$. Find $\det A$ & A^{-1} .

Solution: $\det A = (2)(3) - (6)(-1) = 12$

$$\begin{aligned} A^{-1} &= \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{12} \begin{bmatrix} 3 & -6 \\ 1 & 2 \end{bmatrix} \\ &= \end{aligned}$$

Theorem 2.5

If A is nonsingular, then for each $\mathbf{b} \in \mathbb{R}^n$, $A\mathbf{x} = \mathbf{b}$ is consistent with the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

$$A\mathbf{x} = \mathbf{b}$$

Since A is nonsingular, A^{-1} exists, so:

$$A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}$$

$$(A^{-1}A)\mathbf{x} = A^{-1}\mathbf{b}$$

$$I\mathbf{x} = A^{-1}\mathbf{b}$$

$$\mathbf{x} = A^{-1}\mathbf{b}$$

Example 2.2.2

Let A, B, C, D be invertible $n \times m$ matrices. Solve for C if $A^{-1}B^{-1}ADCD^{-1}B = D$

$$\begin{aligned}
 (A^{-1}A)B^{-1}ADCD^{-1}B &= AD \\
 B^{-1}ADCD^{-1}B &= AD \\
 BB^{-1}ADCD^{-1}B &= BAD \\
 ADCD^{-1}B &= BAD \\
 A^{-1}ADCD^{-1}B &= A^{-1}BAD \\
 DCD^{-1}B &= A^{-1}BAD \\
 D^{-1}DCD^{-1}B &= D^{-1}A^{-1}BAD \\
 CD^{-1}B &= D^{-1}A^{-1}BAD \\
 CD^{-1}BB^{-1} &= D^{-1}A^{-1}BADB^{-1} \\
 CD^{-1} &= D^{-1}A^{-1}BADB^{-1} \\
 CD^{-1}D &= D^{-1}A^{-1}BADB^{-1}D \\
 C &= D^{-1}A^{-1}BADB^{-1}D
 \end{aligned}$$

This tells us that the order in which you multiply makes a difference. Do you multiply on the left or the right?

Theorem 2.6 (Properties of the Inverse)

If A, B are $n \times n$ invertible matrices, then:

1. A^{-1} is invertible, and $(A^{-1})^{-1} = A$
2. AB is invertible, and $(AB)^{-1} = A^{-1}B^{-1}$
3. A^T is invertible, and $(A^T)^{-1} = (A^{-1})^T$

Theorem 2.7

Building on number 2 of the previous theorem, the product of $n \times n$ invertible matrices is invertible, and the inverse is the product of the inverses in reverse order.

eg. $(E_5E_4E_3E_2E_1)^{-1} = E_5^{-1}E_4^{-1}E_3^{-1}E_2^{-1}E_1^{-1}$

This is very similar to the socks and shoes rule: you put on your socks, then put on your shoes. To take them off, you take off your shoe then take off your socks.

Definition 2.10

An elementary matrix is a matrix obtained by performing one Elementary Row Operation on an identity matrix.

Example 2.2.3

$$\begin{aligned}
E_1 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}, \text{ Compute } E_1 \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \\
&= \begin{bmatrix} a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} + g \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & b \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + e \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} + h \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & c \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + f \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} + i \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix} \\
&= \begin{bmatrix} a & b & c \\ d & e & f \\ -2d + g & -2e + h & -2f + i \end{bmatrix}
\end{aligned}$$

$E_1 A$ replaces R_3 with $-2R_2 + R_3$

Elementary matrices are simply a method of completing elementary row operations through matrices. They are very predictable.

Example 2.2.4

Find $E_2 A$ where

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ \& } A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\begin{aligned}
I &\xrightarrow{-3R_2} E_2 \\
A &\xrightarrow{-3R_2} E_2 A
\end{aligned}$$

$$E_2 A = \begin{bmatrix} 1 & 2 & 3 \\ -12 & -15 & -18 \\ 7 & 8 & 9 \end{bmatrix}$$

Elementary matrices can tell you what to do without making any calculations, similar to the idea of how the vertex form of quadratics tell you where the vertex is.

Proposition 2.2

If an ERO is performed on an $m \times n$ matrix A , the resulting matrix can be written EA , where E is the $m \times m$ elementary matrix found by performing the same ERO on I_m .

Proposition 2.3

If E is elementary, then E is invertible. Idea for the following theorem:
 Suppose A is $n \times n$. We want to invert A . Therefore, row reduce $A \rightarrow I$.
 $E_p \dots E_4 E_3 E_2 E_1 A = I \leftarrow$ Identity Matrix
 Where E_p are elementary matrices

$$(E_p \dots E_4 E_3 E_2 E_1 A) = I$$

Thus, $A^{-1} = E_p \dots E_4 E_3 E_2 E_1$

Theorem 2.8

An $n \times n$ matrix A is invertible iff A is row equivalent to I . In this case, any sequence of EROs that reduced $A \rightarrow I$ would also transform $I \rightarrow A^{-1}$

Algorithm for Finding A^{-1}

Start by augmenting $[A \ I]$. To find A^{-1} if A is invertible, $\text{RREF}[A \ I] = [I \ A^{-1}]$

There are other algorithms that are typically computationally longer. The adjoint method (one of these algorithms) is far more work.

Example 2.2.5

Determine if $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & -2 & 3 \end{bmatrix}$ is invertible or singular. If A is invertible, find A^{-1} .

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 1 & -2 & 3 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 0 & -3 & 2 & -1 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -2 & 1 & 0 \\ 0 & -3 & 2 & -1 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & -3 & 2 & -1 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 5 & -3 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 5/2 & -3/2 & 1/2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -3/2 & 3/2 & -1/2 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 5/2 & -3/2 & 1/2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -7/2 & 5/2 & -1/2 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 5/2 & -3/2 & 1/2 \end{bmatrix}$$

$$\text{Therefore, } A^{-1} = \begin{bmatrix} 1 & 0 & 0 & -7/2 & 5/2 & -1/2 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 5/2 & -3/2 & 1/2 \end{bmatrix}$$

Another note: If you cannot row reduce to RREF, then you know that it is not invertible.

2.3 Characterizations of Invertible Matrices

Theorem 2.9 (The (small) Invertible Matrix Theorem)

Suppose A is $n \times n$. The following are equivalent (TFAE):

1. A is invertible
2. A is row equivalent to I .
3. A has n pivot positions.
4. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
5. The columns of A are linearly independent
6. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is 1-1.
7. $A\mathbf{x} = \mathbf{b}$ is consistent with the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.
8. The columns of A span \mathbb{R}^n .
9. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto \mathbb{R}^n
10. There exists an $n \times n$ matrix C such that $CA = I$
11. There exists an $n \times n$ matrix D such that $AD = I$
12. A^T is invertible

There will be additional statements in the future to supplement these initial equivalencies. This theorem above will allow us to **change our goal** to something that is equivalent, which will make solving the problem easier.

Definition 2.11

A linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is invertible if there exists a function $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that

$$\begin{aligned} S(T(\mathbf{x})) &= \mathbf{x} \text{ for all } \mathbf{x} \in \mathbb{R}^n, \text{ and} \\ T(S(\mathbf{x})) &= \mathbf{x} \text{ for all } \mathbf{x} \in \mathbb{R}^n \end{aligned}$$

Theorem 2.10

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be linear with standard matrix A . Then T is invertible iff A is invertible. In this case, the linear transformation S given by $S(\mathbf{x}) = A^{-1}\mathbf{x}$ is the unique function satisfying the invertible definition for T , and $S = T^{-1}$.

Example 2.3.1

Suppose $T : \mathbb{R}^7 \rightarrow \mathbb{R}^6$ is linear and $1 - 1$. Show that T is onto (the range = codomain).

We know T is linear, so it has a standard matrix A , and it is a 6×7 matrix. Since it is not square, the IMT can be used.

Since T defined by $T(\mathbf{x}) = A\mathbf{x}$ is $1 - 1$, we can use the IMT.

By the IMT, T is onto \mathbb{R}^6 .

Chapter 3

Determinants

3.1 Introduction to Determinants

Definition 3.1

For the *uninteresting* 1×1 case, we define $\det A = \det[a_1 1] = a_1 1$

Remark. While this is not used in a practical context, its used to build on this with a recursive definition.

Definition 3.2

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we defined $\det A = ad - bc$.

Consider $A = [a_{ij}]$ is 3×3 with $a_{11} \neq 0$. Then,

$$REF \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & 0 & a_{11}\Delta \end{bmatrix}$$

Where $\Delta = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$

Remark. This is A METHOD, but not a good method. Do not need to memorize this, it is just a step towards a larger definition.

Definition 3.3

For $A = [a_{ij}]$ as a 3×3 matrix, we define $\det A = \Delta$

Determinants help you determine invertability, as if the determinant is 0, then we know it is not invertible. A determinant determines if A is invertible or singular

Definition 3.4

The ij minor matrix A_{ij} , where A_{ij} is the matrix obtained from A by deleting its i^{th} row and j^{th} column.

Example 3.1.1

$$\text{Let } A = \begin{bmatrix} 1 & 0 & -7 & 6 & 1 \\ 2 & 3 & 5 & -1 & 1 \\ 0 & 1 & 1 & 2 & 1 \\ \pi & e & 6 & 4 & 1 \end{bmatrix}. \text{ Find } A_{32}.$$

$$A_{32} = \begin{bmatrix} 1 & -7 & 6 & 1 \\ 2 & 5 & -1 & 1 \\ \pi & 6 & 4 & 1 \end{bmatrix}$$

3.2 Properties of Determinants

Theorem 3.1 (Row Operations & Determinants)

Let A be a square matrix. Let B be obtained from A by a single ERO. Then:

ERO	Effect on \det	$\det B$
Scale by k	Scale by k	$\det A = k \det A$
Interchange	Negates	$\det A = -\det A$
Replacement	Invariant	$\det A = \det A$

Remark. Scaling in determinants is like factoring!

Example 3.2.1

$$\begin{aligned}
 &\text{Find } \left| \begin{array}{cccc} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{array} \right| \\
 &= - \left| \begin{array}{cccc} 1 & -4 & 0 & 6 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 2 & -8 & 6 & 8 \end{array} \right| \\
 &= - \left| \begin{array}{cccc} 1 & -4 & 0 & 6 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 2 & -8 & 6 & 8 \end{array} \right|
 \end{aligned}$$

Row reduce rows below 1 to be 0 in under the first pivot.

$$= - \left| \begin{array}{cccc} 1 & -4 & 0 & 6 \\ 0 & 3 & 5 & -8 \\ 0 & -12 & 1 & 16 \\ 0 & 0 & 6 & -4 \end{array} \right|$$

Reduce row 3 to continue making triangle matrix.

$$= - \left| \begin{array}{cccc} 1 & -4 & 0 & 6 \\ 0 & 3 & 5 & -8 \\ 0 & 0 & 21 & -16 \\ 0 & 0 & 6 & -4 \end{array} \right|$$

Take $\frac{7}{2}$ of row 3.

$$= -\frac{7}{2} \left| \begin{array}{cccc} 1 & -4 & 0 & 6 \\ 0 & 3 & 5 & -8 \\ 0 & 0 & 6 & \frac{-32}{7} \\ 0 & 0 & 6 & -4 \end{array} \right|$$

Eliminate position 3 in row 4 to complete triangle matrix.

$$\begin{aligned}
 &= -\frac{7}{2} \left| \begin{array}{cccc} 1 & -4 & 0 & 6 \\ 0 & 3 & 5 & -8 \\ 0 & 0 & 6 & \frac{-32}{7} \\ 0 & 0 & 0 & \frac{4}{7} \end{array} \right| \\
 &= \frac{7}{2}(1)(3)(6)\left(\frac{4}{7}\right) \\
 &= -36
 \end{aligned}$$