Def: an heart $\mathcal{A}$ of a bounded t-structure on $D^b(X)$ is a full additive subcategory $\mathcal{A} \subseteq D^b(X)$ s.t.

1) $\forall A, B \in \mathcal{A}$ $\quad \mathrm{Hom}(A, B[i]) = 0 \quad \forall i < 0$

2) $\forall E \in D^b(X)$ there exist objects $E_1, \dots, E_m$ of $D^b(X)$, objects A_i $i=1,\dots,m$ of $\mathcal{A}$ and integers $k_1 > \dots > k_m$ s.t. there are triangles

$$\begin{tikzcd}
E_0 = 0 \arrow[r] & E_1 \arrow[r] \arrow[d] & E_2 \arrow[r] \arrow[d] & \dots \arrow[r] & E_m = E \arrow[d] \\
& A_1[k_1] \arrow[ul, dashed] & A_2[k_2] \arrow[ul, dashed] & & A_m[k_m] \arrow[ul, dashed]
\end{tikzcd}$$

Def: A slicing $\mathcal{P}$ of $D^b(X)$ is a $\mathbb{R}$ indexed set of subcategories s.t.

1) $\mathcal{P}(\phi+1) = \mathcal{P}(\phi)[1]$

2) if $\phi_1 > \phi_2$, $E_1 \in \mathcal{P}(\phi_1)$ $E_2 \in \mathcal{P}(\phi_2)$ then $\mathrm{Hom}(E_1, E_2) = 0$

3) $\forall E \in D^b(X)$ there are numbers $\phi_1 > \dots > \phi_m$ and objects E_i, $i=1,\dots,m$ $\quad E_i \in D^b(X)$ and Triangles

$$\begin{tikzcd}
0 \arrow[r] & E_1 \arrow[r] \arrow[d] & E_2 \arrow[r] \arrow[d] & \dots \arrow[r] & E_{m-1} \arrow[r] \arrow[d] & E_m = E \arrow[d] \\
& A_1 \arrow[ul, dashed] & A_2 \arrow[ul, dashed] & & A_{m-1} & A_m \arrow[ul, dashed]
\end{tikzcd}$$

with $A_i \in \mathcal{P}(\phi_i)$

$\phi^+(E) := \phi_1$

$\phi^-(E) := \phi_m$

From a slicing one can construct an heart $\mathcal{A}$ as the extension closure of $\{\mathcal{P}(\phi), \phi \in (0,1]\}$ and actually as e.c. $\{\mathcal{P}(\phi) \ \phi \in (\phi_0, \phi_0+1]\}$. In this sense a slicing is a family of hearts parametrized by $\mathbb{R}$

Example to keep in mind: heart = $\mathrm{Coh}(X)$

slicing: semistable sheaves

Bridgeland stability condition:

We fix Λ a finite rank lattice and $v: K_0(X) \to \Lambda$ a surjection

We fix a norm on Λ (all norms are equivalent so ~~this~~ it's not important which we choose)

Def: A bridgeland stab. condition is a pair $\sigma = (\mathcal{P}, Z)$ where $\mathcal{P}$ is a slicing, $Z : \Lambda \to \mathbb{C}$ additive group homom s.t.

1) $Z(v(E)) \in \mathbb{R}^+ \cdot e^{i\pi\phi} \quad \forall E \in \mathcal{P}(\phi)$

2) $C_\sigma := \inf \left\{ \frac{|Z(v(E))|}{\|v(E)\|} \text{ of } E \in \mathcal{P}(\phi) \right\} > 0$

$\mathcal{P}(\phi)$ = semist. objects of phase ϕ

mass of $E = \sum_i |Z(A_i)|$ where A_i are the HN factors of E

in the case of $Coh(X)$ and $Z = -d + ir$ the mass is the lenght of the perimeter of the polygon forming the HN filtr.

Lemma: Giving a Bridg. stab. cond. $\sigma = (\mathcal{P}, Z)$ on $D^b(X)$ is the same as giving a stab. cond. $(\mathcal{A}, Z)$ where $\mathcal{A}$ is the heart of b. t-structure s.t.

$$C_\sigma := \inf \left\{ \frac{|Z(v(E))|}{\|v(E)\|} \text{ of } E \in \mathcal{A} \text{ semistable} \right\} > 0$$

Proof: From $\sigma = (\mathcal{P}, Z)$ we just define $\mathcal{A} = \mathcal{P}((0,1])$

Converse: $\forall \phi \in (0,1]$ we define $\mathcal{P}(\phi)$ as the category of semist. obj. of phase ϕ in $\mathcal{A}$

we extend the def. of $\mathcal{P}(\phi)$ $\forall \phi \in \mathbb{R}$ using

$$\mathcal{P}(\phi+1) = \mathcal{P}(\phi)[1]$$

$\mathrm{Stab}(X)$ = set of $\overset{\text{Bridg.}}{\check{\ }}$stability condition on $\mathcal{D}^b(X)$ (recall Λ and v fixed)

we put on $\mathrm{Stab}(X)$ the coarsest topology s.T.

$$\left.\begin{array}{l} (\mathcal{A}, Z) \to Z \\ (\mathcal{A}, Z) \to \phi^+(E) \\ (\mathcal{A}, Z) \to \phi^-(E) \end{array}\right\} \text{ are cont. functions } \forall E \in \mathcal{D}^b(X)$$

another way to understand it:

on $\mathrm{Stab}(X)$ we have a generalized metric

$$d(\sigma_1, \sigma_2) = \sup_{\substack{E \in \mathcal{D}^b(X) \\ \neq 0}} \left\{ |\phi^+_{\sigma_1}(E) - \phi^+_{\sigma_2}(E)|, |\phi^-_{\sigma_1}(E) - \phi^-_{2}(E)|, \|Z_1 - Z_2\| \right\}$$

where $\sigma_i = (\mathcal{P}_i, Z_i)$

Action of $\widetilde{GL^+(2,\mathbb{R})}$:

elements of $\widetilde{GL^+(2,\mathbb{R})}$ (univ. cover of $GL^+(2,\mathbb{R})$)

are described as (T, f) where: $f: \mathbb{R} \to \mathbb{R}$ inc. function

$f(\phi+1) = f(\phi) + 1$

$T \in GL^+(2,\mathbb{R})$

$f|_{\mathbb{R}/2\mathbb{Z}} = T|_{S^1}$

(T, f) acts on $(\mathcal{P}, Z)$ via

$(T, f) \cdot (\mathcal{P}, Z) = (\mathcal{P}', Z')$ with $\mathcal{P}'(\phi) = \mathcal{P}(f(\phi))$

$Z' = T^{-1} \circ Z$

what we are doing is applying an orient. preserving autom. of $\mathbb{R}^2$ and changing the phases accordingly

Bridgeland Theorem: The map $\mathcal{Z}: Stab(X) \to Hom(\Lambda, \mathbb{C})$ is a

$$\sigma = (\mathcal{P}, Z) \to Z$$

local homeomorphism

Hence $Stab(X)$ is a complex manifold of dimension $\dim(Stab(X)) = rank(\Lambda)$

Local injectivity: let $(\mathcal{P}_1, Z), (\mathcal{P}_2, Z)$ Bridg. stab. condition with $d(\sigma_1, \sigma_2) < \frac{1}{4}$

$$\Rightarrow |\phi^+_{\sigma_1}(E) - \phi^+_{\sigma_2}(E)| < \frac{1}{4}$$

let $E \in \mathcal{P}_1(\phi)$

let's suppose E is not σ_2 semistable

let A be the object that destabilize it (the first elem. in the $\mathcal{P}_2$ HN filt. of E)

since $\phi^+_{\sigma_1}(E) = \phi \Rightarrow A \in \mathcal{P}_2(\tilde{\phi})$ with $|\phi - \tilde{\phi}| < \frac{1}{4}$

$\Rightarrow A \in \mathcal{P}_1((\phi - 1/2, \phi + 1/2])$ which is an heart and A destabilize E

(against the fact that the obj. of $\mathcal{P}_1(\phi)$ are the semistable ones)

We prove now that given $\sigma = (\mathcal{A}, Z)$ and W close to Z there exist τ close to σ : $\mathcal{Z}(\tau) = W$

we reduce the case in two different settings

1) $\mathcal{R}W = \mathcal{R}Z$

2) $Im\, W = Im\, Z$

$\mathcal{R}$ = real part

Im = Im part

Up to use an element of $\widetilde{GL^+(2,\mathbb{R})}$ we can reduce 1) to the same setting as 2) (by rotating of $\frac{\pi}{2}$)

we claim that in the setting 2)
$(\mathcal{A}, W)$ is still a stab. condition

hyp: $\operatorname{Im} W = \operatorname{Im} Z$

$\|W - Z\| < \varepsilon C_\sigma$

we need to prove exist. of HN filt for $(\mathcal{A}, W)$

<u>Proof</u>: 1) the existence of HN for Z tell us that $\forall F \subset E$

$$\Re Z(F) \geq T_\varepsilon + m_\sigma(F)$$

2) we define F_i as the ~~[illegible]~~ factors in the HN filt. for F w.r.t. Z

by the supp. property

$$|\Re W(F_i) - \Re(Z(F_i))| = |W(F_i) - Z(F_i)| \leq \|W - Z\| \, \|F_i\|$$
$$\leq \varepsilon C_\sigma \|F_i\| \leq \varepsilon |Z(F_i)|$$

hence $|\Re W(F_i) - \Re Z(F_i)| \leq \varepsilon |Z(F_i)|$

in particular $\Re W(F_i) \geq \Re Z(F_i) - \varepsilon |Z(F_i)|$

summing over i

$$\Re W(F) \geq \Re Z(F) - \varepsilon m_\sigma(F) \overset{\text{using (1)}}{\geq} T_\varepsilon + (1-\varepsilon) m_\sigma(F) > T_\varepsilon$$

So we have a bound on the gen. degree

3) if $F \subset E$ is extrem. point for the HN polygon w.r.t. W

$$\Rightarrow \max\{0, \Re W(E)\} > \Re(W(F))$$

$$\Rightarrow m_\sigma(F) \leq \frac{\Re W(F) - T_\varepsilon}{1-\varepsilon} \leq \frac{\max\{0, \Re W(E)\} - T_\varepsilon}{1-\varepsilon} =: T'_\varepsilon$$

4) by taking the HN factor of F w.r.t. Z

from $m_\sigma(F) \leq T'_\varepsilon$ we have $|Z(F_i)| \leq T'_\varepsilon$

$$\Rightarrow \|F_i\| < \frac{T'_\varepsilon}{C_\sigma}$$

$\Rightarrow$ there can be only finitely many classes $[F_i] \in \Lambda$ that can appear as factor of F extremal and hence finitely many vertices

5) support property follows easily

example: $\Lambda = K_{num}(X)$ = Grothendieck numerical group

$K_{num}(X) = \dfrac{K_0(X)}{N(X)}$ $\qquad N(X) = \{ E \in D^b(X) : \langle F, E \rangle = 0 \ \forall F \in D^b(X) \}$

$$\langle F, E \rangle = \sum_j (-1)^j \operatorname{Ext}^j(F, E)$$

$v : K_0(X) \to K_{num}(X)$ natural proj.

Let C be a curve of genus $g \geq 1$

~~[illegible]~~ Claim: $\operatorname{Stab}(C) = \sigma_0 \cdot \widetilde{GL}^+(2, \mathbb{R})$

where σ_0 = usual stab cond.

Proof: 1) $\mathbb{C}(x)$ skyscraper is σ-semist. $\forall x$

we do that by assuming it is not and looking at the triangle $A \to \mathbb{C}(x) \to B \to A[1]$ of its HN filt

and then studying the long exact seq. in cohomology sheaves and obtaining an absurd

2) similar reasoning for line bundles

3) $\forall x \in C$ $\forall A$ line b. we have

$\mathbb{C}(x) \to A[1]$ $\qquad A \to \mathbb{C}(x)$ non zero

hence $\phi_x \in \phi(\mathbb{C}(x))$ $\qquad \phi_A = \phi(A)$

$\phi_x \leq \phi_A + 1$ $\qquad \phi_A \leq \phi_x$ $\quad \Rightarrow \quad \phi_x - 1 \leq \phi_A \leq \phi_x$ $\quad \forall x, \forall A$

$K_{num} \cong \mathbb{R}^2$ $\Rightarrow$ we can act via $\widetilde{GL}^+(2, \mathbb{R})$

s.t. $Z = -d + ir$ and $\exists x : \phi_x = 1$

$\Rightarrow \phi_A \in (0, 1) \ \forall A$

$\Rightarrow \phi_x = 1 \ \forall x$

$\Rightarrow \mathcal{P}((0, 1]) = \operatorname{Coh}(C)$ and up to the action used we have σ_0