Applications of Bessel functions

Introduction

A Bessel beam is the mathematical function which describes a solution of Bessel's differential equation, which itself arises from separable solutions to Laplace's equation as in 1 or the Helmholtz equation in 2 but in cylindrical coordinates where ∇^2 is given by 3.

$$\nabla^2 f = 0 \tag{1}$$

$$\nabla^2 f + k^2 f = 0 \tag{2}$$

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial}{\partial\rho}) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial Z^2} \tag{3}$$

A Bessel beam is a field of electromagnetic, acoustic or even gravitational radiation whose amplitude is described by a Bessel function of the first kind. A true Bessel beam is non-diffractive. This means that as it propagates, it does not diffract and spread out; this is in contrast to the usual behavior of light (or sound), which spreads out after being focused down to a small spot. Bessel beams are also self-healing, meaning that the beam can be partially obstructed at one point, but will re-form at a point further down the beam axis.

As with a plane wave, a true Bessel beam cannot be created, as it is unbounded and would require an infinite amount of energy. Reasonably good approximations can be made, however, and these are important in many optical applications because they exhibit little or no diffraction over a limited distance.

The properties of Bessel beams make them extremely useful for optical tweezing, as a narrow Bessel beam will maintain its required property of tight focus over a relatively long section of beam and even when partially occluded by the dielectric particles being tweezed.we can see different orders of Bessel function in Figure 1

We can derive the Bessel beam solution as follows , if we consider 1-D scalar wave equation

$$\nabla^2 u = \frac{1}{v^2}\frac{\partial^u}{\partial t^2} \tag{4}$$

the trial solution will be in the form

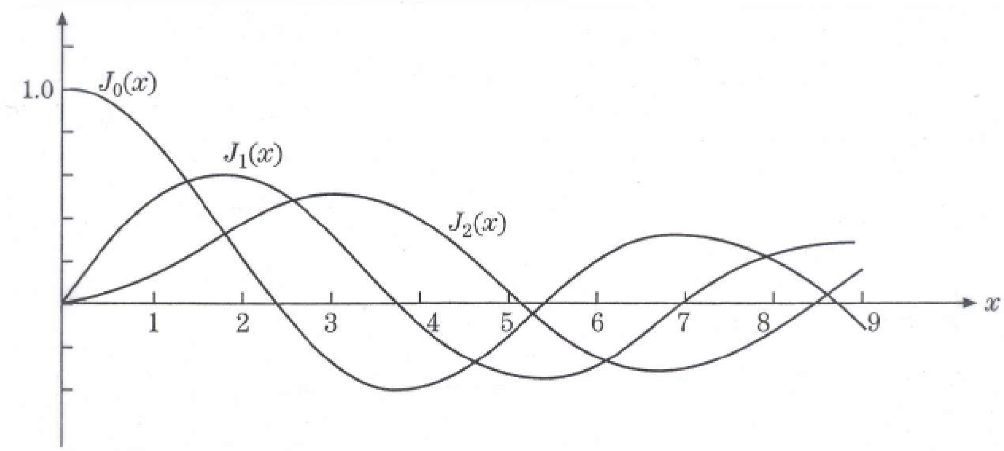

Figure 1: Orders of Bessel function

$$u(x,t) = f(\rho)e^{j(k_3 z - wt)} \tag{5}$$

where k is the wave vector $< k_1, k_2, k_3 >$, so substituting in 4 we reach

$$\rho^2 \frac{d^2 f}{d\rho^2} + \rho \frac{df}{d\rho} + \rho^2 (k^2 + k_3^2) f = 0 \tag{6}$$

by comparing to Bessel's differential equation

$$x^2 \frac{d^2 J_n}{dx^2} + x \frac{dJ_n}{dx} + (x^2 - n^2) J_n = 0 \tag{7}$$

we can find

$$f(\rho) = J_0((k_1^2 + k_2^2)\rho)$$

so the scalar wave equation will be in the form

$$u(x,t) = J_0((k_1^2 + k_2^2)\rho)e^{j(k_3 z - wt)} \tag{8}$$

Application in Optical Fiber

Analysis of the optical fiber is complicated. This is mainly because of the round cross section, along with the fact that it is generally a three-dimensional problem.The simplest fiber configuration is that of a step index, but with the core and cladding indices of values that are very close, that is n1≃ n2. This is the weak guidance condition, whose simplifying effect on the analysis is significant. The core and cladding indices in the slab waveguide need to be very close in value in order to achieve single-mode or few-mode operation. Fiber manufacturers have taken this result to heart, such that the weak-guidance condition is in fact satisfied by most commercial fibers today. the wave equation is the same as 2 but in cylindrical coordinates as in 9 . it is possible to write the x-polarized phasor electric field within a weakly guiding cylindrical fiber as a product of three functions, each of which varies with one of the coordinate variables as in 10

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial E}{\partial\rho})+\frac{1}{\rho^2}\frac{\partial^2 E}{\partial\phi^2}+(k^2-\beta^2)\frac{\partial^2 E}{\partial Z^2} \tag{9}$$

$$E(\rho,\phi,z)=\sum_i R_i(\rho)\Phi_i(\phi)e^{-j\beta_i z} \tag{10}$$

Each term in the summation in 10 is an individual mode of the fiber. Note that the z function is just the propagation term, since we are assuming an infinitely long lossless fiber. We then apply separation of variables and reach the equations below

$$\frac{d^2\Phi}{d\phi^2}+l^2\Phi=0$$

$$\frac{d^2R}{d\rho^2}+\frac{1}{\rho}\frac{dR}{d\rho}+[k^2-\beta^2-\frac{l^2}{\rho^2}]R=0$$

The solution of both the equation will be

$$\Phi(\phi)=cos(l\phi+\alpha)$$

$$R(\rho)=\begin{cases} AJ_l(\beta_t\rho) & \beta_t \text{ is Real} \\ BK_1(|\beta_t|\rho) & \beta_t \text{ is Imaginary} \end{cases}$$

Where α is a constant and $\beta_t=(k^2-\beta^2)^{\frac{1}{2}}$ and K_l is modified Bessel function of second kind

The total solution for electric field wave equation is then given by

$$E(\rho,\phi,z)=\begin{cases} E_0 J_l(\frac{u\rho}{a})cos(l\phi)e^{-j\beta z} & \rho\le a \\ E_0\frac{J_1(u)}{K_1(w)}K_l(\frac{w\rho}{a})cos(l\phi)e^{-j\beta z} & \rho\ge a \end{cases}$$

where a is the cylinder radius , $u=a\sqrt{n_1^2k^2-\beta^2}$ and $w=a\sqrt{\beta^2-n_2^2k^2}$

Application in propagation in Electric Power Line

In the problem of wave propagation in electric power lines , Let v and i denote the voltage and the current in the single-wire system; moreover let L ,C , R and G respectively denote the inductance , capacity, resistance and conductivity. The voltage and current satisfy the following system of equations

$$-\frac{\partial v}{\partial x} = L\frac{\partial i}{\partial t} + Ri$$

$$-\frac{\partial i}{\partial x} = C\frac{\partial v}{\partial t} + Gv$$

If $R = 0$ and $G = 0$, then the transmission line is called non-corrupting and the system of equations leads to one dimensional wave equation.

In general case R and G have a constant values not equal to zero, this will lead to a differential equation of the second order which is sometimes called telegraphers equation

$$\frac{\partial^2 v}{\partial x^2} - LC\frac{\partial^2 v}{\partial t^2} - (RC + LG)\frac{\partial v}{\partial t} - RGv = 0$$

we also can write the same function in terms of i

there are 2 solutions for this equations ; The first one we will assume that the coefficient of first derivative with respect to time is zero then the solution becomes

$$v(x,t) = \begin{cases} aJ_0(a\sqrt{RG(t^2 - \frac{x^2}{a^2}})) & t \geq \frac{x}{a} \\ 0 & t \leq \frac{x}{a} \end{cases}$$

The general solution is when the coefficient of first derivative with respect to time doesn't equal zero

$$v(x,t) = \begin{cases} ae^{-na^2}J_0(\sqrt{(RG - a^2n^2)(t^2 - \frac{x^2}{a^2}})) & t \geq \frac{x}{a} \\ 0 & t \leq \frac{x}{a} \end{cases}$$

where $n = \frac{RC+LG}{2}$ and $a = \frac{1}{\sqrt{LC}}$

Application in Quantum mechanics : Solution to Schroedinger's Equation in a Circular Well

Consider a particle of mass m placed into a two-dimensional potential well, where the potential is zero inside of the radius of the disk, infinite outside of the radius of the disk. In polar coordinates using r, ϕ as representatives of the system, the Laplacian is written:

$$\nabla^2\Psi = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial \Psi}{\partial r}) + \frac{1}{r^2}\frac{\partial^2\Psi}{\partial \phi^2} \tag{11}$$

Which in the Schroedinger equation presents:

$$-\frac{\hbar^2}{2m}[\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial\Psi}{\partial r})+\frac{1}{r^2}\frac{\partial^2\Psi}{\partial\phi^2}]=E\Psi \tag{12}$$

Using the method of separation of variables with a proposed solution $\Psi = R(r)T(\phi)$ and then dividing by Ψ and multiplying by r^2

$$\frac{r}{R}\frac{d}{dr}(r\frac{dR}{dr})+k^2r^2+\frac{1}{T}\frac{d^2T}{d\phi^2}=0 \tag{13}$$

which is fully separated in r and ϕ. To solve, the ϕ dependent portion is set to $-m^2$, yielding the harmonic oscillator equation in $T(\phi)$, which presents the solution:

$$T(\phi)=Ae^{jm\phi} \tag{14}$$

Where A is a constant determined via proper normalization in ϕ , A = $\sqrt{\frac{1}{2\pi}}$

Working now with the r dependent portion of the separated equation, we reach

$$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}+(k^2r^2-m^2)=0 \tag{15}$$

which is the same as Bessel's differential equation.The general solution is of the form of:

$$R(r)=AJ_m(kr)+BY_m(kr) \tag{16}$$

As the solution must be finite at x = 0, and as $Y_m(kr)\to\infty$ as $x\to 0$, this means that the coefficient B = 0, leaving R(r) to be expressed:

$$R(r)=AJ_m(kr) \tag{17}$$

Using the boundary condition that $\Psi = 0$at the radius of the disk, we have the condition that $J_m(kr_b)=0$, which implicitly requires the argument of $J_m$to be a zero of the Bessel function of order m. We now can reach the full solution to the wave equation

$$\Psi = AJ_m(\frac{\alpha_{m,n}r}{r_b})e^{jm\phi} \tag{18}$$

where $\alpha_{m,n}$ is the n^{th} zero of the m^{th} order Bessel function and r_b is the radius of the disk.

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