CL=CSCI 160

CLASS 5

JUSTIFY THE SUBTRACTION ALGORITHM USING THE r'th COMPLEMENT NUMBER REPRESENTATION WE GAVE IN CLASS

In our base r representation, where we allocate n digits to the integer part and m digits to the fraction part, we define the r's complement of a number N as follows:

$$N_{rc} = r^n - N$$

Show that in the above representation, given two non-negative numbers M and N as inputs, the algorithm below computes the value of $M - N$, in the same representation. Compute all steps of the algorithm below in the boxes, as indicated. **Justify each step in its box. Compare $M - N$ to 0 in each of the branches of step 2) below, and show the exact inequality covered by each branch in the places indicated.**

ALGORITHM to compute $M - N$:

1) Compute $M + N_{rc}$

2) (a) If there is an e.a.c. then ignore it. Stop.

(b) If there is no e.a.c. then compute:

$$-(M + N_{rc})_{rc}$$

where $(M + N_{rc})$ is already computed at step 1) above. Stop.

$\boxed{N_{rc} = r^n - N}$

$$N = \underbrace{\overbrace{\qquad\qquad}^{\text{integer part}}}_{n} \bullet \underbrace{\overbrace{\qquad\qquad}^{\text{fraction part}}}_{m}$$

n, m = number of locations

ALGORITHM to compute $M - N$:

1) **Compute $M + N_{rc}$**

2) (a) **If there is an e.a.c. then ignore it. Stop.**
   (b) **If there is no e.a.c. then compute:**

$$-(M + N_{rc})_{rc}$$

where $(M + N_{rc})$ is already computed at step 1) above. Stop.

Justification

1) $M + N_{rc} = M + r^n - N$

2) e. a. c. has the magnitude r^n. On **branch a)** there is an e. a. c., which means we have:

$M + r^n - N \geq r^n \iff \boxed{M - N \geq 0}$. Branch a) says: "if there is an e.a.c. ignore it".

This means we have to subtract the value of the e.a.c., r^n, from the value found at 1); the answer is

$$M + r^n - N - r^n = M - N \geq 0$$

Branch b) is taken when there is no e.a.c., and we know that then $\boxed{M - N < 0}$. Following the algorithm, the answer is: $-(M + N_{rc})_{rc} = -(M + r^n - N)_{rc} = -(r^n - (M + r^n - N)) =$

$$= -(r^n - M - r^n + N) = -(N - M) = -|M - N| < 0 \quad \text{since} \quad N - M > 0.$$

Note In the algorithm using r's complement representation 0 appears in branch (a) as 0, and not in branch (b), as -0, as it appears in the algorithm using (r-1)'s complement representation.

Addendum to Q1.2

Answer this **Addendum to Q1.2** before you answer the other Quiz 1.2 questions.

Suppose M = N.

After we apply the two subtraction algorithms we studied:
1. (r – 1)'s complement representation, and
2. r's complement representation,

which sign (+, or –) should the exact result have in each of the above two algorithms?
Encircle the appropriate sign in spaces below:

1. (r – 1)'s complement

$M - N = + \; - \; 0$

2. r's complement

$M - N = + \; - \; 0$

1.24 Formulate a weighted binary code for the decimal digits, using weights

a) *6, 3, 1, 1
b) 6, 4, 2, 1

Solution

	6 3 1 1	6 4 2 1
0	0 0 0 0	0 0 0 0
1	0 0 0 1(*	0 0 0 1
2	0 0 1 1	0 0 1 0
3	0 1 0 0	0 0 1 1
4	0 1 1 0(*	0 1 0 0
5	0 1 1 1	0 1 0 1
6	1 0 0 0	0 1 1 0(*
7	1 0 1 0(*	1 0 0 1(*
8	1 0 1 1	1 0 1 0
9	1 1 0 0	1 0 1 1

(* Here we have a choice

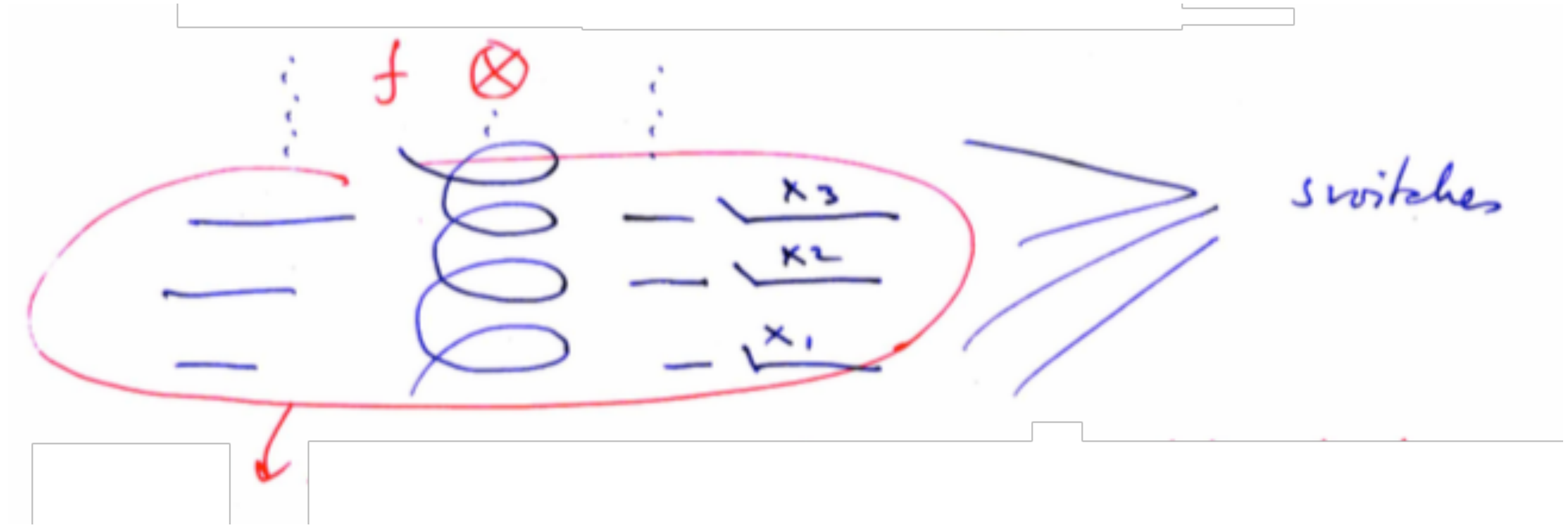
Table
Four Different Binary Codes for the Decimal Digits

Decimal Digit	BCD 8421	2421	Excess-3	8, 4, −2, −1
0	0000	0000	0011	0000
1	0001	0001	0100	0111
2	0010	0010	0101	0110
3	0011	0011	0110	0101
4	0100	0100	0111	0100
5	0101	1011	1000	1011
6	0110	1100	1001	1010
7	0111	1101	1010	1001
8	1000	1110	1011	1000
9	1001	1111	1100	1111
	1010	0101	0000	0001
Unused	1011	0110	0001	0010
bit	1100	0111	0010	0011
combi-	1101	1000	1101	1100
nations	1110	1001	1110	1101
	1111	1010	1111	1110

Note None of these codes is **self-complementary**, i.e. ((r-1)s complements correspond to each other, that is decimal digits complementary in 9's compl. have binary representations that are 1's complements of each other.

<u>Problem of the Independent Switches</u>: example of a Boolean function

There is a staircase leading to **n** floors, all being lit by one lamp, which is independently operated by any of **n** switches, which are placed one at each of the **n** floors.



To simplify, consider **n = 3**; that is there are **3** floors and <u>**3** switches</u>.

Design a function **f** of the 3 switches. **f** represents this lamp. Find its truth table.

x_1	x_2	x_3	f_1	f_2
0	0	0	0	1
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	0

Say: on = 1 and off = 0

<u>Note</u>: the value of **f** for any initial state of the switches is arbitrary, as we can start with any configuration; however once we choose one initial state, the values of **f** for every other configuration of the switches are <u>determined by this initial state</u>.

f_1 and f_2 are **<u>complementary</u>**

<u>Hamming Distance (d_H)</u> between two strings equals the number of locations in which the two strings differ.

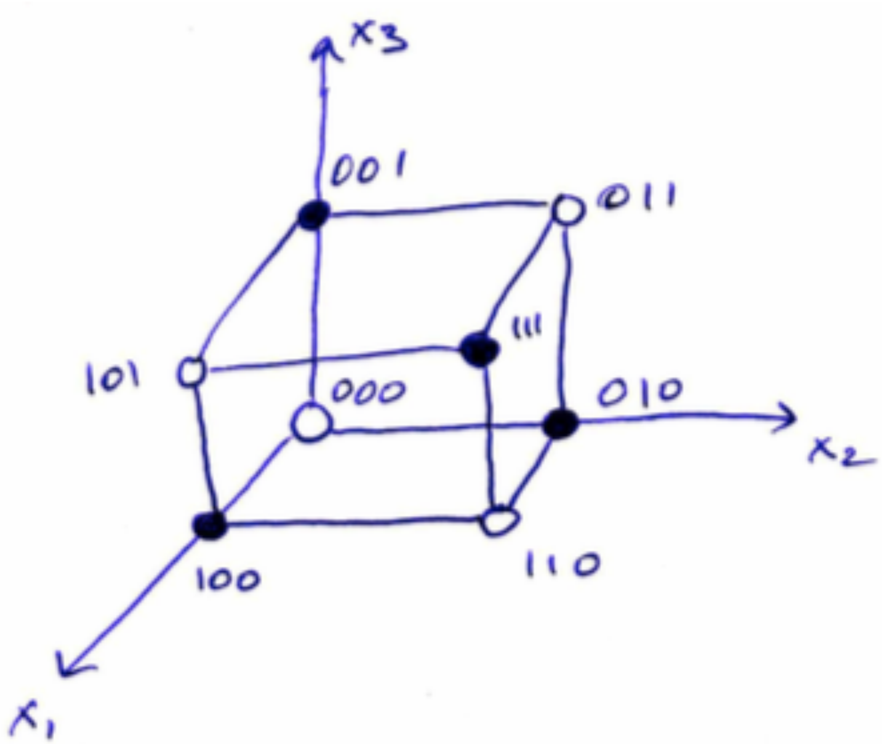
ex: $d_H(1\,\underline{0}\,0,\ 1\,\underline{1}\,0) = \mathbf{1}$; $d_H(\underline{0}\,\underline{0}\,\underline{1},\ \underline{1}\,\underline{1}\,\underline{0}) = \mathbf{3}$

x_1	x_2	x_3	f_1	f_2
0	0	0	0	1
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	0

Hamming Distance (d_H) between two strings equals the number of locations in which the two strings differ.

Note: When the H distance between two $x_1 x_2 x_3$ values is **odd** then the corresponding values for **f** are **different**. Conversely, when the H distance is **even** then the corresponding values for **f** are **the same**.

Let's represent the problem of the 3 independent switches using a 3-D cube.



The vertices of the cube have as coordinates all possible inputs for the 3 switches, while the color of each vertex shows the state of the lamp (**f**).

● = on <--> **f** = 1; ○ = off <--> **f** = 0

HW: Solve the problem of the independent switches for **n** = **4**, which means: 1) Truth table

2) Draw the 4-D cube

Boolean Algebra - Definition

A Boolean algebra is a 6-tuple:

$$(\underbrace{B}_{\text{Set}}, \underbrace{+}_{\text{or}}, \overbrace{\cdot}^{\text{and}}, \underbrace{'}_{\text{complement}}; \overbrace{0}^{\text{identity elem. for }+}, \underbrace{1}_{\text{identity elem. for }\cdot})$$

that follows the next 6 axioms:

1) **Closure** for $+, \cdot$

   if $x, y \in B$, then $x \cdot y,\ x + y \in B,\ \forall x, y \in B$

2) **Identity elements**:

   $x + 0 = 0 + x = x$

   $x \cdot 1 = 1 \cdot x = x,\ \forall x \in B$

3) **$+, \cdot$ Commutative**:

   $x + y = y + x$

   $x \cdot y = y \cdot x, \forall x, y \in B$

4) **$+, \cdot$ Distributive**: one relative to the other

   $(x \cdot y) + z = (x + z)(y + z)$

   $(x + y) \cdot z = x \cdot z + y \cdot z \quad \forall x,y,z \in B$

5) $\forall x \in B, \exists x'$ or $\bar{x} \in B$ such that:

   $x + x' = 1$

   $x \cdot x' = 0$

6) **There are at least 2 elements**:

   $0 \neq 1$.

HW: Write all axioms and properties for the B. algebra of sets:

S = set, $S \neq \varnothing$

$(\underbrace{P(S)}_{\text{Set of all subsets of } S}, \cup, \cap, \complement; \varnothing, S)$.

[Remember: if S has n elements then $P(S)$ has 2^n]

Axioms of the Boolean Algebra ($B, +, \cdot, '; 0, 1$)

1) $+, \cdot$ are **closed operations in B:**
   $x + y \in B$
   $x \cdot y \in B, \quad \forall x, y \in B$

2) **identity elements:**
   $x + 0 = 0 + x = x$
   $x \cdot 1 = 1 \cdot x = x, \quad \forall x \in B$

3) $+, \cdot$ are **commutative:**
   $x + y = y + x$
   $x \cdot y = y \cdot x, \quad \forall x, y \in B$

4) $+, \cdot$ are **distributive** one relative to the other one:
   $(x + y) \cdot z = x \cdot z + y \cdot z$
   $(x \cdot y) + z = (x + z)(y + z), \quad \forall x, y, z \in B$

5) $\forall x \in B, \exists x' \in B$, the **complement** of x, such that:
   $x + x' = 1$ and
   $x\,x' = 0$

6) There are at least two elements in B:
   $\mathbf{0 \neq 1}$

Properties of Boolean Algebras (B)

7) $+, \cdot$ are **idempotent:**

$x + x = x$

$x \cdot x = x, \quad \forall x \in B$

8) $x + 1 = 1,$

$x \cdot 0 = 0, \quad \forall x \in B$

9) $x'' = x, \quad \forall x \in B$

10) **DeMorgan's Law**:

$(x + y)' = x'y',$

$(x \cdot y)' = x' + y', \quad \forall x, y \in B$

11) $+, \cdot$ are **associative:**

$x + (y + z) = (x + y) + z,$

$x \cdot (y \cdot z) = (x \cdot y) \cdot z, \quad \forall x, y, z \in B$

12) **Absorption** for $+, \cdot$:

$x + xy = x$

$x (x + y) = x, \quad \forall x, y \in B$

<u>Axioms + Properties for Boolean Algebra</u> $(B, +, \cdot, '; 0, 1)$

1) $+, \cdot$ are **<u>closed in B</u>**
2) **<u>identity elements:</u>**

   $x + 0 = x = 1 \cdot x, \quad \forall x$

3) $+, \cdot$ are **<u>commutative</u>**
4) $+, \cdot$ are **<u>distributive</u>** one relative to the other one:

   $(x + y) \cdot z = x \cdot z + y \cdot z$

   $\mathbf{(x \cdot y) + z = (x + z)(y + z)}, \quad \forall x, y, z$

5) $\forall x \in B, \exists x' \in B$, the **<u>complement</u>** of x, such that:

   $x + x' = 1$ and $\quad x \cdot x' = 0$

6) $\mathbf{0 \neq 1}$
7) $+, \cdot$ are **<u>idempotent:</u>**

   $x + x = x$ and $x \cdot x = x, \quad \forall x$

8) $x + 1 = 1$ and $x \cdot 0 = 0, \quad \forall x$

9) $x'' = x, \quad \forall x$

10) **<u>DeMorgan's Law</u>**:

    $(x + y)' = x'y'$, and $(x \cdot y)' = x' + y', \quad \forall x, y$

11) $+, \cdot$ are **<u>associative</u>**
12) **<u>Absorption</u>** for $+, \cdot$:

    $x + xy = x$

    $x(x + y) = x, \quad \forall x, y$

HW 3-D

The solutions to the quadratic equation

$x^2 - 11x + 22 = 0$

are $x = 3$ and $x = 6$.

Determine the base of the numbers in the equation.

Solution

use sol. $x = 3$: $3^2 - 11_r \cdot 3 + 22_r = 0 \overset{\text{base } 10}{\Leftrightarrow}$

$\Leftrightarrow \quad 9 - (r+1)\,3 + 2r + 2 = 0 \quad \Leftrightarrow \quad r = 8$

[one could use other solution $r = 6$ and obtain the same].

HW 3-E Convert the hexadecimal number 68BE to binary, and then convert it from binary to octal and then to base 32.

Solution

digit

15	- F
16	- G
17	- H
18	- I
19	- J
20	- K
21	- L
22	- M
23	- N
24	- O
25	- P
26	- Q
27	- R
28	- S
29	- T
30	- U
31	- V

$r = 32$

$$68BE_{hex} = \underbrace{110}_{6}\ \underbrace{1000}_{8}\ \underbrace{1011}_{B}\ \underbrace{1110}_{E}$$

$$\overbrace{110}^{6}\ \overbrace{100}^{4}\ \overbrace{010}^{2}\ \overbrace{111}^{7}\ \overbrace{110}^{6}$$

$$\underbrace{11010}_{26}\ \underbrace{00101}_{5}\ \underbrace{11110}_{30}$$

$$= 64276_8$$

$$= Q5U_{32}$$