Winter 2022 MTH 261 Mini Test 3

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Question 1

Example 0.0.1

Let $A = \begin{bmatrix} 2 & 6 & 14 & 8 \\ 1 & 0 & 0 & 2 \\ -1 & -3 & -5 & 4 \\ -1 & 0 & -7 & 0 \end{bmatrix}$.

1. Compute $\det A$.
2. Determine if A is invertible or singular. Justify your answer as specifically as possible.
3. Are the vectors $\begin{bmatrix} 2 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 14 \\ 0 \\ -5 \\ -7 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ 2 \\ 4 \\ 0 \end{bmatrix}$ linearly independent? Why or why not?
4. Does the set of vectors $\left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 14 \\ 0 \\ -5 \\ -7 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ 2 \\ 4 \\ 0 \end{bmatrix} \right\}$ span $\mathbb{R}^4$? Why or why not?

Part 1

$$\begin{vmatrix} 2 & 6 & 14 & 8 \\ 1 & 0 & 0 & 2 \\ -1 & -3 & -5 & 4 \\ -1 & 0 & -7 & 0 \end{vmatrix}$$

$$= - \begin{vmatrix} 1 & 0 & 0 & 2 \\ 2 & 6 & 14 & 8 \\ -1 & -3 & -5 & 4 \\ -1 & 0 & -7 & 0 \end{vmatrix}$$

$$= - \begin{vmatrix} 1 & 0 & 0 & 2 \\ 0 & 6 & 14 & 4 \\ 0 & -3 & -5 & 6 \\ 0 & 0 & -7 & 2 \end{vmatrix}$$

$$= -2 \begin{vmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 7 & 2 \\ 0 & -3 & -5 & 6 \\ 0 & 0 & -7 & 2 \end{vmatrix}$$

$$= -2\begin{vmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 7 & 2 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & -7 & 2 \end{vmatrix}$$

$$= -4\begin{vmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 7 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & -7 & 2 \end{vmatrix}$$

$$= -4\begin{vmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 7 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 30 \end{vmatrix}$$

$$= -4(1)(3)(1)(30)$$

$$= -360$$

Part 2

Since det A is not equal to 0, we know that A is invertible by the Invertible Matrix Theorem (IMT).

Part 3

By the Invertible Matrix Theorem, we know that because A is invertible, the columns of A are also linearly independent, which happens to be the matrix we took the determinant of. Therefore, we know that the vectors are linearly independent.

Part 4

Yes, it does span $\mathbb{R}^4$, as the invertible matrix theorem states that if the matrix A with size $n \times n$ is invertible, then the matrix also spans $\mathbb{R}^4$.

Question 2

Example 0.0.2

Let D be the set of all diagonal 2×2 matrices. That is

$$D = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| \, a, b \in \mathbb{R} \right\}.$$

Use the subspace test to prove that D is a subspace of $M_{2\times 2}$.

Most notably, we can see that D is a subset of $M_{2\times 2}$

First, we can prove that the set is not empty. We can determine that the set is not empty as $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ exists in the set, and both a and b are real numbers.

Second, we can determine that the set is closed under vector addition:

$$A_1 = \begin{bmatrix} a_1 & 0 \\ 0 & b_2 \end{bmatrix}$$
$$A_2 = \begin{bmatrix} a_2 & 0 \\ 0 & b_2 \end{bmatrix}$$
$$A_1 + A_2 = \begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix} + \begin{bmatrix} a_2 & 0 \\ 0 & b_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & 0 \\ 0 & b_1 + b_2 \end{bmatrix} \epsilon M_{2\times 2}$$

Lastly, we can determine that the set is closed under scalar multiplication:

$$cA = c \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$
$$= c \begin{bmatrix} ca & 0 \\ 0 & cb \end{bmatrix} \epsilon M_{2\times 2}$$

Therefore, it can be determined that the D is a subspace of $M_{2\times 2}$ by the subspace test.

Question 3

Example 0.0.3

Let $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$.

1. What does it mean for $\mathbf{x}$ to be in $ColA$?
2. What does it mean for $\mathbf{x}$ to be in $NulA$?
3. Find a spanning set for $ColA$.
4. Find a spanning set for $NulA$.

Part 1

For $\mathbf{x}$ to be in $ColA$, it means that the vector exists as some linear combination of the spanning set of for $ColA$, which is simply the columns of A.

Part 2

For $\mathbf{x}$ to be in $NulA$, it means that the vector exists as some linear combination of the spanning set of for $NulA$, which can be found by solving the homogenous equation and then using the vectors corresponding to $\mathbf{x}$ in parametric vector form.

Part 3

The spanning set is simply the set of all of the columns in the matrix A. Therefore:

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

is the spanning set.

Part 4

$$A\mathbf{x} = \mathbf{0}$$
$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Creating an augmented matrix, we get:

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Given the identity matrix after row reduction, we know that the resulting $\mathbf{x}$ is simply $\mathbf{0}$. Therefore, the spanning set is simply:

$$\mathbf{x} = \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$