1 Hybridization

Hybridization involves the weakening of Continuity constraints at inter-element boundaries to allow for:

- Larger solution space to achieve approximations in
- Decoupled systems (parallelizable)
- Local computations – better stiffness matrices
- High order approximations through flux recovery

Some of the major hybridized methods are: Primal Hybrid, Dual/Mixed Hybrid, and Lagrangian Discontinuous Galerkin Hybrid.

For a breakdown of many hybridized methods and a unified framework for analyzing some of these, see (Cockburn 1985), (Arnold, Brezzi, Cockburn, Marini 2001/2), (Cockburn et al. 2016). In this work, we will focus on the Primal Hybrid formulation, with a specific post-processing technique for improved accuracy.

The model problem for this study is

$$-d^* du = f.$$

We will look at 2 and 3 dimension versions with d^*d as div grad, grad div, and curl curl.

The primal formulation for this problem is derived as follows: Solve $-d^*du = f$ weakly. i.e. $\langle -d^*du, v\rangle = \langle f, v\rangle$ for all $v \in V$. This, in turn, can be written as a first-order system using the weak form:

$$\langle du, dv\rangle - \langle du, v\rangle_\partial = \langle f, v\rangle$$

where the $\langle , \rangle_\partial$ denotes inner product on the boundaries. The explicit formulation of this term comes from the Divergence/Stokes/Green's theorem and depends on the dimension of the domain and on the derivative d.

2 Postprocessing

With primal hybrid method, we end up with an approximate solution $u_h \in V_h$ to the function $-d^*du = f$. In most physically relevant problems, however, we are looking both for u and for $\sigma = -du$.

Now, rather than evaluating a numerical derivative of the computed solution u_h and leaving it at that, we post-process to achieve a much higher order of accuracy solution σ_h for $-du$.

There are many methods used for improving the approximation for σ_h using the information gained from solving for u_h. See, e.g., (Chou, Kwak, and Kim, 2002). The one that we are considering here uses the trace values as well as computed values for u_h.

So, given the approximation u_h, we then solve the minimization problem: minimize the functional

$$J(\sigma) = \frac{1}{2}\left(||\sigma + du_h||^2 + ||d\sigma - f||^2\right)$$

over all $\sigma \in \Sigma_h$ subject to the constraints:

$$\begin{array}{rcll} \langle du_h, dv\rangle_K + \langle v, \sigma\rangle_{\partial K} & = & \langle f, v\rangle_K & \forall (v,\sigma) \in W_h \times \Sigma_h \\ \langle u, \tau\rangle_{\partial K} & = & 0 & \end{array}$$

Here the equality is taken over each element $K \in T_h$, and W_h the approximation function space for u has weakened continuity constraints on the inter-element boundaries. This allows for a better approximation for σ while the boundary terms introduce lagrange multipliers that insure that u_h satisfies the equations.

3 Lagrange Multipliers

Recall the basic idea of Lagrange Multipliers from Calculus: to minimize $F(x)$ subject to the constraint $G(x) = H(x)$, solve $\nabla F = \lambda \nabla G$

In this case, we want to solve $\nabla J(\sigma) = 0$ while enforcing the original equation constraints on σ. So we solve for (σ_h, u_h):

$$\langle \nabla J(\sigma_h), \tau \rangle = 0 \ \ \forall \tau \in \Sigma_h$$

and at the same time:

$$\langle u_h, v \rangle - \langle v, \sigma_h \rangle_\partial = \langle f, v \rangle, \quad \langle u_h, \tau \rangle_\partial = 0$$