Obtaining Upper Bounds for the Approximation with Radial Basis Functions using Proof Techniques from Statistical Learning Theory

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In statistical learning theory generalization bounds of the form

$$P\left(z_1,\dots,z_n \in Z : \sup_{f\in\mathcal{F}}\left|\int_Z f(z)dP(z) - \frac{1}{n}\sum_{i=1}^{n} f(z_i)\right| \leq \alpha_{n,\delta,C(\mathcal{F})}\right) \geq 1-\delta \tag{0.1}$$

were derived, where Z is a subset of $\mathbb{R}^k$, P is an unknown probability measure, $\mathcal{F} \subset \{f : Z \to [A,B], f \text{ measurable}\}$ is a set of functions and $C(\mathcal{F})$ is a complexity measure of $\mathcal{F}$. In our context $C(\mathcal{F})$ refers to either the growth function or the Rademacher Complexity, where both again can be bounded from above by the VC-Dimension. These bounds can be transferred into approximation theory.

- The unknown probability measure P can be replaced by a known $\mathcal{L}_1$-function λ, used as a probability density function
- The statement "with a probability of at least 1-δ we can find points, such that ..."can be replaced by the statement "For each $\delta \in \{0,1\}$ there exist points, such that ...".
- the loss function $L(\cdot,f)$ can be replaced by the function $f : Z \to \mathbb{R}$.

In my master thesis, we reproduce these steps and state a generalization bound in apporximation theory. Then, we apply this bound to radial basis functions (RBFs). This is done by calculating the VC-Dimension of RBFs, where we give a short outline of the proof. The proof technique derived can also be used to calculate the VC-Dimension of other sets of composed functions $\mathcal{F}\circ\mathcal{G} = \{f\circ g, f\in\mathcal{F}, g\in\mathcal{G}\}$, as long as the VC-Dimension of $\mathcal{G}$ is known. The transfer from statistical learning theory to approximation theory and the calculation of the VC-Dimension of RBFs are the main results of the seminar.