INATTENTIVENESS AND THE TAYLOR PRINCIPLE

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ABSTRACT. We present determinacy bounds on monetary policy in three models of inattentiveness - sticky information, imperfect common knowledge, and arbitrary finite inattentiveness. We find that these bounds are identical across these models as they all share a common vertical long run Phillips curve. The resulting bounds are more conservative than in the standard Calvo sticky price New Keynesian model. Specifically, the Taylor principle is now necessary directly - no amount of output targeting can substitute for the monetary authority's concern for inflation. These determinacy bounds are obtained by appealing to frequency domain and forecasting/prediction innovation techniques that themselves provide novel interpretations of the Phillips curves.

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1. INTRODUCTION

We address the question of bounds on monetary policy to deliver a unique equilibrium when the long run Phillips curve is vertical. We find that when this long run condition holds, only the coefficients in the Taylor rule itself with respect to inflation matter for determinacy. We show this specifically in models of inattentiveness that fulfill the natural rate hypothesis and contrast the results to the canonical sticky price model. If the long run Phillips curve is vertical, no amount of output gap targeting, forward or backward-looking

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inflation targeting can substitute for a more than one-for-one response to current inflation directly. That is, the Taylor principle is necessary in an absolute sense.

Our central contribution is to the understanding of the limits on monetary policy to ensure a unique, stationary or determinant equilibrium following Blanchard and Kahn (1980). In particular we cast our attention to models with information rigidities and add the sticky information model of Mankiw and Reis (2002), the imperfect common knowledge model of Nimark (2008) and our general model of finite attentiveness, specified only by imposing the natural rate hypothesis holding at a finite horizon, to the list of models analyzed via restrictions on coefficients in monetary policy's Taylor (1993) rule following Clarida, Galí, and Gertler (2000) and Woodford (2001b) for the sticky price model.[1] We find that a vertical long run Phillips curve brings a unified perspective across models with imperfect information on determinacy and demands that the more than one-for-one response of the nominal interest rate in response to inflation be in response to current inflation directly. This calls assertions such as Woodford (2003, pp. 254–255) "... indeed, a large enough [response to] *either* [the output gap or inflation] suffices to guarantee determinacy" into question. In the absence of a trade off between inflation and the output gap in the long run - a vertical Phillips curve - we show that no degree of output gap targeting can substitute for a reaction to inflation. This has a very similar flavor to Davig and Leeper (2007) - in their model of regime switching monetary policy, it is the long run characteristics that are decisive for determinacy, their long run Taylor principle - we argue the same logic applies to the Phillips curve, its long run slope being the decisive characteristic. We demonstrate the long run verticality of our three Phillips curves and compare to the dynamic trade off inherent in the Calvo model. The lack of a dynamic structure in inflation in the long run in the three inattentive Phillips curves (it is some imperfect forecast or prediction of inflation in inattentive Phillips curves, whereas the sticky price Phillips curve involve current and future expected inflation) precludes past or future expected inflation targeting to act as substitute for monetary policy's reaction to current inflation. That is, the Taylor principle is necessary in a much stricter sense than

[1]See Bullard and Mitra (2002) and Lubik and Marzo (2007) for compendia of determinacy results in sticky price models. Woodford (2003) and Galí (2008) provide textbook and Clarida, Galí, and Gertler (1999) and Christiano, Trabandt, and Walentin (2011) survey article treatments. Loisel (2022) provides determinacy bounds for a class of models with a wide of a lead-lag dependence. McCallum (1981) is an early reference on determinacy via an interest rate rule. See Benhabib, Schmitt-Grohé, and Uribe (2001) and Cochrane (2011) for critical views on (local) determinacy.

sticky price analyses would otherwise lead one to conclude, closing a gap in the literature by deriving the determinacy bounds in models with vertical long run Phillips curves.

In the sticky information model Mankiw and Reis (2002) assume that firms update their information in an infrequent manner, i.e. firms adjust their prices delayed but optimally in response to their information sets. Once information is updated, it represents the entire state of the economy. Mankiw and Reis (2007), Branch (2007) or Coibion and Gorodnichenko (2015a) and subsequent literature support the sticky information model as it outperforms the standard New Keynesian model and improves the dynamics of macroeconomic responses to monetary policy, precluding disinflationary expansions and attenuated responses to anticipated shocks and persistent zero lower bound episodes.[2] Central to the different role of monetary policy is the sticky information model's vertical long-run Phillips curve, even out of equilibrium, whereas the sticky-price model imposes a systematic relationship between inflation and output, stable even in the long run.[3]

Imperfect common knowledge among firms posits another source of price inertia. Contrary to the sticky information model suppliers in a common knowledge framework choose their prices based on their noisy observations. Woodford (2001a) demonstrates that both unanticipated and anticipated monetary shocks have real effects in the Lucas (1973) island model as higher order expectations - the expectations of others' expectations - persist in the absence of public information.[4] Adam (2007) analyzes optimal monetary policy in a model of nominal demand with imperfect common knowledge and flexible prices and Nimark (2008) combines Calvo price setting with noisy information and derives

[2]Further support comes from empirical evidence on the formation of macroeconomic expectations. Coibion and Gorodnichenko (2015a), Mertens and Nason (2020), Nason and Smith (2021), amongst others, show that stickiness in survey forecasts crucially depends on the inflation process. Andrade and Le Bihan (2013), Roth and Wohlfart (2020), Reis (2020), Cornand and Hubert (2022), Carroll, Crawley, Slacalek, Tokuoka, and White (2020), Link, Peichl, Roth, and Wohlfart (2023) document systematic biases in expectations and disagreement in inflation expectations among various types of agents tracing back to information rigidity. Chou, Easaw, and Minford (2023) estimate different models with incomplete information structures and show that Mankiw and Reis's (2002) sticky information generates a persistent and delayed response of inflation and output gap to a monetary policy shock empirically and An, Abo-Zaid, and Sheng (2023) estimate a sticky information model with endogenous inattention using US survey data and show that monetary policy's impact on the economy is amplified when both firms and households agents are inattentive.

[3]See, e.g., Woodford (2003, p. 254) or Galí (2008, p. 78).

[4]More recently, Acharya, Benhabib, and Huo (2021) and Huo and Takayama (2022) show that changes in agents' beliefs due to information frictions lead to persistent aggregate fluctuations.

an imperfect common knowledge Phillips curve. Nimark's (2008) analysis in a general equilibrium model with New Keynesian IS demand and an interest rate rule is silent about determinacy and this is where our analysis picks up the standard.[5] This model is a poignant alternative specifically as Angeletos and Lian (2018) demonstrate that this imperfect common knowledge approach rectifies the resolution of paradoxical prediction of New Keynesian models with Mankiw and Reis's (2002) sticky information by Chung, Herbst, and Kiley (2015) and Kiley (2016) with the micro data on price stickiness.[6]

Our analysis contributes to the literature on monetary policy in economies with limited information[7] that can provide markedly different policy recommendations than in full information settings like the canonical sticky price framework. Beginning with Ball, Mankiw, and Reis (2005) who consider information stickiness in price setting which leads monetary policy to favor price level over inflation targeting. Angeletos, Iovino, and La'O (2016) show that incomplete information leads to nominal rigidities which can be neutralized by the conduct of monetary policy in the sticky price framework. Paciello and Wiederholt (2014) study optimal policy when firms are rationally inattentive to the state of the economy. Angeletos and La'O (2020) extend the "leaning against the wind" policy to firms' information-dependent actions. Bernstein and Kamdar (2023) and Iovino, La'O, and Mascarenhas (2022) examine the effects of informationally constrained policy makers. Ou, Zhang, and Zhang (2021) find that combining the Calvo friction with imperfect common knowledge leads to two *separate* price dispersion welfare channels associated with each of these friction individually. We maintain the standard concept of determinacy à la Blanchard and Kahn (1980) and while this is not the only concept

[5]Lubik, Matthes, and Mertens (2023) analyze the dynamic effects of sunspot shocks with an imperfectly informed central bank and foreshadow our result that the Taylor principle holds in a strict sense and responses to projections of variables cannot substitute for responses directly to realized inflation.

[6]Angeletos and Huo (2021) further address the potential endogeneity of information in dynamic beauty contests encapsulated in imperfect common knowledge among firms and information theoretic assumptions that relate to Nimark's (2008) assumption of common knowledge of rationality. Indeed they support their dynamic orthogonalization of innovation information by appealing to its nesting of Mankiw and Reis's (2002) sticky information as a special case. This close conceptual relationship is echoed by Chahrour and Jurado (2018) who provide an equivalence between news and noise in agents' beliefs and Coibion and Gorodnichenko (2015b) who find that sticky-information and noisy-information models both point to the same relationship between ex post mean forecast errors and ex ante mean forecast revisions.

[7]See Hellwig, Kohls, and Veldkamp (2012) for a unified framework.

uniqueness,[8] our analysis provides the classical Taylor principle determinacy bounds on monetary policy thus far missing for economies with limited information but is silent as to these alternate perspectives on equilibria.

We also contribute technically to linear DSGE equilibrium principles in the complex analysis and frequency domain following Futia (1981), Whiteman (1983), and Kasa (2000) to address information frictions.[9] Loisel (2022) addresses restrictions on monetary policy via complex analysis also using Roché's theorem, yet remains in the time domain. Tan and Walker (2015), Tan (2021), and Al-Sadoon (2020) on the other hand use frequency domain approaches to solve linear rational expectations models in the vein of Whiteman (1983).[10] Finally, Han, Tan, and Wu (2022) and Jurado (2023) use frequency domain techniques such as Wiener-Hopf prediction and Wiener-Kolmogorov filter to solve models of imperfect information yet the former focuses on the numerical solution and the latter on rational inattention - neither addresses the bounds on monetary policy for determinacy under imperfect information. Specifically, we derive recursive representations of the Phillips curves, in the frequency domain for sticky information and in a higher order expectations operator for imperfect common knowledge, enabling both novel interpretations and the analysis of determinacy in both models. These recursions allow us to separate the long run dynamic restrictions from sequences of prediction/forecast errors responsible for the rich shorter run dynamics, exactly analogously to our generic model of finite inattentiveness. These recursive representations are a first in the literature and allows us to derive our analytic results in a forward-looking environment with a standard New Keynesian IS curve specifying the demand side.

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[8]For example, appealing to coordination concepts of uniqueness, Angeletos and Lian (2023) reformulate the New Keynesian model as a dynamic game under imperfect information and Acharya, Benhabib, and Huo (2021) examine sentiment equilibria in beauty game framework. Angeletos and Huo (2021) spells out specific assumptions that bypass possible endogeneities of information consistent with the exogenous Poisson arrival in sticky information and Nimark's (2008) assumption of common knowledge of rationality.

[9]Empirically, Watson (1993) and Diebold, Ohanian, and Berkowitz (1998) decompose macroeconomic time series data into different frequencies to identify business cycle drivers, King and Rebelo (1993) focuses on low, Beaudry, Galizia, and Portier (2020) on medium-term frequencies. Angeletos, Collard, and Dellas (2020) maps shocks from the frequency domain to address the business cycle, and Rünstler and Vlekke (2018) and Strohsal, Proaño, and Wolters (2019) extend to financial cycles.

[10]Al-Sadoon's (2020) focus is on maintaining continuity in parameters as a fundamental empirical approach. Tan and Walker (2015) and Tan (2021) focus on numerical solution and estimation and like Al-Sadoon (2018) and Onatski (2006) provide determinacy results for linear models with finite lagged expectations precluding their application to the sticky information model.

This paper is structured as follows. In section 2 we review the determinacy bounds on our baseline Taylor rule in the sticky price model and its frictionless counterpart. Next, section 3 introduces the three alternative supply curves we examine here - we relate them to the New Keynesian Phillips curve and the long run neutrality encapsulated in the natural rate hypothesis. In sections 4, 5, and 6 we derive the determinacy restrictions on the three alternative supply curves, which coincide with the determinacy bounds in the frictionless model we began with. We examine the robustness of our findings in section 7 by analyzing the implications of a monetary policy rule extended to arbitrary targeting horizons. Lastly we conclude.

2. Existence and Uniqueness: Frictionless and Sticky Prices

We fix ideas by first reviewing the conditions for determinacy in two basic models from the literature, a frictionless model of Cochrane (2011) and the textbook New Keynesian model, Woodford (2003) or Galí (2008). When there is separation between the nominal and real sides of the economy as in the simplest of models used by Cochrane (2011) in his exposition on determinacy in New Keynesian analysis, a loglinear Fisher equation which is the standard New Keynesian dynamic IS equation with an exogenous real side of the economy, output gap targeting is by constuction irrelevant and the Taylor principle holds in a strict sense. This is no longer the case with the New Keynesian Phillips curve which allows monetary policy to substitute a concern for the output gap for its concern for inflation. The remainder of the paper will argue that the former results hold more generally in models of inattentiveness and that the latter is dubious as it rests on the New Keynesian Phillips curve failing to satisfy the natural rate hypothesis due to it remaining non vertical even in the long run.

We will address determinacy initially in our analysis with the following Taylor rule

$$R_t = \phi_\pi \pi_t + \phi_y y_t \tag{1}$$

where R_t is the nominal interest rate, π_t inflation, and y_t is the output gap. We will assume nonnegative coefficients, $\phi_\pi, \phi_y \geq 0$, unless otherwise noted.

We begin with Cochrane's (2011) simplest model to address determinacy, also used by Davig and Leeper (2007) and Lubik, Matthes, and Mertens (2023), the Fisher equation in a frictionless setting. The loglinear Fisher equation is

$$R_t = rr_t + E_t \pi_{t+1} \tag{2}$$

where R_t is the nominal interest rate, rr_t the real rate, and π_t inflation. In the frictionless setting, the output gap is closed

$$y_t = 0 \tag{3}$$

and the real rate is determined exogenously, e.g., by the expected growth rate of productivity. Thus, without loss of generality, we set it to zero. Note that we get the same condition, $R_t = rr_t + E_t\pi_{t+1}$ with a standard dynamic IS equation in our frictionless setting with (3)

$$y_t = E_t y_{t+1} - \sigma R_t + \sigma E_t \pi_{t+1} \tag{4}$$

Theorem 1 (Frictionless Determinacy). *The frictionless model, given by (4), (3), with the Taylor rule (1), has a unique, stable equilibrium if and only if*

$$\phi_\pi > 1 \tag{5}$$

Proof. Combining (1), (4), and (3) gives

$$\phi_\pi \pi_t = E_t \pi_{t+1} \tag{6}$$

and solving forward, Blanchard (1979)

$$\pi_t = \lim_{j\to\infty} \frac{1}{\phi_\pi^j} E_t \pi_{t+j} \tag{7}$$

delivers a unique, bounded solution for π_t if and only if $1 < \phi_\pi$. □

Hence, the nominal interest rate must move more than one for one with inflation for the equilibrium to be determinate, giving us the celebrated Taylor principle.[11] Importantly,

[11] Notice that we abstract from exogenous shocks - this is without loss of generality, see e.g., Theorem 3.15 of Elaydi (2005) - the solution to a system of difference equations can be split into a particular and a homogenous solution and only the homogenous solution of the system of difference equations is relevant for the examination of determinacy. Following Taylor (1986), the bounded solution will be unique for any given bounded exogenous sequence of shocks if and only if the homogenous solution is uniquely determined by the boundedness conditions on the endogenous variables. Analogous conclusions can be found in Woodford (2003, pp. 252, & 636) and this follows the analysis of Lubik and Marzo (2007) for the sticky price model that follows. To see this consider the frictionless setup of Davig and Leeper (2007) and let $rr_t = \rho rr_{t-1} + \epsilon_t$ with $|\rho| < 1$ and ϵ_t is iid mean zero. The Fisher equation (2) and (1) imply $\phi_\pi \pi_t = E_t \pi_{t+1} + rr_t$. Solving forward gives and solving forward, Blanchard (1979)

$$\pi_t = \lim_{j\to\infty} \frac{1}{\phi_\pi^j} E_t \pi_{t+j} + \sum_{j=0}^{\infty} \frac{1}{\phi_\pi^{j+1}} E_t rr_{t+j} = \lim_{j\to\infty} \frac{1}{\phi_\pi^j} E_t \pi_{t+j} + \frac{1}{\phi_\pi} \sum_{j=0}^{\infty} \left(\frac{\rho}{\phi_\pi}\right)^j rr_t \tag{8}$$

which *again* delivers a unique, bounded solution for π_t if and only if $1 < \phi_\pi$. Now with $\pi_t = \frac{1}{\phi_\pi - \rho} v_t$ instead of $\pi_t = 0$ in theorem 1.

the degree of output gap targeting ϕ_y is irrelevant for determinacy, as the output gap is always zero in the frictionless model and offers no leverage in fulfilling the Taylor principle.

The determinacy situation is dramatically different under the standard linear New Keynesian sticky-price Phillips curve (NKPC) with Calvo (1983)-style overlapping contracts given by[12]

$$\pi_t = \beta E_t \pi_{t+1} + \kappa y_t \tag{9}$$

which replaces the frictionless supply side (3).

Theorem 2 (Sticky Price Determinacy)**.** *The sticky-price model, given by (4), (9), with the Taylor rule (1), has a unique, stable equilibrium if and only if*

$$1 - \frac{1-\beta}{\kappa}\phi_y < \phi_\pi \tag{10}$$

Proof. See the following (cf. time domain results from Woodford (2003), Galí (2008), Bullard and Mitra (2002), or Lubik and Marzo (2007)) □

While the proofs can readily be found elsewhere, it is particularly instructive to repeat them here to make the transition to establishing determinacy in the frequency domain necessary for the sticky information model more straightforward. Combining (1), (4), and (9) gives

$$\begin{bmatrix} -\beta & 0 \\ \sigma & 1 \end{bmatrix} \begin{bmatrix} E_t \pi_{t+1} \\ E_t y_{t+1} \end{bmatrix} = \begin{bmatrix} -1 & \kappa \\ \sigma\phi_\pi & 1+\sigma\phi_y \end{bmatrix} \begin{bmatrix} \pi_t \\ y_t \end{bmatrix} \tag{11}$$

which for $\beta \neq 0$ can be inverted to yield

$$\begin{bmatrix} E_t \pi_{t+1} \\ E_t y_{t+1} \end{bmatrix} = A \begin{bmatrix} \pi_t \\ y_t \end{bmatrix} \tag{12}$$

where $A = \begin{bmatrix} \frac{1}{\beta} & -\frac{\kappa}{\beta} \\ \sigma(\phi_\pi - \frac{1}{\beta}) & 1+\frac{\sigma}{\beta}\kappa + \sigma\phi_y \end{bmatrix}$ is the matrix of coefficients. If both eigenvalues of A lie outside the unit circle, then A can be inverted and solving forward gives

$$\begin{bmatrix} \pi_t \\ y_t \end{bmatrix} = \lim_{j\to\infty} A^{-j} \begin{bmatrix} E_t \pi_{t+j} \\ E_t y_{t+j} \end{bmatrix} \tag{13}$$

unique, bounded solution for π_t and y_t following Blanchard (1979).

[12]See, eg., Woodford (2003, p. 246) or Galí (2008, p. 49).

The Schur-Cohn criteria, (see LaSalle, 1986, p.28), to ascertain whether both eigenvalues indeed do lie outside the unit circle are $|\det(A)| > 1$ and $|\mathrm{tr}(A)| < 1 + \det(A)$. As

$$\det(A) = \frac{1}{\beta}(1 + \sigma\phi_y + \kappa\sigma\phi_\pi) > 1 \text{ and } \mathrm{tr}(A) = \frac{1}{\beta} + \frac{\sigma\kappa}{\beta} + 1 + \sigma\phi_y > 1 \tag{14}$$

the condition $|\det(A)| > 1$ necessarily holds and $|\mathrm{tr}(A)| < 1 + \det(A)$ holds if

$$1 < \frac{1-\beta}{\kappa}\phi_y + \phi_\pi \tag{15}$$

Given the Taylor rule (1), the monetary authority can target inflation as well as the output gap to stabilize the economy - Woodford (2003, pp. 254–255), "... indeed, a large enough [response to] *either* [the output gap or inflation] suffices to guarantee determinacy". Indeed, the real rate can be raised in response to an off equilibrium inflation increase even by responding to output movements alone. Notice that this possibility disappears if $\beta = 1$ - however this is misleading as although an *average* long-run tradeoff disappears in this case, a dynamic one remains $\frac{\pi_t - E_t\pi_{t+1}}{\kappa} = y_t$ which monetary policy needs for its targeting of inflation (or output) as different horizons to translate into a response to current inflation as we will see later in our analysis of extended Taylor rules.

Consider as an intuitive alternative the expectational Phillips curve of Lucas (1973), expressed in terms of inflation and abstracting from shocks

$$y_t = \alpha(\pi_t - E_{t-1}\pi_t) \tag{16}$$

where $\alpha \geq 0$ is the (short run) slope of the Phillips curve that predicts output gaps from unexpected inflation, i.e. forecast errors.

Theorem 3 (Lucas (1973) Determinacy)**.** *The expectational Phillips curve model, given by (4), (16), with the Taylor rule (1), has a unique, stable equilibrium if and only if*

$$\phi_\pi > 1 \tag{17}$$

Proof. Combining the IS curve (4) and the Taylor rule (1)

$$\left(1 + \sigma\phi_y\right) y_t = E_t y_{t+1} - \sigma\phi_\pi\pi_t + \sigma E_t\pi_{t+1} \tag{18}$$

inserting the Phillips curve (16)

$$\left(1 + \sigma\phi_y\right)\alpha(\pi_t - E_{t-1}\pi_t) = \alpha E_t[\pi_{t+1} - E_t\pi_{t+1}] - \sigma\phi_\pi\pi_t + \sigma E_t\pi_{t+1} \tag{19}$$

$$= -\sigma\phi_\pi\pi_t + \sigma E_t\pi_{t+1} \tag{20}$$

and now taking time $t-1$ expectations and recalling the law of iterated expectations

$$0 = -\sigma\phi_\pi E_{t-1}\pi_t + \sigma E_{t-1}E_t\pi_{t+1} \tag{21}$$

or $\phi_\pi \pi_{t|t-1} = E_{t-1}\pi_{t+1|t}$ where $\pi_{t|t-1} \equiv E_{t-1}\pi_t$ and solving forward, Blanchard (1979)

$$\pi_{t|t-1} = \lim_{j\to\infty} \frac{1}{\phi_\pi^j} E_{t-1}\pi_{t+j|t+j-1} = \lim_{j\to\infty} \frac{1}{\phi_\pi^j} E_{t-1}\pi_{t+j} \tag{22}$$

delivers a unique, bounded solution for $\pi_{t|t-1}$ if and only if $1 < \phi_\pi$. This determines the value for $\pi_{t|t-1}$ (and hence $\pi_{t+1|t}$ by the time invariance of the problem) so

$$\pi_t = \frac{1}{\left(1+\sigma\phi_y\right)\alpha + \sigma\phi_\pi}\left(\sigma E_t\pi_{t+1} + \left(1+\sigma\phi_y\right)\alpha E_{t-1}\pi_t\right) \tag{23}$$

and y_t then from (16) and R_t from (1). □

Notice now that despite the fact that we are using a different supply side, the expectational Phillips curve of Lucas (1973), we have the **same** determinacy bounds on monetary policy as in the frictionless case of theorem 1.[13] Both of these models, in contrast to the sticky price model, have vertical long run Phillips curves. Specifically, the frictionless model has a vertical Phillips curve at every horizon - from (3) $y_t = 0$ - and the Lucas (1973) supply side at the one period horizon. To see this, take the $t-1$ expectation of (16)

$$E_{t-1}y_t = \alpha E_{t-1}\left[\pi_t - E_{t-1}\pi_t\right] = 0 \tag{26}$$

We will now show that this equivalence between the determinacy bounds on the frictionless model and the determinacy bounds of Phillips curves that become vertical in the long run holds more generally. We will begin by introducing three different Phillips curves and how they relate to a long run vertical curve before we then turn to their determinacy.

[13] Note that $\left(1+\sigma\phi_y\right)\alpha + \sigma\phi_\pi \neq 0$ if we are to be able to recover π_t from $E_{t-1}\pi_t$. As we only consider $\phi_\pi, \phi_y \geq 0$, this holds here with certainty. As we shall see later, for more general models, this might not hold everywhere, but is simply a singularity in the parameter space that prevents the unique resolution of the prediction error(s). Additionally, in the frictionless model, the output gap was always closed and hence output was determined apart from monetary policy: an indeterminacy would only be a nominal indeterminacy that afflicted π_t and, hence, R_t. This is not true here as an indeterminacy in $E_{t-1}\pi_t$ would lead to an indeterminacy in y_t through the Phillips curve (16)

$$y_t = \alpha\left(\pi_t - E_{t-1}\pi_t\right) = \frac{\alpha}{\left(1+\sigma\phi_y\right)\alpha + \sigma\phi_\pi}\left(\sigma E_t\pi_{t+1} + \left(1+\sigma\phi_y\right)\alpha E_{t-1}\pi_t\right) - \alpha E_{t-1}\pi_t \tag{24}$$

$$= \frac{\alpha\sigma}{\left(1+\sigma\phi_y\right)\alpha + \sigma\phi_\pi}\left(E_t\pi_{t+1} - \phi_\pi E_{t-1}\pi_t\right) \tag{25}$$

3. Phillips Curves of Imperfect Information

In this section, we review two Phillips curves - sticky information and imperfect common knowledge - and introduce a general finite inattentiveness supply side. The first two are examples of each of Angeletos and Lian's (2018) "two leading forms of learning" where agents either become gradually aware of the fundamental or receive private signals about it. We juxtapose these three alongside the canonical sticky price Phillips curve from (27) and relate them to the natural rate hypothesis (NRH).

3.1. Phillips Curves in the Frequency Domain - Sticky Information

The sticky information Phillips curve has an infinite regress of price plans or lagged expectations that cannot be expressed recursively in the time domain,[14] precluding the application of standard DSGE techniques to assess determinacy. We prove in the following, however, that the sticky information Phillips curve does have a recursive representation in the frequency domain, requiring this frequency domain perspective. [15] To this end, we review two Phillips curves in this section - the canonical sticky price and sticky information - and present their frequency domain equivalents. The frequency domain provides a novel, fundamental perspective on the sticky information Phillips curve, while merely providing an alternative representation for the sticky price Phillips curve.

We begin with the standard linear New Keynesian sticky-price Phillips curve (NKPC) with Calvo (1983)-style overlapping contracts given by[16]

$$\pi_t = \beta E_t \pi_{t+1} + \kappa y_t \tag{27}$$

where y_t is the output gap, π_t inflation, $0 < \beta < 1$ is the representative household's idiosyncratic discount factor and $\kappa = \frac{(1-\theta)(1-\beta\theta)}{\theta}\Theta$ is the slope of the short run Phillips curve with $0 < 1-\theta < 1$ being the probability of a price update - that is, θ is the Calvo (1983) sticky price parameter that measures the degree of nominal rigidity - and Θ collects other parameters, such as the link between marginal costs and the output gap, the interaction

[14] In contrast to the sticky price Phillips curve, whose infinite regress of forward-looking price setting behavior can be represented recursively in the time domain.

[15] Our approach does not require us to include shocks explicitly, see also footnote 11, hence we are defining the processes in terms of the kernel of the operator that defines the linear rational expectations model, see Al-Sadoon (2020). See the appendix for an introduction to the frequency domain techniques necessary for our analysis.

[16] See, eg., Woodford (2003, p. 246) or Galí (2008, p. 49).

between returns to scale, etc. Hence, inflation today is a function of current output gap and future expected inflation. Applying the z-transform gives

$$\pi(z) = \beta\frac{1}{z}(\pi(z) - \pi_0) + \kappa y(z) \tag{28}$$

which implies that inflation and the output gap are linked at all frequencies z. To see this, assume that the output gap is a known function in z, $y(z)$, analytic on the unit disk, then

$$\pi(z) = \frac{1}{z-\beta}\left(\kappa z y(z) - \beta\pi_0\right) \tag{29}$$

which uniquely determines inflation as $\pi(z)$ with $\pi(0) = \kappa y(0)$ by continuation over the singularity at $z = \beta$. Conversely, assume that inflation is a known function in z, $\pi(z)$, analytic on the unit disk, then

$$y(z) = \frac{z-\beta}{\kappa z}\pi(z) + \frac{\beta}{\kappa z}\pi_0 \leftrightarrow y(z) = \frac{1}{\kappa}\pi(z) - \frac{\beta}{\kappa z}(\pi(z) - \pi_0) \tag{30}$$

which uniquely determines the output gap as $y(z)$ with $y(0) = \frac{1}{\kappa}\pi(0)$ by continuation again. Hence, we conclude that the sticky price Phillips curve purports an inexorable link between inflation and the output gap at all frequencies.

Sticky information models implement probabilistic contracts of predetermined prices in the vein of Fischer (1977) with the Calvo (1983) mechanism.[17] Mankiw and Reis's (2002) version, the sticky-information model, yields the following aggregate supply equation

$$\pi_t = \frac{1-\lambda}{\lambda}\xi y_t + (1-\lambda)\sum_{i=0}^{\infty}\lambda^i E_{t-i-1}[\pi_t + \xi(y_t - y_{t-1})] \tag{31}$$

where y_t is the output gap, π_t inflation, $\xi > 0$ measures the degree of strategic complementarities, and $0 < 1-\lambda < 1$ is the probability of an information update. The infinite regress of lagged expectations precludes a recursive representation in the time domain.

These lagged expectations $(E_{t-i}[x_t],\ i > 0)$ were dubbed "withholding equations" by Whiteman (1983) and the Wiener-Kolmogorov prediction formula (B.18) provides the representation

$$\mathcal{Z}\{E_{t-i}[x_t]\} = z^i\left[\frac{x(z)}{z^i}\right]_+ = x(z) - \sum_{j=0}^{i} x^j(0)z^j \tag{32}$$

where $x^j(0)$ is the $j'th$ derivative of $x(z)$ evaluated at the origin. These withholding equations by themselves are not sufficient to solve models like those involving the sticky

[17]See Bénassy (2002, Ch. 10), Mankiw and Reis (2002), and Devereux and Yetman (2003).

information Phillips curve (31), as it requires an *infinite* number of withholding equations[18]. Using (32), the sticky information Phillips curve (31) can be expressed as

$$\pi(z) = \frac{1-\lambda}{\lambda}\xi y(z) + (1-\lambda)\sum_{i=0}^{\infty}\lambda^i\left[\pi(z) - \sum_{j=0}^{i}\pi^j(0)z^j + \xi(1-z)\left(y(z) - \sum_{j=0}^{i}y^j(0)z^j\right)\right] \tag{33}$$

The infinite sums in (33) can be resolved by noting that:[19]

$$\sum_{i=0}^{\infty}\lambda^i\left[x(z) - \sum_{j=0}^{i}x_j z^j\right] = \frac{1}{1-\lambda}x(z) - \sum_{i=0}^{\infty}\lambda^i\sum_{j=0}^{i}x_j z^j = \frac{1}{1-\lambda}x(z) - \sum_{j=0}^{\infty}\sum_{i=j}^{\infty}x_j z^j\lambda^i \tag{34}$$

$$= \frac{1}{1-\lambda}x(z) - \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\lambda^i x_j z^j\lambda^j = \frac{1}{1-\lambda}x(z) - \sum_{j=0}^{\infty}\frac{1}{1-\lambda}\lambda^i x_j z^j\lambda^j \tag{35}$$

$$= \frac{1}{1-\lambda}(x(z) - x(\lambda z)) \tag{36}$$

Combining these results we get the following representation of the Phillips curve (31)

$$\pi(z) = \xi\left(\frac{1-\lambda}{\lambda}\right)y(z) + \pi(z) - \pi(\lambda z) + \xi(1-z)y(z) - \xi(1-\lambda z)y(\lambda z) \tag{37}$$

collecting terms gives $\xi(1-\lambda z)y(z) = \lambda\pi(z\lambda) + \xi\lambda(1-\lambda z)y(\lambda z)$ which we rearrange to yield the following representation of the Phillips curve of the sticky information model in the frequency domain

$$\xi\left(\frac{1}{\lambda} - z\right)y(z) = \pi(\lambda z) + \xi(1-\lambda z)y(\lambda z) \tag{38}$$

The output gap at a given frequency, z, depends on inflation and itself at dampened frequencies, λz. Recalling from the previous section and the AR(1) example that $z = Re^{-i\omega}$, where ω is the angular frequency and R is the radius equal to one for unconditional moment or long run statistics and zero for impact or high frequency effects, $\lambda z = \tilde{R}e^{-i\omega}$, $\tilde{R} = \lambda R$ which serves to dampen or scale the variable towards the origin. The parameter λ or probability of *not* receiving an information update introduces a form of stickiness in the frequency domain. If the fraction of firms which get an information update, $1-\lambda$, is low (high) and hence λ closer to one (zero), the output gap is driven more strongly by inflation at low (high) frequencies, that is $\tilde{R} = \lambda R$ is closer to R (zero). However, in the long run, there is no tradeoff between output gap and inflation as the

[18] Tan and Walker (2015, p. 99) claim that their method can be "easily adapted" to models like the sticky information model using withholding equations by "replacing E_t with E_{t-j} for any finite j." This is misleading or incomplete, as the sticky information model involves lagged information that reaches back past any finite j.

[19] The exchange of the order of summation follows from our assumption of processes in the space spanned by time-independent square-summable linear processes. Also note that we provide a different, albeit more lengthy approach in the appendix.

rigidity of information which determines output gap becomes smaller and eventually vanishes. Output gap at a given frequency then only depends on inflation at higher frequencies; i.e., at the lowest frequency $|z| = 1$, the output gap is independent of the lowest frequency or $|z| = 1$ movements in inflation. That is, the sticky information Phillips curve becomes vertical in the long run, as pointed out in the time domain by Mankiw and Reis (2002).

It is the recursivity in the frequency domain implied by (38) that drives this lowest frequency independence and this follows from the properties of scaling in the frequency domain laid out in the previous section. As a result, the output gap can be determined by inflation via the sticky information Phillips curve but not vice versa. This absence of a stable, long-run trade off between inflation and the output gap can be seen through the frequency domain representation by developing (38) further

$$y(z) = \frac{\lambda}{\xi}\frac{1}{1-\lambda z}\pi(\lambda z) + \lambda y(\lambda z) \tag{39}$$

which is recursive in $y(\lambda^i z)$, yielding the following

$$y(z) = \frac{1}{\xi}\sum_{j=1}^{\infty}\frac{\lambda^j}{1-\lambda^j z}\pi(\lambda^j z) + \lim_{j\to\infty}\lambda^j y(\lambda^j z) \tag{40}$$

Defining $\tilde{\pi}(\lambda^j z) \equiv \frac{1}{1-\lambda^j z}\pi(\lambda^j z)$, we get

$$y(z) = \frac{1}{\xi}\sum_{j=1}^{\infty}\lambda^j\tilde{\pi}(\lambda^j z) + \lim_{j\to\infty}\lambda^j y(\lambda^j z) \tag{41}$$

Now take π_t as a given mean zero, linearly regular covariance stationary stochastic process with known Wold representation, i.e., $\pi(z)$ as an analytic function with a region of convergence of at least $|z| \le 1$. Thus, $\pi(\lambda^j z)$ has a region of convergence of at least $|\lambda^j z| \le 1$, which as $0 < \lambda < 1$ is $|z| \le \lambda^{-j}$ and hence $\pi(\lambda^j z)$ has a region of convergence of at least $|z| \le 1$. So $\tilde{\pi}(\lambda^j z)$ will also have a region of convergence of at least $|z| \le 1$ for $0 < \lambda < 1$ as the pole $z \in \mathscr{C} : 1 - \lambda^j z = 0$ is outside the unit circle and the sum is convergent from the λ^j weights. Turning to the limit term, $\lim_{j\to\infty} y(\lambda^j z) = y(0)$, $|y(0)| < \infty$ is the impact response on the output gap, hence $\lim_{j\to\infty}\lambda^j y(\lambda^j z)$ for $0 < \lambda < 1$. That is, given $\pi(z)$, analytic over the unit disk, $y(z)$ is given by

$$y(z) = \frac{1}{\xi}\sum_{j=1}^{\infty}\lambda^j\tilde{\pi}(\lambda^j z) = \frac{1}{\xi}\sum_{j=1}^{\infty}\frac{\lambda^j}{1-\lambda^j z}\pi(\lambda^j z) \tag{42}$$

over the unit disk.

The converse, however, is not true. Instead, now take $y(z)$ as a given mean zero, linearly regular covariance stationary stochastic process with known Wold representation, an analytic function with a region of convergence of at least $|z| \leq 1$. Starting from (39)

$$\pi(\lambda z) = \frac{\xi}{\lambda}(1-\lambda z)(y(z)-\lambda y(\lambda z)) \tag{43}$$

and inflation is given by

$$\pi(z) = \frac{\xi}{\lambda}(1-z)(y(z/\lambda)-\lambda y(z)) \tag{44}$$

Notice that a $\pi(z)$ representation of inflation from this would demand that $y(z/\lambda)$ be analytic with a region of convergence of at least the unit disk. That is, $y(z)$ would need a region of convergence of at least $|z\lambda| \leq 1$ or of at least $|z| \leq 1/\lambda$ for $0 < \lambda < 1$, which of course is outside the unit circle. Thus, knowing $y(z)$ as a given mean zero, linearly regular covariance stationary stochastic process, analytic over the unit disk, is insufficient to determine $\pi(z)$ as an analogously defined process, analytic over the unit disk.

Thus we conclude that the sticky information Phillips curve determines the output gap from inflation and not the other way around. Contrast this with the sticky price Phillips curve (28) rewritten as

$$\pi(z) = \frac{1}{1-\beta/z}\left(\kappa y(z) - \beta/z\pi(0)\right) \tag{45}$$

or

$$y(z) = \frac{1-\beta/z}{\kappa}\pi(z) + \frac{\beta}{\kappa}\frac{1}{z}\pi(0) \tag{46}$$

From (46) it follows directly that assuming π_t is a given mean zero, linearly regular covariance stationary stochastic process with known Wold representation, i.e., $\pi(z)$ as an analytic function with a region of convergence of at least $|z| \leq 1$, that the same holds for $y(z)$. For the converse, notice that as $0 < \beta < 1$ there is a pole $z \in \mathscr{C} : 1-\beta/z = 0$ inside the unit circle. Thus, given a mean zero, linearly regular covariance stationary stochastic process with known Wold representation for $y(z)$, $\pi(z)$ is also an analytic function with a region of convergence of at least $|z| \leq 1$ as the singularity at the pole $z = \beta$ can be removed via

$$\lim_{j\to\infty}\left(1-\beta/z\right)\pi(z) \stackrel{!}{=} 0 = \kappa y(\beta) - \pi(0) \tag{47}$$

Hence, in contrast to the sticky information Phillips curve, the sticky price Phillips curve *does* imply a stable long run tradeoff between inflation and the output gap. This difference is decisive for implications of monetary policy and, in particular, for those of determinacy to which we turn next.

3.2. Phillips Curves under Imperfect Common Knowledge

Instead of examining inattentiveness via outdated information, another strand in the literature examines the consequences of imperfect common information. With firms having only incomplete, noisy signals on the state of the economy, be this assumed exogenously like Nimark (2008) or as the result of a capacity constraint like Adam (2007), they have different information sets and disagree about the state of the economy. Nimark's (2008) shows that the assumption of common knowledge of rationality enables the Phillips curve to be expressed via an infinite cascade of higher order expectations in the otherwise standard time recursive relation between current marginal costs and current and future inflation.[20] We show that this cascade of higher order expectations and the recursivity of the average higher order expectations allows us to express this cascade recursively in the signal space, giving us a compact representation of the imperfect common information Phillips curve.

Nimark (2008) presents a Phillips curve that embeds the standard sticky price approach into this imperfect information setup as follows

$$\pi_t = (1-\theta)\left(1-\beta\theta\right)\sum_{k=0}^{\infty}(1-\theta)^k\, mc_{t|t}^{(k)} + \beta\theta\sum_{k=0}^{\infty}(1-\theta)^k\,\pi_{t+1|t}^{(k+1)} \tag{48}$$

where mc_t are the aggregate marginal costs and the remainder of variable names and parameters are identical to the sticky price Phillips curve in (27). Imperfect knowledge is encompassed in the variables on the right-hand side of (48) where the following notation is used

$$x_{t|t}^{(0)} \equiv x_t \qquad x_{t|s}^{(1)} \equiv \int E\left[x_t | \mathscr{I}_s(j)\right] dj \qquad x_{t|s}^{(k)} \equiv \int E\left[x_{t|s}^{(k-1)} x_t \middle| \mathscr{I}_s(j)\right] dj \tag{49}$$

where $\mathscr{I}_s(j)$ is the atomistic agent j's information set at time s. Hence the imperfect common knowledge Phillips curve in (48) contains the infinite cascade of higher order beliefs or Townsend's (1983) forecasting the forecasts of others. That is, inflation depends not only on marginal costs and future expected inflation, but the average (via the integral over agents) *imperfect* expectation of marginal costs and future expected inflation, the average *imperfect* expectation of the average *imperfect* expectation of marginal costs and future expected inflation, and so forth.

[20] Angeletos and Lian (2018) and Angeletos and Huo (2021) address the potential for dynamic higher order expectations to inhibit this time recursive relation.

We express this Phillips curve recursively by defining the average higher order expectations operator

$$H_s x_t \equiv \int E\left[x_t | \mathscr{I}_s(j)\right] dj \tag{50}$$

and rewriting (48) as

$$\pi_t = (1-\theta)\left(1-\beta\theta\right)\sum_{k=0}^{\infty}(1-\theta)^k H_s^k mc_t + \beta\theta\sum_{k=0}^{\infty}(1-\theta)^k H_s^k H_s \pi_{t+1} \tag{51}$$

$$= \frac{(1-\theta)\left(1-\beta\theta\right)}{1-(1-\theta)H_s} mc_t + \frac{\beta\theta}{1-(1-\theta)H_s} H_s \pi_{t+1} \tag{52}$$

which follows as $0 < 1-\theta < 1$. Multiplying both sides with $1-(1-\theta)H_s$ gives

$$(1-(1-\theta)H_s)\pi_t = (1-\theta)\left(1-\beta\theta\right) mc_t + \beta\theta H_s \pi_{t+1} \tag{53}$$

or

$$\underbrace{(1-H_s)\pi_t}_{\text{A forecast/prediction error}} + \theta H_s \underbrace{\left(\pi_t - \beta\pi_{t+1}\right)}_{\substack{\text{Standard sticky price dynamic} \\ \text{inflation trade off}}} = (1-\theta)\left(1-\beta\theta\right) mc_t \tag{54}$$

Noting further that $H_s E_t \pi_{t+1} = H_s \pi_{t+1}$ due to the law of iterated expectations the Phillips curve can be written as

$$(1-H_s)\pi_t - \theta(1-H_s)\left(\pi_t - \beta E_t \pi_{t+1}\right) + \theta\left(\pi_t - \beta E_t \pi_{t+1}\right) = (1-\theta)\left(1-\beta\theta\right) mc_t \tag{55}$$

or

$$\pi_t = \beta E_t \pi_{t+1} + \frac{(1-\theta)\left(1-\beta\theta\right)}{\theta}\Theta y_t + \zeta_t \tag{56}$$

with marginal costs and the output gap related through Θ as above and where ζ_t is a prediction/forecast error given by

$$\zeta_t = -\frac{1-\theta}{\theta}(1-H_s)\pi_t - \beta(1-H_s)E_t \pi_{t+1} \tag{57}$$

$$= -\frac{1-\theta}{\theta}\int \left(\pi_t - E\left[\pi_t | \mathscr{I}_s(j)\right]\right) dj - \beta\int \left(E_t \pi_{t+1} - E\left[\pi_{t+1} | \mathscr{I}_s(j)\right]\right) dj \tag{58}$$

Comparing (56) with the sticky price Phillips curve in (27), we surmise that imperfect common knowledge introduces forecast/prediction errors that can affect the dynamic, but does not alter the fundamental long run dynamic trade off.

To see the decisiveness of this forecast/prediction error factorization, consider the sticky information Phillips curve (31) again, but we will stay in the time domain now

$$\pi_t - \frac{1-\lambda}{\lambda}\xi y_t = (1-\lambda)\sum_{i=0}^{\infty}\lambda^i E_{t-i-1}[\pi_t + \xi(y_t - y_{t-1})] \tag{59}$$

$$= (1-\lambda)\sum_{i=0}^{\infty}\lambda^i (E_{t-i-1}[\pi_t] - \pi_t + \xi(E_{t-i-1}[y_t] - y_t - E_{t-i-1}[y_{t-1}] + y_{t-1})) \tag{60}$$

$$+ (1-\lambda)\sum_{i=0}^{\infty}\lambda^i (\pi_t + \xi y_t - \xi y_{t-1}) \tag{61}$$

$$= \pi_t + \xi y_t - \xi y_{t-1} - \sum_{i=0}^{\infty}\lambda^i (E_{t-i}[\pi_t] - E_{t-i-1}[\pi_t]) \tag{62}$$

$$-\xi\sum_{i=0}^{\infty}\lambda^i (E_{t-i}[y_t] - E_{t-i-1}[y_t]) + \xi\sum_{i=1}^{\infty}\lambda^i (E_{t-i}[y_{t-1}] - E_{t-i-1}[y_{t-1}]) \tag{63}$$

which can be rearranged to yield

$$y_t = \lambda y_{t-1} + \frac{\lambda}{\xi}\sum_{i=0}^{\infty}\lambda^i (E_{t-i}[\pi_t] - E_{t-i-1}[\pi_t]) \tag{64}$$

$$+\lambda\sum_{i=0}^{\infty}\lambda^i (E_{t-i}[y_t] - E_{t-i-1}[y_t]) - \lambda\sum_{i=1}^{\infty}\lambda^i (E_{t-i}[y_{t-1}] - E_{t-i-1}[y_{t-1}]) \tag{65}$$

or

$$y_t = \lambda y_{t-1} + v_t \tag{66}$$

where $v_t = \frac{\lambda}{\xi}\sum_{i=0}^{\infty}\lambda^i (E_{t-i}[\pi_t] - E_{t-i-1}[\pi_t]) + \lambda\sum_{i=0}^{\infty}\lambda^i (E_{t-i}[y_t] - E_{t-i-1}[y_t]) - \lambda\sum_{i=1}^{\infty}\lambda^i (E_{t-i}[y_{t-1}] - E_{t-i-1}[y_{t-1}])$ is a (or rather three) sequence(s) of forecast errors. Compare this to the frequency domain version in (39), expressed in the time domain using the inverse z-transform[21]

$$y_t = \lambda y_{t-1} + \frac{1}{2\pi i}\oint_{|z|=1} z^{t-1}\left(\frac{\lambda}{\xi}\pi(\lambda z) + \lambda y(\lambda z) - \lambda y(\lambda z)\lambda z\right)dz \tag{67}$$

That is, the long run relation given by the sticky information model is $y_t = \lambda y_{t-1}$ with infinite sequences of forecast errors producing a time varying relation that disappears in the limit. Analogously, the long run relation given by Nimark's (2008) Phillips curve is given by the long run dynamic relation of the standard sticky price model to which he added the information imperfection. The Calvo parameter θ dictates the nominal sticky price rigidity and setting it to zero in (55) gives the Phillips curve with only the information imperfection

$$(1-H_s)\pi_t = mc_t \Rightarrow y_t = \tilde{\zeta}_t \tag{68}$$

[21]See the appendix.

where $\tilde{\zeta}_t = \lim_{\theta \to 0} \frac{\theta}{\Theta} \zeta_t = \frac{1}{\Theta} \int (\pi_t - E[\pi_t | \mathscr{I}_s(j)]) dj$. That is, Nimark's (2008) Phillips curve is non vertical solely due to and identically to the standard sticky price model, the information rigidity itself gives a vertical long run Phillips curve. As Coibion and Gorodnichenko (2015b) demonstrate, the sticky information and noisy information models both relate their respective information rigidities to the same relationship between average forecast errors and prediction revisions, consistent with our assessment that both models share asymptotic properties directed by their common, vertical long run Phillips curve.

3.3. Phillips Curves under Finite Inattentiveness

Both of the above models have the property that the Phillips curve becomes vertical in the forecasting limit and the output gap is necessarily closed, i.e. $\lim_{j \to \infty} E_t [y_{t+j}] = 0$. Instead of limiting our analysis to specific models such as the two above, we will now introduce a Phillips curve that is generic and otherwise unspecified apart from its satisfying the following version of the natural rate hypothesis following Carlstrom and Fuerst (2002)[22]

$$E_{t-k}[y_t] = 0 \; \forall t \tag{69}$$

in such a model, the output gap is necessarily closed on average due to the law of iterated expectations - the Lucas (1973) expectational Phillips curve (16) analyzed above satisfies this at $k = 1$. With the long run setting it at some finite horizon,[23] this ensures that Lucas's (1972) NRH is fulfilled. A supply side that fulfills this condition can be expressed as

$$y_t = \sum_{j=0}^{k-1} \left(E_{t-j}[y_t] - E_{t-j-1}[y_t] \right) \tag{70}$$

Non-zero output gaps can be represented wholly as innovations or forecast errors without making any conjecture as to admissible solutions, in the words of Friedman (1977, p. 456),

[22]Explicit examples are models that implement contracts of predetermined prices in the vein of Fischer (1977) with finite duration, including Andrés, López-Salido, and Nelson's (2005, p. 1034) "Sticky information, staggered á la Taylor," as found also in Koenig (2004), Collard, Dellas, and Smets (2009), and Woodford (2010); the Mussa-McCallum-Barro-Grossmann "P-bar model"—see McCallum (1994) and McCallum and Nelson (2001); models of finitely staggered predetermined prices such as Fischer (1977) and Blanchard and Fischer (1989, pp. 390–394); Carlstrom and Fuerst's (2002, p 81-82) model in this spirit; as well as the expectational Phillips curve of Lucas (1973)—see also Sargent and Wallace (1975)—that formalized the rational expectations revolution here in (16).

[23]It makes no difference for the conclusions that follow whether the long run sets in after four quarters or four millennia: k is completely arbitrary for the analysis so long as it is finite.

"[o]nly surprises matter." Note that the effect of a surprise need not disappear immediately after impacting the output gap, it can have a lasting—but not permanent—effect. That is, there can be a stable short-run tradeoff between the output gap and inflation, but this tradeoff must not be permanent if the model is to satisfy the NRH.

3.4. Long Run Phillips Curves and the Natural Rate Hypothesis

The unspecified supply side (70) is derived from the NRH and here we will provide more insight into this hypothesis and its relevance to the specification of the supply sides above. In particular, how the sticky-price Phillips curve (27) violates the NRH by positing same dynamic tradeoff at every expectational horizon. This is decisive for the determinacy properties and we show how some standard interpretations (e.g., steady state) or alterations (e.g., indexation) to reconcile the sticky-price Phillips curve with the NRH are incomplete and insufficient to remove a trade off that distorts the determinacy limits on monetary policy - the Taylor principle that we will show is otherwise identical across models under the NRH in the next section.

The NRH, succinctly by Friedman (1968, p. 11) "there is always a temporary trade-off between inflation and employment; there is no permanent trade-off[,]" postulates that the output gap is closed on average regardless of monetary policy. The hypothesis and associated vertical Phillips curve are central to the rational expectations revolution.[24] The NRH enjoys near-universal agreement,[25] in part as models that violate the NRH possess the "a priori implausible" implication that there exist inflationary paths on which "a nation can enrich itself in real terms permanently."[26]

McCallum (2004, pp. 21–22) explicitly highlights that the standard New Keynesian Phillips curve (27) violates the NRH and draws a distinction between "Friedman's weaker

[24] See Lucas (1972) and Sargent (1973), with Sargent (1987b, p. 7) calling Friedman's (1968) address the revolution's "opening shot" and Modigliani (1977, p. 5) deeming Lucas's (1972) rational-expectations version the "death blow to the already badly battered Keynesian position."

[25] By the late '70s, Friedman (1977, p. 459) could note that his and Phelps's (1967) NRH was a widely accepted consensus that, as McCallum (2004, p. 21) remarks, "by 1980 even self-styled Keynesian economists were agreeing to." Krugman (1994, p. 52) confirms that "[t]he natural rate hypothesis has received near-universal acceptance" and underlines that it "has a very solid basis in experience," an agreement that Phelps (1994, p. 81) was "delighted to see." Bernanke (2003) went a step further: "Friedman's [point ...] that long run output is determined entirely by real factors [...] is universally accepted today by monetary economists."

[26] McCallum (1998, p. 359)

version" and the "stronger Lucas version" of the NRH. The former states that a higher, but constant, rate of inflation cannot permanently affect output and the latter that no path for prices, inflation, inflation growth, etc., can permanently keep output above its natural level. While the distinction is doubtless appropriate from the perspective of the accelerationist controversy, the nomenclature of McCallum (2004) is perhaps misleading as Friedman (1977, p. 274) himself made explicit that his view of the NRH is not limited to such an accelerationist view: "[S]ome substitute a stable relation between the acceleration of inflation and unemployment for a stable relationship between inflation and unemployment—aware of but not concerned about the possibility that the same logic that drove them to a second derivative will drive them to even higher derivatives." In any event, Lucas's (1972) is the version that McCallum (1994) argues should be upheld by monetary models—the critique repeated more directly in McCallum (1998, p. 359)—and will be the version imposed in my analysis.

First, one can confirm that the standard New Keynesian sticky-price model with Calvo (1983)-style overlapping contracts (27) cannot satisfy Lucas's (1972) NRH by simply taking unconditional expectations

$$E[y_t] = \frac{1}{\kappa}\left(E[\pi_t] - \beta E[\pi_{t+1}]\right) \neq 0 \tag{71}$$

Even in the extreme parameterization of $\beta = 1$, the unconditional expectation of the output gap would still be nonzero ($E[y_t] \neq 0$) with nonstationary inflation. As made explicit by McCallum (1998, p. 359), the NRH requires that "$E[y_t] = 0$ for any monetary policy[;]" i.e., the unconditional expectation of the output gap must be zero for any monetary policy. Nothing in this statement excludes nonstationary policies and, indeed, McCallum and Nelson (2009, p. 7) note that the "Lucas version [...] pertains to inflation paths more general than steady states." Note, furthermore, that (27) posits the same immutable tradeoff at every expectational horizon:

$$E_{t-j}[y_t] = \frac{1}{\kappa}\left(E_{t-j}[\pi_t] - \beta E_{t-j}[\pi_{t+1}]\right), \ \forall j \geq 0 \tag{72}$$

or, alternatively,

$$E_t\left[y_{t+j}\right] = \frac{1}{\kappa}\left(E_t\left[\pi_{t+j}\right] - \beta E_t\left[\pi_{t+j+1}\right]\right), \ \forall j \geq 0 \tag{73}$$

the tradeoff in the sticky-price model is so stable that, from the perspective of today, the same dynamic tradeoff is expected to exist unchanged into the infinite future. The only way for this Phillips curve to satisfy the NRH, is if $\kappa \to \infty$, making the Phillips curve always[27] vertical.

The sticky-price Phillips curve indexed either to steady-state inflation[28]

$$y_t = \frac{1}{\kappa}\left(\pi_t - \bar{\pi} - \beta(E_t[\pi_{t+1}] - \bar{\pi})\right) \tag{74}$$

or past inflation[29]

$$y_t = \frac{1}{\kappa}\left(\pi_t - \beta E_t[\pi_{t+1}] - \gamma\left(\pi_{t-1} - \beta\pi_t\right)\right) \tag{75}$$

still fails to satisfy Lucas's (1972) NRH,[30] for the same reason above. Only those monetary policies that lead to a stationary path for inflation allow the the output gap to be equal, on average, to zero.[31] As above, these Phillips curves can be made to satisfy the NRH, but this requires $\kappa \to \infty$, making them always vertical.

Requiring the equilibrium path of a linearized system to be bounded does not mean that nonstationary paths are inconsequential for local analyses. The local determinacy of a bounded equilibrium path in the linearized system depends crucially on all other potential paths becoming unbounded so that this bounded path is unique. These other paths need never materialize: their mere hypothetical existence in the stead of additional bounded paths that could not be excluded is what renders the single bounded equilibrium unique. This is Cochrane's (2011) assessment of determinacy through a Taylor rule being an off-equilibrium threat and requiring determinacy imposes bounds on coefficients in a policy rule to ensure that there is a unique locally bounded equilibrium. That is, determinacy rests on the ability to convincingly predict when an equilibrium path would become unbounded such that it can be excluded from the class of permissible equilibria; that arguments resting on permanent output-inflation tradeoffs are not convincing is, of course, a central component of the NRH. Reinterpreting the New Keynesian Phillips curves in terms of output gaps driven by inflation makes the violation of the NRH and

[27] I.e., at every expectational horizon.

[28] See Yun (1996).

[29] See Christiano, Eichenbaum, and Evans (2005) for $\gamma = 1$ and Smets and Wouters (2003) for $0 < \gamma \leq 1$.

[30] See McCallum (2004, pp. 21–22) and McCallum and Nelson (2009, pp. 6–7).

[31] Certainly, indexation to steady-state inflation is meaningless, should inflation be nonstationary. As pointed out recently by Nelson (2008), it is monetary policy that determines steady-state inflation, or indeed whether it should exist, and without having specified monetary policy, it is almost vacuous to speak of such a value.

its consequences for determinacy more visible:[32] the right-hand sides of (27), (74), and (75) present a description of the dynamic properties of inflation to achieve any desired dynamic for the output gap. Of course for determinacy analysis, it is unbounded dynamics that are desired for all but one equilibrium paths to render this remaining equilibrium uniquely bounded. Thus, it is not the final equilibrium under study that would display the aberrant behavior implied by an exploitation of these Phillips curves, but rather the hypothetical paths that are being excluded for the sake of equilibrium uniqueness; and these Phillips curves show indeed that inflation and the output gap can work in concert to such an end all the way through to the long run. A NRH model, in contrast, must display a vertical Phillips curve that prevents any such concerted long-run reaction: only unexpected components of inflation cause output gaps.

4. Existence and Uniqueness for Sticky Information

Now we turn to establishing the determinacy bounds with the Taylor rule (1), the dynamic IS equation (4), and supply curves characterized by inattentiveness. Note that the absence of exogenous driving forces is without loss of generality and will remain the same if our systems are appended with stationary driving forces (i.e., we are investigating the properties of the homogenous component of the system of difference equations).[33] For a complete solution, one would then have the additional task of associating the exogenous driving forces with the expectation errors (see, e.g., Sims (2001)). This is precisely the advantage of our analysis, we separate the question of whether there is a unique equilibrium from what this equilibrium is.

We will begin with the sticky information Phillips curve (31). As established in the previous section, this Phillips curve is recursive in the frequency domain and, hence, we will exploit this and establish conditions for its determinacy in the frequency domain. To make the connection to standard, time domain results more clear, we will reestablish the determinacy conditions for the sticky price model of theorem 2 but now in the frequency domain.

[32] Though of no consequence algebraically, reformulating the Phillips curves with the output gap on the left-hand side draws a parallel to Modigliani's (1977, p. 5) assessment that Friedman's (1968) NRH "turns the standard explanation on its head: instead of (excess) employment causing inflation, it is (the unexpected component of) the rate of inflation that causes excess employment."

[33] See footnote 11 and also footnote 15.

Combining the Taylor rule (1), the dynamic IS equation (4) and expressing in the frequency domain gives

$$(1+\sigma\phi_y)zy(z)+\sigma\phi_\pi z\pi(z)=y(z)-y_0+\sigma(\pi(z)-\pi_0) \tag{76}$$

Notice that we are abstracting from shocks and these equations (along with either of the supply curves from the previous section) are entirely homogenous. Thus one solution, the fundamental solution is zero at all frequencies - an inability to rule out nonzero solutions is tantamount to not being able to rule out stable sunspot solutions - i.e. non-uniqueness or indeterminacy.

First, we close the model with (28), the standard sticky-price Phillips curve,

$$\pi(z)=\beta\frac{1}{z}(\pi(z)-\pi_0)+\kappa y(z) \tag{77}$$

and the two foregoing can be summarized in a matrix system as

$$\begin{bmatrix}-\beta & 0\\ \sigma & 1\end{bmatrix}\begin{bmatrix}\pi(z)\\ y(z)\end{bmatrix}=\begin{bmatrix}-1 & \kappa\\ \sigma\phi_\pi & 1+\sigma\phi_y\end{bmatrix}z\begin{bmatrix}\pi(z)\\ y(z)\end{bmatrix}+\begin{bmatrix}-\beta & 0\\ \sigma & 1\end{bmatrix}\begin{bmatrix}\pi_0\\ y_0\end{bmatrix} \tag{78}$$

or equivalently,

$$(I_2-zA)\begin{bmatrix}\pi(z)\\ y(z)\end{bmatrix}=\begin{bmatrix}\pi_0\\ y_0\end{bmatrix} \tag{79}$$

where $A=\begin{bmatrix}\frac{1}{\beta} & -\frac{\kappa}{\beta}\\ \sigma(\phi_\pi-\frac{1}{\beta}) & 1+\frac{\sigma}{\beta}\kappa+\sigma\phi_y\end{bmatrix}$ is the matrix of coefficients. We summarize the condition for determinacy in the following.

Theorem 4 (Sticky Price Determinacy in the Frequency Domain)**.** *The sticky price model, given by (76), (28), with the Taylor rule (1), has a unique, stable equilibrium if and only if*

$$\phi_\pi>1-\frac{1-\beta}{\kappa}\phi_y \tag{80}$$

Proof. See the following (cf. time domain result 2) □

To solve the system of equations in (79) we first decompose the matrix A and then use Cauchy's residue theorem as above to determine π_0 and y_0, the initial conditions for inflation and the output gap. Define ρ_i=$eig(A)$. Iff ρ_i, $i=1,2$ there are two removable singularities. Decompose matrix A into its eigenvalues, and its eigenvector-matrix V as

$$A=V\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}V^{-1}=V\Lambda V^{-1} \tag{81}$$

Similar to Klein (2000) we define

$$\begin{bmatrix} w(z) \\ u(z) \end{bmatrix} = V^{-1} \begin{bmatrix} \pi(z) \\ y(z) \end{bmatrix} \quad \text{for} \quad z = 0, 1, 2 \dots \tag{82}$$

Substituting into our equation system, (79) gives

$$\left(I_2 - zV\Lambda V^{-1}\right) V \begin{bmatrix} w(z) \\ u(z) \end{bmatrix} = V \begin{bmatrix} w_0 \\ u_0 \end{bmatrix} \tag{83}$$

which can be rewritten and redefined as

$$(I_2 - z\Lambda) \begin{bmatrix} w(z) \\ u(z) \end{bmatrix} = \begin{bmatrix} w_0 \\ u_0 \end{bmatrix} \tag{84}$$

The diagonality of the foregoing yields two independent equations

$$(1 - z\rho_1)w(z) = w_0 \text{ and } (1 - z\rho_2)u(z) = u_0 \tag{85}$$

If both eigenvalues, $|\lambda_1|$ and $|\lambda_2| > 1$, we can eliminate the singularities via

$$\lim_{z \to 1/\lambda_1} (1 - z\lambda_1)w(z) = 0 \text{ and } \lim_{z \to 1/\lambda_2} (1 - z\lambda_2)u(z) = 0 \tag{86}$$

pinning down the two conditions $w_0 = 0$ and $u_0 = 0$. From our definition (82) and equation (84) we can therefore deduce

$$\begin{bmatrix} \pi_0 \\ y_0 \end{bmatrix} = V \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{87}$$

uniquely defining $\pi_0 = 0$ and $y_0 = 0$.

The Schur-Cohn criteria can be applied to ascertain whether both eigenvalues, λ_1 and λ_2, indeed do lie outside the unit circle (see LaSalle, 1986, p.28). These criteria, expressed in terms of A are $|\det(A)| > 1$ and $|\mathrm{tr}(A)| < 1 + \det(A)$. As

$$\det(A) = \frac{1}{\beta}(1 + \sigma\phi_y + \kappa\sigma\phi_\pi) > 1 \text{ and } \mathrm{tr}(A) = \frac{1}{\beta} + \frac{\sigma\kappa}{\beta} + 1 + \sigma\phi_y > 1 \tag{88}$$

The condition $|\det(A)| > 1$ necessarily holds and $|\mathrm{tr}(A)| < 1 + \det(A)$ holds if

$$1 < \frac{1-\beta}{\kappa}\phi_y + \phi_\pi. \tag{89}$$

Hence, determinacy in the sticky price model demands

$$1 - \frac{1-\beta}{\kappa}\phi_y < \phi_\pi. \tag{90}$$

Given the Taylor rule (1), the monetary authority can target inflation as well as the output gap to stabilize the economy - Woodford (2003, pp. 254–255), "... indeed, a large enough [response to] *either* [the output gap or inflation] suffices to guarantee determinacy".

Indeed, the real rate can be raised in response to an off equilibrium inflation increase even by responding to output movements alone. Notice that this possibility disappears if $\beta = 1$ - however this is misleading as although an *average* long-run tradeoff disappears in this case, a dynamic one remains $\frac{\pi_t - E_t \pi_{t+1}}{\kappa} = y_t$ which monetary policy needs for its targeting of inflation (or output) at different horizons to translate into a response to current inflation as we will see later in our analysis of extended Taylor rules.

Turning to the sticky information model that was presented in the previous section. In the frequency domain the model is given by the Phillips curve (38)

$$\frac{\xi}{\lambda} y(z) = z\xi y(z) + \pi(\lambda z) + \xi(1 - \lambda z) y(\lambda z) \tag{91}$$

and the IS curve equation with the interest rate rule (1)

$$(1 + \sigma\phi_y) z y(z) + \sigma\phi_\pi z \pi(z) = y(z) - y_0 + \sigma(\pi(z) - \pi_0) \tag{92}$$

We summarize determinacy in the following.

Theorem 5 (Sticky Information Determinacy)**.** *The sticky information model, given by (76), (91), with the Taylor rule (1), has a unique, stable equilibrium if and only if*

$$\phi_\pi > 1 \tag{93}$$

Proof. See the following □

At $z = 0$, define $y(0) = y_0$, $\pi(0) = \pi_0$, the Phillips curve (38) is determined by

$$\xi \frac{1 - \lambda}{\lambda} y_0 = \pi_0 \tag{94}$$

which yields one initial condition: inflation at $z = 0$ is a constant share of output increasing in the share of firms that received an information update in the initial period $1 - \lambda$. The remaining condition at $z = 0$ must follow from the system given by the Phillips curve (38)

$$\frac{\xi}{\lambda} y(z) = z\xi y(z) + \pi(\lambda z) + \xi(1 - \lambda z) y(\lambda z) \tag{95}$$

and the IS curve equation with the interest rate rule (1)

$$(1 + \sigma\phi_y) z y(z) + \sigma\phi_\pi z \pi(z) = y(z) - y_0 + \sigma(\pi(z) - \pi_0) \tag{96}$$

The matrix system is determined by (94), (91) and (96) as

$$\begin{bmatrix} \pi(z) \\ y(z) \end{bmatrix} = \begin{bmatrix} \phi_\pi & \frac{1+\sigma\phi_y-\lambda}{\sigma} \\ 0 & \lambda \end{bmatrix} z \begin{bmatrix} \pi(z) \\ y(z) \end{bmatrix} + \begin{bmatrix} \frac{1-\lambda}{\lambda}\xi + \frac{1}{\sigma} \\ 0 \end{bmatrix} y_0 + \begin{bmatrix} -\frac{\lambda}{\sigma\xi} & -\frac{\lambda}{\sigma}(1-\lambda z) \\ \frac{\lambda}{\xi} & \lambda(1-\lambda z) \end{bmatrix} \begin{bmatrix} \pi(\lambda z) \\ y(\lambda z) \end{bmatrix} \tag{97}$$

If $[\pi(\lambda z), y(\lambda z)]'$ are analytic functions $\forall\ |z| < 1$, then $[\pi(z), y(z)]'$ are analytic functions $\forall\ |z| < \frac{1}{\lambda}$ and as $0 < \lambda < 1$ certainly for $|z| < 1 < \frac{1}{\lambda}$. Similarly to (79) the system of equations can be expressed as

$$(I_2 - zA) \begin{bmatrix} \pi(z) \\ y(z) \end{bmatrix} = \begin{bmatrix} \frac{1-\lambda}{\lambda}\xi \\ 0 \end{bmatrix} y_0 + \begin{bmatrix} -\frac{\lambda}{\sigma\xi} & -\frac{\lambda}{\sigma}(1-\lambda z) \\ \frac{\lambda}{\xi} & \lambda(1-\lambda z) \end{bmatrix} \begin{bmatrix} \pi(\lambda z) \\ y(\lambda z) \end{bmatrix} \tag{98}$$

where $A = \begin{bmatrix} \phi_\pi & \frac{1+\sigma\phi_y-\lambda}{\sigma} \\ 0 & \lambda \end{bmatrix}$. The eigenvalues of matrix A are $\rho_1 = \phi_\pi, \rho_2 = \lambda$ which can be factored as $A = V\Lambda V^{-1}$ where Λ is the matrix of eigenvalues, giving us

$$\begin{bmatrix} w(z) \\ u(z) \end{bmatrix} = V^{-1} \begin{bmatrix} \pi(z) \\ y(z) \end{bmatrix} \tag{99}$$

where $V = \begin{bmatrix} 1 & \frac{1+\sigma\phi_y-\lambda}{\sigma(\lambda-\phi_\pi)} \\ 0 & 1 \end{bmatrix}$ and $V^{-1} = \begin{bmatrix} 1 & -\frac{1+\sigma\phi_y-\lambda}{\sigma(\lambda-\phi_\pi)}. \\ 0 & 1 \end{bmatrix}$.

The system of equations can be diagonalized in $w(z)$ and $u(z)$ as

$$(I_2 - z\Lambda) \begin{bmatrix} w(z) \\ u(z) \end{bmatrix} = \begin{bmatrix} \frac{1-\lambda}{\lambda}\xi + \frac{1}{\sigma} \\ 0 \end{bmatrix} u_0 + \begin{bmatrix} -\frac{\lambda}{\xi}(\frac{1}{\sigma} + v_{12}) & -\frac{\lambda}{\xi}(\frac{1}{\sigma} + v_{12})(\xi + v_{12})(1 - \frac{\xi\lambda}{\xi+v_{12}}z) \\ \frac{\lambda}{\xi} & \frac{\lambda}{\xi}(\frac{1}{\sigma} + v_{12})(1 - \frac{\xi\lambda}{\xi+v_{12}}z) \end{bmatrix} \begin{bmatrix} w(\lambda z) \\ u(\lambda z) \end{bmatrix} \tag{100}$$

The first equation is given by

$$(1 - z\phi_\pi)w(z) = \left(\frac{1-\lambda}{\lambda}\xi + \frac{1}{\sigma}\right)u_0 - \frac{\lambda}{\xi}\left(\frac{1}{\sigma} + u_{12}\right)w(\lambda z) - \frac{\lambda}{\xi}\left(\frac{1}{\sigma} + v_{12}\right)(\xi + v_{12})\left(1 - \frac{\lambda\xi}{\xi + v_{12}}z\right)u(\lambda z). \tag{101}$$

Iff $\phi_\pi > 1$ there is a removable singularity, which provides the additional initial condition

$$\lim_{z \to \frac{1}{\phi_\pi}} (1 - z\phi_\pi)w(z) = 0 \tag{102}$$

which uniquely determines the missing initial condition u_0

$$\left(\frac{1-\lambda}{\lambda}\xi + \frac{1}{\sigma}\right)u_0 = \frac{\lambda}{\xi}\left(\frac{1}{\sigma} + v_{12}\right)\left(w\left(\frac{\lambda}{\phi_\pi}\right) + (\xi + v_{12})\left(1 - \frac{\lambda\xi}{\xi + v_{12}}\frac{1}{\phi_\pi}\right)u\left(\frac{\lambda}{\phi_\pi}\right)\right) \tag{103}$$

from which together with (99) and (94) we can therefore deduce $\pi_0 = 0$ and $y_0 = 0$.[34]

To summarize, $\phi_\pi > 1$ is a necessary condition for determinacy in the sticky information model and not merely sufficient as above in the sticky price model. No amount of output gap targeting can replace a more than one for one response to inflation by the monetary authority. That is, in the absence of a stable long run tradeoff between inflation and output, the Taylor principle as a policy recommendation holds directly.

Under a simple, current inflation-targeting rule, determinacy is obtained if the central bank follows an active monetary policy satisfying the Taylor principle. This holds true for both the sticky price and the sticky information model. Including output gap targeting into the Taylor rule leads to different consequences for monetary policy in the two models. In the presence of sticky prices, the monetary authority can react to inflation and/or the output gap to achieve stability. Output gap movements are translated into inflation movements at a rate of $(1-\beta)/\kappa$ allowing for a feedback effect to inflation, the Phillips curve relationship in the long run. In the sticky information model the monetary authority has fewer options available to stabilize the economy and it should follow an active monetary policy by strongly reacting to inflation - its concern for the output gap is irrelevant for this determinacy consideration. A monetary authority that is uncertain as to whether the sticky price of information paradigm reigns is well advised to simply respond directly to inflation vigorously ($\phi_\pi > 1$) as this will ensure determinacy in both models. Note that this condition is independent of any parameters or their values outside of the monetary authorities own reaction function - no confidence in estimated parameters (such as the

[34] Note that (103) determines u_0 only implicitly, i.e., in dependence of $u\left(\frac{\lambda}{\phi_\pi}\right)$ and $w\left(\frac{\lambda}{\phi_\pi}\right)$. Hence for this homogenous solution where the zero solution is always a solution, uniqueness implies the solution is the zero solution, see footnote 11. As stated above, ascertaining that the equilibrium is unique is different than calculating the equilibrium itself and we proceeded without loss of generality with respect to determinacy in the absence of exogenous shocks. When confronted with exogenous shocks, u_0 would have to be jointly solved with $u\left(\frac{\lambda}{\phi_\pi}\right)$ and $w\left(\frac{\lambda}{\phi_\pi}\right)$ via the system of equations

$$\begin{bmatrix} w(z) \\ u(z) \end{bmatrix} = \begin{bmatrix} (1-\phi_\pi z)^{-1}(\frac{1-\lambda}{\lambda}\xi + \frac{1}{\sigma}) \\ 0 \end{bmatrix} u_0 + \begin{bmatrix} -\frac{\lambda}{\xi}(\frac{1}{\sigma}+v_{12})((1-\phi_\pi z)^{-1}) & -\frac{\lambda}{\xi}(\frac{1}{\sigma}+v_{12})(\xi+v_{12})(1-\frac{\lambda\xi}{\xi+v_{12}}z(1-\phi_\pi z)^{-1} \\ \frac{\lambda}{\xi}((1-\phi_\pi z)^{-1}) & \frac{\lambda}{\xi}(\xi+v_{12})(1-\frac{\lambda\xi}{\xi+v_{12}}z(1-\lambda z)^{-1} \end{bmatrix} \begin{bmatrix} w(\lambda z) \\ u(\lambda z) \end{bmatrix} \tag{104}$$

That is, while we can analytically solve for determinacy conditions in the sticky information model with forward looking demand (4), this approach does not let us analytically solve for, say, impulse responses to inhomogenous shocks.

slope of the Phillips curve to determine an appropriate value for output gap targeting in the sticky price model) is needed.

5. Existence and Uniqueness for Imperfect Common Knowledge

We now turn to the model of imperfect common knowledge by Nimark (2008) that combines an information rigidity with the standard sticky price nominal rigidity. This is a particularly insightful exercise as with both the sticky price and information rigidities, the determinacy bounds will coincide with the sticky price results from theorem 2 and with only the information rigidity with those of the sticky information model of the previous section in theorem 5.

Combining the IS curve (76) and the Taylor rule (1) with the imperfect common knowledge Phillips curve (56)

$$\pi_t = \beta E_t \pi_{t+1} + \kappa y_t + \zeta_t \tag{105}$$

we observe that the model is identical to that of theorem 2 apart from the forecast/prediction error ζ_t that, like exogenous driving forces we have abstracted from, are irrelevant for determinacy.[35] We summarize this in the following.

Theorem 6 (Nimark (2008) Determinacy)**.** *The imperfect common knowledge sticky price model, given by (76), (56), with the Taylor rule (1), has a unique, stable equilibrium if and only if*

$$1 - \frac{1-\beta}{\kappa}\phi_y < \phi_\pi \tag{107}$$

Proof. Following the proof of theorem 2 we can combine (76), (56), with the Taylor rule (1) as

$$\begin{bmatrix} E_t \pi_{t+1} \\ E_t y_{t+1} \end{bmatrix} = A \begin{bmatrix} \pi_t \\ y_t \end{bmatrix} - \begin{bmatrix} \frac{1}{\beta} \\ 0 \end{bmatrix} \zeta_t \tag{108}$$

[35] It is important to reiterate that providing the condition under which we have a determinant solution is not the same as providing the solution. Just as in the sticky information model, see footnote 34, calculating the solution in the presence of exogenous shocks would require us to resolve the prediction/forecast errors in ζ_t. Solving forward (109) yields

$$\begin{bmatrix} \pi_t \\ y_t \end{bmatrix} = \lim_{j\to\infty} A^{-j} \begin{bmatrix} E_t \pi_{t+j} \\ E_t y_{t+j} \end{bmatrix} + \sum_{j=0}^{\infty} A^{-j} \begin{bmatrix} \frac{1}{\beta} \\ 0 \end{bmatrix} E_t \zeta_{t+j} \tag{106}$$

and this sequence of prediction/forecast errors $E_t \zeta_{t+j}$ would have to be resolved consistent with the exogenous shocks and conditioning assumptions on agents' information sets.

where $A = \begin{bmatrix} \frac{1}{\beta} & -\frac{\kappa}{\beta} \\ \sigma(\phi_\pi - \frac{1}{\beta}) & 1 + \frac{\sigma}{\beta}\kappa + \sigma\phi_y \end{bmatrix}$ and $\zeta_t = -\frac{1-\theta}{\theta}\int(\pi_t - E[\pi_t|\mathscr{I}_s(j)])dj - \beta\int(E_t\pi_{t+1} - E[\pi_t|\mathscr{I}_s(j)])dj$ is an increment process of forecast/prediction errors. Determinacy of the inhomogenous system requires determinacy of the homogenous system, which like in theorem 2 requires both eigenvalues of A be outside the unit circle, or $1 - \frac{1-\beta}{\kappa}\phi_y < \phi_\pi$ restricting ourselves to positive coefficients.

Alternatively, we can appeal to frequency domain methods as in theorem 4. Applying the z-transform to the above gives

$$(I_2 - zA)\begin{bmatrix} \pi(z) \\ y(z) \end{bmatrix} = \begin{bmatrix} \pi_0 \\ y_0 \end{bmatrix} - \begin{bmatrix} \frac{1}{\beta} \\ 0 \end{bmatrix}\zeta(z) \tag{109}$$

we require $\pi(z)$, $y(z)$, and $\zeta(z)$ to be analytic on the unit disk and (109), along with the underlying prediction/forecast error definition of $\zeta(z)$, will fail to provide a unique set of restrictions unless we can pin down π_0 and y_0. Precisely when both eigenvalues of A are outside the unit circle, can we appeal as in theorem 4 to Cauchy's residue theorem to provide a unique set of restrictions on π_0 and y_0 such that

$$\begin{bmatrix} \pi(z) \\ y(z) \end{bmatrix} = (I_2 - zA)^{-1}\left(\begin{bmatrix} \pi_0 \\ y_0 \end{bmatrix} - \begin{bmatrix} \frac{1}{\beta} \\ 0 \end{bmatrix} z\zeta(z)\right) \tag{110}$$

continues to be well defined via analytic continuation over these two values of z inside the unit circle. □

The imperfect common knowledge Phillips curve has as a special form (68) with only the information rigidity, i.e. without the sticky price friction of (68)

$$y_t = \tilde{\zeta}_t \tag{111}$$

which is analogous to the (3) in the frictionless model but now with the prediction/forecast innovation $\tilde{\zeta}_t$ and hence we obtain the same determinacy restriction that we summarize in the following.

Theorem 7 (Imperfect Common Knowledge Determinacy). *The imperfect common knowledge model, given by (76), (68), with the Taylor rule (1), has a unique, stable equilibrium if and only if*

$$1 < \phi_\pi \tag{112}$$

Proof. We can let $\kappa \to \infty$ in theorem 6.

Or we can proceed in the time domain following the proof of theorem 1 we can combine (76), (68), with the Taylor rule (1) as

$$\phi_\pi \pi_t = E_t \pi_{t+1} - \frac{1+\sigma\phi_y}{\sigma}\tilde{\zeta}_t + \frac{1}{\sigma}E_t\tilde{\zeta}_{t+1} \tag{113}$$

where $\tilde{\zeta}_t = \frac{1}{\Theta}\int(\pi_t - E\left[\pi_t|\mathscr{I}_s(j)\right])dj$. Solving forward, Blanchard (1979)

$$\pi_t = \lim_{j\to\infty}\frac{1}{\phi_\pi^j}E_t\pi_{t+j} - \sum_{j=0}^{\infty}\frac{1}{\phi_\pi^j}\left(\frac{1+\sigma\phi_y}{\sigma}E_t\tilde{\zeta}_{t+j} - \frac{1}{\sigma}E_t\tilde{\zeta}_{t+j+1}\right) \tag{114}$$

delivers a unique, bounded solution for π_t if and only if $1 < \phi_\pi$. That is, only if $1 < \phi_\pi$ does the sunspot term $\lim_{j\to\infty}\frac{1}{\phi_\pi^j}E_t\pi_{t+j}$ disappear.

Alternatively, we can appeal to frequency domain methods as in theorem 4. Applying the z-transform to the above gives

$$\left(1 - z\phi_\pi\right)\pi(z) = \pi_0 - \frac{1+\sigma\phi_y}{\sigma}z\tilde{\zeta}(z) + \frac{1}{\sigma}\left(\tilde{\zeta}(z) - \tilde{\zeta}(0)\right) \tag{115}$$

We require $\pi(z)$ and $\tilde{\zeta}(z)$ to be analytic on the unit disk and the foregoing, along with the underlying prediction/forecast error definition of $\tilde{\zeta}(z)$, will fail to provide a unique set of restrictions unless we can pin down π_0 uniquely. If $\left|\phi_\pi\right| < 1$, then

$$\pi(z) = \frac{\pi_0 - \frac{1+\sigma\phi_y}{\sigma}z\tilde{\zeta}(z) + \frac{1}{\sigma}\left(\tilde{\zeta}(z) - \tilde{\zeta}(0)\right)}{\left(1 - z\phi_\pi\right)} \tag{116}$$

is analytic on the unit disk for arbitrary finite values of π_0. If however $1 < \left|\phi_\pi\right|$ (or simply $1 < \phi_\pi$ restricting ourselves to positive values of ϕ_π), then $\pi(z)$ has a singularity on the unit disc at $z = 1/\phi_\pi$ that can be removed by setting the residue to zero $\lim_{z\to 1/\phi_\pi}\left(1 - z\phi_\pi\right)\pi(z) = 0$ to continue $\pi(z)$ as an analytic function over this singularity on the unit disk. □

Comparing theorem 7 to theorem 5 allows us to conclude that the determinacy results under sticky information and imperfect common knowledge are identical and comparing these further to theorem 6 and theorem 2 we remark that the instilling a model of imperfect information with a long run tradeoff from, say, a Calvo sticky price friction will instill it with the same determinacy properties of the latter.

6. Existence and Uniqueness for Finite Inattentiveness

To address determinacy of equilibria under the finite inattentiveness supply side in (70), we will assess determinacy in the following extended class of linear rational expectations models that includes our model

$$0 = \sum_{i=0}^{p} \sum_{j=-m}^{n} Q(i,j) E_{t-i} X_{t+j}, \; X_t = \begin{bmatrix} R_t & \pi_t & y_t \end{bmatrix}', \; 0 \le p,m,n < \infty \tag{117}$$

where the $Q(i,j)$'s are matrices of dimensions 3×3. I.e., the model is composed of three structural equations: the supply side, the demand side, and monetary policy. The class encompasses all linear rational expectations models in the three variables of interest that (i) have a finite number of leads (given by n), (ii) have a finite number of lags (given by m), and (iii) have expectations formed at horizons from t into the finite past $t-p$.[36] This captures a wide range of interest rate rules found in the literature, from our standard Taylor rule (1) to the variety of extensions we will examine later.

Theorem 8. *For the system (117) to have a unique stationary solution,*

(1) The model

$$0 = \sum_{j=-m}^{n} \tilde{Q}_j X_{t+j} \tag{118}$$

where $\tilde{Q}_j = \sum_{i=0}^{p} Q(i,j)$, must have a unique saddle-point stable solution.

(2) The square matrix

$$\begin{bmatrix} \mathbf{Q}' & \mathbf{B}' \end{bmatrix}' \tag{119}$$

must be non-singular. $\mathbf{Q}$ and $\mathbf{B}$ are block matrices of dimensions $3p \times 3(p+n)$ and $3n \times 3(p+n)$ respectively with blocks of dimension 3×3. The s^{th} block row of $\mathbf{Q}$ is given by

$$\begin{bmatrix} 0_{max(0,s-1-m)} & \tilde{Q}(s-1, -\min(s-1,m), n) & 0_{p-s} \end{bmatrix} \tag{120}$$

where 0_i is a $3 \times 3i$ block vector of zeros and $\tilde{Q}(a,b,c) = \begin{bmatrix} \tilde{Q}(a,b) & \tilde{Q}(a,b+1) & \ldots \tilde{Q}(a,c) \end{bmatrix}$ with $\tilde{Q}(a,b) = \sum_{i=0}^{\min p,a} Q(i,b)$. The s^{th} block row of $\mathbf{B}$ is given by

$$\begin{bmatrix} 0_{max(0,s+p-m-1)} & -\tilde{B}(\min(p+s-1,m)) & I & 0_{n-s} \end{bmatrix} \tag{121}$$

[36] As discussed previously in the introduction and in footnotes 34 and 35, the absence of exogenous driving forces in (117) is without loss of generality. The conditions for determinacy remain the same if (117) is appended with stationary driving forces (i.e., we am investigating the properties of the homogenous component of the system of difference equations). For a complete solution, one would then have the additional task of associating the exogenous driving forces with the expectation errors (see, e.g., Sims (2001)).

where I is a 3×3 identity matrix and $\tilde{B}(a)$ being the last $3\times3a$ elements of the $3\times3m$ matrix B that forms Anderson's (2010, p. 7) convergent autoregressive solution to (118).

Proof. See Appendix □

The first condition requires that the model is determinate if the information rigidity were removed and the second requires that one can uniquely resolve the prediction errors. The first is Anderson's (2010) extension of the familiar Blanchard and Kahn (1980) result, while the second formalizes Whiteman's (1983, pp. 29–36) insight that resolving lagged expectations, "withholding constraints" in his terminology, is not generally a trivial task. This second restriction will hold generically unless the model of inattentiveness is ill-specified such that at some intermediate forecasting horizon there is an (un)fortuitous collinearity with another equation or forecasting horizon., i.e., due to the non-singularity of the matrix $\begin{bmatrix}\mathbf{Q}' & \mathbf{B}'\end{bmatrix}'$ - see footnote 13 and the simple, univariate example in the appendix. Hence, although this is a non trivial task to resolve these constraints numerically, as the literature on models of this type has clearly demonstrated, it is the first restriction that is the relevant restriction on monetary policy. We can then use theorem 8 to establish the determinacy conditions under the finite inattentiveness supply curve (70) as summarized in the following

Theorem 9 (Determinacy in Models of Inattentiveness)**.** *The finite inattentiveness model, given by (76), (70), with the Taylor rule (1), has a unique, stable equilibrium only if*

$$1 < \phi_\pi \tag{122}$$

Proof. The result is an immediate consequence of the first condition of theorem 8, recognizing that the frictionless equivalent of (70) is given by $y_t = 0$, and appealing to theorem 1. □

Recall from above that the "if" is only missing due to the possibility of a(n) (un)fortuitous collinearity in the exact specification of the information rigidity. Excepting this, we conclude that the determinacy conditions for our model of finite inattentiveness, imperfect common knowledge (*without* additional sticky price rigidities) and sticky information are all identical and coincide with the determinacy results in a frictionless model, i.e. theorem 1.

All three models of inattentiveness examined here share in common that their Phillips curves become vertical in the long run. This means that the more than one for one response of the nominal interest rate in response to inflation of the Taylor principle can only be satisfied directly - the degree of output gap targeting is irrelevant for determinacy. In the sticky-price New Keynesian model, the NRH does not hold at any horizon and its long run Phillips curve posits a stable dynamic tradeoff. As a consequence, the sticky-price model is not even asymptotically isomorphic to its frictionless equivalent: with this permanent link a response of the interest rate to the output gap is equivalent (proportional to the slope of the long run Phillips curve) to a response to inflation. That is, the Taylor principle doesn't need to be satisfied directly and output gap targeting can substitute for inflation targeting with respect to determinacy.

As discussed in section 3.4, several modifications of the standard sticky-price model do satisfy the NRH in equilibrium with stationary inflation. This is insufficient as establishing determinacy requires one to look at all possible equilibria, including explosive equilibria, in the hope that only one is non-explosive. As Cochrane (2011) points out, determinacy via the Taylor principle is an off equilibrium threat. Thus, sticky-price models' violation of the NRH cannot be separated from their results concerning determinacy.

7. Extensions

Here we examine two more general forms of the Taylor rule to capture different forms of interest rate rules. Consider the following rule with arbitrary targeting horizons

$$R_t = \phi_\pi E_t \pi_{t+j} + \phi_y \left(\alpha_y E_t y_{t+m} + \left(1-\alpha_y\right) E_t \Delta y_{t+m}\right) \tag{123}$$

j and m allow us to capture the targeting of inflation and real activity at different horizons and α_y enables us to examine the output gap level ($\alpha_y = 1$) as well as output gap growth ($\alpha_y = 0$) as real activity targeting.

Theorem 10 (Inattentiveness and the General Taylor Rule)**.** *An inattentiveness model, given by (31), (68), or (70) on the supply side; (76) on the demand side; and the general Taylor rule (123) for monetary policy has a unique, stable equilibrium if and only if*

$$\phi_\pi > 1 \, and \, j = 0 \tag{124}$$

Proof. See the appendix. □

Note that theorem 10 contrasts starkly with existing results in sticky prices, see table 1. Examining the table, which contains several different variants of Taylor rules examined for determinacy in the literature as special cases of theorem 10 for models of inattentiveness, the first thing to notice is the utter simplicity of the results under information rigidities. No model specific coefficients, such as subjective discount factors, degree of nominal or informational rigidities, no elasticity of intertemporal substitution is needed to ascertain the restrictions on the monetary policy rule. This is particularly appealing in an uncertain environment, where these parameters are likely to be known only with limited precision. Note, following section 3.4, that setting $\beta = 1$, does not render the bounds identical in the sticky price and information models: a long-run *dynamic* tradeoff remains $\frac{\pi_t - E_t\pi_{t+1}}{\kappa} = y_t$ which opens the possibility of monetary policy targeting past or future inflation (i.e., backward or forward looking targeting) - but even then these are not complete substitutes as they face upper bounds for the reaction to (past/future) inflation. Lubik and Marzo (2007) reconcile this result with non monotonic (e.g., oscillating) sunspot dynamics in the sticky price model - the sticky information model admits no such possibility, just as neither output gap or growth targeting cannot replace a concern for inflation, so too can a concern for past or future inflation not replace the necessity of the monetary authority to vigorously respond to current inflation.

Taking a closer look, the restrictions implied information rigidities only are again more conservative than under sticky prices: if determinacy is given under inattentiveness, it also implies determinacy under sticky prices. Hence, in the face of model uncertainty, a policy maker with a concern for robustness would be well-advised to heed the restrictions we provide here. The restrictions are far from being obscure and in fact are straightforward: the celebrated Taylor principle is necessary and sufficient for determinacy. Yet, it is the Taylor principle in its perhaps simplest, but certainly most direct form that is relevant: the policy rule must posit a contemporaneous, more than one-for-one direct response of the nominal interest rate to inflation. An indirect response via the output gap or its growth rate is insufficient - concern for the real economy can not replace a concern for inflation. This is only possible in the sticky price model as it posits a stable long run tradeoff between inflation and the output gap. This tradeoff is absent in models of inattentiveness as we have reiterated in the analysis above and hence the measure of the monetary authority's rule is in its direct response to current inflation.

Our theorem 10 is directly compatible with Loisel (2022), who provides and analysis of determinacy in a wide set of sticky price models from the Wieland, Cwik, Müller, Schmidt, and Wolters (2012); Wieland, Afanasyeva, Kuete, and Yoo (2016) which includes backward looking New and "Old" Keynesian models with different dynamics in the long run tradeoffs

Taylor Rule	Inattentiveness	Sticky Price: Lower Bound	Sticky Price: Upper Bound
Contemporaneous[a]			
$R_t = \phi_\pi \pi_t$	$1 < \phi_\pi$	$1 < \phi_\pi$	
$R_t = \phi_\pi \pi_t + \phi_y y_t$	$1 < \phi_\pi$	$\max\left\{1 - \frac{1-\beta}{\kappa}\phi_y, 0\right\} < \phi_\pi$	
Forward-looking[b]			
$R_t = \phi_\pi E_t \pi_{t+1}$	$\phi_\pi = \varnothing$	$1 < \phi_\pi$	$\phi_\pi < 1 + 2\frac{1+\beta}{\kappa\sigma}$
$R_t = \phi_\pi E_t \pi_{t+1} + \phi_y y_t$	$\phi_\pi = \varnothing$	$\max\left\{1 - \frac{1-\beta}{\kappa}\phi_y, 0\right\} < \phi_\pi$	$\phi_\pi < 1 + 2\frac{1+\beta}{\kappa\sigma} + \frac{1+\beta}{\kappa}\phi_y$
$R_t = \phi_\pi E_t \pi_{t+1} + \phi_y E_t y_{t+1}$	$\phi_\pi = \varnothing$	$\max\left\{1 - \frac{1-\beta}{\kappa}\phi_y, 0\right\} < \phi_\pi$, $0 \leq \phi_y < 1/\sigma$	$\phi_\pi < 1 + 2\frac{1+\beta}{\kappa\sigma} - \frac{1+\beta}{\kappa}\phi_y$, $0 \leq \phi_y < 1/\sigma$
$R_t = \phi_\pi E_t \pi_{t+1} + \phi_y E_t \Delta y_{t+1}$	$\phi_\pi = \varnothing$	$1 + \phi_y(1+\beta+\kappa) + \frac{1+\kappa+\beta}{\sigma} < \phi_\pi$, $1/\sigma < \phi_y$	$\phi_\pi < 1 + \frac{\kappa+\beta}{\sigma} - \phi_y(1+\kappa+\beta)$, $1/\sigma < \phi_y$
Backward-looking[c]			
$R_t = \phi_\pi \pi_{t-1}$	$\phi_\pi = \varnothing$	$1 < \phi_\pi$	$\phi_\pi < 1 + 2\frac{1+\beta}{\kappa\sigma}$
$R_t = \phi_\pi \pi_{t-1} + \phi_y y_t$	$\phi_\pi = \varnothing$	$1 < \phi_\pi$	
$R_t = \phi_\pi \pi_{t-1} + \phi_y y_{t-1}$	$\phi_\pi = \varnothing$	$\max\left\{1 - \frac{1-\beta}{\kappa}\phi_y, 0\right\} < \phi_\pi$, for $0 \leq \phi_y < \frac{1+\beta}{\sigma\beta}$ or $\max\left\{1 + 2\frac{1+\beta}{\kappa\sigma} - \frac{1+\beta}{\kappa}\phi_y, 0\right\} < \phi_\pi$, for $\frac{1+\beta}{\sigma\beta} < \phi_y$	$\phi_\pi < 1 + 2\frac{1+\beta}{\kappa\sigma} - \frac{1+\beta}{\kappa}\phi_y$, for $0 \leq \phi_y < \frac{1+\beta}{\sigma\beta}$ or $\phi_\pi < \max\left\{1 - \frac{1-\beta}{\kappa}\phi_y, 0\right\}$, for $\frac{1+\beta}{\sigma\beta} < \phi_y$
Interest rate smoothing[d]			
$R_t = \rho_R R_{t-1} + (1-\rho_R)[\phi_\pi E_t \pi_{t+1} + \phi_y y_t]$	$1 < \phi_\pi < \frac{1+\rho_R}{1-\rho_R}$	$\max\left\{1 - \rho_R - \frac{1-\beta}{\kappa}(1-\rho_R)\phi_y, 0\right\} < \phi_\pi$, $0 \leq \rho_R < \beta$	$\phi_\pi < 1 + 2\frac{1+\beta}{\kappa\sigma} + \frac{1+\beta}{\kappa}(1-\rho_R)\phi_y + \left(1 + 2\frac{1+\beta}{\kappa\sigma}\right)\rho_R$, $0 \leq \rho_R < \beta$

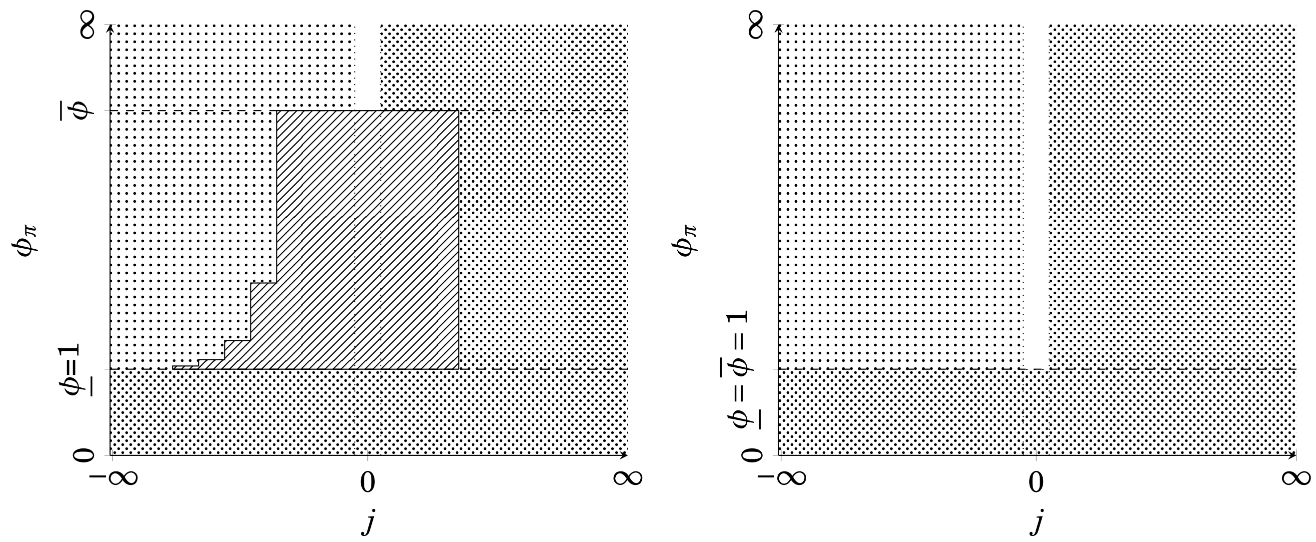
TABLE 1. Determinacy Bounds on Monetary Policy

[a] The bounds on the inattentiveness models follow from theorem 5, 6, 7, 9 and for the sticky price model from Bullard and Mitra (2002) or theorem 4.

[b] For the bounds on the inattentiveness models see theorem 10 and Appendix F and for the sticky price model Bullard and Mitra (2002) or Lubik and Marzo (2007).

[c] For the bounds on the inattentiveness models see theorems 10, 11 and Appendix F and for the sticky price model Bullard and Mitra (2002) or Lubik and Marzo (2007).

[d] The bounds on the inattentiveness models follow from theorem 11 and Appendix G and for the sticky price model from theorem 4 or Lubik and Marzo (2007).



(A) Sticky Price, $\underline{\phi} = 1$, $\overline{\phi} = 1 + 2\frac{1+\beta}{\kappa\sigma}$ (Loisel, 2022) (B) Inattentiveness, $\underline{\phi} = \overline{\phi} = 1$

FIGURE 1. Determinacy Regions

□: Determinacy, ▨: Potential Determinacy, ▩: Indeterminacy, ⚃: Nonexistence

and Taylor rules with arbitrary horizons of inflation targeting. The decisiveness of our restriction on monetary policy is again striking: with any horizon possible and inflation and/or output gap and growth targeting possible, determinacy is obtained if and only if the central bank responds to contemporaneous inflation more than one-for-one. Figure 1 depicts the situation, with our restriction in the lower panel and Loisel's (2022) for purely inflation targeting (again, a fleeting glance at table 1 ought to suffice to convince the reader that simultaneous inflation and output gap targeting at arbitrary horizons is likely to be a very complicated undertaking). It is the intermediate region between $\underline{\phi}$ and $\overline{\phi}$ in the upper panel of Loisel (2022) that constitutes the disagreement. Precisely the varying long run tradeoffs lead to the region of potential determinacy in the interior of the upper panel in his analysis. In models of inattentiveness, these tradeoff disappears entirely in the long run, eliminating this interior region of potentially (dynamically) extended determinacy: only a more than one for one response to current inflation provides determinacy as is depicted in the lower panel.

The determinacy disagreement between sticky prices and inattentiveness hinges on a single parameter - the slope of the long run Phillips cure. The sticky price model possess a vertical long run Phillips curve if and only if $\kappa \to \infty$ (though this also renders its short run slope vertical). Letting κ go to infinity recovers our bounds in the sticky information model from the sticky price restrictions as can be readily seen by setting $\kappa \to \infty$ in our table 1 and comparing the columns. Hence, rejecting our more conservative bounds on monetary policy to deliver a unique, stable equilibrium is not a consequence of preferring one New (or "Old") Kenyesian model over another, but rather of positing a stable long run tradeoff between output and inflation in the derivation of long run consequences of monetary policy.

Consider now the rule with interest rate smoothing

$$R_t = \rho_R R_{t-1} + (1-\rho_R)\left[\phi_\pi E_t \pi_{t+j} + \phi_y \left(\alpha_y E_t y_{t+m} + \left(1-\alpha_y\right) E_t \Delta y_{t+m}\right)\right] \tag{125}$$

$0 \leq \rho_R < 1$ allows for interest rate smoothing along with the generality of varying horizons and measures of real activity in (123).

Theorem 11 (Inattentiveness and the General Taylor Rule with Interest Rate Smoothing)**.** *An inattentiveness model, given by (31), (68), or (70) on the supply side; (76) on the demand side; and the general Taylor rule with interest rate smoothing (125) for monetary policy has a unique, stable equilibrium if and only if*

(1) indeterminacy if $\phi_\pi < 1$
(2) indeterminacy if $\frac{1+\rho_R}{1-\rho_R} < \phi_\pi$ *and* $j > 0$
(3) nonexistence if $\frac{1+\rho_R}{1-\rho_R} < \phi_\pi$ *and* $j < 0$
(4) determinacy if $1 < \phi_\pi$ *and* $j = 0$
(5) determinacy if $1 < \phi_\pi < \frac{1+\rho_R}{1-\rho_R}$ *and* $j = 1$

Proof. See the appendix. □

Again, we see more restrictive bounds on monetary policy than in the sticky price model (see table 1). There is, however, a broadening of the strict interpretation of the Taylor principle as the history dependence of monetary policy through interest rate smoothing implies responses to the contemporaneous inflation rate at differing horizons of inflation targeting. This can be seen via the simplified one period inflation horizon version $R_t = \rho_R R_{t-1} + (1-\rho_R)\left[\phi_\pi E_t \pi_{t+1}\right] = (1-\rho_R)\phi_\pi \left[E_t \pi_{t+1} + \rho_R E_{t-1}\pi_t + ...\right]$ which clearly imparts the interest rate rule with a concern for current inflation (precisely past expectations thereof). This broadening, however, is limited sharply by the degree of history dependance by the upper bound. As in the analysis of the sticky price model by Lubik and Marzo (2007), sunspots need not be monotonic or constant, but may also be oscillating and a too strong a response to future expected inflation in the presence of interest rate smoothing is consistent with such non monotonic sunspots. At higher horizons of future inflation expectations, this window of determinacy collapses.

8. Conclusion

We have derived determinacy bounds on monetary policy when the long run Phillips curve is vertical. In contrast to the sticky price model, we find that only the coefficients in the Taylor rule itself with respect to current inflation matter for determinacy. If the long run Phillips curve is vertical, no amount of output gap targeting, forward or backward-looking inflation targeting can substitute for a more than one-for-one response to current inflation directly. Policy makers with a concern for robustness and who are unwilling to positing a specific, stable long run tradeoff between output and inflation in the derivation of this long run consequence of monetary policy might prefer our conservative bounds. Furthermore, our bounds are simple, also provide determinacy in sticky price models and are well known: heed the Taylor principle and react to current inflation more than one for one.

We have shown this with two specific models, the sticky information model of Mankiw and Reis (2002) and the imperfect common knowledge model of Nimark (2008), the former by formulating it as a recursion in the frequency domain and applying the z-transform proposed by Whiteman (1983) and the latter by identifying a time domain recursion by defining a higher order expectation operator. By doing so we bypassed the need of expanding the model's state space or solving for an infinite sequence of undetermined $MA(\infty)$ coefficients or higher order expectations.The transformations of the models separate the long-run dynamic relationships between variables that establish the determinacy properties from sequences of forecasting errors analogously to the separation of the homogenous component of a difference equation relation from the particular solution. We then examine a non specific model of information rigidity that is specified only as imposing the natural rate hypothesis at some horizon, confirming the generality of our results. The paper thereby has added to the ongoing research on solving macroeconomic models in the frequency domain and policy relevant implications of information frictions.

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Appendix A. Essential Frequency Domain Properties of Discrete Time Series

To lay out the analysis, we present an (incomplete) introduction of the relevant frequency domain properties for our analysis.[37] Whiteman (1983) assumes, and we follow, that solutions for y_t are sought in the space spanned by time-independent square-summable linear combinations of the process(es) fundamental for the driving process, that is H^2 or Hardy space.[38] Let ϵ_t be such a mean zero fundamental process with variance σ_ϵ^2. Then an H^2 solution for an endogenous variable, y_t, is of the form

$$y_t = y(L)\epsilon_t = \sum_{j=0}^{\infty} y_j \epsilon_{t-j} \tag{A.1}$$

with $\sum_{j=0}^{\infty} y_j^2 < \infty$ and L the lag operator $Ly_t = y_{t-1}$.[39] Following, e.g., Sargent (1987a, ch. XI) the Riesz-Fischer Theorem gives an equivalence (a one-to-one and onto transformation) between the space of squared summable sequences $\sum_{j=0}^{\infty} y_j^2 < \infty$ and the space of analytic functions in unit disk $y(z)$ corresponding to the z-transform of the sequence, $y(z) = \sum_{j=0}^{\infty} y_j z^j$.

Given a discrete series y_j its z-transform $y(z)$ is defined as

$$y(z) = \sum_{j=0}^{\infty} y_j z^j \tag{A.2}$$

where z is a complex variable, and the sum extends from 0 to infinity, following the convention used in Hamilton (1994, ch. 6) and Sargent (1987a, ch. XI).[40] By evaluating the z-transform on the unit circle in the complex plane ($z = e^{-i\omega}$, where ω is the angular frequency and i the complex number $\sqrt{-1}$), we obtain the discrete-time Fourier transform

$$y(e^{-i\omega}) = \sum_{j=0}^{\infty} y_j e^{-i\omega j} \tag{A.3}$$

The connection between the autocovariance function and the Fourier transformation of the z-transform evaluated on the unit circle ($z = e^{-i\omega}$)

$$R_y(m) = \frac{\sigma_\epsilon^2}{2\pi} \int_{-\pi}^{\pi} \left| y(e^{-i\omega}) \right|^2 e^{im\omega} d\omega \tag{A.4}$$

This relationship allows us to analyze the temporal dependencies in a time series. By leveraging the z-transform and Fourier transform, along with the calculations of autocovariance and autocorrelation, we

[37] See the online appendix for a more complete representation theorem which we forgo here for expediency.

[38] See, e.g., Han, Tan, and Wu (2022) for a more formal introduction.

[39] Note that we are abusing notation somewhat and choosing to use the same letter y to refer to a discrete time series, y_t, as well as that variable's transform function $y(z)$ or MA representation/response to a fundamental process j periods ago, y_j. This serves to save on the verbosity of notation, which might otherwise read $y_t = \sum_{j=0}^{\infty} \delta_j^y \epsilon_{t-j}$ following, e.g., Meyer-Gohde (2010).

[40] The discrete signal processing and systems theory literature works in negative exponents of z, see Oppenheim, Schafer, and Buck (1999, ch. 3) and Oppenheim, Willsky, and Nawab (1996, ch. 10). Al-Sadoon (2020) follows this convention and interprets the operator being applied as the forward operator. We maintain the more familiar approach in working with the lag operator which results in our use of positive exponents in z.

will uncover the frequency content and temporal dynamics of discrete-time series that are subject to sticky information.

To see the content of the spectral representation and, in particular, how scaling in the z domain affects a series autocovariance, we will examine an AR(1) example[41]

$$y_t = \rho y_{t-1} + \epsilon_t \tag{A.5}$$

where y_t is the current value of the process, y_{t-1} is the previous value, ρ is the autoregressive parameter assumed less than one in absolute value, and ϵ_t is the white noise error term at time t with standard deviation σ_ϵ. The infinite MA representation is given by

$$y_t = \sum_{j=0}^{\infty} \rho^j \epsilon_{t-j} = \left(\sum_{j=0}^{\infty} \rho^j L^j \right) \epsilon_t \tag{A.6}$$

where L is again the lag operator ($L\epsilon_t = \epsilon_{t-1}$). This gives us (A.2) with $y_j = \rho^j$ and $L = z$ an operator defined on the unit circle.

We can use the z-transform and Fourier transformation to calculate the autocovariance of our AR(1) process. Taking the z-transform of both sides of (A.5), we have

$$y(z) = \rho z y(z) + 1 \Rightarrow y(z) = \frac{1}{1-\rho z} \tag{A.7}$$

where $y(z)$ is the z-transform of the AR(1) transfer function. Now, we can calculate the autocovariance using the square of the absolute value of the Fourier transform of the transfer function as in (A.4). Accordingly, $R_y(m)$ can be expressed as

$$R_y(m) = \frac{\sigma_\epsilon^2}{2\pi} \int_{-\pi}^{\pi} \left| y(e^{-i\omega}) \right|^2 e^{im\omega} d\omega = \frac{\sigma_\epsilon^2}{2\pi} \int_{-\pi}^{\pi} \left| \frac{1}{1-\rho e^{-i\omega}} \right|^2 e^{im\omega} d\omega \tag{A.8}$$

which can be written as a contour integral along the unit circle parameterized by $\zeta = e^{i\omega}$

$$R_y(m) = \frac{\sigma^2}{2\pi i} \oint_{|\zeta|=1} \frac{\zeta^{m-1}}{(1-\rho\zeta^{-1})(1-\rho\zeta)} d\zeta = \frac{\sigma^2}{2\pi i} \oint_{|\zeta|=1} \frac{\zeta^m}{(\zeta-\rho)(1-\rho\zeta)} d\zeta \tag{A.9}$$

which can be evaluated by residues[42] for $m \neq 0$. The function $\zeta^{m-1}/\left|1-\rho\zeta^{-1}\right|^2$ has a simple pole inside the contour (unit circle) at $\zeta = \rho$. The residue at $\zeta = \rho$ is:

$$\mathrm{Res}_{\zeta=\rho}\left[\zeta^{m-1}/\left|1-\rho\zeta^{-1}\right|^2\right] = \mathrm{Res}_{\zeta=\rho}\left[\zeta^m/\left((\zeta-\rho)(1-\rho\zeta)\right)\right] = \rho^m/(1-\rho^2) \tag{A.10}$$

which gives the autocovariance function of y_t as

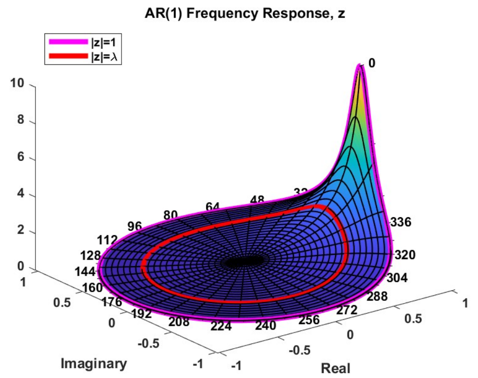
$$R_y(m) = \sigma^2 \times \mathrm{Res}_{\zeta=\rho} = \sigma^2 \rho^m/(1-\rho^2) \tag{A.11}$$

The same value we would obtain using time domain methods.

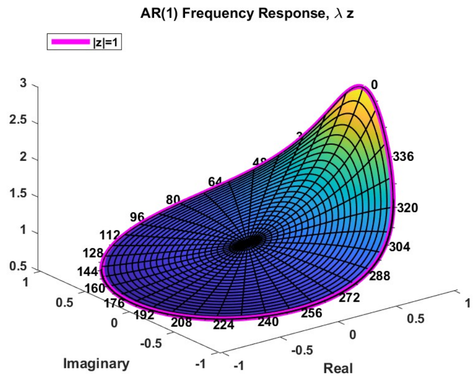
Figure 2 plots the (absolute value of the) transfer function $|y(z)|$, $|z| \leq 1$ for two values of ρ. In figure 2a, the absolute value of the transfer function is plotted with $\rho = 0.9$ and in figure 2b with the autoregressive

[41]See the appendix for an additional ARMA(2,2) example.

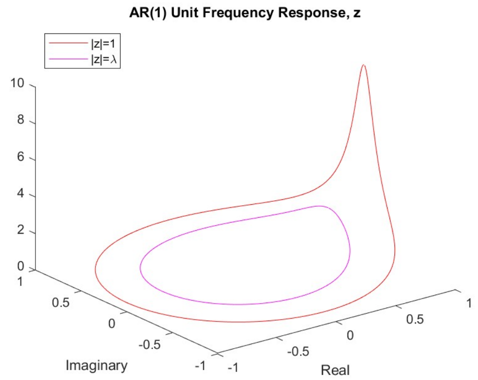
[42]The residue of a function $f(\zeta)$ at a pole ζ_0 of order k is given by $\mathrm{Res}_{\zeta=\zeta_0}[f(\zeta)] = \frac{1}{(k-1)!} \lim_{\zeta\to\zeta_0} \frac{d^{m-1}}{d\zeta^{m-1}} \left((\zeta-\zeta_0)^k f(\zeta)\right)$ and the contour integral along γ is $\frac{1}{2\pi i}\oint_\gamma f(\zeta)d\zeta = \sum_j \mathrm{Res}_{\zeta=\zeta_j}[f(\zeta)]$ where the sum is over all the singularities - ζ_j - enclosed by γ, see Ahlfors (1979, ch. 4.5).



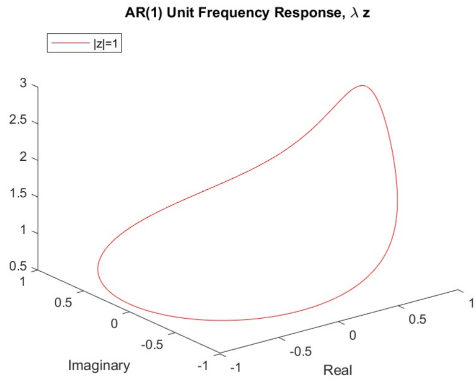
(A) $|y(z)|$: $y_t = \rho y_{t-1} + \epsilon_t$



(B) $|y(z)|$: $y_t = \lambda\rho y_{t-1} + \epsilon_t$



(C) $|y(z)|_{|z|=1}$: $y_t = \rho y_{t-1} + \epsilon_t$



(D) $|y(z)|_{|z|=1}$: $y_t = \lambda\rho y_{t-1} + \epsilon_t$

FIGURE 2. AR(1) - Transfer Functions on the Unit Disk

The values $\rho = 0.9$ and $\lambda = 0.7$ were used

parameter dampened by $\lambda = 0.7$. The values on the unit circle can be found in the lower two panels, figures 2c and 2d, which can be used in (A.8) to determine the autocovariances.

Among the properties of the z-transform - see, e.g., Oppenheim, Schafer, and Buck (1999, ch. 3) and Oppenheim, Willsky, and Nawab (1996, ch. 10), the one that will be both particularly relevant for interpreting sticky information in the next section (and is less known to economists) is that of scaling in the z domain. Proposition 1 tells us that multiplying a sequence with a given region of convergence and set of poles and zeros by an exponential sequence in λ scales the region of convergence and the poles and zeros of y by λ.

Proposition 1 (Scaling in the z domain)**.** *Given a z-transform of a sequence with a region of convergence R*

$$y(z) = \sum_{j=0}^{\infty} y_j z^j \tag{A.12}$$

the scaled sequence

$$y(\lambda z) = \sum_{j=0}^{\infty} y_j \lambda^j z^j \tag{A.13}$$

has a region of convergence $R/|\lambda|$ and if $y(z)$ has a pole (or zero) at a, then $y(\lambda z)$ has a pole (or zero) at λa.

Proof. See Oppenheim, Willsky, and Nawab (1996, p. 768) and note the difference in convention with the signal processing literature developing series in the inverse of z in contrast to the time series literature - e.g., Sargent (1987a, ch.XI) and Hamilton (1994, ch. 6). □

To understand the effects of scaling in the frequency domain, consider the following example. Let A_t be a mean zero, linearly regular covariance stationary stochastic process with known Wold representation given by

$$A_t = A(L)\epsilon_t = \sum_{i=0}^{\infty} a_i \epsilon_{t-i} = \sum_{i=0}^{\infty} a_i L^i \epsilon_t \tag{A.14}$$

Compare this with the process

$$B_t = A(\lambda L)\epsilon_t = \sum_{i=0}^{\infty} a_i \lambda^i \epsilon_{t-i} = \sum_{i=0}^{\infty} a_i (\lambda L)^i \epsilon_t \tag{A.15}$$

The autocovariance of A_t is given by

$$c_A(h) = \sum_{i=-\infty}^{\infty} a_i a_{i+h} \sigma_\epsilon^2 \tag{A.16}$$

and of B_t by

$$c_B(h) = \sum_{i=-\infty}^{\infty} \lambda^i a_i \lambda^{i+h} a_{i+h} \sigma_\epsilon^2 = \lambda^h \sum_{i=-\infty}^{\infty} \lambda^{2i} a_i a_{i+h} \sigma_\epsilon^2 \tag{A.17}$$

Inspection shows that for $0<\lambda<1$, $c_B(h) < c_A(h)$ and that $c_B(h)$ is decreasing in h at a rate λ.

This is directly exemplified by the AR(1) process above. Figure 2 plots $|y(z)|$ for $y_t = \rho y_{t-1} + \epsilon_t$ in the left panels and $|y(\lambda z)|$ on the right. Notice that the entire transfer function inside the closed unit disk for $y_t = \lambda\rho y_{t-1} + \epsilon_t$ can be found as the transfer function of $y_t = \rho y_{t-1} + \epsilon_t$ inside the circle with radius λ. That is, λ scales the transfer function and in this case with $|\lambda| < 1$ towards the origin - that is, away from the unconditional response $|y(1)|$ to shocks at all time horizons and towards the impact response $|y(0)|$ of the process to contemporaneous shocks.

The final and, for our determinacy analysis later, crucial property to observe is that this dampening is not bidirectional. If $|y(z)|$ is well defined (analytic) on the unit disk, so too will $|H(\lambda z)|$ be for $|\lambda| < 1$. Defining $\tilde{z} = \lambda z$, $|y(\tilde{z})|$ being well defined (analytic) on the unit disk does not allow us conclude the same about $|y(\frac{1}{\lambda}\tilde{z})|$ for $|\lambda| < 1$, as $\frac{1}{\lambda}\tilde{z}$ goes past the unit circle. That is, following Proposition 1, λ scales the region of convergence and if the process defined by $y(z)$ has a region of convergence from the origin out to the unit circle, then the process associated with $H(\frac{1}{\lambda}z)$ has a region of convergence out only to $|\lambda| < 1$.

Appendix B. Frequency Domain Solution of Forward-Looking Models

Having laid out the basic properties and paid specific attention to the scaling in the z domain property, we now turn to solving rational expectations models in the frequency domain following Whiteman (1983) - see also Taylor (1986, ch. 2.3) for an approachable introduction with direct comparisons to other methods.

Starting with expectations, the Wiener-Kolmogorov prediction formula gives us $E_t[y_{t+n}] = E_t\left[\sum_{j=0}^{\infty} y_j \epsilon_{t-j+n}\right] = \sum_{j=0}^{\infty} y_{j+n}\epsilon_{t-j}$. The Wiener-Kolmogorov prediction formula of "plussing" gives the frequency domain version

$$\mathcal{Z}\{E_t[x_{t+1}]\} = \left[\frac{x(z)}{z}\right]_+ = \frac{1}{z}(x(z) - x(0)) \tag{B.18}$$

where $_+$ is the annihilation operator, see Sargent (1987a) and Hamilton (1994).

Consider a backward and forward looking model in y_t and ϵ_t

$$aE_t y_{t+1} + by_t + cy_{t-1} + \epsilon_t = 0 \tag{B.19}$$

The same process is presented in the z domain as

$$a\frac{1}{z}(y(z) - y_0) + by(z) + czy(z) + 1 = 0 \tag{B.20}$$

Rearranging allows us to reduce the solution to this model as

$$a(y(z) - y_0) + bzy(z) + cz^2 y(z) + z = 0 \Leftrightarrow (a + bz + z^2)y(z) = ay_0 - z \tag{B.21}$$

$$(a - a(\lambda_1 + \lambda_2)z + a\lambda_1\lambda_2 z^2)y(z) = ay_0 - z \Leftrightarrow (1 - \lambda_1 z)(1 - \lambda_2 z)y(z) = y_0 - \frac{z}{a} \tag{B.22}$$

with the initial condition on y_0 to be determined.

We will require that $y(z)$ be analytic inside the unit disk to give us a stable process y_t causal in ϵ_t. Consider now the following possibilities. If $|\lambda_1|, |\lambda_2| < 1$, then there is no singularity in $y(z)$ inside the unit circle that can be removed to pin down y_0 and, we find that $(1-\lambda_1 L)(1-\lambda_2 L)y_t = \left(y_0 - \frac{L}{a}\right)\epsilon_t$ is necessarily unstable as at most one of the two unstable autoregressive factors $(1-\lambda_k L)$ could be removed by a particular choice of y_0 - that is, we have non existence of a stable solution. If, however, $|\lambda_1|, |\lambda_2| > 1$, there are two singularities in $y(z)$ inside the unit circle and y_0 cannot be uniquely determined so there are multiple stable solutions - that is, we have indeterminacy. If however, $|\lambda_2| < 1 < |\lambda_1|$, there is one singularity in $y(z)$ inside the unit circle, namely at $z = 1/\lambda_1$, and using the residue theorem[43] it can be removed to ensure the analyticity of $y(z)$ over the unit disk by setting the boundary condition on y_0 as

$$\lim_{z \to \frac{1}{\lambda_1}} (1 - \lambda_1 z)(1 - \lambda_2 z)y(z) \stackrel{!}{=} 0 = y_0 - \frac{1}{\lambda_1 a} \Rightarrow y_0 = \frac{1}{\lambda_1 a} \tag{B.23}$$

which determines the unique stable solution for the process on $y(z)$ as

$$y(z) = \frac{1}{1-\lambda_1 z}\frac{1}{1-\lambda_2 z}\frac{1}{a}\left(\frac{1}{\lambda_1} - z\right) = \frac{1}{1-\lambda_2 z}\frac{1}{\lambda_1 a} = \frac{1}{\lambda_1 a}\frac{1}{1-\lambda_2 z} \tag{B.24}$$

Substituting the lag operator for z to express in the time domain gives us

$$y_t = \frac{1}{\lambda_1 a}\frac{1}{1-\lambda_2 L}\epsilon_t \Rightarrow y_t = \lambda_2 y_{t-1} + \frac{1}{\lambda_1 a}\epsilon_t \tag{B.25}$$

Hence our requirement that one root be inside and one outside the unit circle gives us the famed Blanchard and Kahn (1980) condition. Underlining the point that deriving the condition in either time or frequency domain neither alters the model itself or the associated conditions for determinacy, but simply allows us to determine unique solutions and boundary conditions of models with a different tools.

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[43]See Ahlfors (1979, ch. 4).

APPENDIX C. RECURSIVE REPRESENTATION OF THE IMPERFECT COMMON KNOWLEDGE PHILLIPS CURVE

The main text derived the recursive representation using the average higher order expectations operator we defined as $H_s x_t \equiv \int E\left[x_t|\mathscr{I}_s(j)\right]dj$. Here we derive the recursive representation using an alternative route. Begin with the Phillips curve (48)

$$\pi_t = (1-\theta)\left(1-\beta\theta\right)\sum_{k=0}^{\infty}(1-\theta)^k mc_{t|t}^{(k)} + \beta\theta\sum_{k=0}^{\infty}(1-\theta)^k \pi_{t+1|t}^{(k+1)} \tag{C.26}$$

Calculate the average higher order expectation of π_t, $\pi_{t|s}^{(1)} \equiv \int E\left[\pi_t|\mathscr{I}_s(j)\right]dj$

$$\int E\left[\pi_t|\mathscr{I}_s(j)\right]dj = \int E\left[(1-\theta)\left(1-\beta\theta\right)\sum_{k=0}^{\infty}(1-\theta)^k mc_{t|t}^{(k)} + \beta\theta\sum_{k=0}^{\infty}(1-\theta)^k \pi_{t+1|t}^{(k+1)}\middle|\mathscr{I}_s(j)\right]dj \tag{C.27}$$

$$\pi_{t|s}^{(1)} = (1-\theta)\left(1-\beta\theta\right)\sum_{k=0}^{\infty}(1-\theta)^k\int E\left[mc_{t|t}^{(k)}\middle|\mathscr{I}_s(j)\right]dj \tag{C.28}$$

$$+\beta\theta\sum_{k=0}^{\infty}(1-\theta)^k\int E\left[pi_{t+1|t}^{(k+1)}\middle|\mathscr{I}_s(j)\right]dj \tag{C.29}$$

$$\pi_{t|s}^{(1)} = (1-\theta)\left(1-\beta\theta\right)\sum_{k=0}^{\infty}(1-\theta)^k mc_{t|t}^{(k+1)} + \beta\theta\sum_{k=0}^{\infty}(1-\theta)^k \pi_{t+1|t}^{(k+2)} \tag{C.30}$$

multiply with $(1-\theta)$ and compare with (C.26)

$$(1-\theta)\pi_{t|s}^{(1)} = (1-\theta)\left(1-\beta\theta\right)\sum_{k=1}^{\infty}(1-\theta)^k mc_{t|t}^{(k)} + \beta\theta\sum_{k=1}^{\infty}(1-\theta)^k \pi_{t+1|t}^{(k+1)} \tag{C.31}$$

$$= \pi_t - (1-\theta)\left(1-\beta\theta\right)mc_t - \beta\theta\pi_{t+1|t}^{(1)} \tag{C.32}$$

or

$$\pi_t - (1-\theta)\pi_{t|s}^{(1)} = (1-\theta)\left(1-\beta\theta\right)mc_t + \beta\theta\pi_{t+1|t}^{(1)} \tag{C.33}$$

which gives (53) in the main text.

APPENDIX D. PROOF THEOREM 8

By the Wold theorem,[44] any stationary process can be represented as

$$X_t = \sum_{l=0}^{\infty}\theta_l\epsilon_{t-l} + \Xi_t,\ \text{where } E\epsilon_t = 0 \text{ and } E\epsilon_t\epsilon_{t+j}' = 0,\ \forall j \neq 0 \tag{D.34}$$

and Ξ_t is an orthogonal linearly deterministic process, forecastable perfectly from its own history. Starting with the indeterministic part,and inserting into (117)

$$0 = \sum_{j=0}^{n}\left[\sum_{l=0}^{\infty}\left(\sum_{i=0}^{min(p,l)}Q(i,j)\right)\theta_{l+j}\epsilon_{t-l}\right] + \sum_{j=1}^{m}\left[\sum_{l=0}^{\infty}\left(\sum_{i=0}^{min(p,l+j)}Q(i,j)\right)\theta_l\epsilon_{t-l-j}\right] \tag{D.35}$$

Using the definition of $\tilde{Q}(i,j)$ yields

$$0 = \sum_{j=0}^{n}\left[\sum_{l=0}^{\infty}\tilde{Q}(l,j)\theta_{l+j}\epsilon_{t-l}\right] + \sum_{j=1}^{m}\left[\sum_{l=0}^{\infty}\tilde{Q}(l+j,j)\theta_l\epsilon_{t-l-j}\right] \tag{D.36}$$

This must hold for all realizations of ϵ_t. Comparing coefficients yields

$$0 = \sum_{j=0}^{n}\tilde{Q}(l,j)\theta_{l+j} + \sum_{j=1}^{m}\tilde{Q}(l,j)\theta_{l-j} \tag{D.37}$$

[44]See, e.g., Sargent (1987a, pp. 286–290), as well as Priestley (1981, pp. 756–758).

a time-varying system of difference equations with initial conditions $\sum_{j=1}^{m}\theta_{-j}=0$. But as $\tilde{Q}(p+i,j)=\tilde{Q}(p,j)$, $\forall i\geq 0$, the system of difference equations has constant coefficients, after and including p. This system can be written as (118) and coincides with Anderson's (2010) canonical form. If the solution to this system is unique, its stable solution can be written as

$$\theta_l = B\begin{bmatrix}\theta'_{l-m} & \dots & \theta'_{l-1}\end{bmatrix}', \ \forall l\geq p \tag{D.38}$$

The first p (block) equations—remembering the initial conditions—can be gathered into

$$\mathbf{Q}\begin{bmatrix}\theta'_{0} & \dots & \theta'_{n+p-1}\end{bmatrix}'=0 \tag{D.39}$$

giving $3p$ equations in $3(p+n)$ variables. (D.38) yields $3n$ more equations that deliver

$$\mathbf{B}\begin{bmatrix}\theta'_{0} & \vdots & \theta'_{n+p-1}\end{bmatrix}'=0 \tag{D.40}$$

stacking the two yields (119).[45]

The system (D.37) is homogenous. Thus, one stationary solution is given by $\theta_l=0$, $\forall i$, the fundamental solution in the absence of exogenous driving forces. If (119) is invertible and if (118) is saddle-point stable, then this is the only stationary solution.

Only Ξ_t remains. Inserting it into (117), it follows that this can also be written as (118). If there is a unique solution in past values of Ξ_t, the solution can be written in the same form as (D.38), which must be zero when taken to its remote past from the stability of (D.38).

Appendix E. Example of Singular Information Structure in Theorem 8

The first condition theorem 8 requires that the model be determinate if the information rigidity were removed and the second requires that one can uniquely resolve the prediction errors, which would only fail to hold due to the non-singularity of the matrix $\begin{bmatrix}\mathbf{Q}' & \mathbf{B}'\end{bmatrix}'$. While this cannot be guaranteed due to the generality of the class of models specified in (117), there is nothing in the class of models to induce this matrix to be singular in general. Even if one should encounter a parameterization that leads to singularity, a minor pertubation of the model or its parameterization should generally lead to non-singularity.

A simple, univariate example will illustrate. Consider the following system

$$aE_t[\theta_{t+1}]=b\theta_t+cE_{t-1}[\theta_t] \tag{E.41}$$

In the absence of expectations, that is in the form of (118), the equation reduces to

$$a\theta_{t+1}=(b+c)\theta_t \tag{E.42}$$

which is saddle-point stable if $|\frac{b+c}{a}|>1$. But the original equation does have expectations and, indeed, lagged expectations that need to be resolved. Taking expectations of (E.41) at the highest expectational lag (here $t-1$) yields

$$aE_{t-1}[\theta_{t+1}]=(b+c)E_{t-1}[\theta_t] \tag{E.43}$$

[45] This extends Meyer-Gohde's (2010, p. 987) Equation (12) to Anderson's (2010) higher leads and lags.

defining $\tilde{\theta}_{t-1} = E_{t-1}[\theta_t]$, inserting into the above and lagging forward yields

$$aE_t\left[\tilde{\theta}_{t+1}\right] = (b+c)\tilde{\theta}_t \tag{E.44}$$

an equation whose saddle-point properties are the same as (E.42). Thus, if $|\frac{b+c}{a}| > 1$, there is a unique stable solution. As the system is homogenous, this is $\tilde{\theta}_t = 0$. Recalling the definition of $\tilde{\theta}_t$ and inserting into (E.41) yields

$$0 = b\theta_t \tag{E.45}$$

Consider now the special case $b = 0$: the foregoing does not deliver a unique solution for θ_t, even though the condition for saddle-point stability, now $|\frac{c}{a}| > 1$, can still be fulfilled. Of course, $b = 0$ is a special case and it need not hold generally: hence, an isolated singularity.

Appendix F. Proof of Theorem 10

Take the IS equation (4) and express it in the frequency domain

$$y(z) = \frac{1}{z}(y(z) - y_0) - \sigma R(z) + \sigma\frac{1}{z}(\pi(z) - \pi_0) \tag{F.46}$$

do the same with the Taylor rule in (123)

$$R(z) = \phi_\pi z^{-j}\left(\pi(z) - \sum_{k=0}^{j-1}\pi_k z^k\right) + \phi_y z^{-m}\left(\tilde{y}(z) - \sum_{k=0}^{m-1}\tilde{y}_k z^k\right) \tag{F.47}$$

where $\tilde{y}(z) = (1-(1-\alpha)z)y(z)$. Now combine the two

$$\frac{1-z}{z}y(z) - \frac{1}{z}y_0 = \sigma\left(\phi_\pi z^{-j}\left(\pi(z) - \sum_{k=0}^{j-1}\pi_k z^k\right) + z^{-m}\phi_y\left(\tilde{y}(z) - \sum_{k=0}^{m-1}\tilde{y}_k z^k\right)\right) - \sigma\frac{1}{z}(\pi(z) - \pi_0) \tag{F.48}$$

collecting terms

$$\left(\phi_\pi z^{1-j} - 1\right)\pi(z) = \frac{1}{\sigma}(1-z)y(z) - \frac{1}{\sigma}y_0 - \phi_y z^{1-m}\left(z\tilde{y}(z) - \sum_{k=0}^{m-1}\tilde{y}_k z^k\right) + \phi_\pi z^{1-j}\sum_{k=0}^{j-1}\pi_k z^k + \pi_0 \tag{F.49}$$

Now recall that $y(z)$ follows from $\pi(\lambda z)$ and further dampened (as $0 < \lambda < 1$) inflation

$$y(z) = \frac{1}{\xi}\sum_{j=1}^{\infty}\frac{\lambda^j}{1-\lambda^j z}\pi(\lambda^j z) \tag{F.50}$$

Hence, given $\pi(\lambda^j z)$; $j > 0$, $y(z)$ and all $y_k \equiv \left(d^k y(z)/dz^k\right)|_{z=0}$ follow from (F.50).

Note that (F.49) defines $\pi(z)$ with roots $z : \phi_\pi z^{1-j} - 1 = 0$. For a given root, call it $\overline{z^{(1)}}$, (F.49) implies roots for $\pi(\lambda^k z)$ as $z : \phi_\pi\left(\lambda^k z\right)^{1-j} - 1 = 0 \Rightarrow \phi_\pi\lambda^{k(1-j)}z^{1-j} - 1 = 0$. Corresponding to $\overline{z^{(1)}}$ is the root for $\pi(\lambda^k z)$, call it $\overline{\lambda^k z^{(1)}}$. So $\overline{\lambda^k z^{(1)}}$ solves $\phi_\pi\lambda^{k(1-j)}\overline{\lambda^k z^{(1)}}^{1-j} - 1 = 0$ and $\overline{z^{(1)}}$ solves $\phi_\pi\overline{z^{(1)}}^{1-j} - 1 = 0$. Inspection shows that the roots are related via $\overline{\lambda^k z^{(q)}} = \lambda^k\overline{z^{(q)}}$, for $q = 1, 2, ..$ # of roots. Now (F.49) has $\tilde{y}(z)$ and $y(z)$ on the right hand side which, via (F.50) and the definition of $\tilde{y}(z)$, are linear functions of $\pi(\lambda^j z)$; $j > 0$ and it follows that a root $\pi(z)$ on the left hand side, $z : \phi_\pi z^{1-j} - 1 = 0$, corresponds to a root on the right hand side in the terms $\pi(\lambda^j z)$; $j > 0$. That is, extending $\pi(z)$ by removing a singularity at a root $\overline{z^{(q)}}$ removes the corresponding singularity in $\pi(\lambda^k z)$ via $\pi(z)|_{z=\overline{z^{(q)}}} = \pi(\lambda^k z)|_{z=\overline{z^{(q)}}}$ which is evaluating $\pi(\lambda^k z)$ at its root $\overline{\lambda^k z^{(q)}}$ as $\lambda^k\overline{z^{(q)}} = \overline{\lambda^k z^{(q)}}$. Hence, eliminating roots inside the unit circle allows (F.49) to define $\pi(z)$ as an analytic function - and thus also $y(z)$ via (F.50) - over the unit disk. That is, the long run verticality of the Phillips curve (F.50) or independence of $y(z)$ from $\pi(z)$ on the unit circle translates the singularities in $\pi(z)$

to singularities in $y(z)$ - via $\pi(\lambda^k z)$. The elimination of singularities follows thus only via the independent consideration of singularities in $\pi(z)$.

Rewritting (F.49)

$$\left(\phi_\pi z^{1-j} - 1\right)\pi(z) = \phi_\pi z^{1-j}\sum_{k=0}^{j-1}\pi_k z^k - \pi_0 + \text{t.i.d.} \tag{F.51}$$

where t.i.d. refers to "terms independent of determinacy" following the discussion above. This allows us to easily declinate the problem into the number of roots.

For $j < 1$, the summation on the right hand side is empty

$$\left(\phi_\pi z^{1-j} - 1\right)\pi(z) = \pi_0 + \text{t.i.d.} \tag{F.52}$$

therefore only one constant, π_0, needs to be determined. That is, the polynomial $\phi_\pi z^{1-j} - 1 = 0$ must have one and only one z inside the unit circle for the system to be determinate, for π_0 to be set to remove the singularity at the root inside the unit circle so that $\pi(z)$ (and hence $y(z)$) is an analytic function over the unit disk. If there are no roots inside the unit circle, then π_0 cannot be pinned down and the system is indeterminate. If there is more than one root inside the unit circle, then there are not enough constants that can be set to eliminate the singularities to render $\pi(z)$ (and hence $y(z)$) analytic functions over the entire unit disk. The roots are given by

$$z = \left(\frac{1}{\phi_\pi}\right)^{\frac{1}{1-j}} \tag{F.53}$$

If $1 < \phi_\pi$, then all $1-j$ roots are inside the unit circle. If $0 < \phi_\pi < 1$, then all $1-j$ roots are outside the unit circle. This gives the following

$$\begin{cases} \text{for } j = 0, & 1-j = 1 \text{ root inside the unit circle if and only if } 1 < \phi_\pi \\ \text{for } j < 0, & 1-j > 1 \text{ roots inside/outside the unit circle if } 1 < \phi_\pi \ / \ 0 < \phi_\pi < 1 \end{cases} \tag{F.54}$$

For $j \geq 1$, (F.49) becomes

$$\left(\phi_\pi - z^{j-1}\right)\pi(z) = \phi_\pi \sum_{k=0}^{j-1}\pi_k z^k + z^{j-1}\pi_0 + \text{t.i.d.} \tag{F.55}$$

and therefore j constants, $\{\pi_k\}_{k=0,1,\dots,j-1}$, need to be determined. That is, the polynomial $\phi_\pi - z^{j-1} = 0$ must have j roots inside the unit circle for the system to be determinate, for $\{\pi_k\}_{k=0,1,\dots,j-1}$ to be set to remove the singularity at the roots inside the unit circle so that $\pi(z)$ (and hence $y(z)$) is an analytic function over the unit disk. If there are fewer roots inside the unit circle, then not all of $\{\pi_k\}_{k=0,1,\dots,j-1}$ cant be pinned down and the system is indeterminate. If there are more than j roots inside the unit circle, then there are not enough constants that can be set to eliminate the singularities to render $\pi(z)$ (and hence $y(z)$) analytic functions over the entire unit disk. The polynomial $\phi_\pi - z^{j-1} = 0$ is of order $j-1$ and, hence, has $j-1 < j$ roots following from the fundamental theorem of algebra. That is

$$\begin{cases} \text{for } j \geq 1, & \text{less than } j \text{ roots inside the unit circle} \end{cases} \tag{F.56}$$

Summarizing over the cases yields theorem 10 and the lower panel of figure 1.

Appendix G. Proof of Theorem 11

Rouché's theorem, also at the foundation of familiar Schur-Cohn (Woodford, 2003; Lubik and Marzo, 2007) and Jury conditions, will be used in the following and is worth repeating here

Theorem 12 (Rouché's Theorem)**.** *Let f and g be holomorphic in an open region containing the closure of the unit disk, such that g does not vanish on the unit circle. If $|f(z)| < |g(z)|$ on the unit circle, then f and $f+g$ have the same number of zeros, counting multiplicities, inside the unit circle.*

Proof. See Ahlfors (1979, pp. 152-154) □

The Taylor rule in (125) in the frequency domain is

$$\left(1-\rho_R z\right) R(z) = \left(1-\rho_R\right)\left[\phi_\pi z^{-j}\left(\pi(z) - \sum_{k=0}^{j-1} \pi_k z^k\right) + \phi_y z^{-m}\left(\tilde{y}(z) - \sum_{k=0}^{m-1} \tilde{y}_k z^k\right)\right] \tag{G.57}$$

where again $\tilde{y}(z) = (1-(1-\alpha)z)\, y(z)$. Combining this with the IS equation (F.46) then gives

$$\frac{1-z}{z} y(z) - \frac{1}{z} y_0 = \sigma \frac{\left(1-\rho_R\right)}{\left(1-\rho_R z\right)}\left[\phi_\pi z^{-j}\left(\pi(z) - \sum_{k=0}^{j-1} \pi_k z^k\right) + \phi_y z^{-m}\left(\tilde{y}(z) - \sum_{k=0}^{m-1} \tilde{y}_k z^k\right)\right] - \sigma \frac{1}{z}\left(\pi(z) - \pi_0\right) \tag{G.58}$$

collecting terms

$$\left(1-\rho_R z - \left(1-\rho_R\right)\phi_\pi z^{1-j}\right)\pi(z) \tag{G.59}$$

$$= \left(1-\rho_R z\right)\pi_0 - \left(1-\rho_R\right)\phi_\pi z^{1-j} \sum_{k=0}^{j-1} \pi_k z^k - \frac{1-\rho_R z}{\sigma} y_0 - \left(1-\rho_R\right)\phi_y z^{1-m} \sum_{k=0}^{m-1} \tilde{y}_k z^k \tag{G.60}$$

$$+ \left[\left(1-\rho_R z\right)(1-z)\frac{1}{\sigma} + \left(1-\rho_R\right)\phi_y z^{1-m}\left(1-(1-\alpha)z\right)\right] y(z) \tag{G.61}$$

Now recall that $y(z)$ follows from $\pi(\lambda z)$ and further dampened (as $0 < \lambda < 1$) inflation, see (F.50), hence, $y(z)$ and all $y_k \equiv \left(d^k y(z)/dz^k\right)|_{z=0}$ follow from (F.50) given $\pi(\lambda^j z)$; $j > 0$.

Note that (G.59) defines $\pi(z)$ with roots $z : 1-\rho_R z - \left(1-\rho_R\right)\phi_\pi z^{1-j} = 0$. Following the proof of theorem 10 above, extending $\pi(z)$ by removing a singularity at a root $\overline{z^{(q)}}$ removes the corresponding singularity in $\pi(\lambda^k z)$ via $\pi(z)|_{z=\overline{z^{(q)}}} = \pi(\lambda^k z)|_{z=\overline{z^{(q)}}}$ which is evaluating $\pi(\lambda^k z)$ at its root $\overline{\lambda^k z^{(q)}}$ as $\lambda^k \overline{z^{(q)}} = \overline{\lambda^k z^{(q)}}$. Hence, eliminating roots inside the unit circle allows (G.59) to define $\pi(z)$ as an analytic function - and thus also $y(z)$ via (F.50) - over the unit disk. That is, the long run verticality of the Phillips curve (F.50) or independence of $y(z)$ from $\pi(z)$ on the unit circle translates the singularities in $\pi(z)$ to singularities in $y(z)$ - via $\pi(\lambda^k z)$. The elimination of singularities follows thus only via the independent consideration of singularities in $\pi(z)$.

Rewritting (G.59)

$$\left(1-\rho_R z - \left(1-\rho_R\right)\phi_\pi z^{1-j}\right)\pi(z) = \left(1-\rho_R z\right)\pi_0 - \left(1-\rho_R\right)\phi_\pi z^{1-j} \sum_{k=0}^{j-1} \pi_k z^k + \text{t.i.d.} \tag{G.62}$$

where t.i.d. refers to "terms independent of determinacy" following the discussion above. This allows us to easily declinate the problem into the number of roots.

For $j \le 1$, the right hand side is in π_0 (that is, the summation on the right hand side contains at most a term in π_0)

$$\left(1-\rho_R z - \left(1-\rho_R\right)\phi_\pi z^{1-j}\right)\pi(z) = \left[1-\rho_R z - \mathbf{1}_{j=1}\left(1-\rho_R\right)\right]\pi_0 + \text{t.i.d.} \tag{G.63}$$

where $\mathbf{1}_{j=1}$ is the indicator function, equal to 1 if $j = 1$ and 0 otherwise; therefore only one constant, π_0, needs to be determined. That is, the polynomial $1-\rho_R z-(1-\rho_R)\phi_\pi z^{1-j}=0$ must have one and only one z inside the unit circle for the system to be determinate, for π_0 to be set to remove the singularity at the root inside the unit circle so that $\pi(z)$ (and hence $y(z)$) is an analytic function over the unit disk. If there are no roots inside the unit circle, then π_0 cannot be pinned down and the system is indeterminate. If there is more than one root inside the unit circle, then there are not enough constants that can be set to eliminate the singularities to render $\pi(z)$ (and hence $y(z)$) analytic functions over the entire unit disk.

For $j = 1$

For $j = 1$, the polynomial becomes $1-\rho_R z-(1-\rho_R)\phi_\pi = 0$ and the root is given by $z = \frac{1-(1-\rho_R)\phi_\pi}{\rho_R}$. Hence, the system is determinant if $\left|\frac{1-(1-\rho_R)\phi_\pi}{\rho_R}\right| < 1$ or $1 < \phi_\pi < \frac{1+\rho_R}{1-\rho_R}$ and indeterminant otherwise.

For $j = 0$

For $j = 0$, the polynomial becomes $1-\rho_R z-(1-\rho_R)\phi_\pi z = 0$ and the root is given by $z = \frac{1}{\rho_R+(1-\rho_R)\phi_\pi}$. Hence, the system is determinant if $\left|\frac{1}{\rho_R+(1-\rho_R)\phi_\pi}\right| < 1$ or $1 < \phi_\pi$ and indeterminant otherwise.

For $j < 0$

For $j < 0$, the polynomial becomes $1-\rho_R z-(1-\rho_R)\phi_\pi z^k$ for $k = 1-j > 1$. To bound the number of zeros using Rouché's theorem, theorem 12 above, we will factor this polynomial to have the leading term in z^k monic and define its inverse polynomial. Accordingly, (G.63) can be factored as

$$-(1-\rho_R)\phi_\pi\left(z^k+\frac{\rho_R}{1-\rho_R}\frac{1}{\phi_\pi}z-\frac{1}{1-\rho_R}\frac{1}{\phi_\pi}\right)\pi(z) = \left[1-\rho_R z-\mathbf{1}_{j=1}(1-\rho_R)\right]\pi_0+\text{t.i.d.} \tag{G.64}$$

and the relevant polynomial becomes $z^k+\frac{\rho_R}{1-\rho_R}\frac{1}{\phi_\pi}z-\frac{1}{1-\rho_R}\frac{1}{\phi_\pi}$. Define $f(z)\equiv z^k$ and $g(z)\equiv\frac{\rho_R}{1-\rho_R}\frac{1}{\phi_\pi}z-\frac{1}{1-\rho_R}\frac{1}{\phi_\pi}$. The polynomial $f(z)$ has k zeros inside the unit circle (k zeros at the origin to be precise) and as

$$\min|f(z)|_{|z|=1} > \max|g(z)|_{|z|=1} \Rightarrow 1 > \frac{1}{1-\rho_R}\frac{1}{\phi_\pi}\max\left|1-\rho_R z\right|_{|z|=1} \Rightarrow 1 > \frac{1+\rho_R}{1-\rho_R}\frac{1}{\phi_\pi} \tag{G.65}$$

Then for $\phi_\pi > \frac{1+\rho_R}{1-\rho_R}$, the polynomial $f(z)+g(z)$ (our relevant polynomial $z^k+\frac{\rho_R}{1-\rho_R}\frac{1}{\phi_\pi}z-\frac{1}{1-\rho_R}\frac{1}{\phi_\pi}$ above) has the same number of roots as $f(z)$ inside the unit circle by virtue of Rouché's theorem, theorem 12 above. That is, the relevant polynomial has $k = 1-j > 1$ roots inside the unit circle which means there are too many roots inside the unit circle and hence there are not enough constants that can be set to eliminate the singularities to render $\pi(z)$ (and hence $y(z)$) analytic functions over the entire unit disk. We have nonexistence of a stationary solution.

Consider now the system using the reverse polynomial of $1-\rho_R z-(1-\rho_R)\phi_\pi z^k$, i.e., with $\tilde{z}\equiv 1/z$

$$\left(\tilde{z}^k-\rho_R\tilde{z}^{k-1}-(1-\rho_R)\phi_\pi\right)\pi(1/\tilde{z}) = \left[\tilde{z}^k\left(1-\mathbf{1}_{j=1}(1-\rho_R)\right)-\rho_R\tilde{z}^{k-1}\right]\pi_0+\text{t.i.d.} \tag{G.66}$$

For determinacy, we must have one and only one z inside the unit circle which translates to all but one (that is $k-1$) $\tilde{z}$ inside the unit circle. Define $f(\tilde{z})\equiv\tilde{z}^k-\rho_R\tilde{z}^{k-1}=\tilde{z}^{k-1}(\tilde{z}-\rho_R)$. As $|\rho_R|<1$, $f(\tilde{z})$ has k zeros inside the unit circle (one at ρ_R and $k-1$ at the origin). Define as well $g(\tilde{z})\equiv-(1-\rho_R)\phi_\pi$. As $|g(\tilde{z})|=(1-\rho_R)\phi_\pi$ and $\min|f(\tilde{z})|_{|\tilde{z}|=1}=1-\rho_R$ it follows that

$$\min|f(\tilde{z})|_{|\tilde{z}|=1} > \max|g(\tilde{z})|_{|\tilde{z}|=1} \Rightarrow 1-\rho_R > 1-\rho_R\phi_\pi \Rightarrow \phi_\pi < 1 \tag{G.67}$$

Thus for $\phi_\pi < 1$, the polynomial $f(\tilde{z})+g(\tilde{z})$ (our relevant polynomial $\tilde{z}^k-\rho_R\tilde{z}^{k-1}-(1-\rho_R)\phi_\pi$ above) has the same number of roots as $f(\tilde{z})$ inside the unit circle by virtue of Rouché's theorem, theorem 12 above. That is, the relevant polynomial has $k = 1 - j > 1$ roots inside the unit circle which translates (as $\tilde{z} \equiv 1/z$) to no roots inside the unit circle for our original polynomial $1 - \rho_R z - (1-\rho_R)\phi_\pi z^{1-j}$. Thus we have no singularities inside the unit circle that can be removed by pinning down the arbitrary constant π_0 and hence we have indeterminacy.

For $j > 1$

For $j > 1$, define $k = j - 1 > 0$ and (G.62) becomes

$$\left(1 - \rho_R z - (1-\rho_R)\phi_\pi z^{-k}\right)\pi(z) = (1-\rho_R z)\pi_0 - (1-\rho_R)\phi_\pi z^{1-j}\sum_{i=0}^{k}\pi_i z^i + \text{t.i.d.} \tag{G.68}$$

where the right hand side is a function of $\pi_0, \pi_1, \ldots \pi_k$. Hence the system has $k+1$ coefficients to pin down and accordingly the polynomial $1 - \rho_R z - (1-\rho_R)\phi_\pi z^{-k}$ must have $k+1$ roots inside the unit circle for the system to be determinate, for $\{\pi_i\}_{i=0,1,\ldots,k}$ to be set to remove the singularity at the roots inside the unit circle so that $\pi(z)$ (and hence $y(z)$) is an analytic function over the unit disk. If there are fewer roots inside the unit circle, then not all of $\{\pi_i\}_{i=0,1,\ldots,k}$ cant be pinned down and the system is indeterminate. If there are more than $k+1$ roots inside the unit circle, then there are not enough constants that can be set to eliminate the singularities to render $\pi(z)$ (and hence $y(z)$) analytic functions over the entire unit disk. Rewritting the polynomial as $\left(z^k - \rho_R z^{k+1} - (1-\rho_R)\phi_\pi\right)z^{-k}$ and hence determinacy requires $z^k - \rho_R z^{k+1} - (1-\rho_R)\phi_\pi$ to have $k+1$ roots inside the unit circle. The polynomial $z^k - \rho_R z^{k+1} - (1-\rho_R)\phi_\pi$ is of order $k+1$ and, hence, has $k+1$ roots following from the fundamental theorem of algebra and therefore cannot have more than $k+1$ roots. Therefore, the system will be either determinate or indeterminate.

Beginning with $z^k - \rho_R z^{k+1} - (1-\rho_R)\phi_\pi$ and defining $f(z) \equiv z^k - \rho_R z^{k+1}$ and $g(z) \equiv -(1-\rho_R)\phi_\pi$, $\min|f(z)|_{|z|=1} = 1 - \rho_R$ and $\max|g(z)|_{|z|=1} = (1-\rho_R)\phi_\pi$. Noticing that $|\rho_R| < 1$, $f(z)$ has only k zeros inside the unit circle (k at the origin but one at $1/\rho_R$) and

$$\min|f(z)|_{|z|=1} > \max|g(z)|_{|z|=1} \Rightarrow 1 - \rho_R > (1-\rho_R)\phi_\pi \tag{G.69}$$

Then for $\phi_\pi < 1$, the polynomial $f(z) + g(z)$ (our relevant polynomial $z^k - \rho_R z^{k+1} - (1-\rho_R)\phi_\pi$ above) has the same number of roots as $f(z)$ inside the unit circle by virtue of Rouché's theorem, theorem 12 above. That is, the relevant polynomial has only k roots inside the unit circle which means there are too few singularities inside the unit circle that can be removed to pin down all the constants $\{\pi_i\}_{i=0,1,\ldots,k}$. We have indeterminacy or nonuniqueness of the stationary solution.

As above, consider now the reverse polynomial with $\tilde{z} \equiv 1/z$

$$\tilde{z} - \rho_R - (1-\rho_R)\phi_\pi \tilde{z}^{k+1} \Rightarrow -(1-\rho_R)\phi_\pi\left(\tilde{z}^{k+1} - \frac{1}{1-\rho_R}\frac{1}{\phi_\pi}\tilde{z} + \frac{\rho_R}{1-\rho_R}\frac{1}{\phi_\pi}\right) \tag{G.70}$$

For determinacy, we must have $k+1$ roots in z inside the unit circle which translates to zero roots in $\tilde{z}$ inside the unit circle. Define $f(\tilde{z}) \equiv \tilde{z}^{k+1}$, and $f(\tilde{z})$ has $k+1$ zeros inside the unit circle (all at the origin). Define as well $g(\tilde{z}) \equiv -\frac{1}{1-\rho_R}\frac{1}{\phi_\pi}\tilde{z} + \frac{\rho_R}{1-\rho_R}\frac{1}{\phi_\pi} = \frac{1}{1-\rho_R}\frac{1}{\phi_\pi}(\rho_R - \tilde{z})$. As $|f(\tilde{z})|_{|\tilde{z}|=1} = 1$ and $\max|g(\tilde{z})|_{|\tilde{z}|=1} = \frac{1+\rho_R}{1-\rho_R}\frac{1}{\phi_\pi}$, it follows that

$$\min|f(\tilde{z})|_{|\tilde{z}|=1} > \max|g(\tilde{z})|_{|\tilde{z}|=1} \Rightarrow 1 > \frac{1+\rho_R}{1-\rho_R}\frac{1}{\phi_\pi} \Rightarrow \frac{1+\rho_R}{1-\rho_R} < \phi_\pi \tag{G.71}$$

Thus for $\frac{1+\rho_R}{1-\rho_R} < \phi_\pi 1$, the polynomial $f(\tilde{z}) + g(\tilde{z})$ (our relevant polynomial $-(1-\rho_R)\phi_\pi\left(\tilde{z}^{k+1} - \frac{1}{1-\rho_R}\frac{1}{\phi_\pi}\tilde{z} + \frac{\rho_R}{1-\rho_R}\frac{1}{\phi_\pi}\right)$ above) has the same number of roots as $f(\tilde{z})$ inside the unit circle by virtue of Rouché's theorem, theorem 12 above. That is, the relevant polynomial has $k+1$ roots inside the unit circle which translates (as $\tilde{z} \equiv 1/z$) to no roots inside the unit circle for our original polynomial $1-\rho_R z-\left(1-\rho_R\right)\phi_\pi z^{-k}$. Thus we have no singularities inside the unit circle that can be removed by pinning down the arbitrary constants $\{\pi_i\}_{i=0,1,\ldots,k}$ and hence we have indeterminacy.

Appendix H. Determinacy Bounds in Table 1

H.1. *Determinacy bounds for the sticky price model with a forward-looking rule featuring a change in output*

Consider the sticky price model, given by (27), (4) and the following Taylor rule:

$$R_t = \phi_\pi E_t \pi_{t+1} + \Delta y_{t+1} \tag{H.72}$$

We substitute the policy rule into the IS equation (4) and put the system involving the two endogenous variables y_t, π_t in the following form:

$$E_t x_{t+1} = c + A x_t \tag{H.73}$$

where $x_t = [y_t, \pi_t]', c = 0$ and

$$A = \begin{bmatrix} -\frac{\sigma(1-\phi_\pi)}{1-\sigma\phi_y} & \frac{\beta(1+\sigma\phi_y)+\kappa\sigma(1-\phi_\pi)}{\beta(1-\sigma\phi_y)0} \\ 1/\beta & -\kappa/\beta \end{bmatrix}. \tag{H.74}$$

The characteristic equation of a 2×2 system matrix A is given by $p(\lambda) = \lambda^2 - \mathrm{tr}(A)\lambda + \det(A)$. Both roots of the characteristic equation lie outside the unit circle if and only if (see LaSalle, 1986, p.28):

$$|\det(A)| > 1 \qquad \text{and} \qquad |\mathrm{tr}(A)| < 1 + \det(A),$$

where

$$\det(A) = -\frac{(1-\sigma\phi_y)}{\beta(1-\sigma\phi_y)} \tag{H.75}$$

and

$$\mathrm{tr}(A) = -\frac{\sigma(1-\phi_\pi}{\beta(1-\sigma\phi_\pi} - \frac{\kappa}{\beta} \tag{H.76}$$

Over the admissible parameter range, the determinant is strictly above one, if $1/\sigma < \phi_y$, so that the first condition holds. The right-hand-side of the second condition implies that $1+\phi_y(1+\beta+\kappa)+\frac{1+\kappa+\beta}{\sigma} < \phi_\pi$, while the left-hand-side leads to $\phi_\pi < 1+\frac{\kappa+\beta}{\sigma} - \phi_y(1+\kappa+\beta)$ which provides the set of the necessary and sufficient conditions for a unique equilibrium.

H.2. *Determinacy bounds for the sticky information model with a forward-looking rule*

Consider the sticky information model, given by (1), (91) and the following Taylor rule:

$$R_t = \phi_\pi E_t \pi_{t+1} \tag{H.77}$$

Following theorem 11 case (5), the model has a unique, stable equilibrium if and only if

$$1 < \phi_\pi < 1 \tag{H.78}$$

which of course is never true, such that

$$\phi_\pi = \varnothing. \tag{H.79}$$

As determinacy in the model with a forward-looking interest rate is independent of output gap, the result holds also true for other Taylor rules featuring output gap dated at any point in time, i.e. for $R_t = \phi_\pi E_t \pi_{t+1} + y_t$, $R_t = \phi_\pi E_t \pi_{t+1} + y_{t+1}$ and $R_t = \phi_\pi E_t \pi_{t+1} + \Delta y_{t+1}$.

H.3. *Determinacy bounds for the sticky information model with a backward-looking rule*

Consider the sticky information model, given by (1), (91) and the following Taylor rule:

$$R_t = \phi_\pi E_t \pi_{t-1} \tag{H.80}$$

Following theorem 11 case (1), the model features indeterminacy if $\phi_\pi < 1 \; \forall \; j$. Further, according to case (3) the model equilibrium is however nonexistent if $1 < \phi_\pi$, $j = -1$, such that

$$\phi_\pi = \varnothing. \tag{H.81}$$

As these results are independent of output gap, they hold true for other Taylor rules featuring output gap dated at any point in time, i.e. for $R_t = \phi_\pi E_t \pi_{t-1} + y_t$ and $R_t = \phi_\pi E_t \pi_{t-1} + y_{t-1}$.

Appendix I. Online Appendices

I.1. Frequency Domain Representation of Discrete Time Series

Here we present an (incomplete) introduction, following Priestly (1981), Ahlfors (1979), Oppenheim, Schafer, and Buck (1999, ch. 3), Oppenheim, Willsky, and Nawab (1996, ch. 10), Hamilton (1994, ch. 6), Sargent (1987a, ch. XI), and Shumway and Stoffer (2011) to the z-transform and discrete time Fourier transform as it will pertain to our analysis of the determinacy of linear DSGE models. These transforms discern the frequency content and temporal dependencies of a given sequence and, hence, can be used in the analysis of discrete-time series. The autocovariance and autocorrelation functions play the pivotal role in understanding the temporal relationships within a time series and the key element we will introduce here that will be essential for understanding how the sticky information model functions in the frequency domain is the property of scaling in the z-domain.

Our basic assumptions follow, e.g., Priestly (1981, ch. 4.11.) or Shumway and Stoffer (2011, Appendix C), for mean zero, linearly regular covariance stationary stochastic processes with absolutely continuous spectral distribution functions. Let y_t be such a process, then

$$y_t = \int_{-\pi}^{\pi} e^{it\omega} dZ(\omega) \tag{I.82}$$

where $dZ(\omega)$ is a mean zero, random orthogonal increment process with $E\left[|dZ(\omega)|^2\right] = h(\omega)d\omega$ and $E[dZ(\omega_1)dZ(\omega_2)^*] = 0$, for $\omega_1 \neq \omega_2$. Assume that the autocovariance function is absolutely summable

$$\sum_{m=-\infty}^{\infty} \left|R_y(m)\right| < \infty \tag{I.83}$$

where the autocovariance function of a discrete-time series y_t is defined as

$$R_y(m) = \text{Cov}(y_t, y_{t-m}) = E(y_t - \mu_y)(y_{t-m} - \mu_y) \tag{I.84}$$

then the spectral distribution function $Z(\omega)$ is absolutely continuous such that $dZ(\omega) = f_y(\omega)d\omega$ and $f_y(\omega)$ is the spectral density given by

$$f_y(\omega) = \sum_{m=-\infty}^{\infty} R_y(m)e^{-i\omega h}, \quad -\pi \leq \omega \leq \pi \tag{I.85}$$

Whiteman (1983) assumes, and we follow, that solutions for y_t are sought in the space spanned by time-independent square-summable linear combinations of the process(es) fundamental for the driving process, that is H^2 or Hardy space.[46] Let ϵ_t be such a mean zero fundamental process with variance σ_ϵ^2. Its spectral density is thus

$$f_\epsilon(\omega) = \sum_{m=-\infty}^{\infty} R_\epsilon(m)e^{-i\omega h} = \frac{1}{2\pi}\sigma_\epsilon^2 \tag{I.86}$$

[46] See, e.g., Han, Tan, and Wu (2022) for a more formal introduction.

Then an H^2 solution for an endogenous variable, y_t, is of the form $y_t = y(L)\epsilon_t = \sum_{j=0}^{\infty} y_j \epsilon_{t-j}$ with $\sum_{j=0}^{\infty} y_j^2 < \infty$ and L the lag operator $Ly_t = y_{t-1}$.[47] Following, e.g., Sargent (1987a, ch. XI) the Riesz-Fischer Theorem gives an equivalence (a one-to-one and onto transformation) between the space of squared summable sequences $\sum_{j=0}^{\infty} y_j^2 < \infty$ and the space of analytic functions in unit disk $y(z)$ corresponding to the z-transform of the sequence, $y(z) = \sum_{j=0}^{\infty} y_j z^j$.

Given a discrete series y_j with samples taken at equally spaced intervals, its z-transform $y(z)$ is defined in (A.2) as

$$y(z) = \sum_{j=0}^{\infty} y_j z^j \tag{I.87}$$

where z is a complex variable, and the sum extends from 0 to infinity, following the convention used in Hamilton (1994, ch. 6) and Sargent (1987a, ch. XI).[48] By evaluating the z-transform on the unit circle in the complex plane ($z = e^{-i\omega}$, where ω is the angular frequency and i the complex number $\sqrt{-1}$), we obtain the discrete-time Fourier transform (DTFT). The DTFT $y(e^{-i\omega})$ is given by

$$y(e^{-i\omega}) = \sum_{j=0}^{\infty} y_j e^{-i\omega j} \tag{I.88}$$

The DTFT reveals the spectral characteristics of the sequence in terms of its frequency components.

The connection between the autocovariance function and the Fourier transformation of the z-transform evaluated on the unit circle ($z = e^{-i\omega}$) can be established by manipulating the equations

$$R_y(m) = \int_{-\pi}^{\pi} f_y(\omega) e^{im\omega} d\omega \tag{I.89}$$

Hence for our mean zero fundamental process ϵ_t

$$R_\epsilon(m) = \int_{-\pi}^{\pi} f_\epsilon(\omega) e^{im\omega} d\omega = \int_{-\pi}^{\pi} \frac{1}{2\pi} \sigma_\epsilon^2 e^{im\omega} d\omega = \frac{1}{2\pi}\sigma_\epsilon^2 \int_{-\pi}^{\pi} e^{im\omega} d\omega = \begin{cases} \sigma_\epsilon^2 & \text{for } m = 0 \\ 0 & \text{otherwise} \end{cases} \tag{I.90}$$

Now return to $y_t = y(L)\epsilon_t = \sum_{j=0}^{\infty} y_j \epsilon_{t-j}$ and recall $y_t = \int_{-\pi}^{\pi} e^{it\omega} dZ_y(\omega)$ and analogously $\epsilon_t = \int_{-\pi}^{\pi} e^{it\omega} dZ_\epsilon(\omega)$ so therefore it must hold that

$$\int_{-\pi}^{\pi} e^{it\omega} dZ_y(\omega) = \int_{-\pi}^{\pi} y(e^{it\omega}) e^{it\omega} dZ_\epsilon(\omega) \Rightarrow dZ_y(\omega) = y(e^{it\omega}) dZ_\epsilon(\omega) \tag{I.91}$$

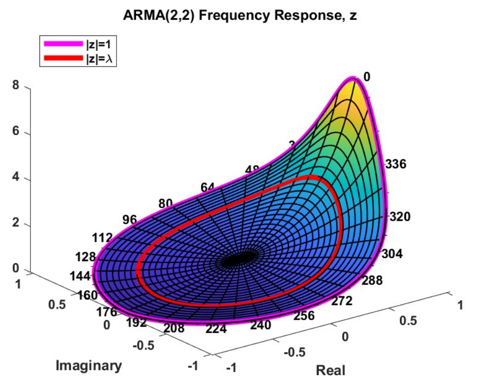
Multiplying both sides by their complex conjugates and taking expectations gives

$$E\left[dZ_y(\omega) dZ_y(\omega)^*\right] = E\left[y(e^{it\omega}) y(e^{it\omega})^* dZ_\epsilon(\omega) dZ_\epsilon(\omega)^*\right] \tag{I.92}$$

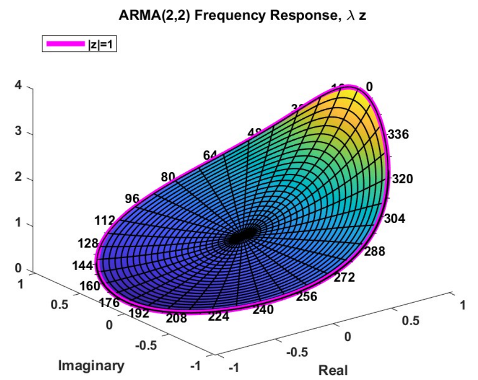
---

[47]Note that we are abusing notation somewhat and choosing to use the same letter y to refer to a discrete time series, y_t, as well as that variable's transform function $y(z)$ or MA representation/response to a fundamental process j periods ago, y_j. This serves to save on the verbosity of notation, which might otherwise read $y_t = \sum_{j=0}^{\infty} \delta_j^y \epsilon_{t-j}$ following, e.g., Meyer-Gohde (2010).

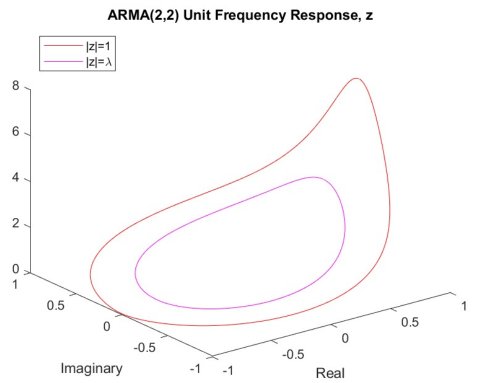
[48]The discrete signal processing and systems theory literature works in negative exponents of z, see Oppenheim, Schafer, and Buck (1999, ch. 3) and Oppenheim, Willsky, and Nawab (1996, ch. 10). Al-Sadoon (2020) follows this convention and interprets the operator being applied as the forward operator. We maintain the more familiar approach in working with the lag operator which results in our use of positive exponents in z.



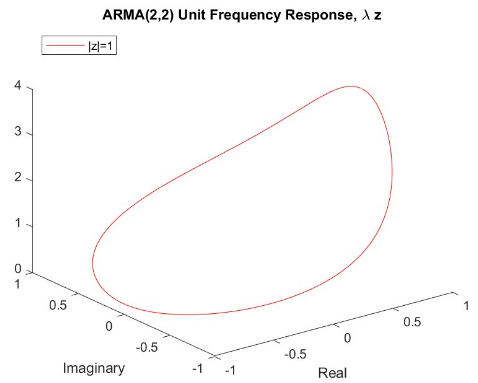
(A) $|y(z)|$: $(1+\rho_1 L+\rho_2 L^2)y_t = (1+\theta_1 L+\theta_2 L^2)\epsilon_t$



(B) $|y(z)|$: $(1+\lambda\rho_1 L+\lambda^2\rho_2 L^2)y_t = (1+\lambda\theta_1 L+\lambda^2\theta_2 L^2)\epsilon_t$



(C) $|y(z)|_{|z|=1}$: $(1+\rho_1 L+\rho_2 L^2)y_t = (1+\theta_1 L+\theta_2 L^2)\epsilon_t$



(D) $|y(z)|_{|z|=1}$: $(1+\lambda\rho_1 L+\lambda^2\rho_2 L^2)y_t = (1+\lambda\theta_1 L+\lambda^2\theta_2 L^2)\epsilon_t$

FIGURE 3. ARMA(2,2) - Transfer Functions on the Unit Disk

The values $\rho_1 = 1.1$, $\rho_2 = -0.28$, $\theta_1 = 0.6$, $\theta_1 = -0.25$, and $\lambda = 0.7$ were used

$$f_y(\omega) = \left|y(e^{it\omega})\right|^2 f_\epsilon(\omega) = \left|y(e^{it\omega})\right|^2 \frac{1}{2\pi}\sigma_\epsilon^2 \tag{I.93}$$

We can insert this directly into (I.89) above to yield (A.4)

$$R_y(m) = \sigma_\epsilon^2 \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|y(e^{-i\omega})\right|^2 e^{im\omega} d\omega \tag{I.94}$$

where $y(e^{-i\omega})$ and $y^*(e^{i\omega})$ denote the DTFT of y_j and its complex conjugate, respectively. This relationship allows us to analyze the temporal dependencies in a time series. By leveraging the z-transform and Fourier transform, along with the calculations of autocovariance and autocorrelation, we will uncover the frequency content and temporal dynamics of discrete-time series that are subject to sticky information.

I.2. AR(2) example of scaling in the z domain

While one might be tempted to dismiss the AR(1) result as a coincidence of the exponential scaling inherently involved with an AR(1) process, examination of a more complicated process, such as an ARMA(2,2) ought to dissuade this temptation

$$y_t + \rho_1 y_{t-1} + \rho_2 y_{t-2} = \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} \tag{I.95}$$

Figure 3 contains the same four panels as for the AR(1) above and, again, the dampening property of $|\lambda| < 1$ is displayed. The transfer function of the ARMA(2,2) is scaled towards the origin by $|\lambda| < 1$. Comparing the upper two panels, the scaling of the z axis instantly reveals the dampening of the associated ARMA(2,2) on the right with L replaced by λL and by noticing that this transfer function is a subset of the original ARMA(2,2) transfer function, out only to $|\lambda|$ instead of 1.

The final, and for our determinacy analysis later crucial, property to observe is that this dampening is not bidirectional. If $|y(z)|$ is well defined (analytic) on the unit disk, so too will $|H(\lambda z)|$ be for $|\lambda| < 1$. Defining $\tilde{z} = \lambda z$, $|y(\tilde{z})|$ being well defined (analytic) on the unit disk does not allow us conclude the same about $|y(\frac{1}{\lambda}\tilde{z})|$ for $|\lambda| < 1$, as $\frac{1}{\lambda}\tilde{z}$ goes past the unit circle. That is, following Proposition 1, λ scales the region of convergence and if the process defined by $y(z)$ has a region of convergence from the origin out to the unit circle, then the process associated with $H(\frac{1}{\lambda}z)$ has a region of convergence out only to $|\lambda| < 1$.

I.3. Additional examples of determinacy in the frequency domain

We briefly demonstrate the requirement of analyticity of the z-transform in the frequency domain in relation to known requirements in the time domain in order to establish intuition. Consider first an autoregressive process of order 1, an AR(1) process:

$$y_t = \rho y_{t-1} + \epsilon_t, \quad \epsilon_t \sim WN(0, \sigma^2) \tag{I.96}$$

which can be rewritten as

$$y_t = \sum_{j=0}^{\infty} L^j y_j \epsilon_t. \tag{I.97}$$

The AR(1) process in the frequency domain, see above, is given by applying the z-transform:

$$y(z) = \rho y(z) + 1 \tag{I.98}$$

$$y(z) = \frac{1}{1 - \rho z} \tag{I.99}$$

y(z) analytic inside the unit disk if $|\rho| < 1$ and determines the solution to the autoregressive process. Now consider a forward-looking process:

$$y_t = \alpha E_t y_{t+1} + \epsilon_t \tag{I.100}$$

whereby the forecast can be rewritten in terms of deviations from the driving process:

$$E_t y_{t+1} = y_{t+1} - y_0 \epsilon_{t+1} = \frac{1}{L}\left(\sum_{j=0}^{\infty} L^j y_j - y_0\right)\epsilon_t. \tag{I.101}$$

In the frequency domain the forward-looking process is described by:

$$y(z) = \alpha \frac{1}{z}(y(z) - y_0) + 1 \tag{I.102}$$

where $y_0 = y(0)$ is the value of the y at frequency 0 and simultaneously presents the initial condition of the stationary process. To determine a solution we solve for $y(z)$:

$$\left(1-\frac{1}{\alpha}z\right)y(z) = y_0 - \frac{z}{\alpha} \tag{I.103}$$

$$y(z) = \left(1-\frac{1}{\alpha}z\right)^{-1}\left(y_0-\frac{z}{\alpha}\right) \tag{I.104}$$

whereby y_0 is not determined yet. If $|\alpha|<1$, then for $z=\alpha$ there is a removable singularity inside the unit disk and we can solve for a boundary condition on y_0:

$$\lim_{z\to\alpha}\left(1-\frac{1}{\alpha}z\right)y(z) = 0 \tag{I.105}$$

giving rise to the initial condition of $y_0 = 1$. The solution to our process in the frequency domain is then determined as:

$$y(z) = \frac{1-\frac{z}{\alpha}}{1-\frac{z}{\alpha}} = 1. \tag{I.106}$$

In the time domain the equivalent unique stationary solution is given by:

$$y_t = \epsilon_t. \tag{I.107}$$

compare with Blanchard (1979).

I.4. Alternative derivation of frequency domain sticky information

We can also derive a recursive representation of the lagged expectations of the endogenous variables in (31) as

$$(1-\lambda)\sum_{i=0}^{\infty}\lambda^i E_{t-i-1}[x_t], \quad x_t = \left(\sum_{j=0}^{\infty}x_j z^j\right)\epsilon_t \tag{I.108}$$

$$= (1-\lambda)\big(E_{t-1}[x_t] + \lambda E_{t-2}[x_t] + \lambda^2 E_{t-3}[x_t] + \ldots\big) \tag{I.109}$$

Applying the Wiener-Kolmogorov prediction formula to the lagged expectations (32), equation (I.109) gives the frequency domain representation as:

$$(1-\lambda)\big(x(z)-x_0+\lambda(x(z)-x_0-zx_1)+\lambda^2(x(z)-x_0-zx_1-z^2x_2)+\ldots\big) \tag{I.110}$$

$$= \quad (1-\lambda)\big(x(z)+\lambda x(z)+\lambda^2 x(z)+\ldots-x_0-\lambda x_0-\lambda^2 x_0\ldots-\lambda z x_1-\lambda^2 z x_1\ldots-\lambda^2 z x_2\ldots\big) \tag{I.111}$$

$$= \quad (1-\lambda)((1+\lambda+\lambda^2+\ldots)x(z)-(1+\lambda+\lambda^2+\ldots)x_0-\lambda z(1+\lambda+\lambda^2+\ldots)x_1 \tag{I.112}$$

$$-\lambda^2 z^2(1+\lambda+\lambda^2+\ldots)x_2-\ldots) \tag{I.113}$$

$$= \quad (1-\lambda)\left(\frac{1}{1-\lambda}x(z)-\frac{1}{1-\lambda}x_0-\frac{\lambda z}{1-\lambda}x_1-\frac{\lambda^2 z^2}{1-\lambda}x_2-\ldots\right) \tag{I.114}$$

$$= \quad x(z)-\sum_{j=0}^{\infty}\lambda^i z^j x_j = x(z)-x(\lambda z) \tag{I.115}$$

Hence, the lagged expectations in (I.109) can be transformed from the time into the frequency domain as:

$$\begin{aligned}(1-\lambda)\sum_{j=0}^{\infty}\lambda^i E_{t-i-1}[x_{t-1}] &= (1-\lambda)\left(\frac{z}{1-\lambda}x(z)-\frac{\lambda z}{1-\lambda}x_0-\frac{(\lambda z)^2}{1-\lambda}x_1-\ldots\right)\\ &= zx(z)-\lambda z x(\lambda z)\end{aligned} \tag{I.116}$$