Mathematics

HANDBOOK

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HANDBOOK OF MATHEMATICS

LOGARITHM

LOGARITHM OF A NUMBER :

The logarithm of the number N to the base 'a' is the exponent indicating the power to which the base 'a' must be raised to obtain the number N. This number is designated as $\log_a N$.

(a) $\log_a N = x$, read as log of N to the base a $\Leftrightarrow a^x = N$
If a = 10 then we write log N or $\log_{10} N$ and if a =e we write ln N or $\log_e N$ (Natural log)

(b) Necessary conditions : $N > 0$; $a > 0$; $a \neq 1$

(c) $\log_a 1 = 0$

(d) $\log_a a = 1$

(e) $\log_{1/a} a = -1$

(f) $\log_a(x.y) = \log_a x + \log_a y$; $x, y > 0$

(g) $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$; $x, y > 0$

(h) $\log_a x^p = p\log_a x$; $x > 0$

(i) $\log_{a^q} x = \frac{1}{q}\log_a x$; $x > 0$

(j) $\log_a x = \frac{1}{\log_x a}$; $x > 0, x \neq 1$

(k) $\log_a x = \log_b x / \log_b a$; $x > 0$, $a, b > 0$, $b \neq 1$, $a \neq 1$

(l) $\log_a b . \log_b c . \log_c d = \log_a d$; $a, b, c, d > 0, \neq 1$

(m) $a^{\log_a x} = x$; $a > 0$, $a \neq 1$

(n) $a^{\log_b c} = c^{\log_b a}$; $a,b,c > 0$; $b \neq 1$

(o) $\log_a x < \log_a y \Leftrightarrow \begin{cases} x < y & \text{if } a > 1 \\ x > y & \text{if } 0 < a < 1 \end{cases}$

(p) $\log_a x = \log_a y \Rightarrow x = y$; $x, y > 0$; $a > 0$, $a \neq 1$

(q) $e^{\ln a^x} = a^x$

(r) $\log_{10} 2 = 0.3010$; $\log_{10} 3 = 0.4771$; $\ell n 2 = 0.693$, $\ell n 10 = 2.303$

(s) If $a > 1$ then $\log_a x < p \Rightarrow 0 < x < a^p$

(t) If $a > 1$ then $\log_a x > p \Rightarrow x > a^p$

(u) If $0 < a < 1$ then $\log_a x < p \Rightarrow x > a^p$

(v) If $0 < a < 1$ then $\log_a x > p \Rightarrow 0 < x < a^p$

TRIGONOMETRIC RATIOS & IDENTITIES

1. RELATION BETWEEN SYSTEM OF MEASUREMENT OF ANGLES :

$$\frac{D}{90} = \frac{G}{100} = \frac{2C}{\pi}$$

1 Radian = $\frac{180}{\pi}$ degree $\approx$ 57°17'15" (approximately)

1 degree = $\frac{\pi}{180}$ radian $\approx$ 0.0175 radian

2. BASIC TRIGONOMETRIC IDENTITIES :

(a) $\sin^2\theta + \cos^2\theta = 1$ or $\sin^2\theta = 1 - \cos^2\theta$ or $\cos^2\theta = 1 - \sin^2\theta$

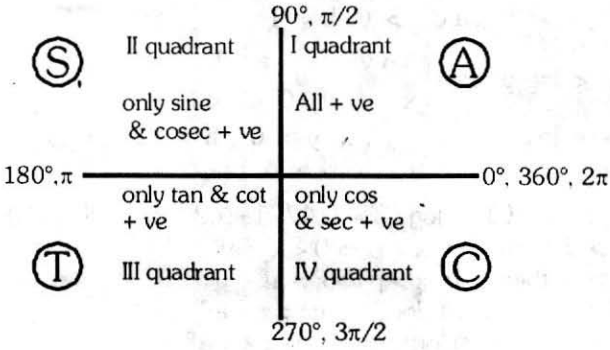
(b) $\sec^2\theta - \tan^2\theta = 1$ or $\sec^2\theta = 1 + \tan^2\theta$ or $\tan^2\theta = \sec^2\theta - 1$

(c) If $\sec\theta + \tan\theta = k \Rightarrow \sec\theta - \tan\theta = \frac{1}{k} \Rightarrow 2\sec\theta = k + \frac{1}{k}$

(d) $\text{cosec}^2\theta - \cot^2\theta = 1$ or $\text{cosec}^2\theta = 1 + \cot^2\theta$ or $\cot^2\theta = \text{cosec}^2\theta - 1$

(e) If $\text{cosec}\theta + \cot\theta = k \Rightarrow \text{cosec}\theta - \cot\theta = \frac{1}{k} \Rightarrow 2\,\text{cosec}\,\theta = k + \frac{1}{k}$

3. SIGNS OF TRIGONOMETRIC FUNCTIONS IN DIFFERENT QUADRANTS :



4. TRIGONOMETRIC FUNCTIONS OF ALLIED ANGLES :

(a) $\sin(2n\pi + \theta) = \sin\theta$, $\cos(2n\pi + \theta) = \cos\theta$, where $n \in I$

(b) $\sin(-\theta) = -\sin\theta$ $\qquad$ $\cos(-\theta) = \cos\theta$

$\sin(90° - \theta) = \cos\theta$ $\qquad$ $\cos(90° - \theta) = \sin\theta$

$\sin(90° + \theta) = \cos\theta$ $\qquad$ $\cos(90° + \theta) = -\sin\theta$

$\sin(180° - \theta) = \sin\theta$ $\qquad$ $\cos(180° - \theta) = -\cos\theta$

$\sin(180° + \theta) = -\sin\theta$ $\qquad$ $\cos(180° + \theta) = -\cos\theta$

$\sin(270° - \theta) = -\cos\theta$ $\qquad$ $\cos(270° - \theta) = -\sin\theta$

$\sin(270° + \theta) = -\cos\theta$ $\qquad$ $\cos(270° + \theta) = \sin\theta$

Note :

(i) $\sin n\pi = 0$; $\cos n\pi = (-1)^n$; $\tan n\pi = 0$ where $n \in I$

(ii) $\sin(2n+1)\dfrac{\pi}{2} = (-1)^n$; $\cos(2n+1)\dfrac{\pi}{2} = 0$ where $n \in I$

5. IMPORTANT TRIGONOMETRIC FORMULAE :

(i) $\sin(A + B) = \sin A\cos B + \cos A\sin B$.

(ii) $\sin(A - B) = \sin A\cos B - \cos A\sin B$.

(iii) $\cos(A + B) = \cos A\cos B - \sin A\sin B$

(iv) $\cos(A - B) = \cos A\cos B + \sin A\sin B$

(v) $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

(vi) $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$

(vii) $\cot(A + B) = \dfrac{\cot B\cot A - 1}{\cot B + \cot A}$

(viii) $\cot(A - B) = \dfrac{\cot B\cot A + 1}{\cot B - \cot A}$

(ix) $2\sin A\cos B = \sin(A + B) + \sin(A - B)$.

(x) $2\cos A\sin B = \sin(A + B) - \sin(A - B)$.

(xi) $2\cos A\cos B = \cos(A + B) + \cos(A - B)$

(xii) $2\sin A\sin B = \cos(A - B) - \cos(A + B)$

(xiii) $\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

(xiv) $\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

(xv) $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

(xvi) $\cos C - \cos D = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)$

(xvii) $\sin 2\theta = 2\sin\theta\cos\theta = \frac{2\tan\theta}{1+\tan^2\theta}$

(xviii) $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$

(xix) $1 + \cos 2\theta = 2\cos^2\theta$ or $\cos\theta = \pm\sqrt{\frac{1+\cos 2\theta}{2}}$

(xx) $1 - \cos 2\theta = 2\sin^2\theta$ or $\sin\theta = \pm\sqrt{\frac{1-\cos 2\theta}{2}}$

(xxi) $\tan\theta = \frac{1-\cos 2\theta}{\sin 2\theta} = \frac{\sin 2\theta}{1+\cos 2\theta} = \pm\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}$

(xxii) $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

(xxiii) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

(xxiv) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

(xxv) $\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$

(xxvi) $\sin^2 A - \sin^2 B = \sin(A+B).\sin(A-B) = \cos^2 B - \cos^2 A$.

(xxvii) $\cos^2 A - \sin^2 B = \cos(A+B).\cos(A-B)$.

(xxviii) $\sin(A+B+C)$

$= \sin A\cos B\cos C + \sin B\cos A\cos C + \sin C\cos A\cos B - \sin A\sin B\sin C$

$= \Sigma \sin A\cos B\cos C - \Pi \sin A$

$= \cos A\cos B\cos C\,[\tan A + \tan B + \tan C - \tan A\tan B\tan C]$

(xxix) $\cos(A+B+C)$

$= \cos A\cos B\cos C - \sin A\sin B\cos C - \sin A\cos B\sin C - \cos A\sin B\sin C$

$= \Pi\cos A - \Sigma \sin A\sin B\cos C$

$= \cos A\cos B\cos C\,[1 - \tan A\tan B - \tan B\tan C - \tan C\tan A]$

(xxx) $\tan(A+B+C)$

$$= \frac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - \tan A\tan B - \tan B\tan C - \tan C\tan A} = \frac{S_1 - S_3}{1 - S_2}$$

(xxxi) $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots \sin(\alpha + \overline{n-1}\,\beta)$

$$= \frac{\sin\left\{\alpha + \left(\frac{n-1}{2}\right)\beta\right\}\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

(xxxii) $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos(\alpha + \overline{n-1}\,\beta)$

$$= \frac{\cos\left\{\alpha + \left(\frac{n-1}{2}\right)\beta\right\}\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

6. VALUES OF SOME T-RATIOS FOR ANGLES 18°, 36°, 15°, 22.5°, 67.5° etc.

(a) $\sin 18^\circ = \frac{\sqrt{5}-1}{4} = \cos 72^\circ = \sin\frac{\pi}{10}$ $\cos 18 = \sqrt{\frac{\sqrt{5}+5}{8}}$

(b) $\cos 36^\circ = \frac{\sqrt{5}+1}{4} = \sin 54^\circ = \cos\frac{\pi}{5}$

(c) $\sin 15^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}} = \cos 75^\circ = \sin\frac{\pi}{12}$

(d) $\cos 15° = \frac{\sqrt{3}+1}{2\sqrt{2}} = \sin 75° = \cos\frac{\pi}{12}$

(e) $\tan\frac{\pi}{12} = 2-\sqrt{3} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = \cot\frac{5\pi}{12}$

(f) $\tan\frac{5\pi}{12} = 2+\sqrt{3} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = \cot\frac{\pi}{12}$

(g) $\tan(22.5°) = \sqrt{2}-1 = \cot(67.5°) = \cot\frac{3\pi}{8} = \tan\frac{\pi}{8}$

(h) $\tan(67.5°) = \sqrt{2}+1 = \cot(22.5°)$

7. MAXIMUM & MINIMUM VALUES OF TRIGONOMETRIC EXPRESSIONS :

(a) $a\cos\theta + b\sin\theta$ will always lie in the interval $[-\sqrt{a^2+b^2}, \sqrt{a^2+b^2}]$ i.e. the maximum and minimum values are $\sqrt{a^2+b^2}$, $-\sqrt{a^2+b^2}$ respectively.

(b) Minimum value of $a^2\tan^2\theta + b^2\cot^2\theta = 2ab$, where $a, b > 0$

(c) $-\sqrt{a^2+b^2+2ab\cos(\alpha-\beta)} \le a\cos(\alpha+\theta) + b\cos(\beta+\theta)$
$\le \sqrt{a^2+b^2+2ab\cos(\alpha-\beta)}$ where α and β are known angles.

(d) Minimum value of $a^2\cos^2\theta + b^2\sec^2\theta$ is either $2ab$ or a^2+b^2, if for some real θ equation $a\cos\theta = b\sec\theta$ is true or not true $\{a, b > 0\}$

(e) Minimum value of $a^2\sin^2\theta + b^2\operatorname{cosec}^2\theta$ is either $2ab$ or a^2+b^2, if for some real θ equation $a\sin\theta = b\operatorname{cosec}\theta$ is true or not true $\{a, b > 0\}$

8. IMPORTANT RESULTS :

(a) $\sin\theta\sin(60°-\theta)\sin(60°+\theta) = \frac{1}{4}\sin 3\theta$

(b) $\cos\theta.\cos(60°-\theta)\cos(60°+\theta) = \frac{1}{4}\cos 3\theta$

(c) $\tan\theta\tan(60^\circ-\theta)\tan(60^\circ+\theta)=\tan 3\theta$

(d) $\cot\theta\cot(60^\circ-\theta)\cot(60^\circ+\theta)=\cot 3\theta$

(e) **(i)** $\sin^2\theta+\sin^2(60^\circ+\theta)+\sin^2(60^\circ-\theta)=\frac{3}{2}$

(ii) $\cos^2\theta+\cos^2(60^\circ+\theta)+\cos^2(60^\circ-\theta)=\frac{3}{2}$

(f) **(i)** If $\tan A+\tan B+\tan C=\tan A\tan B\tan C$,

then $A+B+C=n\pi, n\in I$

(ii) If $\tan A\tan B+\tan B\tan C+\tan C\tan A=1$,

then $A+B+C=(2n+1)\frac{\pi}{2}, n\in I$

(g) $\cos\theta\cos 2\theta\cos 4\theta \ldots \cos(2^{n-1}\theta)=\frac{\sin(2^n\theta)}{2^n\sin\theta}$

(h) $\cot A-\tan A=2\cot 2A$

9. CONDITIONAL IDENTITIES :

If $A+B+C=180^\circ$, then :

(a) $\tan A+\tan B+\tan C=\tan A\tan B\tan C$

(b) $\cot A\cot B+\cot B\cot C+\cot C\cot A=1$

(c) $\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1$

(d) $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

(e) $\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C$

(f) $\cos 2A+\cos 2B+\cos 2C=-1-4\cos A\cos B\cos C$

(g) $\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

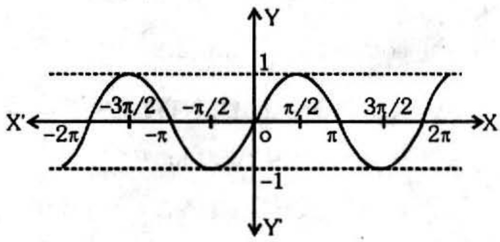
(h) $\cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

10. DOMAINS, RANGES AND PERIODICITY OF TRIGONOMETRIC FUNCTIONS :

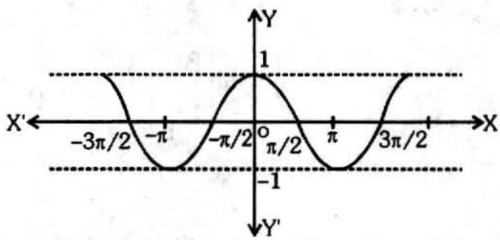
T-Ratio	Domain	Range	Period
sin x	R	$[-1,1]$	2π
cos x	R	$[-1,1]$	2π
tan x	$R-\{(2n+1)\pi/2\ ;\ n\in I\}$	R	π
cot x	$R-\{n\pi : n \in I\}$	R	π
sec x	$R-\{(2n+1)\pi/2 : n \in I\}$	$(-\infty,-1]\cup[1,\infty)$	2π
cosec x	$R-\{n\pi : n \in I\}$	$(-\infty,-1]\cup[1,\infty)$	2π

11. GRAPH OF TRIGONOMETRIC FUNCTIONS :

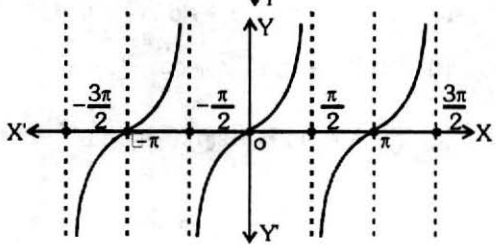
(a) y = sinx



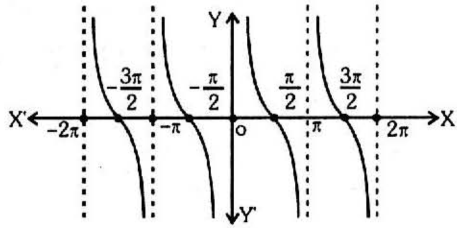
(b) y = cosx



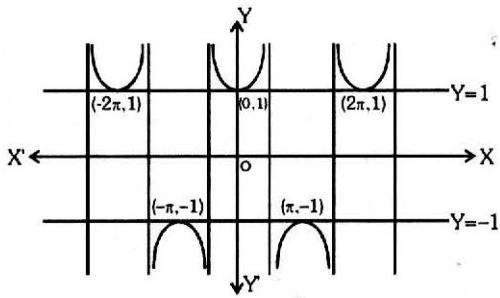
(c) y = tanx



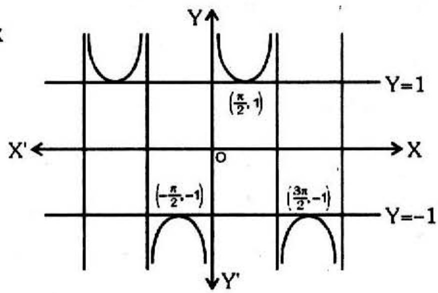
(d) y = cotx



(e) y = secx



(f) y = cosecx



12. IMPORTANT NOTE :

(a) The sum of interior angles of a polygon of n-sides $= (n-2) \times 180° = (n-2)\pi$.

(b) Each interior angle of a regular polygon of n sides

$$= \frac{(n-2)}{n} \times 180° = \frac{(n-2)}{n}\pi.$$

(c) Sum of exterior angles of a polygon of any number of sides $= 360° = 2\pi$.

TRIGONOMETRIC EQUATION

1. TRIGONOMETRIC EQUATION :

An equation involving one or more trigonometrical ratios of unknown angles is called a trigonometrical equation.

2. SOLUTION OF TRIGONOMETRIC EQUATION :

A value of the unknown angle which satisfies the given equations is called a solution of the trigonometric equation.

(a) Principal solution :- The solution of the trigonometric equation lying in the interval $[0, 2\pi]$.

(b) General solution :- Since all the trigonometric functions are many one & periodic, hence there are infinite values of θ for which trigonometric functions have the same value. All such possible values of θ for which the given trigonometric function is satisfied is given by a general formula. Such a general formula is called general solutions of trigonometric equation.

3. GENERAL SOLUTIONS OF SOME TRIGONOMETRIC EQUATIONS (TO BE REMEMBERED) :

(a) If $\sin\theta = 0$, then $\theta = n\pi$, $n \in I$ (set of integers)

(b) If $\cos\theta = 0$, then $\theta = (2n+1)\dfrac{\pi}{2}$ $n \in I$

(c) If $\tan\theta = 0$, then $\theta = n\pi$, $n \in I$

(d) If $\sin\theta = \sin\alpha$, then $\theta = n\pi + (-1)^n\alpha$ where $\alpha \in \left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right]$, $n \in I$

(e) If $\cos\theta = \cos\alpha$, then $\theta = 2n\pi \pm \alpha$, $n \in I$, $\alpha \in [0,\pi]$

(f) If $\tan\theta = \tan\alpha$, then $\theta = n\pi + \alpha$, $n \in I$, $\alpha \in \left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$

(g) If $\sin\theta = 1$, then $\theta = 2n\pi + \dfrac{\pi}{2} = (4n+1)\dfrac{\pi}{2}$, $n \in I$

(h) If $\cos\theta = 1$ then $\theta = 2n\pi,\ n \in I$

(i) If $\sin^2\theta = \sin^2\alpha$ or $\cos^2\theta = \cos^2\alpha$ or $\tan^2\theta = \tan^2\alpha$,

then $\theta = n\pi \pm \alpha,\ n \in I$

(j) For $n \in I$, $\sin n\pi = 0$ and $\cos n\pi = (-1)^n,\ n \in I$

$\sin(n\pi + \theta) = (-1)^n \sin\theta$

$\cos(n\pi + \theta) = (-1)^n \cos\theta$

(k) $\cos n\pi = (-1)^n,\ n \in I$

(l) If n is an odd integer then $\sin\dfrac{n\pi}{2} = (-1)^{\frac{n-1}{2}}, \cos\dfrac{n\pi}{2} = 0$

(m) $\sin\left(\dfrac{n\pi}{2} + \theta\right) = (-1)^{\frac{n-1}{2}}\cos\theta,\quad \cos\left(\dfrac{n\pi}{2} + \theta\right) = (-1)^{\frac{n+1}{2}}\sin\theta$

4. GENERAL SOLUTION OF EQUATION $a\cos\theta + b\sin\theta = c$:

Consider, $a\sin\theta + b\cos\theta = c$ .... (i)

$$\therefore \quad \frac{a}{\sqrt{a^2+b^2}}\sin\theta + \frac{b}{\sqrt{a^2+b^2}}\cos\theta = \frac{c}{\sqrt{a^2+b^2}}$$

equation (i) has the solution only if $|c| \le \sqrt{a^2+b^2}$

let $\dfrac{a}{\sqrt{a^2+b^2}} = \cos\phi,\ \dfrac{b}{\sqrt{a^2+b^2}} = \sin\phi\ \ \&\ \ \phi = \tan^{-1}\dfrac{b}{a}$

by introducing this auxiliary argument ϕ, equation (i) reduces to

$$\sin(\theta + \phi) = \frac{c}{\sqrt{a^2+b^2}}$$

Now this equation can be solved easily.

5. GENERAL SOLUTION OF EQUATION OF FORM :

$a_0\sin^n x + a_1\sin^{n-1}x\cos x + a_2\sin^{n-2} \times \cos^2 x + \ldots\ldots\ldots + a_n\cos^n x = 0$

$a_0, a_1, \ldots\ldots\ldots a_n$ are real numbers

Such an equation is solved by dividing equation by $\cos^n x$.

6. IMPORTANT TIPS :

(a) For equations of the type $\sin\theta = k$ or $\cos\theta = k$, one must check that $|k| \leq 1$.

(b) Avoid squaring the equations, if possible, because it may lead to extraneous solutions.

(c) Do not cancel the common variable factor from the two sides of the equations which are in a product because we may loose some solutions.

(d) The answer should not contain such values of θ, which make any of the terms undefined or infinite.

(e) Check that denominator is not zero at any stage while solving equations.

(f) **(i)** If $\tan\theta$ or $\sec\theta$ is involved in the equations, θ should not be odd multiple of $\frac{\pi}{2}$.

(ii) If $\cot\theta$ or $\operatorname{cosec}\theta$ is involved in the equation, θ should not be integral multiple of π or 0.

(g) If two different trigonometric ratios such as $\tan\theta$ and $\sec\theta$ are involved then after solving we cannot apply the usual formulae for general solution because periodicity of the functions are not same.

(h) If L.H.S. of the given trigonometric equation is always less than or equal to k and RHS is always greater than k, then no solution exists. If both the sides are equal to k for same value of θ, then solution exists and if they are equal for different value of θ, then solution does not exist.

QUADRATIC EQUATION

1. SOLUTION OF QUADRATIC EQUATION & RELATION BETWEEN ROOTS & CO-EFFICIENTS :

(a) The solutions of the quadratic equation, $ax^2 + bx + c = 0$ is given by $\mathbf{x} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

(b) The expression $b^2 - 4\,ac \equiv D$ is called the discriminant of the quadratic equation.

(c) If α & β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then ;

(i) $\alpha + \beta = -b/a$ **(ii)** $\alpha\beta = c/a$ **(iii)** $|\alpha - \beta| = \sqrt{D}/|a|$

(d) Quadratic equation whose roots are α & β is $(x - \alpha)(x - \beta) = 0$ i.e.

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$ i.e. x^2 – (sum of roots) x + product of roots = 0.

2. NATURE OF ROOTS :

(a) Consider the quadratic equation $ax^2 + bx + c = 0$ where a, b, $c \in R$ & $a \neq 0$ then ;

(i) $D > 0 \Leftrightarrow$ roots are real & distinct (unequal).

(ii) $D = 0 \Leftrightarrow$ roots are real & coincident (equal)

(iii) $D < 0 \Leftrightarrow$ roots are imaginary.

(iv) If $p + iq$ is one root of a quadratic equation, then the other root must be the conjugate $p - iq$ & vice versa. $(p, q \in R$ & $i = \sqrt{-1})$.

(b) Consider the quadratic equation $ax^2 + bx + c = 0$ where a, b, $c \in Q$ & $a \neq 0$ then ;

(i) If D is a perfect square, then roots are rational.

(ii) If $\alpha = p + \sqrt{q}$ is one root in this case, (where p is rational & $\sqrt{q}$ is a surd) then other root will be $p - \sqrt{q}$.

3. COMMON ROOTS OF TWO QUADRATIC EQUATIONS

(a) Only one common root.

Let α be the common root of $ax^2 + bx + c = 0$ & $a'x^2 + b'x + c' = 0$ then $a\alpha^2 + b\alpha + c = 0$ & $a'\alpha^2 + b'\alpha + c' = 0$. By Cramer's Rule $\dfrac{\alpha^2}{bc'-b'c} = \dfrac{\alpha}{a'c-ac'} = \dfrac{1}{ab'-a'b}$

Therefore, $\alpha = \dfrac{ca'-c'a}{ab'-a'b} = \dfrac{bc'-b'c}{a'c-ac'}$

So the condition for a common root is

$(ca' - c'a)^2 = (ab' - a'b)(bc' - b'c)$

(b) If both roots are same then $\dfrac{a}{a'} = \dfrac{b}{b'} = \dfrac{c}{c'}$

4. ROOTS UNDER PARTICULAR CASES

Let the quadratic equation $ax^2 + bx + c = 0$ has real roots and

(a) If $b = 0 \Rightarrow$ roots are of equal magnitude but of opposite sign

(b) If $c = 0 \Rightarrow$ one roots is zero other is $-b/a$

(c) If $a = c \Rightarrow$ roots are reciprocal to each other

(d) If $\left.\begin{matrix} a > 0 \ c < 0 \\ a < 0 \ c > 0 \end{matrix}\right\} \Rightarrow$ roots are of opposite signs

(e) If $\left.\begin{matrix} a > 0, b > 0, c > 0 \\ a < 0, b < 0, c < 0 \end{matrix}\right\} \Rightarrow$ both roots are negative.

(f) If $\left.\begin{matrix} a > 0, b < 0, c > 0 \\ a < 0, b > 0, c < 0 \end{matrix}\right\} \Rightarrow$ both roots are positive.

(g) If sign of a = sign of b $\neq$ sign of c $\Rightarrow$ Greater root in magnitude is negative.

(h) If sign of b = sign of c $\neq$ sign of a $\Rightarrow$ Greater root in magnitude is positive.

(i) If $a + b + c = 0 \Rightarrow$ one root is 1 and second root is c/a.

5. MAXIMUM & MINIMUM VALUES OF QUADRATIC EXPRESSION :

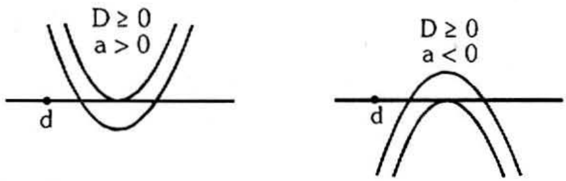
Maximum & Minimum Values of expression $y = ax^2 + bx + c$ is $\frac{-D}{4a}$ which occurs at $x = -(b/2a)$ according as $a < 0$ or $a > 0$.

$$y \in \left[\frac{-D}{4a}, \infty\right) \text{ if } a > 0 \quad \& \quad y \in \left(-\infty, \frac{-D}{4a}\right] \text{ if } a < 0.$$

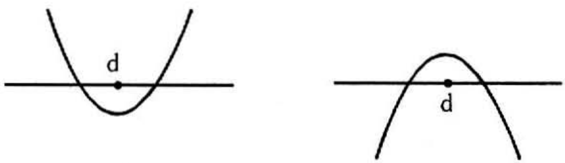
6. LOCATION OF ROOTS :

Let $f(x) = ax^2 + bx + c$, where $a, b, c \in R$, $a \neq 0$

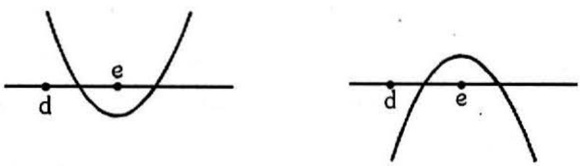
(a) Conditions for both the roots of $f(x) = 0$ to be greater than a specified number 'd' are **$D \geq 0$; $a.f(d) > 0$ & $(-b/2a) > d$.**



(b) Conditions for the both roots of $f(x) = 0$ to lie on either side of the number 'd' in other words the number 'd' lies between the roots of $f(x) = 0$ is **$a.f(d) < 0$**.

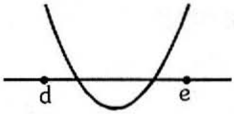


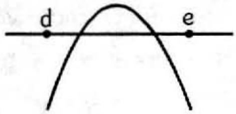
(c) Conditions for exactly one root of $f(x) = 0$ to lie in the interval (d,e) i.e.. $d < x < e$ is **$f(d). f(e) < 0$**



(d) Conditions that both roots of $f(x) = 0$ to be confined between the numbers d & e are (here $d < e$).

$D \geq 0$; $a \cdot f(d) > 0$ & $af(e) > 0$; $d < (-b/2a) < e$





7. GENERAL QUADRATIC EXPRESSION IN TWO VARIABLES :

$f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ may be resolved into two linear factors if ;

$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ OR $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

8. THEORY OF EQUATIONS :

If $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots\ldots \alpha_n$ are the roots of the equation ;

$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots\ldots + a_{n-1} x + a_n = 0$ where a_0, $a_1, \ldots\ldots a_n$ are constants $a_0 \neq 0$ then,

$$\Sigma\alpha_1 = -\frac{a_1}{a_0},\ \Sigma\alpha_1\alpha_2 = +\frac{a_2}{a_0},\ \Sigma\alpha_1\alpha_2\alpha_3$$

$$= -\frac{a_3}{a_0}, \ldots\ldots, \alpha_1\alpha_2\alpha_3\ldots\ldots\alpha_n = (-1)^n \frac{a_n}{a_0}$$

Note :

(i) Every odd degree equation has at least one real root whose sign is opposite to that of its last term, when coefficient of highest degree term is (+)ve {If not then make it (+) ve}.

Ex. $x^3 - x^2 + x - 1 = 0$

(ii) Even degree polynomial whose last term is (–)ve & coefficient of highest degree term is (+)ve has atleast two real roots, one (+)ve & one (–)ve.

(iii) If equation contains only even power of x & all coefficient are (+)ve, then all roots are imaginary.

SEQUENCE & SERIES

1. ARITHMETIC PROGRESSION (AP) :

AP is sequence whose terms increase or decrease by a fixed number. This fixed number is called the **common difference**. If 'a' is the first term & 'd' is the common difference, then AP can be written as $a, a + d, a + 2d, \ldots\ldots\ldots\ldots\ldots a + (n - 1) d, \ldots\ldots\ldots$

(a) n^{th} term of this AP $\boxed{T_n = a + (n-1)d}$, where $d = T_n - T_{n-1}$

(b) The sum of the first n terms : $\boxed{S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}[a + \ell]}$

where ℓ is the last term.

(c) Also n^{th} term $\boxed{T_n = S_n - S_{n-1}}$

Note :

(i) Sum of first n terms of an A.P. is of the form $An^2 + Bn$ i.e. a quadratic expression in n, in such case the common difference is twice the coefficient of n^2. i.e. 2A

(ii) n^{th} term of an A.P. is of the form $An + B$ i.e. a linear expression in n, in such case the coefficient of n is the common difference of the A.P. i.e. A

(iii) Three numbers in AP can be taken as $a - d, a, a + d$; four numbers in AP can be taken as $a - 3d, a - d, a + d, a + 3d$ five numbers in AP are $a - 2d, a - d, a, a + d, a + 2d$ & six terms in AP are $a - 5d, a - 3d, a - d, a + d, a + 3d, a + 5d$ etc.

(iv) If for A.P. p^{th} term is q, q^{th} term is p, then r^{th} term is $= p + q - r$ & $(p + q)^{th}$ term is 0.

(v) If $a_1, a_2, a_3 \ldots\ldots$ and $b_1, b_2, b_3 \ldots\ldots\ldots$ are two A.P.s, then $a_1 \pm b_1, a_2 \pm b_2, a_3 \pm b_3 \ldots\ldots\ldots$ are also in A.P.

(vi) (a) If each term of an A.P. is increased or decreased by the same number, then the resulting sequence is also an A.P. having the same common difference.

(b) If each term of an A.P. is multiplied or divided by the same non zero number (k), then the resulting sequence is also an A.P. whose common difference is kd & d/k respectively, where d is common difference of original A.P.

(vii) Any term of an AP (except the first & last) is equal to half the sum of terms which are equidistant from it.

$$T_r = \frac{T_{r-k} + T_{r+k}}{2},\ k < r$$

2. GEOMETRIC PROGRESSION (GP) :

GP is a sequence of numbers whose first term is non-zero & each of the succeeding terms is equal to the preceeding terms multiplied by a constant. Thus in a GP the ratio of successive terms is constant. This constant factor is called the **COMMON RATIO** of the series & is obtained by dividing any term by the immediately previous term. Therefore a, ar, ar^2, ar^3, ar^4 , ... is a GP with 'a' as the first term & 'r' as common ratio.

(a) n^{th} term $\boxed{T_n = a\,r^{n-1}}$

(b) Sum of the first n terms $\boxed{S_n = \frac{a(r^n - 1)}{r-1},\ \text{if } r \neq 1}$

(c) Sum of infinite GP when $|r| < 1$ & $n \to \infty,\ r^n \to 0$

$$\boxed{S_\infty = \frac{a}{1-r};\ |r| < 1}$$

(d) Any 3 consecutive terms of a GP can be taken as a/r, a, ar; any 4 consecutive terms of a GP can be taken as a/r^3, a/r, ar, ar^3 & so on.

(e) If a, b, c are in GP $\Rightarrow b^2 = ac \Rightarrow$ loga, logb, logc, are in A.P.

Note :

(i) In an G.P. product of k^{th} term from beginning and k^{th} term from the last is always constant which equal to product of first term and last term.

(ii) Three numbers in **G.P.** : $\mathbf{a/r,\ a,\ ar}$

Five numbers in **G.P.** : $\mathbf{a/r^2, a/r, a, ar, ar^2}$

Four numbers in **G.P.** : $\mathbf{a/r^3,\ a/r,\ ar,\ ar^3}$

Six numbers in **G.P.** : $\mathbf{a/r^5, a/r^3,\ a/r,\ ar,\ ar^3, ar^5}$

(iii) If each term of a **G.P.** be raised to the same power, then resulting series is also a **G.P.**

(iv) If each term of a G.P. be multiplied or divided by the same non-zero quantity, then the resulting sequence is also a G.P.

(v) If a_1, a_2, a_3 ..... and b_1, b_2, b_3, ... be two G.P.'s of common ratio r_1 and r_2 respectively, then a_1b_1, a_2b_2 ..... and $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}$ ...... will also form a G.P. common ratio will be $r_1 r_2$ and $\frac{r_1}{r_2}$ respectively.

(vi) In a positive G.P. every term (except first) is equal to square root of product of its two terms which are equidistant from it.

i.e. $\mathbf{T_r} = \sqrt{\mathbf{T_{r-k}T_{r+k}}}$, $k < r$

(vii) If $\mathbf{a_1, a_2, a_3 .....a_n}$ is a **G.P.** of **non zero, non negative terms,** then $\mathbf{\log a_1, \log a_2, ...\log a_n}$ is an **A.P.** and **vice-versa.**

3. HARMONIC PROGRESSION (HP) :

A sequence is said to HP if the reciprocals of its terms are in AP.

If the sequence a_1, a_2, a_3, ..., a_n is an HP then $1/a_1$, $1/a_2$,........, $1/a_n$ is an AP & converse. Here we do not have the formula for the sum of the n terms of an HP. The general form of a harmonic progression is $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, ... \frac{1}{a+(n-1)d}$

Note : No term of any H.P. can be zero. If a, b, c are in

$$\text{HP} \Rightarrow b = \frac{2ac}{a+c} \text{ or } \frac{a}{c} = \frac{a-b}{b-c}$$

4. MEANS

(a) Arithmetic mean (AM) :

If three terms are in AP then the middle term is called the AM between the other two, so if a, b, c are in AP, b is AM of a & c.

n-arithmetic means between two numbers :

If a,b are any two given numbers & a, A_1, A_2, ..., A_n, b are in AP then $A_1, A_2, \ldots A_n$ are the n AM's between a & b, then

$A_1 = a + d,\ A_2 = a + 2d, \ldots\ldots,\ A_n = a + nd$, where $d = \dfrac{b-a}{n+1}$

Note : Sum of n AM's inserted between a & b is equal to n times the single AM between a & b i.e. $\sum_{r=1}^{n} A_r = nA$ where A is the single AM between a & b.

(b) Geometric mean (GM) :

If a, b, c are in GP, b is the GM between a & c, $b^2 = ac$, therefore $b = \sqrt{ac}$

n-geometric means between two numbers :

If a, b are two given positive numbers & a, G_1, G_2, ..., G_n, b are in GP then $G_1, G_2, G_3, \ldots\ldots G_n$ are n GMs between a & b.

$G_1 = ar,\ G_2 = ar^2, \ldots\ldots\ G_n = ar^n$, where $r = (b/a)^{1/n+1}$

Note : The product of n GMs between a & b is equal to nth power of the single GM between a & b i.e. $\prod_{r=1}^{n} G_r = (G)^n$ where G is the single GM between a & b

(c) Harmonic mean (HM) :

If a, b, c are in HP, then b is HM between a & c, then $b = \dfrac{2ac}{a+c}$.

Important note :

(i) If A, G, H, are respectively AM, GM, HM between two positive number a & b then

(a) $G^2 = AH$ (A, G, H constitute a GP) (b) $A \geq G \geq H$

(c) $A = G = H \Rightarrow a = b$

(ii) Let $a_1, a_2, \ldots, a_n$ be n positive real numbers, then we define their arithmetic mean (A), geometric mean (G) and harmonic mean (H) as $A = \dfrac{a_1 + a_2 + \ldots + a_n}{n}$

$G = (a_1 a_2 \ldots\ldots a_n)^{1/n}$ and $H = \dfrac{n}{\left(\dfrac{1}{a_1} + \dfrac{1}{a_2} + \dfrac{1}{a_3} + \ldots \dfrac{1}{a_n}\right)}$

It can be shown that $A \geq G \geq H$. Moreover equality holds at either place if and only if $a_1 = a_2 = \ldots = a_n$

5. ARITHMETICO - GEOMETRIC SERIES :

Sum of First n terms of an Arithmetico-Geometric Series :

Let $S_n = a + (a+d)r + (a+2d)r^2 + \ldots\ldots + [a + (n-1)d]r^{n-1}$

then $S_n = \dfrac{a}{1-r} + \dfrac{dr(1-r^{n-1})}{(1-r)^2} - \dfrac{[a+(n-1)d]\ r^n}{1-r}, \quad r \neq 1$

Sum to infinity :

If $|r| < 1$ & $n \to \infty$ then $\lim\limits_{n\to\infty} r^n = 0 \quad \Rightarrow \quad S_\infty = \dfrac{a}{1-r} + \dfrac{dr}{(1-r)^2}$

6. SIGMA NOTATIONS

Theorems :

(a) $\sum\limits_{r=1}^{n}(a_r \pm b_r) = \sum\limits_{r=1}^{n} a_r \pm \sum\limits_{r=1}^{n} b_r$ **(b)** $\sum\limits_{r=1}^{n} k a_r = k\sum\limits_{r=1}^{n} a_r$

(c) $\sum\limits_{r=1}^{n} k = nk$; where k is a constant.

7. RESULTS

(a) $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$ (sum of the first n natural numbers)

(b) $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$ (sum of the squares of the first n natural numbers)

(c) $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4} = \left[\sum_{r=1}^{n} r\right]^2$ (sum of the cubes of the first n natural numbers)

(d) $\sum_{r=1}^{n} r^4 = \frac{n}{30}(n+1)(2n+1)(3n^2+3n-1)$

PERMUTATION & COMBINATION

1. FUNDAMENTAL PRINCIPLE OF COUNTING (counting without actually counting):

If an event can occur in 'm' different ways, following which another event can occur in 'n' different ways, then the total number of different ways of

(a) Simultaneous occurrence of both events in a definite order is m x n. This can be extended to any number of events (known as multiplication principle).

(b) Happening of exactly one of the events is m + n (known as addition principle).

2. FACTORIAL :

A Useful Notation : $n! = n(n-1)(n-2) \ldots\ldots 3.\ 2.\ 1;$

$n! = n.\ (n-1)!$ where $n \in W$

$0! = 1! = 1$

$(2n)! = 2^n\ .\ n!\ [1.\ 3.\ 5.\ 7 \ldots\ldots (2n-1)]$

Note that factorials of negative integers are not defined.

3. PERMUTATION :

(a) ${}^nP_r$ denotes the number of permutations of n different things, taken r at a time ($n \in N$, $r \in W$, $n \geq r$)

$${}^nP_r = n(n-1)(n-2) \ldots\ldots\ldots\ldots (n-r+1) = \frac{n!}{(n-r)!}$$

(b) The number of permutations of n things taken all at a time when p of them are similar of one type, q of them are similar of second type, r of them are similar of third type and the remaining $n-(p+q+r)$ are all different is : $\dfrac{n!}{p!\ q!\ r!}$.

(c) The number of permutation of n different objects taken r at a time, when a particular object is always to be included is $r!\ .\ {}^{n-1}C_{r-1}$

(d) The number of permutation of n different objects taken r at a time, when repetition be allowed any number of times is $n \times n \times n$ .... r times $= n^r$.

(e) **(i)** The number of circular permutations of n different things taken all at a time is ; $(n-1)! = \frac{^nP_n}{n}$.

If clockwise & anti-clockwise circular permutations are considered to be same, then it is $\frac{(n-1)!}{2}$.

(ii) The number of circular permutation of n different things taking r at a time distinguishing clockwise & anticlockwise arrangement is $\frac{^nP_r}{r}$

4. COMBINATION :

(a) ^nC_r denotes the number of combinations of n different things taken r at a time, and $^nC_r = \frac{n!}{r!(n-r)!} = \frac{^nP_r}{r!}$ where $r \le n$; $n \in N$ and $r \in W$. ^nC_r is also denoted by $\binom{n}{r}$ or A_r^n or $C(n, r)$.

(b) The number of combination of n different things taking r at a time.

(i) when p particular things are always to be included $= {}^{n-p}C_{r-p}$

(ii) when p particular things are always to be excluded $= {}^{n-p}C_r$

(iii) when p particular things are always to be included and q particular things are to be excluded $= {}^{n-p-q}C_{r-p}$

(c) Given n different objects, the number of ways of selecting atleast one of them is, $^nC_1 + {}^nC_2 + {}^nC_3 + \ldots\ldots + {}^nC_n = 2^n - 1$. This can also be stated as the total number of combinations of n distinct things.

(d) (i) Total number of ways in which it is possible to make a selection by taking some or all out of p + q + r +......things, where p are alike of one kind, q alike of a second kind, r alike of third kind & so on is given by : $(p + 1)(q + 1)(r + 1)........-1$.

(ii) The total number of ways of selecting one or more things from p identical things of one kind, q identical things of second kind, r identical things of third kind and n different things is $(p + 1)(q + 1)(r + 1)\,2^n - 1$

5. DIVISORS :

Let $N = p^a . q^b . r^c$ ....where p, q, r........ are distinct primes & a, b, c....... are natural numbers then :

(a) The total numbers of divisors of N including 1 & N is $= (a + 1)(b + 1)(c + 1)......$.

(b) The sum of these divisors is $= (p^0 + p^1 + p^2 + ...+ p^a)(q^0 + q^1 + q^2 + ...+ q^b)(r^0 + r^1 + r^2 + ...+ r^c)...$

(c) Number of ways in which N can be resolved as a product of two factor is =

$\frac{1}{2}(a+1)(b+1)(c+1)......$ if N is not a perfect square

$\frac{1}{2}\left[(a+1)(b+1)(c+1)......+1\right]$ if N is a perfect square

(d) Number of ways in which a composite number N can be resolved into two factors which are relatively prime (or coprime) to each other is equal to 2^{n-1} where n is the number of different prime factors in N.

6. DIVISION AND DISTRIBUTION :

(a) (i) The number of ways in which (m + n) different things can be divided into two groups containing m & n things respectively is : $\frac{(m+n)!}{m!\ n!}$ $(m \neq n)$.

(ii) If m = n, it means the groups are equal & in this case the number of subdivision is $\frac{(2n)!}{n!\ n!\ 2!}$; for in any one way it is possible to inter change the two groups without obtaining a new distribution.

(iii) If 2n things are to be divided equally between two persons then the number of ways $= \frac{(2n)!}{n!\ n!\ (2!)} \times 2!$.

(b) (i) Number of ways in which (m + n + p) different things can be divided into three groups containing m, n & p things respectively is $\frac{(m+n+p)!}{m!\ n!\ p!}$, $m \neq n \neq p$.

(ii) If m = n = p then the number of groups $= \frac{(3n)!}{n!\ n!\ n!\ 3!}$.

(iii) If 3n things are to be divided equally among three people then the number of ways in which it can be done is $\frac{(3n)!}{(n!)^3}$.

(c) In general, the number of ways of dividing n distinct objects into ℓ groups containing p objects each ,m groups containing q objects each is equal to $\frac{n!(\ell+m)!}{(p!)^{\ell}(q!)^{m}\ \ell!m!}$

Here $\ell p + mq = n$

(d) Number of ways in which n distinct things can be distributed to p persons if there is no restriction to the number of things received by them $= p^n$

(e) Number of ways in which n identical things may be distributed among p persons if each person may receive none, one or more things is ; ${}^{n+p-1}C_n$.

7. DEARRANGEMENT :

Number of ways in which n letters can be placed in n directed envelopes so that no letter goes into its own envelope is

$$= n!\left[1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-........+(-1)^n\frac{1}{n!}\right]$$

8. IMPORTANT RESULT :

(a) Number of rectangle of any size in a square of size $n \times n$ is

$\sum_{r=1}^{n} r^3$ & number of square of any size is $\sum_{r=1}^{n} r^2$.

(b) Number of rectangle of any size in a rectangle of size $n \times p$

$(n < p)$ is $\frac{np}{4}(n+1)(p+1)$ & number of squares of any size is

$\sum_{r=1}^{n}(n+1-r)(p+1-r)$

(c) If there are n points in a plane of which $m(<n)$ are collinear :

(i) Total number of lines obtained by joining these points is ${}^nC_2 - {}^mC_2 + 1$

(ii) Total number of different triangle ${}^nC_3 - {}^mC_3$

(d) Maximum number of point of intersection of n circles is ${}^nP_2$ & n lines is ${}^nC_2$.

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BINOMIAL THEOREM

$(x+y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + \ldots.. + {}^nC_rx^{n-r}y^r + \ldots.. + {}^nC_ny^n$

$= \sum_{r=0}^{n} {}^nC_rx^{n-r}y^r$, where $n \in N$.

1. IMPORTANT TERMS IN THE BINOMIAL EXPANSION ARE :

(a) General term: The general term or the $(r+1)^{th}$ term in the expansion of $(x+y)^n$ is given by

$T_{r+1} = {}^nC_r\, x^{n-r}.\, y^r$

(b) Middle term :

The middle term (s) is the expansion of $(x+y)^n$ is (are) :

(i) If n is even, there is only one middle term which is given by $T_{(n+2)/2} = {}^nC_{n/2}.\, x^{n/2}.\, y^{n/2}$

(ii) If n is odd, there are two middle terms which are $T_{(n+1)/2}$ & $T_{[(n+1)/2]+1}$

(c) Term independent of x :

Term independent of x contains no x ; Hence find the value of r for which the exponent of x is zero.

2. If $(\sqrt{A}+B)^n = I + f$, where I & n are positive integers & $0 \le f < 1$, then

(a) $(I+f).\,f = K^n$ if n is odd & $A - B^2 = K > 0$

(b) $(I+f)(1-f) = k^n$ if n is even & $\sqrt{A} - B < 1$

3. SOME RESULTS ON BINOMIAL COEFFICIENTS :

(a) ${}^{n}C_{x} = {}^{n}C_{y} \Rightarrow x = y$ or $x + y = n$

(b) ${}^{n}C_{r-1} + {}^{n}C_{r} = {}^{n+1}C_{r}$

(c) $C_0 + \dfrac{C_1}{2} + \dfrac{C_2}{3} + \ldots\ldots \dfrac{C_n}{n+1} = \dfrac{2^{n+1}-1}{n+1}$

(d) $C_0 - \dfrac{C_1}{2} + \dfrac{C_2}{3} - \dfrac{C_3}{4} \ldots\ldots + \dfrac{(-1)^{n}C_n}{n+1} = \dfrac{1}{n+1}$

(e) $C_0 + C_1 + C_2 + \ldots\ldots = C_n = 2^n$

(f) $C_0 + C_2 + C_4 + \ldots\ldots = C_1 + C_3 + C_5 + \ldots\ldots = 2^{n-1}$

(g) $C_0^{\,2} + C_1^{\,2} + C_2^{\,2} + \ldots\ldots + C_n^{\,2} = {}^{2n}C_n = \dfrac{(2n)!}{n!n!}$

(h) $C_0.C_r + C_1.C_{r+1} + C_2.C_{r+2} + \ldots\ldots + C_{n-r}C_n = \dfrac{(2n)!}{(n+r)!(n-r)!}$

Remember : $(2n)! = 2^n.\ n!\ [1.3.5 \ldots\ldots (2n-1)]$

4. Greatest coefficient & greatest term in expansion of $(x + a)^n$:

(a) If n is even greatest coefficient is ${}^{n}C_{n/2}$

If n is odd greatest coefficient is ${}^{n}C_{\frac{n-1}{2}}$ or ${}^{n}C_{\frac{n+1}{2}}$

(b) For greatest term :

$$\text{Greatest term} = \begin{cases} T_p \ \&\ T_{P+1} \text{ if } \dfrac{n+1}{\left|\frac{x}{a}\right|+1} \text{ is an integer} \\[2ex] T_{q+1} \quad \text{if } \dfrac{n+1}{\left|\frac{x}{a}\right|+1} \text{ is non integer and } \in (q, q+1),\ q \in I \end{cases}$$

5. BINOMIAL THEOREM FOR NEGATIVE OR FRACTIONAL INDICES :

If $n \in Q$, then $(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3$ + ... ∞ provided $|x| < 1$.

Note :

(i) $(1 - x)^{-1} = 1 + x + x^2 + x^3 + \ldots\ldots\ldots\ldots \infty$

(ii) $(1 + x)^{-1} = 1 - x + x^2 - x^3 + \ldots\ldots\ldots\ldots \infty$

(iii) $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots\ldots\ldots \infty$

(iv) $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots\ldots\ldots\ldots \infty$

6. EXPONENTIAL SERIES :

(a) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\infty$; where x may be any real or complex number & $e = \lim\limits_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$

(b) $a^x = 1 + \frac{x}{1!}\ell na + \frac{x^2}{2!}\ell n^2 a + \frac{x^3}{3!}\ell n^3 a + \ldots\ldots\infty$, where $a > 0$.

7. LOGARITHMIC SERIES :

(a) $\ell n\,(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\infty$, where $-1 < x \le 1$

(b) $\ell n\,(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\ldots\infty$, where $-1 \le x < 1$

(c) $\ell n\,\frac{(1+x)}{(1-x)} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\ldots\infty\right)$ $|x| < 1$

COMPLEX NUMBER

1. DEFINITION :

Complex numbers are defined as expressions of the form $a + ib$ where $a, b \in R$ & $i = \sqrt{-1}$. It is denoted by z i.e. $z = a + ib$. 'a' is called real part of z (Re z) and 'b' is called imaginary part of z (Im z).

Every Complex Number Can Be Regarded As

Purely real	Purely imaginary	Imaginary
if $b = 0$	if $a = 0$	if $b \neq 0$

Note :

(i) The set R of real numbers is a proper subset of the Complex Numbers. Hence the Complex Number system is $N \subset W \subset I \subset Q \subset R \subset C$.

(ii) Zero is both purely real as well as purely imaginary but not imaginary.

(iii) $i = \sqrt{-1}$ is called the imaginary unit. Also $i^2 = -1$; $i^3 = -i$; $i^4 = 1$ etc.

(iv) $\sqrt{a}\sqrt{b} = \sqrt{ab}$ only if atleast one of either a or b is non-negative.

2. CONJUGATE COMPLEX :

If $z = a + ib$ then its conjugate complex is obtained by changing the sign of its imaginary part & is denoted by $\bar{z}$. i.e. $\bar{z} = a - ib$.

Note that :

(i) $z + \bar{z} = 2\,\text{Re}(z)$

(ii) $z - \bar{z} = 2i\,\text{Im}(z)$

(iii) $z\bar{z} = a^2 + b^2$ which is real

(iv) If z is purely real then $z - \bar{z} = 0$

(v) If z is purely imaginary then $z + \bar{z} = 0$

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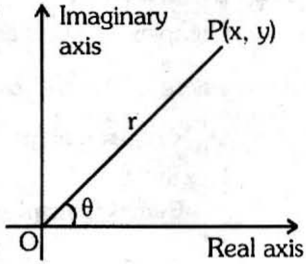
3. REPRESENTATION OF A COMPLEX NUMBER IN VARIOUS FORMS :

(a) Cartesian Form (Geometrical Representation) :

Every complex number $z = x + iy$ can be represented by a point on the cartesian plane known as complex plane (Argand diagram) by the ordered pair (x , y).



Length OP is called **modulus** of the complex number denoted by $|z|$ & θ is called the **argument or amplitude**.

e.g. $|z| = \sqrt{x^2 + y^2}$ & $\theta = \tan^{-1}\dfrac{y}{x}$ (angle made by OP with positive x–axis)

Geometrically $|z|$ represents the distance of point P from origin. ($|z| \geq 0$)

(b) Trigonometric / Polar Representation :

$z = r(\cos\theta + i\sin\theta)$ where $|z| = r$; $\arg z = \theta$; $\bar{z} = r(\cos\theta - i\sin\theta)$

Note : $\cos\theta + i\sin\theta$ is also written as $\text{CiS}\,\theta$.

Euler's formula :

The formula $e^{ix} = \cos x + i\sin x$ is called Euler's formula.

Also $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$ & $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$ are known as Euler's identities.

(c) Exponential Representation :

Let z be a complex number such that $|z| = r$ & $\arg z = \theta$, then $z = r.e^{i\theta}$

4. IMPORTANT PROPERTIES OF CONJUGATE :

(a) $\overline{(\bar{z})} = z$ **(b)** $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(c) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$ **(d)** $\overline{z_1 z_2} = \bar{z}_1 . \bar{z}_2$

(e) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$; $z_2 \neq 0$ **(f)** If $f(\alpha+i\beta) = x + iy \Rightarrow f(\alpha - i\beta) = x - iy$

5. IMPORTANT PROPERTIES OF MODULUS :

(a) $|z| \geq 0$ **(b)** $|z| \geq \text{Re}(z)$ **(c)** $|z| \geq \text{Im}(z)$

(d) $|z| = |\bar{z}| = |-z| = |-\bar{z}|$ **(e)** $z\bar{z} = |z|^2$ **(f)** $|z_1 z_2| = |z_1|.|z_2|$

(g) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$, $z_2 \neq 0$ **(h)** $|z^n| = |z|^n$

(i) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\text{Re}(z_1\bar{z}_2)$

or $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\theta_1 - \theta_2)$

(j) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left[|z_1|^2 + |z_2|^2\right]$

(k) $||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$ **[Triangle Inequality]**

(l) $||z_1| - |z_2|| \leq |z_1 - z_2| \leq |z_1| + |z_2|$ **[Triangle Inequality]**

(m) If $\left|z + \dfrac{1}{z}\right| = 0$ $(a > 0)$, then $\max|z| = \dfrac{a + \sqrt{a^2+4}}{2}$

& $\min|z| = \dfrac{1}{2}\left(\sqrt{a^2+4} - a\right)$

6. IMPORTANT PROPERTIES OF AMPLITUDE :

(a) $\text{amp}(z_1 . z_2) = \text{amp } z_1 + \text{amp } z_2 + 2k\pi$; $k \in I$

(b) $\text{amp}\left(\dfrac{z_1}{z_2}\right) = \text{amp } z_1 - \text{amp } z_2 + 2k\pi$; $k \in I$

(c) $\text{amp}(z^n) = n\,\text{amp}(z) + 2k\pi$, where proper value of k must be chosen so that RHS lies in $(-\pi, \pi]$.

(d) $\log(z) = \log(re^{i\theta}) = \log r + i\theta = \log|z| + i\,\text{amp}(z)$

7. DE'MOIVER'S THEOREM :

The value of $(\cos\theta + i\sin\theta)^n$ is $\cos n\theta + i\sin n\theta$ if 'n' is integer & it is one of the values of $(\cos\theta + i\sin\theta)^n$ if n is a rational number of the form p/q, where p & q are co-prime.

Note : Continued product of roots of a complex quantity should be determined using theory of equation.

8. CUBE ROOT OF UNITY :

(a) The cube roots of unity are 1 , $\omega = \dfrac{-1+i\sqrt{3}}{2} = e^{i2\pi/3}$

& $\omega^2 = \dfrac{-1-i\sqrt{3}}{2} = e^{i4\pi/3}$

(b) $1 + \omega + \omega^2 = 0$, $\omega^3 = 1$, in general

$$1+\omega^r+\omega^{2r} = \begin{cases} 0 & \text{r is not integral multiple of 3} \\ 3 & \text{r is multiple of 3} \end{cases}$$

(c) $a^2 + b^2 + c^2 - ab - bc - ca = (a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$

$a^3 + b^3 = (a + b)(a\omega + b\omega^2)(a\omega^2 + b\omega)$

$a^3 - b^3 = (a - b)(a - \omega b)(a - \omega^2 b)$

$x^2 + x + 1 = (x - \omega)(x - \omega^2)$

9. SQUARE ROOT OF COMPLEX NUMBER :

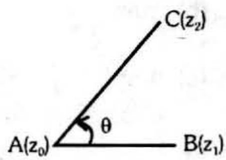
$$\sqrt{a+ib} = \pm\left\{\sqrt{\frac{|z|+a}{2}} + i\sqrt{\frac{|z|-a}{2}}\right\} \text{ for } b > 0$$

& $\pm\left\{\sqrt{\dfrac{|z|+a}{2}} - i\sqrt{\dfrac{|z|-a}{2}}\right\}$ for $b < 0$ where $|z| = \sqrt{a^2+b^2}$.

10. ROTATION :

$$\frac{z_2 - z_0}{|z_2 - z_0|} = \frac{z_1 - z_0}{|z_1 - z_0|}e^{i\theta}$$

Take θ in anticlockwise direction

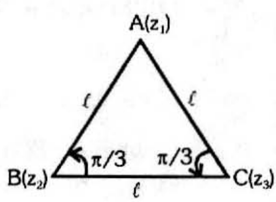


11. RESULT RELATED WITH TRIANGLE :

(a) Equilateral triangle :

$$\frac{z_1 - z_2}{\ell} = \frac{z_3 - z_2}{\ell}e^{i\pi/3} \quad ......(i)$$

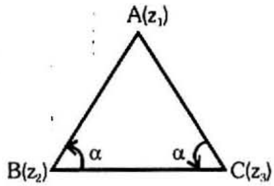
Also $\dfrac{z_2 - z_3}{\ell} = \dfrac{z_1 - z_3}{\ell}.e^{i\pi/3}$ ......(ii)



from (i) & (ii)

$$\Rightarrow \quad z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$

or $$\frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0$$



(b) Isosceles triangle :

$$4\cos^2\alpha\,(z_1 - z_2)(z_3 - z_1) = (z_3 - z_2)^2$$

(c) Area of triangle ΔABC given by modulus of $\frac{1}{4}\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix}$

12. EQUATION OF LINE THROUGH POINTS z_1 & z_2 :

$$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0 \Rightarrow z(\bar{z}_1 - \bar{z}_2) + \bar{z}(z_2 - z_1) + z_1\bar{z}_2 - \bar{z}_1 z_2 = 0$$

$$\Rightarrow \quad z(\bar{z}_1 - \bar{z}_2)i + \bar{z}(z_2 - z_1)i + i(z_1\bar{z}_2 - \bar{z}_1 z_2) = 0$$

Let $(z_2 - z_1)i = a$, then equation of line is $\boxed{\bar{a}z + a\bar{z} + b = 0}$ where $a \in$ C & $b \in$ R.

Note :

(i) Complex slope of line $\bar{a}z + a\bar{z} + b = 0$ is $-\frac{a}{\bar{a}}$

(ii) Two lines with slope μ_1 & μ_2 are parallel or perpendicular if $\mu_1 = \mu_2$ or $\mu_1 + \mu_2 = 0$

(iii) Length of perpendicular from point $A(\alpha)$ to line $\bar{a}z + a\bar{z} + b = 0$ is $\frac{|\bar{a}\alpha + a\bar{\alpha} + b|}{2|a|}$.

13. EQUATION OF CIRCLE :

(a) Circle whose centre is z_0 & radii = r

$$|z - z_0| = r$$

(b) General equation of circle

$z\bar{z}+a\bar{z}+\bar{a}z+b=0$

centre '$-a$' & radii $=\sqrt{|a|^2-b}$

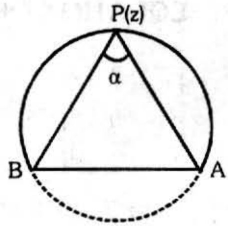
(c) Diameter form $(z-z_1)(\bar{z}-\bar{z}_2)+(z-z_2)(\bar{z}-\bar{z}_1)=0$

or $\arg\left(\dfrac{z-z_1}{z-z_2}\right)=\pm\dfrac{\pi}{2}$

(d) Equation $\left|\dfrac{z-z_1}{z-z_2}\right|=k$ represent a circle if $k\neq 1$ and a straight line if $k=1$.

(e) Equation $|z-z_1|^2+|z-z_2|^2=k$

represent circle if $k\geq\dfrac{1}{2}|z_1-z_2|^2$



(f) $\arg\left(\dfrac{z-z_1}{z-z_2}\right)=\alpha \qquad 0<\alpha<\pi,\ \alpha\neq\dfrac{\pi}{2}$

represent a segment of circle passing through $A(z_1)$ & $B(z_2)$

14. STANDARD LOCI :

(a) $|z-z_1|+|z-z_2|=2k$ (a constant) represent

(i) if $2k>|z_1-z_2| \Rightarrow$ An ellipse

(ii) If $2k=|z_1-z_2| \Rightarrow$ A line segment

(iii) If $2k<|z_1-z_2| \Rightarrow$ No solution

(b) Equation $||z-z_1|-|z-z_2||=2k$ (a constant) represent

(i) If $2k<|z_1-z_2| \Rightarrow$ A hyperbola

(ii) If $2k=|z_1-z_2| \Rightarrow$ A line ray

(iii) $2k>|z_1-z_2| \Rightarrow$ No solution

DETERMINANT

1. MINORS :

The minor of a given element of determinant is the determinant of the elements which remain after deleting the row & the column in which the given element stands.

For example, the minor of a_1 in $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ & the minor of b_2 is $\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$.

Hence a determinant of order three will have " 9 minors".

2. COFACTORS :

If M_{ij} represents the minor of the element belonging to i^{th} row and j^{th} column then the cofactor of that element : $C_{ij} = (-1)^{i+j}.M_{ij}$;

Important Note :

Consider $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Let A_1 be cofactor of a_1, B_2 be cofactor of b_2 and so on, then,

(i) $a_1A_1 + b_1B_1 + c_1C_1 = a_1A_1 + a_2A_2 + a_3A_3 = \ldots\ldots\ldots\ldots = \Delta$

(ii) $a_2A_1 + b_2B_1 + c_2C_1 = b_1A_1 + b_2A_2 + b_3A_3 = \ldots\ldots\ldots\ldots = 0$

3. PROPERTIES OF DETERMINANTS:

(a) The value of a determinants remains unaltered, if the rows & columns are interchanged.

(b) If any two rows (or columns) of a determinant be interchanged, the value of determinant is changed in sign only. e.g.

Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ & $D' = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$ Then $D' = -D$.

(c) If a determinant has any two rows (or columns) identical or in same proportion, then its value is zero.

(d) If all the elements of any row (or columns) be multiplied by the same number, then the determinant is multiplied by that number.

(e) $$\begin{vmatrix} a_1+x & b_1+y & c_1+z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(f) The value of a determinant is not altered by adding to the elements of any row (or column) the same multiples of the corresponding elements of any other row (or column) e.g.

Let $$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$D' = \begin{vmatrix} a_1+ma_2 & b_1+mb_2 & c_1+mc_2 \\ a_2 & b_2 & c_2 \\ a_3+na_1 & b_3+nb_1 & c_3+nc_1 \end{vmatrix}. \text{ Then } D' = D.$$

Note : While applying this property **ATLEAST ONE ROW (OR COLUMN)** must remain unchanged.

(g) If the elements of a determinant Δ are rational function of x and two rows (or columns) become identical when x = a, then x – a is a factor of Δ.

Again, if r rows become identical when a is substituted for x, then $(x-a)^{r-1}$ is a factor of Δ.

(h) If $D(x) = \begin{vmatrix} f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \\ h_1 & h_2 & h_3 \end{vmatrix}$, where f_r, g_r, h_r; r = 1, 2, 3 are three differential function.

then $$\frac{d}{dx}D(x) = \begin{vmatrix} f'_1 & f'_2 & f'_3 \\ g_1 & g_2 & g_3 \\ h_1 & h_2 & h_3 \end{vmatrix} + \begin{vmatrix} f_1 & f_2 & f_3 \\ g'_1 & g'_2 & g'_3 \\ h_1 & h_2 & h_3 \end{vmatrix} + \begin{vmatrix} f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \\ h'_1 & h'_2 & h'_3 \end{vmatrix}$$

4. MULTIPLICATION OF TWO DETERMINANTS :

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix} = \begin{vmatrix} a_1 l_1 + b_1 l_2 & a_1 m_1 + b_1 m_2 \\ a_2 l_1 + b_2 l_2 & a_2 m_1 + b_2 m_2 \end{vmatrix}$$

Similarly two determinants of order three are multiplied.

(a) Here we have multiplied row by column. We can also multiply row by row, column by row and column by column.

(b) If D' is the determinant formed by replacing the elements of determinant D of order n by their corresponding cofactors then $D' = D^{n-1}$

5. SPECIAL DETERMINANTS :

(a) Symmetric Determinant :

Elements of a determinant are such that $a_{ij} = a_{ji}$.

e.g. $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2$

(b) Skew Symmetric Determinant :

If $a_{ij} = -a_{ji}$ then the determinant is said to be a skew symmetric determinant. Here all the principal diagonal elements are zero. The value of a skew symmetric determinant of odd order is zero and of even order is perfect square.

e.g. $\begin{vmatrix} 0 & b & -c \\ -b & 0 & a \\ c & -a & 0 \end{vmatrix} = 0$

(c) Other Important Determinants :

(i) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ac & ab \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

(ii) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

(iii) $\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$

(iv) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^4 & b^4 & c^4 \end{vmatrix} = (a-b)(b-c)(c-a)(a^2+b^2+c^2-ab-bc-ca)$

6. SYSTEM OF EQUATION :

(a) System of equation involving two variable :

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$

- **Consistent** (System of equation has solution)
  - **unique solution**: $\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2}$ or $a_1b_2 - a_2b_1 \neq 0$ (Equations represents intersecting lines)
  - **Infinite solution**: $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$ (Equations represents coincident lines)
- **Inconsistent** (System of equation has no solution): $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} \neq \dfrac{c_1}{c_2}$ (Equations represents parallel disjoint lines)

If $\Delta_1 = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \Delta_2 = \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}, \Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, then $x = \dfrac{\Delta_1}{\Delta}, y = \dfrac{\Delta_2}{\Delta}$

or $\boxed{x = \dfrac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1};\ y = \dfrac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}}$

(b) System of equations involving three variables :

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

To solve this system we first define following determinants

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad \Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix},$$

$$\Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Now following algorithm is used to solve the system.

Check value of Δ

- $\Delta \neq 0$ → Consistent system and has unique solution

  $x = \dfrac{\Delta_1}{\Delta}$, $y = \dfrac{\Delta_2}{\Delta}$, $z = \dfrac{\Delta_3}{\Delta}$

- $\Delta = 0$ → Check the values of Δ_1, Δ_2 and Δ_3
  - Atleast one of Δ_1, Δ_2 and Δ_3 is not zero → Inconsistent system
  - $\Delta_1 = \Delta_2 = \Delta_3 = 0$ → Put z = t and solve any two equations to get the values of x & y in terms of t
    - If these values of x, y and z in terms of t satisfy third equation → Consistent system (System has infinite solutions)
    - The values of x, y, z doesn't satisfy third equation → Inconsistent system

Note :

(i) **Trivial solution :** In the solution set of system of equation if all the variables assumes zero, then such a solution set is called Trivial solution otherwise the solution is called non-trivial solution.

(ii) If $d_1 = d_2 = d_3 = 0$ then system of linear equation is known as system of Homogeneous linear equation which always posses atleast one solution (0, 0, 0).

(iii) If system of homogeneous linear equation posses non-zero/non-trivial solution then $\Delta = 0$.

In such case given system has infinite solutions.

MATRICES

1. INTRODUCTION :

A rectangular array of mn numbers in the form of m horizontal lines (called rows) and n vertical lines (called columns), is called a matrix of order m by n, written as $m \times n$ matrix.

In compact form, the matrix is represented by $A=[a_{ij}]_{m\times n}$.

2. SPECIAL TYPE OF MATRICES :

(a) Row Matrix (Row vector) : $A = [a_{11}, a_{12}, \ldots\ldots\ldots a_{1n}]$ i.e. row matrix has exactly one row.

(b) Column Matrix (Column vector) : $A = \begin{bmatrix} a_{11} \\ a_{21} \\ : \\ a_{m1} \end{bmatrix}$ i.e. column matrix has exactly one column.

(c) Zero or Null Matrix : $(A = O_{m\times n})$, An $m \times n$ matrix whose all entries are zero.

(d) Horizontal Matrix : A matrix of order $m \times n$ is a horizontal matrix if $n > m$.

(e) Vertical Matrix : A matrix of order $m \times n$ is a vertical matrix if $m > n$.

(f) Square Matrix : (Order n) If number of rows = number of column, then matrix is a square matrix.

Note :

(i) The pair of elements a_{ij} & a_{ji} are called Conjugate Elements.

(ii) The elements $a_{11}, a_{22}, a_{33}, \ldots\ldots a_{nn}$ are called Diagonal Elements. The line along which the diagonal elements lie is called "Principal or Leading diagonal. "The quantity Σa_{ii} = trace of the matrix written as, t_r (A)

3. SQUARE MATRICES

SQUARE MATRICES

Triangular Matrix

$$A=\begin{pmatrix}1 & 3 & -2\\ 0 & 2 & 4\\ 0 & 0 & 5\end{pmatrix};\quad B=\begin{pmatrix}1 & 0 & 0\\ 2 & -3 & 0\\ 4 & 3 & 3\end{pmatrix}$$

Upper Triangular: $a_{ij} = 0 \ \forall \ i > j$

Lower Triangular: $a_{ij} = 0 \ \forall \ i < j$

Diagonal Matrix denoted as $A = \text{diag}\,(a_{11}, a_{22}, \ldots\ldots, a_{nn})$ where $a_{ij} = 0$ for $i \neq j$

Scalar Matrix

$$\begin{bmatrix}a & 0 & 0\\ 0 & a & 0\\ 0 & 0 & a\end{bmatrix}$$

If $a_{11} = a_{22} = a_{33} = a$

Unit or Identity Matrix

$$a_{ij} = \begin{cases}1 & \text{if } i = j\\ 0 & \text{if } i \neq j\end{cases}$$

If $a_{11} = a_{22} = a_{33} = 1$

Note :

(i) Minimum number of zeros in triangular matrix of order n $= n(n-1)/2$.

(ii) Minimum number of zeros in a diagonal matrix of order n $= n(n-1)$.

4. EQUALITY OF MATRICES :

Let $A = [a_{ij}]$ & $B = [b_{ij}]$ are equal if,

(a) both have the same order. **(b)** $a_{ij} = b_{ij}$ for each pair of i & j.

5. ALGEBRA OF MATRICES :

Addition : $A + B = [a_{ij} + b_{ij}]$ where A & B are of the same order.

(a) Addition of matrices is commutative : $A + B = B + A$

(b) Matrix addition is associative : $(A + B) + C = A + (B + C)$

6. MULTIPLICATION OF A MATRIX BY A SCALAR :

If $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$, then $kA = \begin{bmatrix} ka & kb & kc \\ kb & kc & ka \\ kc & ka & kb \end{bmatrix}$

7. MULTIPLICATION OF MATRICES (Row by Column) :

Let A be a matrix of order $m \times n$ and B be a matrix of order $p \times q$ then the matrix multiplication AB is possible if and only if $n = p$.

Let $A_{m \times n} = [a_{ij}]$ and $B_{n \times p} = [b_{ij}]$, then order of AB is $m \times p$

& $\boxed{(AB)_{ij} = \sum_{r=1}^{n} a_{ir} b_{rj}}$

8. CHARACTERISTIC EQUATION :

Let A be a square matrix. Then the polynomial $|A - xI|$ is called as characteristic polynomial of A & the equation $|A - xI| = 0$ is called characteristic equation of A.

9. CAYLEY - HAMILTON THEOREM :

Every square matrix A satisfy its characteristic equation

i.e. $a_0x^n + a_1x^{n-1} + \ldots\ldots + a_{n-1}x + a_n = 0$ is the characteristic equation of matrix A, then $a_0A^n + a_1A^{n-1} + \ldots\ldots + a_{n-1}A + a_nI = 0$

10. PROPERTIES OF MATRIX MULTIPLICATION :

(a) $AB = O \not\Rightarrow A = O$ or $B = O$ (in general)

Note :

If A and B are two non-zero matrices such that $AB = O$, then A and B are called the divisors of zero. If A and B are two matrices such that

(i) $AB = BA$ then A and B are said to commute

(ii) $AB = -BA$ then A and B are said to anticommute

(b) Matrix Multiplication Is Associative :

If A, B & C are conformable for the product AB & BC, then $(AB)C = A(BC)$, $(AB)C = A(BC)$

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(c) Distributivity :

$$\left.\begin{array}{l} A(B+C) = AB+AC \\ (A+B)C = AC+BC \end{array}\right]$$ Provided A,B & C are conformable for respective products

11. POSITIVE INTEGRAL POWERS OF A SQUARE MATRIX :

(a) $A^mA^n = A^{m+n}$

(b) $(A^m)^n = A^{mn} = (A^n)^m$

(c) $I^m = I \quad m,n \in N$

12. ORTHOGONAL MATRIX

A square matrix is said to be orthogonal matrix if $A A^T = I$

Note :

(i) The determinant value of orthogonal matrix is either 1 or −1. Hence orthogonal matrix is always invertible

(ii) $AA^T = I = A^TA$ Hence $A^{-1} = A^T$.

13. SOME SQUARE MATRICES :

(a) Idempotent Matrix : A square matrix is idempotent provided $A^2 = A$.

For idempotent matrix note the following :

(i) $A^n = A \quad \forall\ n \geq 2, n \in N$.

(ii) determinant value of idempotent matrix is either 0 or 1

(iii) If idempotent matrix is invertible then its inverse will be identity matrix i.e. I.

(b) Periodic Matrix : A square matrix which satisfies the relation $A^{k+1} = A$, for some positive integer K, is a periodic matrix. The period of the matrix is the least value of K for which this holds true.

Note that period of an idempotent matrix is 1.

(c) Nilpotent Matrix : A square matrix is said to be nilpotent matrix of order m, $m \in N$, if $A^m = O$, $A^{m-1} \neq O$.
Note that a nilpotent matrix will not be invertible.

(d) Involutary Matrix : If $A^2 = I$, the matrix is said to be an involutary matrix.
Note that $A = A^{-1}$ for an involutary matrix.

(e) If A and B are square matrices of same order and AB = BA then
$(A + B)^n = {}^nC_0A^n + {}^nC_1A^{n-1}B + {}^nC_2A^{n-2}B^2 + \ldots\ldots\ldots\ldots + {}^nC_nB^n$

14. TRANSPOSE OF A MATRIX : (Changing rows & columns)

Let A be any matrix of order m × n. Then A^T or $A' = [a_{ij}]$ for $1 \le i \le n$ & $1 \le j \le m$ of order n × m

Properties of transpose :

If A^T & B^T denote the transpose of A and B

(a) $(A+B)^T = A^T+B^T$; note that A & B have the same order.

(b) $(A\,B)^T = B^T A^T$ (Reversal law) A & B are conformable for matrix product AB

(c) $(A^T)^T = A$

(d) $(kA)^T = kA^T$, where k is a scalar.

General : $(A_1 . A_2, \ldots\ldots. A_n)^T = A_n^T . \ldots\ldots . A_2^T . A_1^T$ (reversal law for transpose)

15. SYMMETRIC & SKEW SYMMETRIC MATRIX :

(a) Symmetric matrix :

For symmetric matrix $\mathbf{A = A^T}$.

Note : Maximum number of distinct entries in any symmetric matrix of order n is $\dfrac{n(n+1)}{2}$.

(b) Skew symmetric matrix :

Square matrix $A = [a_{ij}]$ is said to be skew symmetric if $a_{ij} = -a_{ji}\ \forall$ i & j. Hence if A is skew symmetric, then $a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0\ \forall\ i$.

Thus the diagonal elements of a skew square matrix are all zero, but not the converse.

For a skew symmetric matrix $\mathbf{A = -A^T}$.

(c) Properties of symmetric & skew symmetric matrix :

(i) Let A be any square matrix then, $A + A^T$ is a symmetric matrix & $A - A^T$ is a skew symmetric matrix.

(ii) The sum of two symmetric matrix is a symmetric matrix and the sum of two skew symmetric matrix is a skew symmetric matrix.

(iii) If A & B are symmetric matrices then,

(1) AB + BA is a symmetric matrix

(2) AB – BA is a skew symmetric matrix.

(iv) Every square matrix can be uniquely expressed as a sum or difference of a symmetric and a skew symmetric matrix.

$$A = \underbrace{\frac{1}{2}(A + A^T)}_{\text{symmetric}} + \underbrace{\frac{1}{2}(A - A^T)}_{\text{skew symmetric}}$$

and $$A = \frac{1}{2}(A^T + A) - \frac{1}{2}(A^T - A)$$

16. ADJOINT OF A SQUARE MATRIX :

Let $A = [a_{ij}] = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ be a square matrix and let the matrix formed by the cofactors of $[a_{ij}]$ in determinant $|A|$ is

$\begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$. Then (adj A) $= \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$

Note :

If A be a square matrix of order n, then

(i) $A(\text{adj } A) = |A| I_n = (\text{adj } A) . A$

(ii) $|\text{adj } A| = |A|^{n-1}$

(iii) $\text{adj}(\text{adj } A) = |A|^{n-2} A$

(iv) $|\text{adj}(\text{adj } A)| = |A|^{(n-1)^2}$

(v) $\text{adj } (AB) = (\text{adj } B)(\text{adj } A)$

(vi) $\text{adj } (KA) = K^{n-1} (\text{adj } A)$, where K is a scalar

17. INVERSE OF A MATRIX (Reciprocal Matrix) :

A square matrix A said to be invertible (non singular) if there exists a matrix B such that, $AB = I = BA$

B is called the inverse (reciprocal) of A and is denoted by A^{-1}. Thus

$A^{-1} = B \Leftrightarrow AB = I = BA$

We have, $A.(\text{adj } A) = |A| I_n$

$$A^{-1}. A(\text{adj } A) = A^{-1} I_n |A|$$

$$I_n(\text{adj } A) = A^{-1} |A| I_n$$

$$\therefore \quad A^{-1} = \frac{(\text{adj } A)}{|A|}$$

Note : The necessary and sufficient condition for a square matrix A to be invertible is that $|A| \neq 0$

Theorem : If A & B are invertible matrices of the same order, then $(AB)^{-1} = B^{-1}A^{-1}$.

Note :

(i) If A be an invertible matrix, then A^T is also invertible & $(A^T)^{-1} = (A^{-1})^T$.

(ii) If A is invertible, (a) $(A^{-1})^{-1} = A$

(b) $(A^k)^{-1} = (A^{-1})^k = A^{-k}$; $k \in N$

18. SYSTEM OF EQUATION & CRITERIA FOR CONSISTENCY

Gauss - Jordan method :

Example :

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

$$\Rightarrow \begin{bmatrix} a_1x + b_1y + c_1z \\ a_2x + b_2y + c_2z \\ a_3x + b_3y + c_3z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \Rightarrow \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$\Rightarrow AX = B \quad \Rightarrow A^{-1}AX = A^{-1}B$$

$$\Rightarrow X = A^{-1}B = \frac{\text{Adj } A}{|A|}.B$$

Note :

(i) If $|A| \neq 0$, system is consistent having unique solution

(ii) If $|A| \neq 0$ & (adj A) . B $\neq$ O(Null matrix), system is consistent having unique non-trivial solution.

(iii) If $|A| \neq 0$ & (adj A) . B = O (Null matrix), system is consistent having trivial solution.

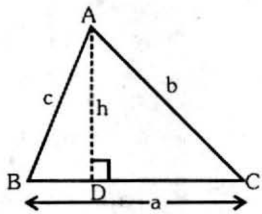
(iv) If $|A| = 0$, then **matrix method fails**

If (adj A) . B = O (null matrix) $\downarrow$ Consistent (infinite solutions)

If (adj A) . B $\neq$ O $\downarrow$ Inconsistent (no solution)

PROPERTIES AND SOLUTIONS OF TRIANGLE

1. SINE FORMULAE :



In any triangle ABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \lambda = \frac{abc}{2\Delta} = 2R$$

where R is circumradius and Δ is area of triangle.

2. COSINE FORMULAE :

(a) $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$ or $a^2 = b^2 + c^2 - 2bc\cos A$

(b) $\cos B = \dfrac{c^2 + a^2 - b^2}{2ca}$

(c) $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

3. PROJECTION FORMULAE :

(a) $b\cos C + c\cos B = a$

(b) $c\cos A + a\cos C = b$

(c) $a\cos B + b\cos A = c$

4. NAPIER'S ANALOGY (TANGENT RULE) :

(a) $\tan\left(\dfrac{B-C}{2}\right) = \dfrac{b-c}{b+c}\cot\dfrac{A}{2}$

(b) $\tan\left(\dfrac{C-A}{2}\right) = \dfrac{c-a}{c+a}\cot\dfrac{B}{2}$

(c) $\tan\left(\dfrac{A-B}{2}\right) = \dfrac{a-b}{a+b}\cot\dfrac{C}{2}$

5. HALF ANGLE FORMULAE :

$$s = \frac{a+b+c}{2} = \text{semi-perimeter of triangle.}$$

(a) (i) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$ **(ii)** $\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$

(iii) $\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$

(b) (i) $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$ **(ii)** $\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$

(iii) $\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$

(c) (i) $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{\Delta}{s(s-a)}$

(ii) $\tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}} = \frac{\Delta}{s(s-b)}$

(iii) $\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = \frac{\Delta}{s(s-c)}$

(d) Area of Triangle

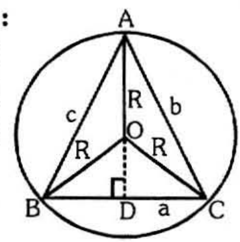
$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \frac{1}{2}ab\sin C$$

$$= \frac{1}{4}\sqrt{2(a^2b^2+b^2c^2+c^2a^2)-a^4-b^4-c^4}$$

6. RADIUS OF THE CIRCUMCIRCLE 'R' :

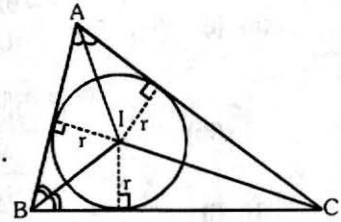
Circumcentre is the point of intersection of perpendicular bisectors of the sides and distance between circumcentre & vertex of triangle is called circumradius 'R'.



$$R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C} = \frac{abc}{4\Delta}.$$

7. RADIUS OF THE INCIRCLE 'r' :

Point of intersection of internal angle bisectors is incentre and perpendicular distance of incentre from any side is called inradius 'r'.



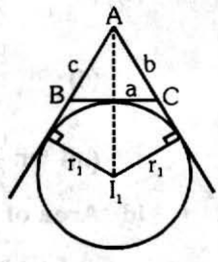
$$r = \frac{\Delta}{s} = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2}$$

$$= (s-c)\tan\frac{C}{2} = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

$$= a\frac{\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}} = b\frac{\sin\frac{A}{2}\sin\frac{C}{2}}{\cos\frac{B}{2}} = c\frac{\sin\frac{B}{2}\sin\frac{A}{2}}{\cos\frac{C}{2}}$$

8. RADII OF THE EX-CIRCLES :

Point of intersection of two external angle and one internal angle bisectors is excentre and perpendicular distance of excentre from any side is called exradius. If r_1 is the radius of escribed circle opposite to angle A of ΔABC and so on then :



(a) $r_1 = \frac{\Delta}{s-a} = s\tan\frac{A}{2} = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$

(b) $r_2 = \frac{\Delta}{s-b} = s\tan\frac{B}{2} = 4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2} = \frac{b\cos\frac{A}{2}\cos\frac{C}{2}}{\cos\frac{B}{2}}$

(c) $r_3 = \frac{\Delta}{s-c} = s\tan\frac{C}{2} = 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2} = \frac{c\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{C}{2}}$

9. LENGTH OF ANGLE BISECTOR, MEDIANS & ALTITUDE :

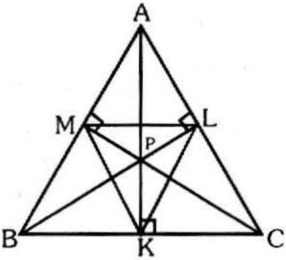
If m_a, β_a & h_a are the lengths of a median, an angle bisector & altitude from the angle A then,

$$\frac{1}{2}\sqrt{b^2+c^2+2bc\cos A}=m_a=\frac{1}{2}\sqrt{2b^2+2c^2-a^2}$$

and $\beta_a=\dfrac{2bc\cos\frac{A}{2}}{b+c}$, $h_a=\dfrac{a}{\cot B+\cot C}$

Note that $m_a^2+m_b^2+m_c^2=\frac{3}{4}(a^2+b^2+c^2)$

10. ORTHOCENTRE AND PEDAL TRIANGLE :



(a) Point of intersection of altitudes is orthocentre & the triangle KLM which is formed by joining the feet of the altitudes is called the pedal triangle.

(b) The distances of the orthocentre from the angular points of the ΔABC are 2R cosA, 2R cosB, & 2R cosC.

(c) The distance of orthocentre from sides are 2R cosB cosC, 2R cosC cosA and 2R cosA cosB

(d) The sides of the pedal triangle are a cos A (= R sin 2A), bcosB(= R sin2B) and c cos C (=R sin 2C) and its angles are $\pi-2A$, $\pi-2B$ and $\pi-2C$

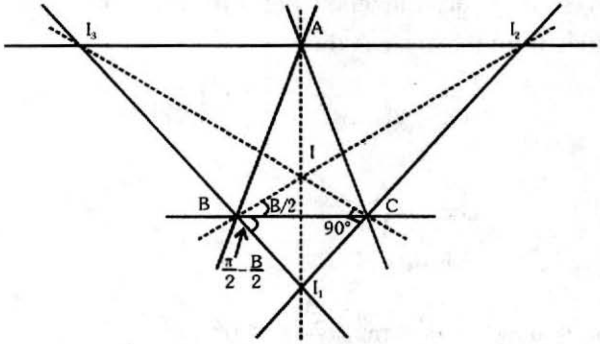
(e) Circumradii of the triangles PBC, PCA, PAB and ABC are equal.

(f) Area of pedal triangle = $2\Delta\cos A\cos B\cos C$

$$=\frac{1}{2}R^2\sin 2A\sin 2B\sin 2C$$

(g) Circumradii of pedal triangle = R/2

11. EX-CENTRAL TRIANGLE :



(a) The triangle formed by joining the three excentres I_1, I_2 and I_3 of Δ ABC is called the excentral or excentric triangle.

(b) Incentre I of ΔABC is the orthocentre of the excentral $\Delta\ I_1I_2I_3$.

(c) ΔABC is the pedal triangle of the $\Delta\ I_1I_2I_3$.

(d) The sides of the excentral triangle are

$$4R\cos\frac{A}{2},\ 4R\cos\frac{B}{2} \text{ and } 4R\cos\frac{C}{2}$$

and its angles are $\frac{\pi}{2}-\frac{A}{2}, \frac{\pi}{2}-\frac{B}{2}$ and $\frac{\pi}{2}-\frac{C}{2}$.

(e) $II_1 = 4R\sin\frac{A}{2}$; $II_2 = 4R\sin\frac{B}{2}$; $II_3 = 4R\sin\frac{C}{2}$.

12. THE DISTANCES BETWEEN THE SPECIAL POINTS :

(a) The distance between circumcentre and orthocentre is $= R\sqrt{1-8\cos A\cos B\cos C}$

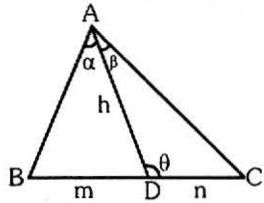
(b) The distance between circumcentre and incentre is $= \sqrt{R^2-2Rr}$.

(c) The distance between incentre and orthocentre is $= \sqrt{2r^2-4R^2\cos A\cos B\cos C}$

(d) The distance between circumcentre & excentre are

$$OI_1 = R\sqrt{1+8\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = \sqrt{R^2+2Rr_1} \text{ \& so on.}$$

13. m-n THEOREM :



$$(m + n) \cot \theta = m \cot \alpha - n \cot \beta$$

$$(m + n) \cot \theta = n \cot B - m \cot C.$$

14. IMPORTANT POINTS :

(a) (i) If $a \cos B = b \cos A$, then the triangle is isosceles.

(ii) If $a \cos A = b\cos B$, then the triangle is isosceles or right angled.

(b) In Right Angle Triangle :

(i) $a^2 + b^2 + c^2 = 8R^2$

(ii) $\cos^2 A + \cos^2 B + \cos^2 C = 1$

(c) In equilateral triangle :

(i) $R = 2r$ **(ii)** $r_1 = r_2 = r_3 = \frac{3R}{2}$

(iii) $r : R : r_1 = 1 : 2 : 3$ **(iv)** area $= \frac{\sqrt{3}a^2}{4}$ **(v)** $R = \frac{a}{\sqrt{3}}$

(d) (i) The circumcentre lies (1) inside an acute angled triangle (2) outside an obtuse angled triangle & (3) mid point of the hypotenuse of right angled triangle.

(ii) The orthocentre of right angled triangle is the vertex at the right angle.

(iii) The orthocentre, centroid & circumcentre are collinear & centroid divides the line segment joining orthocentre & circumcentre internally in the ratio 2 : 1, except in case of equilateral triangle. In equilateral triangle all these centres coincide.

15. REGULAR POLYGON :

Consider a 'n' sided regular polygon of side length 'a'

(a) Radius of incircle of this polygon $r = \frac{a}{2}\cot\frac{\pi}{n}$

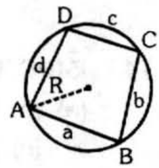
(b) Radius of circumcircle of this polygon $R = \frac{a}{2}\operatorname{cosec}\frac{\pi}{n}$

(c) Perimeter & area of regular polygon

Perimeter $= na = 2nr\tan\frac{\pi}{n} = 2nR\sin\frac{\pi}{n}$

Area $= \frac{1}{2}nR^2\sin\frac{2\pi}{n} = nr^2\tan\frac{\pi}{n} = \frac{1}{4}na^2\cot\frac{\pi}{n}$

16. CYCLIC QUADRILATERAL :



(a) Quadrilateral ABCD is cyclic if $\angle A + \angle C = \pi = \angle B + \angle C$

(opposite angle are supplementary angles)

(b) Area $= \sqrt{(s-a)(s-b)(s-c)(s-d)}$, where $2s = a + b + c + d$

(c) $\cos B = \dfrac{a^2+b^2-c^2-d^2}{2(ab+cd)}$ & similarly other angles

(d) Ptolemy's theorem : If ABCD is cyclic quadrilateral, then $AC \cdot BD = AB \cdot CD + BC \cdot AD$

17. SOLUTION OF TRIANGLE :

Case-I : Three sides are given then to find out three angles use

$\cos A = \dfrac{b^2+c^2-a^2}{2bc}$, $\cos B = \dfrac{c^2+a^2-b^2}{2ac}$ & $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$

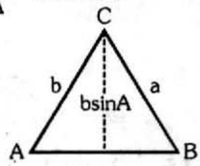
Case-II : Two sides & included angle are given :

Let sides a, b & angle C are given then use $\tan\dfrac{A-B}{2} = \dfrac{a-b}{a+b}\cot\dfrac{C}{2}$

and find value of $A - B$ ....(i)

& $\dfrac{A+B}{2} = 90° - \dfrac{C}{2}$ ....(ii) $c = \dfrac{a\sin C}{\sin A}$ ....(iii)

Case-III :



Two sides a, b & angle A opposite to one of them is given

(a) If $a < b\sin A$ No triangle exist

(b) If $a = b\sin A$ & A is acute, then one triangle exist which is right angled.

(c) $a > b\sin A$, $a < b$ & A is acute, then two triangles exist

(d) $a > b\sin A$, $a > b$ & A is acute, then one triangle exist

(e) $a > b\sin A$ & A is obtuse, then there is one triangle if $a > b$ & no triangle if $a < b$.

18. ANGLES OF ELEVATION AND DEPRESSION :

Let OP be a horizontal line in the vertical plane in which an object R is given and let OR be joined.

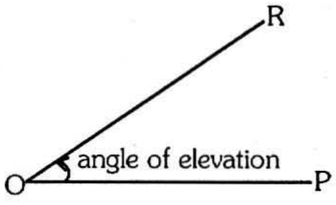


Fig. (a)

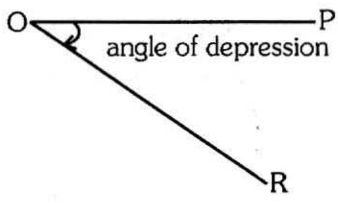


Fig. (b)

In Fig. (a), where the object R is above the horizontal line OP, the angle POR is called the angle of elevation of the object R as seen from the point O. In Fig. (b) where the object R is below the horizontal line OP, the angle POR is called the angle of depression of the object R as seen from the point O.

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STRAIGHT LINE

1. RELATION BETWEEN CARTESIAN CO-ORDINATE & POLAR CO-ORDINATE SYSTEM

If (x,y) are cartesian co-ordinates of a point P, then : $x = r\cos\theta$, $y = r\sin\theta$

and $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

2. DISTANCE FORMULA AND ITS APPLICATIONS :

If $A(x_1,y_1)$ and $B(x_2,y_2)$ are two points, then

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Note :

(i) Three given points A,B and C are collinear, when sum of any two distances out of AB,BC, CA is equal to the remaining third otherwise the points will be the vertices of triangle.

(ii) Let A,B,C & D be the four given points in a plane. Then the quadrilateral will be :

(a) Square if AB = BC = CD = DA & AC = BD ; $AC \perp BD$

(b) Rhombus if AB = BC = CD = DA and AC ≠ BD; $AC \perp BD$

(c) Parallelogram if AB = DC, BC = AD; AC ≠ BD; $AC \not\perp BD$

(d) Rectangle if AB = CD, BC = DA, AC = BD ; $AC \not\perp BD$

3. SECTION FORMULA :

The co-ordinates of a point dividing a line joining the points $A(x_1,y_1)$ and $B(x_2,y_2)$ in the ratio m : n is given by :

(a) For internal division : $\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$

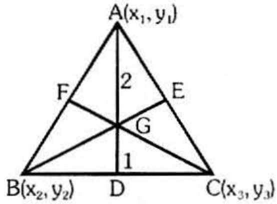
(b) For external division : $\left(\dfrac{mx_2 - nx_1}{m-n}, \dfrac{my_2 - ny_1}{m-n}\right)$

(c) Line $ax + by + c = 0$ divides line joining points $P(x_1 y_1)$ & $Q(x_2, y_2)$ in ratio $= -\dfrac{(ax_1 + by_1 + c)}{(ax_2 + by_2 + c)}$

4. CO-ORDINATES OF SOME PARTICULAR POINTS :

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are vertices of any triangle ABC, then

(a) Centroid :



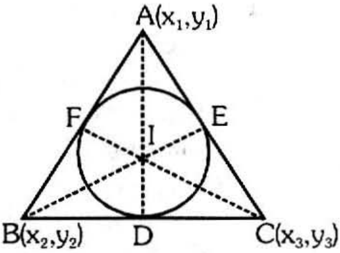
(i) The centroid is the point of intersection of the medians (line joining the mid point of sides and opposite vertices).

(ii) Centroid divides the median in the ratio of 2 : 1.

(iii) Co-ordinates of centroid $G\left(\dfrac{x_1 + x_2 + x_3}{3}, \dfrac{y_1 + y_2 + y_3}{3}\right)$

(iv) If P is any internal point of triangle such that area of ΔAPB, ΔAPC and ΔBPC are same then P must be centroid.

(b) Incenter :



The incenter is the point of intersection of internal bisectors of the angles of a triangle. Also it is a centre of a circle touching all the sides of a triangle.

Co-ordinates of incenter $I\left(\dfrac{ax_1 + bx_2 + cx_3}{a+b+c}, \dfrac{ay_1 + by_2 + cy_3}{a+b+c}\right)$

Where a, b, c are the sides of triangle ABC.

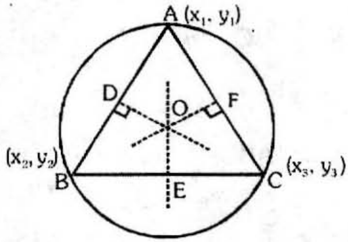
Note :

(i) Angle bisector divides the opposite sides in the ratio of remaining sides. e.g. $\frac{BD}{DC}=\frac{AB}{AC}=\frac{c}{b}$

(ii) Incenter divides the angle bisectors in the ratio $(b+c):a,(c+a):b,(a+b):c$

(c) Circumcenter :

It is the point of intersection of perpendicular bisectors of the sides of a triangle. If O is the circumcenter of any triangle ABC, then $OA^2=OB^2=OC^2$.



Also it is a centre of a circle touching all the vertices of a triangle.

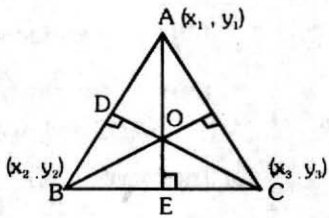
Note :

(i) If a triangle is right angle, then its circumcenter is mid point of hypotenuse.

(ii) Find perpendicular bisector of any two sides and solve them to find circumcentre.

(d) Orthocenter :

It is the point of intersection of perpendicular drawn from vertices on opposite sides of a triangle and can be obtained by solving the equation of any two altitudes.



Note :

If a triangle is right angled triangle, then orthocenter is the point where right angle is formed.

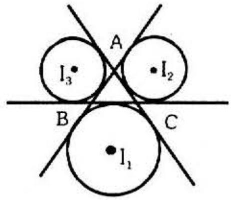
Remarks :

(i) If the triangle is equilateral, then centroid, incentre, orthocenter, circumcenter, coincides.

(ii) Orthocentre, centroid and circumcentre are always collinear and centroid divides the line joining. orthocentre and circumcentre in the ratio 2 : 1

(iii) In an isosceles triangle centroid, orthocentre, incentre, circumcentre lies on the same line.

(e) Ex-centers :



The centre of the circle which touches side BC and the extended portions of sides AB and AC is called the ex-centre of ΔABC with respect to the vertex A. It is denoted by I_1 and its coordinates are

$$I_1 = \left(\frac{-ax_1 + bx_2 + cx_3}{-a+b+c}, \frac{-ay_1 + by_2 + cy_3}{-a+b+c}\right)$$

Similarly ex-centers of ΔABC with respect to vertices B and C are denoted by I_2 and I_3 respectively, and

$$I_2 = \left(\frac{ax_1 - bx_2 + cx_3}{a-b+c}, \frac{ay_1 - by_2 + cy_3}{a-b+c}\right),$$

$$I_3 = \left(\frac{ax_1 + bx_2 - cx_3}{a+b-c}, \frac{ay_1 + by_2 - cy_3}{a+b-c}\right)$$

5. AREA OF TRIANGLE :

Let $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ are vertices of a triangle, then

$$\text{Area of } \Delta ABC = \left|\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right| = \frac{1}{2}\,|[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)]|$$

To remember the above formula, take the help of the following method :

$$= \frac{1}{2}\begin{bmatrix} x_1 & & x_2 & & x_3 & & x_1 \\ & \times & & \times & & \times & \\ y_1 & & y_2 & & y_3 & & y_1 \end{bmatrix} = \left|\frac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)]\right|$$

Remarks :

(i) If the area of triangle joining three points is zero, then the points are collinear.

(ii) Area of Equilateral triangle

If altitude of any equilateral triangle is P, then its area $= \frac{P^2}{\sqrt{3}}$.

If 'a' be the side of equilateral triangle, then its area $= \left(\frac{a^2\sqrt{3}}{4}\right)$

(iii) Area of quadrilateral whose consecutive vertices are (x_1, y_1), (x_2, y_2), (x_3, y_3) & (x_4, y_4) is $\frac{1}{2}\begin{vmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_4 & y_2 - y_4 \end{vmatrix}$

6. CONDITION OF COLLINEARITY FOR THREE POINTS :

Three points (x_1,y_1), (x_2,y_2) and (x_3,y_3) are collinear if any one of the given point lies on the line passing through the remaining two points. Thus the required condition is -

$$\frac{y_3 - y_1}{y_2 - y_1} = \frac{x_3 - x_1}{x_2 - x_1} \quad \text{or} \quad \frac{x_1 - x_2}{x_1 - x_3} = \frac{y_1 - y_2}{y_1 - y_3} \quad \text{or} \quad \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

7. EQUATION OF STRAIGHT LINE :

A relation between x and y which is satisfied by co-ordinates of every point lying on a line is called equation of the straight line. Here remember that every one degree equation in variable x and y always represents a straight line i.e. $ax + by + c = 0$; a & b $\neq 0$ simultaneously.

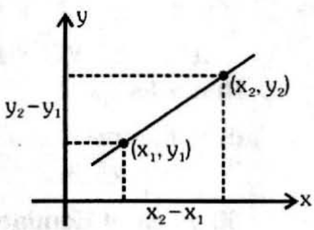
(a) Equation of a line parallel to x-axis at a distance a is $y = a$ or $y = -a$

(b) Equation of x-axis is $y = 0$

(c) Equation of line parallel to y-axis at a distance b is $x = b$ or $x = -b$

(d) Equation of y-axis is $x = 0$

8. SLOPE OF LINE :



If a given line makes an angle θ $(0° \leq \theta < 180°,\ \theta \neq 90°)$ with the positive direction of x-axis, then slope of this line will be $\tan\theta$ and is usually denoted by the letter **m** i.e. $m = \tan\theta$. Obviously the slope of the x-axis and line parallel to it is zero and y-axis and line parallel to it does not exist.

If $A(x_1, y_1)$ and $B(x_2, y_2)$ & $x_1 \neq x_2$ then slope of line AB $= \frac{y_2 - y_1}{x_2 - x_1}$

9. STANDARD FORMS OF EQUATIONS OF A STRAIGHT LINE :

(a) Slope Intercept form : Let m be the slope of a line and c its intercept on y-axis. then the equation of this straight line is written as : $\mathbf{y = mx + c}$

(b) Point Slope form : If m be the slope of a line and it passes through a point (x_1,y_1), then its equation is written as : $\mathbf{y - y_1 = m(x-x_1)}$

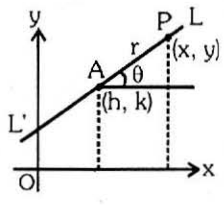
(c) Two point form : Equation of a line passing through two points (x_1,y_1) and (x_2,y_2) is written as :

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \text{ or } \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

(d) Intercept form : If a and b are the intercepts made by a line on the axes of x and y, its equation is written as : $\mathbf{\frac{x}{a} + \frac{y}{b} = 1}$

(e) Normal form : If p is the length of perpendicular on a line from the origin and α the angle which this perpendicular makes with positive x-axis, then the equation of this line is written as : $\mathbf{x\cos\alpha + y\sin\alpha = p}$ (p is always positive), where $0 \le \alpha < 2\pi$.

(f) Parametric form : To find the equation of a straight line which passes through a given point A(h, k) and makes a given angle θ with the positive direction of the x-axis. P(x, y) is any point on the line LAL'. Let AP = r then $x - h = r\cos\theta$, $y - k = r\sin\theta$ & $\frac{x-h}{\cos\theta} = \frac{y-k}{\sin\theta} = r$ is the equation of the straight line LAL'.



Any point P on the line will be of the form $(h + r\cos\theta, k + r\sin\theta)$, where $|r|$ gives the distance of the point P from the fixed point (h, k).

(g) General form : We know that a first degree equation in x and y, **ax + by + c = 0** always represents a straight line. This form is known as general form of straight line.

(i) Slope of this line $= \frac{-a}{b} = -\frac{\text{coeff. of x}}{\text{coeff. of y}}$

(ii) Intercept by this line on x-axis $= -\frac{c}{a}$ and intercept by this line on y-axis $= -\frac{c}{b}$

(iii) To change the general form of a line to normal form, first take c to right hand side and make it positive, then divide the whole equation by $\sqrt{a^2+b^2}$.

10. ANGLE BETWEEN TWO LINES :

(a) If θ be the angle between two lines : $y = m_1x + c$ and $y = m_2x + c_2$, then $\tan\theta = \pm\left(\frac{m_1 - m_2}{1 + m_1m_2}\right)$

(b) If equation of lines are $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$, then these line are -

(i) Parallel $\Leftrightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

(ii) Perpendicular $\Leftrightarrow a_1a_2+b_1b_2=0$

(iii) Coincident $\Leftrightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

(iv) Intersecting $\Leftrightarrow \frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

11. LENGTH OF PERPENDICULAR FROM A POINT ON A LINE :

Length of perpendicular from a point (x_1, y_1) on the line $ax+by+c=0$ is $= \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2+b^2}}\right|$

In particular the length of the perpendicular from the origin on the line $ax + by + c = 0$ is $P = \frac{|c|}{\sqrt{a^2+b^2}}$

12. DISTANCE BETWEEN TWO PARALLEL LINES :

(a) The distance between two parallel lines $ax + by + c_1=0$ and $ax+by+c_2=0$ is $= \frac{|c_1 - c_2|}{\sqrt{a^2+b^2}}$

(**Note :** The coefficients of x & y in both equations should be same)

(b) The area of the parallelogram $= \frac{p_1 p_2}{\sin\theta}$, where p_1 & p_2 are distances between two pairs of opposite sides & θ is the angle between any two adjacent sides. Note that area of the parallelogram bounded by the lines $y = m_1x + c_1$, $y = m_1x + c_2$ and $y = m_2x + d_1$, $y = m_2x + d_2$ is given by $\left|\frac{(c_1-c_2)(d_1-d_2)}{m_1-m_2}\right|$.

13. EQUATION OF LINES PARALLEL AND PERPENDICULAR TO A GIVEN LINE :

(a) Equation of line parallel to line $ax + by + c = 0$

$ax + by + \lambda = 0$

(b) Equation of line perpendicular to line $ax + by + c = 0$

$bx - ay + k = 0$

Here λ, k, are parameters and their values are obtained with the help of additional information given in the problem.

14. STRAIGHT LINE MAKING A GIVEN ANGLE WITH A LINE :

Equations of lines passing through a point (x_1,y_1) and making an angle α, with the line $y=mx+c$ is written as :

$$\mathbf{y - y_1 = \frac{m \pm \tan\alpha}{1 \mp m\tan\alpha}(x - x_1)}$$

15. POSITION OF TWO POINTS WITH RESPECT TO A GIVEN LINE :

Let the given line be $ax + by + c = 0$ and $P(x_1, y_1)$, $Q(x_2, y_2)$ be two points. If the quantities $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have the same signs, then both the points P and Q lie on the same side of the line $ax + by + c = 0$. If the quantities $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have opposite signs, then they lie on the opposite sides of the line.

16. CONCURRENCY OF LINES :

Three lines $a_1x + b_1y + c_1 = 0$; $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$

are concurrent, if $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

Note :

If lines are concurrent then $\Delta = 0$ but if $\Delta = 0$ then lines may or may not be concurrent {lines may be parallel}.

17. REFLECTION OF A POINT :

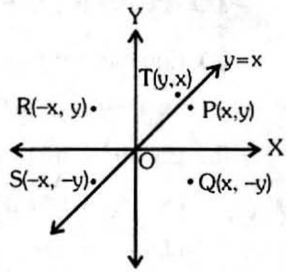
Let P(x,y) be any point, then its image with respect to

(a) x-axis is Q(x, −y)

(b) y-axis is R(−x, y)

(c) origin is S(−x, −y)

(d) line y=x is T(y, x)



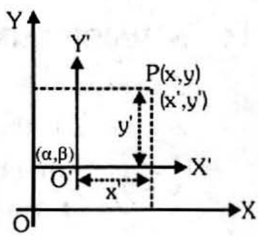
18. TRANSFORMATION OF AXES

(a) Shifting of origin without rotation of axes :

If coordinates of any point P(x, y) with respect to new origin (α, β) will be (x', y')

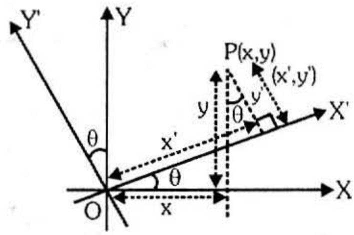
then $x = x' + \alpha$, $\quad y = y' + \beta$

or $\quad x' = x - \alpha$, $\quad y' = y - \beta$



Thus if origin is shifted to point (α, β) without rotation of axes, then new equation of curve can be obtained by putting $x + \alpha$ in place of x and $y + \beta$ in place of y.

(b) Rotation of axes without shifting the origin :



Let O be the origin. Let $P \equiv (x, y)$ with respect to axes OX and OY and let $P \equiv (x', y')$ with respect to axes OX' and OY', where $\angle X'OX = \angle YOY' = \theta$

then $x = x' \cos\theta - y' \sin\theta$

$y = x' \sin\theta + y' \cos\theta$

and $x' = x \cos\theta + y \sin\theta$

$y' = -x \sin\theta + y \cos\theta$

The above relation between (x, y) and (x', y') can be easily obtained with the help of following table

Old / New	$x \downarrow$	$y \downarrow$
$x' \rightarrow$	$\cos\theta$	$\sin\theta$
$y' \rightarrow$	$-\sin\theta$	$\cos\theta$

19. EQUATION OF BISECTORS OF ANGLES BETWEEN TWO LINES :

If equation of two intersecting lines are $a_1x+b_1y+c_1=0$ and $a_2x + b_2y+c_2=0$, then equation of bisectors of the angles between these lines are written as :

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = \pm\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}} \qquad \text{.........(1)}$$

(a) Equation of bisector of angle containing origin :

If the equation of the lines are written with constant terms c_1 and c_2 positive, then the equation of the bisectors of the angle containing the origin is obtained by taking positive sign in (1)

(b) Equation of bisector of acute/obtuse angles :

See whether the constant terms c_1 and c_2 in the two equation are +ve or not. If not then multiply both sides of given equation by -1 to make the constant terms positive

Determine the sign of $a_1a_2 + b_1b_2$

If sign of $a_1a_2 + b_1b_2$	For obtuse angle bisector	For acute angle bisector
+	use + sign in eq. (1)	use – sign in eq. (1)
–	use – sign in eq. (1)	use + sign in eq. (1)

i.e. if $a_1a_2 + b_1b_2 > 0$, then the bisector corresponding to + sign gives obtuse angle bisector

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$

20. FAMILY OF LINES :

If equation of two lines be $P \equiv a_1x + b_1y + c_1=0$ and $Q \equiv a_2x+b_2y + c_2= 0$, then the equation of the lines passing through the point of intersection of these lines is : $P + \lambda Q = 0$ or $a_1x + b_1y + c_1 + \lambda (a_2x + b_2y + c_2) = 0$. The value of λ is obtained with the help of the additional informations given in the problem.

21. GENERAL EQUATION AND HOMOGENEOUS EQUATION OF SECOND DEGREE :

(a) A general equation of second degree $\mathbf{ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0}$ represent a pair of straight lines if $\Delta=abc+2fgh-af^2 - bg^2 - ch^2 = 0$ or

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0$$

(b) If θ be the angle between the lines, then $\tan\theta=\pm\frac{2\sqrt{h^2-ab}}{a+b}$

Obviously these lines are

(i) Parallel, if $\Delta = 0$, $h^2 = ab$ or if $h^2 = ab$ and $bg^2 = af^2$

(ii) Perpendicular, if $a + b =0$ i.e. coeff. of x^2 + coeff. of $y^2 = 0$.

(c) Homogeneous equation of 2[nd] degree $\mathbf{ax^2 + 2hxy + by^2 = 0}$ always represent a pair of straight lines whose equations are

$$y=\left(\frac{-h\pm\sqrt{h^2-ab}}{b}\right)x \equiv y=m_1x \;\&\; y=m_2x$$

and $m_1+m_2=-\dfrac{2h}{b}$; $m_1m_2=\dfrac{a}{b}$

These straight lines passes through the origin and for finding the angle between these lines same formula as given for general equation is used.

The condition that these lines are :

(i) At right angles to each other is $a+b=0$. i.e. co-efficient of x^2 + co-efficient of $y^2=0$.

(ii) Coincident is $h^2=ab$.

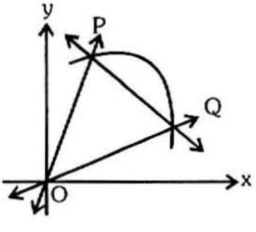
(iii) Equally inclined to the axis of x is $h=0$. i.e. coeff. of $xy=0$.

(d) The combined equation of angle bisectors between the lines represented by homogeneous equation of 2nd degree is given by $\dfrac{x^2-y^2}{a-b}=\dfrac{xy}{h}$, $a\neq b$, $h\neq 0$.

(e) Pair of straight lines perpendicular to the lines $ax^2+2hxy+by^2=0$ and through origin are given by $bx^2-2hxy+ay^2=0$.

(f) If lines $ax^2+2hxy+by^2+2gx+2fy+c=0$ are parallel then distance between them is $=2\sqrt{\dfrac{g^2-ac}{a(a+b)}}$

22. EQUATIONS OF LINES JOINING THE POINTS OF INTERSECTION OF A LINE AND A CURVE TO THE ORIGIN :



Let the equation of curve be :

$ax^2+2hxy+by^2+2gx+2fy+c=0$ ......(i)

and straight line be

$\ell x+my+n=0$ ......(ii)

Now joint equation of line OP and OQ joining the origin and points of intersection P and Q can be obtained by making the equation (i) homogenous with the help of equation of the line. Thus required equation is given by -

$$ax^2+2hxy+by^2+2(gx+fy)\left(\frac{\ell x+my}{-n}\right)+c\left(\frac{\ell x+my}{-n}\right)^2=0$$

23. STANDARD RESULTS :

(a) Area of rhombus formed by lines $a|x|+b|y|+c=0$ or $\pm ax \pm by + c = 0$ is $\frac{2c^2}{|ab|}$.

(b) Area of triangle formed by line $ax+by+c=0$ and axes is $\frac{c^2}{2|ab|}$.

(c) Co-ordinate of foot of perpendicular (h, k) from (x_1, y_1) to the line $ax+by+c=0$ is given by $\frac{h-x_1}{a}=\frac{k-y_1}{b}=\frac{-(ax_1+by_1+c)}{a^2+b^2}$

(d) Image of point (x_1, y_1) w.r.t. to the line $ax+by+c=0$ is given by

$$\frac{h-x_1}{a}=\frac{k-y_1}{b}=\frac{-2(ax_1+by_1+c)}{a^2+b^2}$$

CIRCLE

1. DEFINITION :

A circle is the locus of a point which moves in a plane in such a way that its distance from a fixed point remains constant. The fixed point is called the centre of the circle and the constant distance is called the radius of the circle.

2. STANDARD EQUATIONS OF THE CIRCLE :

(a) Central Form :

If (h, k) is the centre and r is the radius of the circle then its equation is $\mathbf{(x-h)^2+(y-k)^2=r^2}$

(b) General equation of circle :

$\mathbf{x^2+y^2+2gx+2fy+c=0}$, where g, f, c are constants and centre is $(-g, -f)$

i.e. $\left(-\frac{\text{coefficient of } x}{2}, -\frac{\text{coefficient of } y}{2}\right)$

and radius $r=\sqrt{g^2+f^2-c}$

Note : The general quadratic equation in x and y, $ax^2+by^2+2hxy+2gx+2fy+c=0$ represents a circle if :

(i) coefficient of x^2 = coefficient of y^2 or $a=b\neq 0$

(ii) coefficient of $xy=0$ or $h=0$

(iii) $(g^2+f^2-c)\geq 0$ (for a real circle)

(c) Intercepts cut by the circle on axes :

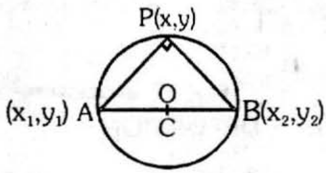
The intercepts cut by the circle $x^2+y^2+2gx+2fy+c=0$ on:

(i) x-axis $=2\sqrt{g^2-c}$ (ii) y-axis $=2\sqrt{f^2-c}$

Note :

Intercept cut by a line on the circle $x^2+y^2+2gx+2fy+c=0$ or length of chord of the circle $=2\sqrt{a^2-P^2}$ where a is the radius and P is the length of perpendicular from the centre to the chord.

(d) Diameter form of circle :



If $A(x_1,y_1)$ and $B(x_2,y_2)$ are the end points of the diameter of the circle then the equation of the circle is given by

$$\mathbf{(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0}$$

(e) The parametric forms of the circle :

(i) The parametric equation of the circle $x^2+y^2 = r^2$ are

$$\mathbf{x = r\cos\theta,\ y = r\sin\theta\ ;\ \theta \in [0, 2\pi)}$$

(ii) The parametric equation of the circle $(x-h)^2 + (y-k)^2 = r^2$ is

$\mathbf{x = h + r\cos\theta,\ y = k + r\sin\theta}$ where θ is parameter.

(iii) The parametric equation of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$

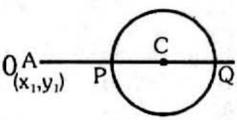
are $\mathbf{x = -g + \sqrt{g^2+f^2-c}\ \cos\theta,\ y = -f + \sqrt{g^2+f^2-c}\ \sin\theta}$

where θ is parameter.

Note that equation of a straight line joining two point α & β on the circle $x^2 + y^2 = a^2$ is

$$x\cos\frac{\alpha+\beta}{2} + y\sin\frac{\alpha+\beta}{2} = a\cos\frac{\alpha-\beta}{2}.$$

3. POSITION OF A POINT W.R.T CIRCLE :



(a) Let the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$ and the point is (x_1,y_1) then :

Point (x_1,y_1) lies out side the circle or on the circle or inside the circle according as

$\Rightarrow x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c >, =, < 0$ or $S_1 >, =, < 0$

(b) The greatest & the least distance of a point A from a circle with centre C & radius r is AC + r & AC − r respectively.

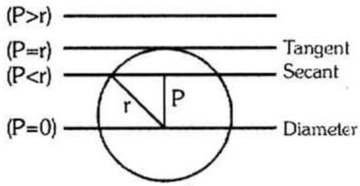
(c) The power of point is given by S_1.

4. TANGENT LINE OF CIRCLE :

When a straight line meet a circle on two coincident points then it is called the tangent of the circle.

(a) Condition of Tangency :

The line L = 0 touches the circle S = 0 if P the length of the perpendicular from the centre to that line and radius of the circle r are equal i.e. P = r.



(b) Equation of the tangent (T = 0) :

(i) Tangent at the point (x_1,y_1) on the circle $x^2+y^2=a^2$ is $\mathbf{xx_1 + yy_1 = a^2}$.

(ii) (1) The tangent at the point (acos t, asin t) on the circle $x^2+y^2=a^2$ is $\mathbf{x\cos t + y\sin t = a}$

(2) The point of intersection of the tangents at the points $P(\alpha)$ and $Q(\beta)$ is $\left(\dfrac{a\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, \dfrac{a\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$.

(iii) The equation of tangent at the point (x_1,y_1) on the circle $x^2+y^2+2gx+2fy+c=0$ is

$$\mathbf{xx_1 + yy_1 + g(x+x_1) + f(y+y_1) + c = 0}$$

(iv) If line y = mx + c is a straight line touching the circle $x^2+y^2=a^2$, then $c=\pm a\sqrt{1+m^2}$ and contact points are $\left(\mp\dfrac{am}{\sqrt{1+m^2}}, \pm\dfrac{a}{\sqrt{1+m^2}}\right)$ or $\left(\mp\dfrac{a^2m}{c}, \pm\dfrac{a^2}{c}\right)$ and equation of tangent is

$$\mathbf{y = mx \pm a\sqrt{1+m^2}}$$

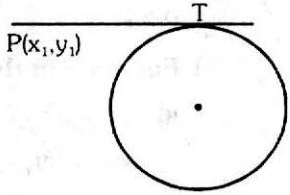
(v) The equation of tangent with slope m of the circle $(x-h)^2+(y-k)^2=a^2$ is

$$\mathbf{(y-k) = m(x-h) \pm a\sqrt{1+m^2}}$$

Note :

To get the equation of tangent at the point (x_1, y_1) on any curve we replace xx_1 in place of x^2, yy_1 in place of y^2, $\frac{x+x_1}{2}$ in place of x, $\frac{y+y_1}{2}$ in place of y, $\frac{xy_1+yx_1}{2}$ in place of xy and c in place of c.

(c) Length of tangent ($\sqrt{S_1}$) :



The length of tangent drawn from point (x_1,y_1) out side the circle

$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is,

$PT = \sqrt{S_1} = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$

(d) Equation of Pair of tangents ($SS_1 = T^2$) :

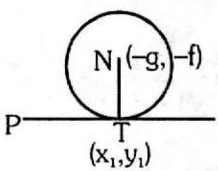
Let the equation of circle $S \equiv x^2 + y^2 = a^2$ and $P(x_1,y_1)$ is any point outside the circle. From the point we can draw two real and distinct tangent PQ & PR and combine equation of pair of tangents is -

$\mathbf{(x^2 + y^2 - a^2)(x_1^2 + y_1^2 - a^2) = (xx_1 + yy_1 - a^2)^2}$ **or** $\mathbf{SS_1 = T^2}$

5. NORMAL OF CIRCLE :

Normal at a point of the circle is the straight line which is perpendicular to the tangent at the point of contact and passes through the centre of circle.

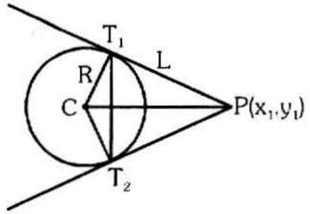
(a) Equation of normal at point (x_1,y_1) of circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is



$$y - y_1 = \left(\frac{y_1+f}{x_1+g}\right)(x - x_1)$$

(b) The equation of normal on any point (x_1,y_1) of circle $x^2 + y^2 = a^2$ is $\left(\frac{y}{x} = \frac{y_1}{x_1}\right)$.

6. CHORD OF CONTACT :



If two tangents PT_1 & PT_2 are drawn from the point $P(x_1, y_1)$ to the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, then the equation of the chord of contact T_1T_2 is :

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ (i.e. $T = 0$ same as equation of tangent).

7. EQUATION OF THE CHORD WITH A GIVEN MIDDLE POINT ($T = S_1$) :

The equation of the chord of the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ in terms of its mid point $M(x_1, y_1)$ is $\mathbf{y - y_1 = -\dfrac{x_1 + g}{y_1 + f}(x - x_1)}$.

This on simplication can be put in the form

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

which is designated by $\mathbf{T = S_1}$.

8. DIRECTOR CIRCLE :

The locus of point of intersection of two perpendicular tangents to a circle is called director circle. Let the circle be $x^2 + y^2 = a^2$, then the equation of director circle is $x^2 + y^2 = 2a^2$.

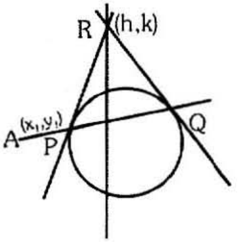
$\therefore$ director circle is a concentric circle whose radius is $\sqrt{2}$ times the radius of the circle.

Note :

The director circle of

$x^2 + y^2 + 2gx + 2fy + c = 0$ is $x^2 + y^2 + 2gx + 2fy + 2c - g^2 - f^2 = 0$

9. POLE AND POLAR :



Let any straight line through the given point $A(x_1,y_1)$ intersect the given circle $S = 0$ in two points P and Q and if the tangent of the circle at P and Q meet at the point R then locus of point R is called polar of the point A and point A is called the pole, with respect to the given circle.

The equation of the polar is the T=0, so the polar of point (x_1,y_1) w.r.t circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $xx_1+yy_1+g(x+x_1) + f(y+y_1)+c=0$

Pole of a given line with respect to a circle

To find the pole of a line we assume the coordinates of the pole then from these coordinates we find the polar. This polar and given line represent the same line. Then by comparing the coefficients of similar terms we can get the coordinates of the pole. The pole of $\ell x + my + n = 0$

w.r.t. circle $x^2 + y^2 = a^2$ will be $\left(\frac{-\ell a^2}{n}, \frac{-ma^2}{n}\right)$

10. FAMILY OF CIRCLES :

(a) The equation of the family of circles passing through the points of intersection of two circles $S_1 = 0$ & $S_2 = 0$ is : $S_1 + K S_2 = 0 \quad (K \neq -1)$.

(b) The equation of the family of circles passing through the point of intersection of a circle $S = 0$ & a line $L = 0$ is given by $S + KL = 0$.

(c) The equation of a family of circles passing through two given points (x_1, y_1) & (x_2, y_2) can be written in the form :

$$(x-x_1)(x-x_2) + (y-y_1)(y-y_2) + K\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$ where K is a parameter.

(d) The equation of a family of circles touching a fixed line $y - y_1 = m(x - x_1)$ at the fixed point (x_1, y_1) is $(x - x_1)^2 + (y - y_1)^2 + K[y - y_1 - m(x - x_1)] = 0$, where K is a parameter.

(e) Family of circles circumscribing a triangle whose sides are given by $L_1 = 0$; $L_2 = 0$ & $L_3 = 0$ is given by ; $L_1L_2 + \lambda L_2L_3 + \mu L_3L_1 = 0$ provided coefficient of $xy = 0$ & coefficient of x^2 = coefficient of y^2.

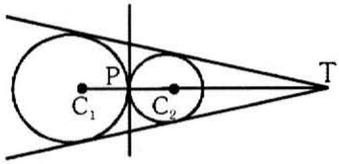
(d) Equation of circle circumscribing a quadrilateral whose side in order are represented by the lines $L_1 = 0$, $L_2 = 0$, $L_3 = 0$ & $L_4 = 0$ are $L_1L_3 + \lambda L_2L_4 = 0$ provided coefficient of x^2 = coefficient of y^2 and coefficient of $xy = 0$.

11. DIRECT AND TRANSVERSE COMMON TANGENTS :

Let two circles having centre C_1 and C_2 and radii, r_1 and r_2 and C_1C_2 is the distance between their centres then :

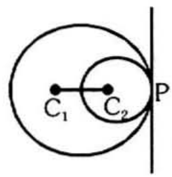
(a) Both circles will touch :

(i) Externally if $C_1C_2 = r_1 + r_2$, point P divides C_1C_2 in the ratio $r_1 : r_2$ (internally).



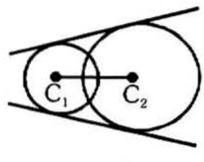
In this case there are **three common tangents.**

(ii) Internally if $C_1C_2 = |r_1 - r_2|$, point P divides C_1C_2 in the ratio $r_1 : r_2$ **externally** and in this case there will be only **one common tangent.**



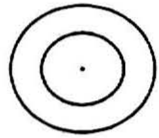
(b) The circles will intersect :

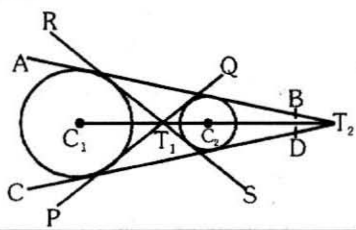
when $|r_1 - r_2| < C_1C_2 < r_1 + r_2$ in this case there are **two common tangents.**



(c) The circles will not intersect :

(i) One circle will lie inside the other circle if $C_1C_2 < |r_1 - r_2|$ In this case there will be no common tangent.





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