TOPICS IN PROPAGATION OF CHAOS

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0. INTRODUCTION

The terminology propagation of chaos comes from Kac. The initial motivation for the subject was to try to investigate the connection between a detailed and a reduced description of particles' evolution. For instance on the one hand one has Liouville's equations:

$$\partial_t u + \sum_1^N v_i \partial_{x_i} u + \sum_{j \neq i} -\nabla V_N(x_i - x_j) \partial_{v_j} u = 0 \ , \tag{0.1}$$

where $u(t, x_1, v_1, \dots, x_n, v_N)$ is the density of presence at time t, assumed to be symmetric of N particles, with position x_i and velocity v_i, and pairwise potential interaction $V_N(\cdot)$.

On the other hand one also has for instance the Boltzmann equation for a dilute gas of hard spheres

$$\partial_t u + v.\nabla_x u = \int_{R^3 \times S_2} (u(x,\tilde{v})u(x,\tilde{v}') - u(x,v)u(x,v'))\, |(v'-v)\cdot n|\; dv' dn \tag{0.2}$$

where $\tilde{v}, \tilde{v}'$ are obtained from v, v' by exchanging the respective components of v and v' in direction n, that is:

$$\begin{aligned} \tilde{v} &= v + (v'-v)\cdot n\; n \\ \tilde{v}' &= v' + (v-v')\cdot n\; n \ , \end{aligned}$$

and $u(t, x, v)$ is the density of presence at time t of particles at location x with velocity v.

One question was to understand the nature of the connection between (0.1) and (0.2). There are now works directly attacking the problem (see Lanford [22], and more recently Uchiyama [54]). However at the time, Kac proposed to get insight into the problem, by studying simpler Markovian models of particles. The forward equations for the Markovian evolutions of N-particle systems come as a substitute for the Liouville equations. They are called master equations. Kac [15] traces the origin of these masters equations back to the forties with works of Norsdieck-Lamb-Uhlenbeck on a problem of cosmic rays, and of Siegert.

In the case of the Boltzmann equation, one can for instance forget about positions of particles and take as master equations:

(0.3)

$$\begin{aligned} &\partial_t u_t^N(v_1, \dots, v_N) \\ &= \frac{1}{(N-1)} \sum_{1 \le i < j \le n} \int_{S_2} (u_t^N(v_1, .., \tilde{v}_i, .., \tilde{v}_j, .., v_N) - u_t^N(v_1, .., v_N)) |(v_i - v_j)\cdot n|) dn \end{aligned}$$

One now tries to connect these master equations with the spatially homogeneous Boltzmann equations for hard spheres:

$$\partial_t u = \int_{\mathbb{R}^3 \times S_2} (u(\tilde{v},t)u(\tilde{v}',t) - u(v,t)u(v',t))|(v'-v)\cdot n|dv'dn \; . \tag{0.4}$$

The idea proposed by Kac, which motivates the terminology "propagation of chaos" is the following. If one picks a "chaotic" initial distribution of particles: $u_N(0,v_1,\dots,v_N) = u_0(v_1)\dots u_0(v_N)$, for fixed N the evolution due to the master equations will in general destroy the independence property of $v_1,\dots,v_N$ at time t. However if one focuses on the reduced distribution at time t of the first k components, it should approximately be given when N is large by $u_t(v_1)\dots u_t(v_k)$, if $u_t(v)$ is the solution of equation (0.3) with initial condition $u_0(\cdot)$. So in this sense independence (or chaos) still propagates and eequation (0.4) emerges.

"Propagation of chaos" deals with symmetric evolution of particles, and this is not an innocent assumption. Among other things it tells us that the probability distribution of the first k particles $u^N(dv_1,\dots,dv_k)$ is the normalized k-particle correlation measure, that is the intensity of the random measure $\frac{1}{N(N-1)\cdots(N-k-1)} \sum_{i_\ell \text{distinct}} \delta_{(v_{i_1},\dots,v_{i_k})}$. The consequence is that the study of one individual gives information on the behavior of the group. We have so far presented the N-particle system and the nonlinear equation. There is a third actor in the play the "nonlinear process". It describes the limit behavior of the trajectory of one individual. It is sometimes called the tagged particle process, however we refrained from using this terminology for it can have different meanings, (see for instance the end of Chapter I). The time marginals of the nonlinear process will evolve according to the nonlinear equation under study, and this motivates the name of "nonlinear process", although of course in some examples, where interactions vanish all can be very linear in fact.

We will present in these notes a selection of topics on "propagation of chaos" which by no means covers the huge literature on the subject.

Let us now close the introduction with an informal discussion, along the lines of the "BBGKY-hierarchical method" for a model of interacting diffusions due to McKean [27]. This will motivate why propagation of chaos should hold in this case.

One looks at N particles on $\mathbb{R}^d$, with initial "chaotic" distribution $u_0^{\otimes N}$, satisfying the S.D.E.:

$$dx_t^i = dB_t^i + \frac{1}{N}\sum_1^N b(x_t^i, x_t^j)\, dt \; , \quad 1 \le i \le N \; , \tag{0.5}$$

where $B^i_.$ are independent Brownian motions, and $b(\cdot,\cdot)$ is for instance smooth compactly supported. We are now going to explain in an informal way how the nonlinear equation

$$\begin{aligned} \partial_t u &= \frac{1}{2}\Delta u - \operatorname{div}\Big(\int b(\cdot,y)\ u(t,y)\ dy\ u\Big) \\ u_{t=0} &= u_0\ , \end{aligned} \tag{0.6}$$

arises in the propagation of chaos effect.

Let us retinterpret equation (0.6). If P^0_t denotes the Brownian transition density, we have the perturbation formula (obtained for instance by differentiating in s, $u_s P^0_{t-s}$):

$$u_t(x) - u_0\ P^0_t(x) = \int_0^t ds_1 \int dx_1 dx_2 u_{s_1}(x_1) u_{s_1}(x_2)\ b(x_1,x_2)\ \nabla_{x_1} P^0_{t-s_1}(x_1,x)\ .$$

Continuing the development of $u_{s_1}(x_1)u_{s_1}(x_2),\dots$, by induction we find:
(0.7)

$$u_t = u_0 P^0_t + \sum_{k=1}^m \int_{0<s_k<\cdots<s_1<t} ds_k..ds_1\ u_0^{\otimes k+1} P^0_{s_k} B\cdot\nabla P^0_{s_{k-1}-s_k}..B\cdot\nabla P^0_{t-s_1} + R_m\ ,$$

$$\text{where } R_m = \int_{0<s_{m+1}<s_m<\cdots<s_1<t} ds_{m+1}..ds_1\ u_{s_{m+1}}^{\otimes m+2}\ B\cdot\nabla P^0_{s_m-s_{m+1}}..\nabla P^0_{t-s_1}\ .$$

Here we have adopted convenient notations, and P^0_t acts naturally in a tensorial way on functions of an arbitrary number of variables, and $B\cdot\nabla$ maps functions of k variables into functions of $(k+1)$ variables, $k\geq 1$, by the formula:

$$[B\cdot\nabla]f(x_1,\dots,x_{k+1}) = \sum_1^k b(x_i,x_{i+1})\ \nabla_i f(x_1,\dots,x_k)\ . \tag{0.8}$$

If we use a similar perturbation method for the time marginals $u_{N,t}(x_1,\dots,x_N)$ of the N-particle system, using the forward equation corresponding to (0.5), we now find:
(0.9)

$$u_{N,t} = u_0^{\otimes N} P^0_t + \sum_{k=1}^m \int_{0<s_k<\cdots<s_1<t} u_0^{\otimes N}\ P^0_{s_k}\ B^N\ \nabla P^0_{s_k-s_{k-1}}\cdots B^N \nabla P^0_{t-s_1} + R^N_m\ ,$$

$$\text{where } R^N_m = \int_{0<s_{m+1}<\cdots<s_1<t} u^N_{s_{m+1}} B^N \nabla P^0_{s_m-s_{m+1}}\cdots B^N\nabla P^0_{t-s}\ , \text{ and}$$

$$[B^N\cdot\nabla]f(x_1,\dots,x_N) = \frac{1}{N}\sum_{i,j=1}^N b(x_i,x_j)\ \nabla_i f(x_1,\dots,x_N)\ . \tag{0.10}$$

Let us now see what happens as N goes to infinity when we consider $\langle u_{N,t}, f\rangle$ for a function $f(x_1)$ depending only on the first variable.

In (0.9), the first term

$$a_0^N = u_0^{\otimes N} P_t^0 f = u_0 P_t^0 f \ ,$$

coincides with the first term a_0 of (0.7).

The second term of (0.9) is

$$a_1^N = \int_0^t ds\ u_0^{\otimes N}\ P_{s_1}^0 B^N \cdot \nabla P_{t-s_1}^0\ f\ ds_1\ .$$

Since $P_{t-s_1}^0 f$ just depends on the x_1 variable

$$a_1^N = \int_0^t ds_1\ u_0^{\otimes N} P_{s_1}^0 \left(\frac{1}{N}\sum b(x_1,x_j)\ \nabla_1(P_{t-s_1}^0 f)(x_1)\right)\ ds_1$$

But $u_0^{\otimes N} P_{s_1}^0$ is symmetric. For $j \neq 1$, one can pick a permutation of $[1,N]$, leaving 1 invariant and mapping j on 2, so

$$a_1^N = \frac{N-1}{N}\int_0^t u_0^{\otimes N} P_{s_1}^0 \{b(x_1,x_2)(\nabla_1 P_{t-s_1}^0 f)(x_1)\}\ ds_1 + o(N)\ ,$$

and in fact $u_0^{\otimes N}$ can be replaced by $u_0 \otimes u_0$ in the last expression so that a_1^N converges to the first term a_1 of (0.7).

The third term a_2^N of (0.9) is

$$a_2^N = \int_{0<s_2<s_1<t} ds_2\ ds_1\ u_0^{\otimes N} P_{s_2}^0 B^N\ \nabla P_{s_1-s_2}^0\ B^N \nabla P_{t-s_1}^0 f\ ds_1 ds_2$$

Denote by ϕ the symmetric distribution $u_0^{\otimes N} P_{s_2}^0$ and by ψ the expression $u_0^{\otimes N} P_{s_2}^0 B^N \cdot \nabla P_{s_1-s_2}^0$. To obtain ψ one lets $P_{s_1-s_2}^0$ act on the right on:

$$-\sum_i \mathrm{div}_i(\phi(x_1,\dots,x_N)\frac{1}{N}\sum_{j=1}^N b(x_i,x_j))\ ,$$

which is also symmetric. Applying the same trick as before, we find:

$$\begin{aligned}
a_2^N &= \frac{N-1}{N}\int_{0<s_2<s_1<t} ds_2 ds_1 \langle \psi\ ,\ b(x_1,x_2)(\nabla_1 P_{t-s_1}^0 f)(x_1)\rangle + o(N) \\
&= \frac{(N-1)(N-2)}{N^2}\int_{0<s_2<s_1<t} ds_2 ds_1 \langle u_0^{\otimes N} P_{s_2}^0, (b(x_1,x_3)\nabla_1 + b(x_2,x_3)\cdot\nabla_2) \\
&\quad \cdot P_{s_1-s_2}^0 (b(x_1,x_2)\nabla_1(P_{t-s}^0 f)(x_1)\rangle \\
&\quad + o(N)\ ,
\end{aligned}$$

and one sees that a_2^N converges to a_2 the third term in (0.7).

In the same way, one sees that there is a term by term convergence of a_k^N to a_k for each k. However, we cannot transform this into a bona fide propagation of chaos result, since we do not have a good control on our series.

Indeed, we have an estimate of the type

$$\|b(x_i,x_j)\nabla_i P_\tau^0\|_{L^\infty \to L^\infty} \leq \frac{c}{\sqrt{\tau}} \ ,$$

and the generic term a_k in (0.7) for instance is naturally estimated in terms of

$$k!c^k \int_{0<s_2<\cdots<s_1<t} ds_k \ldots ds_1 [(t-s_1)\ldots(s_{k-1}-s_k)]^{-1/2} \ ,$$

but this quantity tends to infinity.

Of course this convergence problem comes from the fact that it is unreasonable to use a perturbation series of P_t^0 to take care of the drift term $b(\cdot,\cdot)\cdot\nabla$. Nevertheless this gives a flavor of the propagation of chaos result. In the next section, we will provide a proof by a probabilistic approach.

I. Generalities and first examples.

1) A laboratory example.

We now come back to McKean's example of interacting diffusions, and we are going to use a probabilistic method to attack the problem.

We suppose $b(\cdot,\cdot)$ bounded Lipschitz $R^d \times R^d \to R^d$, and construct on $(R^d \times C_0(R_+, R^d))^{N^*}$, with product measure $(u_0 \otimes W)^{\otimes N^*}$, ($u_0$ probability on R^d, W standard R^d-Wiener measure) the $X^{i,N}_\cdot$, $i = 1, \dots, N$, satisfying

$$(1.1) \qquad \begin{aligned} dX^{i,N}_t &= dw^i_t + \frac{1}{N} \sum_{j=1}^N b(X^{i,N}_t, X^{j,N}_t) dt \ , \quad i = 1, \dots, N \\ X^{i,N}_0 &= x^i_0 \ , \end{aligned}$$

here x^i_0, $(w^i_\cdot)$, $i \geq 1$, are the canonical coordinates on the product space $(R^d \times C_0)^{N^*}$.

We are going to show that when N goes to infinity each $X^{i,N}_\cdot$, has a natural limit $\overline{X}^i$. Each $\overline{X}^i$ will be an independent copy of a new object: "the nonlinear process".

Nonlinear process:

On a filtered probability space $(\Omega, F, F_t, (B_t)_{t \geq 0}, X_0, P)$, endowed with an R^d-valued Brownian motion $(B_t)_{t \geq 0}$, and an u_0-distributed, F_0 measurable R^d-valued random variable X_0, we look at the equation:

$$(1.2) \qquad \begin{aligned} dX_t &= dB_t + \int b(X_t, y) \ u_t(dy) \ dt \\ X_{t=0} &= X_0 \ , \quad u_t(dy) \text{ is the law of } X_t \end{aligned}$$

Theorem 1.1. *There is existence and uniqueness, trajectorial and in law for the solutions of* (1.2).

Remark 1.2. Let us notice that the nonlinear process has time marginals which satisfy in a weak sense the nonlinear equation

$$(1.3) \qquad \partial_t u = \frac{1}{2} \Delta u - \operatorname{div}\Big(\int b(\cdot, y) \ u_t(dy) u \Big) \ .$$

Indeed, for $f \in C^2_b(R^d)$, applying Ito's formula:

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) dB_s + \int_0^t \Big(\frac{1}{2} \Delta f + \int_{R^d} b(X_s, y) \ u_s(dy) \ \nabla f(X_s) \Big) \ ds \ ,$$

after integration this yields a weak version of (1.3).

$\square$

Proof: Let us now turn to the proof of Theorem 1.1. We introduce the Kantorovitch-Rubinstein or Vaserstein metric on the set $M(C)$ of probability measures on $C = C([0,T], R^d)$, defined by

$$
\begin{aligned}
(1.4) \qquad D_T(m_1, m_2) = \inf\{ \int (\sup_{s \le T} |X_s(\omega_1) - X_s(\omega_2)| \wedge 1) \, dm(\omega_1, \omega_2) \, , \\
m \in M(C \times C) \, , \quad p_1 \circ m = m_1 \, , \quad p_2 \circ m = m_2 \} \, .
\end{aligned}
$$

Here X_s is simply the canonical process on C.

Formula (1.4) defines a complete metric on $M(C)$, which gives to $M(C)$ the topology of weak convergence. The proof of this fact can be found in Dobrushin [8].

Take now $T > 0$, and define Φ the map which associates to $m \in M(C([0,T], R^d))$ the law of the solution of

$$
(1.5) \qquad X_t = X_0 + B_t + \int_0^t \Big(\int_C b(X_s, w_s) \, dm(w) \Big) \, ds \, , \quad t \le T \, .
$$

Observe that this law does not depend on the specific choice of the space Ω, we use.

Next observe that if X_t, $t \le T$, is a solution of (1.2), then its law on $C([0,T], R^d)$, is a fixed point of Φ, and conversely if m is such a fixed point of Φ, (1.5) defines a solution of (1.2), up to time T. So now our problem is translated into a fixed point problem for Φ, and one has the contraction lemma:

Lemma 1.3. *For $t \le T$,*

$$
D_t(\Phi(m_1), \Phi(m_2)) \le c_T \int_0^t D_u(m_1, m_2) \, du \, , \quad m_1, m_2 \in M(C_T) \, ,
$$

c_T is a constant and, $D_u(m_1, m_2)$ ($\le D_T(m_1, m_2)$) is the distance between the images of m_1, and m_2 on $C([0,u], R^d)$.

Proof:

$$
\begin{aligned}
X_t^1 &= X_0 + B_t + \int_0^t \Big(\int b(X_s^1, w_s) dm_1(w) \Big) ds \, , \quad t \le T \, , \\
X_t^2 &= X_0 + B_t + \int_0^t \Big(\int b(X_s^2, w_s) dm_2(w) \Big) ds \, , \quad t \le T \, .
\end{aligned}
$$

So we find:

$$
\sup_{s \le t} |X_s^1 - X_s^2| \le \int_0^t ds \Big| \int b(X_s^1(\omega), w_s) dm_1(w) - \int b(X_s^2(\omega), w_s) dm_2(w) \Big| \, .
$$

But

$$
\Big| \int b(x, w_s) dm_1(w) - \int b(y, w_s) dm_2(w) \Big| \le K[|x - y| \wedge 1 + \int |w_s^1 - w_s^2| \wedge 1 \, dm(w^1, w^2)]
$$

where m is any coupling of m_1, m_2 on $C([0,s], \boldsymbol{R}^d)$. From this

$$\sup_{s \le t} |X_s^1 - X_s^2| \le K \int_0^t ds |X_s^1(\omega) - X_s^2(\omega)| \wedge 1 + K \int_0^t D_s(m_1, m_2) ds \ .$$

Using Gronwall's lemma:

$$\sup_{s \le t} |X_s^1 - X_s^2| \wedge 1 \le K \ e^{KT} \int_0^t D_s(m_1, m_2) ds \ ,$$

from which the lemma follows.

□

From the lemma, we can immediately deduce weak and strong uniqueness for the solutions of (1.2). The existence part also follows now from a standard contraction argument.

Namely for $T > 0$, and $m \in M(C_T)$, iterating the lemma:

$$D_T(\Phi^{k+1}(m), \Phi^k(m)) \le c_T^k \frac{T^k}{k!} D_T(\Phi(m), m) \ . \tag{1.6}$$

So $\Phi^k(m)$ is a Cauchy sequence, and converges to a fixed point of $\Phi : P_T$. Now if $T' < T$, the image of P_T on $C([0,T'], \boldsymbol{R}^d)$ is still a fixed point, so the P_T are a consistent family, yielding a P on $C([0,\infty), \boldsymbol{R}^d)$. This provides the required solution.

□

Using Theorem 1.1, we now introduce on $(\boldsymbol{R}^d \times C_0)^{\boldsymbol{N}^*}$, where we have constructed in (1.1) our interacting diffusions $X_.^{i,N}$, $i = 1, \dots, N$, the processes $\overline{X}_.^i$, $i \ge 1$, solution of:

$$\begin{aligned} &\overline{X}_t^i = x_0^i + w_t^i + \int_0^t \int b(\overline{X}_s^i, y) \ u_s(dy) \ ds \ , \\ &u_s(dy) = \text{law}(\overline{X}_s^i) \ . \end{aligned} \tag{1.7}$$

Theorem 1.4. *For any $i \ge 1$, $T > 0$:*

$$\sup_N \sqrt{N} E[\sup_{t \le T} |X_t^{i,N} - \overline{X}_t^i|] < \infty \ . \tag{1.8}$$

Proof: Dropping for notational simplicity the superscript N, we have:

$$\begin{aligned} X_t^i - \overline{X}_t^i &= \int_0^t \Big(\frac{1}{N}\sum_{j=1}^N b(X_s^i, X_s^j) - \int b(\overline{X}_s^i, y) u_s(dy)\Big)\, ds \\ &= \int_0^t ds \frac{1}{N}\sum_{j=1}^N \{(b(X_s^i, X_s^j) - b(\overline{X}_s^i, X_s^j)) + (b(\overline{X}_s^i, X_s^j) - b(\overline{X}_s^i, \overline{X}_s^j)) \\ &\quad + (b(\overline{X}_s^i, \overline{X}_s^j) - \int b(\overline{X}_s^i, y)\, u_s(dy))\} \,. \end{aligned}$$

Writing $b_s(x,x') = b(x,x') - \int b(x,y)\, u_s(dy)$, we see that:

$$E[|X^i - \overline{X}^i|_T^*] \le K \int_0^T ds (E[|X_s^i - \overline{X}_s^i|] + \frac{1}{N}\sum_1^N E[|X_s^j - \overline{X}_s^j|] + E[|\frac{1}{N}\sum_{j=1}^N b_s(\overline{X}_s^i, \overline{X}_s^j)|])$$

Summing the previous inequality over i, and using symmetry, we find:

$$\begin{aligned} NE[|X^1 - \overline{X}^1|_T^*] &= \sum_1^N E[|X^i - \overline{X}^i|_T^*] \\ &\le K' \int_0^T \sum_{i=1}^N (E[|X_s^i - \overline{X}_s^i|] + E[|\frac{1}{N}\sum_{j=1}^N b_s(\overline{X}_s^i, \overline{X}_s^j)|])\; ds \,. \end{aligned}$$

Applying Gronwall's lemma, and symmetry, we find:

$$E[|X^i - \overline{X}^i|_T^*] \le K(T) \int_0^T ds E[|\frac{1}{N}\sum_{j=1}^N b_s(\overline{X}_s^i, \overline{X}_s^j)|] \,.$$

Our claim will follow provided we can show that:

$$E[|\frac{1}{N}\sum_{j=1}^N b_s(\overline{X}_s^i, \overline{X}_s^j)|] \le \frac{C(T)}{\sqrt{N}} \,.$$

But

$$E[(\frac{1}{N}\sum_{j=1}^N b_s(X_s^i, X_s^j))^2] = \frac{1}{N^2} E[\sum_{j,k} b_s(\overline{X}_s^i, \overline{X}_s^j) b_s(\overline{X}_s^i, \overline{X}_s^k)] \,,$$

and because of the centering of $b_s(x,y)$ with respect to its second variable, when $j \neq k$

$$E[b_s(\overline{X}_s^i, \overline{X}_s^j) b_s(\overline{X}_s^i, \overline{X}_s^k)] = 0 \,.$$

The previous sum is then less than const.$/N$. Our claim follows.

□

Let us end this section with some comments.

— We have presented here a simple enough example. It is possible to let basically the same method work in the case of an additional interaction through the diffusion coefficient, see McKean [27], or [41]. The use in this context of the Vasershtein metric, can be found in Dobrushin [9]. Of course one can as well obtain higher moment estimates in (1.8). Theorem 1.4 suggests the possibility of a fluctuation theorem, which indeed exists, see for instance, Tanaka [51], Shiga-Tanaka [38], Kusuoka-Tamura [20], or [40], [41].

— As a result of the probabilistic proof we just described, we see that the introduction of a "nonlinear process", and not only of a nonlinear equation is fairly natural. For each t and k, the joint distribution of $(X_t^{1,N},\dots,X_t^{k,N})$ is converging to $u_t^{\otimes k}$, but in fact we have convergence at the level of processes.

— Let us also mention some interesting examples which arise in possibly singular cases, of the nonlinear equation

$$(1.9) \qquad \partial_t u = \frac{\sigma^2}{2}\Delta u - \operatorname{div}\Big(\int b(\cdot,y)\ u_t(dy)u\Big)\ , \quad (\sigma = \text{constant})$$

a) When $\sigma = 0$, and $\boldsymbol{R}^d = \boldsymbol{R}^3 \times \boldsymbol{R}^3 : (x,v)$, with $b((x,v),(x',v')) = (v, F(x-x'))$, one finds:

$$\partial_t u + v.\nabla_x u + \int F(x-x')\ u(dx',dv')\cdot\nabla_v u = 0$$

This is Vlasov's equation (see Dobrushin [9]).

b) $\boldsymbol{R}^d = \boldsymbol{R}$, $\sigma = 1$, $b(x,y) = c\delta(x-y)$, we have

$$\partial_t u = \frac{1}{2}u'' - c(u^2)'\ ,$$

this is Burger's equation, we will come back to this (singular) example in Chapter II.

c) $\boldsymbol{R}^d = \boldsymbol{R}^2$, $b(x,y) = b(x-y)$ is the Biot-Savart kernel, with $b(z) = \frac{1}{2\pi|z|^2}(-z_2,z_1)$, one now finds:

$$\partial_t u + (b\ *\ u)\nabla u = \frac{\sigma^2}{2}\Delta u\ ,$$

where we use $\operatorname{div}(b\ *\ u) = 0$. If one sets $v = (b\ *\ u)$, curl $v = \partial_2 v_1 - \partial_1 v_2 = u$, and u satisfies the vorticity equation for the Navier-Stokes equation in dimension 2. For this example, see Goodman [12], Marchioro-Pulvirenti [29], Osada [35].

d) Scheutzow [37], gives an example of an equation like (1.3), with polynomial coefficients, in $\boldsymbol{R}^2$, which admits some genuinely periodic solutions.

2) Some generalities

We will now give some definitions, that we will use for the study of propagation of chaos. $M(E)$ denotes here the set of probability measures on E.

Definition 2.1. E a separable metric space, u_N a sequence of symmetric probabilities on E^N. We say that u_N is u-chaotic, u probability on E, if for $\phi_1, \dots, \phi_k \in C_b(E)$, $k \geq 1$,

$$\lim_{N \to \infty} \langle u_N, \phi_1 \otimes \ldots \otimes \phi_k \otimes 1 \cdots \otimes 1 \rangle = \prod_1^k \langle u, \phi_i \rangle \, . \tag{2.1}$$

As mentioned in the introductory chapter, the symmetry assumption on the laws u_N is less innocent than one may think at first. As the next (easy) proposition shows, the notion of u-chaotic means that the empirical measures of the coordinate variables of E^N, under u_N tend to concentrate near u. This is a type of law of large numbers. Condition (2.1) can also be restated as the convergence of the projection of u_N as E^k to $u^{\otimes k}$ when N goes to infinity, see [2], p. 20. In the coming proposition, we suppose u_N symmetric.

Proposition 2.2.

i) *u_N is u-chaotic is equivalent to $\overline{X}_N = \frac{1}{N} \sum_1^N \delta_{X_i}$ ($M(E)$-valued random variables on (E^N, v_N), X_i canonical coordinates on E^N) converge in law to the constant random variable u.* It is also equivalent to condition (2.1), with $k = 2$.

ii) *When E is a Polish space, the $M(E)$-valued variables $\overline{X}_N$ are tight if and only if the laws on E of X_1 under u_N are tight.*

Proof:

i) First suppose u_N satisfies (2.1) with $k = 2$, and consequently with $k = 1$ as well. Take ϕ in $C_b(E)$,

$$E_N[\langle \overline{X}_N - u, \phi \rangle^2] = \frac{1}{N^2} \sum_{i,j=1}^N E_N[\phi(X_i)\phi(X_j)] - \frac{2}{N} \sum_1^N E_N[\phi(X_i)] \langle u, \phi \rangle + \langle u, \phi \rangle^2 \, .$$

Using symmetry we find:

$$\frac{1}{N} E_N[\phi(X_1)^2] + \frac{(N-1)}{N} E_N[\phi(X_1)\phi(X_2)] - 2\langle u, \phi \rangle E_N[\phi(X_1)] + \langle u, \phi \rangle^2$$

which tends to zero by (2.1), with $k = 1, 2$. This implies that $\overline{X}_N$ converges in law to the constant random variable equal to u.

Conversely, suppose $\overline{X}_N$ converge in law to the constant u,

$$\begin{aligned}
&|\langle u_N, \phi_1 \otimes \ldots \otimes \phi_k \otimes 1 \ldots \otimes 1\rangle - \prod_1^k \langle u, \phi_i\rangle| \\
&\le |\langle u_N, \phi_1 \otimes \ldots \otimes \phi_k \otimes 1 \cdots \otimes 1\rangle - \langle u_N, \prod_1^k \langle \overline{X}_N, \phi_i\rangle\rangle| \qquad (2.2)\\
&+ |\langle u_N, \prod_1^k \langle \overline{X}_N, \phi_i\rangle\rangle - \prod_1^k \langle u_0, \phi_i\rangle| \,.
\end{aligned}$$

The second term in the right member of (2.2) goes to zero. The first term using symmetry can be written as:

$$|\langle u_N, \frac{1}{N!} \sum_{\sigma \in S_N} \phi_1(X_{\sigma(1)}) \cdots \phi_k(X_{\sigma(k)}) - \prod_1^k \langle \overline{X}_N, \phi_i\rangle\rangle| \,.$$

Observe now that if $M \ge \|\phi_i\|_\infty$, $1 \le i \le k$.

$$\begin{aligned}
&\sup_{E^N} |\frac{1}{N!} \sum_{\sigma \in \rho_N} \phi_1(X_{\sigma(1)}) \ldots \phi_k(X_{\sigma(k)}) - \prod_1^k \langle \overline{X}_N, \phi_i\rangle| \\
&\le M^k [(\frac{(N-k)!}{N!} - \frac{1}{N^k}) \cdot \frac{N!}{(N-k)!} + \frac{1}{N^k}(N^k - \frac{N!}{(N-k)!})] \qquad (2.3)\\
&= 2M^k(1 - \frac{N!}{N^k(N-k)!}) \to 0 \,.
\end{aligned}$$

Here we simply used that there are $N!/(N-k)!$ injections from $[1,k]$ into $[1,N]$, each of them has weight $(N-k)!/N!$ in the first sum of (2.3) and $1/N^k$ in the second sum, and in the second sum there are also $N^k - N!/(N-k)!$ terms where repetitions of coordinates occur. So we see that the first term of (2.2) goes to zero, and this proves i).
ii) For a probability $Q(dm)$ on $M(E)$, define the intensity $I(Q)$ as the probability measure

$$\langle I(Q), f\rangle = \int_{M(E)} \langle m, f\rangle \, dQ(m) \,, \qquad (2.4)$$

for $f \in bB(E)$.

We will in fact show the more general fact:

(2.5) tightness for a family of measure Q on $M(E)$ is equivalent to the tightness of their intensity measures $I(Q)$ on E.

Now in our situation of ii), by symmetry the intensity measure of the law of $\overline{X}_N$ is just the law of X_1, under E_N, so that our claim ii) follows from (2.5). Now the map

$Q \to I(Q)$ is clearly continuous for the respective weak convergence topologies. So, (2.5) will follow if we prove that whenever $I_n = I(Q_n)$ is tight, then Q_n is tight.

For each $\epsilon > 0$, denote by K_ϵ a compact subset of E, with $I_n(K_\epsilon^c) \le \epsilon$, for every n. Now for $\epsilon, \eta > 0$, and any n, $Q_n(\{m, m(K_{\epsilon\eta}^c) \ge \eta) \le \frac{1}{\eta} I_n(K_{\epsilon\eta}^c) \le \epsilon$.

It follows that:

$$Q_n\Big(\bigcup_{k\ge 1} \{m, m(K^c_{\epsilon 2^{-k}/k}) \ge 1/k\}\Big) \le \sum_{k\ge 1}^{\infty} \epsilon 2^{-k} = \epsilon\,,$$

this means that Q_n puts a mass greater or equal to $1-\epsilon$ on the compact subset of $M(E)$: $\cap_{k\ge 1}\{m, m(K^c_{\epsilon 2^{-k}/k}) \le 1/k\}$. This proves the Q_n are tight, and yields (2.5).

□

Remark 2.3.

1) It is clear from the proof of Proposition 2.2, that thanks to symmetry, the distribution under u_N of k particles chosen with the empirical distribution $\overline{X}_N$ is approximately the same as the law of $(X_1, \dots, X_k)$ under u_N, when N is large. In fact this last distribution is the intensity of the empirical distribution of distinct k-uples: $\overline{X}_{N,k} = \frac{1}{N(N-1)\cdots(N-k+1)} \sum_{i_\ell \text{distinct}} \delta_{(X_{i_1},\dots,X_{i_k})}$.

2) Suppose the empirical measures $\overline{X}_N = \frac{1}{N}\sum_1^N \delta_{X_i}$ converge in law to a constant $u \in M(E)$, with an underlying distribution v_N on E^N non necessarily symmetric. Then the symmetrized distribution u_N on E^N, preserves $\overline{X}_N$, and u_N is then u-chaotic in the sense of Definition 2.1. This remark applies to the case of deterministic sequences X_i, with convergent empirical distributions for example.

□

We will now give a result which will be helpful when transporting results from a space E to a space F. E and F are separable metric spaces, and ϕ is a measurable map from E to F. One also has the natural diagonal or tensor map $\phi^{\otimes N}$ from E^N into F^N. If u_N is a (non necessarily symmetric) probability on E^N, we will write $v_N = \phi^{\otimes N} \circ u_N$.

Proposition 2.4.

(2.6) If u_N is u-chaotic, and the continuity points C_ϕ of ϕ have full u-measure, v_N is v-chaotic if $v = \phi \circ u$.

(2.7) If $\overline{u}_\infty$ is a limit point of the laws of the empirical distributions $\overline{X}_N$ as N tends to infinity, C_ϕ has full measure under the intensity measure $I(\overline{u}_\infty)$ of $\overline{u}_\infty$, and v_N is v-chaotic, then for $\overline{u}_\infty$-a.e. m in $M(E)$, $\phi \circ m = v$.

(2.8) If $F = \boldsymbol{R}$, $Q \in M(M(E))$ and ϕ is bounded and C_ϕ has full $I(Q)$ measure, then for any Q_n converging weakly to Q and any continuous real function $h(\cdot)$,

$$E_Q[h(\langle m, \phi\rangle)] = \lim_n E_{Q_n}[h(\langle m, \phi, \rangle)].$$

Proof: It is not difficult to see that (2.6), (2.7), (2.8) follow from the remark: if $Q \in M(M(E))$ and $I(C_\phi) = 1$, then for any Q-convergent sequence Q_n, $\Psi \circ Q_n$ converges weakly to $\Psi \circ Q$, if $\Psi : M(E) \to M(F)$ is the map: $m \to \phi \circ m$. The proof of this last statement is that $I(C_\phi) = 1$ implies $Q[\{m, \langle, m, C_\phi\rangle = 1\}] = 1$, and when $\langle m, C_\phi\rangle = 1$, then m is a continuity point of Ψ (see Billingsley [2] p. 30). So the continuity points of Ψ have full Q-measure which yield the remark by a second application of the quoted result.

□

We will now give some comments on how the material of this section will be used in a propagation of chaos context.

In several cases one has a symmetric law P_N, on $C(\boldsymbol{R}_+, \boldsymbol{R}^d)^N$ or $D(\boldsymbol{R}_+, \boldsymbol{R}^d)^N$ for instance, describing the interacting particle system. The initial conditions are supposed to be u_0-chaotic, $u_0 \in M(\boldsymbol{R}^d)$, and one tries to prove that the P_N are P-chaotic, for some suitable P on $C(\boldsymbol{R}_+, \boldsymbol{R}^d)$ or $D(\boldsymbol{R}_+, \boldsymbol{R}^d)$ with initial condition u_0, by (2.7).

One way to prove such a statement is to show that the laws $\overline{P}_N$ of the empirical distributions $\overline{X}_N$ converge weakly to δ_P. This can be performed by checking tightness of the $\overline{P}_N$, that is tightness of the laws of X^1 on C or D under P_N, and identifying all possible limit points $\overline{P}_\infty$ as being concentrated on P.

This last step will involve finding some denumerable collection of functionals on probabilities on C or D, $G(m)$ having some continuity property (see (2.8)), for which P is the only common zero. Then we will show that $\overline{P}_\infty$ is concentrated on the zeros of each functional.

The reason for using this method is that in many cases, $G(m)$ will be a quadratic function of m, and we will end up calculating $E_{\overline{P}_\infty}[G(m)^2]$, as $\lim_\ell E_{N_\ell}[G(\overline{X}_{N_\ell})^2]$, with the help of (2.8), for a suitable subsequence N_ℓ. But the study of this last limit when G is quadratic in m, by symmetry basically involves only four particles. In many examples, it yields a line of proof which is more pleasant than working directly with (2.1), even with $k = 2$.

3) Examples.

a) Our laboratory example:

$$dX_t^i = dB_t^i + \frac{1}{N}\sum_1^N b(X_t^i, X_t^j)\ dt\ , \quad i = 1, \dots, N\ ,$$
$$X_0^i : \text{ independent } u\text{-distributed}$$

here the X^i have symmetric laws P_N on $C(R_+, R^d)^N$, which thanks to Theorem 1.4 are P-chaotic, where P is the law of the "nonlinear process" solution of

$$dX_t = dB_t + \int b(X_t, y)\ u_t(dy)\ dt$$
$$X_0 : u - \text{distributed, } u_t(dy) = \text{law of } X_t\ .$$

(A number of assumptions: independence of X_0^i, Lipschitz character of b can in fact be relaxed, see references at the end of section 1).)

b) Uniform measure on the sphere of radius $\sqrt{n}$ in $\boldsymbol{R}^n$.

Proposition 3.1. *The uniform distribution $ds_n(x)$ on the sphere of radius $\sqrt{n}$ in $\boldsymbol{R}^n$ is u-chaotic, if $u = 1/\sqrt{2\pi}\exp\{-x^2/2\}dx$.*

Proof: s_n is clearly symmetric. We will show directly that the projection of s_n on the first k components of $\boldsymbol{R}^n$ converges weakly to $u^{\otimes k}$.

Call $\mu_n(dr)$ the law of the radius under $u^{\otimes n}$, and $s_{n,r}(dx)$ the uniform distribution on the sphere of radius r in $\boldsymbol{R}^n$.

Take now f continuous with compact support on $\boldsymbol{R}^k$. By the law of large numbers for $(x_1^2 + \cdots + x_n^2)/n$, under $u^{\otimes n}$, we know that for $0 < a < 1 < b$,

$$\lim_{n\to\infty} \Big| \int_{\boldsymbol{R}^k} f(x) u^{\otimes k}(dx) - \int_{[a\sqrt{n}, b\sqrt{n}]} d\mu_n(r) \int ds_{n,r}(x)\ f(x_1, \dots, x_k)\Big| = 0\ . \tag{3.1}$$

But one also has:

$$\int_{a\sqrt{n}}^{b\sqrt{n}} \mu_n(dr) \int ds_{n,r}(x)\ f(x_1, \dots, x_k) = \int_{a\sqrt{n}}^{b\sqrt{n}} \mu_n(dr) \int ds_n(x)\ f(x_1 \frac{r}{\sqrt{n}}, \dots, x_k \frac{r}{\sqrt{n}})\ .$$

Now $r/\sqrt{n} \in [a, b]$ in the previous integral, and f is compactly supported and continuous, so that:

$$\lim_{a\to 1, b\to 1} \sup_{n \geq k} \Big| \int_{a\sqrt{n}}^{b\sqrt{n}} \mu_n(dr) \int ds_{n,r}(x) f - \mu_n([a\sqrt{n}, b\sqrt{n}]) \langle s_n, f \otimes 1 \cdots \otimes 1\rangle \Big| = 0\ . \tag{3.2}$$

Now for each $a < 1 < b$, $\mu_n([a\sqrt{n}, b\sqrt{n}]) \to 1$ as n goes to infinity. By (3.1), (3.2), we see that when n goes to infinity $\langle s_n, f \otimes 1 \ldots \otimes 1\rangle$ tends to $\langle u^{\otimes k}, f\rangle$. This finishes the proof of Proposition 3.1.

c) Variation on a theme.

We will now present an example closely related to the previous one, which explains why one had the previous result. In fact one could modify it to include the previous example, however as explained before we are not seeking the maximum generality here.

Take $x_1, \ldots, x_n$, which are iid, with law $\mu(dx) = f(x)\, dx$, on $\mathbb{R}^d$, where $f > 0$, is C^1 and such that

$$\int (f(x) + |\nabla f(x)|)\, e^{\lambda |x|} dx < \infty\ , \qquad \text{for any } \lambda\ . \tag{3.3}$$

Then it is known that for any $a \in \mathbb{R}^d$, there is a unique $\lambda = \lambda_a$ such that $\frac{1}{Z_\lambda} \cdot e^{\lambda x} \mu(dz)$ (Z_λ normalization factor) has mean a. In fact if $I(\cdot)$ is the convex conjugate of the logarithm of the Laplace transform of μ:

$$I(x) = \sup_\lambda (\lambda \cdot x - \log(\int e^{\lambda \cdot y} \mu(dy)))\ ,$$

we have: $\nabla I(a) = \lambda_a$.

Consider $s_n(dx) = \mu^{\otimes n}[(x_1, \ldots, x_n) \in dx / \frac{x_1 + \cdots + x_n}{n} = a]$, the conditional distribution of $(x_1, \ldots, x_n)$ given the mean $x_1 + \cdots + x_n/n = a$.

Proposition 3.2. *$s_n(dx)$ is ν-chaotic, where $\nu = \frac{1}{Z_\lambda} e^{\lambda \cdot x} \mu$ with $\lambda = \lambda_a$ determined by $\int x \cdot d\nu(x) = a$.*

Proof: We have, with obvious notations

(3.4)

$$\begin{aligned} s_n(dx) &= \frac{1}{z_n} f(x_1) \ldots f(x_{n-1})\, f(an - x_1 - \cdots - x_{n-1}) dx_1 \ldots dx_{n-1} \\ &= \frac{1}{e^{\lambda \cdot an} z_n} (e^{\lambda \cdot} f)(x_1) \ldots (e^{\lambda \cdot} f)(x_{n-1})(e^{\lambda \cdot} f)(an - x_1 \ldots - x_{n-1})\, dx_1 \ldots dx_{n-1} \\ &= \nu^{\otimes n}[(x_1, \ldots, x_n) \in dx / x_1 + \cdots x_n = an]\ . \end{aligned}$$

Now looking at $x_i - a = y_i$, we see that we can assume that μ is centered and $a = 0$, and (3.3) holds.

It is now enough to show that for $\phi_1, \ldots, \phi_k \in C_K(\mathbb{R}^d)$,

$$\langle s_n, \phi_1 \otimes \ldots \otimes \phi_k \otimes 1 \ldots \otimes 1\rangle \longrightarrow \prod_1^k \langle \mu, \phi_i\rangle\ , \quad \text{as } N \text{ goes to infinity.} \tag{3.5}$$

Now from (3.4), the expression under study is

$$\begin{aligned}&(\phi_1 f) * \cdots * (\phi_k f) * f^{(n-k)*}(0) \;/\; f^{n*}(0)\\ &= \int \widehat{\phi_1 f}(\xi)\ldots\widehat{\phi_k f}(\xi)\; \hat{f}^{n-k}(\xi)\, d\xi \;/\; \int \hat{f}^n(\xi)\, d\xi\ ,\end{aligned}$$

using Fourier transforms.

Observe that $\int \hat{f}^n(\xi)d\xi = \frac{1}{n^{d/2}}\int \hat{f}^n(\frac{\xi}{\sqrt{n}})\, d\xi$, now $\hat{f}^n(\frac{\xi}{\sqrt{n}}) \overset{n\to\infty}{\longrightarrow} \exp\{-\frac{1}{2}\,{}^t\xi A\xi\}$, for each ξ. On the other hand (3.3) ensures that we have the domination:

$$|\hat{f}(\xi)| \le [1+|\xi|^2]^{-\delta}\ , \quad \xi \in \boldsymbol{R}^d\ ,$$

for a suitable δ. So we have $|\hat{f}^n(\frac{\xi}{\sqrt{n}})| \le [(1+\frac{|\xi|^2}{n})^n]^{-\delta} \le (1+\frac{|\xi|^2}{n_0})^{-n_0\delta}$, when $n \ge n_0$, which is integrable if n_0 is large enough. So we see that as n tends to infinity

$$\int \hat{f}^n(\xi)d\xi \sim (\frac{n}{2\pi})^{-d/2}(\det\, A)^{-1/2} \text{ and } \int_{|\xi|\ge A} n^{d/2}|\hat{f}^{n-k}(\xi)|d\xi \to 0\ , \quad \text{for } A>0\ .$$

So $\langle s_n, \phi_1\otimes\cdots\otimes\phi_k\otimes 1\ldots\otimes 1\rangle$ converges to $\widehat{\phi_1 f}(0)\ldots\widehat{\phi_k f}(0) = \langle \mu,\phi_1\rangle\ldots\langle\mu,\phi_k\rangle$, which yields our claim.

□

d) Symmetric self exclusion process.

In this example there will be a microscopic scale at which particles interact (exclusion condition) and a macroscopic scale. The interaction will in fact disappear in the limit.

Our N particles move on $\frac{1}{N}\boldsymbol{Z} \subset \boldsymbol{R}$. We look at the simple exclusion process for these N particles from the point of view of N particles which evolve in a random medium given as follows: with the bonds $(i, i+\frac{1}{N})$, $i \in \frac{1}{N}\boldsymbol{Z}$, we associate a collection of independent Poisson counting processes $N_t^{i,i+1/N}$, with intensity $\frac{N^2}{2}dt$. The motion of any particle in this random medium is given by the rule:

If a particle is at time t at the location i it remains there until the first of the two Poisson processes $N_\cdot^{i,i+1/N}$ or $N_\cdot^{i-1/N,i}$ has a jump, and then performs a jump across the corresponding bond.

As a consequence of this rule if we consider N particles with N distinct initial conditions, these particles will never collide (self exclusion). Here we pick an initial distribution u_N on $(\frac{1}{N}\boldsymbol{Z})^N \subset \boldsymbol{R}^N$, which is u_0-chaotic, where $u_0 \in M(\boldsymbol{R})$. Using Remark 2.3, we can for instance take the deterministic locations $(\frac{1}{N}, \frac{2}{N}, \ldots, 1)$ and symmetrize them, in this case $u_0(dx) = 1_{[0,1]}dx$. Given the initial conditions and the random medium, this provides us with N trajectories $(X_\cdot^1, \ldots, X_\cdot^N)$ and, we denote by

P_N, the symmetric law on $D(\boldsymbol{R}_+,\boldsymbol{R})^N$, one obtains, when the initial conditions are picked independent of the medium, and u_N distributed.

Theorem 3.3. *P_N is P-chaotic, if P is the law of Brownian motion with initial distribution u_0.*

Corollary 3.4. *For each $s \geq 0$, the law of $(X_s^1, \dots, X_s^N)$ is $u_0 \, * \, p_s$-chaotic if $p_s = (2\pi s)^{-1/2} \exp -\frac{x^2}{2s}$.*

Proof: This is an immediate application of Proposition 3.4, where the map ϕ, is the coordinate at time s on $D(\boldsymbol{R}_+,\boldsymbol{R})$.

□

Before giving the proof of Theorem 3.3, let us give some words of explanation about the point of view we have adopted here. As mentioned before, for the propagation of chaos result one works with symmetric probabilities. This is why we have constructed the self-exclusion process, using the trajectories of particles interacting with a given random medium. If one instead is just interested in "density profiles", that is, the evolution of:

$$\eta_t^N = \frac{1}{N} \sum_1^N \delta_{X_t^i} \in M(\boldsymbol{R}) \ ,$$

one can directly build it as a Markov process evolving on the space of simple point measures, but one loses the notion of particle trajectories, especially when two neighboring particles perform a jump (see Liggett [24], DeMasi-Ianiro-Pellegrinotti-Presutti [7]). Notice however that Corollary 3.4 can be restated purely in terms of η_t^N. Indeed by Proposition 2.2 i), it means that when N tends to infinity η_t^N converges in law to the constant probability $u_0 \, * \, p_t$, that is belongs with vanishing probability to the complement of any neighborhood of $u_0 \, * \, p_t$ for the weak topology.

So we can derive a corollary of Theorem 3.3 purely in terms of "profile measures" η_t^N, stating that if η_0^N is a sum of N distinct atoms of mass $1/N$, and converges in law to the constant probability u_0 on $\boldsymbol{R}$, then the random probabilities η_t^N converge in law to $u_0 \, * \, p_s$. Of course Theorem 3.3 has more in it. From a "profile point of view", we have introduced the symmetric variables through the trajectories $(X_.^1, \dots, X_.^N)$.

Proof of Theorem 3.3: We consider the empirical measures $\overline{X}_N = \frac{1}{N} \sum_1^N \delta_{X_.^i} \in M(D)$, where $D = D(\boldsymbol{R}_+,\boldsymbol{R})$. We will show the $\overline{X}_N$ concentrate their mass around P. We will first show that the laws of the $\overline{X}_N$ are tight, and then we will identify any possible limit point as being δ_P the Dirac mass at P. This will precisely mean the $\overline{X}_N$ converge in law to the constant P.

Tightness:

By Proposition 2.2 ii), it boils down to checking tightness for $X^1_\cdot$, under P_N. But here $X^1_\cdot$ is distributed as a simple random walk on $\frac{1}{N}\mathbf{Z}$ in continuous time with jump intensity $= N^2 dt$, and initial distribution $x^1 \circ u_N$. Classically, the law of $X^1_\cdot$ under P_N converges weakly to P, and this yields tightness.

Identification of limit points:

Take $\overline{P}_\infty$ a limit point of the laws of the $\overline{X}_N$. We already know by the tightness step (see Proposition 2.2 ii) that $I(\overline{P}_\infty)$, the intensity of $\overline{P}_\infty$ is P. By Proposition 2.4, we know that $\overline{P}_\infty$ is concentrated on measures for which X_0 is u_0-distributed. If we introduce $F(m) = E_m[(e^{i\lambda(X_t-X_s)+(\lambda^2/2)(t-s)} - 1)\psi_s(X)]$, where $\lambda \in \mathbf{R}$, $t > s$, and $\psi_s(X) = \phi_1(X_{s_1})\dots\phi_k(X_{s_k})$, with $0 \le s_1 < \dots < s_k \le s$ and $\phi_1,\dots,\phi_k \in C_b(\mathbf{R})$, it is enough to show that $F(m) = 0$, $\overline{P}_\infty$-a.s., for then varying over countable families of λ, $t > s$, s_i, ϕ_i, we will find that $X_t - X_s$ is independent of $\sigma(X_u,\ u \le s)$, and $N(0, t-s)$ distributed, for $\overline{P}_\infty$-a.e. m, which implies $\overline{P}_\infty = \delta_P$.

Using (2.8), since $(e^{i\lambda(X_t-X_s)+\lambda^2(t-s)} - 1)$ has continuity points of full measure under $P = I(\overline{P}_\infty)$, we find

$$
\begin{aligned}
E_{\overline{P}_\infty}[|F(m)|^2] &= \lim_k E_{N_k}[|F(\overline{X}_{N_k})|^2] \\
&= \lim E_{N_k}[|\frac{1}{N}\sum_1^N (e^{i\lambda(X^i_t-X^i_s)+(\lambda^2/2)(t-s)} - 1)\ \psi_s(X^i)|^2]\ ,
\end{aligned}
$$

using symmetry, the latter quantity equals:

(3.6)
$$
\lim E_{N_k}[(e^{i\lambda(X^1_t-X^1_s)+(\lambda^2/2)(t-s)} - 1)(e^{-i\lambda(X^2_t-X^2_s)+(\lambda^2/2)(t-s)} - 1)\ \psi_s(X^1)\ \psi_s(X^2)]\ .
$$

Observe that $E_{N_k}[e^{i\lambda(X^1_t-X^1_s)+(\lambda^2/2)(t-s)}\ \psi_s(X^1)\ \psi_s(X^2)] =$ $E_{N_k}[e^{i\lambda(X^1_t-X^1_s)+(\lambda^2/2)(t-s)}]\ \ E_{N_k}[\psi_s(X^1)\psi_s(X^2)]$, and the limit of the first term of the product is 1, by the weak convergence of the law of X^1 to P. In view of (3.6), to prove tha $F(m) = 0$, $\overline{P}_\infty$ a.s., it is enough to check that:

$$
A_N = E_N[(\exp\{i\lambda[(X^1_t - X^2_t) - (X^1_s - X^2_s)] + \lambda^2(t-s)\} - 1)\ \psi_s(X^1)\psi_s(X^2)]
$$

tends to zero when N goes to infinity. Observe now that (X^1_t, X^2_t) is a pure jump process on $(\frac{1}{N}\mathbb{Z})^2$, with bounded generator:

$$
\begin{aligned}
Lf(x_1,x_2) =&\frac{N^2}{2}(\Delta_N^1+\Delta_N^2)\ f(x_1,x_2)\\
&-\frac{N^2}{2}1\{x_1-x_2=\frac{1}{N}\}\cdot D_1D_2f(x_1-\frac{1}{N},x_2)\\
&-\frac{N^2}{2}1\{x_2-x_1=\frac{1}{N}\}\cdot\ D_2D_1\ f(x_1,x_2-\frac{1}{N})\ ,
\end{aligned}
\tag{3.7}
$$

here Δ^1, Δ^2, D_1, D_2, are the discrete Laplacian and the difference operators with respect to first and second variables. If we pick now $f(x_1,x_2)=\exp\{i\lambda(x_1-x_2)\}$, we see that

$$
\left|\frac{Lf(x_1,x_2)}{f}-N^2(e^{i\lambda/N}+e^{-i\lambda/N}-2)\right|\le \text{const}\ 1\{|x_1-x_2|=\frac{1}{N}\}\ . \tag{3.8}
$$

Using now the fact that $f(X_t^1,X_t^2)\exp-\int_0^t\frac{Lf(X_u)}{f}du$ is a bounded martingale, we have

$A_N=$

$$
E_N[f(X_t^1,X_t^2)/f(X_s^1,X_s^2)\ e^{\lambda^2(t-s)}(1-\exp-\int_s^t(\frac{Lf}{f}+\lambda^2)(X_u)du)\ \psi_s(X^1)\psi_s(X^2)]\ ,
$$

using (3.8) we see that:

$$
|A_N|\le o(N)+\text{const}\ E_N[\int_0^t 1\{|X_s^1-X_s^2|=\frac{1}{N}\}ds]
$$

So Theorem 3.3 will follow from

Lemma 3.5.

$$
\lim_{N\to\infty}E_N[\int_0^\infty e^{-s}1\{|X_s^1-X_s^2|=\frac{1}{N}\}ds]=0\ .
$$

Proof: $Y_s=N|X^1_{s/N^2}-X^2_{s/N^2}|$, as follows from (3.7) is a jump process on $\mathbb{N}^*$, with generator

$$
L'f(k)=1\{k\neq 1\}\Delta f(k)+1\{k=1\}[f(2)-f(1)]\ ,
$$

and the quantity under the limit sign in Lemma 3.5 equals:

$$
E_N[\int_0^\infty e^{-s/N^2}1\{Y_s=1\}\frac{ds}{N^2}]\le\frac{C}{N^2}\sum_{k=0}^\infty Q_2[e^{-\tau/N^2}]^k=\frac{C}{N^2}\ \frac{1}{1-Q_2[e^{-\tau/N^2}]}\ , \tag{3.9}
$$

where τ is the hitting time of 1, and Q_2 is the law of Y, starting from 2. If we now pick $a(N)<1$, such that $a+a^{-1}-2=1/N^2$, using the bounded martingale $a^{Y_{s\wedge\tau}}\ e^{-(s\wedge\tau)/N^2}$, we find

$$E_2[e^{-\tau/N^2}] = a(N) = 1 + \frac{1}{2N^2} - \big((1+\frac{1}{2N^2})^2 - 1\big)^{1/2} = 1 - \frac{1}{N} + o(\frac{1}{N}) \,.$$

It follows that $\frac{C}{N^2}(1-a)^{-1} \sim \frac{C}{N}$ which tends to zero, and in view of (3.9) proves the lemma.

□

e) Reordering of Brownian motions.

We consider N independent Brownian motions $X^1_\cdot, \dots, X^N_\cdot$ on $\boldsymbol{R}$, with initial law u_0, which we suppose atomless. We then introduce the increasing reorderings $Y^1_t \leq \dots \leq Y^N_t$, of the $X^1_\cdot, \dots, X^N_\cdot$, so that $Y^1_t = \inf_i\{X^i_t\}$, $Y^2_t = \sup_{|A|=N-1} \inf_{i\in A}\{X^i_t\}$, etc. Now the processes $(Y^1_\cdot, \dots, Y^N_\cdot)$ are reflected Brownian motions on the convex $\{y_1 \leq y_2 \leq \dots \leq y_N\}$, but they are not symmetric any more, so we consider the symmetrized processes $(Z^1_\cdot, \dots, Z^N_\cdot)$ on the enlarged space $(\mathcal{S}_N \times C^N, d\nu_N \otimes P_{u_0}^{\otimes N})$, where $\mathcal{S}_N$ is the symmetric group on $[1, N]$, $d\nu_N$ the normalized counting measure, and $Z^i_t = Y^{\sigma(i)}_t$, σ being the $\mathcal{S}_N$ valued component on $\mathcal{S}^N \times C^N$.

The interest of this example comes from the fact that on the one hand the $(X^1_\cdot, \dots, X^N_\cdot)$ and $(Z^1_\cdot, \dots, Z^N_\cdot)$ have the same density profile:

$$\frac{1}{N}\sum_1^N \delta_{X^i_t} = \frac{1}{N}\sum_1^N \delta_{Z^i_t} \,, \tag{3.10}$$

but on the other hand we are going to prove that the $Z^i_\cdot$ are Q-chaotic, where Q is a different law from P_{u_0}. Of course the $X^i_\cdot$ are P_{u_0}-chaotic. As a result of (3.10), Q and P_{u_0} will share the same time marginals, namely $u_0 * (\frac{1}{\sqrt{2\pi s}}\exp\{-\frac{x^2}{2s}\})$. This emphasizes the fact that the limit behavior of the profile evolution is not enough to reconstruct the "nonlinear process".

Let us describe the law Q. The distribution function F_t of u_t, is strictly increasing for $t > 0$. If $C = \operatorname{supp} u_0$, we can write the complement of C as a union of disjoint intervals (a_n, b_n). The points of $D = C \setminus \cup\{a_n, b_n\}$ are points of left and right increase of F_0, and D has full u_0 measure.

We define for $x \in D$,

$$\psi_t(x) = F_t^{-1} \circ F_0(x) \,, \tag{3.11}$$

one in fact has $\lim_{t\to 0} \psi_t(x) = x$, and $\psi : x \to (\psi_t(x))_{t\geq 0}$, defines a measurable map from D in $C(\boldsymbol{R}_+, \boldsymbol{R})$.

Theorem 3.6. *The laws of $(Z^1_\cdot, \dots, Z^N_\cdot)$ are Q-chaotic where $Q = \psi \circ u_0$.*

Proof: We first give a lemma, making precise the structure of $Y^1_\cdot, \dots, Y^N_\cdot$ as reflected Brownian motion.

Lemma 3.7. *There are N independent $\sigma(X_u,\ u \le t)$ Brownian motions on $C(\boldsymbol{R}_+, \boldsymbol{R})^N$, $\beta^1_\cdot, \dots, \beta^N_\cdot$, and $(N-1)$ continuous adpated increasing processes $\gamma^1_t, \dots, \gamma^{N-1}_t$ such that*

$$\begin{aligned}
&\gamma^i_t = \int_0^t 1(Y^i_s = Y^{i-1}_s)\, d\gamma^i_s\,, \\
&Y^1_t = Y^1_0 + \beta^1_t - \frac{1}{2}\gamma^1_t\,, \\
&Y^k_t = Y^k_0 + \beta^k_t - \frac{1}{2}\gamma^k_t + \frac{1}{2}\gamma^{k-1}_t \\
&Y^N_t = Y^N_0 + \beta^N_t + \frac{1}{2}\gamma^{N-1}_t\,.
\end{aligned} \tag{3.12}$$

Proof: One uses induction, by stopping the processes at the successive times where distinct X^i, X^j meet, and applies Tanaka's formula. Since these stopping times tend to infinity, one then obtains the lemma. For details see [42].

Let us check tightness of the laws of $Z^1_\cdot$.

Remark 3.8. Before proving this point, let us mention here that the symmetrized sampling of $Y^1_\cdot, \dots, Y^N_\cdot$ by the random permutation σ is crucial for tightness. Suppose u_0 has support in $[0,1]$, if the X^i are constructed on the infinite product space, one knows that for $t > 0$, $\overline{\lim}_{N\to\infty} \frac{M^N_t}{\sqrt{2t\log N}} = 1$, a.s., with $M^N_\cdot = \sup(X^1_\cdot - X^1_0, \dots, X^N_\cdot - X^N_0)$. But $Y^N_t = \sup(X^1_t, \dots, X^N_t) \ge M^N_t$, so one cannot expect tightness for the law of $Y^N_\cdot$. □

By symmetry and (3.10), the law of Z^1_t is u_t. So to prove our claim it is enough to prove

Lemma 3.9.

$$E[\sum_1^N |Y^i_t - Y^i_s|^4] \le 3N(t-s)^2\,.$$

Proof: We apply Ito-Tanaka's formula to

$$\begin{aligned}
\|Y_t - Y_s\|^2 &= \sum_1^N (Y^i_t - Y^i_s)^2 \\
&= 2\sum_i \int_s^t (Y^i_u - Y^i_s)\, d\beta^i_u \\
&\quad + \sum_1^{N-1} \int_s^t [(Y^{i+1}_u - Y^{i+1}_s) - (Y^i_u - Y^i_s)]\, d\gamma^i_u + N(t-s)\,.
\end{aligned}$$

Now by (3.12), $Y_u^{i+1} = Y_u^i \; d\gamma_u^i$-a.s., and $Y_s^i \le Y_s^{i+1}$, so after taking expectations

$$E[\sum_1^N (Y_t^i - Y_s^i)^2] \le N(t-s) \; . \tag{3.13}$$

In the same way, we find:

$$\begin{aligned}\sum_1^N (Y_t^i - Y_s^i)^4 = & 4\sum_1^N \int_s^t (Y_u^i - Y_s^i)^3 d\beta_u^i \\ & + 2\sum_1^{N-1} \int_s^t [(Y_u^{i+1} - Y_s^{i+1})^3 - (Y_u^i - Y_s^i)^3] \; d\gamma_u^i \\ & + 6\sum_1^N \int_s^t (Y_u^i - Y_s^i)^2 du \; ,\end{aligned}$$

and again $(Y_u^{i+1} - Y_s^{i+1})^3 - (Y_u^i - Y_s^i)^3 \le 0$, $d\gamma_u^i$ a.s., so that taking expectation and (3.13),

$$E[\sum_1^N (Y_t^i - Y_s^i)^4] \le 6N \int_s^t (u-s)du = 3N(t-s)^2 \; .$$

□

Let us show now that for fixed k, the law of $(Z_\cdot^1, \dots, Z_\cdot^k)$ converges to $Q^{\otimes k}$. Since for every t, $(Z_t^1, \dots, Z_t^N)$ has the same reordered sequence $(Y_t^1, \dots, Y_t^N)$ as $(X_t^1, \dots, X_t^N)$, symmetry immediately implies that for each t, $(Z_t^1, \dots, Z_t^N)$ has the same distribution as $(X_t^1, \dots, X_t^N)$, that is $u_t^{\otimes N}$. Using this fact for $t = 0$, one sees easily that it is enough to prove that

(3.14) for any limit point $\overline{P}$ of the laws of $Z_\cdot^1$, and $s > 0$, $\overline{P}[Z_s = \psi_s(Z_0)] = 1$,

to deduce that any limit point of the laws of $(Z^1, \dots, Z^k)$ is in fact $Q^{\otimes k}$. To check this statement observe that

$$\begin{aligned} E_{\overline{P}}[|F_0(Z_0) - F_s(Z_s)|] &= \lim_{N_k} E_{N_k}[|F_0(Z_0^1) - F_s(Z_s^1)|] \\ &\le \overline{\lim}_k E_{N_k}[|F_0(Z_0^1) - \frac{1}{N}\sum_{i=1}^N 1(X_0^i \le Z_0^1)|] \\ &\quad + \overline{\lim}_k E_{N_k}[|F_s(Z_s^1) - \frac{1}{N}\sum_{i=1}^N 1(X_s^i \le Z_s^1)|] \\ &\quad + \overline{\lim}_k E_{N_k}[|\frac{1}{N}\sum_{i=1}^N 1(X_0^i \le Z_0^1) - 1(X_s^i \le Z_s^1)|] \; . \end{aligned}$$

The first two terms go to zero. We have in fact

$$\lim E_N[\sup_x |F_0(x) - \frac{1}{N}\sum_i 1(X_0^i \le x)|] = 0 \ ,$$

since u_0 is atomless, and a similar result at time s. On the other hand we know that since $Z_\cdot^1 = Y_\cdot^{\sigma(1)}$,

$$|\sum_i 1(X_0^i \le Z_0^1) - 1(X_s^i \le Z_s^1)| \le 1 \ ,$$

so the last term goes to zero as well.

□

If we set $x_t = \psi_t(x)$, x_t satisfies the differential equation for $t > 0$,

$$\frac{dx_t}{dt} = -\frac{\partial_t F}{\partial_x F}(t, x_t) = -\frac{1}{2}(\log u_t)'(x_t) \ , \tag{3.15}$$

as is seen from the implicit equation $F(t, x_t) = \text{const}$. So Q is the law of a deterministic evolution with initial random distribution u_0, and u_t is a solution of the corresponding forward equation

$$\begin{aligned} &\partial_t u - \partial_x(\frac{1}{2}(\log u_t)'u) = 0 \\ &v(t=0,\cdot) = u_0 \ . \end{aligned}$$

Let us mention finally that when x is a "bad point" in the convex hull of the support of C, that is $x \in [a_n, b_n]$, with $-\infty < a_n < b_n < \infty$, then one sees easily that $\lim_{t\to 0} \psi_t(x) = (a_n + b_n)/2$, on the equation:

$$0 = F_t(x_t) - F_0(x) = u_+ \ * \ p_t(-\infty, x_t] - u_- \ * \ p_t[x_t, \infty) \ , \quad t > 0 \ ,$$

with $u_- = 1_{(-\infty, a_n]} \cdot u_0$, $u_+ = 1_{[b_n, \infty)} \cdot u_0$.

f) Colored particles and nonlinear process.

Now we look at P_N on $C(R_+, R^d)^N$, which are P-chaotic. Let I_0 be a subset of R^d such that $\{X_0 \in I_0\}$ is a continuity set for P, for instance a product of intervals if $u_0 = X_0 \circ P$ has a density. We color in blue the particles which are in I_0 at time zero, and we are interested in the empirical measure at time t of the blue particles:

$$\nu_t^N(dx) = \frac{1}{N}\sum_i 1(X_0^i \in I_0)\delta_{X_t^i} \ . \tag{3.16}$$

Then Proposition 2.4, easily implies that

$$\lim_{N\to\infty} E_N[\nu_t^N \in U^c] = 0 \ ,$$

for any neighborhood for the weak topology on $M_{+,b}(R^d)$ of

$$\nu_t(dx) = P[X_0 \in I_0\ ,\quad X_t \in dx]\ . \tag{3.17}$$

The coloration of particles is one way to recover some trajectorial information, and gain some knowledge on P, if one uses profile measures.

For applications of coloring of particles, in a propagation of chaos context, to stochastic mechanics, we refer the reader to Nagasawa-Tanaka [31].

g) Loss of Markov property and local fluctuations.

The reader might be tempted to think that when the N-particle system follows a symmetric Markovian evolution, which is chaotic, the law P of the "nonlinear process" will inherit a Markov property. We will now give an example showing that this is not the case.

Heuristically, what happens, is that we have an N-particle system having local interactions, but there is no mechanism to "average out" the local fluctuations of the interaction. One interest of the example is that the presence of these local fluctuations does not prevent the chaotic behavior of the N-particle system (so there is propagation of chaos in this sense). The limit law P does not have the Markov property, and the limit of the density profile of particles which is governed by the time marginals of P, does not correspond now to a nonlinear forward equation.

We consider the N-particle system $(Z^1_\cdot, \dots, Z^N_\cdot) \in C(\boldsymbol{R}_+, E)^N$, where $E = \boldsymbol{R}/\boldsymbol{Z} \times \boldsymbol{R}$, and $Z^i_\cdot = (Y^i_\cdot, X^i_\cdot)$. It follows a symmetric Markovian evolution E^N, given as follows:

- The $Y^i_\cdot$ are constant in time, and the Y^i_0 are i.i.d. dx-distributed, on $\boldsymbol{R}/\boldsymbol{Z}$.
- $X^i_t = \sigma_i B^i_t$, $t \geq 0$, $1 \leq i \leq N$, where $B^i_\cdot$, $1 \leq i \leq N$, are i.i.d. standard Brownian motions independent of the $(Y^i_\cdot)$, $1 \leq i \leq N$, and $\sigma_i = (1 + \sum_{j \neq i} 1\{Y_j \in \Delta_N(Y_i)\})^{1/2}$,

$\Delta_N(x)$ denoting the only interval $[\frac{k}{N}, \frac{k+1}{N})$, $0 \leq k < N$, containing $x \in \boldsymbol{R}/\boldsymbol{Z}$.

Let us now describe the law P on $C(\boldsymbol{R}_+, E)$, which appears in the propagation of chaos result. Under P:

- $Y_t \equiv Y_0$ is dy distributed on $\boldsymbol{R}/\boldsymbol{Z}$.
- $X_\cdot$ is independent of $Y_\cdot$ and distributed as a mixture of Brownian motions starting from zero, with trajectorial variance $\sigma^2 = 1 + m$, m being distributed as a Poisson, mean one, variable.

Let us right away observe that the law P is not Markovian. Indeed if (F_t) denotes the natural filtration on $C(\mathbf{R}_+, E)$, $\sigma^2 = \overline{\lim}_{t\to 0} \frac{X_t^2}{2t \log\log 1/t}$ is F_1-measurable, and we have:

$$E^P[X_2^2 \mid F_1] = X_1^2 + \sigma^2 \ ,$$

whereas:

$$\begin{aligned} E^P[X_2^2 \mid X_1] &= X_1^2 + E^P[\sigma^2 \mid X_1] \\ &= X_1^2 + \Big(\sum_{m\geq 0} \frac{(1+m)}{m!} p_{1+m}(X_1)\Big) \ / \ \Big(\sum_{m\geq 0} \frac{1}{m!} p_{1+m}(X_1)\Big) \ . \end{aligned}$$

Here $p_t(x)$ denotes $(2\pi t)^{-1/2} \exp\{-\frac{x^2}{2t}\}$.

We have

Theorem 3.10.

(3.18) The laws P_N of $(Z_\cdot^1, \dots, Z_\cdot^N)$ are P-chaotic.

(3.19) For $t \geq 0$, the random measures $\frac{1}{N}\sum_1^N \delta_{(Y^i, X_t^i)} \in M(\mathbf{R}/\mathbf{Z} \times \mathbf{R})$ ("density profiles") converge in law to the constant $dy \otimes (e^{-1} \sum_{m\geq 0} \frac{1}{m!} p_{t(1+m)}(x) \ dx)$.

Proof: (3.19) is an immediate consequence of (3.18). Let us prove (3.18). In view of Proposition 2.2, we will simply show that the law of $(Z_\cdot^1, Z_\cdot^2)$ converges weakly to $P^{\otimes 2}$. Anyway, the case of $(Z_\cdot^1, \dots, Z_\cdot^k)$ is also obvious from our proof. It is enough to show for $f_i \in C(\mathbf{R}/\mathbf{Z})$, $g_i \in C_b(C(\mathbf{R}_+, \mathbf{R}))$, $i = 1, 2$, that:

$$\lim_N E_N[f_1(Y_\cdot^1) g_1(X_\cdot^1) f_2(Y_0^2) g_2(X_\cdot^2)] = \prod_{i=1}^2 E^P[f_i(Y_0) g_i(X_\cdot)] \ . \tag{3.20}$$

The left member of (3.20) equals

$$\begin{aligned} &\lim_N E_N[f_1(Y_0^1) g_1(\sigma_1 B_\cdot^1) f_2(Y_0^2) g_2(\sigma_2 B_\cdot^2)] \\ &\quad = \lim_N E_N[f_1(Y_0^1) \phi_1(\sigma_1) f_2(Y_0^2) \phi_2(\sigma_2)] \ , \end{aligned} \tag{3.21}$$

where $\phi_i(a) = E[g_i(aB_\cdot)]$, $i = 1, 2$ (Wiener expectation), are continuous bounded functions. The expression inside the limit in (3.21) involves an expectation on the Y^i variables, $1 \leq i \leq N$, alone. To prove (3.20), it suffices to show that conditionally on $Y^1 = y_1$, $Y^2 = y_2$, $y_1 \neq y_2$, the law of $m_i = \sum_{j\neq i} 1\{Y_j \in \Delta_N(Y_i)\}$, $i = 1, 2$, converges weakly to the law of two independent mean one Poisson variables. To check this last point observe that for large N, $\Delta_N(y_1) \cap \Delta_N(y_2)$ is empty. So for $a_1, a_2 > 0$,

$$\begin{aligned}
&\lim_N E_N[e^{-a_1 m_1 - a_2 m_2} / Y_1 = y_1 \ , \ Y_2 = y_2] \\
&= \lim \Big(\int_0^1 \exp\{-a_1 1_{\Delta_N(y_1)}(x) - a_2 1_{\Delta_N(y_2)}(x)\} \, dx \Big)^{N-2} \\
&= \lim_N \big(1 + \frac{1}{N}(e^{-a_1} - 1 + e^{-a_2} - 1)\big)^{N-2} = \exp\{(e^{-a_1} - 1) + (e^{-a_2} - 1)\} \ ,
\end{aligned}$$

from which our claim follows. □

An example of a situation with a loss of the Markov property can also be found in Uchiyama [54].

h) Tagged particle: a counterexample.

In the introduction, we mentioned that we refrained from calling the "nonlinear process", the "tagged particle process", because this expression has a variety of meanings. We will give here an easy example where the tagged particle is the trajectory of the particle with initial starting point nearest to a certain point. It will turn out that the law of the tagged particle will converge to a limit distinct from the natural law of the nonlinear process conditioned to start from this point.

We consider the N-particle system $(Z^1_., \dots, Z^N_.) \in C(\mathbb{R}_+, E)$, where $E = [0,1) \times \mathbb{R}$. The processes $Z^i_. = (Y^i_., X^i_.)$, $1 \le i \le N$, will be independent Markov processes, satisfying:

- The $Y^i_.$ are constant in time, and the Y^i_0 are i.i.d. dy-distributed, on [0,1).
- $X^i_t = \sigma_N(Y^i_0) B^i_t$, $t \ge 0$, $1 \le i \le N$, where $B^i_.$, $1 \le i \le N$, are i.i.d. standard Brownian motions independent of the $(Y^i_.)$, $1 \le i \le N$, and

$$\sigma_N(y) = 1\{y \in \bigcup_{k \ even} [\frac{k}{N} \ , \ \frac{k+1}{N})\} + \sqrt{2}\, 1\{y \in \bigcup_{k \ odd} [\frac{k}{N} \ , \ \frac{k+1}{N})\} \ .$$

The tagged particle process $\overline{Z}^N_.$ will be defined by $\bar{Z}^N_. = (Y^i_., X^i_.)$ on $\{Y^i_0 = \min Y^j_0\}$, $1 \le i \le N$. This defines the tagged particle a.s., with no ambiguity, and since the starting point of $Z^i_.$ is $(Y^i_0, 0)$, the tagged particle corresponds to the particle with initial starting point closest to (0,0) in $[0,1) \times \mathbb{R}$.

If we still denote by dy the measure on trajectories $Y_.$, constant in [0,1), with initial point dy distributed, and by $W^1_.$, $W^2_.$ Wiener measure with respective variance 1 and 2, it is easy to see that the laws of the $Z^i_.$, which are independent, are P-chaotic, if $P = dx \otimes (\frac{1}{2}W^1 + \frac{1}{2}W^2)$ on $C(\mathbb{R}_+, E)$.

Notice, by the way, that the nonlinear process has also lost its Markov property.

We are now going to show:

Proposition 3.11: $\overline{Z}^N_{\cdot}$ *converges in law to* $Q = \delta_0 \otimes ((1+e^{-1})^{-1}W^1 + e^{-1}(1+e^{-1})^{-1}W^2)$ *(here* δ_0 *denotes the Dirac mass on the constant trajectory equal to* 0*).*

Proof: It is enough to show that for $f \in C_b([0,1))$, $g \in C_b(C(\boldsymbol{R}_+, \boldsymbol{R}))$,

$$\lim_N E[f(\overline{Y}^N_0)g(\bar{X}^N_{\cdot})] = f(0) \times \big((1+e^{-1})^{-1}E^{W^1}[g] + e^{-1}(1+e^{-1})E^{W^2}[g]\big) \, . \tag{3.22}$$

Set $\tilde{Y}^N = N\bar{Y}^N$, we have

$$E[f(\bar{Y}^N_0)g(\bar{X}^N_{\cdot})] = \sum_{0 \le k \text{ even } < N} E[f(\frac{\tilde{Y}^N}{N}) \, 1\{k \le \tilde{Y}_N < k+1\}] \, E^{W^1}[g]$$
$$+ \sum_{0 \le k \text{ odd } < N} E[f(\frac{\tilde{Y}^N}{N}) \, 1\{k \le \tilde{Y}_N < k+1\}] \, E^{W^2}[g] \, .$$

Observe now that $\tilde{Y}^N$ converges in law to an exponential variable of parameter 1. Indeed, for $t > 0$, N large,

$$E[\tilde{Y}^N > t] = E[\bigcap_1^N (Y^i > \frac{t}{N}] = (1 - \frac{t}{N})^N \, ,$$

which tends to e^{-t}.

From this it is easy to argue that the expression in (3.23) tends to:

$$\sum_{k \text{ even } \ge 0} (e^{-k} - e^{-(k+1)}) \, f(0) \, E^{W^1}[g] + \sum_{1 \le k \text{ odd}} (e^{-k} - e^{-(k+1)}) \, f(0) E^{W^2}[g] \, ,$$

which is equal to the right member of (3.22). This yields our claim. □

So we see that the limit law for $\bar{Z}^N$ is given by Q which is distinct from the natural conditioning of the nonlinear process to be zero at time zero corresponding to $\delta_0 \otimes (\frac{1}{2}W^1 + \frac{1}{2}W^2)$. We also refer the reader to Guo-Papanicolaou [13], where the limit behavior of a tagged particle process for a system of interacting Brownian motions is studied.

II. A local interaction leading to Burgers' equation

The object of this chapter is to present one model of local interactions, proposed by McKean [27], which leads to Burgers' equation, as the forward equation of the nonlinear process.

The "laboratory example" we discussed in Chapter I, section 1),

$$dX_t^i = dw_t^i + \frac{1}{N}\sum_1^N b(X_t^i, X_t^j)\ dt\ ,\quad i = 1, \dots, N\ .$$

with $b(\cdot,\cdot)$ bounded Lipschitz, has an interaction term $\frac{1}{N}\sum_1^N b(X_t^i, X_t^j)dt$. Such a function b, independent of N and regular, corresponds to an interaction at a macroscopic distance. In this chapter we will be dealing with the one dimensional situation, when $b(\cdot,\cdot)$ =const $\delta(x-y)$, and the interaction will be local in nature. We will start first in section 1) with a warm up calculation, of δ-like interaction terms, for independent particles.

1) A warm up calculation.

In our "laboratory example", as a result of the Lipschitz property of $b(\cdot,\cdot)$, and Theorem 1.4 of Chapter I, one sees easily that for each t,

$$\lim_{N\to\infty} E\Big[\frac{1}{N}\sum_{i=1}^N \Big(\frac{1}{N}\sum_{j=1}^N b(X_t^i, X_t^j) - \int b(X_t^i, y)u_t(dy)\Big)^2\Big] = 0\ .$$

In other words, the instantaneous drift term $\frac{1}{N}\sum_j b(X_t^i, X_t^j)$ seen by particle i, is getting close to the quantity $\int b(X_t^i, y)\ u_t(dy)$ which simply depends on X_t^i.

We are now going to analyze similar quantities when $b(\cdot,\cdot)$ is replaced by $\phi_{N,a}(x-y) = N^{ad}\phi(N^a(x-y))$, with $\phi(\cdot)\geq 0$, smooth, compactly supported, on $\mathbb{R}^d$, $\int\phi(x)dx = 1$. The $X_\cdot^i$ will be independent d-dimensional Brownian motion, with initial distribution $u_0(dx) = u_0(x)dx$ having smooth compactly supported density. The quantities $Z_i = \frac{1}{(N-1)}\sum_{j\neq i}\phi_{N,a}(X_t^i - X_t^j)$ will play the role of an instantaneous "pseudo drift" seen by particle i. We will denote by $p_s(x,y)$ the Gaussian transition density. Now for $0 < a < \infty$, we will look at the $N\to\infty$, behavior of

$$a_N = E\Big[\frac{1}{N}\sum_{i=1}^N\Big(\frac{1}{(N-1)}\sum_{j\neq i}\phi_{N,a}(X_t^i - X_t^j) - u_t(X_t^i)\Big)^2\Big]\ . \tag{1.1}$$

The interpretation of the parameter a, is that now the interaction range between particles is of order N^{-a}. Thanks to symmetry,

$$\begin{aligned} a_N &= E\big[\big(\frac{1}{N-1}\sum_{j=2}^{N}\phi_{N,a}(X_t^1 - X_t^j) - u_t(X_t^1)\big)^2\big] \\ &= E\big[\big(\frac{1}{(N-1)}\sum_{j=2}^{N}\phi_{N,a}(X_t^1 - X_t^j) - \phi_{N,a} \ * \ u_t(X_t^1)\big)^2\big] \\ &\quad + E\big[\big(\phi_{N,a} \ * \ u_t(X_t^1) - u_t(X_t^1)\big)^2\big] \ . \end{aligned}$$

The second term of the previous expression clearly tends to zero. Expanding the square in the first term we find

$$\begin{aligned} a_N &= \frac{1}{(N-1)}E\big[\big(\phi_{N,a}(X_t^1 - X_t^2) - \phi_{N,a} \ * \ u_t(X_t^1)\big)^2\big] + o(N) \\ &= \frac{1}{(N-1)}E\big[\phi_{N,a}^2(X_t^1 - X_t^2)\big] + o(N) \qquad (1.2) \\ &= \frac{1}{(N-1)}N^{ad}E\big[N^{ad}\phi^2(N^a(X_t^1 - X_t^2))\big] + o(N) \ . \end{aligned}$$

So we see that

$$\begin{aligned} &\text{— when} \quad 0 < a < 1/d \ , \quad \lim_N a_N = 0 \ , \\ &\text{— when} \quad a = 1/d \ , \quad \lim_N a_N = \int \phi^2 dx \times \|u_t\|_{L^2}^2 > 0 \ , \qquad (1.3) \\ &\text{— when} \quad a > 1/d \ , \quad \lim_N a_N = \infty \ . \end{aligned}$$

The case $0 < a < 1/d$ corresponds to "moderate interaction" (see Oelschläger [34]). In fact when $a = 1/d$, we are in a "Poisson approximation" regime, and conditionally on $X_t^1 = x$, the sum

$$\frac{1}{(N-1)}\sum_{j=2}^{N}\phi_{N,a}(x - X_t^j) = (1 + o(N)) \times \sum_{j=2}^{N}\phi(N^{1/d}(x - X_t^j)) \ ,$$

converges in law to the distribution of $\int_{\mathbb{R}^d} M(dy)\phi(y)$, with $M(dy)$ Poisson point process of intensity $u_t(x)dy$ (there is no misprint here). So conditioned on X_t^i there is a true fluctuation of the quantity $\frac{1}{(N-1)}\sum_{j\neq i}\phi_{N,1/d}(X_t^i - X_t^j)$, for each i.

When $a > 1/d$, conditionally on $X_t^1 = x$, the quantity $\frac{1}{(N-1)}\sum_{j=2}^{N}\phi_{N,a}(x - X_t^j)$ is zero with a probability going to 1 uniformly in x, but has conditional expectation approximately $u_t(x)$. So now we really have huge fluctuations.

Let us by the way mention that even in the presence of fluctuations a propagation of chaos result may hold. One can in fact see that the symmetric variables $Z_i =$

$\frac{1}{N}\sum_{j\neq i}\phi_{N,a}(X_t^i - X_t^j)$, $s \leq i \leq N$, are v_a-chaotic. Here v_a stands for the law of $u_t(X_t)$, when $a < 1/d$, the law of $\int_{R^d} M(dy)\,\phi(y)$, where conditionally on X_t, M is a Poisson point process with parameter $u_t(X_t)$, when $a = 1/d$, and trivially the Dirac mass in 0 when $a > 1/d$. The case $a = 1/d$ is somewhat comparable to example g) in Chapter I, section 3).

Since we are interested in interactions going very fast to δ (in fact being δ), if we hope to see our "pseudodrift" seen by particle i close for N large to a quantity just depending on particle i, some helping effect has to come to rescue us. This helping effect will be integration over time.

Integration over time as a smoothing effect: We are now going to replace the quantity $\frac{1}{(N-1)}\sum_{j\neq i}\phi_{N,a}(X_t^i - X_t^j)$, for each i, by $\frac{1}{(N-1)}\sum_{j\neq i}\int_0^t \phi_{N,a}(X_s^i - X_s^j)ds$. Correspondingly we are now interested in the limit behavior of:

$$b_N = E\big[\frac{1}{N}\sum_{i=1}^N\big(\frac{1}{(N-1)}\sum_{j\neq i}\int_0^t \phi_{N,a}(X_s^i - X_s^j)\,ds - \int_0^t u_s(X_s^i)ds\big)^2\big]\ . \tag{1.4}$$

An analogous calculation as before yields:

$$b_N = \frac{1}{(N-1)}E\big[\big(\int_0^t ds\phi_{N,a}(X_s^1 - X_s^2)ds\big)^2\big] + o(N)\ .$$

If we introduce $W_s = X_s^1 - X_s^2$, W_s is a Brownian motion with initial distribution $u_0 * \check{u}_0 = v_0$, and transition density $p_{2s}(x,y)$. We find that

(1.5)

$$b_N = \frac{2}{(N-1)}\int v_0(dx)\,dy\,dz\int_0^t du\,p_{2u}(x,y)\,\phi_{N,a}(y)\int_0^{t-u} dv\,p_{2v}(y,z)\,\phi_{N,a}(z) + o(N)\ .$$

It is clear that in dimension $d = 1$, for any value of a, (and formally in(1.5), even if $\phi_{N,a} = \delta$), $\lim_N b_N = 0$. In dimension $d \geq 2$, taking as new variables $N^a y$, and $N^a z$, we find

$$b_N = \frac{2}{(N-1)}\int v_0(dx)dy\,dz\int_0^t du$$

$$p_{2u}(x,\frac{y}{N^a})\,\phi(y)\int_0^{(t-u)} dv\,p_{2v}(\frac{y}{N^a},\frac{z}{N^a})\,\phi(z) + o(N)\ .$$

In dimension $d = 2$, we see again from the logarithmic Green's function singularity appearing in the term $\int_0^{(t-u)} dv\ p_{2v}(y/N^a, z/N^a)$, that for any $a < \infty$, $\lim_N b_N = 0$, and in fact it is clear that b_N will not be vanishing unless the interaction range is exponentially small. In dimension $d \geq 3$, using the transition density scaling

$$b_N = \frac{2}{(N-1)} N^{a(d-2)} \int v_0(dx)dy\ dz \int_0^t du$$
$$p_{2u}(x, \frac{y}{N^a})\ \phi(y) \int_0^{(t-u)N^{2a}} dv\ p_{2v}(y,z)\ \phi(z) + o(N)\ .$$

It is now clear that

$$\begin{aligned} &\text{for} \quad a < \frac{1}{d-2}\,, \quad \lim_N b_N = 0\ ,\\ &\text{for} \quad a = \frac{1}{d-2}\,, \quad \lim b_N = \text{const} > 0\ . \\ &\text{for} \quad a > \frac{1}{d-2}\,, \quad \lim b_N = \infty\ . \end{aligned} \tag{1.6}$$

So we see that the integration over time has removed the existence of a critical exponent $1/d$, in dimension 1. In dimension 2 the new critical regime corresponds to an exponentially small range of interaction. In dimension $d \geq 3$, the critical exponent $a = 1/d$ is raised to $1/(d-2)$.

In Chapter III, we will see that there is a "Poissonian picture" corresponding to these critical regimes in dimension $d \geq 2$.

As for dimension 1, we have seen that $\lim b_N = 0$, for any a. In fact it is the consequence of an even stronger result, namely:

$$\lim E\Big[\frac{1}{N(N-1)} \sum_{i \neq j} \Big(\int_0^t \phi_{N,a}(X_s^i - X_s^j)ds - \frac{1}{2} L^0(X^i - X^j)_t \Big)^2\Big] = 0\ .$$

We will use these type of ideas in the next sections, in our approach to the propagation of chaos result. For other approaches, we refer the reader to Gutkin [14], Kotani-Osada [21].

2. The N-particle system and the nonlinear process.

The N-particle system will be given by the solution

$$\begin{aligned} &dX_t^i = dB_t^i + \frac{c}{N} \sum_{j \neq i} dL^0(X^i - X^j)_t\ , \\ &(X_0^i)_{1 \leq i \leq N} \in \Delta_N\ , \quad u_N - \text{distributed}, \quad c > 0\ . \end{aligned} \tag{2.1}$$

Here $L^0(X^i - X^j)_t$ denotes the symmetric local term in 0 of $X_\cdot^i - X_\cdot^j$, B^i are independent 1-dimensional Brownian motions, Δ_N is the subset of $\boldsymbol{R}^N$ where no three coordinates are equal, and of course the initial conditions X_0^i are independent of the Brownian motions. Existence and uniqueness of solutions of (2.1) holds trajectorically and in law.

The solution is Δ_N valued, strong Markov. It has the Brownian like scaling property: $\lambda X^x_{\cdot/\lambda^2} \overset{\text{law}}{\sim} X^{\lambda x}_{\cdot}$, for an initial starting point $x \in \Delta_N$. The δ-interaction interpretation comes from the fact that the solution of

$$
(2.2) \qquad \begin{aligned} &dX^{i,\epsilon}_t = dB^i_t + \frac{2c}{N} \sum_{j \neq i} \phi_\epsilon(X^{i,\epsilon}_t - X^{j,\epsilon}_t)\, dt \;, \quad \phi_\epsilon(\cdot) = \frac{1}{\epsilon}\phi(\frac{\cdot}{\epsilon}) \;, \\ &X^{i,\epsilon}_0 = X^i_0 \;, \end{aligned}
$$

converge a.s. uniformly on compact intervals to the solution of (2.1), as ϵ goes to zero. Finally if u_N is symmetric the law of the $(X^i_\cdot)$ is symmetric as well. For these results we refer the reader to [42] and [46].

Let us now present the nonlinear process. One might be tempted to define the process as the solution of

$$
(2.3) \qquad \begin{aligned} &dX_t = dB_t + 2c\; u(t, X_t)\; dt \\ &X_0 \;,\; u_0 - \text{distributed}, \end{aligned}
$$

for $u(t,x)$ the density of X_t at time t, from which one would deduce that $u(t,x)$ is the solution of Burgers' equation:

$$
\begin{aligned} &\partial_t u = \frac{1}{2}\partial_x^2 u - 2c\; \partial_x(u^2) \\ &u_{t=0} = u_0 \;, \end{aligned}
$$

However the density of the law at time t of a process is an ill behaved function for the weak topology on $M(C(\check{R}_+, R^d))$, and such a characterization of the nonlinear process is not best suited to apply the strategy explained in Chapter I 2). From the previous section, we know that quantities integrated over time are better behaved, and we are going to characterize the law of the nonlinear process as the unique law of continuous semimartingales $X_\cdot$, on some filtered probability space endowed with a Brownian motion $B_\cdot$, solution of:

(2.4) $X_t = X_0 + B_t + A_t$, X_0, u_0-distributed, A_t continuous adapted, of integrable variation with $A_t = cE_Y[L^0(X-Y)_t]$. $Y_t = Y_0 + \overline{B}_t + \overline{C}_t$ defined on an independent filtered space endowed with a Brownian motion $\overline{B}_\cdot$ has the same law as $(X_\cdot)$, and $\overline{C}$ is a continuous adapted integrable variation process. Here $L^0(X-Y)_\cdot$ denotes the symmetric local time in zero of $X_\cdot - Y_\cdot$, defined on the product space.

For the moment we will be concerned with the uniqueness statement. Indeed the more important role of the nonlinear process, in the proof of "propagation of chaos" comes in the identification of limit points of laws of empirical measures. We will see in the course of the proof that solutions of (2.4) have indeed the structure of (2.3). We will start with some a priori estimates:

Proposition 2.1.

i) *If*

$$X_t = X_0 + B_t + A_t \ ,(B.\ \textit{Brownian motion},\ A.\ \textit{integrable variation}), \tag{2.5}$$

the image of $1_{[0,T]} ds\ dP$ *under the map* $(s,\omega) \to (s, X_s(\omega)) \in [0,T] \times \boldsymbol{R}$, *for* $T < \infty$, *has an* L^2 *density* $u(s,x)$, *with* $\|u\|_{L^2([0,T]\times \boldsymbol{R})} \leq C(E[|A|_T])$.

ii) *If* $Y.$ *on some independent space has a decomposition as in* (2.5), *then*

$$E_Y[L^0(X-Y)_t] = 2\int_0^t u(s,X_s)\ ds \ , \tag{2.6}$$

and this defines a continuous increasing integrable process which only depends on the law of Y. *Here* u *is the density associated to* Y *in* i).

Proof:

i) Take $\phi(\cdot) \geq 0$, smooth, symmetric supported in $[-1/2, 1/2]$, $\int \phi = 1$. Define $\psi = \phi\ *\ \phi$ so that $\psi \geq 0$, is symmetric supported in $[-1,1]$ and $\int \psi = 1$. If we define $\psi_n(\cdot) = n\ \psi(n\cdot)$, $\phi_n(\cdot) = n\phi(n\cdot)$, we have $\psi_n(\cdot) = \phi_n\ *\ \phi_n$. We now use ψ_n to define our test function

$$F_n(x) = \int_{-\infty}^{x}\int_{-\infty}^{u} (\psi_n(v) - \psi_1(v))\ dv\ du \ , \tag{2.7}$$

so that $F_n'(x) = 0$, for x outside $[-1,+1]$, $|F_n'| \leq 1$, and $|F_n| \leq 2$.

Take $X_t(\omega')$ an independent copy of $X_t(\omega)$, and set on the product space $Z_t = X_t(\omega) - X_t(\omega')$. Applying Ito's formula to $F_n(Z_t)$ we find after integration:

$$\begin{aligned} E[F_n(Z_t)] =& E[F_n(Z_0)] + E[\int_0^t F_n'(Z_s)\ d(A_s(\omega) - A_s(\omega'))] \\ &+ E[\int_0^t (\psi_n(Z_s) - \psi_1(Z_s))ds] \ . \end{aligned} \tag{2.8}$$

From this it follows that:

$$E[\int_0^T \psi_n(Z_s)ds] \leq 4 + \|\psi\|_\infty T + 2E[|A|_T] \ . \tag{2.9}$$

But

$$\begin{aligned} E[\int_0^T \psi_n(Z_s)ds] &= \int_0^T \langle u_s \otimes u_s, \int \phi_n(y-z)\phi_n(y'-z)dz\rangle \\ &= \|u_n\|^2_{L^2([0,T]\times \boldsymbol{R})} \ , \end{aligned}$$

where u_s denotes the law of X_s at time s, and $u_n(s,x) = \int u_s(dy)\ \phi_n(x-y)$.

i) easily follows from (2.9) now.

ii) Denote by u and v the densities corresponding to Y and X, using i). The quantity $\int_0^T ds\ u(s,X_s)$ does not depend on the version of u which is used and

$$E\Big[\int_0^T u(s,X_s)ds\Big] = \langle u,v\rangle_{L^2([0,T]\times\boldsymbol{R})} < \infty\ .$$

In our case the corresponding uniform estimate to (2.9), shows that

$$E[L^0(X-Y)_T] < \infty\ ,$$

and in fact working with the limit $F_\infty(\ )$ of the functions F_n, applying to it Tanaka's formula, one finds easily by studying $F_\infty(Z_t) - F_n(Z_t)$ that

$$L^0(X-Y)_t = \lim_{n\to\infty} 2\int_0^t \psi_n(X_s - Y_s)\ ds \quad \text{in } L^1\ . \tag{2.10}$$

The same with approximations of δ supported in $\boldsymbol{R}_+$ or $\boldsymbol{R}_-$ gives the corresponding result for the right and left continuous local times $L^{0,r}(X-Y)$, and $L^{0,\ell}(X-Y)$.. From (2.10), after integration over Y we find, u_n denoting now the ψ_n regularization of u,

$$\lim_n E\Big[\Big|E_Y[L^0(X-Y)_t] - 2\int_0^t u_n(s,X_s)ds\Big|\Big] = 0\ .$$

On the other hand

$$\begin{aligned} lim_n E\Big[\Big|\int_0^t u_n(s,X_s)ds - \int_0^t u(s,X_s)ds\Big|\Big] &\le \lim_n E\Big[\int_0^t |u_n - u|(s,X_s)ds\Big] \\ &\le \lim_n \|u-u_n\|_2\|v\|_2 = 0\ . \end{aligned}$$

ii) easily follows. From our proof it is also clear that similarly

$$E_Y[L^{0,r}(X-Y)_t] = E_Y[L^{0,\ell}(X-Y)_t] = 2\int_0^t u(s,X_s)ds\ , \tag{2.11}$$

the approximating density from the right or from the left just as well converging in L^2 to u.

□

We now state our required uniqueness statement, in fact existence will be proved later.

Theorem 2.2. *Let $S(u_0)$ be the set of laws of solutions of (2.4). $S(u_0)$ has at most one element. If P is the law of a solution of (2.4), $u_t = X_t \circ P$, satisfies*

(2.12) $\exp\{-4cF_t(x)\} = \exp\{-4cF_0\} \; * \; p_t(x)$, F_t distribution function of u_t, (that is u_t satisfies Burgers' equation).

Proof: By Proposition 2.1, we know that $X_t = X_0 + B_t + 2c\int_0^t u(s,X_s)ds$, where $u \in L^2([0,T]\times \boldsymbol{R})$ is the density of the law of X_s for a.e. s. Applying Ito's formula to $f(T,X_T)$, with $f \in C_K^\infty((0,T)\times \boldsymbol{R})$, we see that

$$0 = E[\int_0^T (\partial_s f + \frac{1}{2}\partial_x^2 f + 2cu\partial_x f)(s,X_s)ds] \; ,$$

using the definition of u we deduce that:

(2.13) $-\partial_s u + \frac{1}{2}\partial_x^2 u - 2c\partial_x(u^2) = 0$, in the distribution sense on $(0,T)\times \boldsymbol{R}$.

If we now set: $F(t,x) = \int_{-\infty}^x u_t(dy)$, $-\partial_t F + \frac{1}{2}\partial_x^2 F - 2cu^2$ is a distribution invariant under spatial translations. The value of this distribution tested against $f \in C_K^\infty((0,T)\times \boldsymbol{R})$ is equal to

$$\int_{(0,T)\times \boldsymbol{R}} dt\; dx\; F(t,x-z)(\partial_t f + \frac{1}{2}\partial_x^2 f)(t,x) + 2cu^2(t,x-z)\partial_x f(t,x)$$

for any z in $\boldsymbol{R}$. Letting z tend to $+\infty$, we see this last expression goes to zero, so that

$$-\partial_t F + \frac{1}{2}\partial_x^2 F - 2cu^2 = 0 \text{ in the distribution sense.}$$

Now $w'/w = -4cu$ is a change of unknown function which linearizes Burger's equation into $\partial_t w = \frac{1}{2}\partial_x^2 w$. So we intoduce a regularization by convolution in space time F_λ in $(\epsilon, T-\epsilon)$, and set $w_\lambda(t,x) = \exp\{-4cF_\lambda(t,x)\}$. One now has: $\partial_t w_\lambda - \frac{1}{2}\partial_x^2 w_\lambda = 8c^2 w_\lambda[(u^2)_\lambda - (u_\lambda)^2]$, but as $\lambda \to 0$ $u_\lambda^2 \to u^2$ in $L^1((\epsilon,T-\epsilon),\boldsymbol{R})$, $u_\lambda \to u$ in $L^2((\epsilon,T-\epsilon)\times \boldsymbol{R})$. From this letting λ go to zero we see that $\partial_t w - \frac{1}{2}\partial_x^2 w = 0$, in the distribution sense, so that by hypoellipticity, w is in fact smooth, and bounded, and from this for $0 < s < t < T$, $w_t = w_s \; * \; p_{t-s}$. Letting now s go to zero we find (2.12). So u is in fact the solution of Burgers' equation. Now $u(v,x)$, $v \geq s > 0$, is Lipschitz and bounded, and

$$X_t = X_s + (B_t - B_s) + \int_s^t 2cu(v,X_v)\; dv \; , \quad t \geq s \; ,$$

X_s is $u_s(dx)$ distributed.

It follows that any two solutions of (2.4) generate the same law on $C([s,+\infty),\boldsymbol{R})$. Since s is arbitrary our claim follows.

□

Remark 2.3. When $u_0(dx) = u_0(x)\ dx$, with u_0 bounded measurable, one sees easily that the solution of Burgers' equation given by (2.12) is bounded measurable. Now one has trajectorial uniqueness for the equation

$$X_t = X_0 + B_t + 2c \int_0^t u(s, X_s)\ ds\ ,$$

see Zvonkin [58]. The proof of Theorem 2.2 now yields a trajectorial uniqueness for the solution of (2.4), when X_0, $(B_.)$ are given.

□

3) The Propagation of chaos result.

In this section we will prove the propagation of chaos result:

Theorem 3.1. *If* u_N, *supported on* Δ_N, *is* u_0*-chaotic, then* $(X^1_., \dots, X^N_.)$ *solutions of* (2.1) *are* P_{u_0}*-chaotic, where* P_{u_0} *is the unique element of* $S(u_0)$.

With no restriction of generality we will assume that $c > 0$.

We will use in this section tightness estimates, which will be proved in section 4), namely

Proposition 3.2. *There is a* $K > 0$, *such that for* $N > 2c$, $1 \le i \ne j \le N$, $s \le t$,

$$\begin{aligned} &E[|X^i_t - X^i_s|^4] \le K|t-s|^2 \\ &E[(L^0(X^i - X^j)^t_s)^4] \le K|t-s|^2\ . \end{aligned} \tag{3.1}$$

Theorem 3.1 is in fact the corollary of a stronger statement, that will be presented now. Let us first introduce some notation. We denote by $\tilde{H}$ the closed subset of $C(\boldsymbol{R}_+, R) \times C_0(\boldsymbol{R}_+, \boldsymbol{R})$:

$$\tilde{H} = \{(X_., B_.) \in C \times C_0 : X_. - X_0 - B_. \in C_0^+\}\ .$$

C_0 and C_0^+ are respectively the space of continuous and continuous increasing functions from $\boldsymbol{R}_+$ to $\boldsymbol{R}$, with value zero at time zero. In the course of the proof we will show that in fact $S(u_0)$ has indeed one element. $\tilde{P}_{u_0}$ will stand for the joint distribution of $(X_., B_.)$, if X is the nonlinar process and $B_.$ its driving Brownian motion. Precisely $\tilde{P}_{u_0} \in M(H)$ will be the measure on $\tilde{H}$ image of P_{u_0} under the map: $X_. \in C \to (X_., X_. - X_0 - 2c\int_0^. u(s, X_s)ds) \in \tilde{H}$, where u is the solution of Burgers' equation with initial condition u_0. Let us mention that $u(s,x) \le \text{const}\ s^{-1/2}$, so that the map is in fact continuous. Finally we will consider the law Q image of $\tilde{P}_{u_0} \otimes \tilde{P}_{u_0}$ on the space $H = \tilde{H} \times \tilde{H} \times C_0^+$, under the map:

$$(X^1_., B^1_.)(X^2_., B^2_.) \to (X^1_., B^1_., X^2_., B^2_., L^0(X^1 - X^2)_.)\ ,$$

which is clearly $\tilde{P}_{u_0} \otimes \tilde{P}_{u_0}$ a.s. defined.

Using Proposition 2.4 of Chapter I, Theorem 3.1 is an easy consequence of

Theorem 3.3. *The empirical measures*

$$\overline{Y}_N = \frac{1}{N(N-1)} \sum_{i \neq j} \delta_{(X^i_\cdot, B^i_\cdot, X^j_\cdot, B^j_\cdot, L^0(X^i - X^j)_\cdot)} \in M(H) ,$$

converge in law to the constant Q.

Before embarking on the proof of Theorem 3.3, we are going to give some implications of a result such as Theorem 3.3, in terms of the quantities which were presented in section 1. In our present context the reader who is an afficionado of the "density profile" point of view is in fact interested merely in the behavior for large N of $\langle \eta^N_t, f \rangle = \frac{1}{N} \sum_i f(X^i_t)$, and using Ito's formula, only the bounded variation term $c \int_0^t \frac{1}{N^2} \sum_{i \neq j} f'(X^i_s) dL^0(X^i - X^j)_s$, is really problematic.

So we will give an implication of Theorem 3.3 in terms of this quantity.

Corollary 3.4.

$$\lim_{N \to \infty} E\Big[\frac{1}{N} \sum_i \Big(\frac{1}{N} \sum_{j \neq i} L^0(X^i - X^j)_t - 2 \int_0^t u(s, X^i_s) ds\Big)^2\Big] = 0 \ , \ t \geq 0 \ . \tag{3.2}$$

More generally for f continuous bounded $\boldsymbol{R} \to \boldsymbol{R}$

$$\lim_{N \to \infty} E\Big[\frac{1}{N} \sum_i \Big(\frac{1}{N} \sum_{j \neq i} \int_0^t f(X^i_s) \ dL^0(X^i - X^j)_s - 2 \int_0^t f(X^i_s) u(s, X^i_s) ds\Big)^2\Big] = 0 \ ,$$

and

$$\lim_{N \to \infty} E\Big[\Big| \int_0^t \frac{1}{N^2} \sum_{i \neq j} f(X^i_s) dL^0(X^i - X^j)_s - 2 \int_0^t \int_{\boldsymbol{R}} u^2(s,x) \ f(x) \ ds \ dx \Big|\Big] = 0 \ . \tag{3.4}$$

Proof: First (3.2) is a special case of (3.3), and (3.4) is an immediate consequence of (3.3), and the fact that thanks to Theorem 3.1

$$\lim_N E\Big[\Big|\frac{1}{N} \sum_i \int_0^t f(X^i_s) u(s, X^i_s) ds - \int_0^t \int_{\boldsymbol{R}} u^2(s,x) \ f(x) \ ds \ dx\Big|\Big] = 0 \ .$$

Now to prove (3.3), notice that

$$(X_\cdot, B_\cdot) \in \tilde{H} \to \int_0^t f(X_s) c^{-1} d(X_\cdot - X_0 - B_\cdot)_s \in \boldsymbol{R} \ ,$$

is a continuous map, and the expression under study is

$$E[\frac{1}{N}\sum_i(\int_0^t f(X_s^i)c^{-1}d(X_\cdot^i - X_0^i - B_\cdot^i)_s - 2\int_0^t u(s,X_s^i)ds)^2]\ . \tag{3.5}$$

If we now replace the square in (3.5) by $(\ \ )^2 \wedge A$, using Theorem 3.3, the limit as N goes to infinity of the new quantity is

$$E_{\bar{P}_{u_0}}[(\int_0^t f(X_s)c^{-1}d(X_\cdot - X_0 - B_\cdot)_s - 2\int_0^t f(X_s)u(s,X_s)ds)^2 \wedge A] = 0\ .$$

Using estimates (3.1), to remove the truncation, one easily proves (3.3).

□

Proof of Theorem 3.3: The tightness of the laws of the $\overline{Y}_N$,comes from (3.1), together with the fact that u_N is u_0-chaotic. We now have to prove that any limit point $\overline{Q}_\infty$ of the laws of the $\overline{Y}_N$, is in fact δ_Q. The proof of the following lemma is easy and we refer to [42], for more details. G_t will denote the natural σ-field on H.

Lemma 3.5. *For $\overline{Q}_\infty$ a.e. $m \in M(H)$, $(X_\cdot^1, B_\cdot^1)$ and $(X_\cdot^2, B_\cdot^2)$ are m independent identically distributed, $(B_\cdot^1, B_\cdot^2)$ is a two dimensional G_t-Brownian motion, and the law of X_0^1 (or X_0^2) under m is u_0.*

Let us first introduce some notation. We define the following functions on H:

$A_t^i = X_t^i - X_0^i - B_t^i\ ,\ \ i = 1,2\ , \in C_0^+$ (continuous increasing process),

A_t is the C_0^+ valued component of H ,

$$H_t = |X_t^1 - X_t^2| - |X_0^1 - X_0^2| - \int_0^t \mathrm{sign}^+(X_s^1 - X_s^2)\ dA_s^1 - \int_0^t \mathrm{sign}^+(X_s^2 - X_s^1)dA_s^2 - A_t\ ,$$

$$D_t = |X_t^1 - X_t^2| - |X_0^1 - X_0^2| - \int_0^t \mathrm{sign}^-(X_s^1 - X_s^2)dA_s^1 - \int_0^t \mathrm{sign}^-(X_s^2 - X_s^1)dA_s^2 - A_t\ .$$

Lemma 3.6. *For $\overline{Q}_\infty$ a.e. $m \in M(H)$,*

$$A_t^i = cE_m[A_t/\sigma(X_\cdot^i, B_\cdot^i)]\ ,\quad i = 1,2, \tag{3.6}$$

$$H_t \textit{ is a continuous } G_t - \textit{supermartingale}, \tag{3.7}$$

$$D_t \textit{ is a continuous } G_t - \textit{submartingale}. \tag{3.8}$$

Proof: Let us first prove (3.6). Set $F(m) = \langle m, (cA_t - A_t^1)\; g(X^1_\cdot, B^1)\rangle$, where g is continuous bounded, and define $F_\alpha(m)$ by the same expression, replacing cA_t by $(cA_t)\wedge\alpha$ and A_t^1 by $A_t^1\wedge\alpha$. It is enough to show that $F(m)=0$, $\overline{Q}_\infty$-a.s. Observe now that by (3.1), for $k\leq\infty$,

$$E_{\bar{Q}_{N_k}}[|F(m)-F_\alpha(m)|] \leq \text{const}\; E_{\bar{Q}_{N_k}}[\langle m, (cA_t-\alpha)_+ + (A_t^1-\alpha)_+\rangle] \leq \text{const}\;\alpha^{-1}\,.$$

It follows that

$$\begin{aligned} &E_{\bar{Q}_\infty}[|F(m)|] \leq \underline{\lim}_k E_{\bar{Q}_{N_k}}[|F(m)|] \\ &= c\;\underline{\lim}_k E_{N_k}[|\frac{1}{N(N-1)}\sum_{i\neq j}\{(L^0(X^i-X^j)_t - \frac{1}{N}\sum_{k\neq i} L^0(X^i-X^k)_t)\; g(X^i,B^i)\}|] \\ &= c\;\underline{\lim}_k E_{N_k}[|\frac{1}{N}\sum_i(\frac{1}{N(N-1)}\sum_{j\neq i} L^0(X^i-X^j)_t)\; g(X^i,B^i)|] = 0\,. \end{aligned}$$

Let us now prove (3.7), (3.8) being proved in a similar fashion.

We now introduce for $t>s$

$$F(m) = \langle m, (H_t - H_s)\cdot g_s\rangle\,,$$

where g_s is a nonnegative G_s-measurable continuous bounded function. Take now $K(m)\geq 0$, a continuous bounded function, it is enough to show that

$$E_{\bar{Q}_\infty}[F(m)\;K(m)] \leq 0\,,$$

to be able to conclude that (3.7) holds. Observe now that the functional ($\alpha>0$), on H,

$$-(\int_0^t \text{sign}^+(X_s^1 - X_s^2)\; d(A^1\wedge\alpha)_s + \int_0^t \text{sign}^+(X_s^2 - X_s^1)\; d(A^2\wedge\alpha)_s)$$

is bounded lower semicontinuous. Using very similar truncation arguments, we see that

$$\begin{aligned} &E_{\bar{Q}_\infty}[F(m)\;K(m)] \\ &\leq \underline{\lim}_k E_{N_k}[K(\overline{Y}_N)(\frac{1}{N(N-1)}\sum_{i\neq j}\{(|X_t^i - X_t^j| - |X_s^i - X_s^j| \\ &\quad - c\int_s^t \text{sign}^+(X_u^i - X_u^j)\times\frac{1}{N}\sum_{k\neq i} dL^0(X^i - X^k)_u \\ &\quad - c\int_s^t \text{sign}^+(X_u^j - X_u^i)\times\frac{1}{N}\sum_{h\neq j} dL^0(X^j - X^h)_u \\ &\quad - L^0(X^i - X^j)_t)\; g_s(X^i_\cdot, B^i_\cdot, X^j_\cdot, B^j_\cdot, L^0(X^i-X^j)_\cdot)\})] \end{aligned}$$

Since the process $(X^1_\cdot,\dots,X^N_\cdot)$ is Δ_N valued, we can in fact replace sign^+ by sign in the previous expression, and find using Tanaka's formula:

$$\underline{\lim}_k E_{N_k}[K(\overline{Y}_N)\times \frac{1}{N(N-1)}\sum_{i\neq j}\int_s^t \text{sign}(X^i_u - X^j_u)\ d(B^i_u - B^j_u)\cdot g^{ij}_s]\ ,$$

with obvious notations. This is less than:

$$\text{const}\cdot \underline{\lim}_k E_{N_k}[\{\frac{1}{N(N-1)}\sum_{i\neq j}\int_s^t \text{sign}(X^i_u - X^j_u)\ d(B^i_u - B^j_u)\cdot g^{ij}_s\}^2]^{1/2}\ ,$$

which is easily seen to be zero after expanding the square and using the orthogonality of terms (i,j), (k,ℓ) with $\{i,j\}\cap\{k,\ell\}=\emptyset$.

□

Let us now continue the proof of Theorem 3.3. For an m satisfying the properties of Lemmas 3.5, 3.6, we know that

$$D_t = \int_0^t \text{sign}^+(X^1_s - X^2_s)\ d(B^1 - B^2)_s + L^{0,\ell}(X^1 - X^2)_t - A_t + 2\int_0^t 1(X^1_s = X^2_s)\ dA^1_s\ ,$$

is a G_t-submartingale. From this we deduce that the bounded variation process

$$\text{(3.9)}\qquad K^+_t = L^{0,\ell}(X^1 - X^2)_t - A_t + 2\int_0^t 1(X^1_s = X^2_s)\ dA^1_s\ ,$$

is continuous increasing. Similarly, we see that:

$$\text{(3.10)}\qquad K^-_t = L^{0,\ell}(X^1 - X^2)_t - A_t - 2\int_0^t 1(X^1_s = X^2_s)\ dA^2_s\ ,$$

is a continuous decreasing process. From section 2 (2.11), and the independence under m of $(X^1_\cdot, B^1_\cdot)$, $(X^2_\cdot, B^2_\cdot)$, we know that:

$$E_m[L^{0,\ell}(X^1 - X^2)_t/(X^1, B^1)] = 2\int_0^t u(s, X^1_s)\ ds\ .$$

Conditioning (3.9) with respect to (X^1, B^1), we see that

$$\text{(3.11)}\qquad \frac{1}{c}A^1_t + C^1_t = 2\int_0^t u(s, X^1_s)\ ds + 2\int_0^t p(s, X^1_s)\ dA^1_s\ ,$$

where C^1_t is a continuous increasing process depending on (X^1, B^1), and $p(s,x) = \int 1(X^i_s = x)dm$, for $i = 1,2$. We can write

$$dA^1_t = 1(p(t, X^1_t) < \frac{1}{4c})\ dA^1_t + 1(p(t, X^1_t) \geq \frac{1}{4c})\ dA^1_t\ ,$$

and from equation (3.11), we already know that

$$\left(\frac{1}{c} - 2p(t, X_t^1)\right)\, 1\left(p(t, X_t^1) < \frac{1}{4c}\right)\, dA_t^1$$

is absolutely continuous with respect to Lebesgue measure. Let us now study the measure $1(p(t, X_t^1) \geq \frac{1}{4c}) dA_t^1$. It is supported by the closed set

$$F = \{t \geq 0 \ , \ \exists x \in R \ , \ p(t, x) \geq \frac{1}{4c}\} \ ,$$

which has measure zero, since the law of X_t^1 (or X_t^2) has a density with respect to dx for almost every t.

Let us show that $F \subset \{0\}$. If not there is $I = (a, b) \subset F^c$, with $b < \infty$, $b \in F$. On I, $dA_t^1 = 1(p(t, X_t^1) < \frac{1}{4c}) dA_t^1$, so that $1_I dA_t^1 << dt$. From this it immediately follows that $\int_I 1(X_s^2 = X_s^1) dA_s^1 = \int_I 1(X_s^2 = X_s^1) dA_s^2 = 0$, and by (3.9), (3.10) we get:

$$1_I \cdot dA_t = 1_I \cdot dL^{0,\ell}(X^1 - X^2)_t = 1_I dL^0(X^1 - X^2)_t \ . \tag{3.12}$$

Now the process $\overline{X}_t^1 = X_{t+a^1}^1$, for $a^1 \in I$, and $0 \leq t < b - a^1$, satisfies with obvous notations:

$$\overline{X}_t^1 = \overline{X}_0^1 + \overline{B}_t^1 + c \int_{\tilde{H}} L^0(\overline{X}^1 - \overline{X}^2)_t \ d\tilde{m}(\overline{X}^2, \overline{B}^2) \ .$$

From section 2), Theorem 2.2, this implies that

$$\exp\{-4c\overline{F}_t(x)\} = \exp\{-4c\overline{F}_0\} \ * \ p_t(x) \ , \quad 0 \leq t < b - a^1 \ .$$

So for t near $b - a^1$, $\exp\{-4c\overline{F}_t\}$ is uniformly continuous, bounded above and away from zero,so that $b \notin F$. This shows that $F \subset \{0\}$. Now the same reasoning we just made shows that $A_t = L^{\ell,0}(X^1 - X^2)_t = L^0(X^1 - X^2)_t$, and

$$X_t^1 = X_0^1 + B_t^1 + cE_{\tilde{m}}^2[L^0(X^1 - X^2)_t] \ , \tag{3.13}$$

where $\tilde{m}$ is the law of (X^1, B^1) or $(X^2_., B^2_.)$ under m, and a similar equation for $X^2_.$. So $S(u_0)$ is not empty and $m = Q$. This proves Theorem 3.3.

□

Let us mention that for any $u_0 \in M(R)$ we can find a sequence u_N u_0-chaotic and concentrated in Δ_N, so that we have indeed $S(u_0) = \{P_{u_0}\}$, by any u_0.

From convergence in law to trajectorial convergence:

In the case where $u_0(dx)$ has a bounded density with respect to Lebesgue measure, we can in fact consider the trajectorial solutions

$$(3.14) \qquad \overline{X}_t^i = X_0^i + B_t^i + 2c\int_0^t u(s,\overline{X}_s^i)\,ds\;, \quad 1 \le i \le N\;,$$

where u is the solution of Burgers' equation, with initial condition u_0. As already mentioned in Remark 2.3 we have pathwise uniqueness for the solution of (3.14). We can in fact obtain a trajectorial convergence in the fashion of Theorem 1.4 of Chapter I.

Theorem 3.7. *Suppose the (X_0^i) are independent u_0-distributed, for any i, T,*

$$(3.15) \qquad \lim_{N\to\infty} E[\sup_{t\le T}|X_t^{i,N} - \overline{X}_t^i|] = 0$$

Proof: If one now defines

$$\overline{Z}_N = \frac{1}{N(N-1)}\sum_{i\neq j}\delta_{(X^i_\cdot,B^i_\cdot,\overline{X}^i_\cdot,X^j_\cdot,B^j_\cdot,\overline{X}^j_\cdot,L^0(X^i-X^j)_\cdot)} \in M(\tilde{H}\times C\times\tilde{H}\times C\times C_0^+)\;,$$

by the same proof as before, one sees that $\overline{Z}_N$ converges in law to the Dirac mass on the law of $(\overline{X}^1_\cdot, B^1_\cdot, \overline{X}^1_\cdot, \overline{X}^2_\cdot, B^2_\cdot, \overline{X}^2_\cdot, L^0(\overline{X}^1 - \overline{X}^2)_\cdot)$. Taking $F_\alpha(m) = \langle m, \sup_{s\le T}|X_s^1 - \overline{X}_s^1|\wedge\alpha\rangle$, we see that $\lim_N E[\sup_{s\le T}|X_s^1 - \overline{X}_s^1|\wedge\alpha] = 0$. Estimates (3.1) then allow us to prove (3.15).

□

4) Tightness estimates.

We are now going to explain how one derives the estimates (3.1) of Proposition 3.2. To obtain these estimates, we are going to use the increasing reordering $Y_t^1 \le \ldots \le Y_t^N$ of the processes $(X_t^1,\ldots,X_t^N)$, (see Chapter I section 3) example e)). On the one hand, we will be able to use techniques of reflected processes to derive estimates on $Y^1_\cdot,\ldots,Y^N_\cdot$ and on the other hand the identity $Y_t^1+\cdots+Y_t^N = X_t^1+\cdots+X_t^N$ will yield a piece of information on the bounded variation term of $X_t^1+\cdots+X_t^N$.

In a first step, let us explain how an exponential control on $X_t^1+\cdots+X_t^N$ gives an individual exponential control on the X_t^i, $1\le i\le N$, and yields (3.1).

Proposition 4.1. *Suppose there exist $d_1, d_2 > 0$, such that for $N > 2c$, any $x\in\Delta_N$, and $t>0$,*

$$(4.1) \qquad E_x[\exp\{\frac{d_1}{\sqrt{t}}\sum_{i=1}^N (X_t^i - x^i)\}] \le d_2^N\;,$$

then there exists $d, \overline{d} > 0$, such that for $N > 2c$, $i \in [1, N]$, $t > 0$,

$$E_x[\exp\{\frac{d}{\sqrt{t}}(X_t^i - x^i)\}] \leq \overline{d} \, , \tag{4.2}$$

and (3.1) *holds.*

Proof: Let us first prove (4.2). Using scaling we can assume $t = 1$. From the estimate for $i \in [1, N]$:

$$\begin{aligned} &E[\exp\{d(B_1^i + \frac{c}{N}\sum_{j \neq i} L^0(X^i - X^j)_1)\}] \\ &\leq \exp\{d^2\} \times E[\exp\{\frac{2dc}{N}\sum_{j \neq i} L^0(X^i - X^j)_1\}]^{1/2} \, , \end{aligned}$$

we see it is enough to focus on $E[\exp\{\frac{2d}{N}\sum_{j \neq i} L^0(X^i - X^j)_1\}]$. (4.2) will follow from a link between the individual terms $\frac{1}{N}\sum_{j \neq i} L^0(X^i - X^j)_1$ for $i \in [1, N]$ and the sums $\frac{1}{N}\sum_{1 \leq i \neq j \leq N} L^0(X^i - X^j)_1$. We introduce to this end: f such that f, f' are bounded, $f'' = \delta_0 +$ bounded function:

$$\begin{aligned} &f(y) = [\arctan(y)]_+ \, , \\ &f'(y) = 0, \quad y < 0, 1/2 \, , \quad y = 0, (1 + y^2)^{-1}, \quad y > 0 \, . \\ &g(y) = -\frac{2y_+}{(1 + y^2)^2} \, , \text{ the regular part of } f'' \, . \end{aligned}$$

For simplicity, let us pick $i = 1$ as the individual term to be estimated. Set

(4.3)

$$\begin{aligned} S_t = &\frac{-1}{N}\sum_{j \neq 1}\{f(X_t^1 - X_t^j) - f(X_0^1 - X_0^j)\} + \frac{c}{N^2}\sum_{\substack{j \neq 1 \\ k \neq 1}}\int_0^t f'(X_s^1 - X_s^j) \, dL^0(X^1 - X^k)_s \\ &- \frac{c}{N^2}\sum_{\substack{j \neq 1 \\ k \neq j}}\int_0^t f'(X_s^1 - X_s^j) \, dL^0(X^j - X^k)_s + \frac{1}{N}\sum_{j \neq 1}\int_0^t g(X_s^1 - X_s^j) \, ds \, . \end{aligned}$$

From Tanaka's formula, $S_t + \frac{1}{2N}\sum_{j \neq 1} L^0(X^1 - X^j)_t$ is a martingale with increasing process

$$U_t = \frac{1}{N^2}\sum_{j \neq 1}\int_0^t (f')^2(X_s^1 - X_s^j) \, 2 \, ds + \frac{1}{N^2}\sum_{j \neq k}\int_0^t f'(X_s^1 - X_s^j) \, f'(X_s^1 - X_s^k) \, ds \, . \tag{4.4}$$

We now have for $\lambda > 0$,

(4.4)

$$
\begin{aligned}
&E[\exp\{\frac{\lambda}{2N}\sum_{j\neq 1} L^0(X^1 - X^j)_1\}] \\
&= E[\exp\{\frac{\lambda}{2N}\sum_{j\neq 1} L^0(X^1 - X^j)_1 + \lambda S_1 - \lambda^2 U_1 + \lambda^2 U_1 - \lambda S_1\}] \\
&\leq E[\exp\{\frac{\lambda}{N}\sum_{j\neq 1} L^0(X^1 - X^j)_1 + 2\lambda S_1 - 2\lambda^2 U_1\}]^{1/2} E[\exp\{2\lambda^2 U_1 - 2\lambda S_1\}]^{1/2} .
\end{aligned}
$$

Now by the exponential martingale property the first term of the last expression is smaller than 1. So we have:

$$
E[\exp\{\frac{\lambda}{2N}\sum_{j\neq 1} L^0(X^1 - X^j)_1\}] \leq E[\exp\{2\lambda^2 U_1 - 2\lambda S_1\}]^{1/2} . \tag{4.5}
$$

Observe now that U_1 is bounded, and that the only two dangerous terms of S_1 are the second and third in (4.3). However the second term only comes with a negative sign in $-2\lambda S_1$, so that we only have to worry about the third:

$$
\frac{2\lambda c}{N^2}\sum_{\substack{j\neq 1\\ k\neq j}} \int_0^1 f'(X_s^1 - X_s^j)\, dL^0(X^j - X^k)_s \leq 2\lambda c\frac{\|f'\|_\infty}{N^2} \sum_{1\leq j\neq k\leq N} L^0(X^j - X^k)_1 .
$$

But from the Cauchy-Schwarz inequality:

$$
\begin{aligned}
&E[\exp\{2\lambda\|f'\|_\infty \frac{c}{N^2} \sum_{1\leq j\neq k\leq N} L^0(X^j - X^k)_1\}] \\
&\leq E[\exp\{4\lambda\|f'\|_\infty \frac{1}{N}\sum_{i=1}^N (X_1^i - x^i)\}]^{1/2} E[\exp\{-4\lambda\|f'\|_\infty \frac{1}{N}\sum_i B_1^i\}]^{1/2} \\
&\leq E[\exp\{4\lambda\|f'\|_\infty \sum_{i=1}^N (X_1^i - x^i)\}]^{1/2N} \exp\{4\lambda^2\|f'\|_\infty^2/N\} .
\end{aligned}
$$

Picking λ small enough we see that (4.2) follows from (4.1).

Using scaling and the Cauchy Schwarz inequality, one easily deduces from (4.2), the first estimate $E[|X_t^i - X_s^i|^4] \leq K|t-s|^2$, in (3.1). On Tanaka's formula:

$$
L^0(X^i - X^j)_s^t = |X_t^i - X_t^j| - |X_s^i - X_s^j| - \int_s^t \operatorname{sign}(X_u^i - X_u^j)\, d(X_u^i - X_u^j) ,
$$

it is easy to obtain the second estimate of (3.1)

$$
E[\{L^0(X^i - X^j)_s^t\}^4] \leq K(t-s)^2 .
$$

□

So we have reduced the proof of (3.1) to that of (4.1). Using scaling as before and the Cauchy-Schwarz inequality, it is enough to check that for some $d_1, d_2 > 0$:

$$E[\exp\{\frac{d_1}{N}\sum_{i\neq j} L^0(X^i - X^j)_1\}] \leq d_2^N \ , \tag{4.6}$$

for any initial point $x \in \Delta_N$ and $N > 2c$. As we mentioned before we are now going to use the increasing reordering $Y_t^1 \leq \ldots \leq Y_t^N$ of the processes $X_t^1, \ldots, X_t^N$. The semimartingale structure of the $(Y_\cdot^i)$ processes is given by the following lemma whose proof is very similar to that of Lemma 3.7, once one knows the $(X_\cdot)$ process is Δ_N valued (see [42] for details):

We suppose our process $X_\cdot$ is constructed on some filtered, probability space $(\Omega, F, F_t, (B_t^i), P)$, endowed with F_t-Brownian motions $B_\cdot^i$, $1 \leq i \leq N$.

Lemma 4.2. *There are N independent F_t-Brownian motions $W_t^1, \ldots, W_t^N$, and $(N-1)$ continuous increasing processes $\gamma_t^1, \ldots, \gamma_t^{N-1}$ such that:*

$$\begin{aligned} Y_t^1 &= Y_0^1 + W_t^1 - \frac{1}{2}a\gamma_t^1 \ , \\ Y_t^k &= Y_0^k + W_t^k - \frac{1}{2}a\gamma_t^k + \frac{1}{2}b\gamma^{k-1} \ , \quad 2 \leq k \leq N-1 \ , \\ Y_t^N &= Y_0^N + W_t^N \qquad + \frac{1}{2}b\gamma_t^{N-1} \ , \end{aligned} \tag{4.7}$$

where $a = 1 - 2c/N$, $b = 1 + 2c/N$, and

$$\gamma_t^i = \int_0^t 1(Y_s^i = Y_s^{i+1})\, d\gamma_s^i \ . \tag{4.8}$$

□

The identity $Y_t^1 + \cdots + Y_t^N = X_t^1 + \cdots X_t^N$, now yields

$$\frac{2}{N}(\gamma_t^1 + \cdots + \gamma_t^{N-1}) = \frac{1}{N}\sum_{i\neq j} L^0(X^i - X^j)_t \ . \tag{4.9}$$

So our estimate (4.6) can be rephrased in terms of the $(\gamma_\cdot^i)$ processes. Let us introduce some more convenient processes, namely:

$$\begin{aligned} &D_t^k = b^{-(k-1)}(Y_t^{k+1} - Y_t^k) \ , \quad H_t^k = b^{-(k-1)}(W_t^{k+1} - W_t^k) \ , \\ &C_t^k = b^{-(k-1)}\gamma_t^k \ , \quad 1 \leq k \leq N-1 \ . \end{aligned} \tag{4.10}$$

One can see easily that:

$$
\begin{aligned}
D_t^1 &= D_0^1 + H_t^1 + C_t^1 - \frac{\alpha}{2} C_t^2 \\
D_t^k &= D_0^k + H_t^k + C_t^k - \frac{\alpha}{2} C_t^{k+1} - \frac{1}{2} C_t^{k-1} \ , \quad 2 \le k \le N-2 \\
D_t^{N-1} &= D_0^{N-1} + H_t^{N-1} + C_t^{N-1} \qquad - \frac{1}{2} C_t^{N-2} \ , \quad \text{with } \alpha = ab = 1 - \frac{4c^2}{N^2} \ .
\end{aligned}
$$

Moreover $C_t^i = \int_0^t 1(D_s^i = 0) dC_s^i$. It now follows from the solution to the Skorohod problem that: $(C_.^k)_{1 \le k \le N-1} = F((D_0^k + H_.^k), (C_.^k))$, where $F : (C \times C_0^+)^{N-1} \to (C_0^+)^{N-1}$, is the map:

$$
\begin{aligned}
F(v,c)_t^1 &= \sup_{s \le t} (-v_.^1 + \frac{1}{2} \alpha c_.^2)_+ \ , \\
F(v,c)_t^k &= \sup_{s \le t} (-v_.^k + \frac{1}{2} \alpha c_.^{k+1} + \frac{1}{2} c_.^{k-1})_+ \\
F(v,c)_t^{N-1} &= \sup_{s \le t} (-v_.^k \qquad + \frac{1}{2} c_.^{N-2})_+
\end{aligned}
$$

Set $|w|_t = \sum_{i=1}^{N-1} \sup_{s \le t} |w_s^i|$, for $w \in C^{N-1}$. It is not difficult to prove (see [42]) that:

Lemma 4.3.

$$
|F(v_., c_.) - F(v_., c'_.)|_t \le \frac{1}{2}(1 + |\alpha|) |c_. - c'_.|_t \ . \tag{4.12}
$$

If $c = F(v_., c_.)$, $\bar{c}_. = F(\bar{v}_., \bar{c}_.)$, *then*

$$
|c_. - \bar{c}_.|_t \le \frac{2}{1 - |\alpha|} |v_. - \bar{v}_.|_t \ , \quad t \ge 0 \ , \tag{4.13}
$$

and

$$
v_. \le \bar{v}_. \Rightarrow \bar{c}_. \le c_. \ . \tag{4.14}
$$

□

Because of (4.12), when $N > 2c$, we can consider the fixed point solution $c_. = F(v_., c_.)$ for any $v_. \in C^{N-1}$, which is obtained by iteration.

Now because of (4.14), if one replaces in (4.11) D_0^k by 0, this increases the corresponding $c_.$ processes. If one then replaces D_0^k by $\frac{b}{N}^{-(k-1)}$ we see from (4.13) that the corresponding fixed point $\bar{c}_.$, satisfies

$$
\begin{aligned}
\frac{1}{N}(c_1^1 + \cdots + c_1^{N-1}) &\le \frac{1}{N}(\bar{c}_1^1 + \cdots + \bar{c}_1^{N-1}) + \frac{1}{N} \frac{2}{1-\alpha} \times \sum_{k=1}^{N-1} \frac{b}{N}^{-(k-1)} \\
&\le \frac{1}{N}(\bar{c}_1^1 + \cdots + \bar{c}_1^{N-1}) + \frac{N}{2c^2} \ .
\end{aligned}
$$

The constant b^k, for $k \in [1,N]$ satisfy: $1 \le b^k \le e^{2c}$, and from the previous inequality we simply have to prove

$$(4.15) \qquad E[\exp\{\frac{d}{N}(\gamma_1^1 + \cdots + \gamma_1^{N-1})\}] \le \bar{d}^N \ , \qquad \text{for some } d, \bar{d} > 0 \ ,$$

when the initial point is now $x = (0, \frac{1}{N}, \dots, \frac{N-1}{N})$. Set $\rho = (1-2c/N \ / \ 1+2c/N) = a/b$, one easily sees that for $i \in [1,N]$, $0 < \gamma \le \rho^i \le 1$, where γ is independent of N. By Ito's formula

$$\begin{aligned} \sum_1^N \rho^i (Y_t^i - Y_0^i)^2 &= 2\sum_{i=1}^N \int_0^t \rho^i (Y_u^i - Y_0^i)\ dW_u^i \\ &+ \sum_{i=1}^{n-1} \int_0^t [\rho^{i+1} b (Y_u^{i+1} - Y_0^{i+1}) - \rho^i a (Y_u^i - Y_0^i)]\ d\gamma_u^i \\ &+ t\sum_1^N \rho^i \ . \end{aligned}$$

Using now the fact that $\rho^{i+1} b = \rho^i a$, $Y_u^{i+1} = Y_u^i\ d\gamma_u^i$-a.s., and $Y_0^{i+1} - Y_0^i = \frac{1}{N}$, we find:

$$(4.16) \qquad \sum_1^N \rho^i (Y_t^i - Y_0^i)^2 + \frac{1}{N}\sum_1^{N-1} \rho^i a \gamma_t^i \le 2\sum_{i=1}^N \int_0^t \rho^i (Y_u^i - Y_0^i)\ dW_u^i + Nt \ .$$

So to obtain a control such as (4.15), it is enough to prove

$$E[\exp\{d\sum_1^N \int_0^1 \rho^i (Y_u^i - Y_0^i) dW_u^i\}] \le \bar{d}^N \ , \qquad \text{for some } d, \bar{d} > 0 \ .$$

Using an exponential martingale and Cauchy Schwarz inequality, as we did in (4.5), and then the convexity of the exponential, it is easily seen that it is enough to show that for some $d, \bar{d} > 0$,

$$(4.17) \qquad \int_0^1 E[\exp\{d\sum_1^N (Y_t^i - Y_0^i)^2\}]\ dt \le \bar{d}^N$$

Let us set $|Y_t - Y_0|^2 = \sum_1^N \rho^i (Y_t^i - Y_0^i)^2$, and $U_t = \exp\{\frac{\lambda}{t+1}|Y_t - Y_0|^2\}$. Applying Ito's formula, we find:

$$\begin{aligned}U_t =&1 + \sum_1^N \int_0^t 2\lambda\rho^i \frac{(Y_s^i - Y_0)}{s+1} U_s \, dW_s^i \\ &+ \lambda \sum_{i=1}^{N-1} \int_0^t \{\rho^{i+1} b(Y_s^{i+1} - Y_s^i) - \rho^i a(Y_s^i - Y_0^i)\} \, (s+1)^{-1} \, U_s \, d\gamma_s^i \\ &+ \frac{1}{2} \int_0^t U_s \Big(\frac{4\lambda^2}{(s+1)^2} \sum_i \rho^{2i} (Y_s^i - Y_0^i)^2 - 2\lambda \frac{|Y_s - Y_0|^2}{(s+1)^2}\Big) \, ds \\ &+ \sum_1^N \lambda\rho^i \int_0^t (s+1)^{-1} U_s \, ds \, .\end{aligned}$$

Now by the same reason as in (4.16), the third term of the last expression is nonpositive. For $\lambda \le \frac{1}{2}$ since $\rho \le 1$, the fourth term is as well nonpositive. It then follows that for $\lambda \le 1/2$:

$$U_t \le 1 + \lambda N \int_0^t U_s \, ds + \text{ local martingale.}$$

It then follows using a familiar stopping time argument and Gronwall's inequality, that:

$$E[\exp\{\frac{1}{2(t+1)} \sum_1^N \rho^i (Y_t^i - Y_0^i)^2\}] \le \exp\{\frac{N}{2} t\} \, .$$

From this (4.17) follow, and we get our claim (4.1).

□

5) Reordering of the interacting particle system.

We look now at the same problem in our present context as we did for independent Brownian motions, in section 3 example e) of Chapter I. As we will see a very similar result holds in our case as well. We again suppose u_0 atomless, and u_N u_0-chaotic,Δ_N supported. We introduce the symmetrized, reordered processes $(Z_\cdot^1, \dots, Z_\cdot^N)$ defined by:

(5.1) $$Z_t^i = Y_t^{\sigma(i)} \, , \quad 1 \le i \le N \, ,$$

where σ is a uniformly distributed permutation of $[1, N]$, independent of the space where the $(X_\cdot^1, \dots, X_\cdot^N)$ process is constructed.

We will see that the $(Z^i_\cdot)$ are R-chaotic where R is the law of a "deterministic" type evolution. Let us describe R. more precisely. We define for $x \in D = \operatorname{supp} u_0 \backslash \cup_n \{a_n, b_n\}$, where $(\operatorname{supp} u_0)^c = \cup_n (a_n, b_n)$, (a_n, b_n) disjoint intervals,

$$\psi_t(x) = F_t^{-1} \circ F_0(x) \ , \tag{5.2}$$

where F_t is the distribution function of the solution at time t of Burgers' equation, with initial u_0, that is;

$$\exp\{-4cF_t\} = \exp\{-4cF_0\} \ * \ p_t \ .$$

D has full measure under u_0, and for $x \in D$, $\lim_{t\to 0} \psi_t(x) = x$. The probability R is defined as $\psi \circ u_0$. We have

Theorem 5.1. *The laws $(Z^1_\cdot, \dots, Z^N_\cdot)$ are R-chaotic.*

Proof: The proof is a repetition of the proof of Theorem 3.6 of Chapter I. The only point to explain is the tightness estimate. Recall that $p = \frac{a}{b} = \frac{1-2c/N}{1+2c/N}$, with the notations of section 4. Our required tightness estimates follow from

Lemma 5.2.

$$E[\sum_1^N \rho^i (Y_t^i - Y_s^i)^4] \le 3N(t-s)^2 \ . \tag{5.3}$$

Proof: Basically by the same argument as in (4.16), we know that

$$E[\sum_1^N \rho^i (Y_t^i - Y_s^i)^2] \le N(t-s) \ . \tag{5.4}$$

Applying then Ito's formula to $\sum_1^N \rho^i (Y_t^i - Y_s^i)^4$, we find;

$$\begin{aligned}\sum_1^N \rho^i (Y_t^i - Y_s^i)^4 =& 4\int_s^t \sum_i \rho^i (Y_u^i - Y_s^i)^3 \ dW_u^i \\ &+ 2\sum_1^{N-1} \int_s^t [\rho^{i+1} b (Y_u^{i+1} - Y_s^{i+1})^3 - \rho^i a (Y_u^i - Y_s^i)^3] \ d\gamma_u^i \\ &+ 3\int_s^t \sum_1^N \rho^i (Y_u^i - Y_s^i)^2 \ du \ .\end{aligned}$$

By a now familiar argument the second term of the last expression is nonpositive, and since $\rho \le 1$, from (5.4) after integration we easily find our claim (5.3).

□

The proof of Theorem 5.1 is then basically a repetition of that of Theorem 3.6, of Chapter I.

□

Before closing this section let us mention that now for $t > 0$, $x_t = \psi_t(x)$ satisfies the O.D.E.

$$\dot{x}_t = -\frac{\partial_t F}{\partial_x F}(t, x_t) = -\frac{1}{u}(\frac{1}{2}\partial_x^2 F - 2cu^2) = (-\frac{1}{2}(\log u)' + 2cu)(t, x_t) \ . \tag{5.5}$$

So R is the law of the deterministic solution of O.D.E. (5.5) with initial random condition u_0 distributed.

III. The constant mean free travel time regime

In the warm up calculation of chapter II, section 1, we have seen that the interaction range $N^{-1/d-2}$, in dimension $d \geq 3$, and $\exp\{-\text{const}.N\}$ in dimension $d = 2$, is critical in the study of the quantity:

$$b_N = E\left[\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{(N-1)}\sum_{j\neq i}\int_0^t \phi_{N,a}(X_s^i - X_s^j)\, ds - \int_0^t u_s(X_s^i)\, ds\right)^2\right],$$

for independent d-dimensional Brownian motions X^i, whereas there is no critical regime in dimension 1.

The object of this chapter is to study in more detail this critical regime, which we call "constant mean free travel time regime". The reason for this name is that in the limit regime, a "typical particle" does not feel any influence from the other particles before a positive time. We will see that there is a very algebraic Poissonian picture which governs the limiting regime.

In section 1 we will start with a study of annihilated Brownian spheres, motivated by the work of Lang-Nguyen [23]. However we will follow a different route from the hierarchy method they use (see introduction). Although the results of section 1 answer the "propagation of chaos" question, they somehow miss the deeper Poissonian limit picture which is then explained in sections 2 and 3.

1) Annihilating Brownian spheres

We consider $X_.^1, \dots, X_.^N$, independent Brownian motions with initial distribution u_0, in $\boldsymbol{R}^d$, $d \geq 2$, which are centers of "Brownian soap bubbles" of radius $\frac{1}{2}s_N$. This means that when the centers of two such spheres which are still intact come to a distance smaller than s_N, then both spheres are destroyed. We precisely pick

$$\begin{aligned} s_N &= N^{-1/d-2}\,, \quad d \geq 3\,, \\ &= \exp\{-N\}\,, \quad d = 2\,, \end{aligned} \tag{1.1}$$

and we assume that u_0 has a bounded density V. We will denote by τ_i the death time of the ith sphere. We are going to study the chaotic behavior of the laws of the symmetric variables $(X_.^1, \tau_1), \dots, (X_.^N, \tau_N)$ on $(C(\boldsymbol{R}_+, \boldsymbol{R}^d) \times [0, +\infty])^N$. The result we will obtain will already motivate the terminology of "constant mean free travel time regime".

Theorem 1.1. *The laws of the* $((X^i, \tau_i))_{1\leq i\leq N}$, *are* P*-chaotic, where* P *on* $C \times [0, +\infty]$ *is defined by:*

– $X.$ *is a Brownian motion with initial distribution* u_0.

– $P[\tau > t/X.] = \exp\{-c_d \int_0^t u(s, X_s)ds\}$, *where u is the unique bounded solution of*

$$
(1.2)\qquad \begin{aligned} \partial_t u &= \frac{1}{2}\Delta u - c_d u^2 \\ u_{t=0} &= u_0 \end{aligned}
$$

and

$$
c_d = (d-2)\mathrm{vol}(S_d)\ , \quad d \geq 3\ , \qquad = 2\pi, \qquad d = 2\ .
$$

Remark 1.2. Let us first give some comments,
1) the nonlinear equation (1.2) is understood in the integral form:

$$
u_t = u_0 P_t - c_d \int_0^t u_s^2\ P_{t-s}\ ds\ ,
$$

where P_t is the Brownian semigroup. We let P_t act on the right in the previous integral formula to indicate that we are in fact dealing with a forward equation, although this is somehow obscured here by the self-adjointness of P_t.
2) An application of Ito's formula to $f(t, X_t)\exp\{-c_d \int_0^t u(s, X_s)ds\}$, when $f(s,x) = P_{t-s}\phi(x)$, $\phi \in C_k^\infty(R^d)$, easily shows that the subprobability distribution of the alive particle under P is u_t, that is:

$$
E_P[\phi(X_t)\ 1(\tau > t)] = \int \phi(x)\ u_t(x)\ dx\ .
$$

3) In fact one can get various reinforcements of the basic weak convergence result corresponding to Theorem 1.1. For instance if f_i, $i = 1, \dots, k$ belong to $L^1(\mu)$ (μ: Wiener measure with initial distribution in u_0), and ϕ_i are bounded functions on $[0, +\infty]$, with a set of discontinuities included in $(0, \infty)$ having zero Lebesgue measure, then:

$$
\lim_N E_N[f_1(X^1_.)\phi_1(\tau_1)\dots f_k(X^k_.)\phi_k(\tau_k)] = \prod_1^k E_p[f_i(X)\phi_i(\tau)]\ .
$$

Also if one applies Theorem 2.2 of Baxter-Chacon-Jain [1], to $\tau_1 \wedge \dots \wedge \tau_k$, one can also see that the density of presence of the first k particles, when they are all alive at time t, (that is $\tau_1 \wedge \dots \wedge \tau_k > t$), converges in variation norm to $u(t, x_1)\dots u(t, x_k)\ dx_1 \dots dx_k$.
4) The tightness of the laws of $\overline{X}_N = \frac{1}{N}\sum_i \delta_{(X^i_.,\tau^i)}$ is clearly immediate, by Proposition 2.2 of Chapter I.This is an indication that this point will not be very helpful in the proof. One of the difficulties of the problem is that annihilation induces a fairly complicated and long range dependence structure. The proof has somehow to rely on the treatment of terms like $1\{\tau_i \leq t\}$.

One may be tempted to express an event like $\{\tau_1 \leq t\}$ in terms of the basic independent variables which are the processes $(X^1_., \dots, X^N_.)$, for instance see Lang-Nguyen [23], p. 244.

However we will refrain from doing this. Loosely speaking we will stop at the first step of the unraveling, where the killer of particle 1 is considered: $P_N[\tau_1 \leq t] \cong (N-1)P[X^2$ had a collision with X^1 before time t, both were alive at that time]. Such an equality somehow bootstraps the law of (X^2, τ_2) in the law of $(X^2_., \tau_1)$. The idea is that this should force limit points of empirical laws to be concentrated on measures m satisfying a self-consistent property which will be characteristic of the law P.

5) The collision region between two particles corresponds to the set $|x_1 - x_2| \leq s_N$, in $\mathbb{R}^{2d}$.

It roughly corresponds to the region where the two-particle 1-potential generated by Lebesgue measure sitting on the diagonal of $\mathbb{R}^d \times \mathbb{R}^d$,

$$h(x_1, x_2) = \int_0^\infty e^{-s} ds \int_{\mathbb{R}^d} p_s(x_1, z)\, p_s(x_2, z)\, dz$$

is larger than $c_d^{-1} N$. This notion of level set of potentials is appropriate to find the "right" collision sets in non Brownian situations, see [45]. One can see that if $T_{1,2}$ is the entrance time in the region $|x_1 - x_2| \leq s_N$, then as N goes to infinity for $x_1 \neq x_2 \quad N E_{x_1,x_2}[T_{1,2} \in dt,\ (X^1_{T_{1,2}}, X^2_{T_{1,2}}) \in dx]$ converges vaguely to the measure $c_d p_s(x_1, z)\, p_s(x_2, z)\, dz\, ds$ on $(0, \infty) \times (\mathbb{R}^d)^2$ (see [45]). Here dz stands for Lebesgue measure on the diagonal of $(\mathbb{R}^d)^2$. This also makes plausible that we are dealing with a kind of Poisson limmit theorem.

□

We are going to prove Theorem 1.1, in a number of steps. The first step will be to give another characterization of the law P, given in Theorem 1.1, getting us closer to the actual form we will use to identify limit points of laws of empirical measures.

Lemma 1.2. *P is the only probability m on $C \times [0, \infty]$ such that*

– *X is a Brownian motion with initial distribution u_0,*

$$(1.3) \qquad \text{– for } t \geq 0\,, \quad E_m[1(\tau \leq t) - \int_0^t c_d\, 1(\tau > s)\, u(s, X_s)\, ds / X_.] = 0\,;$$

where $u(s, x) \in L^\infty(ds\, dx)$ is the density of the image of the measure $1(\tau \geq s)\, ds\, dm$ under (s, X_s).

Sketch of Proof: In view of Remark 1.2 2), which identifies the solution of non-linear equation (1.2), with the density of presence of the not yet destroyed particle, for P, one checks readily that P satisfies the required conditions.
Uniqueness:

Let $\nu(X_\cdot, dt)$ be the conditional distribution on $[0,\infty]$ of τ given $X_\cdot$ which is μ-distributed, where μ is Wiener measure with initial condition u_0. Condition (1.3) implies that μ-a.s., for any t:

$$\nu_{X_\cdot}([0,t]) - \int_0^t c_d\ \nu_{X_\cdot}((s,\infty))\ u(s,X_s)\ ds = 0\ ,$$

which implies that $\nu_{X_\cdot}((t,\infty)) = \exp\{-c_d \int_0^t u(s,X_s)ds\}$. It then follows that for $\phi \in C_K^\infty$, and a.e. t:

$$\langle u_t, \phi\rangle = E[\phi(X_t)\ 1(\tau \geq t)] = \langle u_0, P_t\phi\rangle - c_d \int_0^t \langle u_s^2, P_{t-s}\phi\rangle\ ds\ ,$$

so that u is the unique bounded solution of the integral equation corresponding to (1.2). This yields that $m = P$.

□

The present characterization of P has the advantage for us that u does not refer to the solution of (1.2) any more, but can be directly measured on m. We also got rid of the exponential term $\exp\{-c_d \int_0^t u(s,X_s)ds\}$, and deal instead with $c_d \int_0^t 1(\tau > s)\ u(s,X_s)ds$. This will reduce the complexity of computations. Even so $u(s,x)$ is a (very) ill behaved function of m under weak convergence. So later we are in fact going to reinterpret this quantity $u(s,x)$, using a priori knowledge on our possible limit points of empirical measures to obtain continuous enough functionals characterizing P.

Before that we introduce the idea of chain reactions, in order to restore some independence between particles. We set for $1 \leq i \neq j \leq N$

(1.4) $$T_{i,j} = \inf\{s \geq 0\ ,\ |X_s^i - X_s^j| \leq s_N\}\ .$$

The set $A_{N,t}^{i,j}$ will represent the occurrence between the Brownian trajectories $X_\cdot^1, \dots, X_\cdot^N$, and forgetting about any destruction, of a chain reaction leading from i to j before time t. Precisely (for $i = 2$, $j = 1$):

(1.5) $$\begin{aligned} A_{N,t}^{2,1} = \{T_{2,1} \leq t \text{ or } \exists k_1, \dots, k_p \text{ distinct in } [3,N] \text{ such that} \\ S_1 = T_{2,k_1} \leq S_2 = T_{k_1,k_2} \leq \dots \leq S_{p+1} = T_{k_p,1} \leq t\}\ . \end{aligned}$$

One now introduces τ_j^i, the death time of particle j if one does not take into account the trajectory X^i for the determination of collisions. The interest for us of the chain reaction sets $A_{N,t}^{i,j}$ comes from

Lemma 1.3. *For x, y in R^d, $t > 0$:*

$$P_{x,y} - \text{ a.s. } \{t \wedge \tau_j \neq t \wedge \tau_j^i\} \subseteq A_{N,t}^{i,j} .$$

($P_{x,y}$ is the probability on C^N for which the X^ℓ are independent Brownian motions, with initial distribution u_0 for $\ell \neq i, j$, δ_x for $\ell = i$, and δ_y for $\ell = j$).

For the proof of Lemma 1.3, we refer to [43].

Since clearly τ_j^i is independent of X^i, we are now interested in showing that $P_N[A_{N,t}^{i,j}] \to 0$, when N goes to infinity. This will provide us with a tool to restore independence.

Proposition 1.4. *For $\eta > 0$, $\lim_N \sup_{|x-y|>\eta} P_{x,y}[A_{N,t}^{2,1}] = 0$.*

$$\lim_N P_N[A_{N,t}^{2,1}] = 0 . \tag{1.7}$$

Proof: The second statement is an immediate consequence of the first statement since u_0 has a bounded density. Let us prove the first statement. Using symmetry we have

$$P_{x,y}[A_{N,t}^{2,1}] \leq \sum_{p=0}^{N-2} N^p E_{x,y}[S_1 \leq \ldots \leq S_{p+1} \leq t] , \tag{1.8}$$

where now $k_1 = 3, \ldots, k_p = 2 + p \leq N$. For $\lambda > 0$, we set

$$h_N^\lambda(z) = E_{z,0}[\exp\{-\lambda T_{1,2}\}] = (g_\lambda(|z|)/g_\lambda(s_N)) \wedge 1 ,$$

where $g_\lambda(|x-y|)$ is the λ-Green's function for Brownian motion with covariance $2Id$. As $u \to 0$,

$$\begin{aligned} g_\lambda(u) &\sim c_d^{-1} u^{2-d} , \quad d \geq 3 , \\ &\sim (2\pi)^{-1} \log(u^{-1}) , \quad d = 2 , \end{aligned}$$

It follows that for $N \geq N_0(\lambda)$:

$$N h_N^\lambda(z) \leq 2c_d \, g_\lambda(|z|) . \tag{1.9}$$

Set now $a_p^\lambda = N^p E_{x,y}[S_1 \leq \ldots \leq S_p \; e^{-\lambda S_p}]$. We have for $N \geq N_0(\lambda)$:

$$\begin{aligned} a_p^\lambda &\leq N^p \; E_{x,y}[(S_1 \leq \ldots \leq S_{p-1}) \; e^{-\lambda S_{p-1}} \; h_N^\lambda(X_{S_{p-1}}^{p+2} - X_{S_{p-1}}^{p+1})] \\ &\leq N^{p-1} \; E_{x,y}[(S_1 \leq \ldots \leq S_{p-1}) \; e^{-\lambda S_{p-1}} \; 2c_d \; g_\lambda(|X_{S_{p-1}}^{p+2} - X_{S_{p-1}}^{p+1}|)] . \end{aligned}$$

If we integrate over $X_.^{p+2}$ in the last expression, since $\int g_\lambda(|y|)dy = \lambda^{-1}$, we find:

$$a_p^\lambda \le N^{p-1} E_{x,y}[S_1 \le \dots \le S_{p-1}\ e^{-\lambda S_{p-1}}]\ \frac{2c_d\|V\|_\infty}{\lambda}\ ,$$

and now by induction:

$$\text{for } N \ge N_0(\lambda)\ , \quad a_p^\lambda \le \left(\frac{2c_d\|V\|_\infty}{\lambda}\right)^p . \tag{1.10}$$

Denote by a_p the pth term of the series in (1.8). Set $\lambda = 4c_d\|V\|_\infty$. For $N \ge N_0(\lambda)$:

$$a_0 \le e^{\lambda t}\ E_{x,y}[e^{-\lambda T_{2,1}}] \le e^{\lambda t}\frac{2c_d}{N}\ g_\lambda(|x-y|)\ .$$

Consider now $p \ge 1$. Pick $\epsilon \in (0,1)$.

For $N \ge N_0\left(\frac{2c_d\|V\|_\infty}{\epsilon}\right)$, by (1.10):

$$N^p\ E_{x,y}[S_1 \le \dots \le S_p \le \frac{\epsilon}{2c_d\|V\|_\infty}] \le e\epsilon^p\ .$$

It follows that for $N \ge N_0(\lambda) \vee N_0(2c_d\|V\|_\infty/\epsilon)$:

$$\begin{aligned}
&N^p E_{x,y}[S_1 \le \ldots \le S_p \le S_{p+1} \le t] \\
&\quad \le N^p\ E_{x,y}[S_1 \le \ldots \le S_p \le \frac{\epsilon}{2c_d\|V\|_\infty}] \\
&\qquad + e^{\lambda t} N^p E_{x,y}[1(S_1 \le \ldots \le S_p \le S_{p+1})\ 1(S_p \ge \frac{\epsilon}{2c_d\|V\|_\infty})\ e^{-\lambda S_{p+1}}]\ .
\end{aligned}$$

The first term of the last expression is smaller than $e\ \epsilon^p$, and the second is smaller than:

$$e^{\lambda t} N^{p-1} E_{x,y}[(S_1 \le \ldots \le S_p)\ 1(S_p \ge \frac{\epsilon}{2c_d\|V\|_\infty})\ e^{-\lambda S_p} 2c_d g_\lambda(|X^1_{S_p} - X^{2+p}_{S_p}|)] \tag{1.11}$$

The distribution of $X^1_{S_p}$ conditionally on $(X_.^2, \dots, X_.^N)$, is Gaussian with covariance $S_p I_d$, and mean x. It has a density which is uniformly bounded when $S_p \ge \frac{\epsilon}{2c_d\|V\|_\infty}$ by $K = (\frac{c_d\|V\|_\infty}{\pi\epsilon})^{d/2}$.

Integrating in (1.11) over $X_.^1$, we find the upper bound $K\ e^{\lambda t}\ N^{p-1}\ E_{x,y}[(S_1 \le \ldots \le S_p)\ e^{-\lambda S_p}] \le \frac{Ke^{\lambda t}}{N}(\frac{1}{2})^p$, by (1.10).

It follows that for $p \ge 1$, $\epsilon > 0$, and $N \ge N_0(\lambda) \vee N_0(2c_d\|V\|_\infty/\epsilon)$:

$$N^p E_{x,y}[S_1 \le \ldots \le S_p \le S_{p+1} \le t] \le e^{\lambda t}\frac{K}{N}(\frac{1}{2})^p + e\ \epsilon^p\ .$$

It now follows that:

$$P_{x,y}[A^{2,1}_{N,t}] \le e^{\lambda t}(\frac{2c_d}{N} g_\lambda(|x-y|) + \frac{K}{N}) + e\ \frac{\epsilon}{1-\epsilon}\ .$$

From this our claim follows.

□

As we mentioned already the tightness of the laws of the $\overline{X}_N = \frac{1}{N}\sum_i \delta_{(X^i,\tau_i)} \in M(C \times [0,\infty])$ is immediate. Let us now identify the possible limit points. In order to be able to identify such a limit point $\overline{P}_\infty$ as δ_P, we will use the following idea. The density $u(s,x)$ which appears in (1.3), is ill behaved for the weak convergence topology. However we know that for $\overline{P}_\infty$-a.e. m, the $X.$ component will be μ-distributed (that is Brownian motion with initial distribution u_0). So now for such an m:

$$\begin{aligned}\langle u_s, f\rangle &= \langle m, f(X_s)\rangle - \langle m, f(X_s)\ 1(\tau < s)\rangle \\ &= \langle V_s, f\rangle - \langle m, f(X_s)\ 1(\tau < s)\rangle\ ,\end{aligned}$$

where $V(s,x) = u_0\ *\ p_s(x)$. If now we anticipate the fact that for $\overline{P}_\infty$-a.e. m there should be a "Markov property" at time τ, then for such an m, the density $u(s,x)$ should be given by:

$$u(s,x) = V(s,x) - \langle m, p_{s-\tau}(X_\tau - x)\ 1(s > \tau)\rangle\ .$$

So we will interpret u, in (1.3), in terms of this last formula, and then we will check that for $\overline{P}_\infty$-a.e. m this expression defines the density of the image of $1(\tau \geq s)ds\ dm$ under (s, X_s). So the quantity $\int_0^t 1(\tau > s)\ u(s,X_s)ds$ is now replaced by $\int_0^{t\wedge\tau} V(s,X_s)ds - \tilde{E}_m[\int_0^{(t\wedge\tau-\tilde{\tau})_+} p_s(X_{\tilde{\tau}+s} - \tilde{X}_{\tilde{\tau}})ds]$, which has nicer continuity properties. In view of these comments it is natural to try now to obtain

Proposition 1.5. *For* $\overline{P}_\infty$-a.e. m, for $t \geq 0$:

$$(1.12)\quad E_m[1(\tau \leq t) - c_d\int_0^{t\wedge\tau} V(s,X_s)ds + c_d\tilde{E}_m[\int_0^{(t\wedge\tau-\tilde{\tau})_+} p_s(X_{\tilde{\tau}+s} - \tilde{X}_{\tilde{\tau}})ds/X.] = 0$$

Proof: We call D the at most denumerable set of t such that $I(\overline{P}_\infty)\ [\{\tau = t\}] \neq 0$. We then define for $h(\cdot) \in C_b(C)$, $t \notin D$:

$$G(m) = \langle m, \{1(\tau \leq t) - c_d\int_0^{t\wedge\tau} V(s,X_s)ds + \tilde{E}_m[c_d\int_0^{(t\wedge\tau-\tilde{\tau})_+} p_s(X_{\tilde{\tau}+s} - \tilde{X}_{\tilde{\tau}})ds]\}\ h(X)\rangle$$

and we define the smoothed $G_\epsilon(m)$, $\epsilon > 0$, where the last integral in the previous expression is replaced by $\int_0^{(t\wedge\tau-\tilde{\tau}-\epsilon)_+} p_{s+\epsilon}(X_{\tilde{\tau}+s+\epsilon} - \tilde{X}_{\tilde{\tau}})ds$. Since for $\overline{P}_\infty$-a.e. m, the $X.$ component is μ-distributed under m, one easily sees that the expression in the conditional expectation in (1.12) is integrable (and meaningful) and that

$$E_{\overline{P}_\infty}[|G(m)|] \leq \text{const.}\|V\|_\infty\epsilon + \lim_k E_{N_k}[|G_\epsilon(\overline{X}_{N_k})|]\ .$$

Now by considering for $1 \le i \ne j \le N$, separately $X^i_\cdot$, $X^j_\cdot$, and the remaining group of $(N-2)$ particles one sees that

$$1(\tau_i \le t) - \sum_{j \ne i} 1(T_{i,j} \le t)\ 1(\tau^i_j > T_{i,j})\ 1(\tau^j_i > T_{i,j})$$

equals zero except maybe when $\tau_i = 0$, but anyway the quantity remains bounded in absolute value by $\sum_{j\ne i} 1(T_{i,j} = 0)$. Since $N\ s^d_N$ converges to zero, the expectation of this quantity is easily seen to go to zero. From this remark, applying now the Cauchy Schwarz inequality and symmetry, we find:

$$\begin{aligned}
&(\lim_k E_{N_k}[|G_\epsilon(\overline{X}_{N_k})|])^2 \le \\
&\overline{\lim}_N E_N[(N1(T_{1,2} \le t)\ 1(\tau^1_2 > T_{1,2})\ 1(\tau^2_1 > T_{1,2}) - c_d \int_0^{t\wedge\tau_1} V(s, X^1_s)\ ds \\
&\qquad + c_d \int_0^{(t\wedge\tau_1-\tau_2-\epsilon)_+} p_{s+\epsilon}(X^1_{\tau_2+s+\epsilon} - X^2_{\tau_2}) ds)\ h(X^1) \qquad (1.13)\\
&\qquad \text{(same expression with particles 3 and 4) } h(X^3)] \\
&\quad + \overline{\lim}_N O(\frac{1}{N})\ E_N[N^2 1(T_{1,2} \le t)\ 1(T_{2,3} \le t) + N\ 1(T_{1,2} \le t) + 1] \\
&\quad + \overline{\lim} O(\frac{1}{N^2})\ E_N[N^2\ 1(T_{1,2} \le t) + N1(T_{1,2} \le t) + 1]\ .
\end{aligned}$$

Now the last two terms are in fact zero. For instance integrating over X^1_0 and X^3_0, we have:

$$\begin{aligned}
&E_N[N^2\ 1(T_{1,2} \le t)\ 1(T_{2,3} \le t)] \le \\
&\|V\|^2_\infty E_N[N|W_{N,t}(X^2_0 + B^2_\cdot - B^1_\cdot)| \times N|W_{N,t}(X^2_0 + B^2_\cdot - B^3_\cdot)|]\ ,
\end{aligned}$$

where $W_{N,t}$ denotes the Wiener sausage of radius s_N, in time t of the process inside the brackets, and $|\cdot|$ Lebesgue volume. But $N|W_{N,t}(X^2_0 + B^2_\cdot - B^1_\cdot)| = N|W_{N,t}(B^2_\cdot - B^1_\cdot)|$ is bounded in any L^p, $p < \infty$, uniformly in N, using usual estimates on the volume of Wiener sausage. It follows that $\frac{1}{N} E_N[N^2 1(T_{1,2} \le t)\ 1(T_{2,3} \le t)]$ converges to zero. The other terms are treated similarly.

Introduce now $\overline{\tau}_i$ the destruction time of particle i, $1 \le i \le 4$, if one replaces the set $[1, N]$, by $\{i\} \cup [5, N]$, when defining the collisions. Observe now that for instance

$$\{t \wedge \tau^2_1 \ne t \wedge \bar{\tau}_1\} \subseteq A^{3,1}_{N-2,t} \cup A^{4,1}_{N-2,t}\ , \tag{1.14}$$

where the subscript $(N-2)$ refers to the fact that the k_i in definition (1.5) are now supposed to belong to $[5, N]$. Indeed one simply uses the last occurrence of 3, 4, in any chain reaction $\{T_{3,k_1} < \ldots < T_{k_p,1} \le t\}$, or $\{T_{4,k_1} < \ldots < T_{k_p,1} \le t\}$, since on the set where $\{t \wedge \tau_1^2 \neq t \wedge \bar{\tau}_1\}$, one such chain reaction occurs. Similarly

$$\{t \wedge \tau_1 \neq t \wedge \bar{\tau}_1\} \subseteq A^{2,1}_{N-2,t} \cup A^{3,1}_{N-2,t} \cup A^{4,1}_{N-2,t} \,.$$

Let us now consider the quantity obtained by replacing in the first expression in the right member of inequality (1.13), the τ_i^j and τ_i by $\bar{\tau}_i$, i,j in [1,4]. The claim is that once "$\overline{\lim}_N$" is performed one does not change anything. There is a somewhat tricky point about this, that we will describe now. For the full details, however we refer to [43]. The point is that when getting bounds on the difference of the two expressions one obtains terms like:

$$E_N[N^2 1(T_{1,2} \le t)\ 1(T_{3,4} \le t)\ 1_{A^{3,1}_{N-2,t}}]\ ,\ E_N[N\ 1(T_{1,2} \le t)\ 1_{A^{3,1}_{N-2,t}}]\ ,$$
$$\text{or } E_N[N\ 1(T_{3,4} \le t)\ 1_{A^{1,2}_{N-2,t}}]\ .$$

These terms go to zero with N because one can force volume of Wiener sausage in these terms. For instance, we can integrate over X_0^2 and X_0^4 in the first term and get an upper bound by $\|V\|_\infty^2 E_N[N|W_{N,t}(X_0^1 + B_\cdot^1 - B_\cdot^2)\ N|W_{N,t}(X_0^3 + B_\cdot^3 - B_\cdot^4)|\ 1_{A^{1,2}_{N-2,t}}]$, which is easily seen to go to zero,thanks to the usual estimates on the volume of the Wiener sausage and the fact that $E_N[A^{1,2}_{N-2,t}] \to 0$. The point is that in the difference one does not need to generate terms like:

$$E[N^2 1(T_{1,2} \le t)\ 1(T_{3,4} \le t)\ 1_{A^{4,3}_{N-2,t}}]\ ,\ \text{or}$$
$$E[N1(T_{1,2} \le t)\ 1_{A^{2,1}_{N-2,t}}]\ ,$$

for which we cannot use our previous reduction to an estimate on the volume of Wiener sausage. Indeed $\{T_{1,2} \le t\} \subset A^{2,1}_{N-2,t}$, and the last term for instance does not go to zero.

Our last reduction step is to observe (see [43]) that $N\ 1(T_{1,2} \le t)\ 1(\bar{\tau}_1 > T_{1,2})\ 1(\bar{\tau}_2 > T_{1,2}) = N\ 1(T_{1,2} \le t \wedge \bar{\tau}_1) - N\ 1(\bar{\tau}_2 \le T_{1,2} \le t \wedge \bar{\tau}_1)$ a.s. on the set where $\{T_{1,2} > 0\}$, and a similar equality holds for particles 3, 4, on $\{T_{3,4} > 0\}$. With this, one easily concludes that the first expression on the right of inequality (1.13) in fact equals:

$$\overline{\lim}_N E_N[(E_{\mu\otimes\mu}[\tilde{I}])^2]\ ,$$

$$\text{where } \tilde{I} = (N1(T_{1,2} \le t \wedge \bar{\tau}_1) - c_d \int_0^{t\wedge\bar{\tau}_1} V(s, X_s^1)ds)\ h(X^1) \tag{1.15}$$
$$- (N1(\bar{\tau}_2 \le T_{1,2} \le t \wedge \bar{\tau}_1) - c_d \int_0^{(t\wedge\bar{\tau}_1 - \bar{\tau}_2 - \epsilon)_+} p_{s+\epsilon}(X^1_{\bar{\tau}_2+s+\epsilon} - X^2_{\bar{\tau}_2})ds)\ h(X_\cdot^1)$$

In order to study expression (1.15), we now introduce the collision intensity for $w^1 \in C$ and $v \in M(R^d)$, $t \geq 0$:

$$C_t^N(w^1, v) = E_v^2[N\ 1\{T_{1,2} \leq t\}]\ . \tag{1.16}$$

The expectation E_v^2 in (1.16) is performed with respect to the variable w^2, with Wiener measure having initial condition v.

Lemma 1.6. *Let $f(\cdot)$ belong to $C_0(R^d)$, for $T > 0$:*

$$\lim_N \sup_{u \in M(R^d)} E_u^1[\sup_{x \in R^d, t \leq T} |C_t^N(w^1, f(y-x)dy) - c_d \int_0^t f(s, X_s(w^1) - x)ds|] = 0\ ,$$

where $f(s,x) = f\ *\ p_s(x)$.

Let us mention, although we will not really develop this point here, that one interest of the quantity $C_t^N(\cdot, v)$ is that it has nice limit properties for a wide range of processes, and does not really rely for its definition on the additive structure of R^d. For more details see [45].

Proof: Since C_t^N is nondecreasing in t and the limit $c_d \int_0^t f(s, X_s(w^1) - x)ds$ is, uniformly in x, continuous in t, by a well known technique (as in Dini's second theorem), it is enough to show that for fixed t

$$\lim_N \sup_u E_u^1[\sup_x |C_t^N(w^1, f(y-x)dy) - c_d \int_0^t f(s, X_s(w^1) - x)ds|] = 0\ .$$

Now for each N and u the quantity we consider is smaller than:

$$E_u^1 \otimes E_{\delta_0}^2[\sup_x |N \int f(y-x)dy\ 1\{\inf_{0\leq s\leq t} |X_0^1 + B_s^1 - B_s^2 - y| \leq s_N\} - c_d \int_0^t f(X_0^1 + B_s^1 - B_s^2 - x)ds|]$$

where $B_\cdot^i = X_\cdot^i - X_0^i$, $i = 1, 2$. But the last expression equals

$$E_{\delta_0}^1 \otimes E_{\delta_0}^2[\sup_x |N \int f(y-x)dy\ 1\{\inf_{0\leq s\leq t} |B_s^1 - B_s^2 - y| \leq s_N\} - c_d \int_0^t f(B_s^1 - B_s^2 - x)ds|]\ .$$

This latter quantity converges to zero by a refinement on the usual limit result for the Wiener sausage of small radius, for details see [43], Lemma 3.4.

$\square$

Observe now that for any $(X^5_\cdot, \ldots, X^N_\cdot)$:

$$\begin{aligned}
&|E_{\mu\otimes\mu}[(C^N_{t\wedge\bar\tau_1}(X^1,u_0) - c_d\int_0^{t\wedge\bar\tau_1} V(s,X^1_s)ds)\ h(X^1)]| \\
&= |E_\mu[(C^N_{t\wedge\bar\tau_1}(X^1,u_0) - c_d\int_0^{t\wedge\bar\tau_1} V(s,X^1_s)ds)\ h(X^1)]| \\
&\leq E_\mu[\sup_{s\leq t}|C^N_s(X^1,u_0) - c_d\int_0^s V(u,X^1_u)du|]\ \|h\|_\infty
\end{aligned}$$

which converges to zero as N goes to infinity by a slight variant of Lemma 1.6 (here u_0 does not necessarily have a density in $C_0(R^d)$).

So we see that we can replace $E_{\mu\otimes\mu}[\tilde I]$ in (1.15) by

$$E_{\mu\otimes\mu}[(N1(\bar\tau_2 \leq T_{1,2} \leq t\wedge\bar\tau_1) - c_d\int_0^{(t\wedge\bar\tau_1-\bar\tau_2-\epsilon)_+} p_{s+\epsilon}(X^1_{\bar\tau_2+s+\epsilon} - X^2_{\bar\tau_2})ds)\ h(X^1)] \tag{1.17}$$

without changing the limit result. The study of this last term is naturally more delicate. We cannot directly integrate over particle 2, and see a "collision intensity" term appear. The study of this last term will require some "surgery" on trajectories. Of course now $(X^5_\cdot, \ldots, X^N_\cdot)$ are held fixed and represent an outside random medium, and we are going to derive uniform estimates over this random medium. We start with the identity:

$$\begin{aligned}
1\{\bar\tau_2 \leq T_{1,2} \leq t\wedge\bar\tau_1\} =& 1\{\bar\tau_2 \leq T_{1,2} \leq t\wedge\bar\tau_1\} \times 1\{T_{1,2} < \bar\tau_2 + \epsilon\} \\
&+ 1\{\bar\tau_2 + \epsilon \leq T_{1,2} \leq t\wedge\bar\tau_1\}
\end{aligned}$$

Moreoever we can write:

$$\begin{aligned}
1\{\bar\tau_2 + \epsilon \leq T_{1,2} \leq t\wedge\bar\tau_1\} =& (1 - 1\{\bar\tau_2 + \epsilon > t\wedge\bar\tau_1\}) \\
&\times (1 - 1\{t\wedge\bar\tau_2 < T_{1,2} < t\wedge\bar\tau_2 + \epsilon\} - 1\{T_{1,2} \leq t\wedge\bar\tau_2\}) \\
&1\{T_{1,2}\circ\theta_\epsilon\circ\theta_{t\wedge\bar\tau_2} \leq (t\wedge\bar\tau_1 - \bar\tau_2 - \epsilon)_+\}\ .
\end{aligned}$$

Using this decomposition it follows that the absolute value of the expression (1.17) is bounded by:

$$\begin{aligned}
&E_{\mu\otimes\mu}[(N1\{T_{1,2}\circ\theta_\epsilon\circ\theta_{t\wedge\bar\tau_2} \leq (t\wedge\bar\tau_1 - \bar\tau_2 - \epsilon)_+\} \\
&- c_d\int_0^{(t\wedge\bar\tau_1-\bar\tau_2-\epsilon)_+} p_{s+\epsilon}(X^1_{\bar\tau_2+s+\epsilon} - X^2_{\bar\tau_2})ds)h(X^1)]| \\
&+ \|h\|_\infty(2E_{\mu\otimes\mu}[N1\{\bar\tau_2\wedge t \leq T_{1,2} \leq \bar\tau_2\wedge t + \epsilon\}] \\
&\quad + E_{\mu\otimes\mu}[N1\{T_{1,2}\circ\theta_\epsilon\circ\theta_{t\wedge\bar\tau_2} = 0\}] \\
&\quad E_{\mu\otimes\mu}[N1\{T_{1,2}\circ t\wedge\bar\tau_2\}1\{T_{1,2}\circ\theta_{t\wedge\bar\tau_2+\epsilon} \leq t\}])
\end{aligned} \tag{1.18}$$

It is now fairly standard to see that the last two terms converge to zero (uniformly over $X^5_\cdot, \dots, X^N_\cdot$)). As for the second term integrating out the initial condition X^1_0, it is bounded by

$$2\|h\|_\infty E_{\mu\otimes\mu}[N \int u_0(dy)\ 1\{y \in W_{N,\epsilon}(X^2_{t\wedge\bar\tau_2} + \tilde B^2_\cdot - \tilde B^1_\cdot - B^1_{t\wedge\bar\tau_2})\}] ,$$

where $\tilde B^i_\cdot = X^i_{t\wedge\bar\tau_2+\cdot} - X^i_{t\wedge\bar\tau_2}$, $i = 1, 2$, are standard independent Brownian motions. This last quantity is smaller than $2\|h\|_\infty\|V\|_\infty E_{\mu\otimes\mu}\ [N|W_{N,\epsilon}(B^2_\cdot - B^1_\cdot)|]$ which converges to $2\|h\|_\infty\|v\|_\infty c_d\epsilon$, as N converges to infinity.

Let us now look at the first term of (1.18). Conditioning over $X^1_\cdot$, we have

$$\begin{aligned}
&E^2_\mu[N1\{T_{1,2}\circ\theta_\epsilon\circ\theta_{t\wedge\bar\tau_2} \le (t\wedge\bar\tau_1 - \bar\tau_2 - \epsilon)_+\}] \\
&= E^2_\mu[N\tilde E^2_{X^2_{t\wedge\bar\tau_2}}[1\{T_{1,2}\circ\theta_\epsilon(w^1_{t\wedge\bar\tau_2+\cdot}, \tilde w^2_\cdot) \le (t\wedge\bar\tau_1 - \bar\tau_2 - \epsilon)_+\}]] \\
&= E^2_\mu[C^N_{(t\wedge\bar\tau_1-\bar\tau_2-\epsilon)_+}(X^1_{t\wedge\bar\tau_2+\epsilon+\cdot}, p_\epsilon(y - X^2_{t\wedge\bar\tau_2})dy)] .
\end{aligned}$$

After this surgery the first term of (1.18) is now

$$\begin{aligned}
&|E_{\mu\otimes\mu}[(C^N_{t\wedge\bar\tau_1-\bar\tau_2-\epsilon)_+}(X^1_{t\wedge\bar\tau_2+\epsilon+\cdot}, p_\epsilon(y - X^2_{t\wedge\bar\tau_2})dy) \\
&- c_d\int_0^{(t\wedge\bar\tau_1-\bar\tau_2-\epsilon)_+} p_{s+\epsilon}(X^1_{t\wedge\bar\tau_2+s+\epsilon} - X^2_{t\wedge\bar\tau_2})ds)\ h(X^1)]| \\
&\le \|h\|_\infty E_{\mu\otimes\mu}[\sup_{v\le t,x} |C^N_v(X^1_{t\wedge\bar\tau_2+\epsilon+\cdot}, p_\epsilon(y-x)dy) - c_d\int_0^v p_{s+\epsilon}(X^1_{t\wedge\bar\tau_2+s+\epsilon} - x)ds|] \\
&\le \|h\|_\infty \sup_{u\in M(\mathbb{R}^d)} E^1_u[\sup_{v\le t,x} |C^N_v(X^1_\cdot, p_\epsilon(y-x)dy) - c_d\int_0^v p_{s+\epsilon}(X^1_s - x)ds|]
\end{aligned}$$

which is independent of $(X^5_\cdot, \dots, X^N_\cdot)$ and converges to zero as N goes to infinity thanks to Lemma 1.6.

So we have proved that

$$E_{\bar P_\infty}[|G(m)|] \le \text{const. } \|V\|_\infty\epsilon + 2\|h\|_\infty\|V\|_\infty\epsilon .$$

Letting ϵ go to zero, we obtain our claim.

□

As we mentioned previously, $\tilde E_m[\int_0^{(t\wedge\tau-\tilde\tau)_+} p_s(X_{\tilde\tau+s} - \tilde X_{\tilde\tau})ds]$ $= \int_0^t 1(s<\tau)\tilde E_m[1(\tilde\tau < s)\ p_{s-\tilde\tau}(X_s - \tilde X_{\tilde\tau})]ds$. So in view of the characterization of the law P given by Lemma 1.2, the fact that $\overline{P}_\infty = \delta_P$, will follow from

Proposition 1.6. *For $\overline{P}_\infty$-a.e. m, for all $s \ge 0$, and $f \in bB(R^d)$:*

$$\int f(x)\ E_m[p_{s-\tau}(x - X_\tau)\ 1(\tau < s)]\ dx = E_m[f(X_s)\ 1(s > \tau)] . \tag{1.19}$$

Proof: By the usual arguments we simply will check (1.19), for $s \in (0,\infty) \setminus D$, and $f \in C_b(R^d)$. We recall that $D = \{t : I(\overline{P}_\infty)[\{t = \tau\}] \neq 0\}$. Setting $H(m) = \langle m, (P_{(s-\tau)_+} f(X_\tau) - f(X_s)\ 1(\tau < s)\rangle$, we simply want to check that $H(m) = 0$ $\overline{P}_\infty$-a.s. Now

$$E_{\overline{P}_\infty}[H(m)^2] =$$
$$\lim_k E_{N_k}[(P_{(s-\tau_1)_+} f(X^1_{\tau_1}) - f(X^1_s))(P_{(s-\tau_2)_+} f(X^2_{\tau_2}) - f(X^2_s))\ 1(\tau_1 < s)\ 1(\tau_2 < s)]\ .$$

However on $(A^{1,2}_{N,s} \cup A^{2,1}_{N,s})^c$, $\tau_1 \wedge s = \tau_1^2 \wedge s$, and $\tau_2 \wedge s = \tau_2^1 \wedge s$. It follows that

$$E_{\overline{P}_\infty}[H(m)^2] = \lim_k E_{N_k}[E^1_\mu[(P_{(s-\tau_1^2)_+} f(X^1_{\tau_1^2}) - f(X^1_s))1(\tau_1^2 < s)]^2] = 0\ ,$$

by an application of the strong Markov property at time τ_1^2, under the law E^1_μ.

□

This now concludes the proof of Theorem 1.1. In the next section, we are going to give a different line of explanation on why Theorem 1.1 holds.

2. Limit picture for chain reactions in the constant mean free travel time regime.

The result which was presented in the last section is certainly satisfactory from a purely propagation of chaos frame of reference, however it misses the nice limit algebraic picture, corresponding to the "constant mean free travel time" regime, which underlies the result. We will somehow try to motivate and describe this limit picture.

The notion of chain reactions, leading to a specific particle, say particle one, corresponding to definition (1.5) played an important role in the derivation of Theorem 1.1. It simply involves the independent Brownian trajectories and completely forgets about destruction of particles. The idea we will follow here is that one should investigate the limit structure of the chain reactions leading to particle 1, in the constant mean free travel time regime. One then should construct the interaction (annihilation) on the limit object and somehow recapture the result of Theorem 1.1 in this scheme.

As a step in the study of the limit aspect of this somewhat loose notion of chain reaction between independent particles leading to particle 1, one may look at the limiting aspect of the first collision. For each $i \in [1, N]$, we set

$$T_i = \inf_{j \neq i} T_{i,j}\ , \quad \text{when} \quad \inf_{j \neq i} T_{i,j} \leq 1\ , \quad +\infty \text{ otherwise,}$$

and $Y_{\cdot}^i$ "the first colliding trajectory", is defined on the set $0 < T_i \leq 1$, as X^j where j is the only index such that $T_i = T_{i,j}$, and otherwise, it is set to the constant trajectory 0.

Let us quote the following propagation of chaos result from [45].

Theorem 2.1. *The $(X_{\cdot}^i, T_i, Y_{\cdot}^i)_{1\leq i\leq N}$ are Q-chaotic where Q is the law on $C \times ([0,1] \cup \{\infty\}) \times C$ such that under Q*

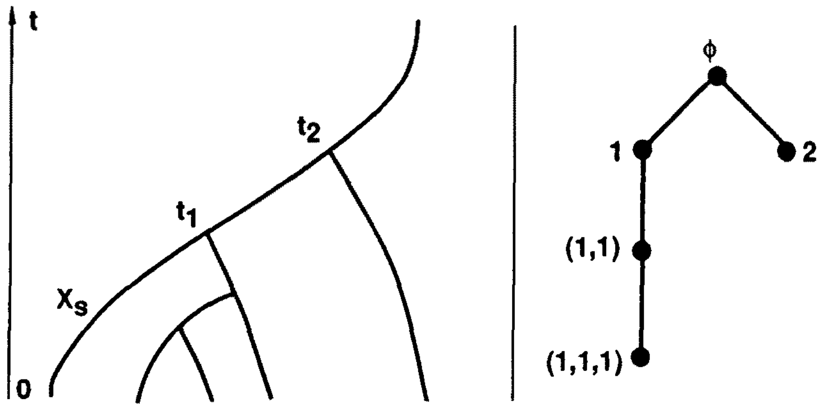
- *X has the distribution of a Brownian motion with initial law u_0.*
- $Q[T > t|X] = \exp\{-c_d \int_0^{t\wedge 1} V(s, X_s)ds\}$, $\quad t \geq 0$.
- *Conditionally on X, T, $0 < T \leq 1$, $Y_{\cdot}$ is distributed as a Brownian motion with initial law u_0 conditioned to be equal to X_T at time T, and otherwise it is the constant trajectory 0.*

This result motivates that the natural limit "chain reaction tree" should be constructed in the following way: One starts with one Brownian particle X_s, the ancestor, running until time t, with initial density u_0 (having the role of X^1). One then constructs the trajectories having a collision with the ancestor: conditionally on X_s, one picks a Poisson distribution of points on $[0,t]$ with intensity $c_d V(s, X_s)ds$. Now conditionally on the times $0 < t_1 < \ldots < t_n < t$, one considers independent Brownian bridges: $W^1, \ldots, W^n$ where W_ℓ for $\ell \leq n$ has the law of Brownian motion in time $[0, t_\ell]$, with initial distribution u_0 conditioned to be equal to X_{t_ℓ} at time t_ℓ. These $W^1, \ldots, W^n$ constitute the first generation in the chain reaction leading to $X_{\cdot}$. Note that we disregard the trajectory of $W_{\cdot}^\ell$, after time t_ℓ, since Theorem 1.1 indicates that there should not be any recollision in the limit picture.

Now one performs the same thing on each of the first generation trajectories W^ℓ, $1 \leq \ell \leq n$, as just described on $X_{\cdot}$. One obtains in this way the second generation of trajectories, and so on.

We are now going to give a precise description of this limit tree of chain reactions. As the explanation above indicates we will build a marked tree. Each "individual" being marked by a trajectory. We use Neveu's notion of tree [33], namely a tree is a subset of the set of finite sequences $U = \cup_{k\geq 0}(\mathbb{N}^*)^k$, (the vertices) containing the sequence $\emptyset$ (only element of $(\mathbb{N}^*)^0$), such that $u \in \pi$ when $uj \in \pi$, $j \in \mathbb{N}^*$, and for $u \in \pi$ there is a $\nu_u \in \mathbb{N}$, with $uj \in \pi$ if and only if $1 \leq j \leq \nu_u$. We will be interested in marked trees that is $\omega = (\pi, (\phi_u, u \in \pi))$, where $\phi_u \in D$, D the set of marks which is for us $\cup_{t>0} C([0,t], \mathbb{R}^d) \sim (0, +\infty) \times C([0,1], \mathbb{R}^d)$.

trajectorial picture: marked tree description:



Following Neveu's notations, for the marked tree ω and $u \in \pi$, $T_u\omega$ is the tree translated at u, G_n is the n^{th} generation of the tree: $\pi \cap (\mathbb{N}^*)^n$, and F_n the σ-field generated by G_k, $0 \le k \le n$, and ϕ^u, $u \in G_k$, $0 \le k \le n$. For a trajectory $\psi_\cdot \in D$, we construct the probability R_ψ on the set of marked trees Ω which satisfies

- $R_\psi[\phi^\emptyset = \psi] = 1$ (the ancestor's mark is ψ).
- For f_u, $u \in U$ a collection of nonnegative measurable functions on Ω:

$$E_\psi[\prod_{u\in G_n} f_u \circ T_u \;/\; F_n] = \prod_{u \in G_n} E_{\phi^u}[f_u]$$

(Branching property).

The last requirement which describes the reproduction law is given in a somewhat more synthetic form than what was explained before:

- The random point measure on D: $\sum_{1\le \ell \le \nu_\emptyset} \delta_{\phi^\ell}$ is a Poisson point process with intensity measure

$$\text{(2.1)} \qquad \int_0^t ds\; V(s,\psi_s)\; P^{s,\psi_s}(d\phi)\,,$$

and $0 < \tau(\phi^1) < \cdots < \tau(\phi^\emptyset) < t = \tau(\psi)$, a.s. Here $\tau(\phi)$ denotes the time duration of a particle ($(0,\infty)$ valued component on D), and $P^{s,x}$ is the law of Brownian motion with initial u_0 on time $[0,s]$, conditioned to be equal to x at time s. We have normalized here c_d as being equal to 1.

Lemma 2.1.

- R_ψ-a.s. *the tree ω is finite.*
- $M(\omega) = \sum_{1\le \ell\le \nu_\emptyset} \delta_{T_\ell(\omega)}$ *(random point measure on Ω)*

is a Poisson point measure with intensity

$$Q_\psi = \int_0^t ds\ V(s,\psi_s)\ R^{s,\psi_s}\ ds\ , \tag{2.2}$$

where $R^{t,x} = \int R_\psi\ P^{t,x}(d\psi)$.

Proof: Let us check the first point: for $0 \le p \le n$, set

$$v_p = E_\psi[\sum_{u\in G_{n-p}} \frac{1}{p!}(c\ \tau(\phi^u))^p]\ , \quad \text{with} \quad c = \|V\|_\infty\ .$$

so that $v_0 = E_\psi[\#G_n]$ and $v_n = \frac{1}{n!}(ct)^n$. Then for $0 \le p < n$:

$$\begin{aligned} v_p &= E_\psi[E_\psi[\sum_{\substack{u\in G_{n-p-1}\\ uj\in G_{n-p}}} \frac{1}{p!}(c\ \tau(\phi^{uj}))^p / F_{n-p-1}]] \\ &= E_\psi[\sum_{u\in G_{n-p-1}} E_{\phi^u}[\sum_{1\le j\le \nu_\emptyset} \frac{1}{p!}(c\ \tau(\phi^j))^p]] \\ &= E_\psi[\sum_{u\in G_{n-p-1}} \int_0^{\tau(\phi^u)} \frac{1}{p!}(ct)^p\ V(t,\phi^u_t)dt] \le v_{p+1}\ . \end{aligned}$$

From this $E_\psi[\#G_n] \le \frac{1}{n!}(ct)^n$, and summing over n, we find the result. As for the second statement, if F is a positive function on Ω:

$$\begin{aligned} E_\psi[\exp\{-\langle M,F\rangle\}] &= E_\psi[\prod_{1\le \ell\le \nu_\emptyset} e^{-F(T_\ell(\omega))}] \\ &= E_\psi[\prod_{1\le \ell\le \nu_\emptyset} R_{\phi^\ell}[e^{-F}]] \quad \text{by the branching property)} \\ &= \exp\{\int_0^t V(s,\psi_s)ds \int dP^{s,\psi_s}(\phi)\ (R_\phi[e^{-F}]-1)) \quad \text{by (2.1)}), \\ &= \exp\{\int dQ(e^{-F}-1)\} \end{aligned}$$

which proves our claim.

□

Our tree ω is R_ψ a.s. finite. Now we can build the interaction on the limit tree corresponding to annihilation by adding a mark Z^u, $u \in \pi$, equal to zero or 1. $Z^u = 0$, will mean that the particle with trajectory ϕ^u is already destroyed by the time $\tau(\phi^u)$ at which it meets its direct ancestor, whereas $Z^u = 1$, will mean that it has not yet been destroyed.

So it is quite natural to impose the following recursive rule, to determine Z^u, $u \in \omega$: If $\nu_u = 0$ (no descendents), $Z^u = 1$. If $Z^{uj} = 1$, for some $1 \le j \le \nu_u$ (one of the direct descendents is alive), then $Z^u = 0$.

We now define $R^t = \int R^{t,x}\ V(t,x)\ dx$, for which the ancestor trajectory is distributed as a Brownian motion in time $[0,t]$, with initial condition u_0. Then we set

$$u(t,x) = V(t,x)\ R^{t,x}[1(Z^\emptyset = 1)]\ , \quad t > 0\ . \tag{2.3}$$

In fact u is the density of presence of ψ_t, under R^t, when the ancestor trajectory $\psi_.$ is not already destroyed at time t.

Lemma 2.2. *For $f \in bB(R^d)$:*

$$\int u(t,x)\ f(x)\ dx = E_{R^t}[f(\psi_t)\ 1(Z^\emptyset = 1)]\ .$$

Proof:

$$\begin{aligned}\int u(t,x)\ f(x) dx &= \int V(t,x)\ R^{t,x}[Z^\emptyset = 1]\ f(x)\ dx \\ &= \int V(t,x) dx \int P^{t,x}(d\psi)\ R_\psi[Z = 1]\ f(\psi_t) \\ &= E_{R^t}[f(\psi_t)\ 1(Z^\emptyset = 1)]\ .\end{aligned}$$

□

The result we show now tells us that one obtains the nonlinear equation (in integral form) (1.2), by constructing the interaction directly on the "limit chain reaction tree".

Theorem 2.3. $u(t,x)$, $0 \le t \le T$, *is the solution in $L^\infty([0,T] \times R^d, ds\ dm)$ of the integral equation:*

$$w(t,x) = V(t,x) - \int_0^t \int w^2(s,y)\ p_{t-s}(y,x)\ ds\ dy\ . \tag{2.4}$$

Proof: The uniqueness is standard, see [44] for details. Let us check that u is a solution of (2.4). Observe that

$$\begin{aligned}R_\psi[Z^\emptyset = 1] &= \exp\{-\int_0^t V(s,\psi_s)\ R^{s,\psi_s}[Z^\emptyset = 1] ds\} \\ &= \exp\{-\int_0^t u(s,\psi_s) ds\}\ .\end{aligned} \tag{2.5}$$

Then

$$
\begin{aligned}
u(t,x) &= V(t,x)\ R^{t,x}[Z=1] \\
&= V(t,x)\Big(1-\int_0^t ds\ P^{t,x}(d\psi)\ u(s,\psi_s)\ \exp\{-\int_0^s u(r,\psi_r)dr\}\Big) \\
&= V(t,x) - \int_0^t ds \int dP^t(\psi) p_{t-s}(\psi_s,x)\ u(s,\psi_s)\ \exp\{-\int_0^s u(r,\psi_r)dr\} \\
&= V(t,x) - \int_0^t ds \int dy\ V(s,y)\ p_{t-s}(y,x)\ u(s,y) \int P^{s,y}(d\psi)\exp\{-\int_0^s u(r,\psi_r)dr\} \\
&= V(t,x) - \int_0^t ds \int dy\ p_{t-s}(y,x)\ u^2(s,y)\ ,
\end{aligned}
$$

which shows that u satisfies (2.4).

□

3) Some comments

Let us finally give some comments on the results presented in sections 1 and 2.

- The results presented in section 2, strongly suggest that in fact when one studies collisions (without any destructions) between independent particles $(X^1_\cdot, \dots, X^N_\cdot)$, a better result than Theorem 2.1 should hold. Somehow one should have a propagation of chaos result at the level of the "trees of chain reactions" leading to each particle X^i, $1 \le i \le N$. The natural "limit" should in fact be precisely the noninteracting marked tree presented in section 2.
- The proof of Theorem 2.3 is very algebraic. The fact that equation (2.4) arises when one constructs the interaction directly on the marked tree remains true when one applies the same construction to a "general Markov process", (see [44]). One then obtains the integral form of the nonlinear equation:

$$\partial_t u = L^* u - u^2 \ , \tag{3.1}$$

where L is the formal generator of the Markov process. In fact by a slight variation on the construction of the marked tree, and of the interaction, one obtains the integral equation corresponding to

$$\partial_t u = L^* u - u^{k+1} \ , \quad (k \ge 1, \text{ integer}). \tag{3.2}$$

- Not only does the construction work for a "general Markov process" but for Brownian motion in one dimension as well! Consider for instance in this case, the random times $0 < t_1 < \cdots < t_n < t$, picked with a Poisson distribution of intensity $V(s, X_s)ds$, on $[0,t]$, when X_s, $s \in [0,t]$ is the ancestor trajectory. They are not at all the first "collision times" of $X_\cdot$ with the first generation trajectories $W^1_\cdot, \dots, W^n_\cdot$. Indeed the $W^\ell_\cdot$ are distributed as independent bridges, being conditioned to be equal to X_{t_ℓ} at time t_ℓ. Of course in dimension $d \ge 2$, the trajectory $W^\ell_\cdot$ meets the ancestor trajectory $X_\cdot$ only at time t_ℓ, but in dimension $d = 1$, this need not be the case.

  This point should be viewed as the fact that the limit structure constructed in section 2, exists very generally whether or not it comes as a limit picture from an approximating constant mean free travel time regime.
- One may wonder what the appropriate collision regime should be if one tries to handle $(k+1)$-particle collisions with possible limit survival equation (3.2). As mentioned previously in Remark 1.2 5), the right guess should not be dictated by

a notion of distance of interaction. Much more naturally it should be picked as a suitable level set in the $(k+1)$ particle configuration space of some potential generated by a measure sitting on the "diagonal" (which is polar) (see [45]). A plausible guess would be to look at the set $h(x_1, \dots, x_{k+1}) \geq N^k$, where

$$h(x_1, \dots, x_{k+1}) = \int_0^\infty e^{-s} ds \int_{R^d} \prod_1^{k+1} p_s(x_i, z)\, dz = g_1^{(kd)}(D)\ .$$

where $g_1^{(kd)}(|x - x'|)$ is the 1-Green's function of Brownian motion in R^{kd} and D is the distance of $(x_1, \dots, x_{k+1})$ to the diagonal $\{(z, \dots, z)\}$ in $(R^d)^{k+1}$. If one wants to benefit from Brownian scaling, in dimension d, with $kd > 2$, one can use instead of h,

$$f(x_1, \dots, x_{k+1}) = \int_0^\infty ds \int_{R^d} \prod_1^{k+1} p_s(x_i, z)\, dz = \frac{2}{c_{kd}} D^{2-kd}\ ,$$

($c_{kd} = (kd-2)vol(S_{kd})$), which is continuous with values in $(0, \infty]$, finite except on the diagonal $x_1 = \cdots = x_{k+1}$, and homogeneous of degree $2 - kd$. The level set $\{f \geq N^k\}$ is then the homothetic of ratio $N^{-1/(d-2/k)}$ of the set $\{f \geq 1\}$.

IV. Uniqueness for the Boltzmann process

In this chapter, we are going to explain how ideas somewhat similar to those used in the "tree construction" of Chapter III, section 2, can be put at work to produce a uniqueness result of the nonlinear process associated to the spatially homogeneous Boltzmann equation for hard spheres.

The spatially homogeneous Boltzmann equation for hard spheres describes the time evolution of the density $u(t,v)$ of particles with velocity v, in a dilute gas, under an assumption of spectral homogeneity and hard spheres collisions as:

$$\begin{aligned}
\partial_t u &= \int_{\boldsymbol{R}^n \times S_n} (u(t,\tilde{v})u(t,\tilde{v}') - u(t,v)\, u(t,v'))\ |(v'-v)\cdot n|\ dv'\ dn \\
&\text{with } \tilde{v} = v + (v'-v)\cdot nn \\
&\qquad \tilde{v}' = v' + (v-v')\cdot nn
\end{aligned}$$

If f is some "nice test function", disregarding integrability problems, a change of variable yields:

$$\partial_t \langle u,\ f\rangle = \langle u\otimes u\ ,\ (f(v+(v'-v)\cdot nn) - f(v))\ |(v'-v)\cdot n|\rangle\ .$$

On this latter form, the equation naturally appears as the forward equation of a "non-linear jump process", with Levy system:

$$\int M_t(v,d\tilde{v})\ h(\tilde{v}) = \int_{\boldsymbol{R}^n\times S_n} h((v'-v)\cdot nn)\ |(v'-v)\cdot n|\ u(t,v')\ dv'\ dn$$

In section 2 we will introduce the nonlinear process as the solution to a certain (non-linear) martingale problem. The unboundedness of the factor $|(v-v')\cdot n|$ will create a serious source of difficulty in seeing that this martingale problem is well posed.

1) Wild's formula

The theme of this section will be a formula of Wild [56], for the solution of the spatially homogeneous Boltzmann equation, which for instance covers the case of "cut-off hard spheres" (that is a collision intensity $|(v'-v)\cdot n|\wedge C$). We will also give some probabilistic interpretations of the formula for closely related to the formula.

The setting is the following: we suppose that we have a Markovian kernel Q_1 : $\boldsymbol{R}^n\times\boldsymbol{R}^n\times\boldsymbol{R}^n$, and for μ_1,μ_2 two probabilities (or two bounded measures) on $\boldsymbol{R}^n$, we define the probability (a bounded measure) $\mu_1\circ\mu_2$ as:

$$(1.1)\qquad \langle \mu_1\circ\mu_2\rangle = \langle \mu_1\otimes\mu_2, Q_1 f\rangle\ .$$

It is clear that for the variation norm we have the estimate $\|u_1 \circ u_2\| \leq \|u_1\| \, \|u_2\|$. Wild's formula will give a series expansion for the solution of

$$\begin{aligned} &\partial_t u = u \circ u - u \\ &u_{t=0} = u_0 \in M(R^n) . \end{aligned} \tag{1.2}$$

The spatially homogeneous Boltzmann equation for hard spheres with cutoff collision kernel $|(v'-v)\cdot n| \wedge 1$, for instance, will correspond to the choice of kernel:

$$\begin{aligned} Q_1 f(v,v') = \int_0^1 d\alpha \int_{S_n} dn[f(v+(v'-v)\cdot nn) \; 1(\alpha \leq |(v'-v)\cdot n| \wedge 1) \\ + f(v) \; 1(\alpha > |(v'-v)\cdot n| \wedge 1)] . \end{aligned}$$

Proposition 1.1. *For any $u_0 \in M(R^n)$, here is a unique strongly continuous solution of*

$$u_t - u_0 = \int_0^t (u_s \circ u_s - u_s) \, ds , \tag{1.3}$$

given by Wild's sum:

$$u_t = e^{-t} \sum_{k \geq 1} (1-e^{-t})^{k-1} \; u^k , \tag{1.4}$$

where $u^1 = u_0$ and $u^{n+1} = \frac{1}{n} \sum u^k \circ u^{n+1-k}$.

Proof: The uniqueness part follows from a classical O.D.E. result. Let us show that (1.4) does provide a solution to (1.3).

Define $u_t^1 \equiv u_0$, and for $n \geq 1$,

$$u_t^{n+1} = e^{-t} u_0 + \int_0^t e^{-(t-s)} \; u_s^n \circ u_s^n \; ds , \tag{1.5}$$

then by induction one sees that for $n \geq 1$:

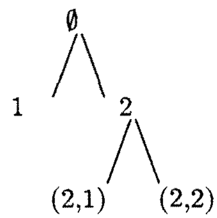
$$u_t^n \geq e^{-t} \sum_1^n (1-e^{-t})^{k-1} \; u^k .$$

From this one easily concludes that u_t^n converges uniformly on bounded time intervals in variation norm to u_t given by (1.4). Because of (1.5), u_t is a solution of (1.3).

□

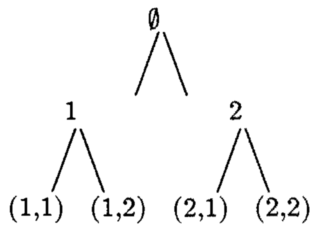
Wild's formula expresses u_t as a barycenter of the sequence u^n, with weights $e^{-t}(1-e^{-t})^{n-1}$. It can in several instances be used to derive asymptotic properties of u_t from those of the sequence u^n, see Ferland-Giroux [10], McKean [25], Tanaka [49], Murata-Tanaka [30].

Wild's formula has a nice interpretation in terms of continuous time binary branching trees, see also Ueno [55], [56]. Indeed $[e^{-t}(1-e^{-t})^{n-1}]_{n\geq 1}$, is the distribution of the total number of particles of such a branching tree at time t, provided each particle branches with unit intensity. For each n, u^n is a convex combination of the various ways of inserting parentheses in a monomial $u_0 \circ \cdots \circ u_0$ of degree n. The ways of inserting parentheses in such a monomial of degree n are in natural correspondence with the tree subsets (in the sense given in section 2 of Chapter III) of the set of vertices $V = \cup_{k\geq 0}\{1,2\}^k$, which possesses exactly n bottom vertices (with no descendants). For instance $u \circ (u \circ u)$ is associated to



$(u \circ u) \circ (u \circ u)$ is associated to

(1.6)



The coefficients appearing in the convex combinations expressing u^n in terms of the various ways of inserting parentheses in a monomial of degree n of u yield precisely the conditional time binary tree, given that it has n bottom vertices at time t.

On the other hand one can also keep track of the time order at which branching occurs. For instance in (1.6) one distinguishes between two ordered trees depending on whether 1 or 2 branched first. Looking at the skeleton of successive jumps, it is easy to see that conditionally on the fact that there are n individuals at time t, there are $(n-1)!$ equally likely such ordered trees.

If one uses a perturbation expansion of

$$u_t = u_0 e^{-t} + \int_0^t e^{-(t-s)}\, u_s \circ u_s \, ds \ ,$$

one finds

(1.7)

$$u_t = u_0 e^{-t} + e^{-t}(1-e^{-t}) u_0 \circ u_0 + \cdots + \frac{e^{-t}}{(n-2)!}(1-e^{-t})^{n-2}\ u_0^{(n-1)}$$
$$+ e^{-t} \int_{0,s_{n-1}<\ldots<s_1<t} ds_{n-1} \ldots ds_1\ u_{s_{n-1}}^{(n)}\ .$$

Here for $v \in M(R^n)$, $v^{(n)}$ is defined as:

$$v^{(n)} = \sum_{\sigma \in S_{n-1}} v \underset{\sigma(1)}{\circ} v \underset{\sigma(2)}{\circ} \cdots \underset{\sigma(n-1)}{\circ} v$$

and the permutation σ dictates the order in which the operation $\circ$ is performed. Observe for instance that $(u \circ u) \circ (u \circ u)$ corresponds both to $u \overset{\circ}{_1} u \overset{\circ}{_3} u \overset{\circ}{_2} u$ and to $u \underset{2}{\circ}\ u \underset{3}{\circ}\ u \underset{1}{\circ}\ u$.

Of course by letting n go to infinity in formula (1.7), one finds (since the last term converges to zero):

$$u_t = \sum_{k \geq 1} \frac{e^{-t}}{(k-1)!}(1-e^{-t})^{k-1}\ u_0^{(n-1)}\ , \tag{1.8}$$

which only differs from Wild's formula because one uses the ordered trees as operation schemes. In Wild's formla one precisely forgets the ordering and simply keeps track of the skeleton subset of $\cup_{k \geq 0}\{1,2\}^k$ induced by the continuous time binary tree.

So we have somewhat informally presented the interpretation of Wild's formula in terms of a continuous time binary branching tree.

We are now going to explain a slightly modified point of view, which will be parallel to the construction of section 2 of Chapter III, and also close in spirit to the uniqueness proof for the Boltzmann process which will be provided in the next section.

First we suppose that we have a measurable map, $\psi(v,v',y)$, where y belongs to an auxiliary Polish space E, and a probability $\nu(dy)$, such that

$$Q_1 f(v,v') = \int_E f(\psi(v,v',y))\ d\nu(y)\ . \tag{1.9}$$

Here the variable y plays the role of a collision parameter.

We basically keep the notations of section 2 Chapter III, and as a first step we will construct a "noninteracting tree". As before we consider Ω the set of marked trees $\omega = (\pi, (\phi^u,\ u \in T))$, where the marks ϕ now take their values in $D = \{(\tau,v,y) \in (0,\infty) \times R^n \times E\}$. Here τ represents a trajectory duration, v an initial velocity and y a collision parameter.

Now for any $(\tau,v,y) \in D$, we consider the probability $R_{(\tau,v,y)}$ on Ω, such that:

- $R_{(\tau,v,y)}[\phi^{\emptyset} = (\tau, v, y)] = 1$ (the ancestor's mark is (τ, v, y)).
- For f_u, $u \in U$ a collection of nonnegative measurable functions on Ω,

$$E_{(\tau,v,y)}[\prod_{u\in G_n} f_u \circ T_u/F_n] = \prod_{u\in G_n} E_{\phi^u}[f_u]$$

(Branching property).
- The reproduction law is given by the fact that the point measure on $D : \sum_{1\le \ell\le \nu_\emptyset} \delta_{\emptyset^\ell}$ is a Poisson point process with intensity measure on D:

$$1(s < \tau)\; ds \otimes u_0(dv) \otimes \nu(dy)\ ,$$

and $R_{(\tau,v,y)}$- a.s., $0 < \tau^1 < \ldots < \tau^{\alpha(1)} < \tau$.

Now very similarly to section 1 of Chapter III, we have:

Lemma 1.2.

- $R_{(\tau,v,y)}$-a.s., ω is a finite tree.
- Under $R_{(\tau,v,y)}$, the random point measure on $\Omega : M(\omega) = \sum_{1\le \ell\le \nu_\phi} \delta_{T_\ell(\omega)}$, is a Poisson point measure with intensity

$$Q_{(\tau,v,y)} = \int_{[0,\tau]\times \boldsymbol{R}^n \times E} d\tau'\, du_0(v')\, d\nu(y')\, R_{\tau',v',y'}\ .$$

We can now construct the interaction on the tree $\omega = (\pi, (\tau^u, v^u, y^u),\ u \in \pi)$ as a supplementary mark x^u, which will represent the velocity of the particle $u \in \pi$, at time τ^u. The mark v^u corresponds to the initial velocity of this particle and the collision parameter y^u, will be used in calculating the effect of paricle u, on its direct ancestor.

More precisely, using the fact that the tree is a.s. finite, we set: $z^u = v^u$ if u has no descendants, $z^u = z_{\nu_u}$, otherwise, where the sequence $z_j,\ 0 \le j \le \nu_u$ is defined by:

$$\begin{aligned} &z_j = \psi(z_j,\ z^{uj}, v^{uj})\ , \quad 1 \le j \le \nu_u \\ &z_0 = v^u\ . \end{aligned} \tag{1.10}$$

We now denote by R^t for $t > 0$, the measure $\int du_0(v)\ d\nu(y)\ R_{t,v,y}$. With this notation, Lemma 1.2 says that $M(\omega)$ under R^t is a Poisson point process with intensity measure $\int_0^t ds\ R^s$.

The corresponding result to Theorem 2.3 of section 2, now tells us that under R^t, the law of the supplementary mark $z^{\emptyset}$, solves the equation (1.3).

Proposition 1.3. *The law u_t of $z^{\emptyset}$ under R^t, satisfies*

$$u_t - u_0 = \int_0^t (u_s \circ u_s - u_s)\ ds\ .$$

Proof: With notation (1.10), working under R^t:

$$z^{\emptyset} = v^{\emptyset}\ 1(\nu_{\emptyset} = 0) + \psi(z_{\nu_{\emptyset}-1}, z^{\nu_{\emptyset}}, y^{\nu_{\emptyset}})\ 1(\nu_{\phi} \geq 1)\ .$$

Now, with the help of Lemma 1.2, conditionally on $\nu_{\emptyset} \geq 1$, and $\tau^{\nu_{\emptyset}} = s$, $z_{\nu_{\emptyset}-1}, z^{\nu_{\emptyset}}, y^{\nu_{\emptyset}}$ are independent, with, $z_{\nu_{\emptyset}-1}$, $z^{\nu_{\emptyset}}$, u_s distributed. So we find that $u_t = u_0 e^{-t} + \int_0^t e^{-(t-s)} u_s \circ u_s\ ds$.

Our claim follows from this.

□

Proposition 1.3 provides a representation formula for the solution of spatially homogeneous Boltzmann equations with cutoff, which is also giving an intuition for the proof off uniqueness of the hard sphere Boltzmann process given in the next section.

2) Uniqueness for the Boltzmann process.

In this section we will look at the Boltzmann process (spatially homogeneous for hard spheres), as the solution of a certain martingale problem. The unboundedness of the collision intensity $|(v' - v) \cdot n|$, is a source of difficulty for the uniqueness of the solution, especially if one does not want to assume too many moment integrability conditions on the solution.

The existence of the solution, in the theorem we are now going to state, comes from a tightness result on the solutions corresponding to the cutoff problems (with collision function $|(v' - v) \cdot n| \wedge N$). For details on the existence part, we refer the reader to [39].

Theorem 2.1. *Let $u_0 \in M(R^n)$, be such that $\int |v|^3\ u_0(dv) < \infty$. There is a unique probability P on $D(R_+, R^n)$, such that*

i) *for $T < \infty$, $\sup_{t \leq T} \int |X_t|^3\ dP < \infty$*

ii) $X_0 \circ P = u_0$

iii) *for $f \in bB(R^n)$,*

$$\begin{aligned} f(X_t) - f(X_0) - \int_0^t \int_{D \times S_n} &[f(X_s + (X_s(\omega') - X_s) \cdot nn) \\ &- f(X_s)]\ |X_s(\omega') - X_s) \cdot n|\ dP(\omega')\ dn\ ds \end{aligned}$$

is a P-martingale.

We are now going to explain in a number of steps the proof of uniqueness.

Let us first recall that for a solution P of such a martingale problem, one can give a trajectorial representation as follows. Denote by $M_p(R_+ \times R_+ \times S_n \times D)$, the set of simple pure point measure on $R_+ \times R_+ \times S_n \times D$, finite on any compact restriction of the first two coordinates (with at mot one atom on each $\{t\} \times R_+ \times S_n \times D$, and none if t = 0).

Now on the product space $R^n \times M_p$ one can put the product of the probabilitiy u_0 and of Poisson measure with intensity $dt \otimes d\theta \otimes dn \otimes dP(\omega')$. P is now the law of the unique solution Z_s of the equation

$$
\begin{aligned}
(2.1)\qquad & Z_t = Z_0 + \int_0^t \int_{R_+ \times S_n \times D} (X_s(\omega') - Z_{s-}) \cdot n\ n\ 1\{\theta < |(Z_{s-} - X_s(\omega')) \cdot n|\} \\
& \qquad N(ds\ d\theta\ dn\ d\omega') \\
& \text{with} \int_0^t \int_{R_+ \times S_n \times D} 1\{\theta < |(Z_{s-} - X_s(\omega')) \cdot n|\}\ N(ds\ d\theta\ dn\ d\omega') < \infty\ , \\
& \text{for all } t < \infty\ ,
\end{aligned}
$$

Here Z_0 is the R^n-valued coordinate on the space $R^n \times M_p$, and N is the canonical Poisson measure induced by the second coordinate.

Consider then P_1 and P_2 two solutions of the martingale problem. We are first going to construct a coupling measure P_0 on $D(R_+, R^n)^2$, of P_1 and P_2, whch will also satisfy a nonlinear martingale problem.

Lemma 2.2. *There exists a probability P_0 on $D(R_+, R^n)^2$ such that*

1) $X^1_\cdot \circ P_0 = P_1$, $X^2_\cdot \circ P_0 = P_2$.

2) *for* $f \in bB(R^n \times R^n)$:

$$
\begin{aligned}
& f(X^1_t, X^2_t) - f(X^1_0, X^2_0) - \int_0^t \int_{D^2 \times S_n} \\
& \quad [f(\tilde{X}^1_s, \tilde{X}^2_s)\ |\ (X^1_s - X^1_s(\omega')) \cdot n| \wedge |(X^2_s - X^2_s(\omega')) \cdot n| \\
& \quad + f(\tilde{X}^1_s, X^2_s)\ (|(X^1_s - X^1_s(\omega')) \cdot n| - |(X^2_s - X^2_s(\omega')) \cdot n|)_+ \\
& \quad + f(X^1_s, \tilde{X}^2_s)\ (|(X^2_s - X^2_s(\omega')) \cdot n| - |(X^1_s - X^1_s(\omega')) \cdot n|)_+ \\
& \quad - f(X^1_s, X^2_s)\ (|(X^2_s - X^1_s(\omega')) \cdot n| \vee |(X^2_s - X^2_s(\omega')) \cdot n|)]\ dn\ dP_0(\omega')\ ds
\end{aligned}
$$

is a P_0-martingale, with the notation for $i = 1, 2$:

$$\tilde{X}^i_s = X^i_s + (X^i_s(\omega') - X^i_s(\omega)) \cdot n\ n$$

3) $(X^2_0, X^2_0) \circ P_0 = \text{diag}\ u_0$, *the diagonal image of* u_0 *on* $R^n \times R^n$.

Proof: The set $\mathcal{C}$ of probabilities on $D(R_+, R^n)^2$ with respective projections on the first and second coordinates givenby P_1 and P_2, is a weakly compact and convex set.

Denote by F the map which associates to $Q \in \mathcal{C}$, the law on $D(\boldsymbol{R}_+, \boldsymbol{R}^n)^2$ of $(Z^1_\cdot, Z^2_\cdot)$ solutions on $\boldsymbol{R}^n \times \boldsymbol{R}^n \times M_p(\boldsymbol{R}_+ \times \boldsymbol{R}_+ \times S_n \times D(\boldsymbol{R}_+, \boldsymbol{R}^n)^2)$ with measure diag $u_0 \otimes$ Poisson $(dt \otimes d\theta \otimes dn \otimes dQ)$ of the equations for $i = 1, 2$.

$$Z^i_t = Z^i_0 + \int_0^t \int_{\boldsymbol{R}_+ \times S_n \times D(\boldsymbol{R}_+, \boldsymbol{R}^n)^2} (X^i_s(\omega') - Z^i_{s-}) \cdot nn \; 1\{\theta < |(Z^i_s - X^i_s(\omega')) \cdot n|\} \\ N(ds \; d\theta \; dn \; d\omega') \, ,$$

with

$$\int_0^t \int 1\{\theta < |(Z^i_{s-} - X^i_s(\omega')) \cdot n|\} \; N(ds \; d\theta \; dn \; d\omega') < \infty \, .$$

In view of representation (2.1) F maps $\mathcal{C}$ into $\mathcal{C}$. Moreover $F(Q)$ is characterized by 1), 2), 3), with the replacement of $dP_0(\omega')$ integration by $dQ(\omega')$ integration. From this one sees that F is weakly continuous, and the existence of a fixed point to the map F now follows from Tychonov's theorem. This yields our claim.

□

Let us now introduce some notations convenient for what follows. We set for $x = (x_1, x_2)$, $x' = (x'_1, x'_2)$:

$$\begin{aligned} &k(x, x') = (1 + |x_i| + |x'_i|) \vee (1 + |x_2| + |x'_2|) \, , \\ &\phi(x) = 9(1 + |x_1|^2 + |x_2|^2) \, , \\ &Qh(x, x') = \int h(x + \Phi(x, x', y), \; x' + \Phi(x', x, y)) \; d\nu(y) \end{aligned}$$

where $y = (\alpha, n) \in [0,1] \times S_n = E$, $d\nu = d\alpha \otimes dn$, and Φ is the map from $\boldsymbol{R}^{2n} \times \boldsymbol{R}^{2n} \times E$ into $\boldsymbol{R}^{2n}$, given by:

$$\begin{aligned} \Phi(x, x', y) &= ((x'_1 - x_1) \cdot nn, \; (x'_2 - x_2) \cdot nn), \;\text{ if } \alpha \leq (a_1 \wedge a_2)/k, \\ &= ((x'_1 - x_1) \cdot nn) \text{ if } a_1/k \geq \alpha > (a_1 \wedge a_2)/k, \\ &= (0, \; (x'_2 - x_2) \cdot nn) \text{ if } a_2/k \geq \alpha > (a_1 \wedge a_2)/k, \\ &= (0, 0) \qquad \text{if } \alpha \geq (a_1 \vee a_2)/k, \end{aligned}$$

with $a_i = |(x_i - x'_i) \cdot n|$, $k = k(x, x')$. We will also write for $f \in bB(\boldsymbol{R}^n)$: $Q_1 f = Q(f \otimes 1)$.

If we put the probability diag $u_0 \otimes$ Poisson$(ds \otimes d\theta \otimes d\nu \otimes dP_0(\omega'))$, on $\boldsymbol{R}^n \times \boldsymbol{R}^n \times M_p(\boldsymbol{R}_+ \times \boldsymbol{R}_+ \times E \times D)$ (with $D = D(\boldsymbol{R}_+, \boldsymbol{R}^n)^2$), P_0 can also be represented as the law of the solution of

$$Z_t(\omega) = Z_0(\omega) + \int_0^t \int_{\boldsymbol{R}_+ \times E \times D} \Phi(Z_{s-}(\omega), \omega'(s), y) \; 1(\theta < k(Z_{s-}(\omega), \omega'(s))) \\ N(\omega, ds \; d\theta \; dy \; d\omega') \, , \tag{2.3}$$

with

$$\int_0^t \int_{R_+\times E\times D} 1(\theta < k(Z_{s^-}(\omega), \omega'(s)))\; N(\omega, ds\; d\theta\; dy\; d\omega')\,.$$

Moreover we have the estimates:

$$\begin{aligned} &k(x,x') \le \phi(x)^{1/2}\;\phi^{1/2}(x')\,, \\ &Q(\phi\oplus\phi) = \phi\oplus\phi \quad (\text{where } \phi\oplus\phi(x,x') = \phi(x)+\phi(x'))\,. \end{aligned} \tag{2.4}$$

Another nice feature of our coupling measure P_0, as seen from (2.2), is that if x and x' are "diagonal", that is $x_1 = x_2$ and $x_1' = x_2'$, then $\Phi(x,x',y)$ is diagonal as well.

The representation formula (2.3) for the coupling measure P_0, should be viewed as a way to keep track thanks to the marks "$\omega'_.$" of the first generation of trajectories, contributing to the determination of the bi-particle trajectory $Z_.$ on $[0,t]$. The scheme is now to construct the successive generations, using first a similar formula to (2.3) for each mark ω' (this yields the second generation), and iterating the procedure. On this "projective object", the problem will now be to see that the ancestor trajectory is in fact the result of a calculation on a finite tree whicch preserves the diagonal property of the initial conditions. This will prove that P_0 is in fact supported by the diagonal, and will yield uniqueness.

The first step (compare with Lemma 1.2, and Lemma 2.1 of Chapter III).

Lemma 2.3. *There is an auxiliary space $(\tilde\Omega, \boldsymbol{P})$, endowed with a Poisson point measure $N(\tilde\omega, ds\; d\theta\; dy\; d\tilde\omega')$ of intensity $ds\otimes d\theta\otimes d\nu(y)\otimes d\boldsymbol{P}(\tilde\omega')$, and with a P_0-distributed process $(Z_.(\omega))$ such that Z_0 is independent of the point measure N and* diag u_0 *distributed and*

$$\text{for } t>0,\; \int_0^t \int_{R_+\times E\times\tilde\Omega} 1\{\theta < k(Z_{s^-}(\tilde\omega),\; Z_s(\tilde\omega'))\}\; N(\tilde\omega, ds\; d\theta\; dy\; d\tilde\omega') < \infty$$

$$\begin{aligned} Z_t(\tilde\omega) = Z_0(\tilde\omega) + \int_0^t \int_{R_+\times E\times\tilde\Omega} &\Phi(Z_{s^-}(\tilde\omega),\; Z_s(\tilde\omega'), y) \\ &1\{\theta < k(Z_{s^-}(\tilde\omega), Z_s(\tilde\omega'))\}\; N(\omega, ds\; d\theta\; dy\; d\omega')\,, \end{aligned}$$

for the proof of this we refer the reader to [39]. The idea of the construction as alluded to before is that (2.3) gives a natural map from $R^{2n}\times M_p(R_+\times R_+\times E\times D)$ into D. Iterating this one has a natural map from $R^{2n}\times M_p(R_+\times R_+\times E\times R^{2n}\times M_p(R_+\times R_+\times E\times D))$ in the space $R^{2n}\times M_p(R_+\times R_+\times E\times D)$. The space $\widetilde\Omega$ essentially arises as a projective limit of this scheme.

Now on our extended space $\tilde{\Omega}$, we can define the number $K_t(\tilde{\omega})$ of generations which contribute to the state of the "ancestor trajectory" $Z_.(\tilde{\omega})$ on time $[0,t]$, by stating:

$$\{K \geq 0\} = R_+ \times \tilde{\Omega}$$

$$\{K \geq 1\} = \{(t,\tilde{\omega})/\exists s \in (0,t],\ N(\tilde{\omega},\{s\} \times R_+ \times E \times \tilde{\Omega}) = 1,$$
$$\text{and } {}^s\theta < k(Z_{s^-}(\tilde{\omega}), Z_s({}^s\tilde{\omega}'))\}$$

here ${}^s\theta$ and ${}^s\tilde{\omega}'$ are the marks of $N(\tilde{\omega},\cdot)$ at time s.

$$\{K \geq n+1\} = \{(t,\omega)/\exists s \in (0,t],\ N(\tilde{\omega},\{s\} \times R \times E \times \tilde{\Omega}\} = 1,$$
$${}^s\theta < k(Z_{s^-}(\tilde{\omega}), Z_s({}^s\tilde{\omega}')) \text{ and } K_s({}^2\tilde{\omega}') \geq n\}\ ,$$

and

$$\{K \geq \infty\} = \bigcap_n \{K \geq n\}\ .$$

Our next step is that only a finite number of generations play a role in the determination of $Z_.(\tilde{\omega})$.

Lemma 2.4. *For $t > 0$, $K_t(\omega) < \infty$, P-a.s.*

Proof: Set $\bar{Z}_t(\tilde{\omega}) = Z_t(\tilde{\omega})$ when $K_t < \infty$, δ otherwise. So $\bar{Z}_.$ is $R^n \cup \{\delta\}$ valued. Now for $f \in bB(R^n)$ (equal to zero on δ):

$$f(\overline{Z}_t) - f(\bar{Z}_0) = \int_0^t \int_{R_+ \times E \times \tilde{\Omega}} [f(Z_s(\tilde{\omega}))\ 1\{K_{s^-}(\tilde{\omega}) \vee K_s(\tilde{\omega}') < \infty\}$$
$$- f(\bar{Z}_{s^-}(\tilde{\omega}))]\ 1\{\theta < k(Z_{s^-}(\tilde{\omega}), Z_s(\tilde{\omega}'))\}\ N(\omega,\ ds\ d\theta\ d\nu(y)\ dP(\tilde{\omega}'))$$

From this after integration, we find:

$$E[f(\bar{Z}_t)] - E[f(Z_0)] = \int_0^t \int_{\Omega\times\Omega} Q_1 f(\bar{Z}_s(\tilde{\omega}), \bar{Z}_s(\tilde{\omega}'))\ k(\bar{Z}_s(\tilde{\omega}), \bar{Z}_s(\tilde{\omega}'))$$
$$- f(\bar{Z}_s(\tilde{\omega}))\ k(\bar{Z}_s(\tilde{\omega}), Z_s(\tilde{\omega}'))\ dP(\tilde{\omega})\ dP(\tilde{\omega}')\ ds \tag{2.5}$$

In view of our integrability assumptions on P_1, and P_2, we can apply (2.5) with ϕ instead of f. And by a very similar argument using $Q(\phi \oplus \phi) = \phi \oplus \phi$ as stated in (2.2), we get:

$$E[\phi(Z_t)] = E[\phi(Z_0)]$$

It then follows that

$$E[\phi(Z_t) - \phi(\bar{Z}_t)] = \int_0^t \int_{\Omega\times\Omega} -\frac{1}{2}[Q(\phi \oplus \phi) - \phi \oplus \phi) - \phi \oplus \phi] \times k(\bar{Z}_s(\tilde{\omega}), \bar{Z}_s(\tilde{\omega}'))$$
$$+ \phi(\bar{Z}_s(\tilde{\omega}))[k(\bar{Z}_s(\tilde{\omega}), Z_s(\tilde{\omega}')) - k(\bar{Z}_s(\tilde{\omega}), \bar{Z}_s(\tilde{\omega}'))]$$
$$dP(\tilde{\omega})\ dP(\tilde{\omega}')\ ds$$

Now the first term in the integral is zero, as for the second, using $k(x,x') \le \phi^{1/2}(n)\ \phi^{1/2}(x')$, we find:

$$E[\phi(Z_t)\ 1\{K_t = \infty\}] \le \int_0^t E[\phi^{3/2}(Z_s)]\ E[\phi^{1/2}(Z_s)\ 1\{K_s = \infty\}]\ ds$$

From Gronwall's lemma, we now find $E[\phi(Z_t)\ 1\{K_t = \infty\}] = 0$, from which our claim follows.

□

We now have the required ingredients to see that our coupling probability P_0 is diagonally supported.

Indeed when $K_t(\tilde{\omega}) = 0$, then $Z_t^1(\tilde{\omega}) = Z_0^1(\tilde{\omega}) = Z_0^2(\tilde{\omega}) = Z_t^2(\tilde{\omega})$, since no jump occurs.

Suppose now that we know that $K_T(\tilde{\omega}) \le n$ implies $Z_t^1(\tilde{\omega}) = Z_t^2(\tilde{\omega})$, for $t \le T$. Take now $\tilde{\omega} \in \widetilde{\Omega}$ with $K_T(\tilde{\omega}) = n+1$, then for $t \le T$:

$$\begin{aligned} Z_t(\tilde{\omega}) = Z_0(\tilde{\omega}) + \int_0^t \int_{R^n \times E \times \widetilde{\Omega}} &\Phi(Z_{s^-}(\tilde{\omega}), Z_s(\tilde{\omega}'), y)\ 1\{\theta < k(Z_{s^-}(\tilde{\omega}), Z_s(\tilde{\omega}'))\} \\ &1\{K_s(\tilde{\omega}') \le n\}\ N(\tilde{\omega}, ds\ d\theta\ dy\ d\tilde{\omega}')\ , \end{aligned}$$

since only mark ${}^s\tilde{\omega}'$ for which $K_s({}^s\tilde{\omega}') \le n$ come in the determination of $Z_t(\tilde{\omega})$, $t \le T$. Now as observed already if x and x' are diagonal so is $\Phi(x,x',y)$. It then follows that $Z_t(\tilde{\omega})$, $t \le T$ is diagonal. Now our claim follows by induction, thanks to Lemma 2.4.

So we obtain that the coupling probability P_0 is diagonally supported, and this proves that $P_1 = P_2$.

□

References

1. Baxter, J. R. - Chacon, R. V. - Jain, N. C.: Weak limits of stopped diffusions, Trans. Amer. Math. Soc., 293, 2, 767-792, (1986).
2. Billingsley, P.: Convergence of Probability Measures, Wiley, New York (1968).
3. Calderoni, P. - Pulvirenti, M.: Propagation chaos for Burger's equation, Ann. Inst. H. Poincaré, série A, N.S. 39, 85-97 (1983).
4. Cercignani, C.: The grad limit for a system of soft spheres, Comm. Pure Appl. Math 26, 4 (1983).
5. Dawson, D. A.: Critical dynamics and fluctuations for a mean field model of cooperative behavior, J. Stat. Phys. 31, 29-85 (1978).
6. Dawson, D. A. - Gärtner, J.: Large deviations from the McKean-Vlasov limit for weakly interacting diffusions, Stochastics 20, 247-308 (1987).
7. De Masi, A. - Ianiro, N. - Pellegrinotti, A. - Pressutti, E.: A survey of the hydrodynamical behavior of many particle systems, in Nonequilibrium phenomena II, Ed.: J. L. Lebowitz, E. W. Montroll, Elsevier (1984).
8. Dobrushin, R. L.: Prescribing a system of random variables by conditional distributions, Th. Probab. and its Applic. 3, 469 (1970).
9. ———— Vlasov equations, Funct. Anal. and Appl. 13, 115 (1979).
10. Ferland, R., Giroux, G.: Cutoff Boltzmann equation: convergence of the solution, Adv. Appl. Math. 8, 98-107 (1987).
11. Funaki, T.: The diffusion approximation of the spatially homogeneous Boltzmann equation, Duke Math. J. 52, 1-23, (1985).
12. Goodman, J., Convergence of the random value method, IMA, vol. 9, G. Papanciolaou ed., Hydrodynamic Behavior and Interacting Particles, 99-106, Springer, Berlin (1987).
13. Guo, M. - Papanicolaou, G. C.: Self diffusion of interacting Brownian particles, Taniguchi Symp., Katata 1985, 113-151.
14. Gutkin, E.: Propagation of chaos and the Hopf-Cole transformation, Adv. Appl. Math. 6, 413-421, 1985.
15. Kac, M.: Foundation of kinetic theory, Proc. Third Berkeley Symp. on Math. Stat. and Probab. 3, 171-197, Univ. of Calif. Press (1956).
16. ———— Some probabilistic aspects of the Boltzmann equation, Acta Physica Austraiaca, suppl. X, Springer, 379-400 (1979).
17. Karandikar, R. L.,Horowitz, J.: Martingale problems associated with the Boltzmann equation, preprint, 1989.
18. Kipnis, C. - Olla, S. - Varadhan, S.R.S.: Hydrodynamics and large deviation for simple exclusion processes Comm. Pure Appl. Math., 42, 115-137, (1989).
19. Kurtz, T.: Approximation of population processes CMBS-NSF Reg. Conf. Sci. Appl. Math. Vol. 36, Society for Industrial and Applied Mathematics, Philadelphia (1981).
20. Kusuoka, S. - Tamura, Y.: Gibbs measures with mean field potentials, J. Fac. Sci. Tokyo Univ., sect. 1A, 31, 1, 223-245 (1984).
21. Kotani, S. - Osada, H.: Propagation of chaos for Burgers' equation, J. Math. Soc. Japan, 37, 275-294 (1985).
22. Lanford, O. E., Time evolution of large classical systems, Lecture Notes in Physics 38, 1-111, Springer, Berlin, (1975).
23. Lang, R. - Nguyen, X.X.: Smoluchowski's theory of coagulation in colloids holds rigorously in the Boltzmann-Grad limit, Z. Wahrscheinlichkeitstheor. Verw. Gebiete 54, 227-280 (1980).

24. Liggett, T. M.: Interacting particle systems, Springer, Berlin (1985).
25. McKean, H. P.: Speed of approach to equilibrium for Kac's caricature of a Maxwellian gas, Arch. Rational Mech. Anal. 21, 347-367 (1966).
26. ———— A class of Markov processes associated with nonlinear parabolic equations, Proc. Nat. Acad. Sci. 56, 1907-1911 (1966).
27. ———— Propagation of chaos for a class of nonlinear parabolic equations, Lecture series in differential equations 7, 41-57, Catholic University, Washington, D. C. (1967).
28. ———— Fluctuations in the kinetic theory of gases, Comm. Pure Appl. Math. 28, 435-455 (1975).
29. Marchioro, C. – Pulvirenti, M.: Hydrodynamics in two dimensions and vortex theory, Comm. Math. Phys. 84, 483-504 (19820.
30. Murata, H. - Tanaka, H.: An inequality for certain functionals of multidimensional probability distributions, Hiroshima Math. J., 4, 75-81 (1974).
31. Nagasawa, M. – Tanaka, H.: Diffusion with interaction and collisions between colored particles and the propagation of chaos, Probab. Th. Rel. Fields 74, 161-198 (1987).
32. ———— On the propagation of chaos for diffusion processes with coefficients not of average form, Tokyo Jour. Math. 10 (2), 403-418 (1987).
33. Neveu,J.: Arbres et processus de Galton-Watson, Ann. Inst. Henri Poincaré Nouv. Ser. B, 22, 2,199-208 (1986).
34. Oelschläger K.: A law of large numbers for moderately interacting diffusion processes, Z. Wahrscheinlichkeitstheor. Verw. Gebeite 69, 279-322 (1985).
35. Osada, H.: Limit points of empirical distributions of vortices with small viscosity, IMA, vol. 9, G. Papanicolaou ed., Hydrodynamic behavior and interacting particles, 117-126, Springer, Berlin (1987).
36. ———— Propagation of chaos for the two dimensional Navier-Stokes equation.
37. Scheutzow, M.: Periodic behavior of the stochastic Brusselator in the mean field limit, Prob. Th. Re. Fields 72, 425-462, (1986).
38. Shiga, T. – Tanaka, H.: Central limit theorem for a system of Markovian particles with mean field interactions, Z. Wahrscheinlichkeitstheor. Verw. Gebiete 69, 439-445 (1985).
39. Sznitman, A. S.: Equations de type Boltzmann spatialement homogènes, Z. Wahrscheinlichkeitstheor. Verw. Gebiete 66, 559-592 (1984).
40. ———— Nonlinear reflecting diffusion process and the propagation of chaos and fluctuations associated, J. Funct. Anal. 56 (3), 311-336 (1984).
41. ———— A fluctuation result for nonlinear diffusions, infinite dimensional analysis, S. Albeverio, ed., 145-160, Pitman, Boston (1985).
42. ———— A propagation of chaos result for Burgers' equation, Z. Wahrscheinlichkeitstheor. Verw. Gebiete 71, 581-613 (1986).
43. ———— Propagation of chaos for a system of annihilating Brownian spheres, Comm. Pure Appl. Math. 60, 663-690 (1987).
44. ———— A trajectorial representation for certain nonlinear equations, Astérisque, 157-158, 363-370 (1988).
45. ———— A limiting result for the structure of collisions between many independent diffusions, Probab. Th. Rel Fields 81, 353-381 (1989).
46. Sznitman, A. S. – Varadhan, S.R.S.: A multidimensional process involving local time, Z. Wahrscheinlichkeitstheor. Verw. Gebiete 71, 553-579 (1986).

47. Spohn, H.: The dynamics of systems with many particles, statistical mechanics of local equilibrium states (preprint).
48. Tamura, Y.: On asymptotic behaviors of the solution of a nonlinear diffusion equation, J. Fac. Sci. Tokyo Univ., sect. IA, 31, 1, 195-221 (1984).
49. Tanaka, H.: Probabilistic treatment of the Boltzmann equation of Maxwellian molecules, Z. Wahrscheinlichkeitstheor. Verw. Gebiete 46, 67-105 (1978).
50. ——————— Some probabilistic problems in the spatially homogeneous Boltzmann equation, Proc. IFIP–ISI conf. on appl. of random fields, Bangalore, Jan. 82.
51. ——————— Limit theorems for certain diffusion processes with interaction, Taniguchi Symp. S. A. Katata, 469-488 (1982).
52. Tanaka, H. – Hitsuda, M.: Centtral limit theorem for a simple model of interacting particles, Hiroshima Math. J. 11, 415-423 (1981).
53. Uchiyama, K.: On the Boltzmann Grad limit for the Broadwell model of the Boltzmann equation, J. Stat. Phys. 52, 331-355 (1988).
54. ——————— Derivation of the Botlzmann equation from particle dynamics, Hiroshima Math. J. 18, 2 (1988).
55. Ueno, T.: A class of Markov processes with bounded nonlinear generators, Japanese J. Math. 38, 19-38 (1968).
56. ——————— A path space and the propagation of chaos for Boltzmann's gas model, Proc. Japan Acad. 6 (47) 529-533 (1971).
57. Wild, E.: On the Boltzmann equation in the kinetic theory of gases, Proc. Cambridge Phil. Soc., 47, 602-609 (1951).
58. Zvonkin, A. K.: A transformation of the phase space of a diffusion process that removes the drift, Math. USSR Sbornik, 22, 1, 129-149, (1974).