1 Hybridization

Hybridization involves the weakening of Continuity constraints at inter-element boundaries to allow for:

- Larger solution space to achieve approximations in
- Decoupled systems (parallelizable)
- Local computations – better stiffness matrices
- High order approximations through flux recovery

Some of the major hybridized methods are: Primal Hybrid, Dual/Mixed Hybrid, and Lagrangian Discontinuous Galerkin Hybrid. The major differences being in the form of the equation used and in the definition of flux terms used to complement the discontinuous function spaces. For a breakdown of many hybridized methods and a unified framework for analyzing some of these, see [1, 3]. In this work, we will focus on the Primal Hybrid formulation, with a specific post-processing technique for improved accuracy.

The model problem for this study is

$$-d^*du = f.$$

We will look at 2 and 3 dimension versions with d^*d as div grad, grad div, and curl curl. The primal formulation for this problem is derived as follows: Solve $-d^*du = f$ weakly. i.e. $\langle -d^*du, v\rangle = \langle f, v\rangle$ for all $v \in V$. This, in turn, can be written as a first-order system using the weak form:

$$\langle du, dv\rangle - \langle du, v\rangle_\partial = \langle f, v\rangle$$

where the $\langle , \rangle_\partial$ denotes inner product on the boundaries. The explicit formulation of this term comes from the Divergence/Stokes/Green's theorem and depends on the dimension of the domain and on the derivative d.

2 Postprocessing

With primal hybrid method, we end up with an approximate solution $u_h \in V_h$ to the function $-d^*du = f$. In most physically relevant problems, however, we are looking both for u and for $\sigma = -du$.

Now, rather than evaluating a numerical derivative of the computed solution u_h and leaving it at that, we post-process to achieve a much higher order of accuracy solution σ_h for $-du$.

There are many methods used for improving the approximation for σ_h using the information gained from solving for u_h. See, e.g., [2]. The one that we are considering here uses the trace values as well as computed values for u_h.

So, given the approximation u_h, we then solve the minimization problem: minimize the functional

$$J(\sigma) = \frac{1}{2}\left(||\sigma + du_h||^2 + ||d\sigma - f||^2\right)$$

over all $\sigma \in \Sigma_h$ subject to the constraint: $a(u_h, v_h) + b(v_h, \sigma) = \langle f, v_h\rangle$ which, when we take the variation of the derivatives of both equations w.r.t. σ_h and combine, is to solve:

$$\begin{aligned} \langle \sigma_h + u_h', \tau\rangle + \langle \sigma_h' - f, \tau'\rangle &= -b(\lambda_h, \tau) \quad && \forall \tau \\ a(u_h, v_h) + b(v_h, \sigma_h) &= \langle f, v_h\rangle && \forall v_h \end{aligned}$$

3 Lagrange Multipliers

Recall the basic idea of Lagrange Multipliers from Calculus: to minimize $F(x)$ subject to the constraint $G(x) = H(x)$, solve $\nabla F = \lambda \nabla G$

In this case, we want to solve $\nabla J(\sigma) = \lambda \nabla b(\sigma, v)$ while enforcing the original equation constraints on σ. So we solve for (σ_h, u_h):

$$\langle \nabla J(\sigma_h), \tau \rangle = \lambda \nabla b(\sigma, v) \quad \forall \tau \in \Sigma_h$$

and at the same time:

$$\langle u_h, v \rangle - \langle v, \sigma_h \rangle_\partial = \langle f, v \rangle, \quad \langle u_h, \tau \rangle_\partial = 0$$

4 grad div

4.1 2D

4.2 3D

5 div grad

The div grad problem does not produce the same superconvergence as the grad div problem.

6 curl curl

The form for the curl curl problem is: find $u \in V$ satisfying

$$\nabla \times (\nabla \times u) = f \quad \text{on } \Omega$$

The primal form is then find u satisfying:

$$\int_\Omega (\nabla \times u) \cdot (\nabla \times v) = \int_\Omega f \cdot v \quad \forall v \in V$$

Note that in most cases, curl curl can be problematic, as the nullspace for curl contains $H(div)$, but if we choose our solution and source term to be divergence free, this causes no issue.

6.1 2D

Although curl is technically defined only for 3D, we can interpret the 2D version as a 3D problem restricted to a plane. In that case, we interpret $\nabla \times u = \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \right) \hat{k}$ as a scalar although it is in fact a vector in the normal direction to the plane.

In this case, we solve the equation:

$$\int_\Omega (\nabla \times u_h) \cdot (\nabla \times v_h) = \int_\Omega f \cdot v_h \quad \forall v_h \in V_h$$

where V_h is now considered as continuous vector-valued 2D functions on the discretized domain, with natural boundary conditions.

To post-process, we weaken the continuity constraint on u_h, which means we have discontinuous vector-valued functions for V_h, and continuous scalar functions for $\sigma = \nabla \times u$. Thus the constrained optimization becomes:
minimize

$$\frac{1}{2} \left(||\sigma - \nabla \times u_h||^2 + ||\nabla \times \sigma - f||^2 \right)$$

subject to the constraint:

$$\begin{array}{rcl} \int_\Omega (\nabla \times u_h) \cdot (\nabla \times v) - \int_{\partial\Omega} v \times (\sigma \cdot n) & = & \int_\Omega f \cdot v \\ \int_{\partial\Omega} u \times (\tau \cdot n) & = & 0 \end{array}$$

Note that this actually gives the rotated version of the 2D grad div problem. This is verified in the curl_curl

6.2 3D

In 3D, the primal form for solving for u_h is unchanged from 2D. However, when it comes to post-processing, we now have to use the vector curl identity

$$(\nabla \times \nabla \times u) \cdot v = \nabla \times u \cdot \nabla \times v + \nabla \cdot (v \times \nabla \times u)$$

Note that, unlike the other weak forms, this integration-by-parts identity does not give a boundary term.

Then to post-process, we minimize the usual functional $J(\sigma)$ subject to the constraint:

$$\int_\Omega (\nabla \times u_h) \cdot (\nabla \times v) - \div(v \times \sigma + u \times \tau) = \int_\Omega f \cdot v$$

This does not give superconvergence.

References

[1] Douglas N. Arnold et al. “Unified analysis of discontinuous Galerkin methods for ellipctic problems”. In: *SIAM J. Numer. Anal.* 39.5 (2001/2), pp. 1749–1779.

[2] So-Hsiang Chou, Do Y. Kwak, and Kwang Y. Kim. “Flux recovery from primal hybrid finite element methods”. In: *SIAM J. Numer. Anal.* 40.2 (2002), pp. 403–415.

[3] Bernardo Cockburn, Jayadeep Gopalakrishnan, and Raytcho Lazarov. “Unified analysis of discontinuous Galerkin, mixed, and continuous Galerkin methods for second order elliptic problems”. In: *SIAM J. Numer. Anal.* 47.2 (2009), pp. 1319–1365.