MTH 261 Lecture Notes

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Chapter 1

Introduction to Linear Algebra

1.1 System of Linear Equations

Linear Algebra is the area of math concerning liner equations/functions that are represented in vector space and through matrices. If Calculus is the foundational language of mathematics, then Linear Algebra is the foundational language of STEM.

> **Remark.** Keep in mind that while Linear Algebra utilizes matrices, it is just a tool to solve problems with. The study is **NOT** of matrices.

Definition 1.1 (Linear Equation)

An equation where $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$, where $a_1, a_2, a_n, b \in$ as constants of $\mathbb{C}$ and in $\mathbb{R}$

Definition 1.2 (Linear System)

A collection of linear equations with the same variable

Definition 1.3 (Solution & Solution Set)

A solution satisfies all equations in a system simultaneously, while a solution set is all possible solutions

Definition 1.4 (Equivalent Solutions)

Two systems with the same identical solution set

Definition 1.5 (Types of Solutions)

There are two major classification with solution, which are broken into three major solutions:

- Inconsistent (no solutions)
- Consistent (at least one solution)
  - Unique Solution
  - Infinite Solutions

Since the systems we are exploring and purely linear, there will not be two, three, or more solutions.

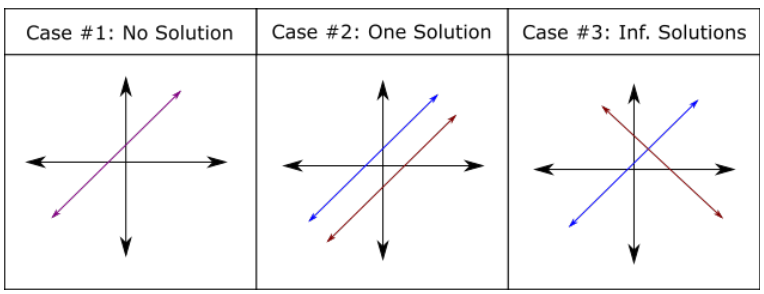

Figure 1.1: Types of solutions in a linear system

> **Remark.** This is true for all linear systems in all space

Definition 1.6 (Matrix)

A rectangular array of numbers, often used to compress systems. Let's say the following system of equations is given:

$$\begin{aligned} x - 7y \qquad + 6t &= 5 \\ z - 2t &= -3 \\ -x + 7y - 4z + 2t &= 7 \end{aligned}$$

It can be reduced into the following matrix:

$$\begin{bmatrix} 1 & -7 & 0 & 6 \\ 0 & 0 & 1 & -2 \\ -1 & 7 & -4 & 2 \end{bmatrix}$$

Definition 1.7 (Augmented Matrix)

A standard matrix which also includes the **b** coefficient in the matrix. The following is the augmented matrix of the system of equations from Definition 1.6:

$$\begin{bmatrix} 1 & -7 & 0 & 6 \\ 0 & 0 & 1 & -2 \\ -1 & 7 & -4 & 2 \end{bmatrix} OR \left[\begin{array}{ccc|c} 1 & -7 & 0 & 6 \\ 0 & 0 & 1 & -2 \\ -1 & 7 & -4 & 2 \end{array}\right]$$

1.2 Row Reduction and Echelon Forms

Every ERO can be undone

1.3 Vector Equations

1.4 The Matrix Equation $A\mathbf{x} = \mathbf{b}$

1.5 Solution Sets of Linear Systems

1.6 Linear Independence

1.7 Introduction to Linear Transformations

Definition 1.8

A transformation T is called linear (linear transformation) if for all $\mathbf{u}$, $\mathbf{v}$ in the domain of T and c $\in \mathbb{R}$,

- $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u} + T(\mathbf{v})$
- $T(c\mathbf{u}) = cT(\mathbf{u}$

Therefore, we can determine that every matrix transformation ($T(\mathbf{x} = A\mathbf{x})$ is a linear transformation

To show that a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear:

1. Introduce $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n andc \in \mathbb{R}$. These must be arbitrary
2. Show that $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u} + T(\mathbf{v}$
3. Show that $T(c\mathbf{u}) = cT(\mathbf{u}$

Turns out, we could show that IF WE ALREADY KNOW that T is linear, then:

- $T(\mathbf{0}) = \mathbf{0}$ and
- $T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v}$

To show that T is NOT linear, either:

- Show $T(\mathbf{0} \neq \mathbf{0})$, or
- $T(\mathbf{u} + \mathbf{v}) \neq T(\mathbf{u}) + T(\mathbf{v})$ for specific vectors u & v, or

1.8 The Matrix of a Linear Transformation

Example 1.8.1

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ that first dilates vectors by a size of 2 then reflects across the line $x_2 = -x_1$. Assuming this transformation is linear, find the standard matrix for T

Solution

1. Dilates (expands) by $2 \to$ Doubles in size.
2. Reflects across $x_1 = -X_1(y = -x)$.

Chapter 2

Matrix Algebra

2.1 Matrix Operations

There are several ways of representing a matrix:

$$\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n\end{bmatrix} = \begin{bmatrix}\mathbf{a}_i j\end{bmatrix} =$$

Definition 2.1

The diagonal entries of the $m \times n$ matrix i

Definition 2.2

The main diagonal is the collection diagonal entries starting from the top left. **NOTE:** This may not include the bottom right due to the size of the matrix

Definition 2.3

Diagonal matrix has the form:

$$\begin{bmatrix}\square & 0 & 0 & 0 \\ 0 & \square & 0 & 0 \\ 0 & 0 & \square & 0\end{bmatrix}$$

Definition 2.4

The zero matrix is any matrix who's entries are all zero.

Basic operations are the same between matrices and vectors (addition, subtraction, multiplication, and equality). Keep in mind, matrices must be the same size for addition/subtraction operations.

Theorem 2.1 (Properties of Matrix Arithmitic)

Let A, B, C, be matrices of the same size, $r, s, \in \mathbb{R}$.

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$
3. $A + O = A$
4. $r(A + B) = rA + rB$
5. $(r + s)A = rA + sA$
6. $r(sA) = (rs)A$

NOTE: O is used a 0 in mathematics.

Matrix Multiplication

When B multiplies a vector $\mathbf{x}$ it starts from $\mathbf{x} \mapsto B\mathbf{x}$

Definition 2.5

If A is $m \times n$ and B is $n \times p$, then the product AB is the $m \times p$ matrix.

$$AB = [A\mathbf{b}_1, A\mathbf{b}_2 \ldots A\mathbf{b}_p]$$

Remark. The idea of non-communicative multiplication is uncommon, however does occur on occasion.

Theorem 2.2

Let A, B, C be matrices so that these are defined, $r \in \mathbb{R}$

1. $A(BC) = (AB)C$
2. $A(B + C) = AB + AC$
3. $(B + C)A = BA + CA$
4. $r(AB) = (rA)B = A(rB)$
5. If A is $m \times n$, then $I_m A = A = AI_n$

A few common pitfalls. In general:

- $AB \neq BA$
- Cancellation does not hold: $AB = AC \not\Rightarrow B = C$
- The Zero Product Principle does not hold: $AB = O \not\Rightarrow A = O$ or $B = O$

Theorem 2.3 (Transpose Properties)
Let A, B, be matrices so that these are defined, $r \in (R)$.

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(rA)^T = rA^T$
4. $(AB)^T = B^T A^T$

> **Remark.** This idea of transposition will come up frequently in Linear Algebra.

2.2 The Inverse of a Matrix

Definition 2.6 (Multiplicative Inverse)
If $c \in \mathbb{R}$ with $c \neq O$,then:

$$c \cdot c^{-1} = 1 \text{ and } c^{-1} \cdot c = 1$$

In other words:

$$A \cdot A^{-1} = I \text{ and } A^{-1} \cdot A = I$$

Definition 2.7
An $n \times n$ matrix A is invertible if there is another $n \times n$ matrix C such that $CA = I$ and $AC = I$. We call C the inverse of A and write $C = A^{-1}$.

Proposition 2.1
The inverse of an invertible matrix is unique.

Definition 2.8

An $n \times n$ matrix that is not invertible is called singular.

$$\begin{aligned} \text{invertible} &= \text{nonsingular} \\ \text{singular} &= \text{noninvertible} \end{aligned}$$

Theorem 2.4

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq O$, then A is invertible, and $A^{-1} = \frac{1}{ad=bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. If $ad - bc = O$, A is singular.

Definition 2.9

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the <u>determinant</u> of A, written $detA$ or $|A|$, is the number $ad - bc$.

Example 2.2.1

Let $A = \begin{bmatrix} 2 & 6 \\ -1 & 3 \end{bmatrix}$. Find $\det A$ & A^{-1}.

Solution: $\det A = (2)(3) - (6)(-1) = 12$

$A^{-1} = \frac{1}{detA} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$= \frac{1}{12} \begin{bmatrix} 3 & -6 \\ 1 & 2 \end{bmatrix}$

$=$

Theorem 2.5

If A is nonsingular, then for each $\mathbf{b} \in \mathbb{R}^n$, $A\mathbf{x} = \mathbf{b}$ is consistent with the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

$A\mathbf{x} = \mathbf{b}$

Since A is singular, A^{-1} exists, so:

$A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}$

$(A^{-1}A)\mathbf{x} = A^{-1}\mathbf{b}$

$I\mathbf{x} = A^{-1}\mathbf{b}$

$\mathbf{x} = A^{-1}\mathbf{b}$

Example 2.2.2

Let A, B, C, D be invertible $n \times m$ matrices. Solve for C if
$A^{-1}B^{-1}ADCD^{-1}B = D$

$$(A^{-1}A)B^{-1}ADCD^{-1}B = AD$$
$$B^{-1}ADCD^{-1}B = AD$$
$$BB^{-1}ADCD^{-1}B = BAD$$
$$ADCD^{-1}B = BAD$$
$$A^{-1}ADCD^{-1}B = A^{-1}BAD$$
$$DCD^{-1}B = A^{-1}BAD$$
$$D^{-1}DCD^{-1}B = D^{-1}A^{-1}BAD$$
$$CD^{-1}B = D^{-1}A^{-1}BAD$$
$$CD^{-1}BB^{-1} = D^{-1}A^{-1}BADB^{-1}$$
$$CD^{-1} = D^{-1}A^{-1}BADB^{-1}$$
$$CD^{-1}D = D^{-1}A^{-1}BADB^{-1}D$$
$$C = D^{-1}A^{-1}BADB^{-1}D$$

This tells us that the order in which you multiply makes a difference. Do you multiply on the left or the right?

Theorem 2.6 (Properties of the Inverse)

If A, B are $n \times n$ invertible matrices, then:

1. A^{-1} is invertible, and $(A^{-1})^{-1} = A$
2. AB is invertible, and $(AB)^{-1} = A^{-1}B^{-1}$
3. A^T is invertible, and $(A^T)^{-1} = (A^{-1})^T$

Theorem 2.7

Building on number 2 of the previous theorem, the product of $n \times n$ invertible matrices is invertible, and the inverse is the product of the inverses in reverse order.
eg. $(E_5E_4E_3E_2E_1)^{-1} = E_5^{-1}E_4^{-1}E_3^{-1}E_2^{-1}E_1^{-1}$

This is very similar to the socks and show rule: you put on your socks, then put on your shoes. To take them off, you take off your shoe then take off your socks.

Definition 2.10

An elemntary matrix is a matrix obtained by performing one Elementary Row Operation on an identity matrix.

Example 2.2.3

$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$, Compute $E_1 \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$= \begin{bmatrix} a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} + g \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & b \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + e \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} + h \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & c \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + f \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} + i \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} a & b & c \\ d & e & f \\ -2d + g & -2e + h & -2f + i \end{bmatrix}$$

$$E_1 A \text{ replaces } R_3 \text{ with } -2R_2 + R_3$$

Elementary matrices are simply a method of completing elementary row operations through matrices. They are very predictable.

Example 2.2.4

Find $E_2 A$ where

$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ & $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

$$I \xrightarrow{-3R_2} E_2$$
$$A \xrightarrow{-3R_2} E_2 A$$

$$E_2 A = \begin{bmatrix} 1 & 2 & 3 \\ -12 & -15 & -18 \\ 7 & 8 & 9 \end{bmatrix}$$

Elementary matrices can tell you what to do without making any calculations, similar to the idea of how the vertex form of quadratics tell you where the vertex is.

Proposition 2.2

If an ERO is performed on an $m \times n$ matrix A, the resulting matrix can be written EA, where E is the $m \times m$ elementary matrix found by preforming the same ERO on I_m.

Proposition 2.3

If E is elementary, then E is invertible. Idea for the following theorem: Suppose A is $n \times n$. We want to invert A. Therefore, row reduce $A \to I$.
$E_p \ldots E_4 E_3 E_2 E_1 A = I \leftarrow$ Identity Matrix
Where E_p are elementary matrices

$$(E_p \ldots E_4 E_3 E_2 E_1 A) = I$$

Thus, $A^{-1} = E_p \ldots E_4 E_3 E_2 E_1$

Theorem 2.8

An $n \times n$ matrix A is invertible iff A is row equivalent to A. In this case, any sequence of EROs that reduced $A \to I$ would also transform $I \to A^{-1}$

Algorithm for Finding A^{-1}

Start by augmenting $[A\ I]$. To find A^{-1} if A is invertible, RREF$[A\ I] = [I\ A^{-1}]$

There are other algorithms that are typically computationally longer. The adjoint method (one of these algorithms) is far more work.

Example 2.2.5

Determine if $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & -2 & 3 \end{bmatrix}$ is invertible or singular. If A is invertible, find A^{-1}.

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 1 & -2 & 3 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 0 & -3 & 2 & -1 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -2 & 1 & 0 \\ 0 & -3 & 2 & -1 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & -3 & 2 & -1 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 5 & -3 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 5/2 & -3/2 & 1/2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -3/2 & 3/2 & -1/2 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 5/2 & -3/2 & 1/2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -7/2 & 5/2 & -1/2 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 5/2 & -3/2 & 1/2 \end{bmatrix}$$

Therefore, $A^{-1} = \begin{bmatrix} 1 & 0 & 0 & -7/2 & 5/2 & -1/2 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 5/2 & -3/2 & 1/2 \end{bmatrix}$

Another note: If you cannot row reduce to RREF, then you know that it is not invertible.

2.3 Characterizations of Invertible Matrices

Theorem 2.9 (The (small) Invertible Matrix Theorem)

Suppose A is $n \times n$. The following are equivalent (TFAE):

1. A is invertible
2. A is row equivalent to I.
3. A has n pivot positions.
4. $A\mathbf{x} = \mathbf{O}$ has only the trivial solution
5. The columns of A are linearly independent
6. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is $1-1$.
7. $A\mathbf{x} = \mathbf{b}$ is consistent with the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.
8. The columns of A span $\mathbb{R}^n$.
9. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto $\mathbb{R}^n$
10. There exists an $n \times n$ matrix C such that $CA = I$
11. There exists an $n \times n$ matrix D suc that $AD = I$
12. A^T is invertible

There will be additional statements in the future to supplement these initial equivalencies. This theorem above will allow us to **change our goal** to something that is equivalent, which will make solving the problem easier.

Definition 2.11

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is invertible if there exists a function $S : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$S(T(\mathbf{x})) \text{ for all } \mathbf{x} \in \mathbb{R}^{\mathbf{2}}, \text{ and}$$
$$T(S(\mathbf{x})) \text{ for all } \mathbf{x} \in \mathbb{R}^{\mathbf{2}}$$

Theorem 2.10

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be linear with standard matrix A. Then T is invertible iff A is invertible. In this case, the linear transformation S given by $S(\mathbf{x}) = A^{-1}\mathbf{x}$ is the unique function satisfying the invertible definition for T, and $S = T^{-1}$.

Example 2.3.1

Suppose $T : \mathbb{R}^7 \to \mathbb{R}^6$ is linear and $1-1$. Show that T is onto (the range = codomain).

We know T is linear, so it has a standard matrix A, and it is a 7×7 matrix. Since it is square, the IMT can be used.

Since T defined by $T(\mathbf{x}) = A\mathbf{x}$ is $1-1$, we can use the IMT.

By the IMT, T is onto $\mathbb{R}^7$

Chapter 3

Determinants

3.1 Introduction to Determinants

Definition 3.1

For the *uninteresting* 1×1 case, we define $detA = det[a_11] = a_11$

> **Remark.** While this is not used in a practical context, its used to build on this with a recursive definition.

Definition 3.2

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we defined $detA = ad - bc$.

Consider $A = [a_ij]$ is 3×3 with $a_11/neqO$. Then,

$$REF \begin{bmatrix} a_11 & a_12 & a_13 \\ a_21 & a_22 & a_23 \\ a_31 & a_32 & a_33 \end{bmatrix} = \begin{bmatrix} a_11 & a_12 & a_13 \\ 0 & a_11a_22 - a_12a_21 & a_11a_23 - a_13a_21 \\ 0 & 0 & a_11\Delta \end{bmatrix}$$

Where $\Delta = a_11a_22a_33 + a_12a_23a_31 + a_13a_21a_32 - a_11a_23a_32 - a_12a_21a_33 - a_13a_22a_31$

> **Remark.** This is A METHOD, but not a good method. Do not need to memorize this, it is just a step towards a larger definition.

Definition 3.3

For $A = a_ij]$ as a 3×3 matrix, we define $detA = \Delta$

Determinants help you determine invertability, as if the determinant is 0, the we know it is not invertible. A determinant determines if A is invertible or singular

Definition 3.4

The ij minor matrix A_ij, where A_ij is the matrix obtained from A by deleting its i^th row nd j^th column.

Example 3.1.1

Let $A = \begin{bmatrix} 1 & 0 & -7 & 6 & 1 \\ 2 & 3 & 5 & -1 & 1 \\ 0 & 1 & 1 & 2 & 1 \\ \pi & e & 6 & 4 & 1 \end{bmatrix}$. Find A_{32}.

$A_{32} = \begin{bmatrix} 1 & -7 & 6 & 1 \\ 2 & 5 & -1 & 1 \\ \pi & 6 & 4 & 1 \end{bmatrix}$

3.2 Properties of Determinants