Problem Set 8

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1 Introduction

This report explores the dynamics of mechanical systems under various physical conditions through numerical simulations. The first part investigates oscillatory motion by simulating the behavior of harmonic, anharmonic, and van der Pol oscillators. These systems highlight the effects of nonlinearity, damping, and initial conditions on oscillatory motion. The second part focuses on projectile motion, specifically examining the effects of air resistance and mass on the trajectory of a cannonball. Using systems of ordinary differential equations (ODEs), the simulations provide insights into real-world deviations from idealized behavior, such as sinusoidal oscillations and parabolic trajectories in vacuum conditions.

The full code used for these analyses is available in the GitHub repository: `https://github.com/maryam-manz/phys-ga2000/tree/main/PS9`. The final solutions are in python files eg: Question1ab.py

2 Question 1

2.1 Part a

I had to rewrite the second-order differential equation for a harmonic oscillator into a system of first-order differential equations, then solve them numerically for the case where $\omega = 1$, and plot the displacement $x(t)$ as a function of time over the interval $t \in [0, 50]$.

2.1.1 Harmonic Oscillator Equation

The harmonic oscillator is described by the second-order differential equation:

$$\frac{d^2x}{dt^2} = -\omega^2 x,$$

where:

- $x(t)$ is the displacement at time t,
- ω is the angular frequency, and
- $\frac{dx}{dt}$ is the velocity.

To solve this equation numerically, it is rewritten as a system of first-order equations:

$$\frac{dx}{dt} = v,$$
$$\frac{dv}{dt} = -\omega^2 x,$$

where $v = \frac{dx}{dt}$.

The system of equations was solved numerically using the `solve_ivp` function from the SciPy library. The parameters and initial conditions for this problem were:

- Angular frequency: $\omega = 1$,
- Time range: $t \in [0, 50]$,
- Initial conditions: $x(0) = 1$ and $\frac{dx}{dt}(0) = 0$.

The numerical solution provides the displacement $x(t)$ at various points in the specified time range.

2.1.2 Results

The plot below shows the displacement $x(t)$ as a function of time for the given parameters:

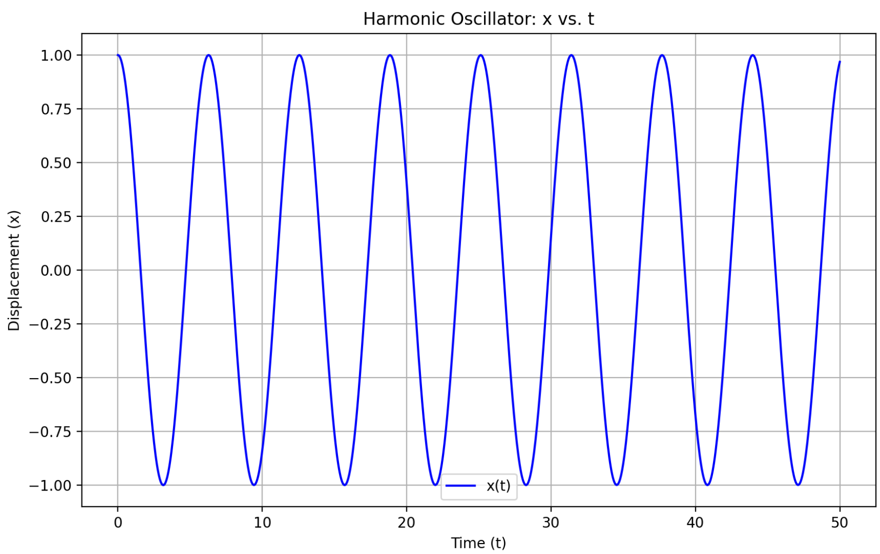

Figure 1: Displacement $x(t)$ of the harmonic oscillator with $\omega = 1$ over $t \in [0, 50]$.

The solution exhibits sinusoidal behavior, consistent with the expected motion of a simple harmonic oscillator. The displacement $x(t)$ oscillates with a constant amplitude and period determined by ω.

2.2 Part b

In this part, the goal is to investigate how increasing the initial amplitude of oscillations affects the period of the harmonic oscillator. Specifically, the initial displacement $x(0)$ is increased from 1 to 2, and the resulting oscillations are compared to determine whether the period remains constant.

The initial conditions are modified to:

- Case 1: $x(0) = 1$, $\frac{dx}{dt}(0) = 0$,
- Case 2: $x(0) = 2$, $\frac{dx}{dt}(0) = 0$.

The system is solved numerically for both cases using the `solve_ivp` function, and the displacement $x(t)$ is plotted for comparison.

2.2.1 Results

The plot below shows the displacement $x(t)$ for initial amplitudes $x(0) = 1$ and $x(0) = 2$:

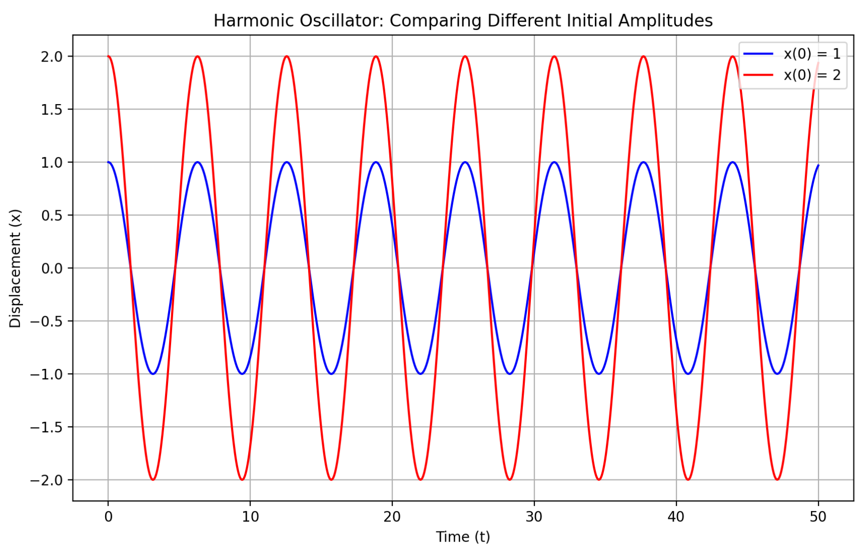

Figure 2: Displacement $x(t)$ for initial amplitudes $x(0) = 1$ and $x(0) = 2$.

As seen in Figure 2, increasing the initial amplitude of the oscillations affects the maximum displacement but does not alter the period of the oscillations. Both cases exhibit the same periodicity, consistent with the properties of the simple harmonic oscillator:

$$T = \frac{2\pi}{\omega}.$$

2.3 Part c

The goal of Part C is to modify the program to solve for the motion of an anharmonic oscillator described by the equation:

$$\frac{d^2x}{dt^2} = -\omega x^3,$$

with $\omega = 1$. The oscillator's motion is simulated for different initial amplitudes, and the effect of amplitude on oscillation frequency is analyzed.

2.3.1 Anharmonic Oscillator Equation

The second-order differential equation for the anharmonic oscillator is:

$$\frac{d^2x}{dt^2} = -\omega x^3.$$

This equation is rewritten as a system of first-order equations:

$$\frac{dx}{dt} = v,$$
$$\frac{dv}{dt} = -\omega x^3,$$

where $v = \frac{dx}{dt}$.

The equations were solved numerically for three different initial conditions:

- $x(0) = 1$, $\frac{dx}{dt}(0) = 0$,
- $x(0) = 2$, $\frac{dx}{dt}(0) = 0$,
- $x(0) = 0.5$, $\frac{dx}{dt}(0) = 0$.

The `solve_ivp` function from SciPy was used to integrate the system over $t \in [0, 50]$. The solutions were then plotted to compare the motion for different amplitudes.

2.3.2 Results

The plot below shows the displacement $x(t)$ for initial amplitudes $x(0) = 1$, $x(0) = 2$, and $x(0) = 0.5$:

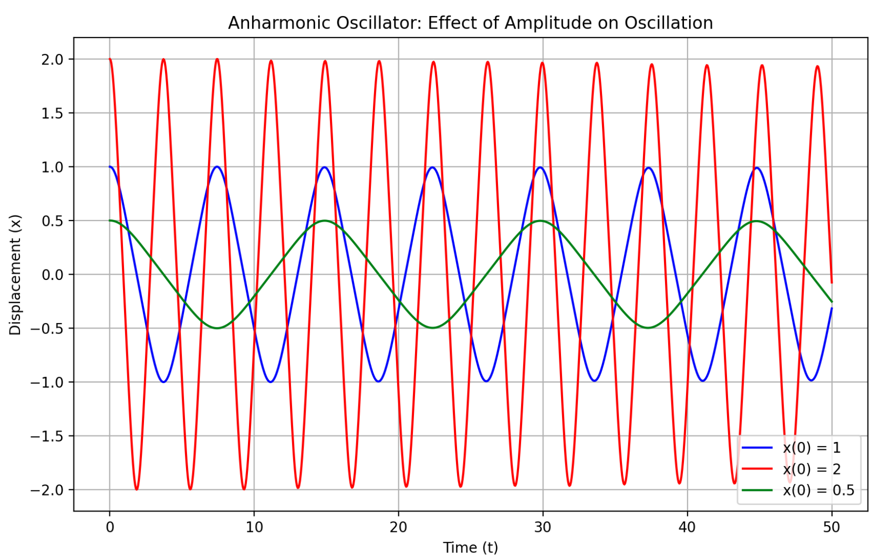

Figure 3: Displacement $x(t)$ of the anharmonic oscillator for different initial amplitudes.

The plot in Figure 3 demonstrates that the frequency of oscillations increases with higher amplitudes. This behavior is characteristic of anharmonic oscillators, where the restoring force is not linear but depends on x^3. In contrast to the simple harmonic oscillator, the period of oscillation is not constant and decreases as the initial amplitude increases.

2.4 Part d

The purpose of this section is to analyze the phase space of the anharmonic oscillator. The phase space diagram shows the relationship between the position x and velocity $\frac{dx}{dt}$, providing insights into the dynamics of the system for different initial amplitudes.

Results

The phase space diagram for the anharmonic oscillator is shown in Figure 4:

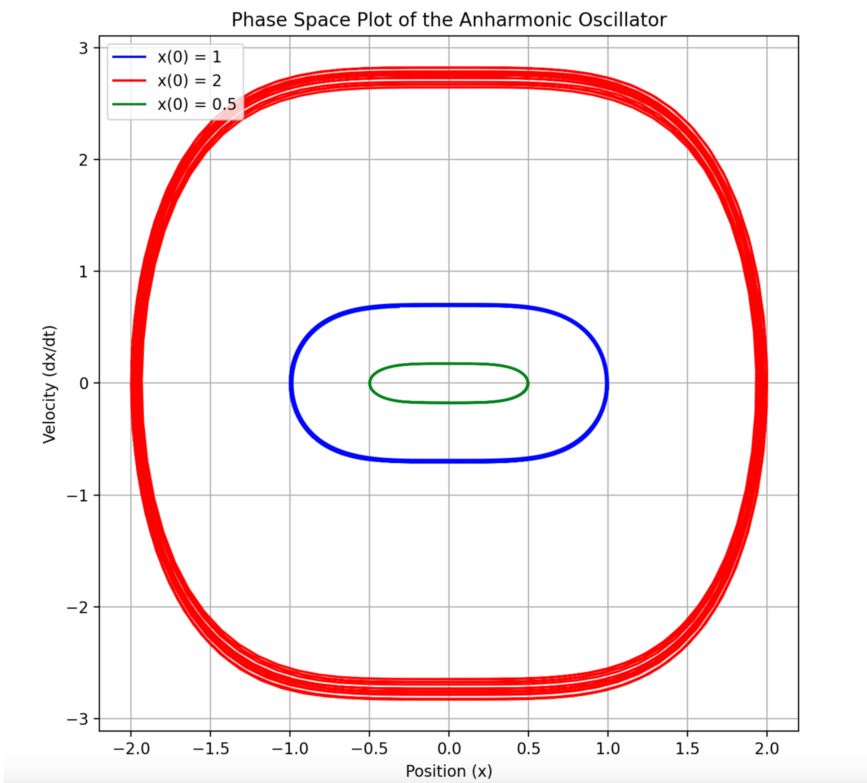

Figure 4: Phase space plot of the anharmonic oscillator for different initial amplitudes.

The phase space trajectories in Figure 4 exhibit the following characteristics:

- Each trajectory forms a closed curve, indicating periodic motion.
- Larger initial amplitudes ($x(0)$) result in trajectories with larger areas in the phase space, reflecting the increased energy of the system.
- The non-linear nature of the restoring force ($-\omega x^3$) causes deviations from the elliptical phase space trajectories characteristic of simple harmonic oscillators.

- Small differences in the initial conditions amplify as the system evolves, leading to larger deviations in the phase space trajectories; this effect is particularly noticeable for x(0)=2, where the "thickness" of the line reflects the system's sensitivity to initial conditions at higher amplitudes.

2.5 Part e

The objective of this section is to analyze the phase space of the van der Pol oscillator, a system that appears in electronic circuits and laser physics. The van der Pol oscillator is described by the equation:

$$\frac{d^2x}{dt^2} - \mu(1 - x^2)\frac{dx}{dt} + \omega^2 x = 0,$$

where μ is the damping parameter, and ω is the angular frequency. The system's behavior is simulated for different values of μ, and the corresponding phase space trajectories are plotted.

The second-order differential equation is rewritten as a system of first-order equations:

$$\frac{dx}{dt} = v,$$
$$\frac{dv}{dt} = \mu(1 - x^2)v - \omega^2 x,$$

where $v = \frac{dx}{dt}$.

The parameters and initial conditions used in the simulation are:

- Angular frequency: $\omega = 1$,
- Time range: $t \in [0, 20]$,
- Initial conditions: $x(0) = 1$, $\frac{dx}{dt}(0) = 0$,
- Damping parameter: $\mu = 1, 2, 4$.

The system was solved numerically using the `solve_ivp` function from SciPy, with a small time step (`max_step=0.01`) to ensure smooth and accurate phase space plots. The trajectories for different values of μ were plotted in the x-v plane.

2.5.1 Results

The phase space diagram for the van der Pol oscillator is shown in Figure 5:

The phase space trajectories reveal the following characteristics:

- For $\mu = 1$, the trajectory exhibits small deviations from a closed circular loop, indicating weak nonlinearity and damping.
- For $\mu = 2$, the nonlinearity becomes more pronounced, and the trajectory begins to exhibit the characteristic relaxation oscillation shape.
- For $\mu = 4$, the relaxation oscillations dominate, with rapid changes in velocity during portions of the cycle, resulting in a sharp phase space trajectory.

The system's dynamics change significantly with increasing μ, transitioning from nearly harmonic oscillations ($\mu = 1$) to strongly nonlinear relaxation oscillations ($\mu = 4$).

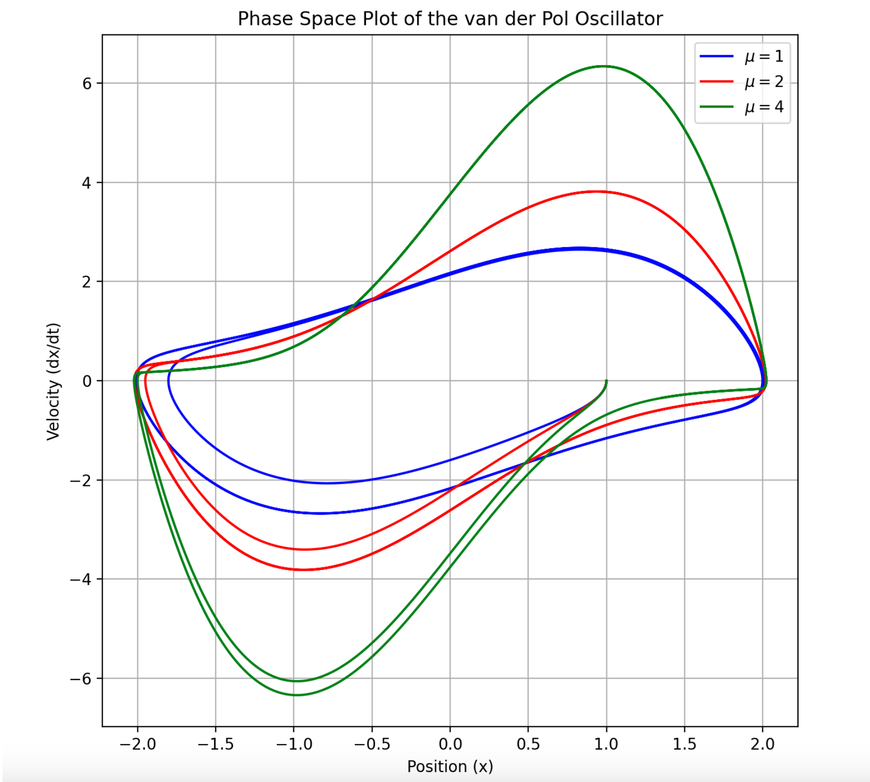

Figure 5: Phase space plot of the van der Pol oscillator for different values of μ.

3 Question 2

3.1 Mathematical Formulation

Consider a spherical cannonball of radius R, mass m, shot through the air of density ρ, experiencing air resistance proportional to its velocity. The equations of motion are given by Newton's second law:

$$m\frac{d^2\vec{x}}{dt^2} = -mg\hat{j} - \frac{1}{2}C\rho\pi R^2|\vec{v}|\vec{v},$$

where $\vec{x} = (x, y)$ is the position, $\vec{v} = \frac{d\vec{x}}{dt} = (v_x, v_y)$ is the velocity, and $|\vec{v}| = \sqrt{v_x^2 + v_y^2}$. The parameter C is the coefficient of drag, a property of the sphere.

Separating into components, we have:

$$m\frac{d^2x}{dt^2} = -\frac{1}{2}C\rho\pi R^2|\vec{v}|v_x,$$
$$m\frac{d^2y}{dt^2} = -mg - \frac{1}{2}C\rho\pi R^2|\vec{v}|v_y.$$

To nondimensionalize, we introduce characteristic timescale T, length scale L, and velocity scale V. Rescale the variables as:

$$t' = \frac{t}{T}, \quad x' = \frac{x}{L}, \quad y' = \frac{y}{L}, \quad v'_x = \frac{v_x}{V}, \quad v'_y = \frac{v_y}{V},$$

where $L = VT$ and $V = gT$. Setting $T = \sqrt{\frac{m}{R^2 \rho C g}}$, the length scale becomes $L = \sqrt{\frac{mg}{R^2 \rho C}}$.

Using these scales, the equations of motion transform into:

$$\frac{d^2 x'}{dt'^2} = -\kappa |\vec{v}'| v'_x,$$
$$\frac{d^2 y'}{dt'^2} = -1 - \kappa |\vec{v}'| v'_y,$$

where the dimensionless parameter κ is given by:

$$\kappa = \frac{R^2 \rho C T^2 g}{m}.$$

This nondimensionalized system demonstrates that the motion depends on a single dimensionless parameter κ, which encapsulates the effects of drag, gravity, and the properties of the cannonball and air.

3.2 Rewriting Second-Order Equations as First-Order Equations

To rewrite the second-order equations into a system of first-order equations, we introduce the following variables:

$$x' = v_x, \quad y' = v_y.$$

The second-order equations:

$$x'' = -kx' \sqrt{x'^2 + y'^2},$$
$$y'' = -g - ky' \sqrt{x'^2 + y'^2},$$

become:

$$\frac{dx}{dt} = v_x,$$
$$\frac{dy}{dt} = v_y,$$
$$\frac{dv_x}{dt} = -kv_x \sqrt{v_x^2 + v_y^2},$$
$$\frac{dv_y}{dt} = -g - kv_y \sqrt{v_x^2 + v_y^2}.$$

Here, the constant k is given by:

$$k = \frac{1}{2} \frac{C \rho \pi R^2}{m}.$$

3.3 Effect of Mass on the Distance Traveled by a Cannonball

When air resistance is ignored, the horizontal distance traveled by a projectile does not depend on its mass, as predicted by Newtonian mechanics. However, in real-world scenarios where air resistance plays a significant role, the mass of the projectile can influence its trajectory.

To explore this effect, we used the program developed earlier to simulate the motion of a cannonball under the influence of gravity and air resistance. The cannonball was launched with an initial velocity of $v_0 = 100\,\text{m/s}$ at an angle $\theta = 30°$, with a radius $R = 0.08\,\text{m}$, an air density $\rho = 1.22\,\text{kg/m}^3$, and a drag coefficient $C = 0.47$. The mass of the cannonball was varied between $0.5\,\text{kg}$ and $10\,\text{kg}$, and the total horizontal distance traveled (range) was computed for each case.

3.4 Part b

The effects of air resistance are included to observe how it modifies the ideal parabolic trajectory.

We calculated a numerical solution using the following initial conditions:

$$\begin{aligned} x(0) &= 0, \\ y(0) &= 0, \\ v_x(0) &= v_0 \cos(\theta), \\ v_y(0) &= v_0 \sin(\theta), \end{aligned}$$

where $v_0 = 100\,\text{m/s}$ is the initial velocity, and $\theta = 30°$ is the launch angle.

The system of ODEs was solved using the `solve_ivp` function from SciPy, with a time range of $t \in [0, 20]$ seconds. The trajectory was computed and filtered to include only points where the height $y \geq 0$, representing the flight path until the cannonball hits the ground.

3.4.1 Results

The trajectory of the cannonball, accounting for air resistance, is shown in Figure 6.

The trajectory deviates significantly from the ideal parabolic path due to the inclusion of air resistance:

- The horizontal range is reduced compared to the ideal case, as the drag force opposes the motion.
- The maximum height achieved is also lower, as air resistance reduces the vertical velocity more rapidly.
- The cannonball exhibits a steeper descent near the end of its trajectory, characteristic of drag-dominated motion.

3.5 Part c

The goal of this simulation is to investigate how varying the mass of a cannonball affects its range when air resistance is included in the motion. The drag force, which depends on the mass through the drag constant k, plays a significant role in determining the trajectory.

For each mass, the drag constant k was recalculated, and the trajectory was simulated using `solve_ivp`. The horizontal range was determined as the distance at which the cannonball hits the ground ($y \geq 0$).

3.5.1 Results

The plot below shows the relationship between the mass of the cannonball and its range:

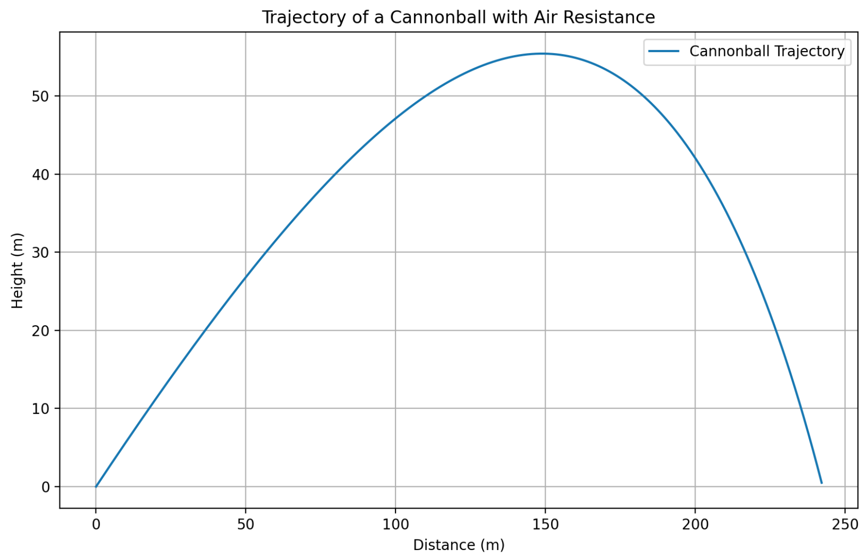

Figure 6: Trajectory of a cannonball under the influence of gravity and air resistance.

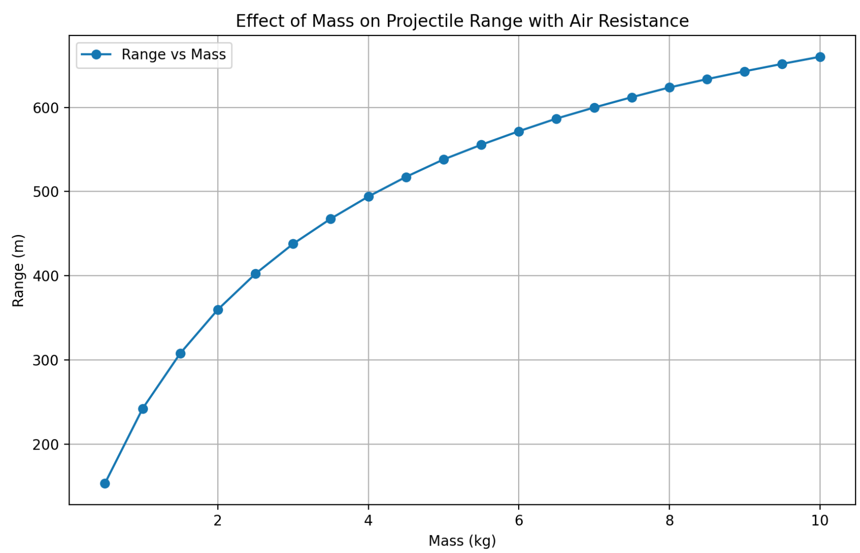

Figure 7: Effect of mass on the projectile range of a cannonball with air resistance.

3.5.2 Discussion

1. **Trajectories for Different Masses:** Simulations for cannonballs of different masses showed that heavier cannonballs experienced less deceleration due to air resistance and traveled further. Lighter

cannonballs, on the other hand, were significantly slowed by drag forces, resulting in shorter ranges.

2. **Graph of Distance vs. Mass:** A plot of horizontal distance traveled as a function of mass revealed that:
   - For very small masses, the range increases sharply with mass because air resistance dominates the motion.
   - As the mass increases, the range approaches an asymptotic limit, indicating that the influence of air resistance becomes negligible for sufficiently large masses.
3. **Key Observation:** Heavier projectiles are less affected by air resistance due to the inverse dependence of the acceleration from drag on the mass, as seen in the term $\frac{1}{m}$ in the equations of motion.