Definition. *(Henselian ring) Let $(R, \mathfrak{m}, k)$ be a local ring. Hensel's lemma states that for any monic polynomials $F \in R[t]$ and $g, h \in k[t]$ such that $(g, h) = (1)$ and $\overline{F} = gh$, there are monic polynomials $G, H \in R[t]$ such that $F = GH$ and $\overline{G} = g, \overline{H} = h$. If Hensel's lemma holds, we say that R is Henselian. If in addition k is separably closed, we say that R is strictly Henselian.*

Theorem 0.1. *Let $(R, \mathfrak{m}, k)$ be a local ring. Then the following conditions are equivalent:*
(i) *R is Henselian;*
(i)' *for a monic polynomial $F(t) \in R[t]$ and an $x \in k$ such that $\overline{F}(x) = 0$ and $\overline{F'}(x) \neq 0$, there is an $a \in R$ such that $\overline{a} = x$ and $F(a) = 0$;*
(ii) *every finite R-algebra A is a finite product of local rings;*
(ii)' (ii) *holds for $A = R[t]/(F(t))$ for a monic polynomial $F(t)$;*
(iii) *for every finite R-algebra A, $A \to A/\mathfrak{m}A$ gives a bijection between idempotent elements;*
(iii)' (iii) *holds for $A = R[t]/(F(t))$ for a monic polynomial $F(t)$;*
(iv) *for every etale morphism $X \to \operatorname{Spec} R$, the section of the closed fiber $X_s \to \operatorname{Spec} k$ can be lifted to a section of $X \to \operatorname{Spec} R$.*

Remark 0.1. *If R is a Henselian ring and A is a finite R-algebra, then $A = \prod_{\mathfrak{m}} A_{\mathfrak{m}}$ where $\mathfrak{m}$ runs over (finitely many) maximal ideals of A.*

Theorem 0.2. *A complete local ring is Henselian.*

Theorem 0.3. *Let $(R, \mathfrak{m}, k)$ be a local ring. Then the following conditions are equivalent:*
(i) *R is strictly Henselian;*
(ii) *R is Henselian and every finite etale scheme over $S = \operatorname{Spec} R$ is a finite disjoint union of copies of S;*
(iii) *for every etale morphism $f : X \to S$ and a point $x \in X$ which is mapped to the closed point s of x, there is a section of f which maps s to x.*

Definition. *Let A, A' be local rings and $\phi : A \to A'$ a local homomorphism. We say that A' is essentially etale over A if there is an etale A-algebra B and a prime ideal P of B lying over the maximal ideal of A such that A' is isomorphic to B_P over A. If in addition the homomorphism between the residue field is an isomporphism, we say that A' is strictly essentially etale over A.*

Theorem 0.4. *Compositions of essentially etale morphisms is again essentially etale.*

(Proof). Let $A \to A' \to A''$ be two essentially etale morphisms. There are etale A-algebra B, A'-algebra B' and a prime ideal P of B lying over the maximal ideal of A, a prime ideal P' of B' lying over the maximal ideal of A' such that A' is isomorphic to B_P over A, A'' is isomorphic to $B'_{P'}$.
Since $B_P = \varinjlim_{f \notin P} B_f$ and each B_f has finitely presentation over A, by the "direct limit of schemes," there is an etale B_f-algebra C such that $B' = C \otimes_{B_f} B_P$ over B_f $(f \notin P)$.

Replacing B by B_f, we have etale morphisms $A \to B \to C$ such that $B_P = A'$ and $C_{P'} = A''$. $\square$

Theorem 0.5. *Let $(A, \mathfrak{m}, k) \to (A', \mathfrak{m}', k')$ be an essentially etale homomorphism and let $(R, \mathfrak{n}, l)$ be a Henselian ring over A. Given a k-homomorphism $k' \to l$, this can be lifted to a unique A-homomorphism $A' \to R$.*

Theorem 0.6. *Let $(A, \mathfrak{m}, k)$ be a local ring and let K/k be a field extension. Let $\mathcal{C}_K$ be the category whose objects are pairs $((A', \mathfrak{m}', k'), \beta')$ of essentially etale A-algebra and a k-homomorphism $\beta : k' \to K$ and whose morphisms $((A', \mathfrak{m}', k'), \beta') \to ((A'', \mathfrak{m}'', k''), \beta'')$ are local A-homomorphisms $\phi : A' \to A''$ such that $\beta''\overline{\phi} = \beta'$ where $\overline{\phi} : k' \to k''$ is induced from ϕ.*
Then $\mathcal{C}$ is a directed category, i.e.,
(i) *for any two objects λ and μ, there is an object ν together with morphisms $\lambda \to \nu$ and $\mu \to \nu$;*
(ii) *for any objects λ and μ, there exist at most one morphism $\lambda \to \mu$.*

Definition. *Let $(A, \mathfrak{m}, k)$ be a local ring and let K/k be a separable field extension. We write*

$$A^K = \varinjlim_{A' \in \mathcal{C}_K} A'.$$

We define the Henselization to be $A^h = A^k$ and the strict Henselization to be $A^{hs} = A^{k^{\mathrm{sep}}}$.

Remark 0.2. *If A' is an essentially etale A-algebra and $\alpha : k(A') \to K$ is a $k(A)$-homomorphism, there is an A-homomorphism $A' \to A^K$ which lifts α.*

Theorem 0.7. *Let $(A_\lambda, \phi_{\lambda\mu})$ be a directed system of local rings $(A_\lambda, \mathfrak{m}_\lambda, k_\lambda)$ and local homomorphisms $\phi_{\lambda\mu} : A_\lambda \to A_\mu$. Let $A = \varinjlim_\lambda A_\lambda$. Then:*
(i) *A is a local ring with the maximal ideal $= \varinjlim_\lambda \mathfrak{m}_\lambda$ and the residue filed $k = \varinjlim_\lambda k_\lambda$;*
(ii) *if A_μ is flat over A_λ for every $\mu \geq \lambda$, then A is flat over A_λ;*
(iii) *if $\mathfrak{m}_\mu = A_\lambda \mathfrak{m}_\lambda$ for every $\mu \geq \lambda$, then $\mathfrak{m} = A\mathfrak{m}_\lambda$;*
(iv) *if* (ii) *and* (iii) *hold and A_λ is Noetherian, then so is A.*

Theorem 0.8. *Let $(A, \mathfrak{m}, k)$ be a local ring and let K/k be a separable field extension. Then:*
(i) *A^K is a faithfully flat Henselian local ring over A with the maximal ideal $A^K\mathfrak{m}$ and the residue field K;*
(ii) *for a Henselian ring $(R, \mathfrak{n}, l)$ and a local homomorphism $f : A \to R$ and a homomorphism $\alpha : K \to l$ which agrees with $\overline{f} : k \to l$ on k, there is a unique local homomorphism $F : A^K \to R$ extending f such that $\overline{F} = \alpha$; in particular* $\mathrm{Aut}(A^K/A) = \mathrm{Gal}(K/k)$; *note that $\square^K$ defines a functor on the category local rings and local homomorphisms;*
(iii) *if A is Noetherian, then so is A^K;*
(iv) *$A^h \simeq \widehat{A^h}$;*
(v)? *for a deirected system of local rings and local homomorphisms (R_λ), we have $(\varinjlim_\lambda R_\lambda)^K \simeq \varinjlim_\lambda R_\lambda^K$;*

(Proof). (i): Let $f : X \to S = \operatorname{Spec} A^K$ be an etale morphism and let $s \in S$ be the closed point and let $g : \operatorname{Spec} K \to X_s$ be a section of the closed fiber f_s. Shrinking X and using direct limit descent, we may assume that there is an etale morphism $f' : X' \to S' = \operatorname{Spec} A'$ for an $A' \in \mathcal{C}_K$ such that f is the base change of f' to S. We may assume that $X' = \operatorname{Spec} B'$ is affine. Set $B = B' \otimes_{A'} A^K$ and $X = \operatorname{Spec} B$. Consider $\operatorname{Spec} K \xrightarrow{g} X_s \to X \to X'$. Let $x = g(s)$ and $x' \in X'$ be the image of s to X'. Then this corresponds to a $k(A)$-homomorphism $\alpha : k(P') \to K$. Therefore we have a A-homomorphism $\phi : B'_{P'} \to A^K$ such that $\overline{\phi} = \alpha$. This gives a morphism $S \to X'$ and consider the graph $\Gamma : S \to X' \times_{S'} S = X$. This is a section of f. This agrees with g_s on the closed fiber. Γ brings s to the point of X over x' and s; it induces a homomorphism $\Gamma^\sharp : k(P') \otimes_{k(A')} K \to K$. g also induces a homomorphism $g^\sharp : k(P') \otimes_{k(A')} K \to K$. By $\overline{\phi} = \alpha$, $\Gamma^\sharp = \alpha \times 1_K = g^\sharp$. Therefore $\Gamma(s) = g(s)$. The section of an etale morphism is determined by a single point, so $\Gamma = g$.

(ii): It suffices to show that if $A \to A'$ is an essentially etale homomorphism, R is a Henselian ring and $\alpha : k(A') \to k(R)$ is a $k(A)$-homomorphism, then there exists a unique A-homomorphism $\phi : A' \to R$ such that $\overline{\phi} = \alpha$.
Let $A' = B_P$ for an etale A-algebra B and a prime ideal P over the maximal ideal of A. Consider $k(R) \otimes_A B \to k(R) \otimes_{k(A)} k(A') \to k(R)$ and the corresponding morphism $\operatorname{Spec} k(R) \to \operatorname{Spec} k(R) \otimes_A B$. This is a section of the closed fiber of the etale morphism $\operatorname{Spec} R \otimes_A B \to \operatorname{Spec} R$. Lifting to $\operatorname{Spec} R \to B$, we obtain a homomorphism $B \to R$. This gives the existence of $A' \to R$. To show the uniqueness, construct the section and use the fact that sections of etale morphisms is determined by a single point.

(iv): Note that A^K is faithfully flat over A.
$\mathfrak{m}^n A^h / \mathfrak{m}^{n+1} A^h = \mathfrak{m}^n / \mathfrak{m}^{n+1} \otimes_A A^h = \mathfrak{m}^n / \mathfrak{m}^{n+1} \otimes_{A/\mathfrak{m}} A^h / \mathfrak{m} A^h = \mathfrak{m}^n / \mathfrak{m}^{n+1}$.
Therefore $A^h / \mathfrak{m}^n A^h = A / \mathfrak{m}^n$. $\square$

Corollary 0.9. *Let $(A, \mathfrak{m}, k)$ be a local ring.*
(i) *$A^{h(s)}$ is a faithfully flat Henselian local ring over A with the maximal ideal $A^{h(s)}\mathfrak{m}$ and the residue field $k^{(\mathrm{sep})}$;*
(ii) *for a Henselian ring R,* $\mathrm{loc.Hom}(A, R) \simeq \mathrm{loc.Hom}(A^h, R)$ *and* $\mathrm{Aut}(A^{hs}/A)$ *is the absolute Galois group of k;*
(ii)' *the functor $\square^{h(s)}$ is the left adjoint to the inclusion of the full subcategory of (strict) Henselian rings into the category of local rings and local homomorphisms;*
(ii)' *the functor $\square^{h(s)}$ is the left adjoint to the inclusion of the full subcategory of (strict) Henselian rings into the category of local rings and local homomorphisms;*

Theorem 0.10. *Let R be a Henselian ring and let $S = \operatorname{Spec} R$. for every etale morphism $X \to S$, the section of the closed fiber $X_s \to \operatorname{Spec} k$ can be lifted to a section of $X \to S$.*

Theorem 0.11. *Let A be a local ring, B a finite A-algebra with maximal ideals $\mathfrak{n}_1, \cdots, \mathfrak{n}_k$. Then $B \otimes_A A^h = (B_{\mathfrak{n}_1})^h \times \cdots \times (B_{\mathfrak{n}_k})^h$.*

Theorem 0.12. *Let A be a local ring, B a finite A-algebra. Let $\mathfrak{n}'$ be a maximal ideal of $B' = B \otimes_A A^{hs}$. Then $\mathfrak{n} = \mathfrak{n}' \cap B$ is a maximal ideal of B and $B'/\mathfrak{n}' = k(\mathfrak{n}')$ is a separable closure of $B/\mathfrak{n} = k(\mathfrak{n})$ and $B'_{\mathfrak{n}}$ is the strict Henselization of $B_{\mathfrak{n}}$.*

1 References

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