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Take home exam

Find the maximum likelihood estimate of the unknown parameters , a1, a2,c0, c1, and c2, for the equation:

$$X(n) = a_0\sqrt{x(n-1)} + \sqrt{h(n)}\epsilon(n)$$

where:

- $h(n) = c_0 e^{x(n\text{-}1)}$.

Where $\varepsilon(n)$ is zero mean unit variance Gaussian random variable.

Solution:

$f(X(n), X(n-1), \ldots\ldots\ldots\ldots, X(1), X(0)) =.$

$= f(X(n-1), X(n-2), \ldots\ldots\ldots\ldots, X(0)) *$

$f(X(n-1)|x(n-2), \ldots\ldots\ldots\ldots, X(0) \ldots\ldots\ldots\ldots\ldots\ldots f(X(0))f(X(0))$

Now we calculate first: Assuming zero initial conditions

$f(X(0))$ is Gaussian distribution with zero mean and variance $h(0)$ where $h(0) = c_o$.

Therefore:

$$f(X(0)) = \frac{1}{\sqrt{2\pi\, c0}}\, exp(\frac{-1}{2}\, \frac{X^2(0)}{c0})$$

Then we calculate $f(X(1)\mid X(0))$:

$f(X(1))$ is Gaussian distribution with mean equal $a_0\sqrt{X(0)}$ and variance $h(1)$ where: $h(1) = c_o\, exp(X(0))$.

Therefore:

$$f(X(1)\mid X(0)) = \frac{1}{\sqrt{2\pi\, h(1)}}\, exp(\frac{-1}{2}\, \frac{(X(1) - a_0\sqrt{X(0)}\ )^2}{h(1)})$$

Then we calculate $f(X(2)|X(1), X(0)\ )$:

$f(X(2)|X(1), X(0)\ )$ is Gaussian distribution with mean equal $a_0\sqrt{X(1)}$ and variance $h(2)$ where $h(2) = c_o\, exp(\ X(1)\ )$.

Therefore:

$$f(X(2)|X(1), X(0)\ ) = \frac{1}{\sqrt{2\pi\, h(2)}}\, exp(\frac{-1}{2}\, \frac{(X(2) - a_0\sqrt{X(1)}\ )^2}{h(2)})$$

.

.

And so on finally we get:

$$f(X(n), X(n-1), X(n-2), \ldots\ldots, X(0)) =$$

$$f(X(0)) \prod_{n=1}^{n=N} \frac{1}{\sqrt{2\pi\, h(n)}}\ exp(\frac{-1}{2}\ \frac{(X(n) - a_0 \sqrt{X(n-1)}\ )^2}{h(n)})$$

Taking “ln” to the base e of both sides we get:

$$ln(F) = ln\, f(X(0)) - \frac{(N-1)}{2}\ln(2\pi) - \frac{1}{2}\ ln\, h(n) - \frac{1}{2} \sum_{n=1}^{n=N} \frac{\left(X(n) - a_0 \sqrt{X(n-1)}\right)^2}{h(n)} ..\ eqn.1$$

$$ln\, f(X(0)) = ln\, [\frac{1}{\sqrt{2\pi\, c0}}\ exp(\frac{-1}{2}\ \frac{X^2(0)}{c0})] = -\frac{1}{2}ln(2\pi) - \frac{1}{2}ln(c0) - \frac{1}{2}\frac{X^2(0)}{c0}\ ..\ eqn.2$$

Substitute eqn.2 in eqn.1 and then apply Parameter Estimation:

$$\frac{\partial F}{\partial a_0} = 0$$

❖ $$\sum_{n=1}^{n=N} \frac{\left(X(n)\sqrt{X(n-1)} - a_0\, X(n-1)\right)}{h(n)} = 0\ \ ....\ eqn.3$$

$$\frac{\partial F}{\partial c_o} = \frac{\partial F}{\partial h(n)} * \frac{\partial h(n)}{\partial c_o} = 0,$$

❖ $$-\sum_{n=1}^{n=N} \frac{exp(X(n-1))}{2h(n)} + \frac{1}{2}\sum_{n=1}^{n=N} \frac{\left(X(n) - a_0\sqrt{X(n-1)}\right)^2}{h(n)^2} exp(X(n-1)) = 0.......\ eqn.4$$

➔*From eqn.3 and eqn.4 :*

- These are two independent nonlinear equations in two unknowns.
- It is more complex to solve these equations by hand, So it needs mathematical program to solve them such as matlab
- After solving these equations, we get the unknown parameters which are required.

(2) We have two sets of signal (First Set and Second Set) and we need to select the best model for each set from the following models, AR (5) , MA (4) , and ARMA (3, 4) . Now, for a new test signal, we need to determine to which of the above models it belongs i.e. Does it belong to the model of the First Set or the model of the Second Set?.

Hint: Hypothesis Testing.

index	First Set	Second Set	Test signal
0	0.866236	0.35940	0.65163
1	0.742524	0.3101	0.07668
2	-0.40805	-0.6747	0.992269
3	-0.99432	-0.76200	-0.114643
4	-0.20552	-0.00327	-0.786694
5	0.867497	0.91948	0.923796
6	0.740828	-0.3192	-0.990731
7	-0.41035	0.17053	0.549820
8	-0.99404	-0.5523	0.737389
9	-0.20304	0.50253	-0.831128
10	0.868752	-0.4898	-0.2004347
11	0.739127	0.01191	-0.13717
12	-0.41266	0.39815	0.600137
13	-0.99377	0.78180	-0.48026
14	-0.20057	0.9185	-0.20043
15	0.870002	0.09443	0.634606

Solution:

- The Code included in a separate file .

Using Matlab:

1) **Signal Models for each set:**
   a. Models of First set:

AR(5):

$-0.0450\,X(n) = 1.2687\,X(n-1) - 1.3387\,X(n-2) + 0.3009\,X(n-3) + 0.2557\,X(n-4) - 0.3116\,X(n-5)$, Where $\sigma_\varepsilon^2 = 4.7571 * 10^{-12}$

MA(4):

$0.0278\,X(n) = 1.0918\,X(n-1) - 0.1753\,X(n-2) - X(n-3) - 0.5832\,X(n-4)$

Where $\sigma_\varepsilon^2 = 0.09173$

ARMA(3,4):

AR (3): $-0.0537\,X(n) = 0.9046\,X(n-1) - 1.1774\,X(n-2) + 0.2875\,X(n-3)$

Where $\sigma_\varepsilon^2 = 2 * 10^{-7}$

MA (4): $-0.0537\,X(n) = 0.4638\,X(n-1) - 0.6234\,X(n-2) - 0.0218\,X(n-3) + 0.4061X(n-4)$

Where $\sigma_\varepsilon^2 = 2 * 10^{-7}$

   b. Models of Second Set:

AR(5):

$0.0835\,X(n) = 0.1804\,X(n-1) + 0.1283\,X(n-2) - 0.2671\,X(n-3) + 0.5999X(n-4) + 0.3586\,X(n-5)$, Where $\sigma_\varepsilon^2 = 0.1967$

MA(4):

$-0.0694\,X(n) = 0.1159\,X(n-1) - 0.1368\,X(n-2) - 0.9054\,X(n-3) - 0.0738\,X(n-4)$

Where $\sigma_\varepsilon^2 = 0.16381$

ARMA(3,4):

AR (3): $0.0299\,X(n) = 0.3943\,X(n-1) - 0.0682\,X(n-2) - 0.7382\,X(n-3)$

Where $\sigma_\varepsilon^2 = 0.1508$

MA (4): $0.0299\,X(n) = -0.4489\,X(n-1) + 0.1764\,X(n-2) - 0.2989\,X(n-3) + 0.4574X(n-4)$, Where $\sigma_\varepsilon^2 = 0.1508$

2) <u>The best model for each set:</u>

a. First Set:

Min_AIC = -359.5856 , and this is for <u>AR(5) model</u>

b. Second Set

Min_AIC = 24.4609 , and this is for <u>MA(4) model</u>

3) <u>Hypothesis testing:</u>

By Applying Hypothesis Test:

- **MAX_Likilihood** = -64.1308 and this is for for the second set so, test signal is belong to Second set

```
Command Window

testDataClass =

     2
```

The Selected Models are:

- For First Data Set AR(5)
- For Second Data Set MA(4)

```
    ARIMA(5,0,0) Model:
    --------------------
    Distribution: Name = 'Gaussian'
               P: 5
               D: 0
               Q: 0
        Constant: -0.0449902
              AR: {1.26871 -1.33866 0.300911 0.255708 -0.311571} at Lags [1 2 3 4 5]
             SAR: {}
              MA: {}
             SMA: {}
        Variance: 4.75707e-12
    ARIMA(0,0,4) Model:
    --------------------
    Distribution: Name = 'Gaussian'
               P: 0
               D: 0
               Q: 4
```