1 Hilbert Polynomial

Let X be a closed subscheme of a projective space $\mathbb{P}^N_k$ over a filed k. It is well-known that the cohomology groups $H^i(X,\mathcal{F})$ of a coherent sheaf $\mathcal{F}$ on X is a finite-dimensional vector space. Therefore we can define the Euler characteristic $\chi(\mathcal{F}) := \sum_{i=0}^{\infty}(-1)^i \dim_k H^i(X,\mathcal{F})$.
Fix a very ample invertible sheaf $\mathcal{O}_X(1)$. We will show that $\phi_{\mathcal{F}}(n) = \chi(\mathcal{F}(n))$ is a polynomial in $n \in \mathbb{N}$, which is called the Hilbert polynomial.

First of all, one can reduce to the case where $X = \mathbb{P}^N_k$. Indeed, letting $\iota : X \to \mathbb{P}^N_k$ be the closed immersion, we have $H^i(X,\mathcal{F}(n)) = H^i(\mathbb{P}^N_k, (\iota_*\mathcal{F}(n))) = H^i(\mathbb{P}^N_k, (\iota_*\mathcal{F})(n))$ and the sheaf $\iota_*\mathcal{F}$ is coherent on $\mathbb{P}^N_k$. The last equation comes from the projection formula. In what follows in this section we let $X = \mathbb{P}^N_k$.
Let $S = k[t_0, \cdots, t_N]$. Induction on N. If $N = 0$, then $\mathbb{P}^N_k = \operatorname{Spec} k$ and it is obvious. Consider the morphism $\mathcal{F}(-1) \xrightarrow{\cdot t_N} \mathcal{F}$, and let $\mathcal{K}$ and $\mathcal{C}$ be its kernel and cokernel. There is an exact sequence

$$0 \to \mathcal{K}(n) \to \mathcal{F}(n-1) \to \mathcal{F}(n) \to \mathcal{C}(n) \to 0.$$

Since $\mathcal{K}$ and $\mathcal{C}$ vanishes by the multiplication by t_N, it can be considered as a coherent morphism on $H = V_+(t_N) \simeq \mathbb{P}^{N-1}_k$ (precisely, letting $\iota : H \to \mathbb{P}^N_k$ be the closed immersion, there is a coherent sheaf $\mathcal{K}'$ and $\mathcal{C}'$ on H such that $\iota_*\mathcal{K}' = \mathcal{K}$ and $\iota_*\mathcal{C}' = \mathcal{C}$).
Take the Euler characteristic we get $\chi(\mathcal{F}(n)) - \chi(\mathcal{F}(n-1)) = \chi(\mathcal{C}(n)) - \chi(\mathcal{K}(n))$. The right-hand side is a polynomial in n by inductive hypothesis. As $\sum_{m=0}^{n} m^i$ is written in a polynomial in n of degree $i+1$, we get the result. $\square$

By the Serre's theorem, $\chi(\mathcal{F}(n)) = \dim_k H^0(X,\mathcal{F}(n))$ for sufficiently large n. The right-hand side is called the Hilbert function. Letting $M = \Gamma_*(\mathcal{F})$, we have $\dim_k H^0(X,\mathcal{F}(n)) = \dim_k M_n$. Hence it is the Hilbert function of the graded S-module M.
By the dimension theory, we have $\deg \phi_{\mathcal{F}} \leq \dim \operatorname{Supp} \mathcal{F}$. When $\mathcal{F} = \iota_* O_Y$ is the structure sheaf of Y pushed forward into X via the closed immersion $\iota : Y \to X$, then the equality holds.
In order to show this, we need following two theorems:

Theorem 1.1 ([Mat 13.7]). *Let $A = \bigoplus_{n\geq 0} A_n$ be a graded Noetherian ring. Then:*

(i) for a homogeneous ideal I, its prime divisors are also homogeneous;

(ii) for a homogeneous ideal P of height r, there is a prime chain $P = P_0 \supsetneq P_1 \supsetneq \cdots \supsetneq P_r$ consisting of only homogeneous prime ideals.

Theorem 1.2 ([Mat 13.8])**.** *Let k be a field and $R = k[\xi_1, \cdots, \xi_n]$ be a graded k-algebra generated by homogeneous elements ξ_i of degree 1. Let $\mathfrak{m} = R_+ = (\xi_1, \cdots, \xi_n)$ and let $h(n) = \dim_k R_n$ be the Hilbert function of R. Then we have* $\dim R = \mathrm{ht}\ \mathfrak{m} = \deg h + 1$.

Using these we get $\dim Y = \dim(\mathrm{Proj}\, S/I) = \mathrm{ht}\ (t_0, \cdots, t_n)(S/I) - 1 = \deg\ \phi_{\mathcal{O}_Y}$. For a coherent sheaf $\mathcal{F}$ with support Y, we have an exact sequence

$$0 \to \mathcal{K} \to \bigoplus_{i=1}^{r} \mathcal{O}_Y(q_i) \to \mathcal{F} \to 0$$

for some q_i. Tensoring with $\mathcal{O}_X(n)$ for a sufficiently large and take the Euler charactersitic, we get $\phi_{\mathcal{F}}(n) \leq \sum_{i=1}^{r} \phi_{\mathcal{O}_Y}(n + q_i)$. This shows $\deg \phi_{\mathcal{F}} \leq \dim Y$.

2 Flatness

Flatness is a purely algebraic notion, but it plays an important role in algebraic geometry.

Theorem 2.1 ([Har III.9])**.** *Let X be a projective scheme over a connected Noetherian scheme S. For a coherent sheaf $\mathcal{F}$ on X, if $\mathcal{F}$ is flat over S, then the Hilbert polynomial of $\mathcal{F}_s$ on the fiber $X_s \subseteq \mathbb{P}^n_{k(s)}$ is independent of $s \in S$. Furthermore if S is integral, then the converse is true.*

By this theorem we can define the Hilbert polynomial for a coherent sheaf $\mathcal{F}$ flat over S.
The aim of this chapter is to show the following theorem:

Theorem 2.2 (Flatteness Stratification [Mum Lecture 8])**.** *Let X be a projective scheme over a Noetherian scheme S and let $\mathcal{F}$ be a coherent sheaf on X. There are finitely many disjoint locally closed subschemes $S_1, \cdots, S_m$ of S such that $S = \bigcup S_i$ with the following property:*
for a morphism $g : T \to S$ with T Noetherian, let $g_T : X_T = X \times_S T \to T$ be the base change; then $g_T^* \mathcal{F}$ on X_T is flat over T if and only if $g : T \to S$ factors through $\coprod S_i \to S$.

We follow the method of [Mum]. Note that we can reduce to the case where $X = \mathbb{P}^n_S$. In order to prove this, we need several facts.
For a point $s \in S$, we let $\mathcal{F}_s$ be the pull-back of $\mathcal{F}$ on $\mathbb{P}^n_{k(s)}$. By the Serre's theorem, for any $s \in S$, there is an $m_0 > 0$, depending on s, such that $H^i(\mathbb{P}^n_{k(s)}, \mathcal{F}_s(m)) = 0$ for all $m \geq m_0$ and $i > 0$. First thing is to take a uniform m_0, that is, an m_0 independent of $s \in S$.
Since S is quasi-compact, we only have to show that an m_0 for s propagates for points around s. When $\mathcal{F}$ is flat over S, this works:

Theorem 2.3 (Upper Semicontinuity [Har III.12.8]). *If $\mathcal{F}$ is flat over S, then the map*

$$h^i(s) = \dim_{k(s)} H^i(X_s, \mathcal{F}_s)$$

is upper semicontinuous, i.e., for any $s \in S$, there is an open neighborhood U of s such that $h^i(s') \leq h^i(s)$ for $s' \in U$.

For the general case, let us show the following lemma:

Lemma 2.4. *Let $f : X \to S$ be a morphism of finite type between Noetherian schemes and let $\mathcal{F}$ be a coherent sheaf on X. There exist finitely many irreducible locally closed subsets $Y_1, \cdots, Y_r$ of S that cover S and that if we endow Y_i with the reduced subscheme structure, then $\iota^*\mathcal{F}$ is flat over Y_i, where $\iota : f^{-1}(Y_i) \to X$ is the base change of the canonical immersion $Y_i \to S$.*

To show this, we need some algebraic result:

Lemma 2.5 ([Mat 7.9]). *Let*

$$0 \to M' \to M \to M'' \to 0$$

be an exact sequence of A-modules. If M' and M'' are flat, then so is M flat.

Lemma 2.6. *Let A be a Noetherian integral domain, let B be a finitely generated A-algebra and let M be a finite B-module. There is a nonzero $f \in A$ such that M_f is free over A_f.*

(Proof). First, there is a sequence of sub-B-modules

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$$

such that $M_i/M_{i-1} \simeq B/P_i$ for some prime ideal P_i of B [Mat 6.4].
Now we may suppose that $M = B$ and B is an integral domain. Let K and L be the field of fractions of A and B, resp. Let $B = A[x_1, \cdots x_n]$ and $B' = K[x_1, \cdots, x_n] \subseteq L$. Apply Noether's normalization lemmato B', there exist $z_1, \cdots, z_r \in B$ algebraically independent over K such that B' is integral over $K[z_1, \cdots, z_r]$ ($r = \text{tr.deg}_K L$).
Each x_i has an algebraic relation by a monic polynomial with coefficients in $K[z_1, \cdots, z_r]$. If we let f be the product of all the denominators of these coefficients, each x_i is integral over $A_f[z_1, \cdots, z_r]$.
Therefore, $B_f = A_f[x_1, \cdots, x_n]$ is integral over $A' := A_f[z_1, \cdots, z_r]$. It is also finitely generated, so B_f is a finite A'-module. Write $B_f = A'y_1 + \cdots + A'y_m$. Let $K' = K(z_1, \cdots, z_r)$ be the field of fractions of A' and let $y_1, \cdots, y_s$ be the basis for the module $K'y_1 + \cdots + K'y_m$.
Then there is an exact sequence of A_f-modules

$$0 \to A'^s \to B_f \to D \to 0$$

with D annihilated by some nonzero element $g \in A'$.

Now we use the same method to show that D'_f is free over A'_f for some f'. Then the rings that appear has transcendental degree $<r$. By induction on r, we get some f' and then $B_{ff'}$ is free over $A_{ff'}$ (note that A' is free over A_f).
It now suffices to show the case where $r = 0$. B is integral over A and L/K is a finite field extension. Let $d_i = [K(x_1, \cdots, x_i) : K(x_1, \cdots, x_{i-1})]$. Then $\{\prod_e x_i^{e_i} : 0 \le e_i < d_i\}$ is a basis for B over A. □

Now we can prove (2.4):

(Proof). We endow each irreducible components Z of S the reduced induced closed subscheme structure. It suffices to show that the lemma holds for each Z, so we may assume that S is integral.
By (2.6), there is a nonempty open set V of Y such that $\mathcal{F}|_{f^{-1}(V)}$ is flat over $f^{-1}(V)$.
Now by the Noetherian induction of closed subsets we get the lemma. □

Now by (2.4) and by the flat case, we have a unifrom $m_0 > 0$ depending only on $\mathcal{F}$ such that $H^i(\mathbb{P}^n_{k(s)}, \mathcal{F}_s(m)) = 0$ for every $s \in S$, $m \ge m_0$ and $i > 0$.
By (2.1) it suffices to see the Hilbert polynomial of $\mathcal{F}$. In order to show (2.2) we may assume that S is connected. For $m \ge m_0$, we have $\phi_{\mathcal{F}}(m) = \dim H^0(\mathbb{P}^n_{k(s)}, \mathcal{F}_s(m))$ for some $s \in S$.
Let $p : \mathbb{P}^n_S \to S$ be the structure morphism and let $\mathcal{E}_m = p_* \mathcal{F}(m)$. Then we have a useful relation between $\mathcal{E}_m \otimes k(s)$ and $H^0(\mathbb{P}^n_{k(s)}, \mathcal{F}_s)$.

Theorem 2.7 (Grauert's theorem [Har III.12.9]). *Let S be an integral Noetherian scheme, let $f : X \to S$ be a projective morphism and let $\mathcal{F}$ be a coherent sheaf on X flat over S. If the map $s \to \dim_{k(s)} H^i(X_s, \mathcal{F}_s)$ is constant for $s \in S$, then $R^i f_*(\mathcal{F}) \otimes k(s) \to H^i(X_s, \mathcal{F}_s)$ is an isomorphism for every $s \in S$.*

Let us fix an irreducible locally closed subscheme Y of S, and consider the following diagram:

$$\begin{array}{ccc} \mathbb{P}^n_Y & \xrightarrow{h} & \mathbb{P}^n_S \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{g} & S \end{array}$$

Suppose that $h^*\mathcal{F}$ is flat over Y. The Hilbert polynomial of $(h^*\mathcal{F})_s = \mathcal{F}_s$ for $s \in Y$, which is equal to $\dim H^0(\mathbb{P}^n_{k(s)}, \mathcal{F}(m)_s)$ for $m \ge m_0$, is independent of s. By Grauert's theorem we have the isomorphism $q_* h^* \mathcal{F}(m) \otimes k(s) \to H^i(\mathbb{P}^n_{k(s)}, \mathcal{F}(m)_s)$.
There is a canonical morphism $g^*\mathcal{E}_m = g^* p_* \mathcal{F}(m) \to q_* h^* \mathcal{F}(m)$. This is an isomorphism of sufficiently large m [Mum Lecture 7, 3°, (i)].
Thus for sufficiently large m, we have an isomorphism $\mathcal{E}_m \otimes k(s) = g^*\mathcal{E}_m \otimes k(s) \to H^0(\mathbb{P}^n_{k(s)}, \mathcal{F}(m)_s)$ for $s \in S$. Since one can cover S with finite number of Y's, $\mathcal{E}_m \otimes k(s) \to H^0(\mathbb{P}^n_{k(s)}, \mathcal{F}(m)_s)$ is an isomorphism for every $s \in S$ and $m \ge m_1$ for some $m_1 \ge m_0$.

Theorem 2.8 (Cohomology and Base Change [Har III.12.11])**.** *Let $f : X \to S$ be a projective morphism of Noetherian schemes and let $\mathcal{F}$ be a coherent sheaf on X flat over S. For a point $s \in S$ we let $\psi_s^i : R^i f_*(\mathcal{F}) \otimes k(s) \to H^i(X_s, \mathcal{F}_s)$ be the canonical morphism. Suppose that ψ_s^i is surjective. Then:*

(i) ψ_s^i is an isomorphism, and the same holds for s' around s;

(ii) ψ_s^{i-1} is surjective if and only if $R^i f_*(\mathcal{F})$ is locally free in a neighborhood of s.

Theorem 2.9. *Let S be a Noetherian scheme, let $p : \mathbb{P}^n_S \to S$ be the projective space and let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n_S$. Then $\mathcal{F}$ is flat over S if and only if $p_*\mathcal{F}(m)$ is finite locally free for sufficiently large m.*

(Proof). Take an $M > 0$ such that $R^i f_*(\mathcal{F}(m)) = 0$ for $i > 0$ and $m \geq M$. Then (2.8) implies that $p_*\mathcal{F}(m) \otimes k(s) \to H^0(X_s, \mathcal{F}(m)_s)$ is an isomorphism.
If $\mathcal{F}$ is flat over S, then the Hilbert polynomial of $\mathcal{F}_s$ is locally constant for $s \in S$ (2.1). For sufficiently large m, the Hilbert polynomial is equal to $\dim_{k(s)} H^0(X_s, \mathcal{F}(m)_s)$, so $p_*\mathcal{F}(m)$ is finite locally free [Stacks 05P2].
Conversely suppose that $p_*\mathcal{F}(m)$ is locally free for $m \geq m_0$. We may assume that $S = \operatorname{Spec} R$ is an affine spectrum of a Noetherian ring R. $H^0(S, p_*\mathcal{F}(m)) = H^0(X, \mathcal{F}(m))$ is a finite free A-module. Let $M = \bigoplus_{m \geq m_0} H^0(X, \mathcal{F}(m))$. Then $\widetilde{M} \simeq \mathcal{F}$ is flat over S, as easily seen. $\square$

Now we are ready to show (2.2).
We have an $m_1 > 0$ such that $\mathcal{E}_m \otimes k(s) \to H^0(\mathbb{P}^n_{k(s)}, \mathcal{F}(m)_s)$ is an isomorphism and $H^i(\mathbb{P}^n_{k(s)}, \mathcal{F}(m)_s) = 0$ for every $i > 0, s \in S$ and $m \geq m_1$.
Let $g : T \to S$ be a morphism of Noetherian schemes, and consider the following diagram:

$$\begin{array}{ccc} \mathbb{P}^n_T & \xrightarrow{h} & \mathbb{P}^n_S \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ T & \xrightarrow{g} & S \end{array}$$

(*): Let $\mathcal{F}_T = h^*\mathcal{F}$. For $m \geq m_1$, we have a canonical isomorphism $g^*\mathcal{E}_m \to q_*\mathcal{F}_T(m)$ [Mum Lecture 7, 3°, Corollary 2]. If $\mathcal{F}_T$ is flat over T, then $g^*\mathcal{E}_m$ is finite locally free for $m \geq m_1$ by using (2.8) and the proof in (2.9). Conversely if $g^*\mathcal{E}_m$ is finite locally free for all $m \geq m_1$, then $\mathcal{F}_T$ is flat over T by (2.9). So let us investigate when $g^*\mathcal{E}$ is finite locally free.

Next we consider the flattening stratification in the case $X = S$. Let $\mathcal{E}$ be a coherent sheaf on S and $h(s) = \dim_{k(s)} \mathcal{E} \otimes_{\mathcal{O}_S} k(s)$. For any $s \in S$, let $m = h(s)$; there is an open neighborhood U of s and an exact sequence

$$\mathcal{O}_U^n \xrightarrow{\phi} \mathcal{O}_U^m \xrightarrow{\psi} \mathcal{E} \to 0$$

by Nakayama's lemma. We let U_s be such a U. The set $Z_m = \{s \in S : h(s) = m\}$ is a locally closed subset of S, since $Z_m \cap U_s$ is the set of loci where ϕ vanish. We let $Y_s = Z_m \cap U_s$. We endow Y_s with the closed subscheme structure of U_s, defined by the ideal generated by all components of the matrix (a_{ij}) defining ϕ. We claim the following property:

(**) for any morphism $h : T \to U_s$ of Noetherian schemes, $h^*\mathcal{E}$ is locally free of rank m if and only if h factors through Y_s.

Indeed, h factors through Y_s if and only if all $h^\sharp(a_{ij})$ vanish, i.e., $h^*\phi = 0$.
The sequence

$$\mathcal{O}_T^n \xrightarrow{h^*\phi} \mathcal{O}_T^m \xrightarrow{h^*\psi} h^*\mathcal{E} \to 0$$

is exact, so it is equivalent to saying that $h^*\psi$ is isomorphism; hence $h^*\mathcal{E}$ is locally free of rank m.
Conversely if $h^*\mathcal{E}$ is locally free of rank m, let $\mathcal{K}$ be the kernel of $h^*\phi$ and we have an exact sequence

$$0 = \mathrm{Tor}_1(h^*\mathcal{E}, k(t)) \to \mathcal{E} \otimes k(t) \to k(t)^m \to h^*\mathcal{E} \otimes k(t) \to 0.$$

for $t \in T$. Since the last term is a vector space over $k(t)$ of dimension m, we get $\mathcal{K} \otimes k(t) = 0$. By Nakayama's lemma we get $\mathcal{K} = 0$. $h^*\psi$ is an isomorphism and $h^*\phi = 0$. h factors through Y_s. $\square$

Now the property (**) characterizes the locally closed subscheme Y_s in $Z_m \cap U_s$. That is, for two points $s_1, s_2 \in Z_m$, two locally closed subschemes Y_{s_1} and Y_{s_2} restricted on the open set $U_{s_1} \cap U_{s_2}$ are equal. Thus one can glue to get a locally closed subscheme structure on Z_m. The property (*) gives that for any morphism $h : T \to S$ of Noetherian schemes and a coherent sheaf $\mathcal{E}$ on T, $g^*\mathcal{E}$ is finite locally free (of rank m) on T if and only if $g : T \to S$ factors through $\coprod_j Z_j \to S$ $(Z_m \to S)$. This is the flattening stratification of $\mathcal{E}$.

We can finally prove (2.2). Take $Y_1, \cdots, Y_r$ of S as in (2.4) and flattening stratification Z_j^m of $\mathcal{E}_m$ for $m \geq m_1$. Let P_i be the Hilbert polynomial of $\mathcal{F}$ over Y_i. We can take the union of two Y_i's that having the same Hilbert polynomials (2.1) to assume that P_i's are pairwise distinct.

We claim that set-theoretically $Y_i = \bigcap_{m=m_1}^{m_1+n} Z^m_{P_i(m)}$. For an $s \in S$ let P_j be the Hilbert polynomial of $\mathcal{F}_s$; s is contained in the right-hand side if and only if $P_j(m) = \dim_{k(s)} H^0(\mathbb{P}^n_{k(s)}, \mathcal{F}_s) = \dim_{k(s)} \mathcal{E}_m \otimes k(s) = P_i(m)$ for $m = m_1, \cdots, m_1 + n$. Hilbert polynomials have the degree $\leq n$, so we have $P_i = P_j$ and that $s \in Y_i$. The converse inclusion is obvious.
Now endow the locally closed subsets Y_i (which has the reduced subscheme structure) with the intersection structure of $\bigcap_{m=m_1}^{\infty} Z^m_{P_i(m)}$, and denote this by

S_i. This is possible since it is the limit of locally closed subschemes with a fixed underlying space.
These S_i's are the flattening stratification of $\mathcal{F}$. If $T \to S$ is a morphism of Noetherian schemes, let us consider the following diagram:

$$\begin{array}{ccc} \mathbb{P}^n_T & \xrightarrow{h} & \mathbb{P}^n_S \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ T & \xrightarrow{g} & S \end{array}$$

Suppose that $\mathcal{F}_T$ is flat over T. Then $g^*\mathcal{E}_m$ is finite locally free on T for every $m \geq m_1$ as we have seen in (*). For an $m \geq m_1$, $g^*\mathcal{E}_m$ is finite locally free if and only if g factors through $\coprod_e Z_e^m$. As we have seen, there is a canonical isomorphism $g^*\mathcal{E}_m \to q_*\mathcal{F}_T(m)$. Let us show that the dimension of $q_*\mathcal{F}_T(m) \otimes k(t)$ over $k(t)$ is equal to $P_i(m)$ where $s = g(t) \in S_i$.
First, we have a commutative diagram:

$$\begin{array}{ccc} \mathbb{P}^n_{k(t)} & \longrightarrow & \mathbb{P}^n_{k(s)} \\ \big\downarrow & & \big\downarrow \\ \operatorname{Spec} k(t) & \longrightarrow & \operatorname{Spec} k(s) \end{array}$$

The bottom horizontal morphism is flat, so we can use the flat base change theorem [Stacks 02KH] and $H^1(\mathbb{P}^n_{k(t)}, \mathcal{F}_T(m)_t) = H^1(\mathbb{P}^n_{k(s)}, \mathcal{F}(m)_s) \otimes k(t) = 0$. By (2.8) and the flat base change theorem we get $q_*\mathcal{F}_T(m) \otimes k(t) \simeq H^0(\mathbb{P}^n_{k(t)}, \mathcal{F}(m)_t) = H^0(\mathbb{P}^n_{k(s)}, \mathcal{F}(m)_s) \otimes k(t)$, which has the dimension $P_i(m)$.
Hence g factors through $\coprod_i S_i$. The converse is trivial. $\square$

3 Grassmannian

Let $0 \leq k \leq n$. We define the Grassmannian functor

$\mathrm{Gr}(n,k)(S) = \{$ all equivalence classes $q : \mathcal{O}_S^{\oplus n} \to \mathcal{F}$ with $\mathcal{F}$ locally free of rank $n-k$ $\}$,

$$\mathrm{Gr}(n,k)(f : S' \to S) = f^*$$

Here $q : \mathcal{O}_S^{\oplus n} \to \mathcal{F}$ and $q' : \mathcal{O}_S^{\oplus n} \to \mathcal{F}'$ is equivalent if there is an isomorphism $f : \mathcal{F} \to \mathcal{F}'$ such that $f \circ q = q'$.

For $I = i_1, \cdots, i_{n-k}$ with $1 \leq i_1 < \cdots < i_{n-k} \leq n$, we define $s_I : \mathcal{O}_S^{\oplus n} \to \mathcal{O}_S^{\oplus n-k}$ by $s_I(e_k) = e_{i_k}$. Let $F_I(S) = \{q \in \mathrm{Gr}(n,k)(S) : q \circ s_I$ is surjective (hence isomorphic)$\}$. This is a sub-functor of $F = \mathrm{Gr}(n,k)$ and $F_I \subseteq F$ is represented by an open immersion.
According to [Stacks 089T], F is a representable functor. Let $G(n,k)$ be the scheme representing F. $G(n,k)$ is called the Grassmannian scheme. Each F_I is represented by an open subscheme U_I of $G(n,k)$ and $U_I \simeq \mathbb{A}_{\mathbb{Z}}^{n(n-k)}$'s cover

$G(n,k)$. From this fact we know that $G(n,k)$ is smooth over $\mathbb{Z}$ and the dimension of the fiber over $\mathbb{Z}$ is $n(n-k)$.

It is also known that $G(n,k)$ is projective. Let $A = \bigwedge^{n-k} \mathbb{Z}^{\oplus n}$ be an Ableian group, and let $P = \operatorname{Proj}_{\mathbb{Z}} \operatorname{Sym}(A)$.
P represents the functor

$S \mapsto$ equivalence classes of surjective morphism $\bigwedge^{n-k} \mathcal{O}_S^{\oplus n} \to \mathcal{L}$ with $\mathcal{L}$ invertible. Therefore we can define the morphism of functors $\Phi : F \to h_P = \operatorname{Hom}(\cdot, P)$ by

$$(q : \mathcal{O}_S^{\oplus n} \to \mathcal{F}) \mapsto (q : \bigwedge^{n-k} \mathcal{O}_S^{\oplus n} \to \bigwedge^{n-k} \mathcal{F}).$$

When we restrict Φ to the subfunctor F_I, this is injective by an easy argument. This means that if we let $f : G(n,k) \to P$ be the morphism corresponding to Φ, $f|U_I$ is a monomorphism. Since U_I's cover $G(n,k)$, this means that f is a monomorphism. Since $P \simeq \mathbb{P}_{\mathcal{Z}}^{N-1}$ where $N = \binom{n}{n-k}$ and by the fact that $G(n,k)$ is proper over $\mathbb{Z}$ (whose proof is postponed), by the separated cancelation we know that f is proper. A proper monomorphism is a closed immersion; hence f is a closed immersion and $G(n,k)$ is projective.

4 Hilbert scheme and Quot scheme

The Grassmannian $G(k,n)$ is, unformally speaking, a scheme that parametrizes all the free subgroups of $\mathbb{Z}^n$ of rank k. If we base change to some field K, then it is just the Grassmannian in the context of geometry.

The Hilbert scheme is a scheme that parametrizes all the closed subschemes $Z \subseteq \mathbb{P}_S^n$ that is flat over a fixed base scheme S. Use the functor, we can say that the Hilbert scheme is a scheme that represents the functor $T \mapsto$ closed subschemes $Z \subseteq \mathbb{P}_T^n$ which is flat over T.

Combining these two functors, we can define the following functor. Let $X \to S$ be a projective morphism of Noetherian schemes.

$$\operatorname{Quot}_{X/S,\mathcal{E}}(T) = \{\text{ isomorphism classes of surjections } q : \mathcal{E}_T \to \mathcal{F} \text{ with } \mathcal{F} \text{ flat over } T\},$$

$$\operatorname{Quot}_{X/S,\mathcal{E}}(f : T' \to T) = f_X^*.$$

Here $\mathcal{E}_T$ is the pull-back of $\mathcal{E}$ by the fiber change $X \times_S T \to T$.

The Hilbert functor is:

$$\operatorname{Hilb}_{X/S}(T) = \{\text{ closed subschemes of } X \times_S T \text{ flat over } T\},$$

$$\operatorname{Hilb}_{X/S,\mathcal{E}}(f : T' \to T) = f_X^*.$$

It is obvious that $\mathrm{Hilb}_{X/S} = \mathrm{Quot}_{X/S,\mathcal{O}_X}$ and that $\mathrm{Gr}(n,k)$ is a sub-functor of $\mathrm{Quot}_{\mathrm{Spec}\,\mathbb{Z}/\,\mathrm{Spec}\,\mathbb{Z},\mathcal{O}^{\oplus n}_{\mathrm{Spec}\,\mathbb{Z}}}$.
Our main topic is the representability of the Quot functor.

[Insert the division of functors by the Hilbert polynomials]

5 Castelnuovo-Mumford Regularity

The regularity for a coherent sheaf on a projective space is introduced in the book of Mumford [Mum]. We fix a Noetherian ring R.

Theorem 5.1. *Let $\mathcal{F}$ be a coherent sheaf on the projective space $\mathbb{P}^n_R$. $\mathcal{F}$ is m-regular if $H^i(\mathbb{P}^n_R, \mathcal{F}(m-i)) = 0$ for all $i > 0$.*

As a simple remark, by the Serre's theorem every coherent sheaf is m-regular for some sufficiently large m.

Theorem 5.2. *m-regularity impiles $m+1$-regularity.*

(Proof). Induction on n. $n = 0$ is obvious since every coherent sheaf is m-regular for all m.
Let $n > 0$. Since $\mathcal{F}$ is a coherent sheaf on a Noetherian scheme P^n_R, there are only finitely many associated primes of $\mathcal{F}$. Let $\mathfrak{p}_1, \cdots, \mathfrak{p}_r$ be the corresponding prime ideals in the graded R-algebra $S = R[x_0, \cdots, x_n]$. If all the linear form is contained in some $\mathfrak{p}_i$, i.e., $S_1 \subseteq \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_r$, then letting $M_i = \mathfrak{p}_i \cap S_1$, $M_1, \cdots, M_r$ are finite R-modules and $S_1 = M_1 \cup \cdots \cup M_r$ $\square$

Theorem 5.3. *If $\mathcal{F}$ is m-regular, then the canonical map*

$$H^0(\mathbb{P}^n_R, \mathcal{F}(m)) \otimes H^0(\mathbb{P}^n_R, \mathcal{O}(1)) \to H^0(\mathbb{P}^n_R, \mathcal{F}(m+1))$$

(Proof). $\square$

Theorem 5.4. *For all $n \geq 0$, there is a polynomial $F(x_0, \cdots, x_n) \in \mathbb{Q}[x]$ with the following property:*
for a coherent ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_{\mathbb{P}^n_R}$ and $a_0, \cdots, a_n$ defined by

$$\chi(\mathcal{I}(m)) = \sum_{i=0}^{n} a_i \binom{m}{i},$$

$\mathcal{I}$ is $F(a_0, \cdots, a_n)$-regular.

(Proof). $\square$

6 References

[Har] Hartshorne, R. *Algebraic Geometry* (Springer)

[Mat] Matsumura, H. *Commutative Ring Theory* (Cambridge University Press, Cambridge)

[Mum] Mumford, D. *Lectures on Curves on an Algebraic Surface* (Princeton University Press, United States of America)

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