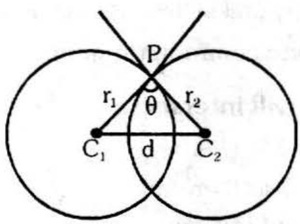
(ii) When circle are apart from each other then $C_1C_2 > r_1 + r_2$ and in this case there will be **four common tangents.** Lines PQ and RS are called **transverse** or **indirect** or **internal** common tangents and these lines meet line C_1C_2 on T_1 and T_1 divides the line C_1C_2 in the ratio $r_1 : r_2$ internally and lines AB & CD are called **direct** or **external** common tangents. These lines meet C_1C_2 produced on T_2. Thus T_2 divides C_1C_2 externally in the ratio $r_1 : r_2$.

Note : Length of direct common tangent = $\sqrt{(C_1C_2)^2 - (r_1 - r_2)^2}$

Length of transverse common tangent = $\sqrt{(C_1C_2)^2 - (r_1 + r_2)^2}$

12. THE ANGLE OF INTERSECTION OF TWO CIRCLES :

Definition : The angle between the tangents of two circles at the point of intersection of the two circles is called angle of intersection of two circles.



then $\cos\theta = \dfrac{2g_1g_2 + 2f_1f_2 - c_1 - c_2}{2\sqrt{g_1^2 + f_1^2 - c_1}\ \sqrt{g_2^2 + f_2^2 - c_2}}$ or $\boxed{\cos\theta = \left(\dfrac{r_1^2 + r_2^2 - d^2}{2r_1r_2}\right)}$

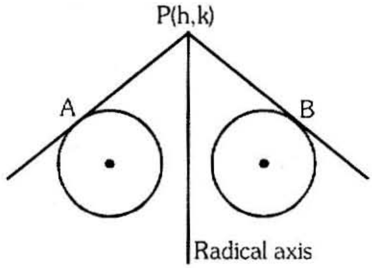
Here r_1 and r_2 are the radii of the circles and d is the distance between their centres.

If the angle of intersection of the two circles is a right angle then such circles are called **"Orthogonal circles"** and conditions for the circles to be orthogonal is

$$\mathbf{2g_1g_2 + 2f_1f_2 = c_1 + c_2}$$

13. RADICAL AXIS OF THE TWO CIRCLES ($S_1 - S_2 = 0$) :

Definition : The locus of a point, which moves in such a way that the length of tangents drawn from it to the circles are equal is called the radical axis. If two circles are -



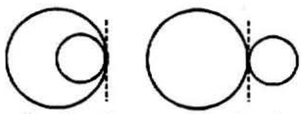
$$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1 y + c_1 = 0$$

$$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$$

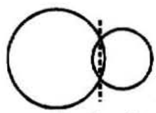
Then the equation of radical axis is given by $\mathbf{S_1 - S_2 = 0}$

Note :

(i) If two circles touches each other then common tangent is radical axis.



(ii) If two circles cuts each other then common chord is radical axis.



(iii) If two circles cuts third circle orthogonally then radical axis of first two is locus of centre of third circle.

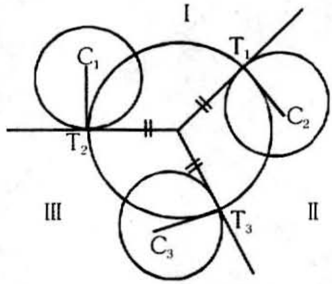
(iv) The radical axis of the two circles is perpendicular to the line joining the centre of two circles but not always pass through mid point of it.

14. Radical centre :

The radical centre of three circles is the point from which length of tangents on three circles are equal i.e. the point of intersection of radical axis of the circles is the radical centre of the circles.

Note :

(i) The circle with centre as radical centre and radius equal to the length of tangent from radical centre to any of the circle, will cut the three circles orthogonally.



(ii) If three circles are drawn on three sides of a triangle taking them as diameter then its orthocenter will be its radical centre.

PARABOLA

1. CONIC SECTIONS :

A conic section, or conic is the locus of a point which moves in a plane so that its distance from a fixed point is in a constant ratio to its perpendicular distance from a fixed straight line.

(a) The fixed point is called the FOCUS.

(b) The fixed straight line is called the DIRECTRIX.

(c) The constant ratio is called the ECCENTRICITY denoted by e.

(d) The line passing through the focus & perpendicular to the directrix is called the AXIS.

(e) A point of intersection of a conic with its axis is called a VERTEX.

2. GENERAL EQUATION OF A CONIC : FOCAL DIRECTRIX PROPERTY :

The general equation of a conic with focus (p, q) & directrix $lx + my + n = 0$ is :

$$(l^2 + m^2)\,[(x - p)^2 + (y - q)^2] = e^2\,(lx + my + n)^2$$
$$\equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

3. DISTINGUISHING BETWEEN THE CONIC :

The nature of the conic section depends upon the position of the focus S w.r.t. the directrix & also upon the value of the eccentricity e. Two different cases arise.

Case (i) When the focus lies on the directrix :

In this case $D \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ & the general equation of a conic represents a pair of straight lines and if :

$e > 1$, $h^2 > ab$ the lines will be real & distinct intersecting at S.

$e = 1$, $h^2 = ab$ the lines will coincident.

$e < 1$, $h^2 < ab$ the lines will be imaginary.

[redacted]ase (ii) When the focus does not lie on the directrix :

The conic represents :

a parabola	an ellipse	a hyperbola	a rectangular hyperbola
$e = 1; D \neq 0$ $h^2 = ab$	$0 < e < 1; D \neq 0$ $h^2 < ab$	$D \neq 0 ; e > 1$ $h^2 > ab$	$e > 1 ; D \neq 0$ $h^2 > ab ; a + b = 0$

4. PARABOLA :

A parabola is the locus of a point which moves in a plane, such that its distance from a fixed point (focus) is equal to its perpendicular distance from a fixed straight line (directrix).

Standard equation of a parabola is $y^2 = 4ax$. For this parabola :

(i) Vertex is (0, 0)

(ii) Focus is (a, 0)

(iii) Axis is $y = 0$

(iv) Directrix is $x + a = 0$

(a) Focal distance :

The distance of a point on the parabola from the focus is called the FOCAL DISTANCE OF THE POINT.

(b) Focal chord :

A chord of the parabola, which passes through the focus is called a FOCAL CHORD.

(c) Double ordinate :

A chord of the parabola perpendicular to the axis of the symmetry is called a DOUBLE ORDINATE with respect to axis as diameter.

(d) Latus rectum :

A focal chord perpendicular to the axis of parabola is called the LATUS RECTUM. For $y^2 = 4ax$.

(i) Length of the latus rectum = 4a.

(ii) Length of the semi latus rectum = 2a.

(iii) Ends of the latus rectum are $L(a, 2a)$ & $L'(a, -2a)$

Note that :

(i) Perpendicular distance from focus on directrix = half the latus rectum.

(ii) Vertex is middle point of the focus & the point of intersection of directrix & axis.

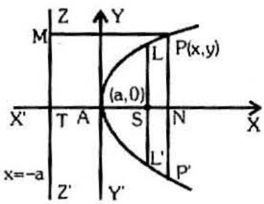
(iii) Two parabolas are said to be equal if they have latus rectum of same length.

5. PARAMETRIC REPRESENTATION :

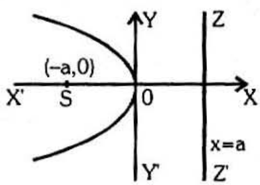
The simplest & the best form of representing the co-ordinates of a point on the parabola $y^2 = 4ax$ is $(at^2, 2at)$. The equation $x = at^2$ & $y = 2at$ together represents the parabola $y^2 = 4ax$, t being the parameter.

6. TYPE OF PARABOLA :

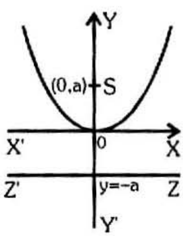
Four standard forms of the parabola are $y^2 = 4ax$; $y^2 = -4ax$; $x^2 = 4ay$; $x^2 = -4ay$



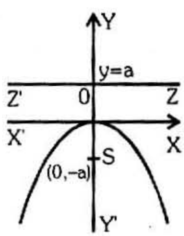
$y^2 = 4ax$



$y^2 = -4ax$



$x^2 = 4ay$



$x^2 = -4ay$

Parabola	Vertex	Fous	Axis	Directrix	Length of Latus rectum	Ends of Latus rectum	Parametric equation	Focal length
$y^2 = 4ax$	(0,0)	(a,0)	y=0	x=–a	4a	(a, ±2a)	$(at^2, 2at)$	x+a
$y^2 = -4ax$	(0,0)	(–a,0)	y=0	x=a	4a	(–a, ±2a)	$(-at^2, 2at)$	x–a
$x^2 = +4ay$	(0,0)	(0,a)	x=0	y=–a	4a	(±2a, a)	$(2at, at^2)$	y+a
$x^2 = -4ay$	(0,0)	(0,–a)	x=0	y=a	4a	(±2a, –a)	$(2at, -at^2)$	y–a
$(y-k)^2=4a(x-h)$	(h,k)	(h+a,k)	y=k	x+a–h=0	4a	(h+a, k±2a)	$(h+at^2, k+2at)$	x–h+a
$(x-p)^2=4b(y-q)$	(p,q)	(p, b+q)	x=p	y+b–q=0	4b	(p±2a,q+a)	$(p+2at, q+at^2)$	y–q+b

7. POSITION OF A POINT RELATIVE TO A PARABOLA :

The point (x_1, y_1) lies outside, on or inside the parabola $y^2 = 4ax$ according as the expression $y_1^2 - 4ax_1$ is positive, zero or negative.

8. CHORD JOINING TWO POINTS :

The equation of a chord of the parabola $y^2 = 4ax$ joining its two points $P(t_1)$ and $Q(t_2)$ is $y(t_1 + t_2) = 2x + 2at_1t_2$

Note :

(i) If PQ is focal chord then $t_1t_2 = -1$.

(ii) Extremities of focal chord can be taken as $(at^2, 2at)$ & $\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$

(iii) If $t_1t_2 = k$ then chord always passes a fixed point $(-ka, 0)$.

9. LINE & A PARABOLA :

(a) The line $y = mx + c$ meets the parabola $y^2 = 4ax$ in two points real, coincident or imaginary according as $a >=< cm$

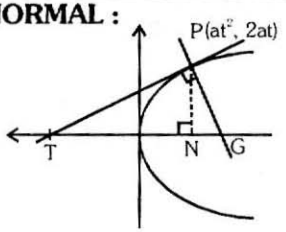
$\Rightarrow$ condition of tangency is, $c = \dfrac{a}{m}$.

Note : Line $y = mx + c$ will be tangent to parabola $x^2 = 4ay$ if $\mathbf{c = -am^2}$.

(b) Length of the chord intercepted by the parabola $y^2 = 4ax$ on the line $y = mx + c$ is : $\left(\frac{4}{m^2}\right)\sqrt{a(1+m^2)(a-mc)}$.

Note : length of the focal chord making an angle α with the x-axis is $4a\,\text{cosec}^2\,\alpha$.

10. LENGTH OF SUBTANGENT & SUBNORMAL :



PT and PG are the tangent and normal respectively at the point P to the parabola $y^2 = 4ax$. Then

TN = length of subtangent = twice the abscisaa of the point P

(Subtangent is always bisected by the vertex)

NG = length of subnormal which is constant for all points on the parabola & equal to its semilatus rectum (2a).

11. TANGENT TO THE PARABOLA $y^2 = 4ax$:

(a) Point form :

Equation of tangent to the given parabola at its point (x_1, y_1) is

$yy_1 = 2a(x + x_1)$

(b) Slope form :

Equation of tangent to the given parabola whose slope is 'm', is

$$y = mx + \frac{a}{m}, (m \neq 0)$$

Point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

(c) Parametric form :

Equation of tangent to the given parabola at its point P(t), is -

$ty = x + at^2$

Note : Point of intersection of the tangents at the point t_1 & t_2 is $[at_1 t_2, a(t_1 + t_2)]$. (i.e. G.M. and A.M. of abscissae and ordinates of the points)

12. NORMAL TO THE PARABOLA $y^2 = 4ax$:

(a) Point form :

Equation of normal to the given parabola at its point (x_1, y_1) is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1)$$

(b) Slope form :

Equation of normal to the given parabola whose slope is 'm', is

$y = mx - 2am - am^3$ foot of the normal is $(am^2, -2am)$

(c) Parametric form :

Equation of normal to the given parabola at its point P(t), is

$y + tx = 2at + at^3$

Note :

(i) Point of intersection of normals at t_1 & t_2 is $(a(t_1^2 + t_2^2 + t_1t_2 + 2), - at_1t_2(t_1 + t_2))$.

(ii) If the normal to the parabola $y^2 = 4ax$ at the point t_1, meets the parabola again at the point t_2, then $t_2 = -\left(t_1 + \dfrac{2}{t_1}\right)$.

(iii) If the normals to the parabola $y^2 = 4ax$ at the points t_1 & t_2 intersect again on the parabola at the point 't_3' then $t_1t_2 = 2$; $t_3 = -(t_1 + t_2)$ and the line joining t_1 & t_2 passes through a fixed point $(-2a, 0)$.

13. PAIR OF TANGENTS :

The equation of the pair of tangents which can be drawn from any point $P(x_1, y_1)$ outside the parabola to the parabola $y^2 = 4ax$ is given by : $SS_1 = T^2$, where :

$S \equiv y^2 - 4ax$; $S_1 \equiv y_1^2 - 4ax_1$; $T \equiv yy_1 - 2a(x + x_1)$.

14. CHORD OF CONTACT :

Equation of the chord of contact of tangents drawn from a point $P(x_1, y_1)$ is $yy_1 = 2a(x + x_1)$

Remember that the area of the triangle formed by the tangents from the point (x_1, y_1) & the chord of contact is $\dfrac{(y_1^2 - 4ax_1)^{3/2}}{2a}$. Also note that the chord of contact exists only if the point P is not inside.

$S_1^{3/2}/2a$

15. CHORD WITH A GIVEN MIDDLE POINT :

Equation of the chord of the parabola $y^2 = 4ax$ whose middle point is (x_1, y_1) is $y - y_1 = \dfrac{2a}{y_1}(x - x_1)$.

This reduced to $T = S_1$

where $T \equiv yy_1 - 2a(x + x_1)$ & $S_1 \equiv y_1^2 - 4ax_1$.

16. DIAMETER :

The locus of the middle points of a system of parallel chords of a Parabola is called a DIAMETER. Equation to the diameter of a parabola is $y = 2a/m$, where m = slope of parallel chords.

17. CONORMAL POINTS :

Foot of the normals of three concurrent normals are called conormals point.

(i) Algebraic sum of the slopes of three concurrent normals of parabola $y^2 = 4ax$ is zero.

(ii) Sum of ordinates of the three conormal points on the parabola $y^2 = 4ax$ is zero.

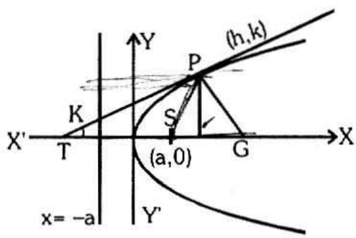
(iii) Centroid of the triangle formed by three co-normal points lies on the axis of parabola.

(iv) If $27ak^2 < 4(h - 2a)^3$ satisfied then three real and distinct normal are drawn from point (h, k) on parabola $y^2 = 4ax$.

(v) If three normals are drawn from point (h, 0) on parabola $y^2 = 4ax$, then $h > 2a$ and one of the normal is axis of the parabola and other two are equally inclined to the axis of the parabola.

18. IMPORTANT HIGHLIGHTS :

(a) If the tangent & normal at any point 'P' of the parabola intersect the axis at T & G then ST = SG = SP where 'S' is the focus. In other words the tangent and the normal at a point P on the parabola are the bisectors of the angle between



the focal radius SP & the perpendicular from P on the directrix. From this we conclude that all rays emanating from S will become parallel to the axis of the parabola after reflection.

(b) The portion of a tangent to a parabola cut off between the directrix & the curve subtends a right angle at the **focus**.

(c) The tangents at the extremities of a focal chord intersect at right angles on the **directrix,** and a circle on any focal chord as diameter touches the directrix. Also a circle on any focal radii of a point P $(at^2, 2at)$ as diameter touches the tangent at the vertex and intercepts a chord of length $a\sqrt{1+t^2}$ on a normal at the point P.

(d) Any tangent to a parabola & the perpendicular on it from the focus meet on the tangent at the vertex.

(e) Semi latus rectum of the parabola $y^2 = 4ax$, is the harmonic mean between segments of any focal chord

i.e. $2a = \dfrac{2bc}{b+c}$ or $\dfrac{1}{b}+\dfrac{1}{c}=\dfrac{1}{a}$.

(f) Image of the focus lies on diretrix with respect to any tangent of parabola $y^2 = 4ax$.

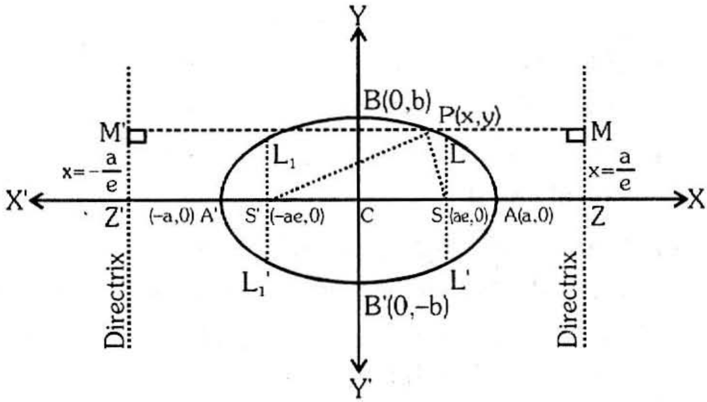
ELLIPSE

1. STANDARD EQUATION & DEFINITION :

Standard equation of an ellipse referred to its principal axis along the co-ordinate axis is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. where a > b & $b^2 = a^2(1-e^2)$

$\Rightarrow a^2 - b^2 = a^2 e^2$.

where e = eccentricity (0 < e < 1).



FOCI :S $\equiv$ (ae, 0) & S' $\equiv$ (− ae, 0).

(a) Equation of directrices :

$$x=\frac{a}{e} \ \& \ x=-\frac{a}{e}.$$

(b) Vertices :

A' $\equiv$ (−a, 0) & A $\equiv$ (a, 0).

(c) Major axis : The line segment A' A in which the foci S' & S lie is of length 2a & is called the **major axis** (a > b) of the ellipse. Point of intersection of major axis with directrix is called **the foot of the directrix (Z)** $\left(\pm\frac{a}{e},0\right)$.

(d) Minor Axis : The y-axis intersects the ellipse in the points $B' \equiv (0,-b)$ & $B \equiv (0, b)$. The line segment B'B of length 2b (b < a) is called the **Minor Axis** of the ellipse.

(e) Principal Axis : The major & minor axis together are called **Principal Axis** of the ellipse.

(f) Centre : The point which bisects every chord of the conic drawn through it is called the **centre** of the conic. $C \equiv (0,0)$ the origin is the centre of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

(g) Diameter : A chord of the conic which passes through the centre is called a **diameter** of the conic.

(h) Focal Chord : A chord which passes through a focus is called a **focal chord.**

(i) Double Ordinate : A chord perpendicular to the major axis is called a **double ordinate** with respect to major axis as diameter.

(j) Latus Rectum : The focal chord perpendicular to the major axis is called the **latus rectum**.

(i) Length of latus rectum

$$(LL') = \frac{2b^2}{a} = \frac{(\text{minor axis})^2}{\text{major axis}} = 2a(1-e^2)$$

(ii) Equation of latus rectum : $x = \pm ae$.

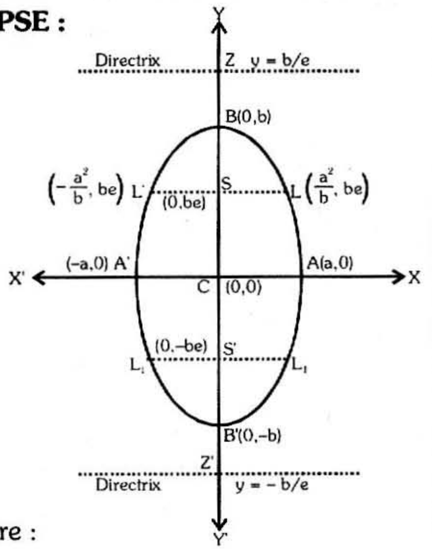
(iii) Ends of the latus rectum are $L\left(ae, \frac{b^2}{a}\right)$, $L'\left(ae, -\frac{b^2}{a}\right)$,

$L_1\left(-ae, \frac{b^2}{a}\right)$ and $L_1'\left(-ae, -\frac{b^2}{a}\right)$.

(k) Focal radii : SP= a −ex and S 'P = a + ex
$\Rightarrow$ SP + S 'P = 2a = Major axis.

(l) Eccentricity : $e = \sqrt{1-\frac{b^2}{a^2}}$

2. ANOTHER FORM OF ELLIPSE :



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \ (a < b)$$

(a) AA' = Minor axis = 2a

(b) BB' = Major axis = 2b

(c) $a^2 = b^2(1-e^2)$

(d) Latus rectum

$$LL' = L_1L_1' = \frac{2a^2}{b}.$$

equation $y = \pm be$

(e) Ends of the latus rectum are :

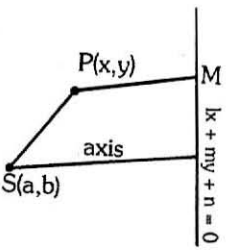
$$L\left(\frac{a^2}{b}, be\right), L'\left(-\frac{a^2}{b}, be\right), L_1\left(\frac{a^2}{b}, -be\right), L_1'\left(-\frac{a^2}{b}, -be\right)$$

(f) Equation of directrix $y = \pm \frac{b}{e}$.

(g) Eccentricity : $e = \sqrt{1 - \frac{a^2}{b^2}}$

3. GENERAL EQUATION OF AN ELLIPSE :

Let (a,b) be the focus S, and $lx + my + n = 0$ is the equation of directrix. Let P(x,y) be any point on the ellipse. Then by definition.



$\Rightarrow$ SP = e PM (e is the eccentricity)

$$\Rightarrow (x-a)^2 + (y-b)^2 = e^2 \frac{(lx+my+n)^2}{(l^2+m^2)}$$

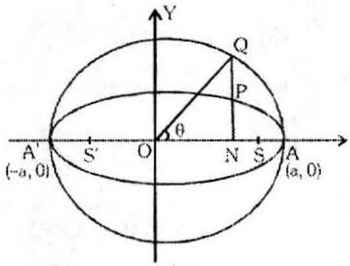
$$\Rightarrow (l^2 + m^2)\{(x-a)^2 + (y-b)^2\} = e^2\{lx + my + n\}^2$$

4. POSITION OF A POINT W.R.T. AN ELLIPSE :

The point $P(x_1, y_1)$ lies outside, inside or on the ellipse according as ; $\frac{x_1^{\ 2}}{a^2} + \frac{y_1^{\ 2}}{b^2} - 1 > <$ or $= 0$.

5. AUXILLIARY CIRCLE/ECCENTRIC ANGLE :

A circle described on major axis as diameter is called the **auxiliary circle.** Let Q be a point on the auxiliary circle $x^2 + y^2 = a^2$ such that QP produced is perpendicular to the x-axis then P & Q are called as the CORRESPONDING POINTS on the ellipse & the auxiliary circle respectively.'θ' is called the **ECCENTRIC ANGLE** of the point P on the ellipse $(0 \leq 0 < 2\pi)$.



Note that $\frac{l(PN)}{l(QN)} = \frac{b}{a} = \frac{\text{Semi minor axis}}{\text{Semi major axis}}$

Hence "If from each point of a circle perpendiculars are drawn upon a fixed diameter then the locus of the points dividing these perpendiculars in a given ratio is an ellipse of which the given circle is the auxiliary circle".

6. PARAMATRIC REPRESENTATION :

The equations $x = a\cos\theta$ & $y = b\sin\theta$ together represent the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

where θ is a parameter (eccentric angle).

Note that if $P(\theta) \equiv (a\cos\theta, b\sin\theta)$ is on the ellipse then ;

$Q(\theta) \equiv (a\cos\theta, a\sin\theta)$ is on the auxiliary circle.

7. LINE AND AN ELLIPSE :

The line $y = mx + c$ meets the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in two real points, coincident or imaginary according as c^2 is $<$ = or $>$ $a^2m^2 + b^2$.

Hence $y = mx + c$ is tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if $c^2 = a^2m^2 + b^2$.

The equation to the chord of the ellipse joining two points with eccentric angles α & β is given by $\frac{x}{a}\cos\frac{\alpha+\beta}{2} + \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha-\beta}{2}$.

8. TANGENT TO THE ELLIPSE $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$:

(a) Point form :

Equation of tangent to the given ellipse at its point (x_1, y_1) is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

(b) Slope form :

Equation of tangent to the given ellipse whose slope is 'm', is

$$y = mx \pm \sqrt{a^2m^2 + b^2}$$

Point of contact are $\left(\frac{\pm a^2m}{\sqrt{a^2m^2+b^2}}, \frac{\mp b^2}{\sqrt{a^2m^2+b^2}}\right)$

(c) Parametric form :

Equation of tangent to the given ellipse at its point $(a\cos\theta, b\sin\theta)$, is $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$

9. NORMAL TO THE ELLIPSE $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$:

(a) Point form : Equation of the normal to the given ellipse at (x_1, y_1) is $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2 = a^2e^2$.

(b) Slope form : Equation of a normal to the given ellipse whose slope is 'm' is $y = mx \mp \frac{(a^2 - b^2)m}{\sqrt{a^2 + b^2m^2}}$.

(c) Parametric form : Equation of the normal to the given ellipse at the point $(a\cos\theta, b\sin\theta)$ is $ax.\sec\theta - by.\operatorname{cosec}\theta = (a^2 - b^2)$.

10. CHORD OF CONTACT :

If PA and PB be the tangents from point $P(x_1,y_1)$ to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

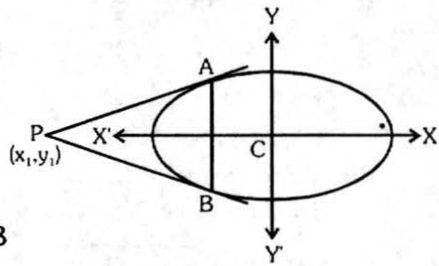
then the equation of the chord of contact AB is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ or $T = 0$ at (x_1,y_1)

11. PAIR OR TANGENTS :

If $P(x_1,y_1)$ be any point lies outside the ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

and a pair of tangents PA, PB can be drawn to it from P.



Then the equation of pair of tangents of PA and PB is $SS_1 = T^2$

where $S_1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$, $T = \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1$

i.e. $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1\right) = \left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1\right)^2$

12. DIRECTOR CIRCLE :

Locus of the point of intersection of the tangents which meet at right angles is called the **Director Circle.** The equation to this locus is $x^2 + y^2 = a^2 + b^2$ i.e. a circle whose centre is the centre of the ellipse & whose radius is the length of the line joining the ends of the major & minor axis.

13. EQUATION OF CHORD WITH MID POINT (x_1,y_1) :

The equation of the chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$,

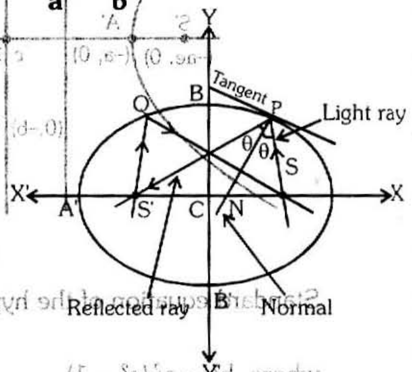
whose mid-point be (x_1,y_1) is $T = S_1$

where $T=\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1,\ S_1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$

i.e. $\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)=\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)$

14. IMPORTANT HIGHLIGHTS for $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$:

(I) The tangent & normal at a point P on the ellipse bisect the external & internal angles between the focal distances of P. This refers to the well known reflection property of the ellipse which states that rays from one focus are reflected through other focus & vice-versa.



(II) Point of intersection of the tangents at the point α & β is $\left(a\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}},\ b\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$

(III) If $A(\alpha)$, $B(\beta)$, $C(\gamma)$ & $D(\delta)$ are conormal points then sum of their eccentric angles is odd multiple of π. i.e. $\alpha+\beta+\gamma+\delta = (2n+1)\pi$.

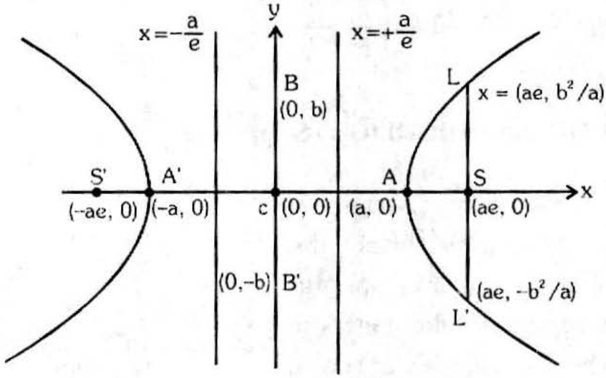
(IV) If $A(\alpha)$, $B(\beta)$, $C(\gamma)$ & $D(\delta)$ are four concyclic points then sum of their eccentric angles is even multiple of π.
i.e. $\alpha+\beta+\gamma+\delta = 2n\pi$

(V) The product of the length's of the perpendicular segments from the foci on any tangent to the ellipse is b^2 and the feet of these perpendiculars lie on its auxiliary circle.

HYPERBOLA

The **Hyperbola** is a conic whose eccentricity is greater than unity. $(e > 1)$.

1. STANDARD EQUATION & DEFINITION(S) :



Standard equation of the hyperbola is $\frac{\mathbf{x^2}}{\mathbf{a^2}} - \frac{\mathbf{y^2}}{\mathbf{b^2}} = \mathbf{1}$,

where $b^2 = a^2(e^2 - 1)$

or $a^2e^2 = a^2 + b^2$ i.e. $e^2 = 1 + \frac{b^2}{a^2} = 1 + \left(\frac{\text{Conjugate Axis}}{\text{Transverse Axis}}\right)^2$

(a) Foci :

$S \equiv (ae, 0)$ & $S' \equiv (-ae, 0)$.

(b) Equations of directrices :

$x = \frac{a}{e}$ & $x = -\frac{a}{e}$.

(c) Vertices :

$A \equiv (a, 0)$ & $A' \equiv (-a, 0)$.

(d) Latus rectum :

(i) Equation : $x = \pm ae$

(ii) Length $= \dfrac{2b^2}{a} = \dfrac{(\text{Conjugate Axis})^2}{(\text{Transverse Axis})} = 2a\,(e^2 - 1)$

$= 2e(\text{distance from focus to directrix})$

(iii) Ends : $\left(ae, \dfrac{b^2}{a}\right), \left(ae, \dfrac{-b^2}{a}\right)$; $\left(-ae, \dfrac{b^2}{a}\right), \left(-ae, \dfrac{-b^2}{a}\right)$

(e) (i) Transverse Axis :

The line segment A′A of length 2a in which the foci S′ & S both lie is called the Transverse Axis of the Hyperbola.

(ii) Conjugate Axis :

The line segment B′B between the two points $B' \equiv (0, -b)$ & $B \equiv (0, b)$ is called as the Conjugate Axis of the Hyperbola.

The Transverse Axis & the Conjugate Axis of the hyperbola are together called the Principal axis of the hyperbola.

(f) Focal Property :

The difference of the focal distances of any point on the hyperbola is constant and equal to transverse axis i.e. $\left| |PS| - |PS'| \right| = 2a$. The distance SS′ = focal length.

(g) Focal distance :

Distance of any point P(x, y) on hyperbola from foci $PS = ex - a$ & $PS' = ex + a$.

2. CONJUGATE HYPERBOLA :

Two hyperbolas such that transverse & conjugate axis of one hyperbola are respectively the conjugate & the transverse axis of the other are called **Conjugate Hyperbolas** of each other. eg. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ &

$-\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ are conjugate hyperbolas of each other.

Note that :

(i) If e_1 & e_2 are the eccentricities of the hyperbola & its conjugate then $e_1^{-2} + e_2^{-2} = 1$.

(ii) The foci of a hyperbola and its conjugate are concyclic and form the vertices of a square.

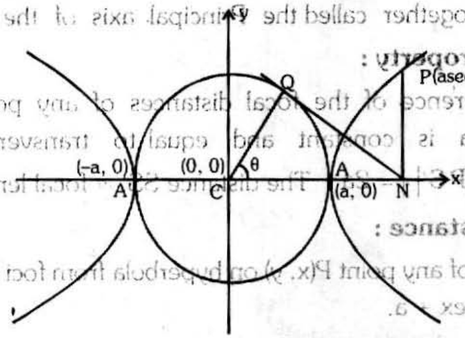
(iii) Two hyperbolas are said to be similar if they have the same eccentricity.

3. RECTANGULAR OR EQUILATERAL HYPERBOLA :

The particular kind of hyperbola in which the lengths of the transverse & conjugate axis are equal is called an **Equilateral Hyperbola**.

Note that the eccentricity of the rectangular hyperbola is $\sqrt{2}$ and the length of it's latus rectum is equal to it's transverse or conjugate axis.

4. AUXILIARY CIRCLE :



A circle drawn with centre C & transverse axis as a diameter is called the **Auxiliary Circle** of the hyperbola. Equation of the auxiliary circle is $x^2 + y^2 = a^2$.

Note from the figure that P & Q are called the **"Corresponding Points"** on the hyperbola & the auxiliary circle. 'θ' is called the eccentric angle of the point 'P' on the hyperbola. $(0 \leq \theta < 2\pi)$.

Parametric Equation :

The equations $x = a \sec\theta$ & $y = b \tan\theta$ together represents the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where θ is a parameter.

5. POSITION OF A POINT 'P' w.r.t. A HYPERBOLA :

The quantity $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$ is positive, zero or negative according as the points (x_1, y_1) lies within, upon or outside the curve.

6. LINE AND A HYPERBOLA :

The straight line $y = mx + c$ is a secant, a tangent or passes outside the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ according as : $c^2 > = < a^2 m^2 - b^2$.

Equation of a chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ joining its two points $P(\alpha)$ & $Q(\beta)$ is $\frac{x}{a}\cos\frac{\alpha - \beta}{2} - \frac{y}{b}\sin\frac{\alpha + \beta}{2} = \cos\frac{\alpha + \beta}{2}$

7. TANGENT TO THE HYPERBOLA $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$:

(a) Point form : Equation of the tangent to the given hyperbola at the point (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.

Note : In general two tangents can be drawn from an external point $(x_1 y_1)$ to the hyperbola and they are $y - y_1 = m_1 (x - x_1)$ & $y - y_1 = m_2 (x - x_2)$, where m_1 & m_2 are roots of the equation $(x_1^2 - a^2) m^2 - 2 x_1 y_1 m + y_1^2 + b^2 = 0$. If $D < 0$, then no tangent can be drawn from $(x_1 y_1)$ to the hyperbola.

(b) Slope form : The equation of tangents of slope m to the given hyperbola is $y = m x \pm \sqrt{a^2m^2 - b^2}$. Point of contact are $\left(\frac{\pm a^2 m}{\sqrt{a^2m^2 - b^2}}, \frac{\pm b^2}{\sqrt{a^2m^2 - b^2}}\right)$

Note that there are two parallel tangents having the same slope m.

(c) Parametric form : Equation of the tangent to the given hyperbola at the point $(a \sec\theta, b \tan\theta)$ is

$$\frac{x \sec\theta}{a} - \frac{y \tan\theta}{b} = 1.$$

Note : Point of intersection of the tangents at θ_1 & θ_2 is

$$x = a\frac{\cos\left(\frac{\theta_1 - \theta_2}{2}\right)}{\cos\left(\frac{\theta_1 + \theta_2}{2}\right)}, \quad y = b\tan\left(\frac{\theta_1 + \theta_2}{2}\right)$$

8. NORMAL TO THE HYPERBOLA $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$:

(a) Point form : Equation of the normal to the given hyperbola at the point $P(x_1, y_1)$ on it is $\frac{a^2 x}{x_1} + \frac{b^2 y}{y_1} = a^2 + b^2 = a^2 e^2$.

(b) Slope form : The equation of normal of slope m to the given hyperbola is $y = mx \mp \frac{m(a^2 + b^2)}{\sqrt{(a^2 - m^2b^2)}}$ foot of normal are

$$\left(\pm\frac{a^2}{\sqrt{(a^2 - m^2b^2)}}, \mp\frac{mb^2}{\sqrt{(a^2 - m^2b^2)}}\right).$$

(c) Parametric form : The equation of the normal at the point $P(a \sec\theta, b \tan\theta)$ to the given hyperbola is

$$\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2 = a^2 e^2.$$

9. DIRECTOR CIRCLE :

The locus of the intersection of tangents which are at right angles is known as the **Director Circle** of the hyperbola. The equation to the director circle is : $x^2 + y^2 = a^2 - b^2$.

If $b^2 < a^2$ this circle is real ; if $b^2 = a^2$ the radius of the circle is zero & it reduces to a point circle at the origin. In this case the centre is the only point from which the tangents at right angles can be drawn to the curve.

If $b^2 > a^2$, the radius of the circle is imaginary, so that there is no such circle & so no tangents at right angle can be drawn to the curve.

10. CHORD OF CONTACT :

If PA and PB be the tangents from point $P(x_1,y_1)$ to the Hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then the equation of the chord of contact AB is

$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$ or $T = 0$ at (x_1,y_1)

11. PAIR OR TANGENTS :

If $P(x_1,y_1)$ be any point lies outside the Hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and a pair of tangents PA, PB can be drawn to it from P. Then the equation of pair of tangents of PA and PB is $SS_1 = T^2$

where $S_1=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$, $T=\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1$

i.e. $\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1\right)=\left(\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1\right)^2$

12. EQUATION OF CHORD WITH MID POINT (x_1,y_1) :

The equation of the chord of the ellipse $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$,

whose mid-point be (x_1,y_1) is $T = S_1$

where $T=\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1$, $S_1=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$

i.e. $\left(\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1\right)=\left(\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1\right)$

13. ASYMPTOTES :

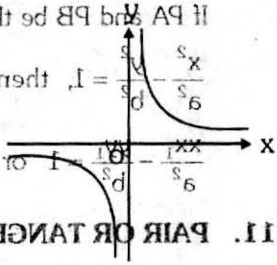
Definition : If the length of the perpendicular let fall from a point on a hyperbola to a straight line tends to zero as the point on the hyperbola moves to infinity along the hyperbola, then the straight line is called the **Asymptote of the Hyperbola**.

Combined equation of asymptotes of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ will be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$$

14. RECTANGULAR HYPERBOLA :

Rectangular hyperbola referred to its asymptotes as axis of coordinates.



(a) Equation is $xy = c^2$ with parametric representation $x = ct, y = c/t, t \in R - \{0\}$.

(b) Equation of a chord joining the points $P(t_1)$ & $Q(t_2)$ is $x + t_1 t_2 y = c(t_1 + t_2)$ with slope $m = -\frac{1}{t_1 t_2}$

(c) Equation of the tangent at $P(x_1, y_1)$ is $\frac{x}{x_1} + \frac{y}{y_1} = 2$

& at $P(t)$ is $\frac{x}{t} + ty = 2c$.

(d) Equation of normal is $y - \frac{c}{t} = t^2(x - ct)$

(e) Chord with a given middle point as (h, k) is $kx + hy = 2hk$.

15. IMPORTANT HIGHLIGHTS :

(i) The tangent and normal at any point of a hyperbola bisect the angle between the focal radii.

(ii) Reflection property of the hyperbola : An incoming light ray aimed towards one focus is reflected from the outer surface of the hyperbola towards the other focus.

FUNCTION

1. DEFINITION :

If to every value (considered as real unless other-wise stated) of a variable x, which belongs to a set A, there corresponds one and only one finite value of the quantity y which belong to set B, then y is said to be a function of x and written as $f : A \to B$, $y = f(x)$, x is called argument or independent variable and y is called dependent variable.

Pictorially : $\xrightarrow[\text{input}]{x} \boxed{f} \xrightarrow[\text{output}]{f(x) = y}$

y is called the image of x & x is the pre-image of y, under f.

Every function $f : A \to B$ satisfies the following conditions.

(i) $f \subset A \times B$ **(ii)** $\forall a \in A \;\; \exists \;\; b \in B$ such that $(a, b) \in f$ and

(iii) If $(a, b) \in f$ & $(a, c) \in f \;\; \Rightarrow \;\; b = c$

2. DOMAIN, CO-DOMAIN & RANGE OF A FUNCTION :

Let $f : A \to B$, then the set A is known as the domain of 'f' & the set B is known as co-domain of 'f'. The set of all f images of elements of A is known as the range of 'f'. Thus

Domain of $f = \{x \mid x \in A, (x, f(x)) \in f\}$

Range of $f = \{f(x) \mid x \in A, f(x) \in B\}$

Range is a subset of co-domain.

3. IMPORTANT TYPES OF FUNCTION :

(a) Polynomial function :

If a function 'f' is called by $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots + a_{n-1}x + a_n$ where n is a non negative integer and $a_0, a_1, a_2, \ldots. a_n$ are real numbers and $a_0 \neq 0$, then f is called a polynomial function of degree n.

Note :

(i) A polynomial of degree one with no constant term is called an odd linear function. i.e. $f(x) = ax, a \neq 0$

(ii) There are two polynomial functions, satisfying the relation ; $f(x).\ f(1/x) = f(x) + f(1/x)$. They are :

(a) $f(x) = x^n + 1$ &

(b) $f(x) = 1 - x^n$, where n is a positive integer.

(iii) Domain of a polynomial function is R

(iv) Range of odd degree polynomial is R whereas range of an even degree polynomial is never R.

(b) Algebraic function :

A function 'f' is called an algebraic function if it can be constructed using algebraic operations (such as addition, subtraction, multiplication, division, and taking radicals) starting with polynomials.

(c) Rational function :

A rational function is a function of the form $y = f(x) = \dfrac{g(x)}{h(x)}$, where g(x) & h(x) are polynomials & $h(x) \neq 0$,

Domain : $R - \{x \mid h(x) = 0\}$

Any rational function is automatically an algebraic function.

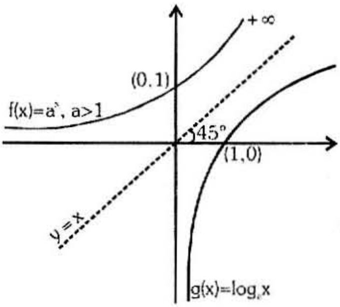
(d) Exponential and Logarithmic Function :

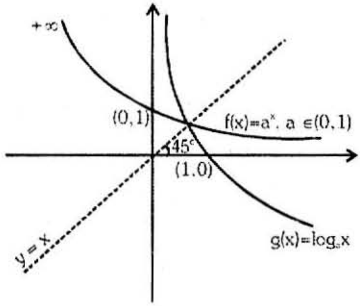
A function $f(x) = a^x (a > 0), a \neq 1, x \in R$ is called an exponential function. The inverse of the exponential function is called the logarithmic function, i.e. $g(x) = \log_a x$.

Note that f(x) & g(x) are inverse of each other & their graphs are as shown. (Functions are mirror image of each other about the line y = x)

Domain of a^x is R **Range** R^+

Domain of $\log_a x$ is R^+ **Range** R



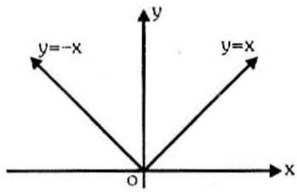


(e) Absolute value function :

It is defined as : $y = |x|$

$$\Rightarrow \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$

Also defined as $\max\{x, -x\}$



Domain : R **Range :** $[0, \infty)$

Note : $f(x)=\dfrac{1}{|x|}$ **Domain :** $R-\{0\}$ **Range :** R^+

Properties of modulus function :

For any x, y, a ∈ R.

(i) $|x| \geq 0$

(ii) $|x| = |-x|$

(iii) $|xy| = |x|\,|y|$

(iv) $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}$; $b \neq 0$

(v) $|x| = a \Rightarrow x = \pm a$

(vi) $\sqrt{x^2} = |x|$

(vii) $|x| \geq a \Rightarrow x \geq a$ or $x \leq -a$. where a is positive.

(viii) $|x| \leq a \Rightarrow x \in [-a, a]$. where a is positive

(ix) $|x| > |y| \Rightarrow x^2 > y^2$

(x) $\big||x|-|y|\big| \leq |x|+|y| = \begin{cases} (a)\,|x|+|y|=|x+y| & \Rightarrow \quad xy \geq 0 \\ (b)\,|x|+|y|=|x-y| & \Rightarrow \quad xy \leq 0 \end{cases}$

(f) Signum function :

Signum function y = sgn (x) is defined as follows

$$y=\begin{cases}\dfrac{|x|}{x}, x\neq 0\\ 0, x=0\end{cases}=\begin{cases}1 & \text{for } x>0\\ 0 & \text{for } x=0\\ -1 & \text{for } x<0\end{cases}$$

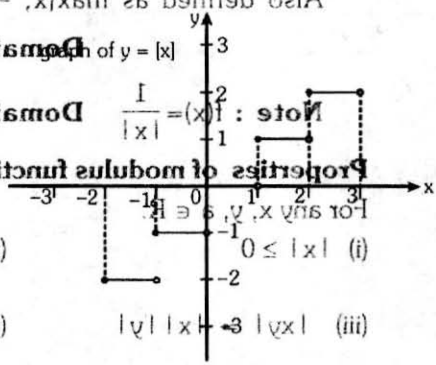
Domain : R

Range: {–1, 0, 1}

(g) Greatest integer or step up function :

The function y = f(x) = [x] is called the greatest integer function where [x] denotes the greatest integer less than or equal to x. Note that for :

x	[x]
[–2,–1)	–2
[–1,0)	–1
[0,1)	0
[1,2)	1



Domain : R

Range : I

Properties of greatest integer function :

(i) $[x] \le x < [x] + 1$ and $x - 1 < [x] \le x$, $0 \le x - [x] < 1$

(ii) $[x+y]=\begin{cases}[x]+[y] & \{x\}+\{y\}\in[0,1)\\ [x]+[y]+1 & \{x\}+\{y\}\in[1,2)\end{cases}$

(iii) $[x]+[-x]=\begin{cases}0, & x\in I\\ -1, & x\notin I\end{cases}$

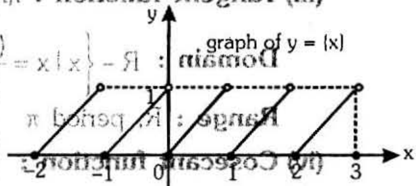
Note : $f(x) = \dfrac{1}{[x]}$

Domain : $R - [0, 1)$ **Range :** $\left\{\dfrac{1}{n},\ n \in I - \{0\}\right\}$

(h) Fractional part function :

It is defined as : $g(x) = \{x\} = x - [x]$.

x	{x}
[–2,–1)	x+2
[–1,0)	x+1
[0,1)	x
[1,2)	x–1

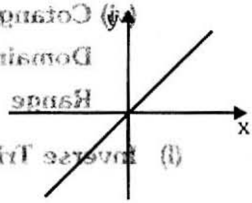


Domain : R **Range :** [0,1) **Period :** 1

Note : $f(x) = \dfrac{1}{\{x\}}$ **Domain :** $R - I$ **Range :** $(1, \infty)$

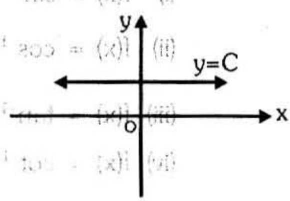
(i) Identity function :

The function $f : A \to A$ defined by $f(x) = x\ \forall\ x \in A$ is called the identity function on A and is denoted by I_A.



(j) Constant function :

$f : A \to B$ is said to be constant function if every element of A has the same f image in B. Thus $f : A \to B$; $f(x) = c,\ \forall\ x \in A$, $c \in B$ is constant function.



Domain : R **Range :** {c}

(k) Trigonometric functions :

(i) Sine function : $f(x) = \sin x$

Domain : R **Range :** $[-1, 1]$, period 2π

(ii) Cosine function : $f(x) = \cos x$

Domain : R **Range :** $[-1, 1]$, period 2π

(iii) Tangent function : $f(x) = \tan x$

Domain : $R - \left\{x \mid x = \frac{(2n+1)\pi}{2}, n \in I\right\}$

Range : R, period π

(iv) Cosecant function : $f(x) = \operatorname{cosec} x$

Domain : $R - \{x \mid x = n\pi, n \in I\}$

Range : $R - (-1, 1)$, period 2π

(v) Secant function : $f(x) = \sec x$

Domain : $R - \{x \mid x = (2n + 1)\pi/2 : n \in I\}$

Range : $R - (-1, 1)$, period 2π

(vi) Cotangent function : $f(x) = \cot x$

Domain : $R - \{x \mid x = n\pi, n \in I\}$

Range : R, period π

(l) Inverse Trigonometric function :

(i) $f(x) = \sin^{-1} x$ **Domain :** $[-1, 1]$ **Range :** $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(ii) $f(x) = \cos^{-1} x$ **Domain :** $[-1, 1]$ **Range :** $[0, \pi]$

(iii) $f(x) = \tan^{-1} x$ **Domain :** R **Range :** $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(iv) $f(x) = \cot^{-1} x$ **Domain :** R **Range :** $(0, \pi)$

(v) $f(x) = \operatorname{cosec}^{-1} x$ **Domain :** $R - (-1, 1)$ **Range :** $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

(vi) $f(x) = \sec^{-1} x$ **Domain :** $R - (-1, 1)$ **Range :** $[0, \pi] - \left\{\frac{\pi}{2}\right\}$

4. EQUAL OR IDENTICAL FUNCTION :

Two function f & g are said to be equal if :

(a) The domain of f = the domain of g

(b) The range of f = range of g and

(c) f(x) = g (x), for every x belonging to their common domain (i.e. should have the same graph)

5. ALGEBRAIC OPERATIONS ON FUNCTIONS :

If f & g are real valued functions of x with domain set A, B respectively, f + g, f − g, (f . g) & (f/g) as follows :

(a) $(f \pm g)(x) = f(x) \pm g(x)$ domain in each case is $A \cap B$

(b) $(f.g)(x) = f(x).g(x)$ domain is $A \cap B$

(c) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ domain $A \cap B - \{x \mid g(x) = 0\}$

6. CLASSIFICATION OF FUNCTIONS :

(a) One-One function (Injective mapping) :

A function $f : A \to B$ is said to be a one-one function or injective mapping if different elements of A have different f images in B. Thus for $x_1, x_2 \in A$ & $f(x_1), f(x_2) \in B$, $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$ or $x_1 \neq x_2 \Leftrightarrow f(x_1) \neq f(x_2)$.

Note:

(i) Any continuous function which is entirely increasing or decreasing in whole domain is one-one.

(ii) If a function is one-one, any line parallel to x-axis cuts the graph of the function at atmost one point

(b) Many-one function :

A function $f : A \to B$ is said to be a many one function if two or more elements of A have the same f image in B.

Thus $f : A \to B$ is many one if $\exists\ x_1, x_2 \in A$, $f(x_1) = f(x_2)$ but $x_1 \neq x_2$

Note : If a continuous function has local maximum or local minimum, then f(x) is many-one because atleast one line parallel to x-axis will intersect the graph of function atleast twice.

Total number of functions

= number of one-one functions + number of many-one function

(c) Onto function (Surjective) :

If range = co-domain, then f(x) is onto.

(d) Into function :

If $f : A \to B$ is such that there exists atleast one element in co-domain which is not the image of any element in domain, then f(x) is into.

Note :

(i) If 'f' is both injective & surjective, then it is called a **Bijective** mapping. The bijective functions are also named as invertible, non singular or biuniform functions.

(ii) If a set A contains n distinct elements then the number of different functions defined from $A \to A$ is n^n & out of it n! are one one and rest are many one.

(iii) $f : R \to R$ is a polynomial

(a) Of even degree, then it will neither be injective nor surjective.

(b) Of odd degree, then it will always be surjective, no general comment can be given on its injectivity.

7. COMPOSITE OF UNIFORMLY & NON-UNIFORMLY DEFINED FUNCTION :

Let $f : A \to B$ & $g : B \to C$ be two functions. Then the function gof : $A \to C$ defined by $(gof)(x) = g(f(x)) \ \forall \ x \in A$ is called the composite of the two functions f & g.

Hence in gof(x) the range of 'f' must be a subset of the domain of 'g'. $x \longrightarrow \boxed{f} \longrightarrow \boxed{g} \longrightarrow g(f(x))$

Properties of composite functions:

(a) In general composite of functions is not commutative i.e. gof $\neq$ fog.

(b) The composite of functions is associative i.e. if f, g, h are three functions such that fo(goh) & (fog)oh are defined, then fo(goh) = (fog)oh.

(c) The composite of two bijections is a bijection i.e. if f & g are two bijections such that gof is defined, then gof is also a bijection.

(d) If gof is one-one function then f is one-one but g may not be one-one.

8. HOMOGENEOUS FUNCTIONS :

A function is said to be homogeneous with respect to any set of variables when each of its terms is of the same degree with respect to those variables.

For examples $5x^2 + 3y^2 - xy$ is homogenous in x & y. Symbolically if, $f(tx, ty) = t^n f(x, y)$, then f(x, y) is homogeneous function of degree n.

9. BOUNDED FUNCTION :

A function is said to be bounded if $|f(x)| \leq M$, where M is a finite quantity.

10. IMPLICIT & EXPLICIT FUNCTION :

A function defined by an equation not solved for the dependent variable is called an **implicit function**. e.g. the equation $x^3 + y^3 = 1$ defines y as an implicit function. If y has been expressed in terms of x alone then it is called an **Explicit function.**

11. INVERSE OF A FUNCTION :

Let $f : A \to B$ be a one-one & onto function, then their exists a unique function $g : B \to A$ such that $f(x) = y \Leftrightarrow g(y) = x$, $\forall\ x \in A$ & $y \in B$. Then g is said to be inverse of f.

Thus $g = f^{-1} : B \to A = \{(f(x), x)) \mid (x, f(x)) \in f\}$

Properties of inverse function :

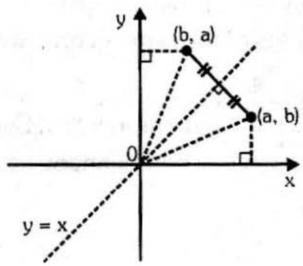
(a) The inverse of a bijection is unique.

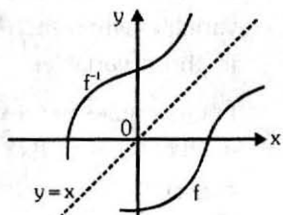
(b) If $f : A \rightarrow B$ is a bijection & $g : B \rightarrow A$ is the inverse of f, then $fog = I_B$ and $gof = I_A$, where I_A & I_B are identity functions on the sets A & B respectively. If $fof = I$, then f is inverse of itself.

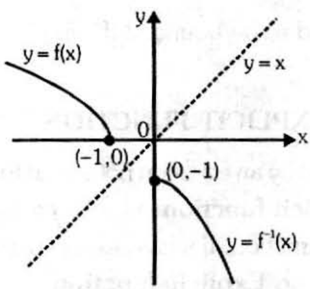
(c) The inverse of a bijection is also a bijection.

(d) If f & g are two bijections $f : A \rightarrow B$, $g : B \rightarrow C$ & gof exist, then the inverse of gof also exists and $(gof)^{-1} = f^{-1}og^{-1}$.

(e) Since $f(a) = b$ if and only if $f^{-1}(b) = a$, the point (a, b) is on the graph of 'f' if and only if the point (b, a) is on the graph of f^{-1}. But we get the point (b, a) from (a, b) by reflecting about the line $y = x$.







The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

12. ODD & EVEN FUNCTIONS :

If a function is such that whenever 'x' is in it's domain '–x' is also in it's domain & it satisfies

$f(-x) = f(x)$ it is an even function

$f(-x) = -f(x)$ it is an odd function

Note :

(i) A function may neither be odd nor even.

(ii) Inverse of an even function is not defined, as it is many-one function.

(iii) Every even function is symmetric about the y-axis & every odd function is symmetric about the origin.

(iv) Every function which has '$-x$' in it's domain whenever 'x' is in it's domain, can be expressed as the sum of an even & an odd function.

$$\text{e.g. } f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{EVEN}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{ODD}}$$

(v) The only function which is defined on the entire number line & even and odd at the same time is $f(x) = 0$

(vi) If $f(x)$ and $g(x)$ both are even or both are odd then the function $f(x) \,.\, g(x)$. will be even but if any one of them is odd & other is even, then $f.g$ will be odd.

13. PERIODIC FUNCTION :

A function $f(x)$ is called periodic if there exists a positive number $T (T > 0)$ called the period of the function such that $f(x + T) = f(x)$, for all values of x within the domain of $f(x)$.

Note :

(i) Inverse of a periodic function does not exist.

(ii) Every constant function is periodic, with no fundamental period.

(iii) If $f(x)$ has a period T & $g(x)$ also has a period T then it does not mean that $f(x) + g(x)$ must have a period T. e.g. $f(x) = |\sin x| + |\cos x|$.

(iv) If f(x) has period p and g(x) has period q, then period of f(x) + g(x) will be LCM of p & q provided f(x) & g(x) are non interchangeable. If f(x) & g(x) can be interchanged by adding a least positive number r, then smaller of LCM & r will be the period.

(v) If f(x) has period p, then $\frac{1}{f(x)}$ and $\sqrt{f(x)}$ also has a period p.

(vi) If f(x) has period T then f(ax + b) has a period T/a (a > 0).

(vii) $|\sin x|, |\cos x|, |\tan x|, |\cot x|, |\sec x|$ & $|\text{cosec}x|$ are periodic function with period π.

(viii) $\sin^n x, \cos^n x, \sec^n x, \text{cosec}^n x$, are periodic function with period 2π when 'n' is odd or π when n is even.

(ix) $\tan^n x, \cos^n x$ are periodic function with period π.

14. GENERAL :

If x, y are independent variables, then :

(a) $f(xy) = f(x) + f(y) \Rightarrow f(x) = k\ell n\ x$

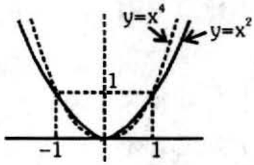
(b) $f(xy) = f(x) . f(y) \Rightarrow f(x) = x^n, n \in R$ or $f(x) = 0$

(c) $f(x + y) = f(x) . f(y) \Rightarrow f(x) = a^{kx}$ or $f(x) = 0$

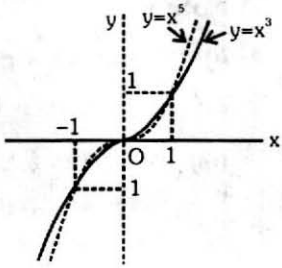
(d) $f(x + y) = f(x) + f(y) \Rightarrow f(x) = kx$, where k is a constant.

15. SOME BASIC FUNCTION & THEIR GRAPH :

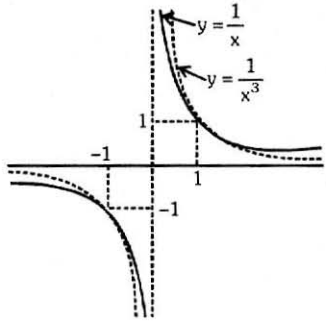
(a) $y = x^{2n}$, where $n \in N$



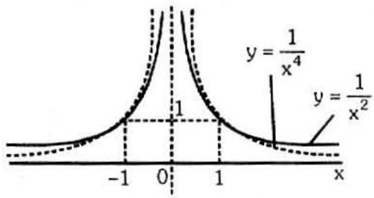
(b) $y = x^{2n+1}$, where $n \in N$



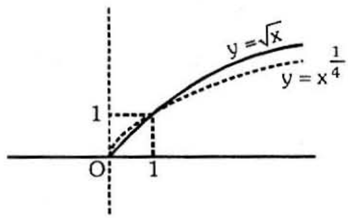
(c) $y = \dfrac{1}{x^{2n-1}}$, where $n \in N$



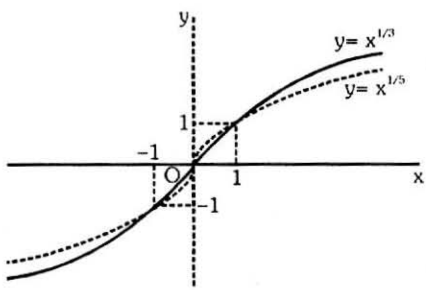
(d) $y = \dfrac{1}{x^{2n}}$, where $n \in N$



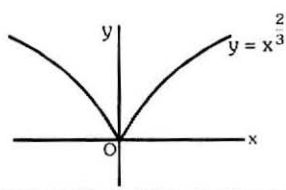
(e) $y = x^{\frac{1}{2n}}$, where $n \in N$



(f) $y = x^{\frac{1}{2n+1}}$, where $n \in N$

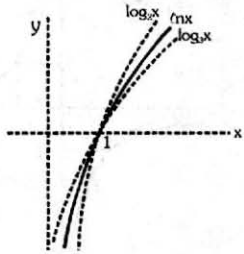


Note : $y = x^{2/3}$

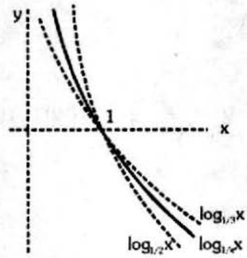


(g) $y = \log_a x$

when $a > 1$

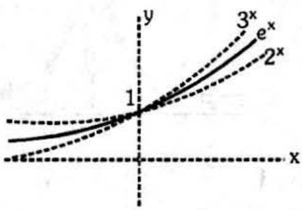


when $0 < a < 1$

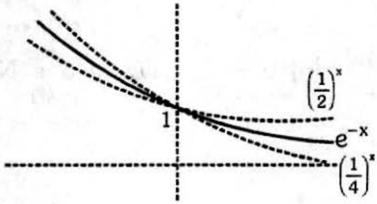


(h) $y = a^x$

$a > 1$

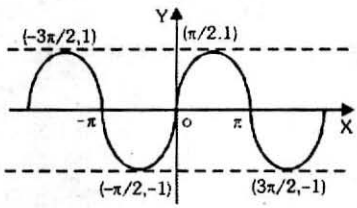


$0 < a < 1$

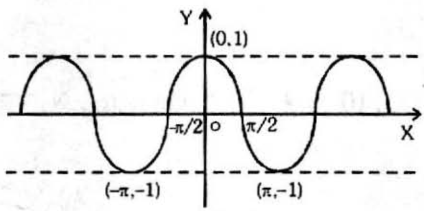


(i) Trigonometric functions :

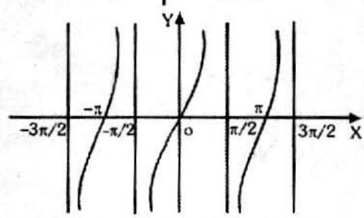
$y = \sin x$



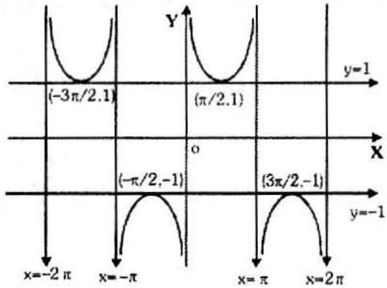
$y = \cos x$



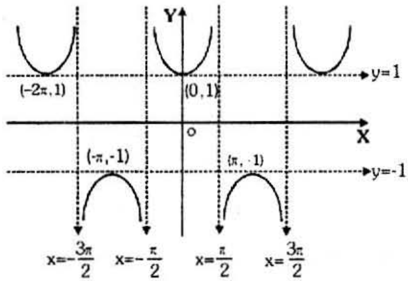
$y = \tan x$



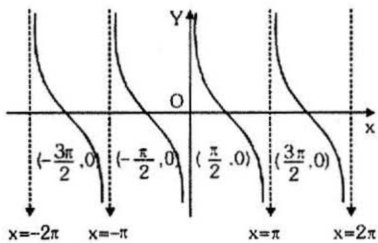
$y = \operatorname{cosec} x$



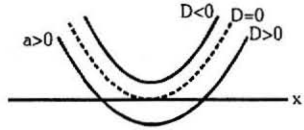
$y = \sec x$



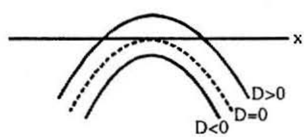
$y = \cot x$



(j) $y = ax^2 + bx + c$



vertex $\left(-\dfrac{b}{2a}, -\dfrac{D}{4a}\right)$



where $D = b^2 - 4ac$

16. TRANSFORMATION OF GRAPH :

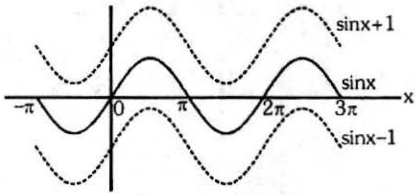
(a) when $f(x)$ transforms to $f(x) + k$

if $k > 0$ then shift graph of $f(x)$ upward through k

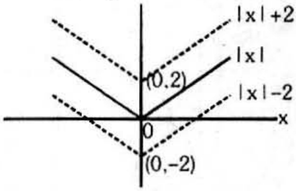
if $k < 0$ then shift graph of $f(x)$ downward through k

Examples :

1.



2.



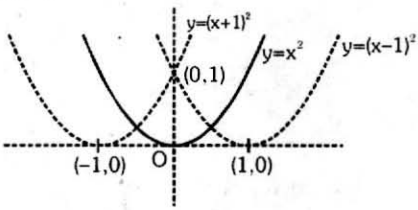
(b) $f(x)$ transforms to $f(x + k)$:

if $k > 0$ then shift graph of $f(x)$ through k towards left.

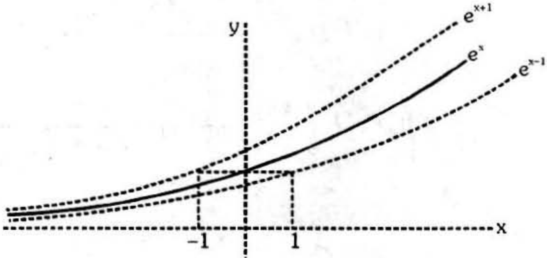
if $k < 0$ then shift graph of $f(x)$ through k towards right.

Examples :

1.



2.

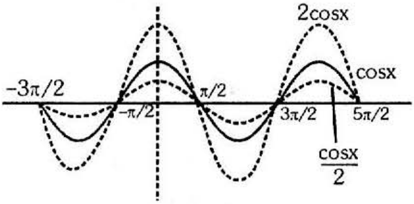


(c) $f(x)$ transforms to $kf(x)$:

if $k > 1$ then stretch graph of $f(x)$ k times along y-axis

if $0 < k < 1$ then shrink graph of $f(x)$, k times along y-axis

Examples :

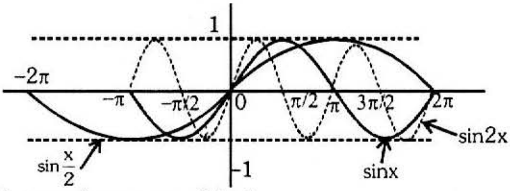


(d) $f(x)$ transforms to $f(kx)$:

if $k > 1$ then shrink graph of $f(x)$, 'k' times along x-axis

if $0 < k < 1$ then stretch graph of $f(x)$, 'k' times along x-axis

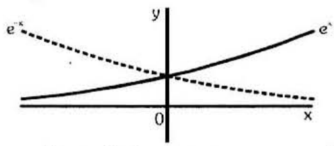
Examples :

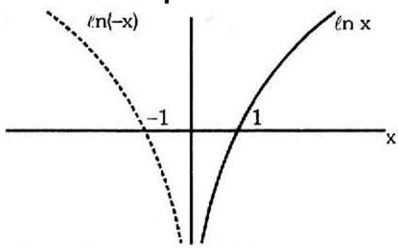


(e) $f(x)$ transforms to $f(-x)$:

Take mirror image of the curve $y = f(x)$ in y-axis as plane mirror

Example :

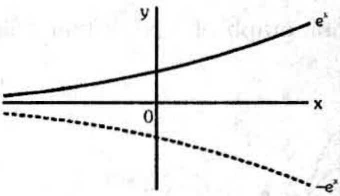
1. 

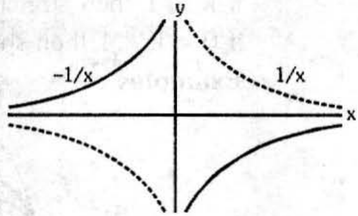
2. 

(f) $f(x)$ transforms to $-f(x)$:

Take image of $y = f(x)$ in the x axis as plane mirror

Examples :

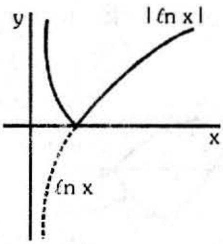


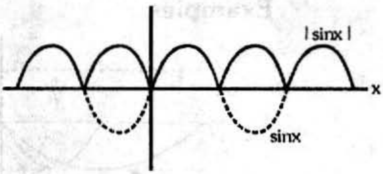


(g) $f(x)$ transforms to $|f(x)|$:

Take mirror image (in a axis) of the portion of the graph of $f(x)$ which lies below x-axis.

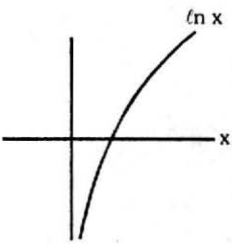
Examples :

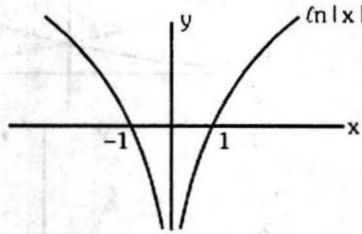


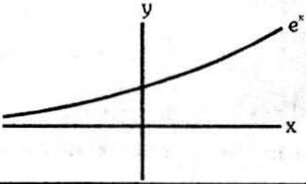


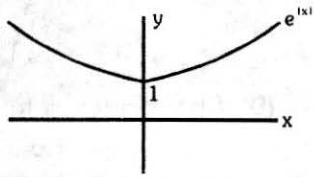
(h) $f(x)$ transforms to $f(|x|)$:

Neglect the curve for $x < 0$ and take the image of curve for $x \geq 0$ about y-axis.





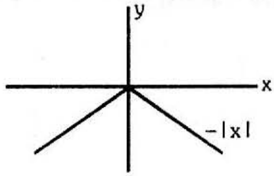


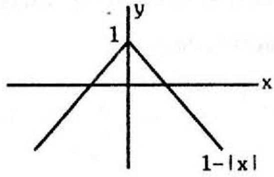


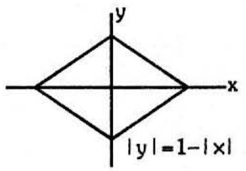
(i) $y = f(x)$ transforms to $|y| = f(x)$:

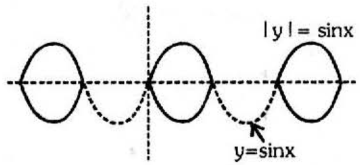
Remove the portion of graph which lies below x-axis & then take mirror image {in x axis} of remaining portion of graph

Examples :







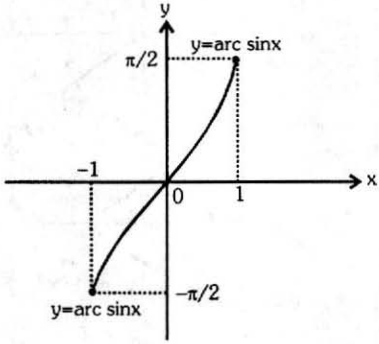


INVERSE TRIGONOMETRIC FUNCTION

1. DOMAIN, RANGE & GRAPH OF INVERSE TRIGONOMETRIC FUNCTIONS :

(a) $f^{-1} : [-1, 1] \to [-\pi/2, \pi/2]$

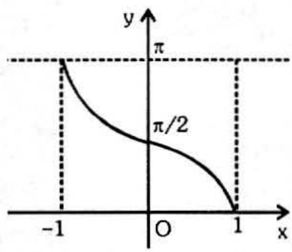
$f^{-1}(x) = \sin^{-1}(x)$



$(y = \sin^{-1}x)$

(b) $f^{-1} : [-1, 1] \to [0, \pi]$

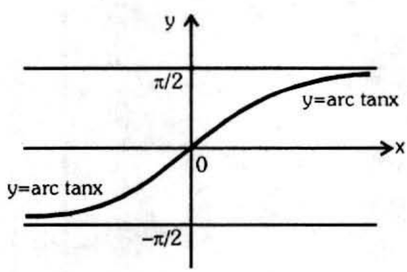
$f^{-1}(x) = \cos^{-1} x$



$(y = \cos^{-1}x)$

(c) $f^{-1} : R \to (-\pi/2, \pi/2)$

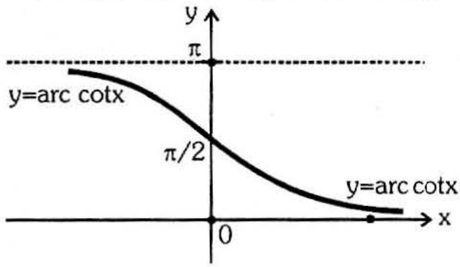
$f^{-1}(x) = \tan^{-1} x$



$(y = \tan^{-1}x)$

(d) $f^{-1} : R \to (0, \pi)$

$f^{-1}(x) = \cot^{-1} x$

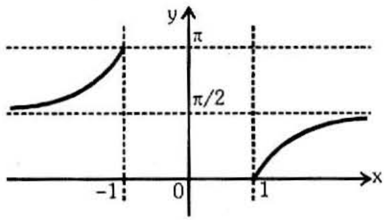


$(y = \cot^{-1}x)$

(e) $f^{-1} : (-\infty, -1] \cup [1, \infty)$

$\to [0, \pi/2) \cup (\pi/2, \pi]$

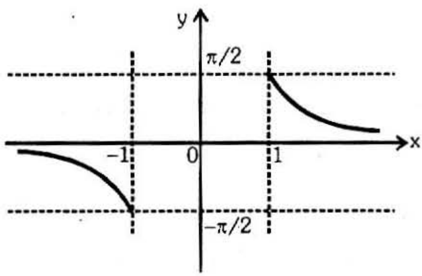
$f^{-1}(x) = \sec^{-1} x$



(f) $f^{-1} : (-\infty, -1] \cup [1, \infty)$

$\to [-\pi/2, 0) \cup (0, \pi/2]$

$f^{-1}(x) = \text{cosec}^{-1} x$



2. PROPERTIES OF INVERSE CIRCULAR FUNCTIONS :

P-1 :

(i) $y = \sin(\sin^{-1}x) = x$, $x \in [-1,1]$, $y \in [-1,1]$, y is aperiodic

(ii) $y = \cos(\cos^{-1} x) = x$, $x \in [-1,1]$, $y \in [-1,1]$, y is aperiodic

(iii) $y = \tan(\tan^{-1} x) = x$, $x \in R$, $y \in R$, y is aperiodic

(iv) $y = \cot(\cot^{-1} x) = x$, $x \in R$, $y \in R$, y is aperiodic

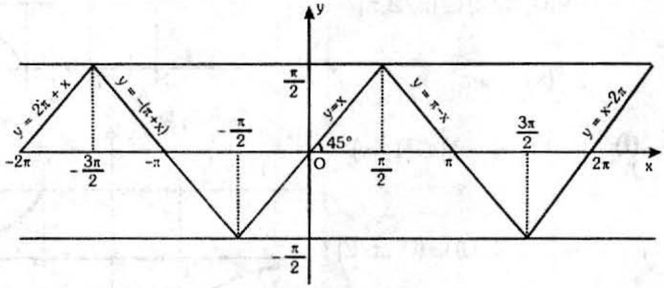
(v) $y = \text{cosec}(\text{cosec}^{-1} x) = x$, $|x| \ge 1$, $|y| \ge 1$, y is aperiodic

(vi) $y = \sec(\sec^{-1} x) = x$, $|x| \ge 1$; $|y| \ge 1$, y is aperiodic

P-2 :

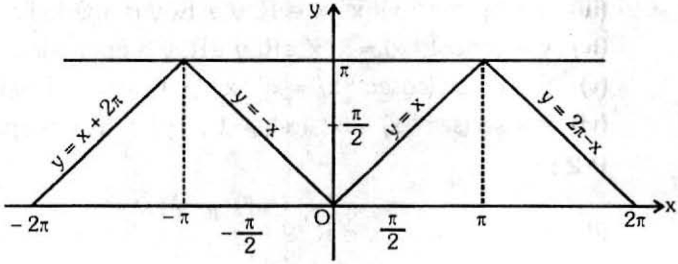
(i) $y = \sin^{-1}(\sin x)$, $x \in R$, $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Periodic with period 2π.

$$\sin^{-1}(\sin x)=\begin{cases}-\pi-x, & -\frac{3\pi}{2}\le x\le-\frac{\pi}{2}\\ x, & -\frac{\pi}{2}\le x\le\frac{\pi}{2}\\ \pi-x, & \frac{\pi}{2}\le x\le\frac{3\pi}{2}\\ x-2\pi, & \frac{3\pi}{2}\le x\le\frac{5\pi}{2}\\ 3\pi-x, & \frac{5\pi}{2}\le x\le\frac{7\pi}{2}\\ x-4\pi, & \frac{7\pi}{2}\le x\le\frac{9\pi}{2}\end{cases}$$



(ii) $y = \cos^{-1}(\cos x)$, $x \in R$, $y \in [0,\pi]$, periodic with period 2π

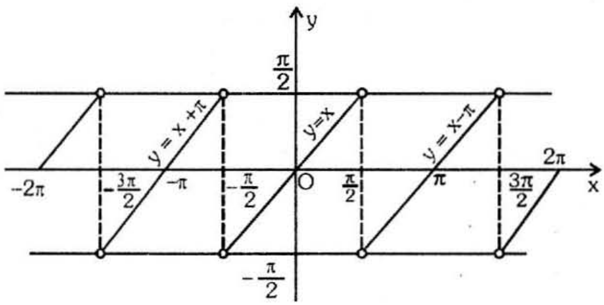
$$\cos^{-1}(\cos x)=\begin{cases}-x, & -\pi\le x\le 0\\ x, & 0\le x\le\pi\\ 2\pi-x, & \pi\le x\le 2\pi\\ x-2\pi, & 2\pi\le x\le 3\pi\\ 4\pi-x, & 3\pi\le x\le 4\pi\end{cases}$$



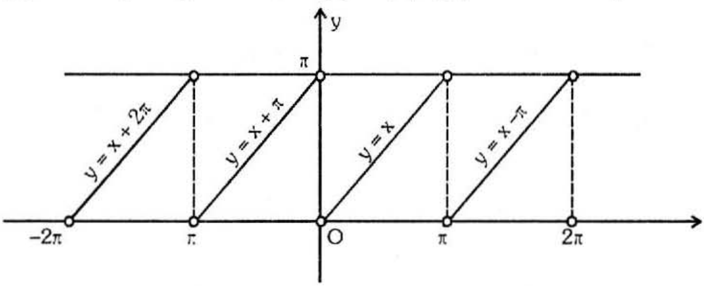
(iii) $y = \tan^{-1}(\tan x)$

$$x \in R - \left\{(2n-1)\frac{\pi}{2}, n \in I\right\};\quad y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),\text{ periodic with period } \pi$$

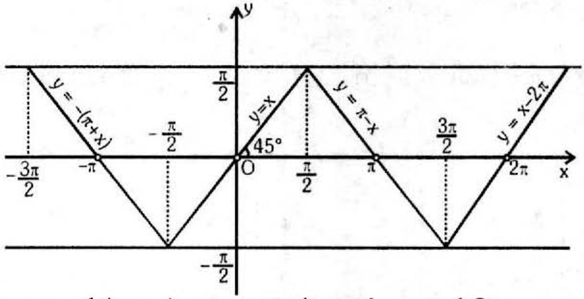
$$\tan^{-1}(\tan x) = \begin{cases} x+\pi & , \quad -\dfrac{3\pi}{2} < x < -\dfrac{\pi}{2} \\ x & , \quad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2} \\ x-\pi & , \quad \dfrac{\pi}{2} < x < \dfrac{3\pi}{2} \\ x-2\pi & , \quad \dfrac{3\pi}{2} < x < \dfrac{5\pi}{2} \\ x-3\pi & , \quad \dfrac{5\pi}{2} < x < \dfrac{7\pi}{2} \end{cases}$$



(iv) $y = \cot^{-1}(\cot x)$, $x \in R - \{n\pi\}$, $y \in (0, \pi)$, periodic with period π

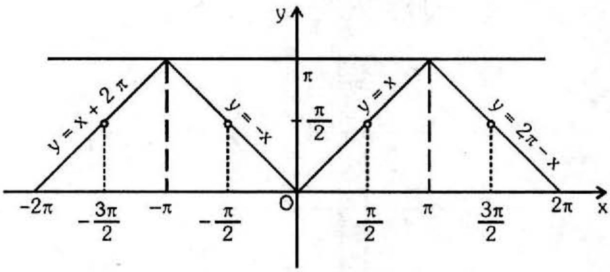


(v) $y = \text{cosec}^{-1}(\text{cosec } x)$, $x \in R - \{n\pi, n \in I\}$ $y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ y is periodic with period 2π.



(vi) $y = \sec^{-1}(\sec x)$, y is periodic with period 2π

$$x \in R - \left\{(2n-1)\frac{\pi}{2} n \in I\right\},\ \ y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$$



P-3 :

(i) $\text{cosec}^{-1} x = \sin^{-1}\frac{1}{x}$; $x \le -1, x \ge 1$

(ii) $\sec^{-1} x = \cos^{-1}\frac{1}{x}$; $x \le -1, x \ge 1$

(iii) $\cot^{-1}x = \tan^{-1}\frac{1}{x}$; $x > 0$

$= \pi + \tan^{-1}\frac{1}{x}$; $x < 0$

P-4 :

(i) $\sin^{-1}(-x) = -\sin^{-1} x$, $-1 \le x \le 1$

(ii) $\tan^{-1}(-x) = -\tan^{-1} x$, $x \in R$

(iii) $\cos^{-1}(-x) = \pi - \cos^{-1} x$, $-1 \le x \le 1$

(iv) $\cot^{-1}(-x) = \pi - \cot^{-1} x$, $\quad x \in R$

(v) $\sec^{-1}(-x) = \pi - \sec^{-1} x$, $\quad x \le -1$ or $x \ge 1$

(vi) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x$, $\quad x \le -1$ or $x \ge 1$

P-5 :

(i) $\sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2} \qquad -1 \le x \le 1$

(ii) $\tan^{-1} x + \cot^{-1} x = \dfrac{\pi}{2} \qquad x \in R$

(iii) $\operatorname{cosec}^{-1} x + \sec^{-1} x = \dfrac{\pi}{2} \qquad |x| \ge 1$

P-6 :

(i) $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\dfrac{x+y}{1-xy}$ where $x > 0, y > 0$ & $xy < 1$

$= \pi + \tan^{-1}\dfrac{x+y}{1-xy}$, where $x > 0, y > 0$ & $xy > 1$

$= \dfrac{\pi}{2}$, where $x > 0, y > 0$ & $xy = 1$

(ii) $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\dfrac{x-y}{1+xy}$ where $x > 0$, $y > 0$

(iii) $\sin^{-1} x + \sin^{-1} y = \sin^{-1}[x\sqrt{1-y^2} + y\sqrt{1-x^2}]$,
where $x > 0, y > 0$ & $(x^2 + y^2) < 1$

Note that : $x^2 + y^2 < 1 \Rightarrow 0 < \sin^{-1} x + \sin^{-1} y < \dfrac{\pi}{2}$

(iv) $\sin^{-1} x + \sin^{-1} y = \pi - \sin^{-1}[x\sqrt{1-y^2} + y\sqrt{1-x^2}]$,
where $x > 0, y > 0$ & $x^2 + y^2 > 1$

Note that : $x^2 + y^2 > 1 \Rightarrow \dfrac{\pi}{2} < \sin^{-1} x + \sin^{-1} y < \pi$

(v) $\sin^{-1} x - \sin^{-1} y = \sin^{-1}[x\sqrt{1-y^2} - y\sqrt{1-x^2}]$ where $x > 0, y > 0$

(vi) $\cos^{-1}x + \cos^{-1}y = \cos^{-1}[xy - \sqrt{1-x^2}\sqrt{1-y^2}]$, where $x > 0, y > 0$

(vii) $\cos^{-1} x - \cos^{-1} y = \begin{cases} \cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right) & ; \quad x < y, \quad x, y > 0 \\ -\cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right) & ; \quad x > y, \quad x, y > 0 \end{cases}$

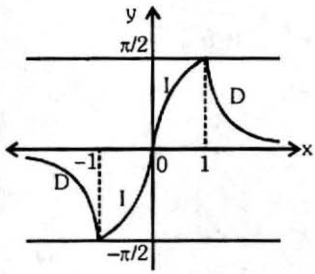
(viii) $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left[\dfrac{x+y+z-xyz}{1-xy-yz-zx}\right]$

if $x > 0, y > 0, z > 0$ & $xy + yz + zx < 1$

Note : In the above results x & y are taken positive. In case if these are given as negative, we first apply P-4 and then use above results.

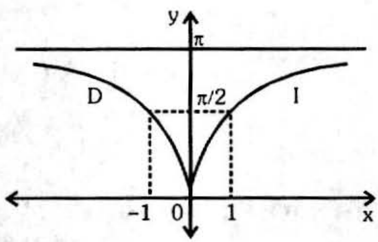
3. SIMPLIFIED INVERSE TRIGONOMETRIC FUNCTIONS :

(a) $y = f(x) = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$



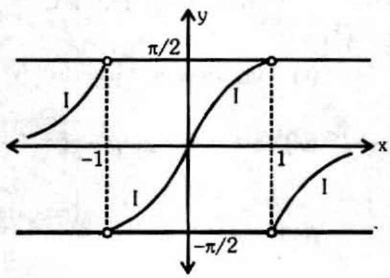
$$= \begin{cases} 2\tan^{-1}x & \text{if } |x| \le 1 \\ \pi - 2\tan^{-1}x & \text{if } x > 1 \\ -(\pi + 2\tan^{-1}x) & \text{if } x < -1 \end{cases}$$

(b) $y = f(x) = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$



$$= \begin{cases} 2\tan^{-1}x & \text{if } x \ge 0 \\ -2\tan^{-1}x & \text{if } x < 0 \end{cases}$$

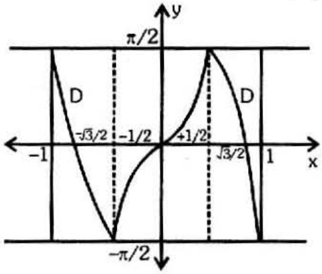
(c) $y = f(x) = \tan^{-1}\dfrac{2x}{1-x^2}$



$$= \begin{cases} 2\tan^{-1}x & \text{if } |x| < 1 \\ \pi + 2\tan^{-1}x & \text{if } x < -1 \\ -(\pi - 2\tan^{-1}x) & \text{if } x > 1 \end{cases}$$

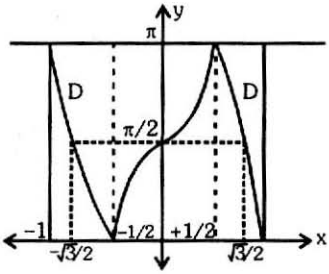
(d) $y = f(x) = \sin^{-1}(3x - 4x^3)$

$$= \begin{cases} -(\pi + 3\sin^{-1} x) & \text{if} \quad -1 \le x \le -\dfrac{1}{2} \\ 3\sin^{-1} x & \text{if} \quad -\dfrac{1}{2} \le x \le \dfrac{1}{2} \\ \pi - 3\sin^{-1} x & \text{if} \quad \dfrac{1}{2} \le x \le 1 \end{cases}$$



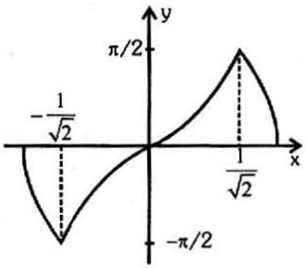
(e) $y = f(x) = \cos^{-1}(4x^3 - 3x)$

$$= \begin{cases} 3\cos^{-1} x - 2\pi & \text{if} \quad -1 \le x \le -\dfrac{1}{2} \\ 2\pi - 3\cos^{-1} x & \text{if} \quad -\dfrac{1}{2} \le x \le \dfrac{1}{2} \\ 3\cos^{-1} x & \text{if} \quad \dfrac{1}{2} \le x \le 1 \end{cases}$$



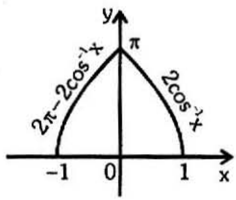
(f) $\sin^{-1}\left(2x\sqrt{1 - x^2}\right)$

$$= \begin{cases} -(\pi + 2\sin^{-1} x) & -1 \le x \le -\dfrac{1}{\sqrt{2}} \\ 2\sin^{-1} x & -\dfrac{1}{\sqrt{2}} \le x \le \dfrac{1}{\sqrt{2}} \\ \pi - 2\sin^{-1} x & \dfrac{1}{\sqrt{2}} \le x \le 1 \end{cases}$$



(g) $\cos^{-1}(2x^2 - 1)$

$$= \begin{cases} 2\cos^{-1} x & 0 \le x \le 1 \\ 2\pi - 2\cos^{-1} x & -1 \le x \le 0 \end{cases}$$



LIMIT

1. DEFINITION :

Let f(x) be defined on an open interval about 'a' except possibly at 'a' itself. If f(x) gets arbitrarily close to L (a finite number) for all x sufficiently close to 'a' we say that f(x) approaches the limit L as x approaches 'a' and we write $\lim\limits_{x \to a} f(x) = L$ and say "the limit of f(x), as x approaches a, equals L".

2. LEFT HAND LIMIT & RIGHT HAND LIMIT OF A FUNCTION :

Left hand limit (LHL) = $\lim\limits_{x \to a^-} f(x) = \lim\limits_{h \to 0} f(a - h)$, $h > 0$.

Right hand limit (RHL) = $\lim\limits_{x \to a^+} f(x) = \lim\limits_{h \to 0} f(a + h)$, $h > 0$.

Limit of a function f(x) is said to exist as, $x \to a$ when $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x) =$ Finite quantity.

Important note :

In $\lim\limits_{x \to a} f(x)$, $x \to a$ necessarily implies $x \neq a$. That is while evaluating limit at x = a, we are not concerned with the value of the function at x = a. In fact the function may or may not be defined at x = a.

Also it is necessary to note that if f(x) is defined only on one side of **'x = a'**, one sided limits are good enough to establish the existence of limits, & if f(x) is defined on either side of **'a'** both sided limits are to be considered.

3. FUNDAMENTAL THEOREMS ON LIMITS :

Let $\lim\limits_{x\to a} f(x) = l$ & $\lim\limits_{x\to a} g(x) = m$. If l & m exists finitely then :

(a) Sum rule : $\lim\limits_{x\to a}\left[f(x)+g(x)\right] = l + m$

(b) Difference rule : $\lim\limits_{x\to a}\left[f(x)-g(x)\right] = l - m$

(c) Product rule : $\lim\limits_{x\to a} f(x).g(x) = l.m$

(d) Quotient rule : $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{l}{m}$, provided $m \neq 0$

(e) Constant multiple rule : $\lim\limits_{x\to a} kf(x) = k\lim\limits_{x\to a} f(x)$; where k is constant.

(f) Power rule : If m and n are integers, then $\lim\limits_{x\to a}\left[f(x)\right]^{m/n} = l^{m/n}$

provided $l^{m/n}$ is a real number.

(g) $\lim\limits_{x\to a} f\left[g(x)\right] = f\left(\lim\limits_{x\to a} g(x)\right) = f(m)$; provided f(x) is continuous at

x = m.

For example : $\lim\limits_{x\to a} \ell n(f(x)) = \ell n[\lim\limits_{x\to a} f(x)]$; provided ℓnx is defined

at $x = \lim\limits_{t\to a} f(t)$.

4. INDETERMINATE FORMS :

$\dfrac{0}{0}, \dfrac{\infty}{\infty}, \infty-\infty, 0\times\infty, 1^{\infty}, 0^0, \infty^0$.

Note :

We cannot plot ∞ on the paper. Infinity (∞) is a symbol & not a number. It does not obey the laws of elementary algebra,

5. GENERAL METHODS TO BE USED TO EVALUATE LIMITS:

(a) Factorization :

Important factors :

(i) $x^n - a^n = (x - a)(x^{n-1} + ax^{n-2} + ... + a^{n-1}),\ n \in N$

(ii) $x^n + a^n = (x + a)(x^{n-1} - ax^{n-2} + ... + a^{n-1})$, n is an odd natural number.

Note : $\lim\limits_{x\to a} \dfrac{x^n - a^n}{x - a} = na^{n-1}$

(b) Rationalization or double rationalization :

In this method we rationalise the factor containing the square root and simplify.

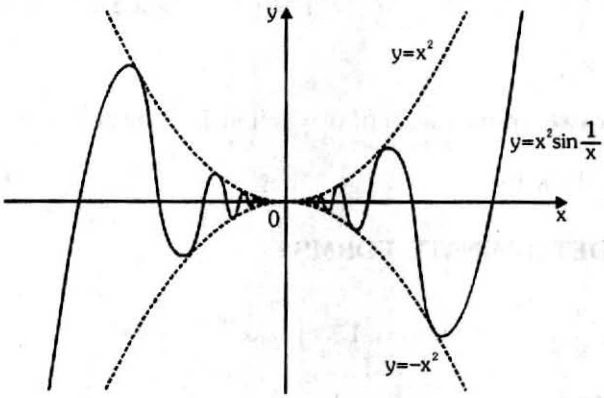
(c) Limit when $x \to \infty$:

(i) Divide by greatest power of x in numerator and denominator.

(ii) Put $x = 1/y$ and apply $y \to 0$

(d) Squeeze play theorem (Sandwich theorem) :

If $f(x) \le g(x) \le h(x);\ \forall x$ & $\lim\limits_{x\to a} f(x) = \ell = \lim\limits_{x\to a} h(x)$ then $\lim\limits_{x\to a} g(x) = \ell$,



for example : $\lim\limits_{x\to 0} x^2 \sin\dfrac{1}{x} = 0$, as illustrated by the graph given.

6. LIMIT OF TRIGONOMETRIC FUNCTIONS :

$\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1 = \lim\limits_{x\to 0} \dfrac{\tan x}{x} = \lim\limits_{x\to 0} \dfrac{\tan^{-1} x}{x} = \lim\limits_{x\to 0} \dfrac{\sin^{-1} x}{x}$ [where x is measured in radians]

(a) If $\lim\limits_{x\to a} f(x) = 0$, then $\lim\limits_{x\to a} \dfrac{\sin f(x)}{f(x)} = 1$.

(b) Using substitution $\lim\limits_{x\to a} f(x) = \lim\limits_{h\to 0} f(a-h)$ or $\lim\limits_{h\to 0} f(a+h)$ i.e.

by substituting x by a – h or a + h

7. LIMIT OF EXPONENTIAL FUNCTIONS :

(a) $\lim\limits_{x\to 0} \dfrac{a^x - 1}{x} = \ell n a \,(a > 0)$ In particular $\lim\limits_{x\to 0} \dfrac{e^x - 1}{x} = 1$.

In general if $\lim\limits_{x\to a} f(x) = 0$,then $\lim\limits_{x\to a} \dfrac{a^{f(x)} - 1}{f(x)} = \ell n a,\ a > 0$

(b) $\lim\limits_{x\to 0} \dfrac{\ell n(1+x)}{x} = 1$

(c) $\lim\limits_{x\to 0} (1+x)^{1/x} = e = \lim\limits_{x\to \infty} \left(1 + \dfrac{1}{x}\right)^x$

(**Note :** The base and exponent depends on the same variable.)

In general, if $\lim\limits_{x\to a} f(x) = 0$, then $\lim\limits_{x\to a} (1 + f(x))^{1/f(x)} = e$

(d) If $\lim\limits_{x\to a} f(x) = 1$ and $\lim\limits_{x\to a} \phi(x) = \infty$,

then $\lim\limits_{x\to a} [f(x)]^{\phi(x)} = e^k$ where $k = \lim\limits_{x\to a} \phi(x)\,[f(x) - 1]$

(e) If $\lim\limits_{x\to a} f(x) = A > 0$ & $\lim\limits_{x\to a} \phi(x) = B$ (a finite quantity),

then $\lim\limits_{x\to a} [f(x)]^{\phi(x)} = e^{B \ln A} = A^B$

8. LIMIT USING SERIES EXPANSION :

Expansion of function like binomial expansion, exponential & logarithmic expansion, expansion of sinx, cosx, tanx should be remembered by heart which are given below :

(a) $a^x = 1 + \frac{x \ell na}{1!} + \frac{x^2 \ell n^2 a}{2!} + \frac{x^3 \ell n^3 a}{3!} + \text{..........} a > 0$

(b) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \text{..........}$

(c) $\ell n(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \text{.......for } -1 < x \le 1$

(d) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \text{..........}$

(e) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \text{..........}$

(f) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \text{.......}$

(g) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \text{.......}$

(h) $\sin^{-1} x = x + \frac{1^2}{3!}x^3 + \frac{1^2.3^2}{5!}x^5 + \frac{1^2.3^2.5^2}{7!}x^7 + \text{.......}$

(i) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \text{...........} n \in Q$

CONTINUITY

1. CONTINUOUS FUNCTIONS :

A function f(x) is said to be continuous at x = a, if $\lim\limits_{x\to a} f(x) = f(a)$.

Symbolically f is continuous at x = a if $\lim\limits_{h\to 0} f(a-h) = \lim\limits_{h\to 0} f(a+h) = f(a)$.

2. CONTINUITY OF THE FUNCTION IN AN INTERVAL :

(a) A function is said to be continuous in (a,b) if f is continuous at each & every point belonging to (a, b).

(b) A function is said to be continuous in a closed interval [a,b] if :

- f is continuous in the open interval (a,b)
- f is right continuous at 'a' i.e. $\lim\limits_{x\to a^+} f(x) = f(a) =$ a finite quantity
- f is left continuous at 'b' i.e. $\lim\limits_{x\to b^-} f(x) = f(b) =$ a finite quantity

Note :

(i) All Polynomials, Trigonometrical functions, exponential & Logarithmic functions are continuous in their domains.

(ii) If f & g are two functions that are continuous at x = c then the function defined by : $F_1(x) = f(x) \pm g(x)$; $F_2(x) = K f(x)$, K any real number $F_3(x) = f(x).g(x)$ are also continuous at x = c. Further, if g(c) is not zero, then $F_4(x) = \frac{f(x)}{g(x)}$ is also continuous at x = c.

(iii) If f and g are continuous then fog and gof are also continuous.

(iv) If f and g are discontinuous at x = c, then f + g, f − g, f.g may still be continuous.

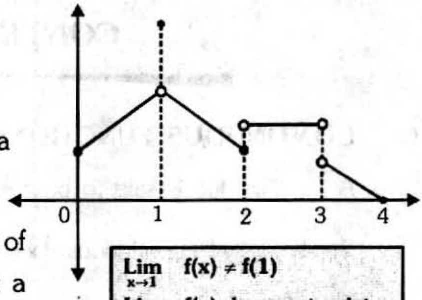
3. REASONS OF DISCONTINUITY :

(a) Limit does not exist

i.e. $\lim\limits_{x\to a^-} f(x) \neq \lim\limits_{x\to a^+} f(x)$

(b) f(x) is not defined at x = a

(c) $\lim\limits_{x\to a} f(x) \neq f(a)$



$\lim\limits_{x\to 1} f(x) \neq f(1)$

$\lim\limits_{x\to 2} f(x)$ does not exist

f(x) is not defined at x = 3

Geometrically, the graph of the function will exhibit a break at x = a, if the function is discontinuous at x = a. The graph as shown is discontinuous at x = 1 , 2 and 3.

4. TYPES OF DISCONTINUITIES :

Type-1 : (Removable type of discontinuities) : In case $\lim\limits_{x\to a} f(x)$ exists but is not equal to f(a) then the function is said to have a removable discontinuity or discontinuity of the first kind. In this case we can redefine the function such that $\lim\limits_{x\to a} f(x) = f(a)$ & make it continuous at x = a. Removable type of discontinuity can be further classified as:

(a) Missing point discontinuity :

Where $\lim\limits_{x\to a} f(x)$ exists finitely but f(a) is not defined.

(b) Isolated point discontinuity :

Where $\lim\limits_{x\to a} f(x)$ exists & f(a) also exists but; $\lim\limits_{x\to a} f(x) \neq f(a)$.

Type-2 : (Non-Removable type of discontinuities) :-

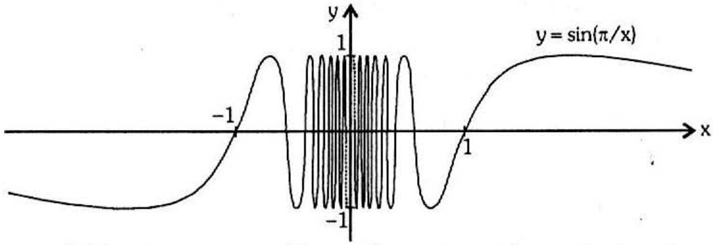
In case $\lim\limits_{x\to a} f(x)$ does not exist then it is not possible to make the function continuous by redefining it. Such a discontinuity is known as non-removable discontinuity or discontinuity of the 2nd kind. Non-removable type of discontinuity can be further classified as :

(a) Finite type discontinuity : In such type of discontinuity left hand limit and right hand limit at a point exists but are not equal.

(b) Infinite type discontinuity : In such type of discontinuity atleast one of the limit viz. LHL and RHL is tending to infinity.

(c) Oscillatory type discontinuity :

$$f(x) = \sin\frac{\pi}{x}$$

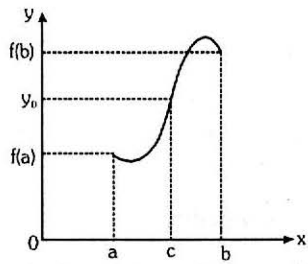


f(x) has non removable oscillatory type discontinuity at x = 0

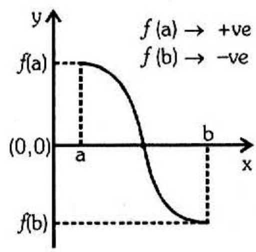
Note : In case of non-removable (finite type) discontinuity the non-negative difference between the value of the RHL at x = a & LHL at x = a is called THE JUMP OF DISCONTINUITY. A function having a finite number of jumps in a given interval I is called a PIECE WISE CONTINUOUS or SECTIONALLY CONTINUOUS function in this interval.

5. THE INTERMEDIATE VALUE THEOREM :

Suppose f(x) is continuous on an interval I and a and b are any two points of I. Then if y_0 is a number between f(a) and f(b), their exists a number c between a and b such that $f(c) = y_0$



The function f, being continuous on [a,b] takes on every value between f(a) and f(b)



$f(x) = 0$ in (a, b)

Note that a function f which is continuous in [a,b] possesses the following properties :

(a) If f(a) & f(b) posses opposite signs, then there exists atleast one solution of the equation f(x) = 0 in the open interval (a,b).

(b) If K is any real number between f(a) & f(b), then there exists atleast one solution of the equation f(x) = K in the open interval (a,b).

DIFFERENTIABILITY

1. INTRODUCTION :

The derivative of a function 'f' is function ; this function is denoted by symbols such as

$$f'(x),\ \frac{df}{dx},\ \frac{d}{dx}f(x)\ \text{or}\ \frac{df(x)}{dx}$$

The derivative evaluated at a point a, can be written as :

$$f'(a), \left[\frac{df(x)}{dx}\right]_{x=a}, f'(x)_{x=a},\ \text{etc.}$$

2. RIGHT HAND & LEFT HAND DERIVATIVES :

(a) Right hand derivative :

The right hand derivative of f(x) at x = a denoted by $f'(a^+)$ is defined as :

$$f'(a^+) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h},$$ provided the limit exists & is finite.

(b) Left hand derivative :

The left hand derivative of f(x) at x = a denoted by $f'(a^-)$ is defined

as : $$f'(a^-) = \lim_{h\to 0}\frac{f(a-h)-f(a)}{-h},$$ provided the limit exists & is finite.

(c) Derivability of function at a point :

If $f'(a^+) = f'(a^-)$ = finite quantity, then f(x) is said to be **derivable or differentiable at x = a**. In such case $f'(a^+) = f'(a^-) = f'(a)$ & it is called derivative or differential coefficient of f(x) at x = a.

Note :

(i) All polynomial, trigonometric, inverse trigonometric, logarithmic and exponential function are continuous and differentiable in their domains, except at end points.

(ii) If f(x) & g(x) are derivable at x = a then the functions f(x) + g(x), f(x) – g(x), f(x). g(x) will also be derivable at x = a & if $g(a) \neq 0$ then the function f(x)/g(x) will also be derivable at x = a.

3. IMPORTANT NOTE :

(a) Let $f'(a^+) = p$ & $f'(a^-) = q$ where p & q are finite then :

(i) $p = q \Rightarrow$ f is derivable at x = a $\Rightarrow$ f is continuous at x = a

(ii) $p \neq q \Rightarrow$ f is not derivable at x = a

It is very important to note that 'f' may be still continuous at x = a

In short, for a function 'f' :

Differentiable $\Rightarrow$ **Continuous ;**

Not Differentiable $\not\Rightarrow$ **Not Continuous**

But Not Continuous $\Rightarrow$ **Not Differentiable**

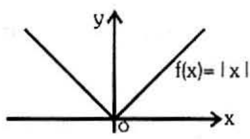
Continuous $\Rightarrow$ **May or may not be Differentiable**

(b) Geometrical interpretation of differentiability :

(i) If the function y = f(x) is differentiable at x = a, then a unique tangent can be drawn to the curve y = f(x) at P(a, f(a)) & f'(a) represent the slope of the tangent at point P.

(ii) If LHD and RHD are finite but unequal then it geometrically implies a sharp corner at x = a.

e.g. f(x) = | x | is continuous but not differentiable at x = 0. A sharp corner is seen at x = 0 in the graph of f(x) = | x |.



(iii) If a function has vertical tangent at x = a then also it is nonderivable at x = a.

(c) Vertical tangent :

If for $y = f(x)$,

$f'(a^+) \to \infty$ and $f'(a^-) \to \infty$ or $f'(a^+) \to -\infty$ and $f'(a^-) \to -\infty$

then at $x = a$, $y = f(x)$ has vertical tangent but $f(x)$ is not differentiable at $x = a$

4. DERIVABILITY OVER AN INTERVAL :

(a) $f(x)$ is said to be derivable over an open interval (a, b) if it is derivable at each & every point of the open interval (a, b).

(b) $f(x)$ is said to be derivable over the closed interval $[a, b]$ if :

(i) $f(x)$ is derivable in (a, b) &

(ii) for the points a and b, $f'(a^+)$ & $f'(b^-)$ exist.

Note :

(i) If $f(x)$ is differentiable at $x = a$ & $g(x)$ is not differentiable at $x = a$, then the product function $F(x)=f(x).g(x)$ can still be differentiable at $x = a$.

(ii) If $f(x)$ & $g(x)$ both are not differentiable at $x = a$ then the product function; $F(x)=f(x).g(x)$ can still be differentiable at $x = a$.

(iii) If $f(x)$ & $g(x)$ both are non-derivable at $x=a$ then the sum function $F(x)=f(x)+g(x)$ may be a differentiable function.

(iv) If $f(x)$ is derivable at $x = a$ $\not\Rightarrow$ $f'(x)$ is continuous at $x = a$.

METHODS OF DIFFERENTIATION

1. DERIVATIVE OF f(x) FROM THE FIRST PRINCIPLE :

Obtaining the derivative using the definition

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} = f'(x) = \frac{dy}{dx}$$ is called calculating

derivative using first principle or ab initio or delta method.

2. FUNDAMENTAL THEOREMS :

If f and g are derivable function of x, then,

(a) $\frac{d}{dx}(f \pm g) = \frac{df}{dx} \pm \frac{dg}{dx}$

(b) $\frac{d}{dx}(cf) = c\frac{df}{dx}$, where c is any constant

(c) $\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$ known as **"PRODUCT RULE"**

(d) $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\left(\frac{df}{dx}\right) - f\left(\frac{dg}{dx}\right)}{g^2}$,

where $g \neq 0$ known as **"QUOTIENT RULE"**

(e) If $y = f(u)$ & $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ known as **"CHAIN RULE"**

Note : In general if $y = f(u)$, then $\frac{dy}{dx} = f'(u) \cdot \frac{du}{dx}$.

3. DERIVATIVE OF STANDARD FUNCTIONS :

	f(x)	f'(x)
(i)	x^n	nx^{n-1}
(ii)	e^x	e^x
(iii)	a^x	$a^x \ell na, a > 0$
(iv)	ℓnx	$1/x$
(v)	$\log_a x$	$(1/x) \log_a e, a > 0, a \neq 1$
(vi)	sinx	cosx
(vii)	cosx	– sinx
(viii)	tanx	$\sec^2 x$
(ix)	secx	secx tanx
(x)	cosecx	– cosecx . cotx
(xi)	cotx	$-\text{cosec}^2 x$
(xii)	constant	0
(xiii)	$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}, -1 < x < 1$
(xiv)	$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}, -1 < x < 1$
(xv)	$\tan^{-1} x$	$\frac{1}{1+x^2}, x \in R$
(xvi)	$\sec^{-1} x$	$\frac{1}{\lvert x \rvert \sqrt{x^2-1}}, \lvert x \rvert > 1$
(xvii)	$\text{cosec}^{-1} x$	$\frac{-1}{\lvert x \rvert \sqrt{x^2-1}}, \lvert x \rvert > 1$
(xviii)	$\cot^{-1} x$	$\frac{-1}{1+x^2}, x \in R$

4. LOGARITHMIC DIFFERENTIATION :

To find the derivative of :

(a) A function which is the product or quotient of a number of function or

(b) A function of the form $[f(x)]^{g(x)}$ where f & g are both derivable, it is convenient to take the logarithm of the function first & then differentiate.

5. DIFFERENTIATION OF IMPLICIT FUNCTION :

(a) Let function is $\phi(x, y) = 0$ then to find dy /dx, in the case of implicit functions, we differentiate each term w.r.t. x regarding y as a functions of x & then collect terms in dy / dx together on one side to finally find dy / dx

OR $\frac{dy}{dx} = \frac{-\partial\phi/\partial x}{\partial\phi/\partial y}$ where $\frac{\partial\phi}{\partial x}$ & $\frac{\partial\phi}{\partial y}$ are partial differential coefficient of $\phi(x, y)$ w.r.to x & y respectively.

(b) In answers of dy/dx in the case of implicit functions, both x & y are present.

6. PARAMETRIC DIFFERENTIATION :

If $y = f(\theta)$ & $x = g(\theta)$ where θ is a parameter, then $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$.

7. DERIVATIVE OF A FUNCTION W.R.T. ANOTHER FUNCTION :

Let $y = f(x)$; $z = g(x)$, then $\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{f'(x)}{g'(x)}$

8. DERIVATIVE OF A FUNCTION AND ITS INVERSE FUNCTION :

If inverse of $y = f(x)$

$x = f^{-1}(y)$ is denoted by $x = g(y)$ then $g(f(x)) = x$

$g'(f(x))f'(x) = 1$

9. HIGHER ORDER DERIVATIVE :

Let a function $y = f(x)$ be defined on an open interval (a, b). It's derivative, if it exists on (a, b) is a certain function $f'(x)$ [or (dy / dx) or y'] & it is called the first derivative of y w. r. t. x. If it happens that the first derivative has a derivative on (a , b) then this derivative is called second derivative of y w.r.t. x & is denoted by $f''(x)$ or (d^2y/dx^2) or y''. Similarly, the 3rd order derivative of y w.r.t x, if it exists, is defined by $\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$. It is also denoted by $f'''(x)$ or y''' & so on.

10. DIFFERENTIATION OF DETERMINANTS :

If $F(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$, where f, g, h. l, m, n, u, v, w are differentiable functions of x, then

$$F'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l'(x) & m'(x) & n'(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$$

11. L' HÔPITAL'S RULE :

(a) Applicable while calculating limits of indeterminate forms of the type $\frac{0}{0}, \frac{\infty}{\infty}$. If the function f(x) and g(x) are differentiable in certain neighbourhood of the point a, except, may be, at the point a itself, and $g'(x) \neq 0$, and if

$\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = 0$ or $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = \infty$,

then $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$

provided the limit $\lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$ exists (L' Hôpital's rule). The point 'a' may be either finite or improper $+\infty$ or $-\infty$.

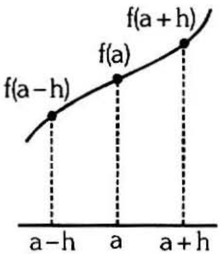
(b) Indeterminate forms of the type $0.\ \infty$ or $\infty - \infty$ are reduced to forms of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by algebraic transformations.

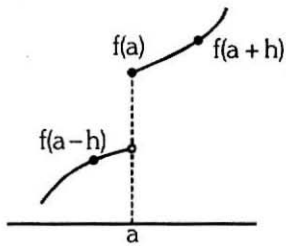
(c) Indeterminate forms of the type 1^∞, ∞^0 or 0^0 are reduced to forms of the type $0.\ \infty$ by taking logarithms or by the transformation $[f(x)]^{\phi(x)} = e^{\phi(x).\ell nf(x)}$.

MONOTONICITY

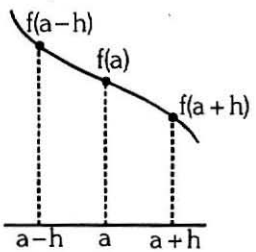
1. MONOTONICITY AT A POINT :

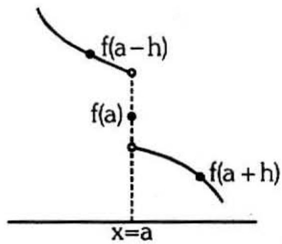
(a) A function f(x) is called an increasing function at point x = a, if in a sufficiently small neighbourhood of x = a ; $f(a-h) < f(a) < f(a+h)$



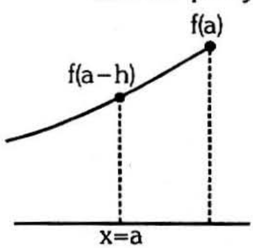


(b) A function f(x) is called a decreasing function at point x = a if in a sufficiently small neighbourhood of x= a ; $f(a-h) > f(a) > f(a+h)$



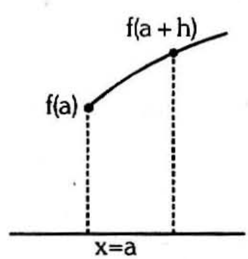


Note : If x = a is a boundary point then use the appropriate one sides inequality to test Monotonicity of f(x).



$f(a) > f(a-h)$

increasing at x = a



$f(a+h) > f(a)$

(c) Derivative test for increasing and decreasing functions at a point :

(i) If $f'(a) > 0$ then $f(x)$ is increasing at $x = a$.

(ii) If $f'(a) < 0$ then $f(x)$ is decreasing at $x = a$.

(iii) If $f'(a) = 0$ then examine the sign of $f'(a^+)$ and $f'(a^-)$.

(1) If $f'(a^+) > 0$ and $f'(a^-) > 0$ then increasing

(2) If $f'(a^+) < 0$ and $f'(a^-) < 0$ then decreasing

(3) Otherwise neither increasing nor decreasing.

Note : Above rule is applicable only for functions that are differentiable at $x = a$.

2. MONOTONICITY OVER AN INTERVAL :

(a) A function $f(x)$ is said to be monotonically increasing (MI) in (a, b) if $f'(x) \geq 0$ where equality holds only for discrete values of x i.e. $f'(x)$ does not identically become zero for $x \in (a, b)$ or any sub interval.

(b) $f(x)$ is said to be monotonically decreasing (MD)in (a, b) if $f'(x) \leq 0$ where equality holds only for discrete values of x i.e. $f'(x)$ does not identically become zero for $x \in (a, b)$ or any sub interval.

- By discrete points, we mean that points where $f'(x) = 0$ does not form an interval.

Note : A function is said to be monotonic if it's either increasing or decreasing.

3. SPECIAL POINTS :

(a) Critical points : The points of domain for which $f'(x)$ is equal to zero or doesn't exist are called critical points.

(b) Stationary points: The stationary points are the points of domain where $f'(x) = 0$.

Every stationary point is a critical point.

4. ROLLE'S THEOREM :

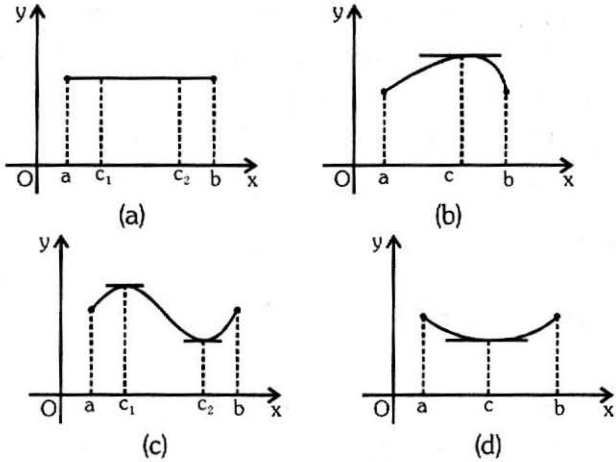
Let f be a function that satisfies the following three hypotheses :

(a) f is continuous in the closed interval $[a, b]$.

(b) f is differentiable in the open interval (a, b)

(c) $f(a) = f(b)$

Then there is a number c in (a, b) such that f'(c) = 0.



Conclusion : If f is a differentiable function then between any two consecutive roots of f(x) = 0, there is atleast one root of the equation f'(x) = 0.

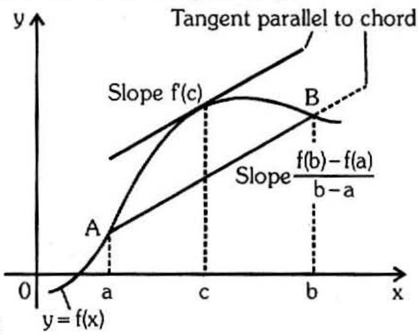
5. LAGRANGE'S MEAN VALUE THEOREM (LMVT) :

Let f be a function that satisfies the following hypotheses:

(i) f is continuous in a closed interval [a, b]

(ii) f is differentiable in the open interval (a, b).

Then there is a number c in (a, b) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$



(a) Geometrical Interpretation :

Geometrically, the Mean Value Theorem says that somewhere between A and B the curve has at least one tangent parallel to chord AB.

(b) Physical Interpretations :

If we think of the number $(f(b) - f(a))/(b - a)$ as the average change in f over $[a, b]$ and $f'(c)$ as an instantaneous change, then the Mean Value Theorem says that at some interior point the instantaneous change must equal the average change over the entire interval.

6. SPECIAL NOTE :

Use of Monotonicity in identifying the number of roots of the equation in a given interval. Suppose a and b are two real numbers such that,

(a) $f(x)$ & its first derivative $f'(x)$ are continuous for $a \le x \le b$.

(b) $f(a)$ and $f(b)$ have opposite signs.

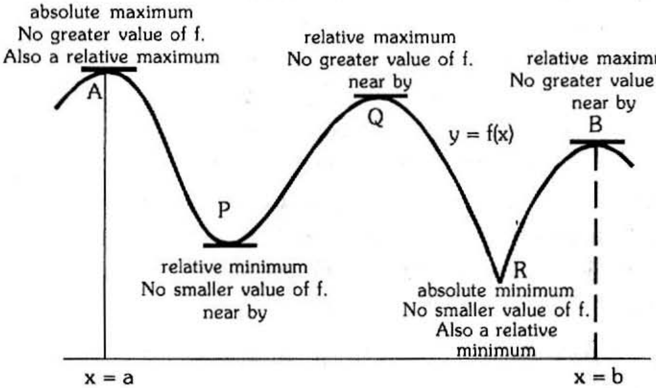
(c) $f'(x)$ is different from zero for all values of x between a & b.

Then there is one & only one root of the equation $f(x) = 0$ in (a, b).

MAXIMA-MINIMA

1. INTRODUCTION :

(a) Maxima (Local maxima) :



A function f(x) is said to have a maximum at $x = a$ if there exist a neighbourhood $(a - h, a + h) - \{a\}$ such that

$f(a) > f(x) \;\; \forall \;\; x \in (a - h, a + h) - \{a\}$

(b) Minima (Local minima) :

A function f(x) is said to have a minimum at $x = a$ if there exist a neighbourhood $(a - h, a + h) - \{a\}$ such that

$f(a) < f(x) \;\; \forall \;\; x \in (a - h, a + h) - \{a\}$

(c) Absolute maximum (Global maximum) :

A function f has an absolute maximum (or global maximum) at c if $f(c) \geq f(x)$ for all x in D, where D is the domain of f. The number f(c) is called the maximum value of f on D.

(d) Absolute minimum (Global minimum) :

A function f has an absolute minimum at c if $f(c) \leq f(x)$ for all x in D and the number f(c) is called the minimum value of f on D. The maximum and minimum values of f are called the **extreme values** of f.

Note that :

(i) the maximum & minimum values of a function are also known as **local/relative maxima or local/relative minima** as these are the greatest & least values of the function relative to some neighbourhood of the point in question.

(ii) the term 'extremum' or 'turning value' is used both for maximum or a minimum value.

(iii) a maximum (minimum) value of a function may not be the greatest (least) value in a finite interval.

(iv) a function can have several maximum & minimum values & a minimum value may be greater than a maximum value.

(v) local maximum & local minimum values of a continuous function occur alternately & between two consecutive local maximum values there is a local minimum value & vice versa.

(vi) Monotonic function do not have extreme points.

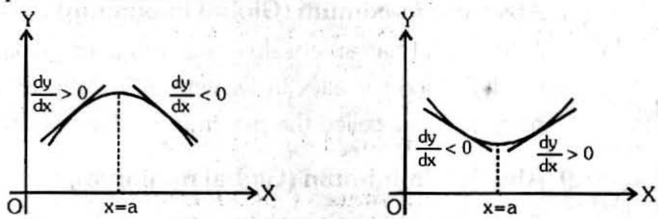
2. DERIVATIVE TEST FOR ASCERTAINING MAXIMA/MINIMA :

(a) First derivative test :

Find the point (say x = a) where $f'(x) = 0$ and

(i) If $f'(x)$ changes sign from positive to negative while graph of the function passes through x = a then x = a is said to be a point of **local maxima.**

(ii) If $f'(x)$ changes sign from negative to positive while graph of the function passes through x = a then x = a is said to be a point of **local minima.**



Note : If $f'(x)$ does not change sign i.e. has the same sign in a certain complete neighbourhood of a, then f(x) is either strictly increasing or decreasing throughout this neighbourhood implying that f(a) is not an extreme value of f.

(b) Second derivative test :

If f(x) is continuous and differentiable at $x = a$ where $f'(a) = 0$ and $f''(a)$ also exists then for ascertaining maxima/minima at $x = a$, 2^{nd} derivative test can be used -

(i) If $f''(a) > 0$ $\Rightarrow x = a$ is a point of local minima

(ii) If $f''(a) < 0$ $\Rightarrow x = a$ is a point of local maxima

(iii) If $f''(a) = 0$ $\Rightarrow$ second derivative test fails. To identify maxima/minima at this point either first derivative test or higher derivative test can be used.

(c) n^{th} derivative test :

Let f(x) be a function such that $f'(a) = f''(a) = f'''(a) = \ldots\ldots = f^{n-1}(a) = 0$ & $f^n(a) \neq 0$, then

(i) If n is even & $f^n(a) > 0 \Rightarrow$ Minima, $f^n(a) < 0 \Rightarrow$ Maxima

(ii) If n is odd, then neither maxima nor minima at $x = a$

3. USEFUL FORMULAE OF MENSURATION TO REMEMBER:

(a) Volume of a cuboid = ℓbh.

(b) Surface area of a cuboid = $2(\ell b + bh + h\ell)$.

(c) Volume of a prism = area of the base x height.

(d) Lateral surface area of prism = perimeter of the base x height.

(e) Total surface area of a prism = lateral surface area + 2 area of the base (Note that lateral surfaces of a prism are all rectangles).

(f) Volume of a pyramid $= \frac{1}{3}$ area of the base x height.

(g) Curved surface area of a pyramid $= \frac{1}{2}$ (perimeter of the base) x slant height.
(Note that slant surfaces of a pyramid are triangles).

(h) Volume of a cone $= \frac{1}{3}\pi r^2 h$.

(i) Curved surface area of a cylinder = $2\pi rh$.

(j) Total surface area of a cylinder = $2\pi rh + 2\pi r^2$.

(k) Volume of a sphere = $\frac{4}{3}\pi r^3$.

(l) Surface area of a sphere = $4\pi r^2$.

(m) Area of a circular sector = $\frac{1}{2} r^2 \theta$, when θ is in radians.

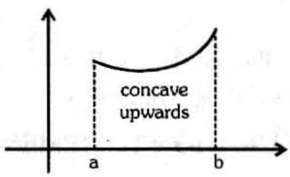
(n) Perimeter of circular sector = $2r + r\theta$.

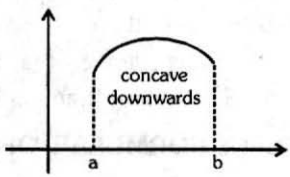
4. SIGNIFICANCE OF THE SIGN OF 2ND ORDER DERIVATIVE AND POINT OF INFLECTION :

The sign of the 2nd order derivative determines the concavity of the curve.

If $f''(x) > 0 \ \forall \ x \in (a, b)$ then graph of f(x) is concave upward in (a, b).

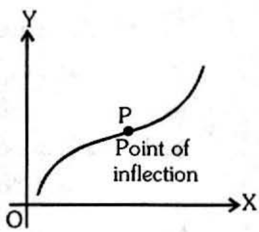
Similarly if $f''(x) < 0 \ \forall \ x \in (a, b)$ then graph of f(x) is concave downward in (a, b).





Point of inflection :

A point where the graph of a function has a tangent line and where the concavity changes is called a point of inflection.



For finding point of inflection of any function, compute the solutions of $\frac{d^2y}{dx^2} = 0$ or does not exist. Let the solution is x = a, if sign of $\frac{d^2y}{dx^2}$ changes about this point then it is called point of inflection.

Note : If at any point $\frac{d^2y}{dx^2}$ does not exist but sign of $\frac{d^2y}{dx^2}$ changes about this point then it is also called point of inflection.

5. SOME STANDARD RESULTS :

(a) Rectangle of largest area inscribed in a circle is a square.

(b) The function $y = \sin^m x \cos^n x$ attains the max value at $x = \tan^{-1}\sqrt{\frac{m}{n}}$

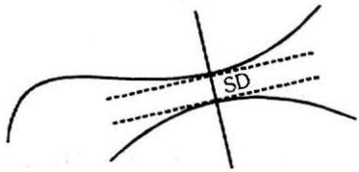
(c) If $0 < a < b$ then $|x-a| + |x-b| \geq b - a$ and equality hold when $x \in [a, b]$.

If $0 < a < b < c$ then $|x-a| + |x-b| + |x-c| \geq c - a$ and equality hold when $x = b$

If $0 < a < b < c$ then $|x-a| + |x-b| + |x-c| + |x-d| \geq d - a$ and equality hold when $x \in (b,c)$.

6. SHORTEST DISTANCE BETWEEN TWO CURVES :

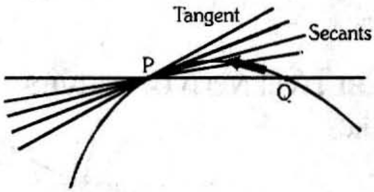
Shortest distance between two non-intersecting curves always along the common normal. (Wherever defined)

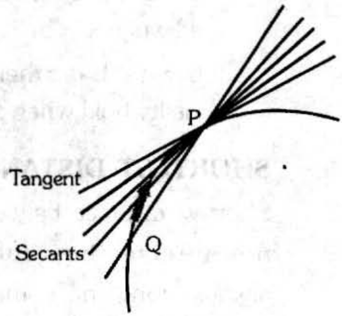


TANGENT & NORMAL

1. TANGENT TO THE CURVE AT A POINT :

The tangent to the curve at 'P' is the line through P whose slope is limit of the secant's slope as Q → P from either side.





2. NORMAL TO THE CURVE AT A POINT :

A line which is perpendicular to the tangent at the point of contact is called normal to the curve at that point.

3. THINGS TO REMEMBER :

(a) The value of the derivative at P (x_1, y_1) gives the slope of the tangent to the curve at P. Symbolically

$$f'(x_1) = \left.\frac{dy}{dx}\right]_{(x_1, y_1)} = \text{Slope of tangent at } P(x_1, y_1) = m(\text{say}).$$

(b) Equation of tangent at (x_1, y_1) is ;

$$y - y_1 = \left.\frac{dy}{dx}\right]_{(x_1, y_1)} (x - x_1)$$

(c) Equation of normal at (x_1, y_1) is; $y - y_1 = -\dfrac{1}{\left.\frac{dy}{dx}\right]_{(x_1, y_1)}} (x - x_1)$.

Note :

(i) The point P (x_1, y_1) will satisfy the equation of the curve & the equation of tangent & normal line.

(ii) If the tangent at any point P on the curve is parallel to the axis of x then dy/dx = 0 at the point P.

(iii) If the tangent at any point on the curve is parallel to the axis of y, then dy/dx is not defined or dx/dy = 0 at that point.

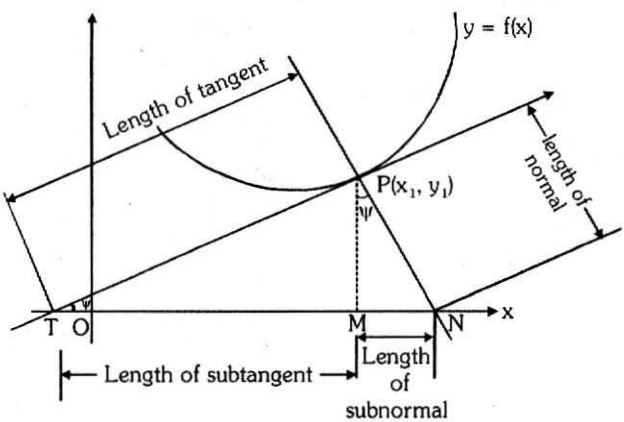
(iv) If the tangent at any point on the curve is equally inclined to both the axes then dy/dx = ±1.

(v) If a curve passing through the origin be given by a rational integral algebraic equation, then the equation of the tangent (or tangents) at the origin is obtained by equating to zero the terms of the lowest degree in the equation. e.g. If the equation of a curve be $x^2 - y^2 + x^3 + 3x^2y - y^3 = 0$, the tangents at the origin are given by $x^2 - y^2 = 0$ i.e. $x + y = 0$ and $x - y = 0$

4. ANGLE OF INTERSECTION BETWEEN TWO CURVES :

Angle of intersection between two curves is defined as the angle between the two tangents drawn to the two curves at their point of intersection. If the angle between two curves is 90° then they are called **ORTHOGONAL** curves.

5. LENGTH OF TANGENT, SUBTANGENT, NORMAL & SUBNORMAL :



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(a) Length of the tangent (PT) = $\frac{y_1\sqrt{1+[f'(x_1)]^2}}{f'(x_1)}$

(b) Length of Subtangent (MT) = $\frac{y_1}{f'(x_1)}$

(c) Length of Normal (PN) = $y_1\sqrt{1+\left[\ f'(x_1)\ \right]^2}$

(d) Length of Subnormal (MN) = $y_1 f'(x_1)$

6. DIFFERENTIALS :

The differential of a function is equal to its derivative multiplied by the differential of the independent variable. Thus if, y = tanx then dy = $\sec^2$x dx. In general dy = f'(x)dx or df(x) = f'(x)dx

Note :

(i) d(c) = 0 where 'c' is a constant

(ii) d(u + v) = du + dv

(iii) d(uv) = udv + vdu

(iv) d(u − v) = du − dv

(v) $d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2}$

(vi) For the independent variable 'x', increment Δx and differential dx are equal but this is not the case with the dependent variable 'y' i.e. $\Delta y \neq dy$.

$\therefore$ Approximate value of y when increment Δx is given to independent variable x in y = f(x) is $y + \Delta y = f(x + \Delta x) = f(x) + \frac{dy}{dx}.\Delta x$

(vii) The relation dy = f'(x) dx can be written as $\frac{dy}{dx} = f'(x)$; thus the quotient of the differentials of 'y' and 'x' is equal to the derivative of 'y' w.r.t. 'x'.

INDEFINITE INTEGRATION

If f & F are function of x such that F' (x) = f(x) then the function F is called a **PRIMITIVE OR ANTIDERIVATIVE OR INTEGRAL** of f(x) w.r.t. x and is written symbolically as

$\int f(x)dx = F(x) + c \Leftrightarrow \frac{d}{dx}\{F(x)+c\} = f(x)$, where c is called the **constant of integration.**

1. STANDARD RESULTS :

(i) $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c;\ n \neq -1$

(ii) $\int \frac{dx}{ax+b} = \frac{1}{a}\ell n|ax+b| + c$

(iii) $\int e^{ax+b}dx = \frac{1}{a}e^{ax+b} + c$

(iv) $\int a^{px+q}dx = \frac{1}{p}\frac{a^{px+q}}{\ell na}(a>0) + c$

(v) $\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b) + c$

(vi) $\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b) + c$

(vii) $\int \tan(ax+b)dx = \frac{1}{a}\ell n\sec(ax+b) + c$

(viii) $\int \cot(ax+b)dx = \frac{1}{a}\ell n\sin(ax+b) + c$

(ix) $\int \sec^2(ax+b)dx = \frac{1}{a}\tan(ax+b) + c$

(x) $\int \mathrm{cosec}^2(ax+b)dx = -\frac{1}{a}\cot(ax+b) + c$

(xi) $\int \mathrm{cosec}(ax+b).\cot(ax+b)dx = -\frac{1}{a}\mathrm{cosec}(ax+b) + c$