PhD AMA II.I

Prof. Meyer-Gohde

Exercise #4

Real Business Cycle Model

1) HH maximization problem

$$\max_{\{C_t, N_t, K_t\}_{t=0}^{\infty}} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \left(\ln C_t - \phi N_t \right) \right]$$

s.t. $K_t + C_t = (1-\delta) K_{t-1} + R_t K_{t-1} + W_t N_t + F_t, \quad \forall t \geq 0$

given K_{-1} and $\{R_t, W_t, F_t\}_{t=0}^{\infty}$

2) Lagrangian

$$\mathcal{L} = \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \left(\ln C_t - \phi N_t - \lambda_t \left(K_t + C_t - (1-\delta) K_{t-1} - R_t K_{t-1} - W_t N_t - F_t \right) \right] \right.$$

- at time t, the hh reoptimizes:

$$\mathcal{L} = \mathbb{E}_t \left[\sum_{s=0}^{\infty} \beta^s \left(\ln C_{t+s} - \phi N_{t+s} - \lambda_{t+s} \left(K_{t+s} + C_{t+s} - (1-\delta) K_{t+s-1} - R_{t+s} K_{t+s-1} - W_{t+s} N_{t+s} - F_{t+s} \right) \right) \right]$$

- "telescope the Lagrangian"

$$\mathcal{L} = \mathbb{E}_t \Big[\ln C_t - \phi N_t - \lambda_t \left(K_t + C_t - (1-\delta) K_{t-1} - R_t K_{t-1} - W_t N_t - F_t \right)$$
$$+ \beta \left(\ln C_{t+1} - \phi N_{t+1} - \lambda_{t+1} \left(K_{t+1} + C_{t+1} - (1-\delta) K_t - R_{t+1} K_t - W_{t+1} N_{t+1} - F_{t+1} \right) \right.$$
$$\vdots$$
$$\Big]$$

3) FOC, EE, LS

$$\frac{\partial \mathcal{L}}{\partial c_t} \stackrel{!}{=} 0 = \mathbb{E}_t\left[\frac{1}{c_t} - \lambda_t\right] \Rightarrow \lambda_t = \frac{1}{c_t} \quad (1)$$

$$\frac{\partial \mathcal{L}}{\partial N_t} \stackrel{!}{=} 0 = \mathbb{E}_t\left[-\phi + \lambda_t w_t\right] \Rightarrow \lambda_t w_t = \phi \quad (2)$$

$$\frac{\partial \mathcal{L}}{\partial K_t} \stackrel{!}{=} 0 = \mathbb{E}_t\left[-\lambda_t + \beta(1-\delta)\lambda_{t+1} + \lambda_{t+1} R_{t+1})\right] \quad (3)$$

$$\lambda_t = \beta \mathbb{E}_t\left[\lambda_{t+1}(R_{t+1} + 1 - \delta)\right]$$

No! $\frac{\lambda_t}{\beta \mathbb{E}_t(\lambda_{t+1})} \neq \mathbb{E}_t(R_{t+1}) + 1 - \delta$ ↑ not the same

$$\text{Var}\, x = \mathbb{E}(x - \mathbb{E}x)^2 = \mathbb{E}(x^2 - 2x\mathbb{E}x + \mathbb{E}_x\mathbb{E}_x) = \mathbb{E}(x^2) - \mathbb{E}(x)\mathbb{E}(x)$$

Combine (1) and (3):

$$\boxed{\frac{1}{c_t} = \beta \mathbb{E}_t\left[\frac{1}{c_{t+1}}(R_{t+1} + 1 - \delta)\right]} \quad \text{EE}$$

combine (1) and (2): $\boxed{\frac{w_t}{c_t} = \phi}$ LS

4) Firm maximization

Profits: $Y_t - R_t K_{t-1} - W_t N_t$, $Y_t = e^{z_t} K_{t-1}^{\alpha} N_t^{1-\alpha}$, $0 < \alpha \leq 1$

$$\max_{K_{t-1}, N_t} \quad e^{z_t} K_{t-1}^{\alpha} N_t^{1-\alpha} - R_t K_{t-1} - W_t N_t$$

- $\frac{\partial \cdot}{\partial K_{t-1}} \overset{!}{=} 0 = \alpha \frac{Y_t}{K_{t-1}} - R_t \Rightarrow \boxed{R_t = \alpha \frac{Y_t}{K_{t-1}}}$ MPK

- $\frac{\partial \cdot}{\partial N_t} \overset{!}{=} 0 = (1-\alpha) \frac{Y_t}{N_t} - W_t \Rightarrow \boxed{W_t = (1-\alpha) \frac{Y_t}{N_t}}$ MPL

5) Resource constraint of economy $\Rightarrow C_t + I_t = Y_t$

- $Y_t = \underbrace{\alpha Y_t}_{R_t K_{t-1}} + \underbrace{(1-\alpha) Y_t}_{N_t W_t} \Leftrightarrow Y_t = R_t K_{t-1} + N_t W_t$

↳ Profits are zero, F_t for hhs thus $= 0 \ \forall t$

- $K_t + C_t = (1-\delta) K_{t-1} + \underbrace{\underbrace{R_t K_{t-1}}_{\alpha Y_t} + \underbrace{W_t N_t}_{(1-\alpha) Y_t}}_{Y_t}$

If cap. market clears, R_t = marg. prod. of K_{t-1}

If the labour market clears, W_t = marg. prod. of N_t

Thus:

$\Leftrightarrow C_t + \underbrace{K_t - (1-\delta) K_{t-1}}_{I_t} = Y_t$

$\Leftrightarrow \boxed{C_t + I_t = Y_t}$ ← Resource constraint of the economy, i.e. goods market clearing condition

↳ Was implied by hh's BC given that all other markets (capital + labor) clear.

Collect equations of the model

Add Def. of eq'm!
↳ Eq'm is when all prices clear market, and S=D.

Variables

C K Y W N R

z

1. $C_t + K_t = Y_t + (1-\delta) K_{t-1}$
2. $\frac{W_t}{C_t} = \phi$
3. $W_t = (1-\alpha) \frac{Y_t}{N_t}$
4. $\frac{1}{C_t} = \beta \mathbb{E}_t \left[\frac{1}{C_{t+1}} (R_{t+1} + 1 - \delta) \right]$
5. $R_t = \alpha \frac{Y_t}{K_{t-1}}$
6. $Y_t = e^{z_t} K_{t-1}^{\alpha} N_t^{1-\alpha}$ → exogenous

check: as many equations as variables.
6 equations, 6 variables ✓

what about investment?
- add an equation with investment
  $I_t + C_t = Y_t$ or $K_t = (1-\delta) K_{t-1} + I_t$
- what about adding both equations? ↯ 8 eq. in 7 variables! No!

Write the model without factor prices

1. $C_t + K_t = Y_t + (1-\delta) K_{t-1}$
2. $(1-\alpha) \frac{Y_t}{C_t N_t} = \phi$
3. $\frac{1}{C_t} = \beta \mathbb{E}_t \left[\frac{1}{C_{t+1}} \left(\alpha \frac{Y_{t+1}}{K_t} + 1 - \delta \right) \right]$
4. $Y_t = e^{z_t} K_{t-1}^{\alpha} N_t^{1-\alpha}$

→ Different ways to write the same model!
- can add equations to track additional variables (like invest above).

The Method of Undetermined Coefficients

What does it mean to solve a model?

→ Finding policy functions: $C_t = C(K_{t-1}, Z_t)$

$K_t = K(\underbrace{K_{t-1}, Z_t}_{})$

these are our state variables, i.e. variables, whose values at t are given.

↳ Ev. impossible to find these functions in general

Case 1: $\delta = 1$

6) Now eq. 1 and 3 become:

1. $C_t + K_t = Y_t$

3. $\frac{1}{C_t} = \beta \mathbb{E}_t \left[\frac{1}{C_{t+1}} \cdot \alpha \frac{Y_{t+1}}{K_t} \right]$

undetermined coeff.

Hint from problem statement: $C_t = \gamma Y_t$

→ using the guess/hint, verify it by determining γ

1) $\gamma Y_t + K_t = Y_t \Rightarrow K_t = (1-\gamma) Y_t$

2) $(1-\alpha) \frac{Y_t}{\gamma Y_t N_t} = \phi \Rightarrow N_t = \frac{1-\alpha}{1-\gamma}$, hours worked are constant!

3) $\frac{1}{\gamma Y_t} = \beta \mathbb{E}_t \left[\frac{1}{\gamma Y_{t+1}} \cdot \alpha \frac{Y_{t+1}}{(1-\gamma) Y_t} \right]$

$\Leftrightarrow 1 = \beta \frac{\alpha}{1-\gamma} \Rightarrow 1 - \gamma = \alpha\beta \Rightarrow \boxed{\gamma = 1 - \alpha\beta}$ ✓

4) $Y_t = e^{Z_t} K_{t-1}^{\alpha} \left(\frac{1-\alpha}{\phi(1-\alpha\beta)} \right)^{1-\alpha} = Y(K_{t-1}, Z_t)$! Policy Function.

$$\boxed{\begin{aligned}
Y_t &= Y(K_{t-1}, Z_t) = e^{Z_t} K_{t-1}^{\alpha} \left(\frac{1-\alpha}{\phi(1-\alpha\beta)} \right)^{1-\alpha} \\
K_t &= K(K_{t-1}, Z_t) = \alpha\beta e^{Z_t} K_{t-1}^{\alpha} \left(\frac{1-\alpha}{\phi(1-\alpha\beta)} \right)^{1-\alpha} \\
C_t &= C(K_{t-1}, Z_t) = (1-\alpha\beta) e^{Z_t} K_{t-1}^{\alpha} \left(\frac{1-\alpha}{\phi(1-\alpha\beta)} \right)^{1-\alpha} \\
N_t &= N(K_{t-1}, Z_t) = \frac{1-\alpha}{\phi(1-\alpha\beta)}
\end{aligned}}$$

7.) Confirm that $\hat{c}_t (= \ln c_t - \ln \bar{c})$ + $\hat{k}_t$ are linear functions of z_t and $\hat{k}_{t-1}$

- $$\bar{k} = \alpha\beta e^{\bar{z}} \bar{k}^{\alpha} \left(\frac{1-\alpha}{\phi(1-\alpha\beta)}\right)^{1-\alpha}$$

$$\ln k_t = \ln\left(\alpha\beta\left(\frac{1-\alpha}{\phi(1-\alpha\beta)}\right)^{1-\alpha}\right) + z_t + \alpha \ln k_{t-1}$$

$$\underbrace{\ln k_t - \ln \bar{k}}_{\hat{k}_t} = \underbrace{z_t - \bar{z}}_{z_t - \bar{z}} + \alpha \underbrace{(\ln k_{t-1} - \ln \bar{k})}_{\hat{k}_{t-1}}$$

- $$\bar{c} = (1-\alpha\beta) e^{\bar{z}} \bar{k}^{\alpha} \left(\frac{1-\alpha}{\phi(1-\alpha\beta)}\right)^{1-\alpha}$$

$$\ln c_t = \ln\left((1-\alpha\beta)\left(\frac{1-\alpha}{\phi(1-\alpha\beta)}\right)^{1-\alpha}\right) + z_t + \alpha \ln k_{t-1}$$

$$\ln c_t - \ln \bar{c} = z_t - \bar{z} + \alpha \underbrace{(\ln k_{t-1} - \ln \bar{k})}_{\hat{k}_{t-1}}$$

SUMMARY:

In the special case $\sigma = 1$, we can determine the policy fcts. in closed form and these functions are loglinear!

For capital: $\hat{k}_t = z_t + \alpha \hat{k}_{t-1}$ (set $\bar{z} = 0$ for convenience)

1. - If $|\alpha| < 1$ then if $z_t = 0\ \forall t$, $\hat{k}_t$ will converge to 0 for any $\hat{k}_{-1}$ (any initial value)
   - If $|\alpha| > 1$ then if $z_t = 0\ \forall t$, $\hat{k}_t$ will diverge to $\pm\infty$ for any $\hat{k}_{-1} \neq 0$ $\Rightarrow$ economic nonsense!
   - fortunately, $0 < \alpha < 1$ and hence $|\alpha| < 1$.

   Our solution is (asymptotically) stable.
2. If $z_t \sim \mathcal{N}(0, \sigma^2)$ $\rightarrow$ all of our variables in log-deviations will be normally distributed as well.

- Why do we care about all of this?

Given, say, K_{-1} and $\{z_t\}_{t=0}^{T}$ we can now compute $\{Y_t, K_t, C_t, N_t\}_{t=1}^{T}$.
We have solved the model by finding recursive fcts. that map the state variables into our endogenous variables.

Our policy fcts. are loglinear (linear in the logs of our var.).
Particularly:

$$\ln K_t = \alpha \ln K_{t-1} + z_t + \ln(\alpha\beta\theta^{1-\alpha})$$

Solve for $\ln \bar{K}$ if $z_t = 0$:

$$\ln \bar{K} = \alpha \ln \bar{K} + \ln(\alpha\beta\theta^{1-\alpha})$$

$$\Leftrightarrow \ln \bar{K} = \frac{1}{1-\alpha} \ln(\alpha\beta\theta^{1-\alpha})$$

$$\Rightarrow \boxed{\ln K_t - \ln \bar{K} = \alpha(\ln K_{t-1} - \ln \bar{K}) + z_t}$$

- If $|\alpha| < 1$, $\ln K_t \to \ln \bar{K}$ for any K_{-1} (holding $\{z_t\}_{t=0}^{\infty} = 0$)
- If $|\alpha| > 1$, $\ln K_t \to \pm\infty$ for any $\ln K_{-1} \neq \ln \bar{K}$ (holding $\{z_t\}_{t=0}^{\infty}$)

Fortunately, $0 < \alpha < 1$.

Case 2: $\delta \neq 1$

· Try $C_t = \upsilon Y_t$

1) $K_t + \upsilon Y_t = Y_t + (1-\delta) K_{t-1} \Rightarrow K_t = (1-\upsilon) Y_t + (1-\delta) K_{t-1}$

Subst. into 3)

$$\frac{1}{\upsilon Y_t} = \beta \, \mathbb{E}_t \left[\frac{1}{\upsilon Y_{t+1}} \left(\alpha \frac{Y_{t+1}}{(1-\upsilon) Y_t + (1-\delta) K_{t-1}} + 1 - \delta \right) \right]$$

↳ not going to simplify .... ×

Could try: $C_t = \upsilon \dfrac{Y_t}{K_{t-1}}$, ... → not going to work ×

....

Way forward?

- If $\delta = 1$, solution was loglinear.
- If $\delta \neq 1$, model is "close" to $\delta = 1$ model
  - loglinearize the equations and guess loglinear solutions

8)

State the problem

1) $C_t + K_t = Y_t + (1-\delta) K_{t-1}$

2) $\phi C_t = (1-\alpha) \dfrac{Y_t}{N_t}$ ⟶ $N_t = \dfrac{(1-\alpha)}{\phi} \dfrac{Y_t}{C_t}$ → insert into 4)

3) $\dfrac{1}{C_t} = \beta \, \mathbb{E}_t \left[\dfrac{1}{C_{t+1}} \left(\alpha \dfrac{Y_{t+1}}{K_t} + 1 - \delta \right) \right]$

4) $Y_t = e^{z_t} K_{t-1}^{\alpha} N_t^{1-\alpha}$ ⟶ $Y_t = e^{z_t} K_{t-1}^{\alpha} \left(\dfrac{(1-\alpha) Y_t}{\phi C_t} \right)^{1-\alpha}$

Solve for Y_t:

$$Y_t = \left[e^{z_t} K_{t-1}^{\alpha} \left(\frac{1-\alpha}{\phi} \right)^{1-\alpha} C_t^{\alpha - 1} \right]^{1/\alpha}$$

Insert this into 1) and 3):

$$\boxed{\begin{aligned} &1) \quad C_t + K_t = \left[e^{z_t} K_{t-1}^{\alpha} \left(\frac{1-\alpha}{\phi} \right)^{1-\alpha} C_t^{\alpha-1} \right]^{1/\alpha} + (1-\delta) K_{t-1} \\ &2) \quad \frac{1}{C_t} = \beta \, \mathbb{E}_t \left[\frac{1}{C_{t+1}} \left(\alpha \frac{\left[e^{z_{t+1}} K_t^{\alpha} \left(\frac{1-\alpha}{\phi} \right)^{1-\alpha} C_{t+1}^{\alpha-1} \right]^{1/\alpha}}{K_t} + 1 - \delta \right) \right] \end{aligned}}$$

a) Log-linearization

"Simplify" for log-linearization → distribute exponents and multiply products out.

1) $C_t + K_t = e^{z_t} K_{t-1} \left(\frac{1-\alpha}{\phi}\right)^{\frac{1-\alpha}{\alpha}} C_t^{\frac{\alpha-1}{\alpha}} + (1-\delta) K_{t-1}$

2) $C_t^{-1} = \beta \mathbb{E}_t \left[C_{t+1}^{-1/\alpha} \, \alpha e^{\frac{1}{\alpha} z_{t+1}} \left(\frac{1-\alpha}{\phi}\right)^{\frac{1-\alpha}{\alpha}} \right] + \beta(1-\delta) \mathbb{E}_t \left[C_{t+1}^{-1} \right]$

? $C_{t+1}^{-1} C_{t+1}^{\frac{\alpha-1}{\alpha}} = C_{t+1}^{\frac{\alpha-1}{\alpha}-1} = C_{t+1}^{-1/\alpha}$ and also: $K_t^{\alpha-\alpha} = 1$

In a steady state:

1) $\bar{C} + \bar{K} = \bar{K} \left(\frac{1-\alpha}{\phi}\right)^{\frac{1-\alpha}{\alpha}} \bar{C}^{\frac{\alpha-1}{\alpha}} + (1-\delta)\bar{K}$

2) $\bar{C}^{-1} = \beta \bar{C}^{-1/\alpha} \alpha \left(\frac{1-\alpha}{\phi}\right)^{\frac{1-\alpha}{\alpha}} + \beta(1-\delta)\bar{C}^{-1}$

(can solve ~~for~~ 2) for $\bar{C}$ and then 1) for $\bar{K}$)
we'll take $\bar{C}$ and $\bar{K}$ as given.

Log-linearize

i) Insert $X_t = \bar{X} e^{\hat{x}_t}$

1) $\bar{C} e^{\hat{c}_t} + \bar{K} e^{\hat{k}_t} = e^{1/\alpha z_t} \bar{K} e^{\hat{k}_{t-1}} \left(\frac{1-\alpha}{\phi}\right)^{\frac{1-\alpha}{\alpha}} \left(\bar{C} e^{\hat{c}_t}\right)^{\frac{\alpha-1}{\alpha}} + (1-\delta) \bar{K} e^{\hat{k}_{t-1}}$

ii) distribute exponents and gather products of e
(remember $e^x e^y = e^{x+y}$)

$\bar{C} e^{\hat{c}_t} + \bar{K} e^{\hat{k}_t} = e^{1/\alpha z_t + \hat{k}_{t-1} + \frac{\alpha-1}{\alpha} \hat{c}_t} \bar{K} \left(\frac{1-\alpha}{\phi}\right)^{\frac{1-\alpha}{\alpha}} \bar{C}^{\frac{\alpha-1}{\alpha}} + (1-\delta) \bar{K} e^{\hat{k}_{t-1}}$

iii) Approximate $e^{\hat{x}_t} \approx 1 + \hat{x}_t$

$\underline{\bar{C}} (1 + \hat{c}_t) + \underline{\bar{K}} (1 + \hat{k}_t) = (\underline{1} + \frac{1}{\alpha} z_t + \hat{k}_{t-1} + \frac{\alpha-1}{\alpha} \hat{c}_t) \underline{\bar{K} \left(\frac{1-\alpha}{\phi}\right)^{\frac{1-\alpha}{\alpha}} \bar{C}^{\frac{\alpha-1}{\alpha}}}$
$+ (\underline{1-\delta}) \underline{\bar{K}} (\underline{1} + \hat{k}_{t+1})$.

From our st. st., free components drop out.

Then:

$$\bar{C}\hat{c}_t + \bar{K}\hat{k}_t = (\bar{C}+\delta\bar{K})\left(\frac{1}{\alpha}z_t + \hat{k}_{t-1} + \frac{\alpha-1}{\alpha}\hat{c}_t\right) + (1-\delta)\bar{K}\hat{k}_{t-1}$$

- Gather like terms together

$$\boxed{\bar{K}\hat{k}_t = \frac{\bar{C}+\delta\bar{K}}{\alpha}z_t + \left(\frac{\alpha-1}{\alpha}(\bar{C}+\delta\bar{K}) - \bar{C}\right)\hat{c}_t + (\bar{C}+\bar{K})\hat{k}_{t-1}} \quad \text{(I)}$$

2) $$c_t^{-1} = \alpha\beta\left(\frac{1-\alpha}{\phi}\right)^{\frac{1-\alpha}{\alpha}} \mathbb{E}_t\left[e^{\frac{1}{\alpha}z_{t+1}} c_{t+1}^{-1/\alpha}\right] + \beta(1-\delta)\mathbb{E}_t\left[c_{t+1}^{-1}\right]$$

i) insert $x_t = \bar{x}e^{\hat{x}_t}$

$$(\bar{c}e^{\hat{c}_t})^{-1} = \alpha\beta\left(\frac{1-\alpha}{\phi}\right)^{\frac{1-\alpha}{\alpha}} \mathbb{E}_t\left[e^{1/\alpha\, z_{t+1}} (\bar{c}e^{\hat{c}_{t+1}})^{-1/\alpha}\right] + \beta(1-\delta)\mathbb{E}_t\left[(\bar{c}e^{\hat{c}_{t+1}})^{-1}\right]$$

ii) distribute exponents and gather products of e

$$\bar{c}^{-1}e^{-\hat{c}_t} = \alpha\beta\left(\frac{1-\alpha}{\phi}\right)^{\frac{1-\alpha}{\alpha}} \bar{c}^{-1/\alpha}\mathbb{E}_t\left[e^{1/\alpha\, z_{t+1} - 1/\alpha\, \hat{c}_{t+1}}\right] + \beta(1-\delta)\bar{c}^{-1}\mathbb{E}_t(e^{-\hat{c}_{t+1}}$$

iii) approx. $e^{\hat{x}_t} \approx 1 + \hat{x}_t$.

$$\bar{c}^{-1}(1-\hat{c}_t) = \alpha\beta\left(\frac{1-\alpha}{\phi}\right)^{\frac{1-\alpha}{\alpha}} \bar{c}^{-1/\alpha}\mathbb{E}_t\left[1 + \frac{1}{\alpha}z_{t+1} - \frac{1}{\alpha}\hat{c}_{t+1}\right] + \beta(1-\delta)\bar{c}^{-1}\mathbb{E}_t\left[1-\hat{c}_{t+1}\right]$$

From st.st.: cancel free constants.

Then:

$$-\bar{c}^{-1}\hat{c}_t = \bar{c}^{-1}(1-\beta(1-\delta))\mathbb{E}_t\left[\frac{1}{\alpha}z_{t+1} - \frac{1}{\alpha}\hat{c}_{t+1}\right] - \beta(1-\delta)\bar{c}^{-1}\mathbb{E}_t\left[\hat{c}_{t+1}\right]$$

$$-\hat{c}_t = \frac{(1-\beta(1-\delta))}{\alpha}\mathbb{E}_t z_{t+1} - \underbrace{\frac{1}{\alpha} + \beta(1-\delta)\left(1-\frac{1}{\alpha}\right)}_{1/\alpha\,(1-\beta(1-\delta)(1-\alpha))}\mathbb{E}_t\left[\hat{c}_{t+1}\right]$$

$$\Rightarrow \boxed{-\hat{c}_t = \frac{(1-\beta(1-\delta))}{\alpha}\mathbb{E}_t\left[z_{t+1}\right] - \frac{1}{\alpha}(1-\beta(1-\delta)(1-\alpha))\mathbb{E}_t\left[\hat{c}_{t+1}\right]} \quad \text{(II)}$$

10) Guess and Verify

Assume that $z_{t+1} = \rho z_t + \varepsilon_{t+1}$, $|\rho| < 1$, $\varepsilon_t \overset{iid}{\sim} (0, \sigma^2)$

$$\hat{c}_t = \eta_{ck}\hat{k}_{t-1} + \eta_{cz}z_t$$

$$\hat{k}_t = \eta_{kk}\hat{k}_{t-1} + \eta_{kz}z_t.$$

Inserting into (I)

$$0 = -\bar{K}\eta_{kk}\hat{k}_{t-1} - \bar{K}\eta_{kz}z_t + \frac{\bar{c}+\delta\bar{K}}{\alpha}z_t + \left(\frac{\alpha-1}{\alpha}\delta\bar{K} - \frac{1}{\alpha}\bar{c}\right)\eta_{ck}\hat{k}_{t-1}$$
$$+\left(\frac{\alpha-1}{\alpha}\delta\bar{K} - \frac{1}{\alpha}\bar{c}\right)\eta_{cz}z_t + (\bar{c}+\bar{K})\hat{k}_{t-1}$$

Gather coeffs on z_t & $\hat{k}_{t-1}$

$$0 = \overbrace{\left(-\bar{K}\eta_{kk} + \left(\frac{\alpha-1}{\alpha}\delta\bar{K} - \frac{1}{\alpha}\bar{c}\right)\eta_{ck} + \bar{c} + \bar{K}\right)}^{=0}\hat{k}_{t-1}$$
$$+\underbrace{\left(-\bar{K}\eta_{kz} + \frac{\bar{c}+\delta\bar{K}}{\alpha} + \left(\frac{\alpha-1}{\alpha}\delta\bar{K} - \frac{1}{\alpha}\bar{c}\right)\eta_{cz}\right)}_{=0}z_t$$

Terms in parantheses must be equal to zero for our equation to hold for any values of z_t & $\hat{k}_{t-1}$.

Gives us 2 equations in $\underbrace{\eta_{kk}, \eta_{ck}, \eta_{kz}, \eta_{cz}}_{4 \text{ variables}}$.

(II) $$0 = \eta_{ck}\hat{k}_{t-1} + \eta_{cz}z_t + \frac{1-\beta(1-\delta)}{\alpha}\rho z_t$$
$$- \frac{1+(\alpha-1)\beta(1-\delta)}{\alpha}\eta_{ck}\boxed{\hat{k}_t} - \frac{1+(\alpha-1)\beta(1-\delta)}{\alpha}\eta_{cz}\rho z_t$$

replace with $\eta_{kk}\hat{k}_{t-1} + \eta_{kz}z_t$

$$= \eta_{ck}\hat{k}_{t-1} + \eta_{cz}z_t + \frac{1-\beta(1-\delta)}{\alpha}\rho z_t - \frac{1+(\alpha-1)\beta(1-\delta)}{\alpha}\eta_{cz}\rho z_t$$
$$- \frac{1+(\alpha-1)\beta(1-\delta)}{\alpha}\eta_{ck}\eta_{kk}\hat{k}_{t-1} - \frac{1+(\alpha-1)\beta(1-\delta)}{\alpha}\eta_{ck}\eta_{kz}z_t$$

Gather coeffs on $\hat{k}_{t-1}$ & z_t:

$$0 = \overbrace{\left(\eta_{ck} - \frac{1+(\alpha-1)\beta(1-\delta)}{\alpha}\eta_{ck}\eta_{kk}\right)}^{=0}\hat{k}_{t-1}$$
$$+ \underbrace{\left(\eta_{cz} + \frac{1-\beta(1-\delta)}{\alpha}\rho - \frac{1+(\alpha-1)\beta(1-\delta)}{\alpha}(\eta_{ck}\eta_{kz} + \eta_{cz}\rho)\right)}_{=0}z_t.$$

Focus on solving for $\eta_{ck}+\eta_{kk}$ (with them, η_{cz}, η_{kz} can be solved from the remaining eqs.)

1) $$0=\eta_{ck}-\underbrace{\frac{1+(\alpha-1)\beta(1-\delta)}{\alpha}}_{=\gamma_3}\eta_{ck}\eta_{kk}$$

$$0=\eta_{ck}+\gamma_3\eta_{ck}\eta_{kk}$$

2) $$0=-\bar{k}\eta_{kk}+\left(\frac{\alpha-1}{\alpha}\delta\bar{k}-\frac{1}{\alpha}\bar{c}\right)\eta_{ck}+\bar{c}+\bar{k} \quad | \text{ solve for } \eta_{ck}$$

$$\eta_{ck}=\gamma_1\eta_{kk}-\gamma_2, \quad \gamma_1=\frac{\bar{k}}{\frac{\alpha-1}{\alpha}\delta\bar{k}-\frac{1}{\alpha}\bar{c}}, \quad \gamma_2=\frac{\bar{c}+\bar{k}}{\frac{\alpha-1}{\alpha}\delta\bar{k}-\frac{1}{\alpha}\bar{c}}$$

Insert into 1):

$$0=\gamma_1\eta_{kk}-\gamma_2+\gamma_3\gamma_1\eta_{kk}^2-\gamma_2\gamma_3\eta_{kk}$$

$$0=\gamma_3\gamma_1\eta_{kk}^2+(\gamma_1-\gamma_3\gamma_2)\eta_{kk}-\gamma_2$$ ← Quadratic eq. in η_{kk}!

$$\Rightarrow \eta_{kk_{1,2}}=\frac{\gamma_3\gamma_2-\gamma_1\pm\sqrt{(\gamma_1-\gamma_3\gamma_2)^2+4\gamma_1\gamma_2\gamma_3}}{2\gamma_3\gamma_1}$$

Let's check it out:

$$\gamma_1^2-2\gamma_1\gamma_3\gamma_2+(\gamma_3\gamma_2)^2+4\gamma_1\gamma_2\gamma_3=(\gamma_1+\gamma_2\gamma_3)^2 \quad | \text{ Subst. into above expression}$$

$$\eta_{kk_{1,2}}=\frac{\gamma_3\gamma_2-\gamma_1\pm(\gamma_1+\gamma_2\gamma_3)}{2\gamma_3\gamma_1}$$

$$\nearrow \frac{\gamma_3\gamma_2-\gamma_1+\gamma_1+\gamma_2\gamma_3}{2\gamma_3\gamma_1}=\frac{\gamma_2}{\gamma_1}$$

$$\searrow \frac{\gamma_3\gamma_2-\gamma_1-\gamma_1-\gamma_2\gamma_3}{2\gamma_3\gamma_1}=-\frac{1}{\gamma_3}$$

$$\eta_{kk_1}=\frac{\bar{c}+\bar{k}}{\bar{k}}>1$$ → choosing this would lead to an instable $\hat{k}_t=\eta_{kk}\hat{k}_{t-1}$

$$\eta_{kk_2}=\frac{\alpha}{1+(\alpha-1)\beta(1-\delta)}=\frac{\alpha}{1-(1-\alpha)\beta(1-\delta)}$$ ← Hopefully less than 1, in absolute value

when $\delta = 1$, $\eta_{kk2} = \alpha$, the solution we want in closed above

11)

- η_{kk2} in the general case is inside the unit circle

$$0 < \alpha, \beta, \delta < 1$$

$$1 > (1-\alpha)\beta(1-\delta)$$

$$\Leftrightarrow \quad -1 \overset{?}{<} \frac{\alpha}{1-(1-\alpha)\beta(1-\delta)} \overset{?}{<} 1$$

$$\Leftrightarrow \underbrace{(1-\alpha)\beta(1-\delta)-1}_{<0,\text{ but } \alpha>0} \overset{?}{<} \alpha \overset{?}{<} 1-(1-\alpha)\beta(1-\delta)$$

So, $|\eta_{kk2}| < 1$ boils down to: $\alpha \overset{?}{<} 1-(1-\alpha)\beta(1-\delta)$

$$\Leftrightarrow (1-\beta(1-\delta))\,\alpha \overset{?}{<} 1-\beta(1-\delta)$$

$$\Leftrightarrow \quad \alpha \overset{?}{<} 1 \; ✓$$

So, $|\eta_{kk1}| > 1$ and $|\eta_{kk2}| < 1$

we choose the stable solution

12) η_{ck}, η_{cz} and η_{kz} follow by solving linear equations.