(xii) $\int \sec(ax+b).\tan(ax+b)dx = \frac{1}{a}.\sec(ax+b)+c$

(xiii) $\int \sec x\,dx = \ell n|\sec x + \tan x| + c$

OR $\int \sec x dx = \ell n \tan\left|\frac{\pi}{4}+\frac{x}{2}\right| + c$

(xiv) $\int \operatorname{cosec} x\,dx = \ell n|\operatorname{cosec} x - \cot x| + c$

OR $\int \operatorname{cosec} x\,dx = \ell n\left|\tan\frac{x}{2}\right| + c$ OR $-\ell n(\operatorname{cosec} x + \cot x) + c$

(xv) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$

(xvi) $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

(xvii) $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + c$

(xviii) $\int \frac{dx}{\sqrt{x^2+a^2}} = \ell n\left[x+\sqrt{x^2+a^2}\right] + c$

(xix) $\int \frac{dx}{\sqrt{x^2-a^2}} = \ell n\left[x+\sqrt{x^2-a^2}\right] + c$

(xx) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ell n\left|\frac{a+x}{a-x}\right| + c$

(xxi) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ell n\left|\frac{x-a}{x+a}\right| + c$

(xxii) $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

(xxiii) $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ell n\left(x+\sqrt{x^2+a^2}\right) + c$

(xxiv) $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ell n\left(x+\sqrt{x^2-a^2}\right) + c$

(xxv) $\int e^{ax}.\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c$

(xxvi) $\int e^{ax}.\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c$

2. TECHNIQUES OF INTEGRATION :

(a) Substitution or change of independent variable :

Integral I = $\int f(x)\,dx$ is changed to $\int f(\phi(t))\phi'(t)\,dt$, by a suitable substitution $x = \phi(t)$ provided the later integral is easier to integrate.

Some standard substitution :

(1) $\int [f(x)]^n f'(x)dx$ OR $\int \frac{f'(x)}{[f(x)]^n}dx$ put $f(x) = t$ & proceed.

(2) $\int \frac{dx}{ax^2+bx+c}, \int \frac{dx}{\sqrt{ax^2+bx+c}}, \int \sqrt{ax^2+bx+c}\;dx$

Express $ax^2 + bx + c$ in the form of perfect square & then apply the standard results.

(3) $\int \frac{px+q}{ax^2+bx+c}dx, \int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$

Express $px + q = A$ (differential coefficient of denominator) + B.

(4) $\int e^x[f(x)+f'(x)]dx = e^x.f(x)+c$

(5) $\int [f(x)+xf'(x)]dx = x\,f(x)+c$

(6) $\int \frac{dx}{x(x^n+1)}$ $n \in N$, take x^n common & put $1 + x^{-n} = t$.

(7) $\int \frac{dx}{x^2\left(x^n+1\right)^{(n-1)/n}}$ $n \in N$, take x^n common & put $1 + x^{-n} = t^n$

(8) $\int \frac{dx}{x^n(1+x^n)^{1/n}}$, take x^n common and put $1 + x^{-n} = t$.

(9) $\int \frac{dx}{a+b\sin^2 x}$ OR $\int \frac{dx}{a+b\cos^2 x}$

OR $\int \frac{dx}{a\sin^2 x + b\sin x\cos x + c\cos^2 x}$

Multiply N^r & D^r by $\sec^2 x$ & put $\tan x = t$.

(10) $\int \frac{dx}{a+b\sin x}$ OR $\int \frac{dx}{a+b\cos x}$ OR $\int \frac{dx}{a+b\sin x+c\cos x}$

Convert sines & cosines into their respective tangents of half the angles, put $\tan\frac{x}{2}=t$

(11) $\int \frac{a.\cos x+b.\sin x+c}{p.\cos x+q.\sin x+r}dx.$

Express Numerator $(N^r) \equiv \ell(D^r)+m\frac{d}{dx}(D^r)+n$ & proceed.

(12) $\int \frac{x^2+1}{x^4+Kx^2+1}dx$ OR $\int \frac{x^2-1}{x^4+Kx^2+1}dx$,

where K is any constant.

Divide Nr & Dr by x^2, then put $x-\frac{1}{x}=t$ OR $x+\frac{1}{x}=t$ respectively & proceed

(13) $\int \frac{dx}{(ax+b)\sqrt{px+q}}$ & $\int \frac{dx}{(ax^2+bx+c)\sqrt{px+q}}$; put $px+q=t^2$

(14) $\int \frac{dx}{(ax+b)\sqrt{px^2+qx+r}}$, put $ax+b=\frac{1}{t}$;

$\int \frac{dx}{(ax^2+bx+c)\sqrt{px^2+qx+r}}$, put $x=\frac{1}{t}$

(15) $\int \sqrt{\frac{x-\alpha}{\beta-x}}dx$ OR $\int \sqrt{(x-\alpha)(\beta-x)}$; put $x=\alpha\cos^2\theta+\beta\sin^2\theta$

$\int \sqrt{\frac{x-\alpha}{x-\beta}}dx$ OR $\int \sqrt{(x-\alpha)(x-\beta)}$; put $x=\alpha\sec^2\theta-\beta\tan^2\theta$

$\int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}$; put $x-\alpha=t^2$ or $x-\beta=t^2$.

(16) To integrate $\int \sin^m x \cos^n x\, dx$.

(i) If m is odd positive integer put cos x = t.

(ii) If n is odd positive integer put sin x = t

(iii) If m + n is negative even integer then put tan x = t.

(iv) If m and n both even positive integer then use

$$\sin^2 x = \frac{1-\cos 2x}{2}, \cos^2 x = \frac{1+\cos 2x}{2}$$

(b) Integration by part : $\int u.v\, dx = u\int v\, dx - \int\left[\frac{du}{dx}.\int v\, dx\right] dx$

where u & v are differentiable functions.

Note : While using integration by parts, choose u & v such that

(i) $\int v dx$ & (ii) $\int\left[\frac{du}{dx}.\int v\, dx\right] dx$ is simple to integrate.

This is generally obtained, by keeping the order of u & v as per the order of the letters in **ILATE**, where; I-Inverse function, L-Logarithmic function, A-Algebraic function, T-Trigonometric function & E-Exponential function.

(c) Partial fraction : Rational function is defined as the ratio of two polynomials in the form $\frac{P(x)}{Q(x)}$, where P(x) and Q(x) are polynomials in x and Q(x) ≠ 0. If the degree of P(x) is less than the degree of Q(x), then the rational function is called proper, otherwise, it is called improper. The improper rational function can be reduced to the proper rational functions by long division process. Thus, if $\frac{P(x)}{Q(x)}$ is improper, then $\frac{P(x)}{Q(x)} = T(x) + \frac{P_1(x)}{Q(x)}$, where T(x) is a polynomial in x and $\frac{P_1(x)}{Q(x)}$ is proper rational function. It is always possible to write the integrand as a sum of simpler rational functions by a method called partial fraction decomposition. After this, the integration can be carried out easily using the already known methods.

S. No.	Form of the rational function	Form of the partial fraction
1.	$\dfrac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\dfrac{A}{x-a}+\dfrac{B}{x-b}+\dfrac{C}{x-c}$
2.	$\dfrac{px^2+qx+r}{(x-a)^2(x-b)}$	$\dfrac{A}{x-a}+\dfrac{B}{(x-a)^2}+\dfrac{C}{x-b}$
3.	$\dfrac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\dfrac{A}{x-a}+\dfrac{Bx+C}{x^2+bx+c}$
	where x^2+bx+c cannot be factorised further	

Note :

In competitive exams, partial fraction are generally found by inspection by noting following fact :

$$\frac{1}{(x-\alpha)(x-\beta)}=\frac{1}{(\alpha-\beta)}\left(\frac{1}{x-\alpha}-\frac{1}{x-\beta}\right).$$

It can be applied to the case when x^2 or any other function is there in place of x.

Example :

(1) $\dfrac{1}{(x^2+1)(x^2+3)}=\dfrac{1}{2}\left(\dfrac{1}{t+1}-\dfrac{1}{t+3}\right)$ {take $x^2=t$}

(2) $\dfrac{1}{x^4(x^2+1)}=\dfrac{1}{x^2}\left(\dfrac{1}{x^2}-\dfrac{1}{x^2+1}\right)=\dfrac{1}{x^4}-\left(\dfrac{1}{x^2}-\dfrac{1}{x^2+1}\right)$

(3) $\dfrac{1}{x^3(x^2+1)}=\dfrac{1}{x}\left(\dfrac{1}{x^2}-\dfrac{1}{x^2+1}\right)=\dfrac{1}{x^3}-\dfrac{1}{x(x^2+1)}$

DEFINITE INTEGRATION

1. (a) The Fundamental Theorem of Calculus, Part 1 :

If f is continuous on [a, b], then the function g defined by

$$g(x) = \int_a^x f(t)dt \qquad a \le x \le b$$

is continuous on [a, b] and differentiable on (a, b), and $g'(x) = f(x)$.

(b) The Fundamental Theorem of Calculus, Part 2 :

If f is continuous on [a, b], then $\int_a^b f(x)dx = F(b) - F(a)$ where F is any antiderivative of f, that is, a function such that $F' = f$.

Note : If $\int_a^b f(x)dx = 0 \Rightarrow$ then the equation $f(x) = 0$ has atleast one root lying in (a,b) provided f is a continuous function in (a,b).

2. A definite integral is denoted by $\int_a^b f(x)dx$ which represent the area bounded by the curve $y = f(x)$, the ordinates $x = a$, $x = b$ and the x-axis. ex. $\int_0^{2\pi} \sin x\, dx = 0$

3. PROPERTIES OF DEFINITE INTEGRAL :

(a) $\int_a^b f(x)dx = \int_a^b f(t)\,dt \Rightarrow \int_0^b f(x)\,dx$ does not depend upon x. It is a numerical quantity.

(b) $\int_a^b f(x)dx = -\int_b^a f(x)\,dx$

(c) $\int_a^b f(x)dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where c may lie inside or outside the interval [a, b]. This property to be used when f is piecewise continuous in (a, b).

(d) $\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} [f(x) + f(-x)]dx = \begin{cases} 0 & ;\text{if } f(x) \text{ is an odd function} \\ 2\int_{0}^{a} f(x)dx & ;\text{if } f(x) \text{ is an even function} \end{cases}$

(e) $\int_{a}^{b} f(x)dx = \int_{a}^{b} f(a+b-x)\,dx$, In particular $\int_{0}^{a} f(x)dx = \int_{0}^{a} f(a-x)\,dx$

(f) $\int_{0}^{2a} f(x)dx = \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(2a-x)\,dx = \begin{cases} 2\int_{0}^{a} f(x)\,dx & ; \text{ if } f(2a-x) = f(x) \\ 0 & ; \text{ if } f(2a-x) = -f(x) \end{cases}$

(g) $\int_{0}^{nT} f(x)dx = n\int_{0}^{T} f(x)\,dx$, $(n \in I)$; where 'T' is the period of the function i.e. $f(T + x) = f(x)$

Note that : $\int_{x}^{T+x} f(t)dt$ will be independent of x and equal to $\int_{0}^{T} f(t)dt$

(h) $\int_{a+nT}^{b+nT} f(x)dx = \int_{a}^{b} f(x)dx$ where f(x) is periodic with period T & $n \in I$.

(i) $\int_{ma}^{na} f(x)dx = (n-m)\int_{0}^{a} f(x)dx$, $(n, m \in I)$ if f(x) is periodic with period 'a'.

4. WALLI'S FORMULA :

(a) $\int_{0}^{\pi/2} \sin^n x\,dx = \int_{0}^{\pi/2} \cos^n x\,dx = \dfrac{(n-1)(n-3).....(1 \text{ or } 2)}{n(n-2).....(1 \text{ or } 2)} K$

where $K = \begin{cases} \pi/2 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$

(b) $\int_{0}^{\pi/2} \sin^n x . \cos^m x\,dx$

$= \dfrac{[(n-1)(n-3)(n-5)....1 \text{ or } 2][(m-1)(m-3)....1 \text{ or } 2]}{(m+n)(m+n-2)(m+n-4)....1 \text{ or } 2} K$

$$\text{Where } K = \begin{cases} \dfrac{\pi}{2} & \text{if both m and n are even } (m, n \in N) \\ 1 & \text{otherwise} \end{cases}$$

5. DERIVATIVE OF ANTIDERIVATIVE FUNCTION (Newton-Leibnitz Formula) :

If h(x) & g(x) are differentiable functions of x then,

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)dt = f[h(x)].h'(x) - f[g(x)].g'(x)$$

6. DEFINITE INTEGRAL AS LIMIT OF A SUM :

$$\int_a^b f(x)dx = \lim_{n\to\infty} h[f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f\left(a + \overline{n-1}h\right)]$$

$$\lim_{h\to\infty} h\sum_{r=0}^{n-1} f(a+rh) = \int_0^1 f(x)dx \quad \text{where } b - a = nh$$

If $a = 0$ & $b = 1$ then, $\lim\limits_{n\to\infty} h\sum\limits_{r=0}^{n-1} f(rh) = \int_0^1 f(x)dx$; where $nh = 1$

OR $\lim\limits_{n\to\infty}\left(\frac{1}{n}\right)\sum\limits_{r=1}^{n-1} f\left(\frac{r}{n}\right) = \int_0^1 f(x)dx$.

7. ESTIMATION OF DEFINITE INTEGRAL :

(a) If f(x) is continuous in [a, b] and it's range in this interval is [m, M], then $m(b-a) \le \int_a^b f(x)dx \le M(b-a)$

(b) If $f(x) \le \phi(x)$ for $a \le x \le b$ then $\int_a^b f(x)dx \le \int_a^b \phi(x)dx$

(c) $\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$.

(d) If $f(x) \geq 0$ on the interval $[a,b]$, then $\int_a^b f(x)dx \geq 0$.

(e) f(x) and g(x) are two continuous function on [a, b] then

$$\left|\int_a^b f(x)\,g(x)\,dx\right| \leq \sqrt{\int_a^b f^2(x)dx \int_a^b g^2(x)\,dx}$$

8. SOME STANDARD RESULTS :

(a) $\displaystyle\int_0^{\pi/2} \log\sin x\,dx = -\frac{\pi}{2}\log 2 = \int_0^{\pi/2} \log\cos x\,dx$

(b) $\displaystyle\int_a^b \{x\}\,dx = \frac{b-a}{2}$ a, b $\in$ I

(c) $\displaystyle\int_a^b \frac{|x|}{x}dx = |b| - |a|$.

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DIFFERETIAL EQUATION

1. DIFFERENTIAL EQUATION :

An equation that involves independent and dependent variables and the derivatives of the dependent variables is called a DIFFERENTIAL EQUATION.

2. SOLUTION (PRIMITIVE) OF DIFFERENTIAL EQUATION :

Finding the unknown function which satisfies given differential equation is called SOLVING OR INTEGRATING the differential equation. The solution of the differential equation is also called its PRIMITIVE, because the differential equation can be regarded as a relation derived from it.

3. ORDER OF DIFFERENTIAL EQUATION :

The order of a differential equation is the order of the highest differential coefficient occurring in it.

4. DEGREE OF DIFFERENTIAL EQUATION :

The degree of a differential equation which can be written as a polynomial in the derivatives is the degree of the derivative of the highest order occurring in it, after it has been expressed in a form free from radicals & fractions so far as derivatives are concerned, thus the differential equation :

$f(x,\ y)\left[\frac{d^m y}{dx^m}\right]^p + \phi(x,y)\left[\frac{d^{m-1}(y)}{dx^{m-1}}\right]^q + \ldots\ldots\ldots = 0$ is of order m & degree p.

Note that in the differential equation $e^{y'''} - xy'' + y = 0$ order is three but degree doesn't exist.

5. FORMATION OF A DIFFERENTIAL EQUATION :

If an equation in independent and dependent variables having some arbitrary constant is given, then a differential equation is obtained as follows :

(a) Differentiate the given equation w.r.t the independent variable (say x) as many times as the number of arbitrary constants in it.

(b) Eliminate the arbitrary constants.

The eliminant is the required differential equation.

Note : A differential equation represents a family of curves all satisfying some common properties. This can be considered as the geometrical interpretation of the differential equation.

6. GENERAL AND PARTICULAR SOLUTIONS :

The solution of a differential equation which contains a number of independent arbitrary constants equal to the order of the differential equation is called the GENERAL SOLUTION (OR COMPLETE INTEGRAL OR COMPLETE PRIMITIVE). A solution obtainable from the general solution by giving particular values to the constants is called a PARTICULAR SOLUTION.

7. ELEMENTARY TYPES OF FIRST ORDER & FIRST DEGREE DIFFERENTIAL EQUATIONS :

(a) Variables separable :

TYPE-1 : If the differential equation can be expressed as ; $f(x)dx + g(y)dy = 0$ then this is said to be variable – separable type.

A general solution of this is given by $\int f(x)dx + \int g(y)dy = c$;

where c is the arbitrary constant. Consider the example $(dy/dx) = e^{x-y} + x^2.e^{-y}$.

TYPE-2 : Sometimes transformation to the polar co-ordinates facilitates separation of variables. In this connection it is convenient to remember the following differentials. If $x = r\cos\theta$, $y = r\sin\theta$ then,

(i) $xdx + ydy = rdr$

(ii) $dx^2 + dy^2 = dr^2 + r^2d\theta^2$

(iii) $xdy - ydx = r^2\, d\theta$

If $x = r\sec\theta$ & $y = r\tan\theta$ then

$x\,dx - y\,dy = r\,dr$ and $x\,dy - y\,dx = r^2\sec\theta\, d\theta$.

TYPE – 3 : $\dfrac{dy}{dx} = f(ax+by+c),\ b \neq 0$

To solve this, substitute $t = ax + by + c$. Then the equation reduces to separable type in the variable t and x which can be solved.

Consider the example $(x+y)^2\dfrac{dy}{dx} = a^2$

(b) Homogeneous equations :

A differential equation of the form $\dfrac{dy}{dx} = \dfrac{f(x,y)}{\phi(x,y)}$, where $f(x, y)$ & $\phi(x, y)$ are homogeneous functions of x & y and of the same degree, is called HOMOGENEOUS. This equation may also be reduced to the form $\dfrac{dy}{dx} = g\left(\dfrac{x}{y}\right)$ & is solved by putting $y = vx$ so that the dependent variable y is changed to another variable v, where v is some unknown function, the differential equation is transformed to an equation with variables separable. Consider the example $\dfrac{dy}{dx} + \dfrac{y(x+y)}{x^2} = 0$

(c) Equations reducible to the homogeneous form :

If $\dfrac{dy}{dx} = \dfrac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$; where $a_1b_2 - a_2b_1 \neq 0$, i.e. $\dfrac{a_1}{b_1} \neq \dfrac{a_2}{b_2}$

then the substitution $x = u + h$, $y = v + k$

transform this equation to a homogeneous type in the new variables u and v where h and k are arbitrary constants to be chosen so as to make the given equation homogeneous.

(i) If $a_1b_2 - a_2b_1 = 0$, then a substitution $u = a_1x + b_1y$ transforms the differential equation to an equation with variables separable.

(ii) If $b_1 + a_2 = 0$, then a simple cross multiplication and substituting d(xy) for xdy + ydx & integrating term by term yields the result easily.

Consider the examples $\frac{dy}{dx} = \frac{x-2y+5}{2x+y-1}$; $\frac{dy}{dx} = \frac{2x+3y-1}{4x+6y-5}$

& $\frac{dy}{dx} = \frac{2x-y+1}{6x-5y+4}$

(iii) In an equation of the form : $yf(xy)dx + xg(xy)dy = 0$ the variables can be separated by the substitution $xy = v$.

8. LINEAR DIFFERENTIAL EQUATIONS :

A differential equation is said to be linear if the dependent variable & its differential coefficients occur in the first degree only and are not multiplied together.

The nth order linear differential equation is of the form ;

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots\dots\dots + a_n(x).$$

$y = \phi(x)$, where $a_0(x)$, $a_1(x)$ .... $a_n(x)$ are called the coefficients of the differential equation.

(a) Linear differential equations of first order :

The most general form of a linear differential equations of first order is $\frac{dy}{dx} + Py = Q$, where P & Q are functions of x.

To solve such an equation multiply both sides by $e^{\int P dx}$. Then the solution of this equation will be $ye^{\int Pdx} = \int Qe^{\int Pdx}dx + c$

(b) Equations reducible to linear form :

The equation $\frac{dy}{dx} + Py = Q.y^n$ where P & Q are function, of x, is reducible to the linear form by dividing it by y^n & then substituting $y^{-n+1} = Z$. Consider the example $(x^3y^2+xy)dx = dy$.

The equation $\frac{dy}{dx} + Py = Qy^n$ is called BERNOULI'S EQUATION.

9. TRAJECTORIES :

A curve which cuts every member of a given family of curves according to a given law is called a Trajectory of the given family.

Orthogonal trajectories :

A curve making at each of its points a right angle with the curve of the family passing through that point is called an orthogonal trajectory of that family.

We set up the differential equation of the given family of curves. Let it be of the form $F(x, y, y') = 0$

The differential equation of the orthogonal trajectories is of the form

$$F\left(x, y, \frac{-1}{y'}\right) = 0$$

The general integral of this equation $\phi_1(x, y, C) = 0$ gives the family of orthogonal trajectories.

Note :

Following exact differentials must be remembered :

(i) $xdy + y\,dx = d(xy)$

(ii) $\frac{xdy - ydx}{x^2} = d\left(\frac{y}{x}\right)$

(iii) $\frac{ydx - xdy}{y^2} = d\left(\frac{x}{y}\right)$

(iv) $\frac{xdy + ydx}{xy} = d(\ell n xy)$

(v) $\frac{dx + dy}{x + y} = d(\ell n(x + y))$

(vi) $\frac{xdy - ydx}{xy} = d\left(\ell n \frac{y}{x}\right)$

(vii) $\dfrac{ydx - xdy}{xy} = d\left(\ell n \dfrac{x}{y}\right)$ **(viii)** $\dfrac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-1}\dfrac{y}{x}\right)$

(ix) $\dfrac{ydx - xdy}{x^2 + y^2} = d\left(\tan^{-1}\dfrac{x}{y}\right)$ **(x)** $\dfrac{xdx + ydy}{x^2 + y^2} = d\left[\ell n\sqrt{x^2 + y^2}\right]$

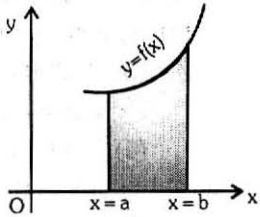
(xi) $d\left(-\dfrac{1}{xy}\right) = \dfrac{xdy + ydx}{x^2y^2}$ **(xii)** $d\left(\dfrac{e^x}{y}\right) = \dfrac{ye^x dx - e^x dy}{y^2}$

(xiii) $d\left(\dfrac{e^y}{x}\right) = \dfrac{xe^y dy - e^y dx}{x^2}$

AREA UNDER THE CURVE

1. The area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$ & $x = b$ is given by,

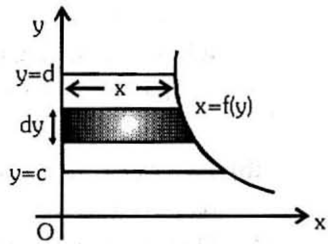
$$A = \int_a^b f(x)dx = \int_a^b y\,dx.$$



2. If the area is below the x-axis then A is negative. The convention is to consider the magnitude only i.e. $A = \left|\int_a^b y\,dx\right|$ in this case.

3. The area bounded by the curve $x = f(y)$, y-axis & abscissa $y = c$, $y = d$ is given by,

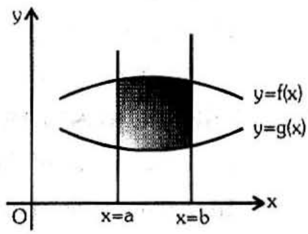
$$\text{Area} = \int_c^d x\,dy = \int_c^d f(y)dy$$



4. Area between the curves $y = f(x)$ & $y = g(x)$ between the ordinates $x = a$ & $x = b$ is given by,

$$A = \int_a^b f(x)dx - \int_a^b g(x)dx$$

$$= \int_a^b [f(x) - g(x)]dx$$



5. Average value of a function $y = f(x)$ w.r.t. x over an interval $a \le x \le b$ is defined as : $y(av) = \dfrac{1}{b-a}\int_a^b f(x)dx$.

6. CURVE TRACING :

The following outline procedure is to be applied in Sketching the graph of a function $y = f(x)$ which in turn will be extremely useful to quickly and correctly evaluate the area under the curves.

(a) Symmetry : The symmetry of the curve is judged as follows :

(i) If all the powers of y in the equation are even then the curve is symmetrical about the axis of x.

(ii) If all the powers of x ar even, the curve is symmetrical about the axis of y.

(iii) If powers of x & y both are even, the curve is symmetrical about the axis of x as well as y.

(iv) If the equation of the curve remains unchanged on interchanging x and y, then the curve is symmetrical about $y = x$.

(v) If on interchanging the signs of x & y both the equation of the curve is unaltered then there is symmetry in opposite quadrants.

(b) Find dy/dx & equate it to zero to find the points on the curve where you have horizontal tangents.

(c) Find the points where the curve crosses the x-axis & also the y-axis.

(d) Examine if possible the intervals when f(x) is increasing or decreasing. Examine what happens to 'y' when $x \to \infty$ or $-\infty$.

7. USEFUL RESULTS :

(a) Whole area of the ellipse, $x^2/a^2 + y^2/b^2 = 1$ is πab.

(b) Area enclosed between the parabolas $y^2 = 4ax$ & $x^2 = 4by$ is $16ab/3$.

(c) Area included between the parabola $y^2 = 4ax$ & the line $y = mx$ is $8a^2/3m^3$.

VECTORS

1. Physical quantities are broadly divided in two categories viz (a) Vector Quantities & (b) Scalar quantities.

(a) Vector quantities :

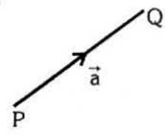
Any quantity, such as velocity, momentum, or force, that has both magnitude and direction and for which vector addition is defined and meaningful; is treated as vector quantities.

(b) Scalar quantities :

A quantity, such as mass, length, time, density or energy, that has size or magnitude but does not involve the concept of direction is called scalar quantity.

2. REPRESENTATION :

Vectors are represented by directed straight line segment



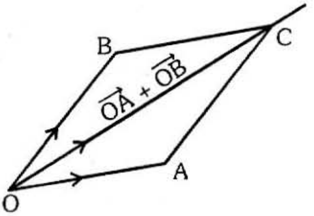
magnitude of $\vec{a} = |\vec{a}|$ = length PQ

direction of $\vec{a}$ = P to Q.

3. ADDITION OF VECTORS :

(a) It is possible to develop an Algebra of Vectors which proves useful in the study of Geometry, Mechanics and other branches of Applied Mathematics.

(i) If two vectors $\vec{a}$ & $\vec{b}$ are represented by $\overrightarrow{OA}$ & $\overrightarrow{OB}$, then their sum $\vec{a}+\vec{b}$ is a vector represented by $\overrightarrow{OC}$, where OC is the diagonal of the parallelogram OACB.



(ii) $\vec{a}+\vec{b}=\vec{b}+\vec{a}$ (commutative)

(iii) $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$ (associativity)

(b) Multiplication of vector by scalars :

(i) $m(\vec{a}) = (\vec{a})m = m\vec{a}$ **(ii)** $m(n\vec{a}) = n(m\vec{a}) = (mn)\vec{a}$

(iii) $(m+n)\vec{a} = m\vec{a} + n\vec{a}$ **(iv)** $m(\vec{a}+\vec{b}) = m\vec{a} + m\vec{b}$

4. **(a) ZERO VECTOR OR NULL VECTOR :**

A vector of zero magnitude i.e. which has the same initial & terminal point is called a ZERO VECTOR. It is denoted by $\vec{O}$.

(b) UNIT VECTOR :

A vector of unit magnitude in direction of a vector $\vec{a}$ is called unit vector along $\vec{a}$ and is denoted by $\hat{a}$ symbolically $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$.

(c) COLLINEAR VECTORS

Two vectors are said to be collinear if their supports are parallel disregards to their direction. Collinear vectors are also called **Parallel vectors**. If they have the same direction they are named as **like vectors** otherwise **unlike vectors.**

Symbolically two non zero vectors $\vec{a}$ & $\vec{b}$ are collinear if and only if, $\vec{a} = K\vec{b}$, where $K \in R$

(d) COPLANAR VECTORS

A given number of vectors are called coplanar if their supports are all parallel to the same plane.

Note that "TWO VECTORS ARE ALWAYS COPLANAR".

(e) EQUALITY OF TWO VECTORS :

Two vectors are said to be equal if they have

(i) the same length,

(ii) the same or parallel supports and

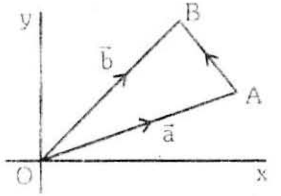
(iii) the same sense.

(f) Free vectors : If a vector can be translated anywhere in space without changing its magnitude & direction, then such a vector is called free vector. In other words, the initial point of free vector can be taken anywhere in space keeping its magnitude & direction same.

(g) Localized vectors : For a vector of given magnitude and direction, if its initial point is fixed in space, then such a vector is called localised vector. Unless & until stated, vectors are treated as free vectors.

5. POSITION VECTOR :

Let O be a fixed origin, then the position vector of a point P is the vector $\overrightarrow{OP}$. If $\vec{a}$ & $\vec{b}$ are position vectors of two point A and B, then,



$\overrightarrow{AB} = \vec{b} - \vec{a}$ = pv of B – pv of A.

6. SECTION FORMULA :

If $\vec{a}$ & $\vec{b}$ are the position vectors of two points A & B then the p.v. of a point which divides AB in the ratio m : n is given by :

$\vec{r} = \dfrac{n\vec{a} + m\vec{b}}{m+n}$.

7. VECTOR EQUATION OF A LINE :

Parametric vector equation of a line passing through two point $A(\vec{a})$ & $B(\vec{b})$ is given by, $\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$ where t is a parameter. If the line pass through the point $A(\vec{a})$ & is parallel to the vector $\vec{b}$ then its equation is $\vec{r} = \vec{a} + t\vec{b}$

8. TEST OF COLLINEARITY OF THREE POINTS :

(a) Three points A, B, C with position vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ respectively are collinear, if & only if there exist scalars x, y, z not all zero simultaneously such that ; $x\vec{a}+y\vec{b}+z\vec{c}=0$, where $x+y+z=0$

(b) Three points A, B, C are collinear, if any two vectors $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CA}$ are parallel.

9. SCALAR PRODUCT OF TWO VECTORS (DOT PRODUCT):

(a) $\vec{a}.\vec{b}=|\vec{a}||\vec{b}|\cos\theta \ (0\le\theta\le\pi)$, θ is angle between $\vec{a}$ & $\vec{b}$.

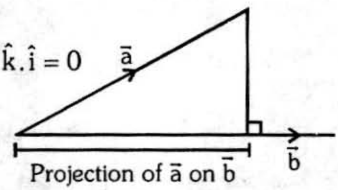
Note that if θ is acute then $\vec{a}.\vec{b}>0$ & if θ is obtuse then $\vec{a}.\vec{b}<0$

(b) $\vec{a}.\vec{a}=|\vec{a}|^2=\vec{a}^2$, $\vec{a}.\vec{b}=\vec{b}.\vec{a}$ (commutative) $\vec{a}.(\vec{b}+\vec{c})=\vec{a}.\vec{b}+\vec{a}.\vec{c}$ (distributive)

(c) $\vec{a}.\vec{b}=0 \Leftrightarrow \vec{a}\perp\vec{b}$; $(\vec{a},\vec{b}\neq 0)$

(d) $\hat{i}.\hat{i}=\hat{j}.\hat{j}=\hat{k}.\hat{k}=1$; $\hat{i}.\hat{j}=\hat{j}.\hat{k}=\hat{k}.\hat{i}=0$

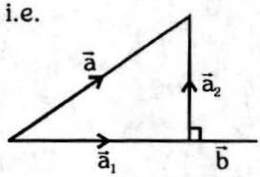
(e) Projection of $\vec{a}$ on $\vec{b}=\dfrac{\vec{a}.\vec{b}}{|\vec{b}|}$.



Note :

(i) The vector component of $\vec{a}$ along $\vec{b}$ i.e.

$\vec{a}_1=\left(\dfrac{\vec{a}.\vec{b}}{\vec{b}^2}\right)\vec{b}$ and perpendicular



to $\vec{b}$ i.e. $\vec{a}_2=\vec{a}-\left(\dfrac{\vec{a}.\vec{b}}{\vec{b}^2}\right)\vec{b}$ $\qquad (\vec{a}=\vec{a}_1+\vec{a}_2)$

(ii) The angle ϕ between $\vec{a}$ & $\vec{b}$ is given by

$$\cos\phi=\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} \quad 0\le\phi\le\pi$$

(iii) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ & $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then

$$\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}, \ |\vec{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$$

(iv) $-|\vec{a}||\vec{b}| \le \vec{a}.\vec{b} \le |\vec{a}||\vec{b}|$

(v) Any vector $\vec{a}$ can be written as, $\vec{a} = (\vec{a}.\hat{i})\hat{i} + (\vec{a}.\hat{j})\hat{j} + (\vec{a}.\hat{k})\hat{k}$

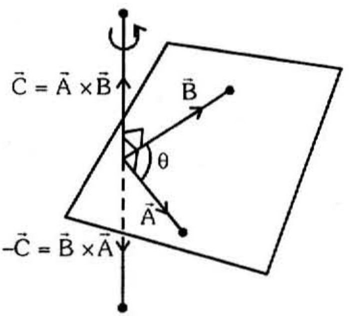
(vi) A vector in the direction of the bisector of the angle between the two vectors $\vec{a}$ & $\vec{b}$ is $\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{b}|}$. Hence bisector of the angle between the two vectors $\vec{a}$ & $\vec{b}$ is $\lambda(\hat{a} + \hat{b})$, where $\lambda \in R^+$. Bisector of the exterior angle between $\vec{a}$ & $\vec{b}$ is $\lambda(\hat{a} - \hat{b})$, $\lambda \in R^+$

(vii) $|\vec{a} \pm \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 \pm 2\vec{a}.\vec{b}$

(viii) $|\vec{a} + \vec{b} + \vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a})$

10. VECTOR PRODUCT OF TWO VECTORS (CROSS PRODUCT) :

(a) If $\vec{a}$ & $\vec{b}$ are two vectors & θ is the angle between them, then $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta\hat{n}$, where $\hat{n}$ is the unit vector perpendicular to both $\vec{a}$ & $\vec{b}$ such that $\vec{a}$, $\vec{b}$ & $\vec{n}$ forms a right handed screw system .



(b) Lagranges Identity : For any two vectors $\vec{a}$ & $\vec{b}$;

$$(\vec{a} \times \vec{b})^2 = |\vec{a}|^2|\vec{b}|^2 - (\vec{a}.\vec{b})^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} \\ \vec{a}.\vec{b} & \vec{b}.\vec{b} \end{vmatrix}$$

(c) Formulation of vector product in terms of scalar product : The vector product $\vec{a} \times \vec{b}$ is the vector $\vec{c}$, such that

(i) $|\vec{c}| = \sqrt{\vec{a}^2\vec{b}^2 - (\vec{a}.\vec{b})^2}$ **(ii)** $\vec{c}.\vec{a} = 0;\ \vec{c}.\vec{b} = 0$ and

(iii) $\vec{a}$, $\vec{b}$, $\vec{c}$ form a right handed system

(d) $\vec{a} \times \vec{b} = 0 \Leftrightarrow \vec{a}$ & $\vec{b}$ are parallel (collinear) $(\vec{a} \neq 0,\ \vec{b} \neq 0)$

i.e. $\vec{a} = K\vec{b}$, where K is a scalar

(i) $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$ (not commutative)

(ii) $(m\vec{a}) \times \vec{b} = \vec{a} x (m\vec{b}) = m(\vec{a} \times \vec{b})$ where m is a scalar.

(iii) $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$ (distributive)

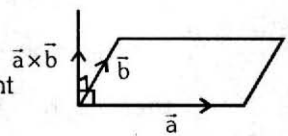
(vi) $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$

$\hat{i} \times \hat{j} = \hat{k},\ \hat{j} \times \hat{k} = \hat{i},\ \hat{k} \times \hat{i} = \hat{j}$

(e) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ & $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

(f) Geometrically $|\vec{a} \times \vec{b}|$ = area of the parallelogram whose two adjacent sides are represented by $\vec{a}$ & $\vec{b}$.



(g) **(i)** Unit vector perpendicular to the plane of $\vec{a}$ & $\vec{b}$ is

$$\hat{n} = \pm \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$$

(ii) A vector of magnitude 'r' & perpendicular to the plane of $\vec{a}$ & $\vec{b}$ is $\pm \dfrac{r(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|}$

(iii) If θ is the angle between $\vec{a}$ & $\vec{b}$ then $\sin\theta = \dfrac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}$

(h) Vector area :

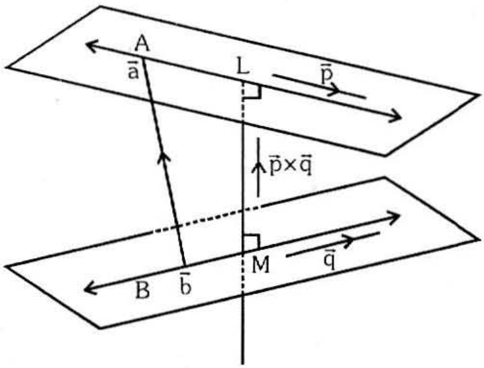
(i) If $\vec{a}$, $\vec{b}$ & $\vec{c}$ are the pv's of 3 points A, B & C then

the vector area of triangle ABC $= \frac{1}{2}\left[\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}\right]$.

The points A, B & C are collinear if $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}=0$

(ii) Area of any quadrilateral whose diagonal vectors are $\vec{d}_1$ & $\vec{d}_2$ is given by $\frac{1}{2}\left|\vec{d}_1\times\vec{d}_2\right|$. Area of $\Delta = \frac{1}{2}|\vec{a}\times\vec{b}|$

11. SHORTEST DISTANCE BETWEEN TWO LINES :



Lines which do not intersect & are also not parallel are called skew lines. In other words the lines which are not coplanar are skew lines. For Skew lines the direction of the shortest distance vector would be perpendicular to both the lines. The magnitude of the shortest distance vector would be equal to that of the projection of $\vec{AB}$ along the direction of the line of shortest distance, $\vec{LM}$ is parallel to $\vec{p}\times\vec{q}$

i.e. $\vec{LM}$ = | Projection of $\vec{AB}$ on $\vec{LM}$ |

= | Projection of $\vec{AB}$ on $\vec{p}\times\vec{q}$ |

$$= \left|\frac{\vec{AB}.(\vec{p}\times\vec{q})}{\vec{p}\times\vec{q}}\right| = \left|\frac{(\vec{b}-\vec{a}).(\vec{p}\times\vec{q})}{|\vec{p}\times\vec{q}|}\right|$$

(a) The two lines directed along $\vec{p}$ & $\vec{q}$ will intersect only if shortest distance = 0

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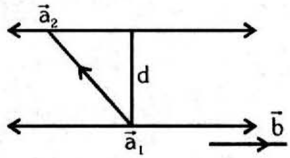
i.e. $(\vec{b}-\vec{a}).(\vec{p}\times\vec{q}) = 0$ i.e. $(\vec{b}-\vec{a})$ lies in the plane containing $\vec{p}$ & $\vec{q}$ $\Rightarrow \left[(\vec{b}-\vec{a})\ \vec{p}\ \vec{q}\right] = 0$

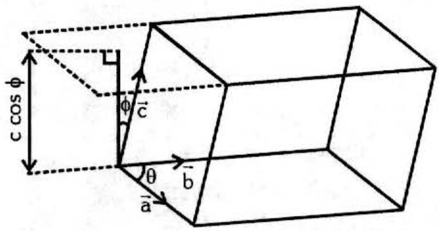
(b) If two lines are given by $\vec{r}_1 = \vec{a}_1 + K_1\vec{b}$ & $\vec{r}_2 = \vec{a}_2 + K_2\vec{b}$ i.e. they are parallel then, $d = \left|\frac{\vec{b}\times(\vec{a}_2-\vec{a}_1)}{|\vec{b}|}\right|$



12. SCALAR TRIPLE PRODUCT / BOX PRODUCT / MIXED PRODUCT :

(a) The scalar triple product of three vectors $\vec{a}, \vec{b}$ & $\vec{c}$ is defined as: $(\vec{a}\times\vec{b}).\vec{c} = |\vec{a}||\vec{b}||\vec{c}|\sin\theta\cos\phi$

where θ is the angle between $\vec{a}$ & $\vec{b}$ & ϕ is the angle between $\vec{a}\times\vec{b}$ & $\vec{c}$. It is also defined as $[\vec{a}\ \vec{b}\ \vec{c}]$, spelled as box product.



(b) In a scalar triple product the position of dot & cross can be interchanged i.e. $\vec{a}.(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b}).\vec{c}$ OR $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$

(c) $\vec{a}.(\vec{b}\times\vec{c}) = -a.(\vec{c}\times\vec{b})$ i.e. $[\vec{a}\ \vec{b}\ \vec{c}] = -[\vec{a}\,\vec{c}\,\vec{b}]$

(d) If $\vec{a}, \vec{b}, \vec{c}$ are coplanar $\Leftrightarrow [\vec{a}\ \vec{b}\ \vec{c}] = 0 \Rightarrow \vec{a}, \vec{b}, \vec{c}$ are linearly dependent.

(e) Scalar product of three vectors, two of which are equal or parallel is 0 i.e. $[\vec{a}\ \vec{b}\ \vec{c}] = 0$

(f) $[i\,j\,k] = 1; [K\vec{a}\ \vec{b}\ \vec{c}] = K[\vec{a}\ \vec{b}\ \vec{c}]; [(\vec{a}+\vec{b})\ \vec{c}\ \vec{d}] = [\vec{a}\,\vec{c}\,\vec{d}] + [\vec{b}\,\vec{c}\,\vec{d}]$

(g) **(i)** The Volume of the tetrahedron OABC with O as origin & the pv's of A, B and C being $\vec{a}, \vec{b}$ & $\vec{c}$ are given by $V = \frac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}]$

(ii) Volume of parallelopiped whose co-terminus edges are $\vec{a}, \vec{b}$ & $\vec{c}$ is $[\vec{a}\ \vec{b}\ \vec{c}]$.

(h) Remember that :

(i) $[\vec{a}-\vec{b}\ \ \vec{b}-\vec{c}\ \ \vec{c}-\vec{a}]=0$

(ii) $[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}] = 2[\vec{a}\ \vec{b}\ \vec{c}]$

(iii) $[\vec{a}\ \vec{b}\ \vec{c}]^2 = [\vec{a}\times\vec{b}\ \ \vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}] = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} & \vec{a}.\vec{c} \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} & \vec{b}.\vec{c} \\ \vec{c}.\vec{a} & \vec{c}.\vec{b} & \vec{c}.\vec{c} \end{vmatrix}$

13. VECTOR TRIPLE PRODUCT :

Let $\vec{a}$, $\vec{b}$ & $\vec{c}$ be any three vectors, then that expression $\vec{a}\times(\vec{b}\times\vec{c})$ is a vector & is called a vector triple product.

(a) $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}.\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c}$

(b) $(\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}\ .\ \vec{c})\vec{b}-(\vec{b}\ .\ \vec{c})\vec{a}$

(c) $(\vec{a}\times\vec{b})\times\vec{c} \neq \vec{a}\times(\vec{b}\times\vec{c})$

14. LINEAR COMBINATIONS / LINEAR INDEPENDENCE AND DEPENDENCE OF VECTORS :

Linear combination of vectors :

Given a finite set of vectors $\vec{a}, \vec{b}, \vec{c}$,........... then the vector $\vec{r} = x\vec{a}+y\vec{b}+z\vec{c}+$........... is called a linear combination of $\vec{a}, \vec{b}, \vec{c}$,........ for any x, y, z .... $\in R$. We have the following results :

(a) If $\vec{x}_1, \vec{x}_2,....\vec{x}_n$ are n non zero vectors, & $k_1, k_2,....k_n$ are n scalars & if the linear combination $k_1\vec{x}_1+k_2\vec{x}_2+.....k_n\vec{x}_n=0$ $\Rightarrow k_1=0, k_2=0.....k_n=0$ then we say that vectors $\vec{x}_1, \vec{x}_2,....\vec{x}_n$ are **linearly independent vectors**

(b) If $\vec{x}_1, \vec{x}_2,.......\vec{x}_n$ are **not linearly independent** then they are said to be linear dependent vectors. i.e. if $k_1\vec{x}_1, k_2\vec{x}_2+.......k_n\vec{x}_n=0$ & if there exists at least one $k_r \neq 0$ then $\vec{x}_1, \vec{x}_2,...\vec{x}_n$ are said to be **linearly dependent.**

(c) Fundamental theorem in plane : let $\vec{a}, \vec{b}$ be non zero, non collinear vectors. then any vector $\vec{r}$ coplanar with $\vec{a}, \vec{b}$ can be expressed uniquely as a linear combination of $\vec{a}, \vec{b}$ i.e. there exist some unique $x, y \in R$ such that $x\vec{a} + y\vec{b} = \vec{r}$

(d) Fundamental theorem in space : let $\vec{a}, \vec{b}, \vec{c}$ be non-zero, non-coplanar vectors in space. Then any vector $\vec{r}$, can be uniquely expressed as a linear combination of $\vec{a}, \vec{b}, \vec{c}$ i.e. There exist some unique $x, y, z \in R$ such that $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c}$.

15. COPLANARITY OF FOUR POINTS :

Four points A, B, C, D with position vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ respectively are coplanar if and only if there exist scalars x, y, z, w not all zero simultaneously such that $x\vec{a} + y\vec{b} + z\vec{c} + w\vec{d} = 0$
where, $x + y + z + w = 0$

16. RECIPROCAL SYSTEM OF VECTORS :

If $\vec{a}, \vec{b}, \vec{c}$ & $\vec{a}', \vec{b}', \vec{c}'$ are two sets of non coplanar vectors such that $\vec{a} . \vec{a}' = \vec{b} . \vec{b}' = \vec{c} . \vec{c}' = 1$ then the two systems are called Reciprocal System of vectors.

Note : $\vec{a}' = \dfrac{\vec{b} \times \vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}$; $\vec{b}' = \dfrac{\vec{c} \times \vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}$; $\vec{c}' = \dfrac{\vec{a} \times \vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$

17. TETRAHEDRON :

(i) Lines joining the vertices of a tetrahedron to the centroids of the opposite faces are concurrent and this point of concurrecy is called the centre of the tetrahedron.

(ii) In a tetrahedron, straight lines joining the mid points of each pair of opposite edges are also concurrent at the centre of the tetrahedron.

(iii) The angle between any two plane faces of regular tetrahedron is $\cos^{-1}\dfrac{1}{3}$

3D-COORDINATE GEOMETRY

1. DISTANCE FORMULA :

The distance between two points A (x_1, y_1, z_1) and B (x_2, y_2, z_2) is given by $AB = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$

2. SECTION FORMULAE :

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points and let R (x, y, z) divide PQ in the ratio $m_1 : m_2$. Then R is

$$(x, y, z) = \left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}, \frac{m_1z_2 + m_2z_1}{m_1 + m_2}\right)$$

If (m_1/m_2) is positive, R divides PQ internally and if (m_1/m_2) is negative, then externally.

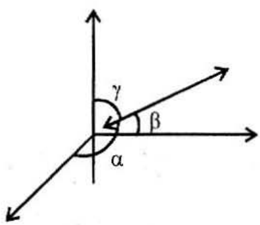
Mid point of PQ is given by $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$

3. CENTROID OF A TRIANGLE :

Let $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ be the vertices of a triangle ABC. Then its centroid G is given by

$$G = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$$

4. DIRECTION COSINES OF LINE :



If α, β, γ be the angles made by a line with x-axis, y-axis & z-axis respectively then $\cos\alpha$, $\cos\beta$ & $\cos\gamma$ are called direction cosines of a line, denoted by l, m & n respectively and the relation between ℓ, m, n is given by $\ell^2 + m^2 + n^2 = 1$

D. cosine of x-axis, y-axis & z-axis are respectively
1, 0, 0; 0, 1, 0; 0, 0, 1

5. DIRECTION RATIOS :

Any three numbers a, b, c proportional to direction cosines ℓ, m, n are called direction ratios of the line.

i.e. $\frac{\ell}{a} = \frac{m}{b} = \frac{n}{c}$

It is easy to see that there can be infinitely many sets of direction ratios for a given line.

6. RELATION BETWEEN D.C'S & D.R'S :

$$\frac{\ell}{a} = \frac{m}{b} = \frac{n}{c}$$

$$\therefore \quad \frac{\ell^2}{a^2} = \frac{m^2}{b^2} = \frac{n^2}{c^2} = \frac{\ell^2 + m^2 + n^2}{a^2 + b^2 + c^2}$$

$$\therefore \quad \ell = \frac{\pm a}{\sqrt{a^2 + b^2 + c^2}}; \quad m = \frac{\pm b}{\sqrt{a^2 + b^2 + c^2}}; \quad n = \frac{\pm c}{\sqrt{a^2 + b^2 + c^2}}$$

7. DIRECTION COSINE OF AXES :

Direction ratios and Direction cosines of the line joining two points :

Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be two points, then d.r.'s of AB are $x_2 - x_1, y_2 - y_1, z_2 - z_1$ and the d.c.'s of AB are $\frac{1}{r}(x_2 - x_1)$, $\frac{1}{r}(y_2 - y_1)$, $\frac{1}{r}(z_2 - z_1)$ where $r = \sqrt{[\Sigma(x_2 - x_1)^2]} = |\overrightarrow{AB}|$

8. PROJECTION OF A LINE ON ANOTHER LINE :

Let PQ be a line segment with $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ and let L be a straight line whose d.c.'s are l, m, n. Then the length of projection of PQ on the line L is $|\ell(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)|$

9. ANGLE BETWEEN TWO LINES :

Let θ be the angle between the lines with d.c.'s l_1, m_1, n_1 and l_2, m_2, n_2 then $\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$. If a_1, b_1, c_1 and a_2, b_2, c_2 be D.R.'s of two lines then angle θ between them is given by

$$\cos\theta = \frac{(a_1a_2 + b_1b_2 + c_1c_2)}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}\sqrt{(a_2^2 + b_2^2 + c_2^2)}}$$

10. PERPENDICULARITY AND PARALLELISM :

Let the two lines have their d.c.'s given by l_1, m_1, n_1 and l_2, m_2, n_2 respectively then they are perpendicular if $\theta = 90°$ i.e. $\cos\theta = 0$, i.e. $l_1 l_2 + m_1m_2 + n_1n_2 = 0$.

Also the two lines are parallel if $\theta = 0$ i.e. $\sin\theta = 0$, i.e. $\frac{\ell_1}{\ell_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$

Note:
If instead of d.c.'s, d.r.'s a_1, b_1, c_1 and a_2, b_2, c_2 are given, then the lines are perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$ and parallel if $a_1/a_2 = b_1/b_2 = c_1/c_2$.

11. EQUATION OF A STRAIGHT LINE IN SYMMETRICAL FORM :

(a) One point form : Let $A(x_1, y_1, z_1)$ be a given point on the straight line and l, m, n the d.c's of the line, then its equation is

$$\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r \quad \text{(say)}$$

It should be noted that $P(x_1 + lr, y_1 + mr, z_1 + nr)$ is a general point on this line at a distance r from the point $A(x_1, y_1, z_1)$ i.e. $AP = r$. One should note that for $AP = r$; l, m, n must be d.c.'s not d.r.'s. If a, b, c are direction ratios of the line, then equation of the line is $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = r$ but here $AP \neq r$

(b) Equation of the line through two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$

is $\dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1} = \dfrac{z - z_1}{z_2 - z_1}$

12. FOOT, LENGTH AND EQUATION OF PERPENDICULAR FROM A POINT TO A LINE :

Let equation of the line be

$$\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r \qquad \text{(say)} \qquad \text{.......... (i)}$$

and A (α, β, γ) be the point. Any point on the line (i) is

$P(\ell r + x_1, mr + y_1, nr + z_1)$ .... (ii)

If it is the foot of the perpendicular, from A on the line, then AP is $\perp$ to the line, so $\ell(\ell r + x_1 - \alpha) + m(mr + y_1 - \beta) + n(nr + z_1 - \gamma) = 0$

i.e. $r = (\alpha - x_1)\ell + (\beta - y_1)m + (\gamma - z_1)n$

since $\ell^2 + m^2 + n^2 = 1$

Putting this value of r in (ii), we get the foot of perpendicular from point A to the line.

Length : Since foot of perpendicular P is known, length of perpendicular,

$$AP = \sqrt{[(\ell r + x_1 - \alpha)^2 + (mr + y_1 - \beta)^2 + (nr + z_1 - \gamma)^2]}$$

Equation of perpendicular is given by

$$\frac{x - \alpha}{\ell r + x_1 - \alpha} = \frac{y - \beta}{mr + y_1 - \beta} = \frac{z - \gamma}{nr + z_1 - \gamma}$$

13. EQUATIONS OF A PLANE :

The equation of every plane is of the first degree i.e. of the form $ax + by + cz + d = 0$, in which a, b, c are constants, where $a^2 + b^2 + c^2 \neq 0$ (i.e. a, b, c $\neq 0$ simultaneously).

(a) Vector form of equation of plane :

If $\vec{a}$ be the position vector of a point on the plane and $\vec{n}$ be a vector normal to the plane then it's vectorial equation is given by $(\vec{r} - \vec{a}).\vec{n} = 0 \Rightarrow \vec{r}.\vec{n} = d$ where $d = \vec{a}.\vec{n}$ = constant.

(b) Plane Parallel to the Coordinate Planes :

(i) Equation of y-z plane is $x = 0$.

(ii) Equation of z-x plane is $y = 0$.

(iii) Equation of x-y plane is $z = 0$.

(iv) Equation of the plane parallel to x-y plane at a distance c is $z = c$. Similarly, planes parallel to y-z plane and z-x plane are respectively $x = c$ and $y = c$.

(c) Equations of Planes Parallel to the Axes :

If $a = 0$, the plane is parallel to x-axis i.e. equation of the plane parallel to x-axis is $by + cz + d = 0$.

Similarly, equations of planes parallel to y-axis and parallel to z-axis are $ax + cz + d = 0$ and $ax + by + d = 0$ respectively.

(d) Equation of a Plane in Intercept Form :

Equation of the plane which cuts off intercepts a, b, c from the axes is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

(e) Equation of a Plane in Normal Form :

If the length of the perpendicular distance of the plane from the origin is p and direction cosines of this perpendicular are (l, m, n), then the equation of the plane is $lx + my + nz = p$.

(f) Vectorial form of Normal equation of plane :

If $\hat{n}$ is a unit vector normal to the plane from the origin to the plane and d be the perpendicular distance of plane from origin then its vector equation is $\vec{r}.\hat{n} = d$.

(g) Equation of a Plane through three points :

The equation of the plane through three non-collinear points $(x_1, y_1, z_1), (x_2, y_2, z_2)\ (x_3, y_3, z_3)$ is $\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$

14. ANGLE BETWEEN TWO PLANES :

Consider two planes $ax + by + cz + d = 0$ and $a'x + b'y + c'z + d' = 0$. Angle between these planes is the angle between their normals.

$$\cos\theta = \frac{aa' + bb' + cc'}{\sqrt{a^2 + b^2 + c^2}\sqrt{a'^2 + b'^2 + c'^2}}$$

$\therefore$ Planes are perpendicular if $aa' + bb' + cc' = 0$ and they are parallel if $a/a' = b/b' = c/c'$.

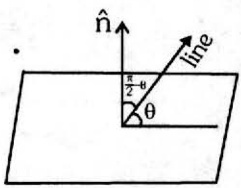
Planes parallel to a given Plane :

Equation of a plane parallel to the plane $ax + by + cz + d = 0$ is $ax + by + cz + d' = 0$. d' is to be found by other given condition.

15. ANGLE BETWEEN A LINE AND A PLANE :

Let equations of the line and plane be $\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ and $ax + by + cz + d = 0$ respectively and θ be the angle which line makes with the plane. Then $(\pi/2 - \theta)$ is the angle between the line and the normal to the plane.

So $\sin\theta = \frac{a\ell + bm + cn}{\sqrt{(a^2 + b^2 + c^2)}\sqrt{(\ell^2 + m^2 + n^2)}}$



Line is parallel to plane if $\theta = 0$

i.e. if $al + bm + cn = 0$.

Line is $\perp$ to the plane if line is parallel to the normal of the plane

i.e. if $\frac{a}{\ell} = \frac{b}{m} = \frac{c}{n}$.

16. CONDITION IN ORDER THAT THE LINE MAY LIE ON THE GIVEN PLANE :

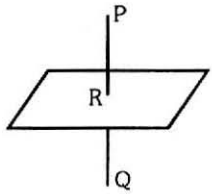
The line $\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ will lie on the plane $Ax + By + Cz + D = 0$

if **(a)** $A\ell + Bm + Cn = 0$ and **(b)** $Ax_1 + By_1 + Cz_1 + D = 0$

17. POSITION OF TWO POINTS W.R.T. A PLANE :

Two points $P(x_1, y_1, z_1)$ & $Q(x_2, y_2, z_2)$ are on the same or opposite sides of a plane $ax + by + cz + d = 0$ according to $ax_1 + by_1 + cz_1 + d$ & $ax_2 + by_2 + cz_2 + d$ are of same or opposite signs.

18. IMAGE OF A POINT IN THE PLANE :



Let the image of a point $P(x_1, y_1, z_1)$

in a plane $ax + by + cz + d = 0$ is

$Q(x_2, y_2, z_2)$ and foot of perpendicular

of point P on plane is $R(x_3, y_3, z_3)$, then

(a) $$\frac{x_3 - x_1}{a} = \frac{y_3 - y_1}{b} = \frac{z_3 - z_1}{c} = -\left(\frac{ax_1 + by_1 + cz_1 + d}{a^2 + b^2 + c^2}\right)$$

(b) $$\frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = \frac{z_2 - z_1}{c} = -2\left(\frac{ax_1 + by_1 + cz_1 + d}{a^2 + b^2 + c^2}\right)$$

19. CONDITION FOR COPLANARITY OF TWO LINES :

Let the two lines be

$$\frac{x - \alpha_1}{\ell_1} = \frac{y - \beta_1}{m_1} = \frac{z - \gamma_1}{n_1} \qquad \text{.......... (i)}$$

and $$\frac{x - \alpha_2}{\ell_2} = \frac{y - \beta_2}{m_2} = \frac{z - \gamma_2}{n_2} \qquad \text{.......... (ii)}$$

These lines will coplanar if $\begin{vmatrix} \alpha_2 - \alpha_1 & \beta_2 - \beta_1 & \gamma_2 - \gamma_1 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix} = 0$

the plane containing the two lines is $\begin{vmatrix} x - \alpha_1 & y - \beta_1 & z - \gamma_1 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix} = 0$

20. PERPENDICULAR DISTANCE OF A POINT FROM THE PLANE :

Perpendicular distance p, of the point $A(x_1, y_1, z_1)$ from the plane $ax + by + cz + d = 0$ is given by

$$p = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{(a^2 + b^2 + c^2)}}$$

Distance between two parallel planes $ax + by + cz + d_1 = 0$

& $ax + by + cz + d_2 = 0$ is - $\left|\frac{d_1 - d_2}{\sqrt{a^2 + b^2 + c^2}}\right|$

21. A PLANE THROUGH THE LINE OF INTERSECTION OF TWO GIVEN PLANES :

Consider two planes

$u \equiv ax + by + cz + d = 0$ and $v \equiv a'x + b'y + c'z + d' = 0$.

The equation $u + \lambda v = 0$, λ a real parameter, represents the plane passing through the line of intersection of given planes and if planes are parallel, this represents a plane parallel to them.

22. BISECTORS OF ANGLES BETWEEN TWO PLANES :

Let the equations of the two planes be $ax + by + cz + d = 0$ and $a_1x + b_1y + c_1z + d_1 = 0$.

Then equations of bisectors of angles between them are given by

$$\frac{ax + by + cz + d}{\sqrt{(a^2 + b^2 + c^2)}} = \pm \frac{a_1x + b_1y + c_1z + d_1}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}}$$

(a) Equation of bisector of the angle containing origin : First make both constant terms positive. Then +ve sign give the bisector of the angle which contains the origin.

(b) Bisector of acute/obtuse angle : First making both constant terms positive,

$aa_1 + bb_1 + cc_1 > 0 \quad \Rightarrow$ origin lies in obtuse angle

$aa_1 + bb_1 + cc_1 < 0 \quad \Rightarrow$ origin lies in acute angle

PROBABILITY

1. SOME BASIC TERMS AND CONCEPTS

(a) **An Experiment :** An action or operation resulting in two or more outcomes is called an experiment.

(b) **Sample Space :** The set of all possible outcomes of an experiment is called the sample space, denoted by S. An element of S is called a sample point.

(c) **Event :** Any subset of sample space is an event.

(d) **Simple Event :** An event is called a simple event if it is a singleton subset of the sample space S.

(e) **Compound Events :** It is the joint occurrence of two or more simple events.

(f) **Equally Likely Events :** A number of simple events are said to be equally likely if there is no reason for one event to occur in preference to any other event.

(g) **Exhaustive Events :** All the possible outcomes taken together in which an experiment can result are said to be exhaustive or disjoint.

(h) **Mutually Exclusive or Disjoint Events :** If two events cannot occur simultaneously, then they are mutually exclusive.
If A and B are mutually exclusive, then $A \cap B = \phi$.

(i) **Complement of an Event :** The complement of an event A, denoted by $\bar{A}$, A' or A^C, is the set of all sample points of the space other then the sample points in A.

2. MATHEMATICAL DEFINITION OF PROBABILITY

Let the outcomes of an experiment consists of *n* exhaustive mutually exclusive and equally likely cases. Then the sample spaces S has *n* sample points. If an event A consists of m sample points, $(0 \le m \le n)$, then the probability of event A, denoted by P(A) is defined to be m/n i.e. $P(A) = m/n$.

Let $S = a_1, a_2, \ldots\ldots, a_n$ be the sample space

(a) $P(S) = \frac{n}{n} = 1$ corresponding to the certain event.

(b) $P(\phi) = \frac{0}{n} = 0$ corresponding to the null event ϕ or impossible event.

(c) If $A_i = \{a_i\}, i = 1, \ldots\ldots, n$ then A_i is the event corresponding to a single sample point a_i. Then $P(A_i) = \frac{1}{n}$.

(d) $0 \leq P(A) \leq 1$

3. ODDS AGAINST AND ODDS IN FAVOUR OF AN EVENT :

Let there be m + n equally likely, mutually exclusive and exhaustive cases out of which an event A can occur in m cases and does not occur in n cases. Then by definition of probability of occurrences

$$= \frac{m}{m+n}$$

The probability of non-occurrence $= \frac{n}{m+n}$

$\therefore\ P(A) : P(A') = m : n$

Thus the odd in favour of occurrences of the event A are defined by m : n i.e. P(A) : P(A'); and the odds against the occurrence of the event A are defined by n : m i.e. P(A') : P(A).

4. ADDITION THEOREM

(a) If A and B are any events in S, then

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Since the probability of an event is a nonnegative number, it follows that

$P(A \cup B) \leq P(A) + P(B)$

For three events A, B and C in S we have

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$.

General form of addition theorem

For n events $A_1, A_2, A_3, \ldots\ldots A_n$ in S, we have

$P(A_1 \cup A_2 \cup A_3 \cup A_4 \ldots\ldots\ldots \cup A_n)$

$$= \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) \ldots\ldots$$

$$+ (-1)^{n-1} P(A_1 \cap A_2 \cap A_3 \ldots\ldots \cap A_n)$$

(b) If A and B are mutually exclusive, then $P(A \cap B) = 0$ so that $P(A \cup B) = P(A) + P(B)$.

5. MULTIPLICATION THEOREM

Independent event :

So if A and B are two independent events then happening of B will have no effect on A.

Difference between independent & mutually exclusive event :

(i) Mutually exclusiveness is used when events are taken from same experiment & independence when events one takes from different experiment.

(ii) Independent events are represented by word "and" but mutually exclusive events are represented by word "OR".

(a) When events are independent :

$P(A/B) = P(A)$ and $P(B/A) = P(B)$, then

$P(A \cap B) = P(A).\ P(B)$ OR $P(AB) = P(A)\ .\ P(B)$

(b) When events are not independent

The probability of simultaneous happening of two events A and B is equal to the probability of A multiplied by the conditional probability of B with respect to A (or probability of B multiplied by the conditional probability of A with respect to B) i.e

$P(A \cap B) = P(A)\ .\ P(B/A)$ or $P(B)\ .\ P(A/B)$

OR

$P(AB) = P(A)\ .\ P(B/A)$ or $P(B)\ .\ P(A/B)$

(c) Probability of at least one of the n Independent events

If p_1 , p_2 , p_3 , ... p_n are the probabilities of n independent events A_1 , A_2, A_3 ..... A_n then the probability of happening of at least one of these event is

$1-[(1-p_1)(1-p_2)\,.....\,(1-p_n)]$

$P(A_1+A_2+A_3+\,.....\,+A_n)=1-P(\bar{A}_1)P(\bar{A}_2)P(\bar{A}_3)\,....\,P(\bar{A}_n)$

6. CONDITIONAL PROBABILITY :

If A and B are any events in S then the conditional probability of B relative to A is given by

$$P(B/A)=\frac{P(B\cap A)}{P(A)},\quad \text{If } P(A)\neq 0$$

7. BAYE'S THEOREM OR INVERSE PBOBABILITY :

Let A_1, A_2, ..., A_n be n mutually exclusive and exhaustive events of the sample space S and A is event which can occur with any of the events then $P\left(\frac{A_i}{A}\right)=\frac{P(A_i)P(A/A_i)}{\sum_{i=1}^{n}P(A_i)P(A/A_i)}$

8. BINOMIAL DISTRIBUTION FOR REPEATED TRIALS

Binomial Experiment : Any experiment which has only two outcomes is known as binomial experiment.

Outcomes of such an experiment are known as success and failure. Probability of success is denoted by p and probability of failure by q.

$\therefore\ p+q=1$

If binomial experiment is repeated n times, then

$(p+q)^n={}^nC_0\,q^n+{}^nC_1pq^{n-1}+{}^nC_2p^2q^{n-2}+\,.....\,+{}^nC_r\,p^r\,q^{n-r}+\,.......\,+{}^nC_n\,p^n=1$

(a) Probability of exactly r successes in n trials $={}^nC_rp^rq^{n-r}$

(b) Probability of at most r successes in n trails $=\sum_{\lambda=0}^{r}{}^nC_\lambda p^\lambda q^{n-\lambda}$

(c) Probability of atleast r successes in n trails $=\sum_{\lambda=r}^{n}{}^nC_\lambda p^\lambda q^{n-\lambda}$

(d) Probability of having I^{st} success at the r^{th} trials $=p\,q^{r-1}$.

The mean, the variance and the standard deviation of binomial distribution are np, npq, $\sqrt{npq}$.

9. SOME IMPORTANT RESULTS

(a) Let A and B be two events, then

(i) $P(A) + P(\bar{A}) = 1$

(ii) $P(A + B) = 1 - P(\bar{A}\bar{B})$

(iii) $P(A/B) = \dfrac{P(AB)}{P(B)}$

(iv) $P(A + B) = P(AB) + P(\bar{A}B) + P(A\bar{B})$

(v) $A \subset B \Rightarrow P(A) \le P(B)$

(vi) $P(\bar{A}B) = P(B) - P(AB)$

(vii) $P(AB) \le P(A)\,P(B) \le P(A + B) \le P(A) + P(B)$

(viii) $P(AB) = P(A) + P(B) - P(A + B)$

(ix) P(Exactly one event) $= P(A\bar{B}) + P(\bar{A}B)$

$= P(A) + P(B) - 2P(AB) = P(A + B) - P(AB)$

(x) P(neither A nor B) $= P(\bar{A}\bar{B}) = 1 - P(A + B)$

(xi) $P(\bar{A} + \bar{B}) = 1 - P(AB)$

(b) Number of exhaustive cases of tossing n coins simultaneously (or of tossing a coin n times) $= 2^n$

(c) Number of exhaustive cases of throwing n dice simultaneously (or throwing one dice n times) $= 6^n$

(d) Playing Cards :

(i) Total Cards : 52(26 red, 26 black)

(ii) Four suits : Heart, Diamond, Spade, Club - 13 cards each

(iii) Court Cards : 12 (4 Kings, 4 queens, 4 jacks)

(iv) Honour Cards : 16 (4 aces, 4 kings, 4 queens, 4 jacks)

(e) Probability regarding n letters and their envelopes :

If n letters corresponding to n envelopes are placed in the envelopes at random, then

(i) Probability that all letters are in right envelopes $= \frac{1}{n!}$.

(ii) Probability that all letters are not in right envelopes $= 1 - \frac{1}{n!}$

(iii) Probability that no letters is in right envelopes

$$= \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots\ldots + (-1)^n \frac{1}{n!}$$

(iv) Probability that exactly r letters are in right envelopes

$$= \frac{1}{r!}\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots\ldots + (-1)^{n-r} \frac{1}{(n-r)!}\right]$$

STATISTICS

MEASURES OF CENTRAL TENDENCY :

An average value or a central value of a distribution is the value of variable which is representative of the entire distribution, this representative value are called the measures of central tendency.

Generally the following five measures of central tendency.

(a) Mathematical average

(i) Arithmetic mean (ii) Geometric mean

(iii) Harmonic mean

(b) Positional average

(i) Median (ii) Mode

1. ARITHMETIC MEAN :

(i) For ungrouped dist. : If $x_1, x_2, \ldots\ldots x_n$ are n values of variate x_i then their A.M. $\bar{x}$ is defined as

$$\bar{x} = \frac{x_1 + x_2 + \ldots\ldots + x_n}{n} = \frac{\sum_{i=1}^{n} x_i}{n}$$

$\Rightarrow \Sigma x_i = n\bar{x}$

(ii) For ungrouped and grouped freq. dist. : If $x_1, x_2, \ldots. x_n$ are values of variate with corresponding frequencies $f_1, f_2, \ldots f_n$ then their A.M. is given by

$$\bar{x} = \frac{f_1x_1 + f_2x_2 + \ldots. + f_nx_n}{f_1 + f_2 + \ldots. + f_n} = \frac{\sum_{i=1}^{n} f_i x_i}{N}, \quad \text{where } N = \sum_{i=1}^{n} f_i$$

(iii) By short method :

Let $d_i = x_i - a$

$$\therefore \bar{x} = a + \frac{\Sigma f_i d_i}{N}, \quad \text{where a is assumed mean}$$

(iv) By step deviation method :

Let $u_i = \frac{d_i}{h} = \frac{x_i - a}{h}$

$\therefore \quad \bar{x} = a + \left(\frac{\Sigma f_i u_i}{N}\right) h$

(v) Weighted mean : If $w_1, w_2, \ldots\ldots w_n$ are the weights assigned to the values $x_1, x_2, \ldots\ldots x_n$ respectively then their weighted mean is defined as

$$\text{Weighted mean} = \frac{w_1 x_1 + w_2 x_2 + \ldots\ldots + w_n x_n}{w_1 + \ldots\ldots + w_n} = \frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i}$$

(vi) Combined mean : If $\bar{x}_1$ and $\bar{x}_2$ be the means of two groups having n_1 and n_2 terms respectively then the mean (combined mean) of their composite group is given by combined mean

$$= \frac{n_1 \bar{x}_1 + n_2 \bar{x}_2}{n_1 + n_2}$$

If there are more than two groups then,

$$\text{combined mean} = \frac{n_1 \bar{x}_1 + n_1 \bar{x}_2 + n_3 \bar{x}_3 + \ldots.}{n_1 + n_2 + n_3 + \ldots.}$$

(vii) Properties of Arithmetic mean :

- Sum of deviations of variate from their A.M. is always zero i.e. $\Sigma(x_i - \bar{x}) = 0$, $\Sigma f_i (x_i - \bar{x}) = 0$
- Sum of square of deviations of variate from their A.M. is minimum i.e. $\Sigma(x_i - \bar{x})^2$ is minimum
- If $\bar{x}$ is the mean of variate x_i then A.M. of $(x_i + \lambda) = \bar{x} + \lambda$

  A.M. of $(\lambda x_i) = \lambda \bar{x}$

  A.M. of $(ax_i + b) = a\bar{x} + b$ (where λ, a, b are constant)
- A.M. is independent of change of assumed mean i.e. it is not effected by any change in assumed mean.

2. MEDIAN :

The median of a series is the value of middle term of the series when the values are written in ascending order. Therefore median, divided an arranged series into two equal parts.

Formulae of median :

(i) For ungrouped distribution : Let n be the number of variate in a series then

$$\text{Median} = \begin{cases} \left(\dfrac{n+1}{2}\right)^{th} \text{term}, \ (\text{when } n \text{ is odd}) \\ \text{Mean of} \left(\dfrac{n}{2}\right)^{th} \text{and} \left(\dfrac{n}{2}+1\right)^{th} \text{terms}, \ (\text{when } n \text{ is even}) \end{cases}$$

(ii) For ungrouped freq. dist. : First we prepare the cumulative frequency (c.f.) column and Find value of N then

$$\text{Median} = \begin{cases} \left(\dfrac{N+1}{2}\right)^{th} \text{term}, \ (\text{when } N \text{ is odd}) \\ \text{Mean of} \left(\dfrac{N}{2}\right)^{th} \text{and} \left(\dfrac{N}{2}+1\right)^{th} \text{terms}, \ (\text{when } N \text{ is even}) \end{cases}$$

(iii) For grouped freq. dist : Prepare c.f. column and find value of $\dfrac{N}{2}$ then find the class which contain value of c.f. is equal or just greater to N/2, this is median class

$$\therefore \ \text{Median} = \ell + \frac{\left(\dfrac{N}{2} - F\right)}{f} \times h$$

where ℓ — lower limit of median class

f — freq. of median class

F — c.f. of the class preceeding median class

h — Class interval of median class

3. MODE :

In a frequency distribution the mode is the value of that variate which have the maximum frequency

Method for determining mode :

(i) **For ungrouped dist. :** The value of that variate which is repeated maximum number of times

(ii) **For ungrouped freq. dist. :** The value of that variate which have maximum frequency.

(iii) **For grouped freq. dist. :** First we find the class which have maximum frequency, this is model calss

$$\therefore \text{ Mode} = \ell + \frac{f_0 - f_1}{2f_0 - f_1 - f_2} \times h$$

where ℓ — lower limit of model class

f_0 — freq. of the model class

f_1 — freq. of the class preceeding model class

f_2 — freq. of the class succeeding model class

h — class interval of model class

4. RELATION BETWEEN MEAN, MEDIAN AND MODE :

In a moderately asymmetric distribution following relation between mean, median and mode of a distribution. It is known as imprical formula.

Mode = 3 Median – 2 Mean

Note : (i) Median always lies between mean and mode

(ii) For a symmetric distribution the mean, median and mode are coincide.

5. MEASURES OF DISPERSION :

The dispersion of a statistical distribution is the measure of deviation of its values about the their average (central) value.

Generally the following measures of dispersion are commonly used.

(i) Range

(ii) Mean deviation

(iii) Variance and standard deviation

(i) **Range :** The difference between the greatest and least values of variate of a distribution, are called the range of that distribution. If the distribution is grouped distribution, then its range is the difference between upper limit of the maximum class and lower limit of the minimum class.

Also, coefficient of range = $\dfrac{\text{difference of extreme values}}{\text{sum of extreme values}}$

(ii) **Mean deviation (M.D.) :** The mean deviation of a distribution is, the mean of absolute value of deviations of variate from their statistical average (Mean, Median, Mode).

If A is any statistical average of a distribution then mean deviation about A is defined as

Mean deviation = $\dfrac{\sum\limits_{i=1}^{n}|x_i - A|}{n}$ (for ungrouped dist.)

Mean deviation = $\dfrac{\sum\limits_{i=1}^{n}f_i\,|x_i - A|}{N}$ (for freq. dist.)

Note :- is minimum when it taken about the median

Coefficient of Mean deviation = $\dfrac{\text{Mean deviation}}{A}$

(where A is the central tendency about which Mean deviation is taken)

(iii) **Variance and standard deviation :** The variance of a distribution is, the mean of squares of deviation of variate from their mean. It is denoted by σ^2 or var(x).

The positive square root of the variance are called the standard deviation. It is denoted by σ or S.D.

Hence standard deviation = $+\sqrt{\text{variance}}$

Formulae for variance :

(i) for ungrouped dist. :

$$\sigma_x^2 = \frac{\Sigma(x_i - \overline{x})^2}{n}$$

$$\sigma_x^2 = \frac{\Sigma x_i^2}{n} - \overline{x}^2 = \frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2$$

$$\sigma_d^2 = \frac{\Sigma d_i^2}{n} - \left(\frac{\Sigma d_i}{n}\right)^2, \quad \text{where } d_i = x_i - a$$

(ii) For freq. dist. :

$$\sigma_x^2 = \frac{\Sigma f_i(x_i - \overline{x})^2}{N}$$

$$\sigma_x^2 = \frac{\Sigma f_i x_i^2}{N} - (\overline{x})^2 = \frac{\Sigma f_i x_i^2}{N} - \left(\frac{\Sigma f_i x_i}{N}\right)^2$$

$$\sigma_d^2 = \frac{\Sigma f_i d_i^2}{N} - \left(\frac{\Sigma f_i d_i}{N}\right)^2$$

$$\sigma_u^2 = h^2\left[\frac{\Sigma f_i u_i^2}{N} - \left(\frac{\Sigma f_i u_i}{N}\right)^2\right] \quad \text{where } u_i = \frac{d_i}{h}$$

(iii) Coefficient of S.D. $= \frac{\sigma}{\overline{x}}$

Coefficient of variation $= \frac{\sigma}{\overline{x}} \times 100$ (in percentage)

Note :- $\sigma^2 = \sigma_x^2 = \sigma_d^2 = h^2 \sigma_u^2$

6. MEAN SQUARE DEVIATION :

The mean square deviation of a distrubution is the mean of the square of deviations of variate from assumed mean. It is denoted by S^2

Hence $S^2 = \dfrac{\Sigma(x_i - a)^2}{n} = \dfrac{\Sigma d_i^2}{n}$ (for ungrouped dist.)

$S^2 = \dfrac{\Sigma f_i(x_i - a)^2}{N} = \dfrac{\Sigma f_i d_i^2}{N}$ (for freq. dist.), where $d_i = (x_i - a)$

7. RELATION BETWEEN VARIANCE AND MEAN SQUARE DEVIATION :

$\because \quad \sigma^2 = \dfrac{\Sigma f_i d_i^2}{N} - \left(\dfrac{\Sigma f_i d_i}{N}\right)^2$

$\Rightarrow \quad \sigma^2 = s^2 - d^2$, where $d = \bar{x} - a = \dfrac{\Sigma f_i d_i}{N}$

$\Rightarrow \quad s^2 = \sigma^2 + d^2 \Rightarrow s^2 \geq \sigma^2$

Hence the variance is the minimum value of mean square deviation of a distribution

8. MATHEMATICAL PROPERTIES OF VARIANCE :

- $Var.(x_i + \lambda) = Var.(x_i)$
  $Var.(\lambda x_i) = \lambda^2 . Var(x_i)$
  $Var(ax_i + b) = a^2 . Var(x_i)$
  where λ, a, b, are constant

- If means of two series containing n_1, n_2 terms are $\bar{x}_1, \bar{x}_2$ and their variance's are σ_1^2, σ_2^2 respectively and their combined mean is $\bar{x}$ then the variance σ^2 of their combined series is given by following formula

$$\sigma^2 = \frac{n_1(\sigma_1^2 + d_1^2) + n_2(\sigma_2^2 + d_2^2)}{(n_1 + n_2)} \text{ where } d_1 = \bar{x}_1 - \bar{x},\ d_2 = \bar{x}_2 - \bar{x}$$

$$\text{i.e. } \sigma^2 = \frac{n_1\sigma_1^2 + n_2\sigma_2^2}{n_1 + n_2} + \frac{n_1 n_2}{(n_1 + n_2)^2}(\bar{x}_1 - \bar{x}_2)^2$$

MATHEMATICAL REASONING

1. STATEMENT :

A sentence which is either true or false but cannot be both are called a statement. A sentence which is an exclamatory or a wish or an imperative or an interrogative can not be a statement.

If a statement is true then its truth value is T and if it is false then its truth value is F

2. SIMPLE STATEMENT :

Any statement whose truth value does not depend on other statement are called simple statement.

3. COMPOUND STATEMENT :

A statement which is a combination of two or more simple statements are called compound statement

Here the simple statements which form a compound statement are known as its sub statements

4. LOGICAL CONNECTIVES :

The words or phrases which combined simple statements to form a compound statement are called logical connectives.

S.N.	Connectives	Symbol	Use	Operation
1.	and	$\wedge$	$p \wedge q$	conjunction
2.	or	$\vee$	$p \vee q$	disjunction
3.	not	$\sim$ or '	$\sim p$ or p'	negation
4.	If .... then	$\Rightarrow$ or $\rightarrow$	$p \Rightarrow q$ or $p \rightarrow q$	Implication or conditional
5.	If and only if (iff)	$\Leftrightarrow$ or $\leftrightarrow$	$p \Leftrightarrow q$ or $p \leftrightarrow q$	Equivalence or Bi-conditional

5. TRUTH TABLE :

Conjunction

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Negation

p	$(\sim p)$
T	F
F	T

Conditional

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Biconditional

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$ or $p \leftrightarrow q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Note : If the compound statement contain n sub statements then its truth table will contain 2^n rows.

6. LOGICAL EQUIVALENCE :

Two compound statements $S_1(p, q, r...)$ and $S_2(p, q, r ....)$ are said to be logically equivalent or simply equivalent if they have same truth values for all logically possibilities

Two statements S_1 and S_2 are equivalent if they have identical truth table i.e. the entries in the last column of their truth table are same. If statements S_1 and S_2 are equivalent then we write $S_1 \equiv S_2$

i.e. $\boxed{p \rightarrow q \equiv \sim p \vee q}$

7. TAUTOLOGY AND CONTRADICTION :

(i) Tautology : A statement is said to be a tautology if it is true for all logical possibilities

i.e. its truth value always T. it is denoted by t.

(ii) Contradiction : A statement is a contradiction if it is false for all logical possibilities.

i.e. its truth value always F. It is denoted by c.

Note : The negation of a tautology is a contradiction and negation of a contradiction is a tautology

8. DUALITY :

Two compound statements S_1 and S_2 are said to be duals of each other if one can be obtained from the other by replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$

If a compound statement contains the special variable t (tautology) and c (contradiction) then obtain its dual we replaced t by c and c by t in addition to replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$.

Note :

(i) the connectives $\wedge$ and $\vee$ are also called dual of each other.

(ii) If $S^*(p,q)$ is the dual of the compound statement $S(p,q)$ then

(a) $S^*(\sim p, \sim q) \equiv \sim S(p, q)$ (b) $\sim S^*(p, q) \equiv S(\sim p, \sim q)$

9. CONVERSE, INVERSE AND CONTRAPOSITIVE OF THE CONDITIONAL STATEMENT $(p \to q)$:

(i) Converse : The converse of the conditional statement $p \to q$ is defined as $q \to p$

(ii) Inverse : The inverse of the conditional statement $p \to q$ is defined as $\sim p \to \sim q$

(iii) Contrapositive : The contrapositive of conditional statement $p \to q$ is defined as $\sim q \to \sim p$

10. NEGATION OF COMPOUND STATEMENTS :

If p and q are two statements then

(i) Negation of conjunction : $\sim(p \wedge q) \equiv \sim p \vee \sim q$

(ii) Negation of disjunction : $\sim(p \vee q) \equiv \sim p \wedge \sim q$

(iii) Negation of conditional : $\sim(p \to q) \equiv p \wedge \sim q$

(iv) Negation of biconditional : $\sim (p \leftrightarrow q) \equiv (p \wedge \sim q) \vee (q \wedge \sim p)$

we know that $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

$$\therefore\ \sim(p \leftrightarrow q) \equiv \sim[(p \to q) \wedge (q \to p)]$$
$$\equiv \sim(p \to q) \vee \sim(q \to p)$$
$$\equiv (p \wedge \sim q) \vee (q \wedge \sim p)$$

Note : The above result also can be proved by preparing truth table for $\sim(p \leftrightarrow q)$ and $(p \wedge \sim q) \vee (q \wedge \sim p)$

11. ALGEBRA OF STATEMENTS :

If p, q, r are any three statements then the some low of algebra of statements are as follow

(i) Idempotent Laws :

(a) $p \wedge p \equiv p$ (b) $p \vee p \equiv p$

(ii) Comutative laws :

(a) $p \wedge q \equiv q \wedge p$ (b) $p \vee q \equiv q \vee p$

(iii) Associative laws :

(a) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

(b) $(p \vee q) \vee r \equiv p \vee (q \vee r)$

(iv) Distributive laws :

(a) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

(b) $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge (p \wedge r)$

(c) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

(d) $p \vee (q \vee r) \equiv (p \vee q) \vee (p \vee r)$

(v) De Morgan Laws :

(a) $\sim(p \wedge q) \equiv \sim p \vee \sim q$

(b) $\sim(p \vee q) \equiv \sim p \wedge \sim q$

(vi) Involution laws (or Double negation laws) : $\sim(\sim p) \equiv p$

(vii) Identity Laws : If p is a statement and t and c are tautology and contradiction respectively then

(a) $p \wedge t \equiv p$ (b) $p \vee t \equiv t$ (c) $p \wedge c \equiv c$ (d) $p \vee c \equiv p$

(viii) Complement Laws :

(a) $p \wedge (\sim p) \equiv c$ (b) $p \vee (\sim p) \equiv t$

(c) $(\sim t) \equiv c$ (d) $(\sim c) \equiv t$

(ix) Contrapositive laws : $p \to q \equiv \sim q \to \sim p$

12. QUANTIFIED STATEMENTS AND QUANTIFIERS :

The words or phrases "All", "Some", "None", "There exists a" are examples of quantifiers.

A statement containing one or more of these words (or phrases) is a quantified statement.

Note : Phrases "There exists a" and "Atleast one" and the word "some" have the same meaning.

NEGATION OF QUANTIFIED STATEMENTS :

(1) '**None**' is the negation of '**at least one**' or '**some**' or '**few**'
Similarly negation of '**some**' is '**none**'

(2) The negation of "**some A are B**" or "**There exist A which is B**" is "**No A are (is) B**" or "**There does not exist any A which is B**".

(3) Negation of "**All A are B**" is "**Some A are not B**".

SETS

SET :

A set is a collection of well defined objects which are distinct from each other

Set are generally denoted by capital letters A, B, C, ... etc. and the elements of the set by a, b, c .... etc.

If a is an element of a set A, then we write $a \in A$ and say a belongs to A.

If a does not belong to A then we write $a \notin A$,

SOME IMPORTANT NUMBER SETS :

N = Set of all natural numbers

$= \{1, 2, 3, 4, ...\}$

W = Set of all whole numbers

$= \{0, 1, 2, 3, ...\}$

Z or I set of all integers

$= \{.... -3, -2, -1, 0, 1, 2, 3, ...\}$

Z^+ = Set of all +ve integers

$= \{1, 2, 3, ...\} = N.$

Z^- = Set of all –ve integers

$= \{-1, -2, -3, ...\}$

Z_0 = The set of all non-zero integers.

$= \{\pm 1, \pm 2, \pm 3, ...\}$

Q = The set of all rational numbers.

$$= \left\{\frac{p}{q} : p, q \in I, q \neq 0\right\}$$

R = the set of all real numbers.

R–Q = The set of all irrational numbers

METHODS TO WRITE A SET :

(i) Roster Method : In this method a set is described by listing elements, separated by commas and enclose then by curly brackets

(ii) Set Builder From : In this case we write down a property or rule p Which gives us all the element of the set

$$A = \{x : P(x)\}$$

TYPES OF SETS :

Null set or Empty set : A set having no element in it is called an Empty set or a null set or void set it is denoted by ϕ or { }

A set consisting of at least one element is called a non-empty set or a non-void set.

Singleton : A set consisting of a single element is called a singleton set.

Finite Set : A set which has only finite number of elements is called a finite set.

Order of a finite set : The number of elements in a finite set is called the order of the set A and is denoted O(A) or n(A). It is also called cardinal number of the set.

Infinite set : A set which has an infinite number of elements is called an infinite set.

Equal sets : Two sets A and B are said to be equal if every element of A is a member of B, and every element of B is a member of A. If sets A and B are equal. We write A = B and A and B are not equal then $A \neq B$

Equivalent sets : Two finite sets A and B are equivalent if their number of elements are same

i.e. $n(A) = n(B)$

Note : Equal set always equivalent but equivalent sets may not be equal

Subsets : Let A and B be two sets if every element of A is an element B, then A is called a subset of B if A is a subset of B. we write $A \subseteq B$

Proper subset : If A is a subset of B and $A \neq B$ then A is a proper subset of B. and we write $A \subset B$

Note-1 : Every set is a subset of itself i.e. $A \subseteq A$ for all A

Note-2 : Empty set ϕ is a subset of every set

Note-3 : Clearly $N \subset W \subset Z \subset Q \subset R \subset C$

Note-4 : The total number of subsets of a finite set containing n elements is 2^n

Universal set : A set consisting of all possible elements which occur in the discussion is called a Universal set and is denoted by U

Note : All sets are contained in the universal set

Power set : Let A be any set. The set of all subsets of A is called power set of A and is denoted by P(A)

Some Operation on Sets :

(i) **Union of two sets :** $A \cup B = \{x : x \in A \text{ or } x \in B\}$

(ii) **Intersection of two sets :** $A \cap B = \{x : x \in A \text{ and } x \in B\}$

(iii) **Difference of two sets :** $A - B = \{x : x \in A \text{ and } x \notin B\}$

(iv) **Complement of a set :** $A' = \{x : x \notin A \text{ but } x \in U\} = U - A$

(v) **De-Morgan Laws :** $(A \cup B)' = A' \cap B'$; $(A \cap B)' = A' \cup B'$

(vi) $A - (B \cup C) = (A - B) \cap (A - C)$; $A - (B \cap C) = (A - B) \cup (A - C)$

(vii) **Distributive Laws :** $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$; $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(viii) **Commutative Laws :** $A \cup B = B \cup A$; $A \cap B = B \cap A$

(ix) **Associative Laws :** $(A \cup B) \cup C = A \cup (B \cup C)$; $(A \cap B) \cap C = A \cap (B \cap C)$

(x) $A \cap \phi = \phi$; $A \cap U = A$

$A \cup \phi = A$; $A \cup U = U$

(xi) $A \cap B \subseteq A$; $A \cap B \subseteq B$

(xii) $A \subseteq A \cup B$; $B \subseteq A \cup B$

(xiii) $A \subseteq B \Rightarrow A \cap B = A$

(xiv) $A \subseteq B \Rightarrow A \cup B = B$

Disjoint Sets :

IF $A \cap B = \phi$, then A, B are disjoint.

Note : $A \cap A' = \phi \quad \therefore$ A, A' are disjoint.

Symmetric Difference of Sets :

$A \Delta B = (A - B) \cup (B - A)$

- $(A')' = A$
- $A \subseteq B \Leftrightarrow B' \subseteq A'$

If A and B are any two sets, then

(i) $A - B = A \cap B'$

(ii) $B - A = B \cap A'$

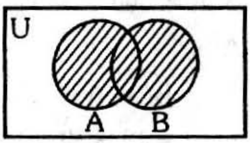
(iii) $A - B = A \Leftrightarrow A \cap B = \phi$

(iv) $(A - B) \cup B = A \cup B$

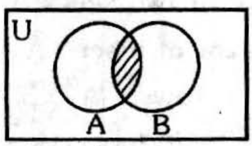
(v) $(A - B) \cap B = \phi$

(vi) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

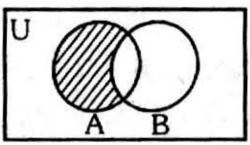
Venn Diagrame :



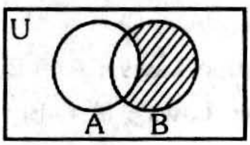
$A \cup B$



$A \cap B$

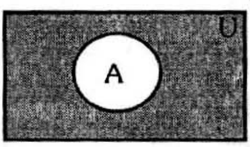


$A - B$

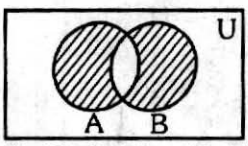


$B - A$

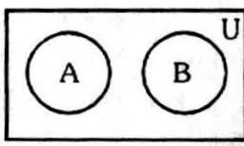
Clearly $(A - B) \cup (B - A) \cup (A \cup B) = A \cup B$



A'



$(A\Delta B) = (A-B) \cup (B-A)$



Disjoint Sets

Note : $A \cap A' = \phi$, $A \cup A' = U$

SOME IMPORTANT RESULTS ON NUMBER OF ELEMENTS IN SETS :

If A, B and C are finite sets, and U be the finite universal set, then

(i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(ii) $n(A \cup B) = n(A) + n(B) \Leftrightarrow$ A, B are disjoint non-void sets.

(iii) $n(A - B) = n(A) - n(A \cap B)$ i.e. $n(A - B) + n(A \cap B) = n(A)$

(iv) $n(A \Delta B)$ = No. of elements which belong to exactly one of A or B

$= n((A - B) \cup (B - A))$

$= n(A - B) + n(B - A)$ [$\because$ (A – B) and (B – A) are disjoint]

$= n(A) - n(A \cap B) + n(B) - n(A \cap B)$

$= n(A) + n(B) - 2n(A \cap B)$

$= n(A) + n(B) - 2n(A \cap B)$

(v) $n(A \cup B \cup C)$

$= n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

(vi) Number of elements in exactly two of the sets A, B, C

$= n(A \cap B) + n(B \cap C) + n(C \cap A) - 3n(A \cap B \cap C)$

(vii) number of elements in exactly one of the sets A, B, C

$= n(A) + n(B) + n(C) - 2n(A \cap B) - 2n(B \cap C) - 2n(A \cap C) + 3n(A \cap B \cap C)$

(viii) $n(A' \cup B') = n((A \cap B)') = n(U) - n(A \cap B)$

(ix) $n(A' \cap B') = n((A \cup B)') = n(U) - n(A \cup B)$

(x) If A_1, A_2 .... A_n are finite sets, then

$$n\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} n(A_i) - \sum_{1 \le i < j \le n} n(A_i \cap A_j)$$

$$+ \sum_{1 \le i < j < k \le n} n(A_i \cap A_j \cap A_k) - .... + (-1)^{n-1} n(A_1 \cap A_2 \cap .....A_n)$$

RELATIONS

INTRODUCTION :

Let A and B be two sets. Then a relation R from A to B is a subset of $A \times B$. thus, R is a relation from A to B $\Leftrightarrow R \subseteq A \times B$.

Total Number of Realtions : Let A and B be two non-empty finite sets consisting of m and n elements respectively. Then $A \times B$ consists of mn ordered pairs. So total number of subsets of $A \times B$ is 2^{mn}.

Domain and Range of a relation : Let R be a relation from a set A to a set B. Then the set of all first components or coordinates of the ordered pairs belonging to R is called to domain of R, while the set of all second components or coordinates of the ordered pairs in R is called the range of R.

Thus, Domain (R) = $\{a : (a, b) \in R\}$

and, Range (R) = $\{b : (a, b) \in R\}$

It is evident from the definition that the domain of a relation from A to B is a subset of A and its range is a subset of B.

Inverse Relation : Let A, B be two sets and let R be a relation from a set A to a set B. Then the inverse of R, denoted by R^{-1}, is a relation from B to A and is defined by

$$R^{-1} = \{(b, a) : (a, b) \in R\}$$

Clearly, $(a, b) \in R \Leftrightarrow (b, a) \in R^{-1}$

Also, $\text{Dom}(R) = \text{Range}(R^{-1})$ and $\text{Range}(R) = \text{Dom}(R^{-1})$

TYPES OF RELATIONS :

In this section we intend to define various types of relations on a given set A.

Void Relation : Let A be a set. Then $\phi \subseteq A \times A$ and so it is a relation on A. This relation is called the void or empty relation on A.

Universal Relation : Let A be a set. Then $A \times A \subseteq A \times A$ and so it is a relation on A. This relation is called the universal relation on A.

Identity Relation : Let A be a set. Then the relation $I_A = \{(a, a) : a \in A\}$ on A is called the identity relation on A.

In other words, a relation I_A on A is called the identity relation if every element of A is related to itself only.

Reflexive Relation : A relation R on a set A is said to be reflexive if every element of A is related to itself.

Thus, R on a set A is not reflexive if there exists an element $A \in A$ such that $(a, a) \notin R$.

Every Identity relation is reflexive but every reflexive ralation is not identity.

Symmetric Relation : A relation R on a set A is said to be a symmetric relation iff

$(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$

i.e. $aRb \Rightarrow bRa$ for all $a, b, \in A$.

Transitive Relation : Let A be any set. A relation R on A is said to be a transitive relation iff

$(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$

i.e. aRb and $bRc \Rightarrow aRc$ for all $a, b, c \in A$

Antisymmetric Relation : Let A be any set. A relation R on set A is said to be an antisymmetric relation iff

$(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b$ for all $a, b \in A$

Equivalence Relation : A relation R on a set A is said to be an equivalence relation on A iff

(i) it is reflexive i.e. $(a, a) \in R$ for all $a \in A$

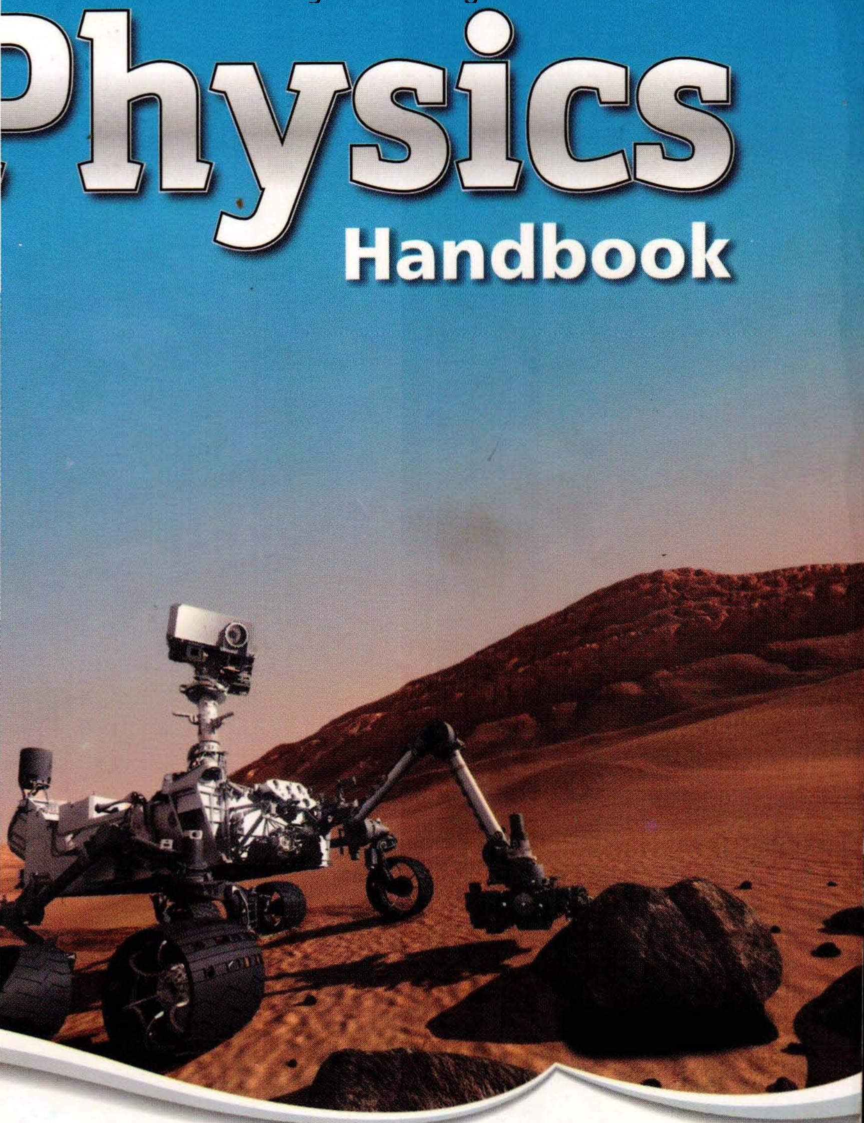
(ii) it is symmetric i.e. $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$

(iii) it is transitive i.e. $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$.

It is not neccessary that every relation which is symmetric and transitive is also reflexive.

Important Notes

Physics
Handbook



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Contents

1 Basic Mathematics used in Physics

- **Quadratic equation**

Roots of $ax^2 + bx + c = 0$ are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Sum of roots $x_1 + x_2 = -\frac{b}{a}$; Product of roots $x_1 x_2 = \frac{c}{a}$

- **Binomial theorem**

$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \frac{n(n-1)(n-2)}{6}x^3 + \ldots$

$(1-x)^n = 1 - nx + \frac{n(n-1)}{2}x^2 - \frac{n(n-1)(n-2)}{6}x^3 + \ldots$

If $x \ll 1$ then $(1+x)^n \approx 1 + nx$ & $(1-x)^n \approx 1 - nx$

- **Logarithm**

$\log mn = \log m + \log n$ $\qquad$ $\log \frac{m}{n} = \log m - \log n$

$\log m^n = n \log m$ $\qquad$ $\log_e m = 2.303 \log_{10} m$

$\log 2 = 0.3010$ $\qquad$ $\log 3 = 0.4771$

- **Componendo and dividendo theorem**

If $\frac{p}{q} = \frac{a}{b}$ then $\frac{p+q}{p-q} = \frac{a+b}{a-b}$

- **Arithmetic progression-AP**

a, a+d, a+2d, a+3d, ...a+(n – 1)d here d = common difference

Sum of n terms $S_n = \frac{n}{2}[2a+(n-1)d]$

Note : (i) $1+2+3+4+5 \ldots + n = \frac{n(n+1)}{2}$

(ii) $1^2+2^2+3^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$

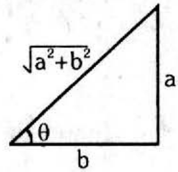
- **Geometrical progression-GP**

a, ar, ar^2, ar^3, ... here, r = common ratio

Sum of n terms $S_n = \frac{a(1-r^n)}{1-r}$ $\qquad$ Sum of ∞ terms $S_\infty = \frac{a}{1-r}$

- **Trigonometry**

2π radian $= 360° \Rightarrow 1$ rad $= 57.3°$



$$\sin\theta = \frac{\text{perpendicular}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{base}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{perpendicular}}{\text{base}} \qquad \cot\theta = \frac{\text{base}}{\text{perpendicular}}$$

$$\sec\theta = \frac{\text{hypotenuse}}{\text{base}} \qquad \text{cosec}\,\theta = \frac{\text{hypotenuse}}{\text{perpendicular}}$$

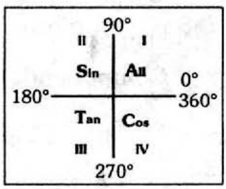
$$\sin\theta = \frac{a}{\sqrt{a^2+b^2}} \qquad \cos\theta = \frac{b}{\sqrt{a^2+b^2}} \qquad \tan\theta = \frac{a}{b}$$

$$\text{cosec}\,\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \tan^2\theta = \sec^2\theta \qquad 1 + \cot^2\theta = \text{cosec}^2\theta$$

$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B \qquad \cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$$



$$\sin 2A = 2\sin A\cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1$$

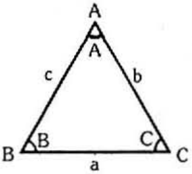
$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

$\sin(90°+\theta) = \cos\theta$	$\sin(180°-\theta) = \sin\theta$	$\sin(-\theta) = -\sin\theta$	$\sin(90°-\theta) = \cos\theta$
$\cos(90°+\theta) = -\sin\theta$	$\cos(180°-\theta) = -\cos\theta$	$\cos(-\theta) = \cos\theta$	$\cos(90°-\theta) = \sin\theta$
$\tan(90°+\theta) = -\cot\theta$	$\tan(180°-\theta) = -\tan\theta$	$\tan(-\theta) = -\tan\theta$	$\tan(90°-\theta) = \cot\theta$
$\sin(180°+\theta) = -\sin\theta$	$\sin(270°-\theta) = -\cos\theta$	$\sin(270°+\theta) = -\cos\theta$	$\sin(360°-\theta) = -\sin\theta$
$\cos(180°+\theta) = -\cos\theta$	$\cos(270°-\theta) = -\sin\theta$	$\cos(270°+\theta) = \sin\theta$	$\cos(360°-\theta) = \cos\theta$
$\tan(180°+\theta) = \tan\theta$	$\tan(270°-\theta) = \cot\theta$	$\tan(270°+\theta) = -\cot\theta$	$\tan(360°-\theta) = -\tan\theta$

$\theta \rightarrow$	0° (0)	30° $\left(\frac{\pi}{6}\right)$	45° $\left(\frac{\pi}{4}\right)$	60° $\left(\frac{\pi}{3}\right)$	90° $\left(\frac{\pi}{2}\right)$	120° $\left(\frac{2\pi}{3}\right)$	135° $\left(\frac{3\pi}{4}\right)$	150° $\left(\frac{5\pi}{6}\right)$	180° (π)	270° $\left(\frac{3\pi}{2}\right)$	360° (2π)
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1	0	1
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0	∞	0

- **sine law**

$$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$$



- **cosine law**

$$\cos A=\frac{b^2+c^2-a^2}{2bc},\ \cos B=\frac{c^2+a^2-b^2}{2ca},\ \cos C=\frac{a^2+b^2-c^2}{2ab}$$

- **For small θ**

$\sin\theta \approx \theta$ $\cos\theta \approx 1$ $\tan\theta \approx \theta$ $\sin\theta \approx \tan\theta$

- **Differentiation**

- $y=x^n \rightarrow \frac{dy}{dx}=nx^{n-1}$
- $y=\ell nx \rightarrow \frac{dy}{dx}=\frac{1}{x}$
- $y=\sin x \rightarrow \frac{dy}{dx}=\cos x$
- $y=\cos x \rightarrow \frac{dy}{dx}=-\sin x$
- $y=e^{\alpha x+\beta} \rightarrow \frac{dy}{dx}=\alpha e^{\alpha x+\beta}$
- $y=uv \rightarrow \frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$
- $y=f(g(x)) \Rightarrow \frac{dy}{dx}=\frac{df(g(x))}{dg(x)}\times\frac{d(g(x))}{dx}$
- $y=k=$ constant $\Rightarrow \frac{dy}{dx}=0$
- $y=\frac{u}{v} \Rightarrow \frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

- **Integration**

- $\int x^n dx=\frac{x^{n+1}}{n+1}+C, n\neq -1$
- $\int \frac{1}{x}dx=\ell nx+C$
- $\int \sin x dx=-\cos x+C$
- $\int \cos x dx=\sin x+C$
- $\int e^{\alpha x+\beta}dx=\frac{1}{\alpha}e^{\alpha x+\beta}+C$
- $\int(\alpha x+\beta)^n dx=\frac{(\alpha x+\beta)^{n+1}}{\alpha(n+1)}+C$

- **Maxima & Minima of a function y = f(x)**

  - For maximum value $\frac{dy}{dx} = 0$ & $\frac{d^2y}{dx^2} = -ve$

  - For minimum value $\frac{dy}{dx} = 0$ & $\frac{d^2y}{dx^2} = +ve$

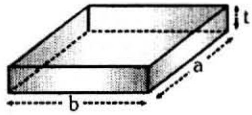
- **Average of a varying quantity**

If y = f(x) then $< y > = \bar{y} = \frac{\int_{x_1}^{x_2} ydx}{\int_{x_1}^{x_2} dx} = \frac{\int_{x_1}^{x_2} ydx}{x_2 - x_1}$

Formulae for determination of area

- Area of a square = $(\text{side})^2$
- Area of rectangle = length $\times$ breadth
- Area of a triangle = $\frac{1}{2} \times$ base $\times$ height
- Area of a trapezoid = $\frac{1}{2} \times$ (distance between parallel sides) $\times$ (sum of parallel sides)
- Area enclosed by a circle = πr^2 (r = radius)
- Surface area of a sphere = $4\pi r^2$ (r = radius)
- Area of a parallelogram = base $\times$ height
- Area of curved surface of cylinder = $2\pi r\ell$ (r = radius and ℓ = length)
- Area of whole surface of cylinder = $2\pi r (r + \ell)$ (ℓ = length)
- Area of ellipse = π ab (a & b are semi major and semi minor axis respectively)
- Surface area of a cube = $6(\text{side})^2$
- Total surface area of a cone = $\pi r^2 + \pi r\ell$ where $\pi r\ell = \pi r\sqrt{r^2 + h^2}$ = lateral area

Formulae for determination of volume :



- Volume of a rectangular slab = length × breadth × height = abt
- Volume of a cube = $(\text{side})^3$
- Volume of a sphere = $\frac{4}{3}\pi r^3$ (r = radius)
- Volume of a cylinder = $\pi r^2 \ell$ (r = radius and ℓ = length)
- Volume of a cone = $\frac{1}{3}\pi r^2 h$ (r = radius and h = height)

KEY POINTS:

- To convert an angle from degree to radian, we have to multiply it by $\frac{\pi}{180°}$ and to convert an angle from radian to degree, we have to multiply it by $\frac{180°}{\pi}$.
- By help of differentiation, if y is given, we can find $\frac{dy}{dx}$ and by help of integration, if $\frac{dy}{dx}$ is given, we can find y.
- The maximum and minimum values of function $A\cos\theta + B\sin\theta$ are $\sqrt{A^2+B^2}$ and $-\sqrt{A^2+B^2}$ respectively.
- $(a+b)^2 = a^2 + b^2 + 2ab$ $(a-b)^2 = a^2 + b^2 - 2ab$

  $(a+b)(a-b) = a^2 - b^2$ $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$

  $(a-b)^3 = a^3 - b^3 - 3ab(a-b)$

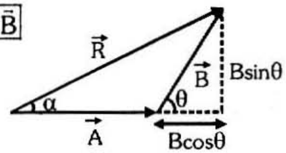
Important Notes

2 Vectors

- **Vector Quantities**

A physical quantity which requires magnitude and a particular direction, when it is expressed.

- **Triangle law of Vector addition** $\boxed{\vec{R}=\vec{A}+\vec{B}}$



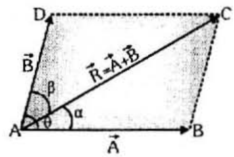
$$R=\sqrt{A^2+B^2+2AB\cos\theta}$$

$\tan\alpha=\dfrac{B\sin\theta}{A+B\cos\theta}$ If A = B then $R=2A\cos\dfrac{\theta}{2}$ & $\alpha=\dfrac{\theta}{2}$

R_{max} = A+B for θ=0° ; R_{min} = A–B for θ=180°

- **Parallelogram Law of Addition of Two Vectors**

If two vectors are represented by two adjacent sides of a parallelogram which are directed away from their common point then their sum (i.e. resultant vector) is given by the diagonal of the parallelogram passing away through that common point.



$$\overline{AB}+\overline{AD}=\overline{AC}=\vec{R} \text{ or } \vec{A}+\vec{B}=\vec{R}\Rightarrow R=\sqrt{A^2+B^2+2AB\cos\theta}$$

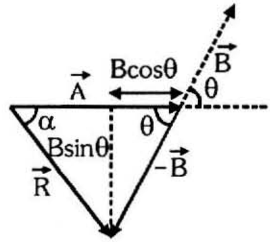
$$\tan\alpha=\frac{B\sin\theta}{A+B\cos\theta} \quad \text{and} \quad \tan\beta=\frac{A\sin\theta}{B+A\cos\theta}$$

- **Vector subtraction**

$\boxed{\vec{R}=\vec{A}-\vec{B}}\Rightarrow\vec{R}=\vec{A}+(-\vec{B})$

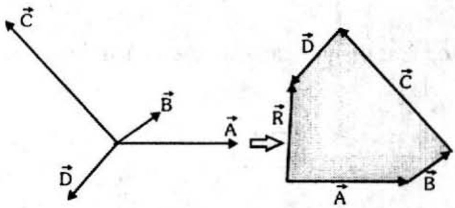
$R=\sqrt{A^2+B^2-2AB\cos\theta}$, $\tan\alpha=\dfrac{B\sin\theta}{A-B\cos\theta}$

If A = B then $R=2A\sin\dfrac{\theta}{2}$



- **Addition of More than Two Vectors (Law of Polygon)**

If some vectors are represented by sides of a polygon in same order, then their resultant vector is represented by the closing side of polygon in the opposite order.



$$\vec{R} = \vec{A} + \vec{B} + \vec{C} + \vec{D}$$

- **Rectangular component of a 3–D vector**

□ $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

Angle made with x-axis $\quad \cos\alpha = \dfrac{A_x}{A} = \dfrac{A_x}{\sqrt{A_x^2 + A_y^2 + A_z^2}} = \ell$

Angle made with y-axis $\quad \cos\beta = \dfrac{A_y}{A} = \dfrac{A_y}{\sqrt{A_x^2 + A_y^2 + A_z^2}} = m$

Angle made with z-axis $\quad \cos\gamma = \dfrac{A_z}{A} = \dfrac{A_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}} = n$

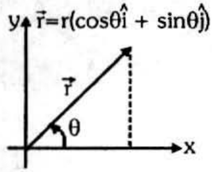
□ ℓ, m, n are called direction cosines

$$\ell^2 + m^2 + n^2 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{A_x^2 + A_y^2 + A_z^2}{\left(\sqrt{A_x^2 + A_y^2 + A_z^2}\right)^2} = 1$$

or $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$

- **General Vector in x-y plane**

$$\vec{r} = x\hat{i} + y\hat{j} = r\left(\cos\theta\hat{i} + \sin\theta\hat{j}\right)$$



Examples

1. Construct a vector of magnitude 6 units making an angle of 60° with x-axis.

Sol. $\vec{r} = r(\cos 60\hat{i} + \sin 60\hat{j}) = 6\left(\frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}\right) = 3\hat{i} + 3\sqrt{3}\hat{j}$

2. Construct an unit vector making an angle of 135° with x axis.

Sol. $\hat{r} = 1(\cos 135°\hat{i} + \sin 135°\hat{j}) = \frac{1}{\sqrt{2}}(-\hat{i} + \hat{j})$

- **Scalar product (Dot Product)**

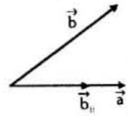
  - $\vec{A}.\vec{B} = AB\cos\theta \Rightarrow$ $\boxed{\text{Angle between two vectors } \theta = \cos^{-1}\left(\frac{\vec{A}\cdot\vec{B}}{AB}\right)}$

  - If $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ & $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$ then

    $\vec{A}.\vec{B} = A_xB_x + A_yB_y + A_zB_z$ and angle between $\vec{A}$ & $\vec{B}$ is given by

    $$\cos\theta = \frac{\vec{A}.\vec{B}}{AB} = \frac{A_xB_x + A_yB_y + A_zB_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}\sqrt{B_x^2 + B_y^2 + B_z^2}}$$

  - $\hat{i}.\hat{i} = 1$, $\hat{j}.\hat{j} = 1$, $\hat{k}.\hat{k} = 1$, $\hat{i}.\hat{j} = 0$, $\hat{i}.\hat{k} = 0$, $\hat{j}.\hat{k} = 0$

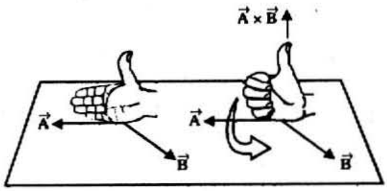


  - Component of vector $\vec{b}$ along vector $\vec{a}$, $\vec{b}_{||} = (\vec{b}\,.\,\hat{a})\hat{a}$

  - Component of $\vec{b}$ perpendicular to $\vec{a}$, $\vec{b}_{\perp} = \vec{b} - \vec{b}_{||} = \vec{b} - (\vec{b}\cdot\hat{a})\hat{a}$

- **Cross Product (Vector product)**

  - $\vec{A} \times \vec{B} = AB\sin\theta\,\hat{n}$ where $\hat{n}$ is a vector perpendicular to $\vec{A}$ & $\vec{B}$ or their plane and its direction given by right hand thumb rule.



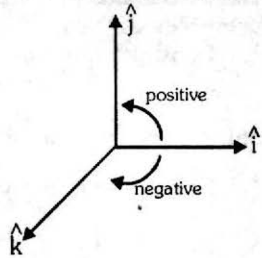
  - $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{i}(A_yB_z - A_zB_y) - \hat{j}(A_xB_z - B_xA_z) + \hat{k}(A_xB_y - B_xA_y)$

  - $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

- $(\vec{A} \times \vec{B}).\vec{A} = (\vec{A} \times \vec{B}).\vec{B} = 0$
- $\hat{i} \times \hat{i} = \vec{0}$, $\hat{j} \times \hat{j} = \vec{0}$, $\hat{k} \times \hat{k} = \vec{0}$
- $\hat{i} \times \hat{j} = \hat{k}$; $\hat{j} \times \hat{k} = \hat{i}$,

  $\hat{k} \times \hat{i} = \hat{j}$; $\hat{j} \times \hat{i} = -\hat{k}$

  $\hat{k} \times \hat{j} = -\hat{i}$, $\hat{i} \times \hat{k} = -\hat{j}$



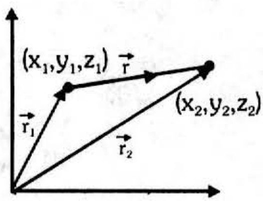
- **Differentiation**

- $\dfrac{d}{dt}(\vec{A}.\vec{B}) = \dfrac{d\vec{A}}{dt}.\vec{B} + \vec{A}.\dfrac{d\vec{B}}{dt}$
- $\dfrac{d}{dt}(\vec{A} \times \vec{B}) = \dfrac{d\vec{A}}{dt} \times \vec{B} + \vec{A} \times \dfrac{d\vec{B}}{dt}$

- **When a particle moved from (x_1, y_1, z_1) to (x_2, y_2, z_2) then its displacement vector**

$$\vec{r} = \vec{r}_2 - \vec{r}_1 = (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k})$$

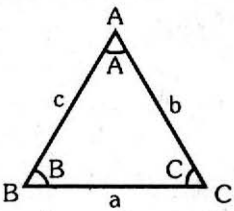
$$= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$



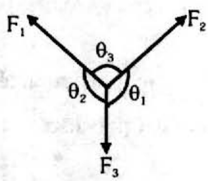
Magnitude

$$r = |\vec{r}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

- **Lami's theorem**

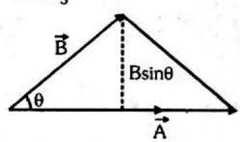


$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

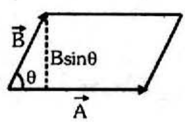


$$\frac{F_1}{\sin\theta_1} = \frac{F_2}{\sin\theta_2} = \frac{F_3}{\sin\theta_3}$$

- **Area of triangle** Area $= \dfrac{|\vec{A} \times \vec{B}|}{2} = \dfrac{1}{2}AB\sin\theta$



- **Area of parallelogram** Area $= |\vec{A} \times \vec{B}| = AB\sin\theta$



- **For Parallel vectors** $\vec{A} \times \vec{B} = \vec{0}$
- **For perpendicular vectors** $\vec{A}.\vec{B} = 0$
- **For coplanar vectors** $\vec{A}.(\vec{B} \times \vec{C}) = 0$

Examples of dot products :

- Work, $W = \vec{F}.\vec{d} = Fd\cos\theta$ where $F \to$ force, $d \to$ displacement
- Power, $P = \vec{F}.\vec{v} = Fv\cos\theta$ where $F \to$ force, $v \to$ velocity
- Electric flux, $\phi_E = \vec{E}.\vec{A} = EA\cos\theta$ where $E \to$ electric field, $A \to$ Area
- Magnetic flux, $\phi_B = \vec{B}.\vec{A} = BA\cos\theta$ where $B \to$ magnetic field, $A \to$ Area
- Potential energy of dipole in where $p \to$ dipole moment,

  uniform field, $U = -\vec{p}.\vec{E}$ where $E \to$ Electric field

Examples of cross products :

- Torque $\vec{\tau} = \vec{r} \times \vec{F}$ where $r \to$ position vector, $F \to$ force
- Angular momentum $\vec{J} = \vec{r} \times \vec{p}$ where $r \to$ position vector, $p \to$ linear momentum
- Linear velocity $\vec{v} = \vec{\omega} \times \vec{r}$ where $r \to$ position vector, $\omega \to$ angular velocity
- Torque on dipole placed in electric field $\vec{\tau} = \vec{p} \times \vec{E}$

  where $p \to$ dipole moment, $E \to$ electric field

KEY POINTS :

- **Tensor :** A quantity that has different values in different directions is called tensor.

  Ex. Moment of Inertia

  In fact tensors are merely a generalisation of scalars and vectors; a scalar is a zero rank tensor, and a vector is a first rank tensor.
- Electric current is not a vector as it does not obey the law of vector addition.
- A unit vector has no unit.
- To a vector only a vector of same type can be added and the resultant is a vector of the same type.
- A scalar or a vector can never be divided by a vector.

Important Notes

3 Units, Dimension, Measurements and Practical Physics

- **Fundamental or base quantities :**

  The quantities which do not depend upon other quantities for their complete definition are known as *fundamental or base quantities.*

  e.g. : length, mass, time, etc.

- **Derived quantities :**

  The quantities which can be expressed in terms of the fundamental quantities are known as *derived quantities* .e.g.

  Speed (=distance/time), volume, acceleration, force, pressure, etc.

- **Units of physical quantities**

  The chosen reference standard of measurement in multiples of which, a physical quantity is expressed is called the *unit* of that quantity.

  Physical Quantity = Numerical Value × Unit

Systems of Units

	MKS	CGS	FPS	MKSQ	MKSA
(i)	Length (m)	Length (cm)	Length (ft)	Length (m)	Length (m)
(ii)	Mass (kg)	Mass (g)	Mass (pound)	Mass (kg)	Mass (kg)
(iii)	Time (s)	Time (s)	Time (s)	Time (s)	Time (s)
(iv)	–	–	–	Charge (Q)	Current (A)

Fundamental Quantities in S.I. System and their units

S.N.	Physical Qty.	Name of Unit	Symbol
1	Mass	kilogram	kg
2	Length	meter	m
3	Time	second	s
4	Temperature	kelvin	K
5	Luminous intensity	candela	Cd
6	Electric current	ampere	A
7	Amount of substance	mole	mol

SI Base Quantities and Units

Base Quantity	SI Units		
	Name	Symbol	Definition
Length	meter	m	The meter is the length of the path traveled by light in vacuum during a time interval of 1/(299, 792, 458) of a second (1983)
Mass	kilogram	kg	The kilogram is equal to the mass of the international prototype of the kilogram (a platinum-iridium alloy cylinder) kept at International Bureau of Weights and Measures, at Sevres, near Paris, France. (1889)
Time	second	s	The second is the duration of 9, 192, 631, 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium-133 atom (1967)
Electric Current	ampere	A	The ampere is that constant current which, if maintained in two straight parallel conductors of infinite length, of negligible circular cross-section, and placed 1 metre apart in vacuum, would produce between these conductors a force equal to 2 x 10^{-7} Newton per metre of length. (1948)
Thermodynamic Temperature	kelvin	K	The kelvin, is the fraction 1/273.16 of the thermodynamic temperature of the triple point of water. (1967)
Amount of Substance	mole	mol	The mole is the amount of substance of a system, which contains as many elementary entities as there are atoms in 0.012 kilogram of carbon-12. (1971)
Luminous Intensity	candela	Cd	The candela is the luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency 540 x 10^{12} hertz and that has a radiant intensity in that direction of 1/683 watt per steradian (1979).

- **Supplementary Units**
  - Radian (rad) - for measurement of plane angle
  - Steradian (sr) - for measurement of solid angle
- **Dimensional Formula**

  Relation which express physical quantities in terms of appropriate powers of fundamental units.

- **Use of dimensional analysis**
  - To check the dimensional correctness of a given physical relation
  - To derive relationship between different physical quantities
  - To convert units of a physical quantity from one system to another

$$n_1u_1 = n_2u_2 \Rightarrow n_2 = n_1\left(\frac{M_1}{M_2}\right)^a\left(\frac{L_1}{L_2}\right)^b\left(\frac{T_1}{T_2}\right)^c \quad \text{where } u = M^aL^bT^c$$

- **Limitations of this method :**
  - In Mechanics the formula for a physical quantity depending on more than three other physical quantities cannot be derived. It can only be checked.
  - This method can be used only if the dependency is of multiplication type. The formulae containing exponential, trigonometrical and logarithmic functions can't be derived using this method. Formulae containing more than one term which are added or subtracted like $s = ut + \frac{1}{2}at^2$ also can't be derived.
  - The relation derived from this method gives no information about the dimensionless constants.
  - If dimensions are given, physical quantity may not be unique as many physical quantities have the same dimensions.
  - It gives no information whether a physical quantity is a scalar or a vector.

SI PREFIXES

The magnitudes of physical quantities vary over a wide range. The CGPM recommended standard prefixes for magnitude too large or too small to be expressed more compactly for certain powers of 10.

Prefixes used for different powers of 10

Power of 10	Prefix	Symbol	Power of 10	Prefix	Symbol
10^{18}	exa	E	10^{-1}	deci	d
10^{15}	peta	P	10^{-2}	centi	c
10^{12}	tera	T	10^{-3}	milli	m
10^{9}	giga	G	10^{-6}	micro	μ
10^{6}	mega	M	10^{-9}	nano	n
10^{3}	kilo	k	10^{-12}	pico	p
10^{2}	hecto	h	10^{-15}	femto	f
10^{1}	deca	da	10^{-18}	atto	a

Units of important Physical Quantities

Physical quantity	Unit	Physical quantity	Unit
Angular acceleration	rad s^{-2}	Frequency	hertz
Moment of inertia	kg – m^2	Resistance	kg m^2 A^{-2} s^{-3}
Self inductance	Henry	Surface tension	newton/m
Magnetic flux	Weber	Universal gas constant	joule K^{-1} mol^{-1}
Pole strength	A–m	Dipole moment	Coulomb–meter
Viscosity	Poise	Stefan constant	watt m^{-2} K^{-4}
Reactance	Ohm	Permittivity of free space (ε_0)	$Coulomb^2$/N–m^2
Specific heat	J/kg°C	Permeability of free space (μ_0)	Weber/A-m
Strength of magnetic field	newton A^{-1} m^{-1}	Planck's constant	joule–sec
Astronomical distance	Parsec	Entropy	J/K