

# Algorithm Efficiency (More formally)

# Today's Plan



Algorithm Efficiency

# What is CSCI 235?

Programming => Software Analysis and Design  
Expected professional and responsible attitude

Think like a Computer Scientist:

Design and maintain complex programs

Software Engineering, Abstraction, OOP

Design and represent data and its management

Abstract Data Types

Implement data representation and operations

Data Structures

Algorithms

Analyze Algorithms and their Efficiency



# Algorithm Efficiency

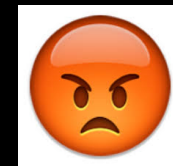
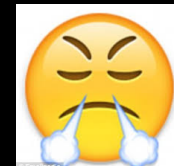
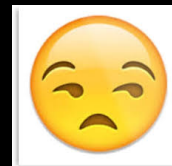
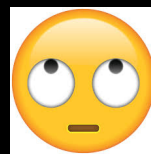
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whatever it is doing it's taking way too long...

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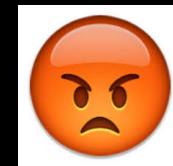
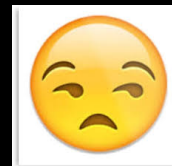
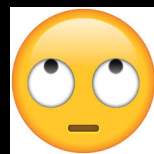
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ridiculously frustrated?



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You are using an application and suddenly it stalls...  
whatever it is doing it's taking way too long...

how "*long*" does that have to be for you to become  
ridiculously frustrated?



... probably not that long

# Scenario 2

At your next super high-end job with the company/research-center of your dreams you are given a very difficult problem to solve.

You work hard on it, find a solution, code it up and it works!!!!

Proudly you present it the next day  
but...



Given some new (large) input it keeps stalling...

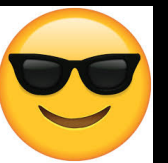


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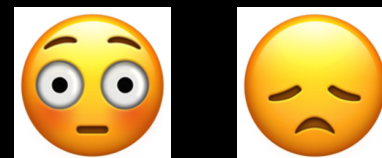
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Given some new (large) input it keeps stalling...

Well... sorry but your solution is no good!!!



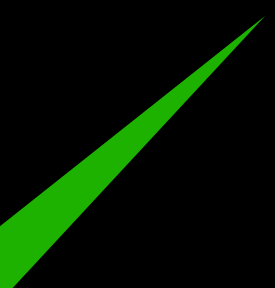
You need to have a means to estimate/predict the efficiency of your algorithms on **unknown input**.

What is a good solution?

How can we compare solutions  
to a problem? (Algorithms)

# What is a good solution?

Correct



If it's not  
correct it is not  
a solution at all

# What is a good solution?

Correct

Efficient

Time

Space

# What is a good solution?

Correct

Efficient

Time

Space

We are going to  
focus on time

How can we measure time  
efficiency?

How can we measure time  
efficiency?

**Runtime?**



# Problems with actual runtime for comparison

What computer are you using?

Runtime is highly sensitive to hardware

# Problems with actual runtime for comparison

What computer are you using?

Runtime is highly sensitive to hardware

What implementation are you using?

Implementation details may affect runtime but are not reflective of algorithm efficiency

How should we measure  
execution time?

Constant

How should we measure  
execution time?

Number of "steps" or "operations"  
as a function of the size of the input

Variable

```
template<class T>
void List<T>::traverse()
{
    for(Node<T>* ptr = first; ptr != nullptr; ptr = ptr->getNext())
    {
        std::cout << ptr->getItem() << endl;
    }
}
```

What are the operations?  
Let  $n$  be the number of nodes

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1 node instantiation and assignment  
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$K_0$

1 node instantiation and assignment  
upon entering the loop

$K_1$

call to getNext ( )

$K_2$

pointer assignment

$K_3$

pointer comparison

$K_4$

call to getItem ( )

$K_5$

write to the console

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$$\text{Operations} = K_0 + n(K_1 + K_2 + K_3 + K_4 + K_5)$$

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write to the console

$$\text{Operations} = K_0 + nK_6$$

# Lecture Activity

Identify the steps and write down an expression for execution time

```
bool linearSearch(const string& str, char ch)
{
    for (int i = 0; i < str.length(); i++)
    {
        if (str[i] == ch) {
            return true;
        }
    }
    return false;
}
```

# Lecture Activity

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}
```



Was this tricky?



n here is the length of the string

```
bool linearSearch(const string& str, char ch)
{
    // 1 int assignment upon entering loop
    for (int i = 0; i < str.length(); i++)
    { // call to length() and increment
        if (str[i] == ch) { // Comparisons
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    return false; //return operation, maybe
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```

Maybe stop in the middle

Maybe stop at end of loop

**n** here is the length of the string

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```

In the  
**WORST CASE**

Execution completes in **at most:**

$k_0n + k_1$  operations

# Types of Analysis

**Best case analysis:** running time under best input (e.g., in linear search item we are looking for is the first) - not reflective of overall performance)

**Average case analysis:** assumes equal probability of input (usually **not** the case)

**Expected case analysis:** assumes probability of occurrence of input is known or can be estimated, and if it were possible may be too expensive



**Worst case analysis:** running time under worst input, gives upper bound, it can't get worse, good for sleeping well at night!

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Some constant number  
of operations repeated  
inside the loop

Some constant number  
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```

The number of times the loop is repeated, i.e. the size of `str`

Execution completes in **at most:**

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Some constant number of operations repeated inside the loop

Some constant number of operations performed outside the loop

# Observation

Don't need to explicitly compute the constants  $k_i$

$$4n + 1000$$

$$n + 137$$

***Dominant term*** is sufficient to explain overall behavior (in this case linear)

# Big-O Notation

Ignores everything except **dominant term**

Examples:

Notation: describes the overall behavior

$$T(n) = 4n + 4 = O(n)$$

$$T(n) = 164n + 35 = O(n)$$

$$T(n) = n^2 + 35n + 5 = O(n^2)$$

$$T(n) = 2n^3 + 98n^2 + 210 = O(n^3)$$

$$T(n) = 2^n + 5 = O(2^n)$$



# Big-O Notation

$T(n)$  is the running time

$n$  is the size of the input

Ignores everything except **dominant term**

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Big-O describes the overall behavior

Let  $T(n)$  be the **running time** of an algorithm measured as number of operations given **input of size  $n$** .

$T(n)$  is  $O(f(n))$

if it grows **no faster** than  $f(n)$

# Big-O Notation

But  
 $164n+35 > n$



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Notation: describes the overall behavior

More formally:

**$T(n)$  is  $O(f(n))$**

if there exist constants  **$k$**  and  **$n_0$**   
such that for all  **$n \geq n_0$**

$$T(n) \leq k f(n)$$

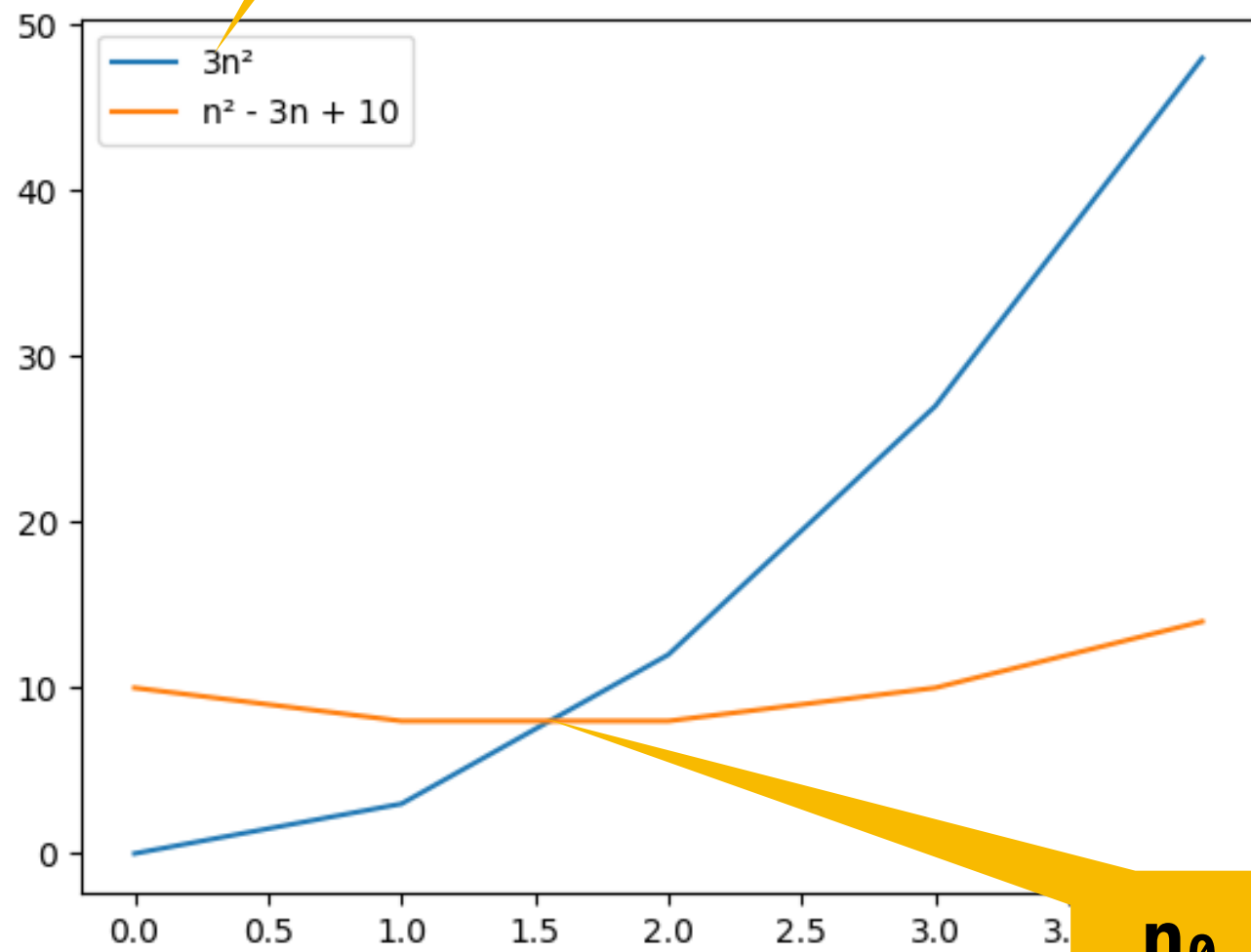


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**$k = 3$**



**$n_0$**

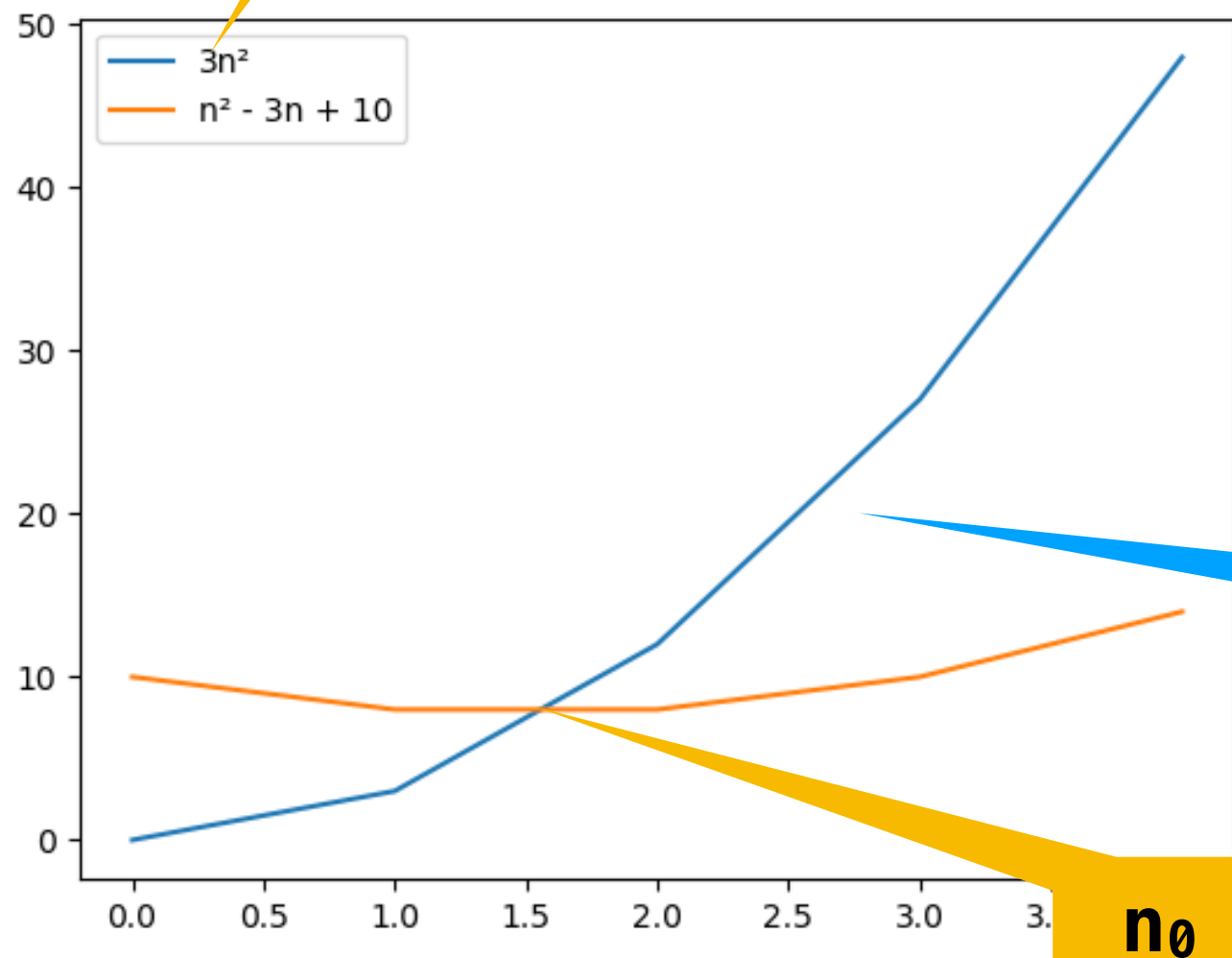
$T(n) = n^2 - 3n + 10$   
 $T(n)$  is  $O(n^2)$   
For  $k=3$  and  $n \geq 1.5$

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$T(n) = n^2 - 3n + 10$   
 $T(n)$  is  $O(n^2)$   
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This is why we can  
look at **dominant**  
**term only** to explain  
behavior

Big-O describes the overall growth rate of an algorithms for **large n**

# Proving Big-O Relationship

Apply definition of Big-O to prove that  $T(n)$  is  $O(f(n))$  for particular functions  $T$  and  $f$

Do so by choosing  $k$  and  $n_0$  s.t. for all  $n \geq n_0$ ,  
 $T(n) \leq kf(n)$



# Proving Big-O Relationship

## Example:

Suppose  $T(n) = (n+1)^2$

We can say that  $T(n)$  is  $O(n^2)$

To prove it must find  $k$  and  $n_0$  s.t. for all  $n \geq n_0$ ,

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Expand  $(n+1)^2 = n^2 + 2n + 1$

Observe that, as long as  $n \geq 1$ ,  $n \leq n^2$  and  $1 \leq n^2$

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**Thus** if we choose  $n_0 = 1$  and  $k = 4$  we have

$$n^2 + 2n + 1 \leq n^2 + 2n^2 + n^2 = 4n^2$$

$T(n)$

$k$

$f(n)$

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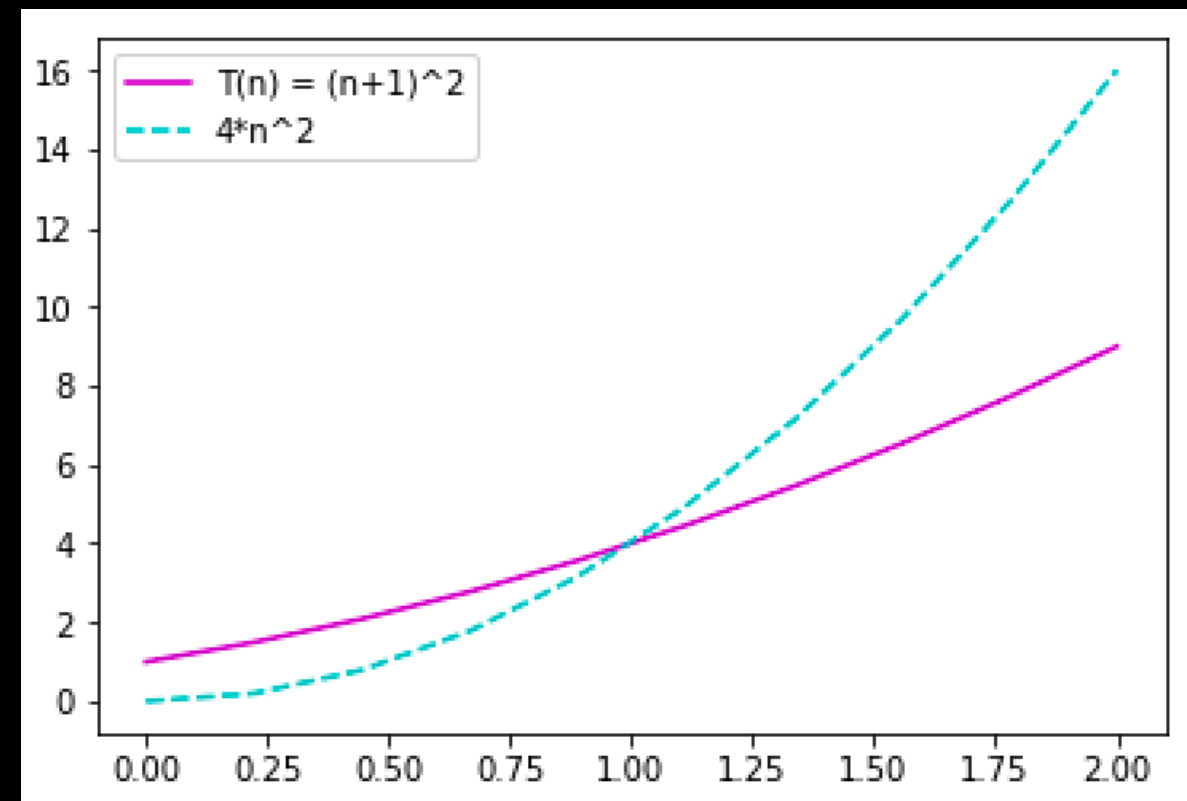
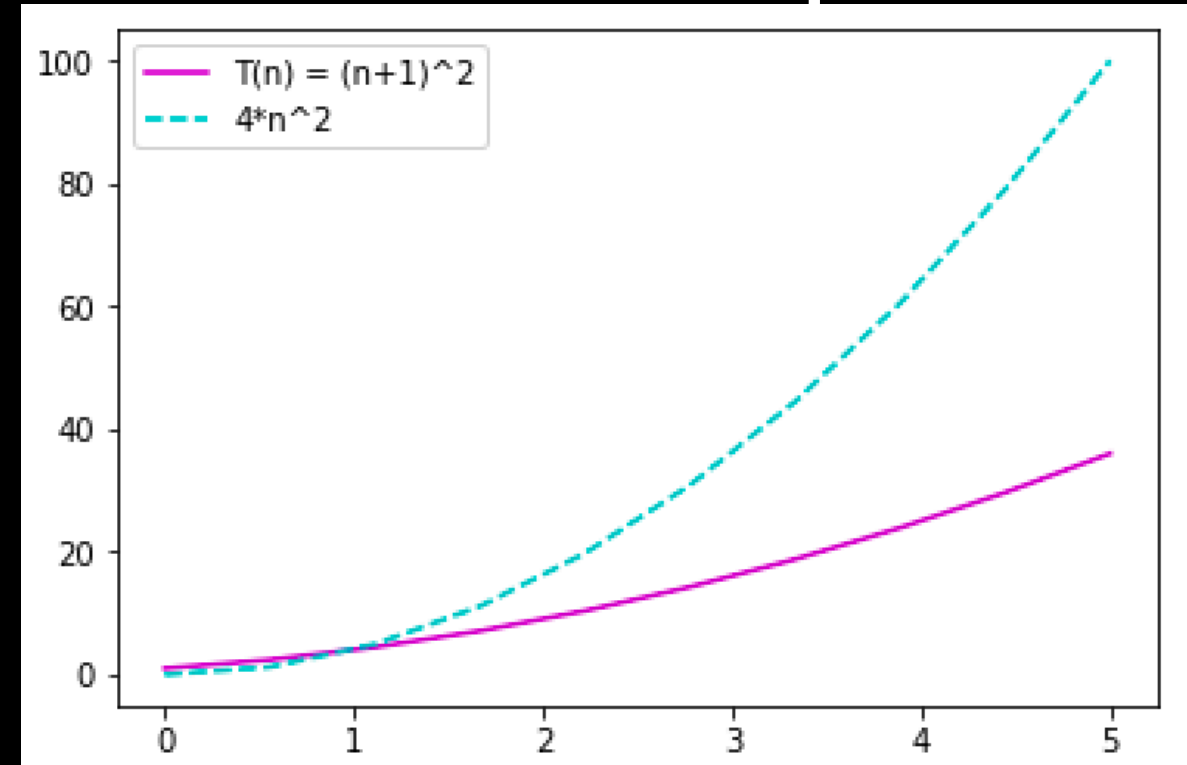
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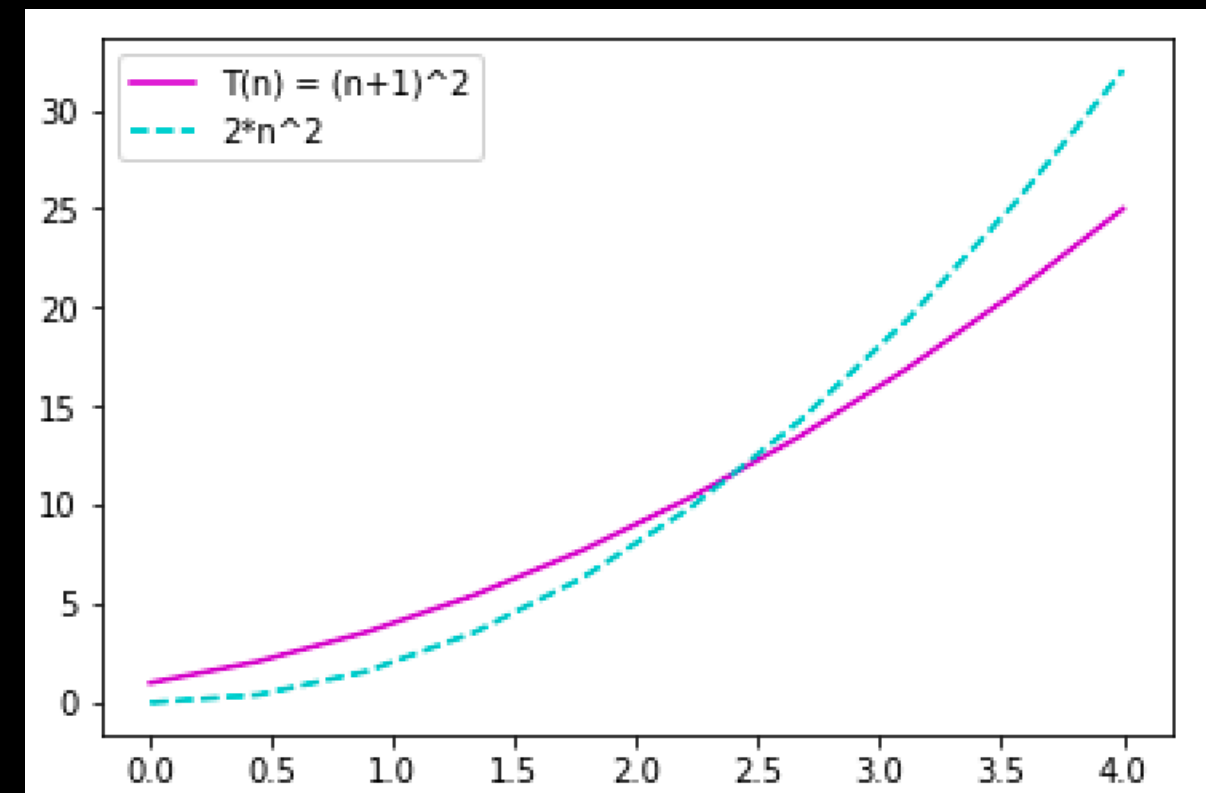
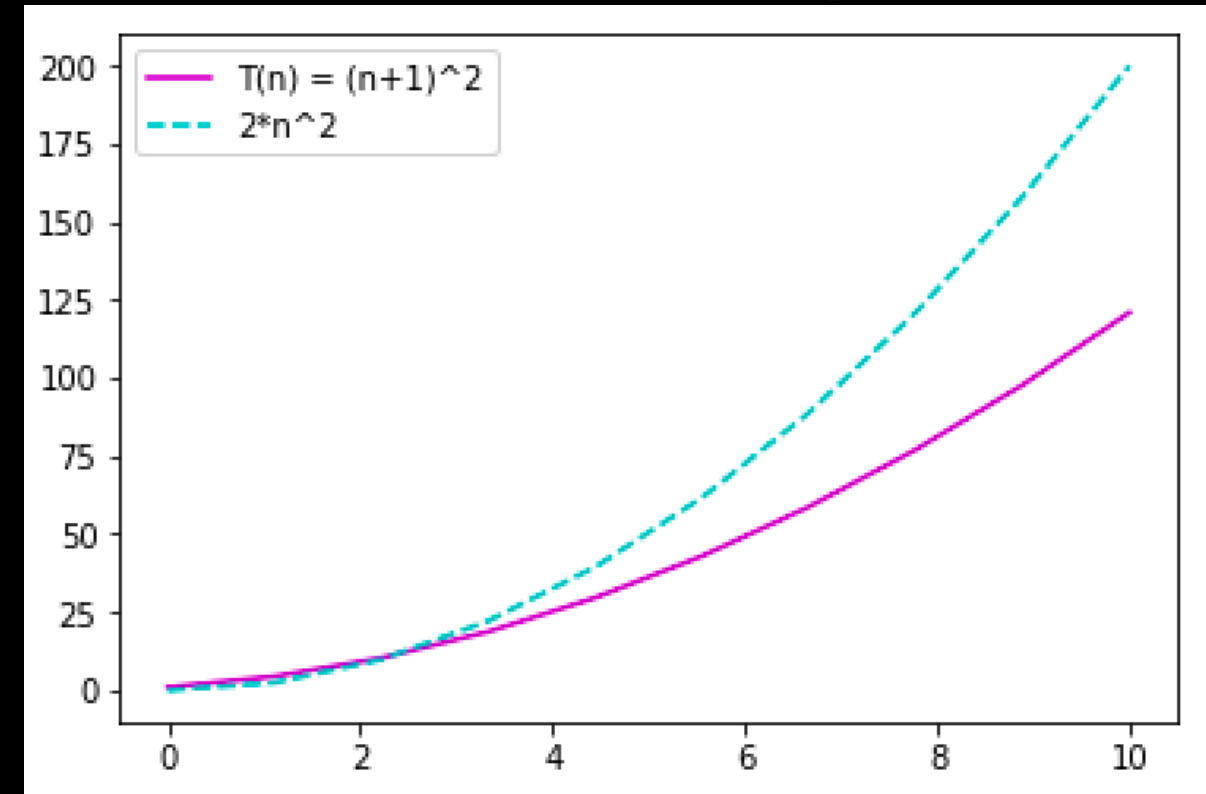
# Proving Big-O Relationship

**Not Unique:**

Could also choose  $n_0 = 3$  and  $k = 2$  because

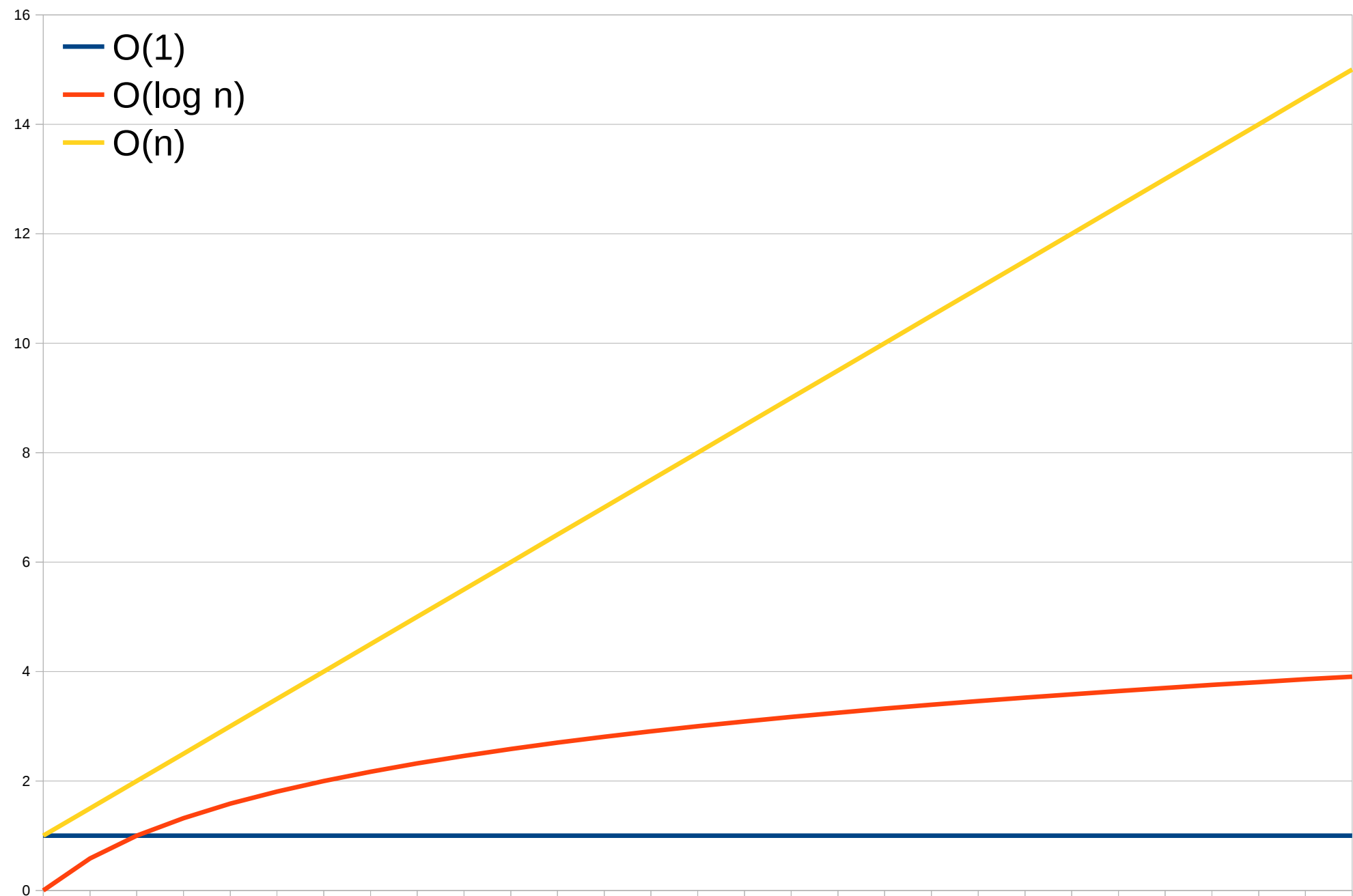
$$(n+1)^2 \leq 2n^2 \text{ for all } n \geq 3$$

For proof one is enough

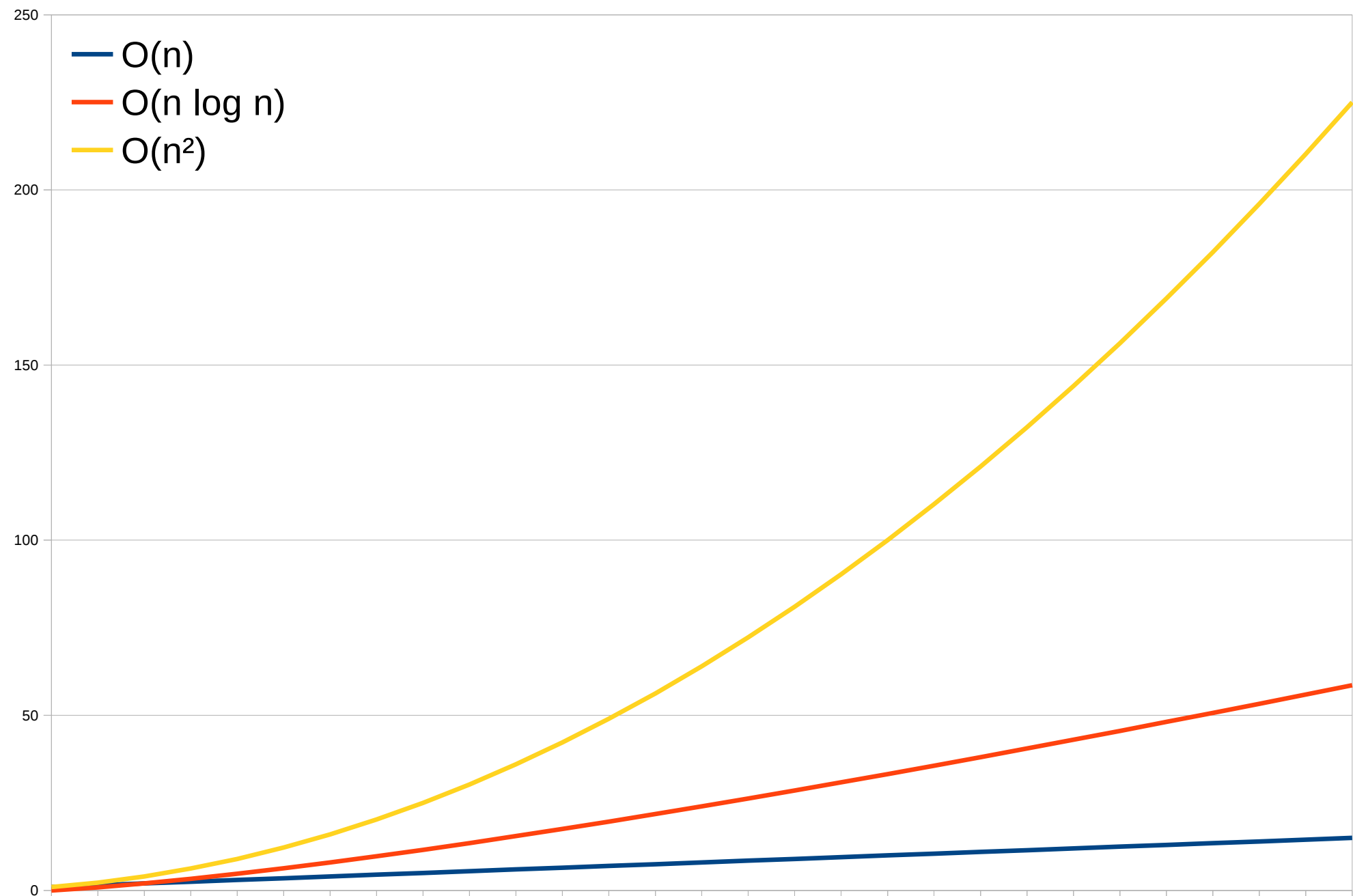


# A visual comparison of growth rates

## Growth Rates, Part One

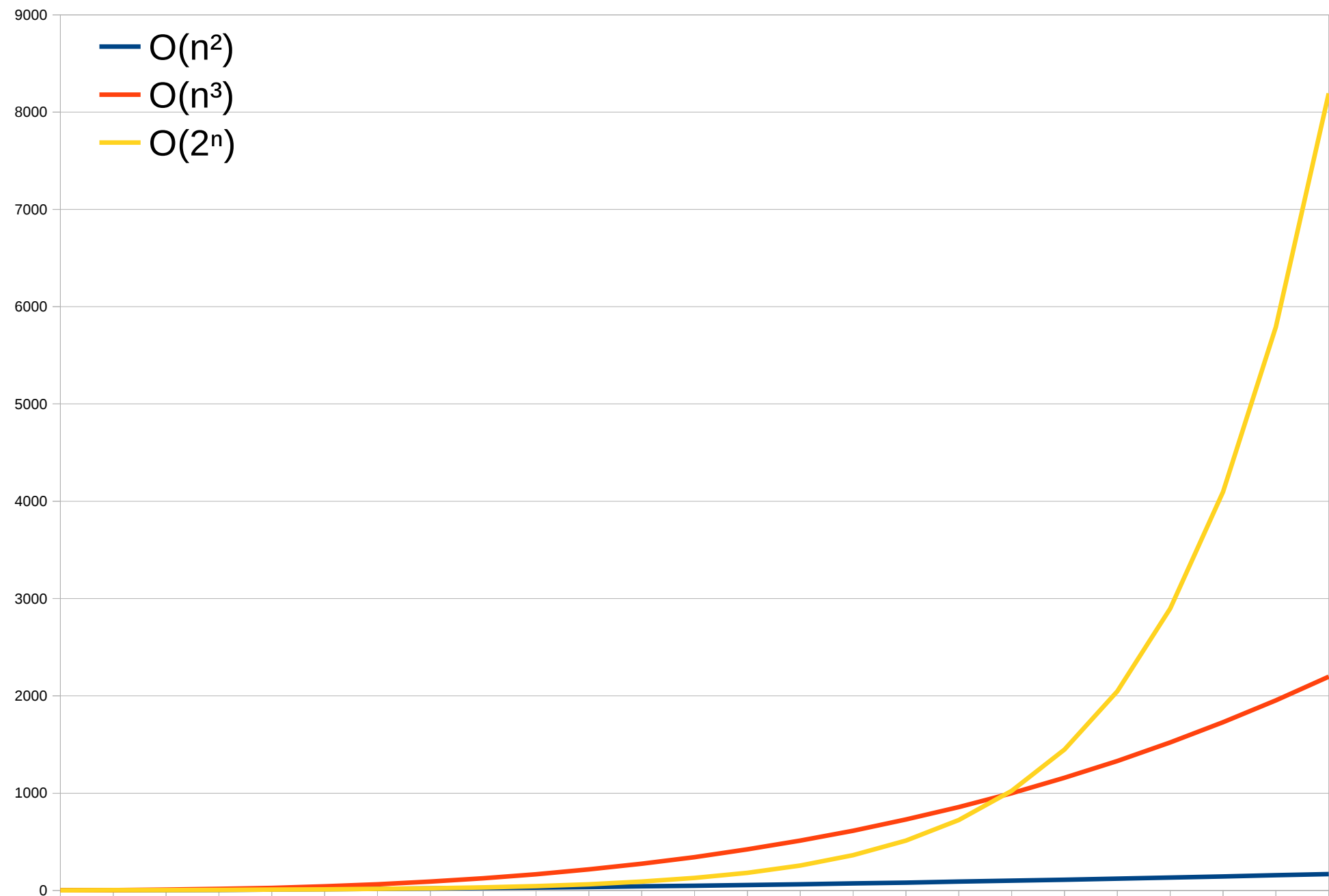


## Growth Rates, Part Two

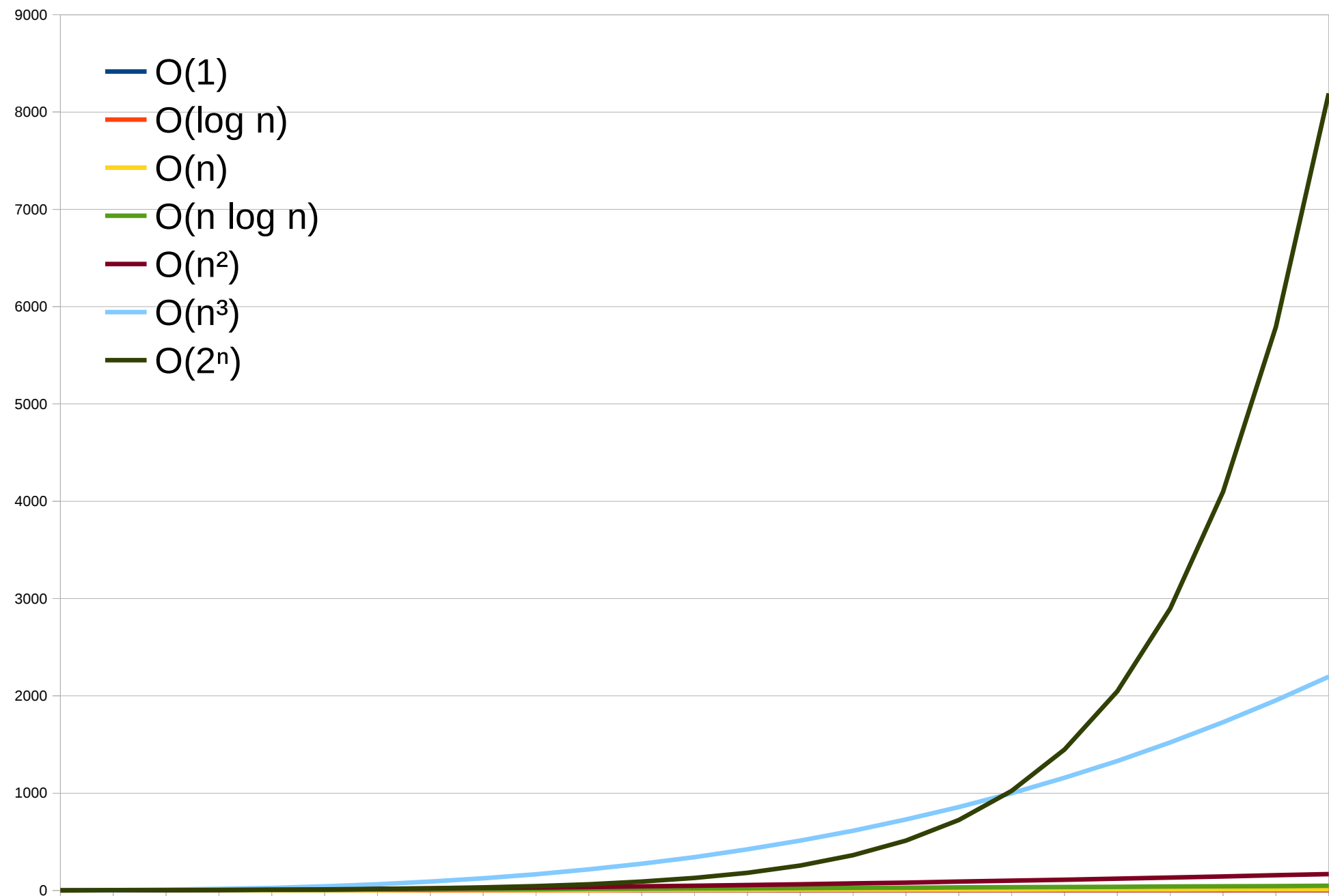




## Growth Rates, Part Three



To Give You A Better Sense...



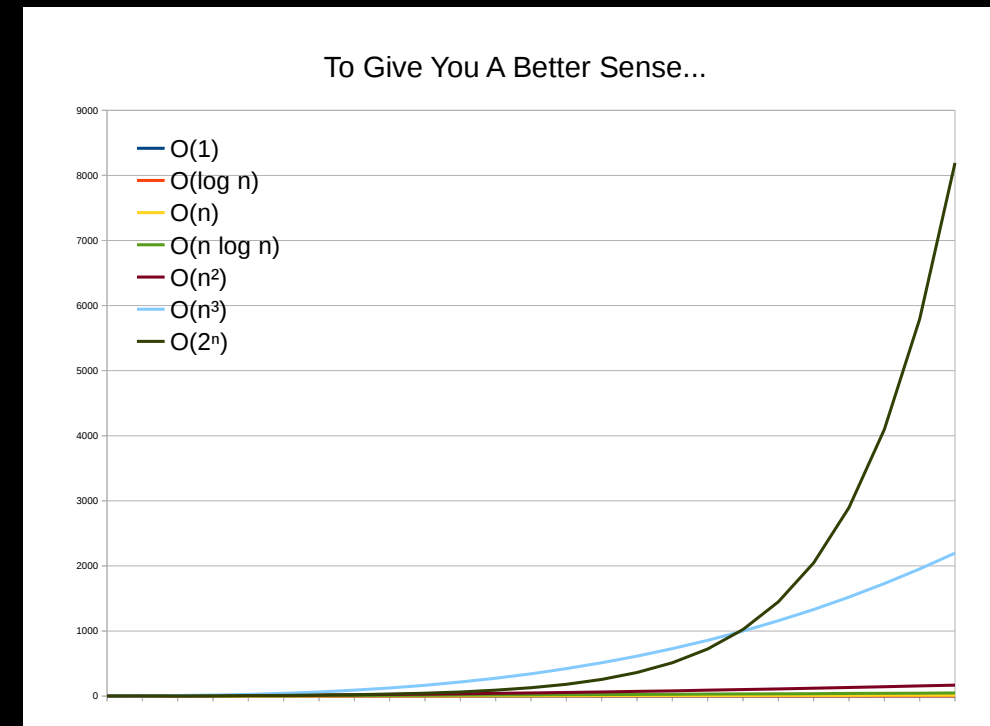
# Tight is more meaningful

If  $T(n)$  is  $O(n)$

It is also true that  $T(n)$  is  $O(n^3)$

And it is also true that  $T(n)$  is  $O(2^n)$

But what does it mean???



The closest Big-O is the most descriptive of the overall worst-case behavior

# Tightening the bounds

Big-O: upper bound

$T(n)$  is  $O(f(n))$

if there exist constants  $k$  and  $n_0$  such that for all  $n \geq n_0$   $T(n) \leq k f(n)$

Grows no faster than  $f(n)$

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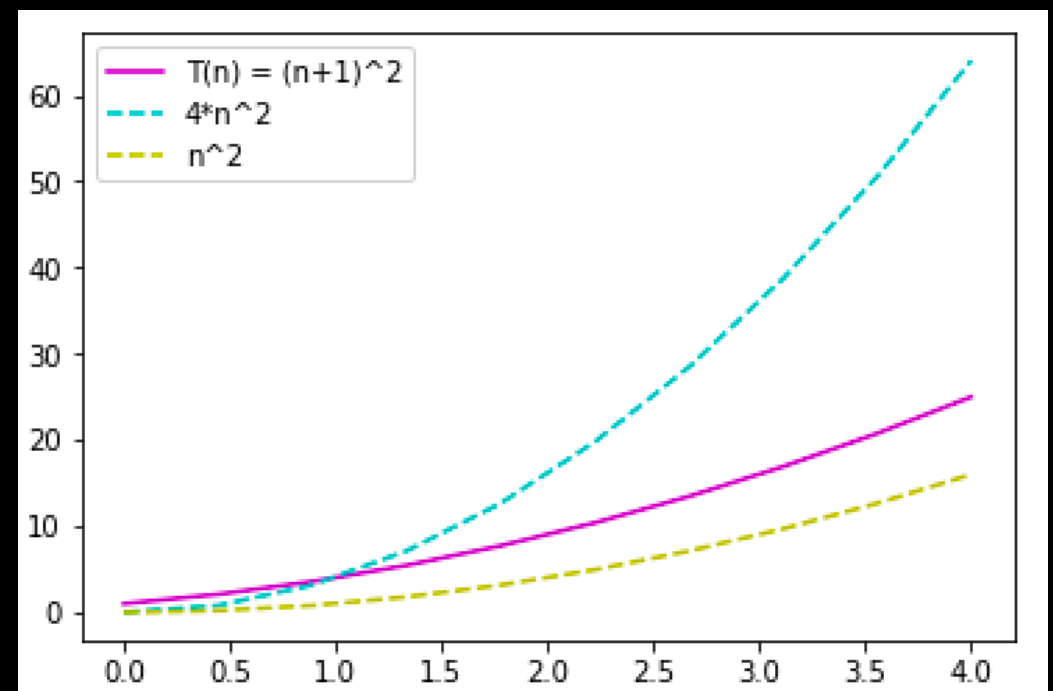
Grows no faster than  $f(n)$

Omega: lower bound

$T(n)$  is  $\Omega(f(n))$

if there exist constants  $k$  and  $n_0$  such that for all  $n \geq n_0$   $T(n) \geq k f(n)$

Grows at least as fast as  $f(n)$

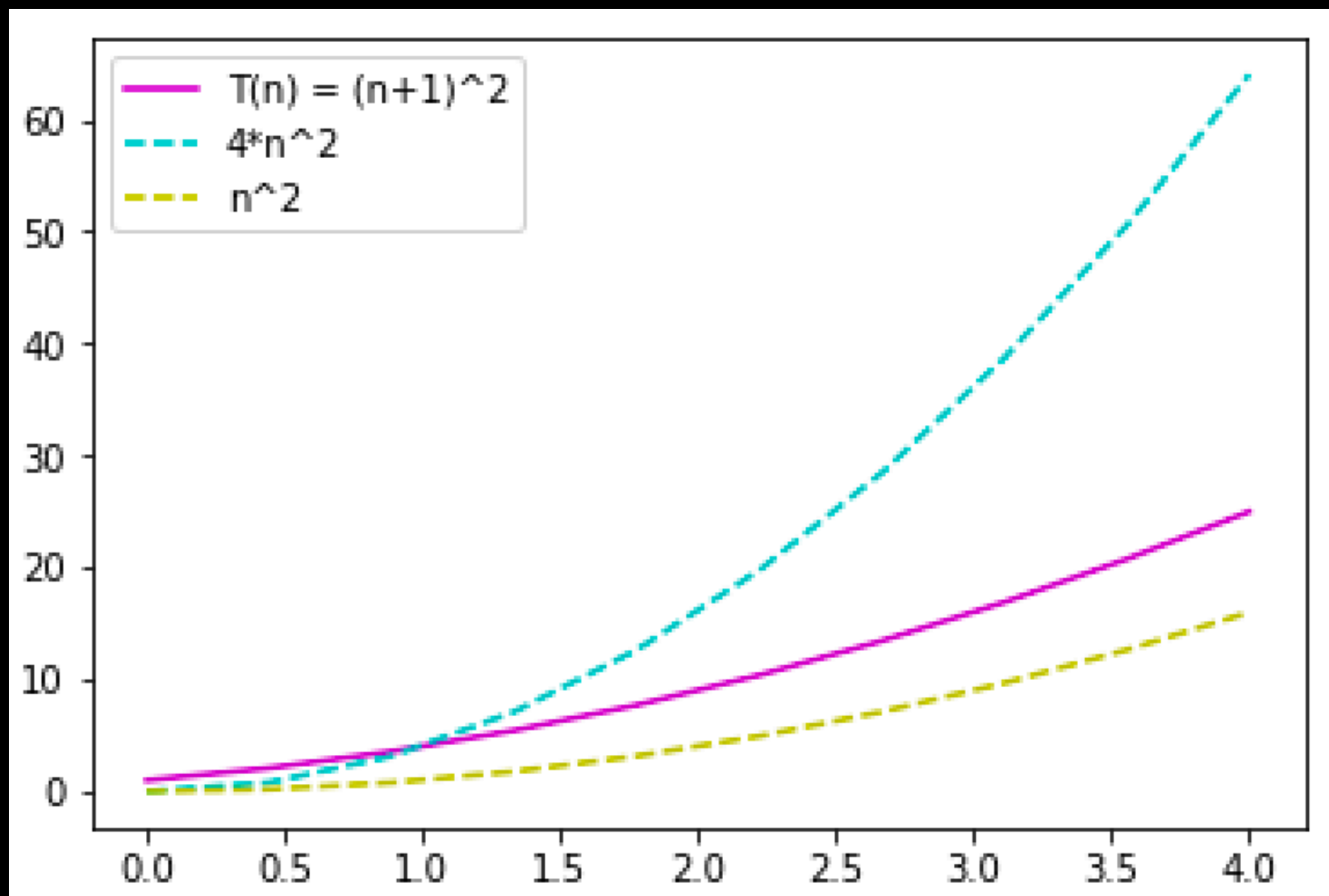


# Tightening the bounds

Theta: tight bound

$T(n)$  is  $\Theta(f(n))$

Grows at the same rate as  $f(n)$  : iff both  $T(n)$  is  $O(f(n))$  and  $\Omega(f(n))$



# A numerical comparison of growth rates

<b>f(n) \ n</b>	<b>10</b>	<b>100</b>	<b>1,000</b>	<b>10,000</b>	<b>100,000</b>	<b>1,000,000</b>
<b>1</b>	1	1	1	1	1	1
<b>log<sub>2</sub>n</b>	3	6	9	13	16	19
<b>n</b>	10	10 <sup>2</sup>	10 <sup>3</sup>	10 <sup>4</sup>	10 <sup>5</sup>	10 <sup>6</sup>
<b>n * log<sub>2</sub>n</b>	30	664	9,965	10 <sup>5</sup>	10 <sup>6</sup>	10 <sup>7</sup>
<b>n<sup>2</sup></b>	10 <sup>2</sup>	10 <sup>4</sup>	10 <sup>6</sup>	10 <sup>8</sup>	10 <sup>10</sup>	10 <sup>12</sup>
<b>n<sup>3</sup></b>	10 <sup>3</sup>	10 <sup>6</sup>	10 <sup>9</sup>	10 <sup>12</sup>	10 <sup>15</sup>	10 <sup>18</sup>
<b>2<sup>n</sup></b>	10 <sup>3</sup>	10 <sup>30</sup>	10 <sup>301</sup>	10 <sup>3,010</sup>	10 <sup>30,103</sup>	10 <sup>301,030</sup>





# What **does** Big-O describe?

“Long term” behavior of a function

Compare behavior  
of 2 algorithms

If algorithm A has runtime  $O(n)$  and algorithm B has runtime  $O(n^2)$ , **for large inputs** A will always be faster.

If algorithm A has runtime  $O(n)$ , doubling the size of the input will double the runtime

Analyze algorithm behavior  
with growing input

# What **can't** Big-O describe?

The actual runtime of an algorithm

$$10^{100}n = O(n)$$

$$10^{-100}n = O(n)$$

How an algorithm behaves on small input

$$n^3 = O(n^3)$$

$$10^6 = O(1)$$

# To summarize Big-O

It is a means of describing the growth rate of a function

It ignores all but the dominant term

It ignores constants

Allows for quantitative ranking of algorithms

Allows for quantitative reasoning about algorithms