Real Analysis Notebook

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1 Preliminaries

Definition 1.0.1. *A set A is **finite** if there is a 1-1 mapping of some set $\{1, 2, 3, ..., n\}$ of positive integers onto A. A is **countably infinite** if there is a 1-1 mapping of the set $\mathbb{Z}^+$ of positive integers onto A. A is **countable** if either finite or countably infinite. Otherwise A is **uncountable**.*

1.0.2
(deMorgan laws) If $\{E_\lambda : \lambda \in \Lambda\}$ is any indexed collection of subsets of some set E, then

$$\left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^c = \bigcap_{\lambda\in\Lambda} E_\lambda^c \quad \text{and} \quad \left(\bigcap_{\lambda\in\Lambda} E_\lambda\right)^c = \bigcup_{\lambda\in\Lambda} E_\lambda^c$$

where A^c denotes the complement of A in E.

Proof.

$$\left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^c = \{x : x \in E_\lambda \text{ for some } \lambda \in \Lambda\}^c = \{x : x \in E_\lambda^c \forall \lambda \in \Lambda\} = \bigcap_{\lambda\in\Lambda} E_\lambda^c$$

$$\left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^c = \{x : x \in E_\lambda \text{ for some } \lambda \in \Lambda\}^c = \{x : x \in E_\lambda^c \forall \lambda \in \Lambda\} = \bigcap_{\lambda\in\Lambda} E_\lambda^c$$

□

Remark 1.0.3. *In the previous problem the index set Λ need not be countable. One could imagine indexing a collection of sets by the real numbers, for instance (e.g. E_x is the interval of length 1 centered at x.)*

1.0.4
Any countable union of sets of real numbers can be expressed as a disjoint union: $E \cup F = E \cup (F \setminus E)$ or $\bigcup_{k=1}^{\infty} E_k = \bigcup_{k=1}^{\infty} F_k$ where $F_1 = E_1$, $F_k = E_k \setminus \bigcup_{j=1}^{k-1} E_j$. Here $A \setminus B = A \cap B^c$.

Proof. Let E, F be sets of real numbers. Then for any $x \in E \cup F$, if $x \in E$, then $x \in E \subset \{E \cup (F \setminus E)\}$. Similarly, if $x \notin E$ but $x \in F$, then $x \in (F \setminus E) \subset \{E \cup (F \setminus E)\}$. Thus

$$E \cup F \ \subset \ E \cup (F \setminus E).$$

On the other hand, for any $x \in E \cup (F \setminus E)$, if $x \in E$ then $x \in E \cup F$. Also, if $x \in F \setminus E$ then $x \in F \subset (E \cup F)$. Thus

$$E \cup (F \setminus E) \subset E \cup F.$$

$$E \cup F \subsetneqq E \cup (F \setminus E) \quad \Longrightarrow \quad E \cup F = E \cup (F \setminus E).$$

For the more general case, take any $x \in \bigcup_{j=1}^{\infty} F_k$, then there is some F_j such that $x \in F_j$. Since each $F_j \subset E_j$, then $x \in E_j \subset \bigcup_{j=1}^{\infty} E_j$. Therefore $\bigcup_{j=1}^{\infty} F_j \subset \bigcup_{j=1}^{\infty} E_j$.

On the other hand, for any $x \in \bigcup_{j=1}^{\infty} E_j$. Then there is some k such that $x \in E_k$. Let a be the smallest index such that $x \in E_a$. Then $x \in E_a$ and $\forall b < a$, $x \notin E_b$. Therefore $x \in F_a$, and so $x \in \bigcup_{j=1}^{\infty} F_j$. Therefore, since we have shown subset containment in both directions,

$$\bigcup_{j=1}^{\infty} E_j = \bigcup_{j=1}^{\infty} F_j.$$

□

1.0.5
If a and b are real numbers, then $a \leq b$ iff for every $\epsilon > 0$, $a \leq b + \epsilon$. Similarly, $a \geq b$ iff for every $\epsilon > 0$, $a \geq b - \epsilon$.

Proof. Let $a, b \in \mathbb{R}$ and $a \leq b$. Then for every $\epsilon > 0$, $a \leq b < b + \epsilon$, and therefore $a \leq b + \epsilon$. Now, let $a, b \in \mathbb{R}$, and assume that for any $\epsilon > 0$, $a \leq b + \epsilon$. Then if $a > b$, there is some $\epsilon_1 \geq 0$ such that $a = b + \epsilon_1$. But since $a \leq b + \epsilon$ for all $\epsilon > 0$, we have $a \leq b + \frac{\epsilon}{2} < b + \epsilon_1 = a$ which is a contradiction. Therefore $a \leq b$.

Now let $a \geq b$. This can be re-written as $b \leq a$. By the first part of the problem, $b \leq a$ iff $\forall \epsilon > 0$, $b \leq a + \epsilon$. This is true iff $b - \epsilon \leq a$. □

Remark 1.0.6. *This problem may seem trivial, but we will use it over and over again during the quarter. It says that we can give a little something away, and then take it back.*

Definition 1.0.7.

(a) *A set E of real numbers is **bounded above** if there exists a real number u such that for every $x \in E$, $x \leq u$. we call u an **upper bound** for E. Similarly, E is **bounded below** if there exists a real number w such that for every $x \in E$, $w \leq x$, and then w is a **lower bound** for E. E is **bounded**, if bounded both above and below.*

(b) *If E is bounded above and nonempty, the **supremum** (or least upper bound) of E, sup E, is the unique real number s such that (i) s is an upper bound for E and (ii) $s < u$ for any other upper bound u for E. Similarly, if E is bounded below and nonempty, the **infimum** (or greatest lower bound) of E, inf E, is the unique real number t such that (i) t is a lower bound for E and (ii) $t > w$ for any other lower bound w for E. (Recall that the Least Upper Bound Axiom guarantees the existence of the supremum and infimum.)*

(c) *A real number c is a **cluster point** of a set E of real numbers if for every $\epsilon > 0$ there is $y \in E$ such that $0 < |c - y| < \epsilon$.*

1.0.8
Let v be a lower bound for a set E of real numbers, where $v \notin E$. Then $v = \inf E$ iff v is a cluster point of E.

Proof. Let $v = \inf E$, $v \notin E$. Then for any $\epsilon > 0$, there is $y \in E$ such that $y < v + \epsilon$, as otherwise $v + \epsilon$ would be a lower bound of E greater than $v = \inf E$. Now, for any $\epsilon > 0$, we have $y \in E$ such that

$$v < y < v + \epsilon$$

Subtracting v from each inequality, this can be re-written as

$$0 < y - v < \epsilon$$

and since $y > v$, this is

$$0 < |y - v| < \epsilon \quad \Leftrightarrow \quad 0 < |v - y| < \epsilon$$

Thus v is a cluster point of E.

Now, assume that v is a cluster point of E, $v \notin E$, and v is a L.B. for E.

For any $w > v$, $w = v + \epsilon$ for some $\epsilon > 0$, there is $y \in E$ such that $v < y < v + \epsilon$. Therefore w is not a lower bound of E. Thus $v = \inf E$. $\square$

Definition 1.0.9.

(a) *A* ***sequence*** *of real numbers* $\{a_k\}_{k=1}^{\infty}$ *is a function from the set of positive integers* $\mathbb{Z}^+$ *into the real numbers. (More generally, the domain of a sequence can be any set of the form* $\{k : k \geq k_0\}$ *for some integer* k_0*.) The sequence is* ***bounded*** *(or bounded above or ...) if its range is bounded (or ...).*

(b) *The sequence* $\{a_k\}_{k=1}^{\infty}$ ***converges*** *to the* ***limit*** a *if for each* $\epsilon > 0$ *there is a positive integer* K *such that* $|a_k - a| < \epsilon$ *for all* $k \geq K$*.*

(c) *A* ***subsequence*** $\{a_{k_j}\}_{j=1}^{\infty}$ *of a sequence* $\{a_k\}_{k=1}^{\infty}$ *is the composition of* $\{a_k\}_{k=1}^{\infty}$ *with an increasing sequence* $\{k_j\}_{j=1}^{\infty}$ *of integers*

1.0.10

Let $\{a_k\}_{k=1}^{\infty}$ be a non-decreasing sequence of real numbers. Then $\{a_k\}_{k=1}^{\infty}$ is bounded above iff $\{a_k\}_{k=1}^{\infty}$ converges to the least upper bound of the set $\{a_k : k \in \mathbb{Z}^+\}$. (Of course there is a symmetric result for non-increasing sequences.)

Proof. Let $\{a_k\}_{k=1}^{\infty}$ be bounded above. Now, let L be the least upper bound of the set $\{a_k : k \in \mathbb{Z}^+\}$. Since $\{a_k\}$ is a non-decreasing sequence, for each $k \in \mathbb{Z}^+$, $a_k \leq a_{k+1} \leq a_{k+2} \leq ...$ And also, for all $k \in \mathbb{Z}^+$, $a_k \leq L$ since L is the least upper bound. Therefore for all $k \in \mathbb{Z}^+$,

$$a_k \leq a_{k+1} \leq \cdots \leq L$$

For any $\epsilon > 0$, let $a^* = L - \epsilon < L$. Then since L is the least upper bound of the set $\{a_k : k \in \mathbb{Z}^+\}$, we know that $a^* \leq a_j$ for some $j \in \mathbb{Z}^+$. Therefore (using the fact that the sequence is non-decreasing) for all $k \geq j$,

$$L \geq a_k \geq L - \epsilon$$

which can be re-written as

$$0 \leq |L - a_k| \leq \epsilon$$

Thus $\{a_k\}_{k=1}^{\infty}$ converges to the least upper bound

Now, if $\{a_k\}_{k=1}^{\infty}$ converges to the L.U.B L of $\{a_k : k \in \mathbb{Z}^+\}$ and $a_k \leq a_{k+1}$ for all $k \in \mathbb{Z}^+$, then for all $k \in \mathbb{Z}^+$, we have

$$a_k \leq a_{k+1} \leq \cdots \leq L$$

Thus $\{a_k\}_{k=1}^{\infty}$ is bounded above. $\square$

Remark 1.0.11.

1. *Note the distinction between the sequence* $\{a_k\}_{k=1}^{\infty}$*, which as a function is a set of ordered pairs of real numbers, and the set* $\{a_k : k \in \mathbb{Z}^+\} \subset \mathbb{R}$ *which is the range of that function.*
2. *If* $\{a_k\}_{k=1}^{\infty}$ *is non-decreasing and not bounded above, we often write* $\lim_{k\to\infty} a_k = \infty$ *and speak as though* ∞ *were a number "way out there at the end of the number line." We do not, however, do arithmetic with* ∞*.*

Definition 1.0.12.

1. $\{a_k\}_{k=1}^{\infty}$ *be a sequence of real numbers. The numbers* $s_n = \sum_{k=1}^{n} a_k$ *are the* ***partial sums of the infinite series*** $\sum_{k=1}^{\infty} a_k$*. The series* ***converges*** *if* $\lim_{n\to\infty} s_n$ *exists (as a real number). Otherwise the series* ***diverges****. If the series converges, the number* $s = \lim_{n\to\infty} s_n$ *is called the* ***sum of the infinite series*** *and is denoted* $\sum_{k=1}^{\infty} a_k$*.*
2. *More generally, if* S *is any set of positive integers, and for any positive integer* n*,* $S_n = \{k \in \mathbb{Z}^+ : k \in S \text{ and } k \leq n\}$*, we define the finite sums* $s_n = \sum_{k\in S_n} a_k$ *to be the* ***partial sums*** *of the series* $\sum_{k\in S} a_k$*, and say that the series* ***converges*** *if* $\lim_{n\to\infty} s_n$ *exists (as a real number). Otherwise the series* ***diverges****. If the series converges, the number* $s = \lim_{n\to\infty} s_n$ *is called the* ***sum of the series*** *and is denoted* $\sum_{k\in S} a_k$*.*

1.0.13

Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of non-negative reals. Let $\{F_j\}_{j=1}^{\infty}$ be any nested sequence of sets of positive integers such that $F_1 \subset F_2 \subset ...$ and $\bigcup_{j=1}^{\infty} F_j = \mathbb{Z}^+$. For each j let $f_j = \sum_{k\in F_j} a_k$. Then $\sup_j f_j = \sum_{k=1}^{\infty} a_k$. (This means that if either number is finite, then they both are and and then they are equal. Notice that it is not assumed that the sets F_j are finite, so the f_j need not be finite.)

Proof. $\sum_{k=1}^{\infty} a_k = \sum_{k\in\cup_{j=1}^{\infty} F_j} a_k \in \{f_j\}_{j=1}^{\infty}$ And so, by definition

$$\sum_{k=1}^{\infty} a_k \quad \leq \quad \sup_j f_j$$

Now, since $\forall j \in \mathbb{Z}^+$, $F_j \subset F_{j+1}$, then

$$f_j = \sum_{k\in F_j} a_k \leq \sum_{k\in F_{j+1}} a_k = f_{j+1}$$

Now, if the F_j are finite, then using the fact that $\{f_j\}_{j=1}^{\infty}$ is a sequence of non-decreasing positive real values, we have

$$f_j = \sum_{k \in F_j} a_k \leq \lim_{n\to\infty} f_n = \sum_{k=1}^{\infty} a_k \quad \forall j \in \mathbb{Z}^+$$

On the other hand, if the F_j are not necessarily finite, the values can be re-ordered in increasing order, and then each f_j can be written as the limit of a sequence of partial sums $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} a_{j(k)}$ where $F_j = \{j(k)\}_{k=1}^{\infty}$. Then all of the partial sums are bounded above by $\sum\limits_{k=1}^{\infty} a_k$, and so $f_j \leq \sum\limits_{k=1}^{\infty} a_k$ for all j and irrespective whether the F_j are finite or infinite.

Therefore $\sum\limits_{k=1}^{\infty} a_k$ is an upper bound of $\{f_j\}_{j=1}^{\infty}$, and so

$$\sup_j f_j \leq \sum_{k \in F_j} a_k$$

□

1.0.14
Consider the double indexed set $\{a_{j,k}\}_{j,k=1}^{\infty}$ of non-negative numbers. Set $S_n = \sum\limits_{j,k=1}^{n} a_{j,k}$. (This means the sum of the n^2 terms $a_{j,k}$ where $1 \leq j \leq n$ and $1 \leq k \leq n$.) Let $A_1 = \lim\limits_{n\to\infty} S_n$, $A_2 = \sum\limits_{j=1}^{\infty}\sum\limits_{k=1}^{\infty} a_{j,k}$, $A_3 = \sum\limits_{k=1}^{\infty}\sum\limits_{j=1}^{\infty} a_{j,k}$. Then $A_1 = A_2 = A_3$. This means that if any one of these three numbers is finite, then all of them are and they are equal. (This is, of course, the equivalent for series of the fact that a double integral can be evaluated as an iterated integral. For instance, A_2 if it exists, is the limit as $n \to \infty$ of $\sum\limits_{j=1}^{\infty}\sum\limits_{k=1}^{\infty} a_{j,k}$.)

Proof. Now, as a sum of non-negative numbers for any n,

$$S_n = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{j,k} \leq \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{j,k} = A_2$$

which implies that, since this is true for any n, $A_1 \leq A_2$.

Now to show that $A_2 \leq A_1$, we will consider the sequence $\{T_n\}_{n=1}^{\infty} = \{\sum\limits_{j=1}^{n}\sum\limits_{k=1}^{\infty} a_{j,k}\}_{n=1}^{\infty}$. This is a sequence of non-decreasing positive values with $\lim\limits_{n\to\infty} T_n = A_2$. Also for all $n \in \mathbb{Z}^+$,

$$\sum_{j=1}^{n}\sum_{k=1}^{\infty} a_{j,k} \leq A_1$$

And so $A_2 \leq A_1$.

Note that $S_n = \sum_{j=1}^{n} \sum_{k=1}^{n} a_{j,k} = \sum_{k=1}^{n} \sum_{j=1}^{n} a_{j,k}$ As a finite sum, and that, since this is true for all n, then showing that $A_1 = A_2$ is equivalent to showing $A_1 = A_3$

□

Definition 1.0.15.

*(a) An **open ball** about a real number x of radius r is the set $B_r(x) = (x-r, x+r)$.*

*(b) A set G of real numbers is **open** if it contains an open ball about each of its points.*

*(c) A set F of real numbers is **closed** if every cluster point of F is contained in F.*

1.0.16
Every open subset G of real numbers can be expressed as a countable union of pairwise disjoint open intervals. (Think about the rationals in G.)

Proof. Consider all of the the rational numbers in G. This is a countable set, so indexing by any subset of this set will also be countable.

Now, for all rational $x \in G$, let I_x be the largest open interval in G containing x. For any $y \in G$ such that y is not rational, Because G is open, we can find $\epsilon > 0$ such that $y \in (x - \epsilon, x + \epsilon) \subset G$ for some rational $x \in G$. Then the set $\{I_x : x \in G \text{ is rational}\}$ is an open cover of G.

To find a subset that are pairwise disjoint and that covers all of G, consider any two rational numbers $x, y \in G$. If $I_x \cap I_y \neq \emptyset$, then by the maximality of I_x and I_y $x \in I_x \cup I_y \subset G$, and so $I_x = I_x \cup I_y = I_y$. Thus every pair of intervals I_x, I_y is either disjoint or identical intervals. □

Definition 1.0.17. *The set E of real numbers is **compact** if every sequence $\{a_k\}_{k=1}^{\infty}$ of elements of E has a subsequence that converges to an element of E.*

Fact 1.0.18. *These properties are equivalent for a set E of real numbers.*

(a) E is compact.

(b) E is closed and bounded.

(c) Every infinite subset of E has a cluster point in E.

*(d) If $\{G_\lambda : \lambda \in \Lambda\}$ is a collection of open sets such that $E \subset \bigcup_{\lambda \in \Lambda} G_\lambda$ (an **open cover** of E), then some finite subcollection is also an open cover of E.*

2 Measure on $[0, 1]$

Definition 2.0.1. *For any subset E of $[0, 1]$, the* ***outer measure*** *of E is*

$$m^*(E) = \inf \left\{ \sum_{j=1}^{\infty} \ell(G_j) : E \subset \bigcup_{j=1}^{\infty} G_j \right\}$$

where each $G_j = (a_j, b_j)$ is an open interval in $\mathbb{R}$ and $\ell(G_j) = b_j - a_j$. (Note: this includes finite sums by setting all $G_j = \emptyset$ from some point on. Also note the G_j need not be contained in $[0, 1]$, e.g. in considering $m^*(\{1\})$).

Remark 2.0.2.

(a) We could use coverings by closed intervals and get the same result. It is clear that we could get numbers not larger than we have. (Replace $G_j = (a_j, b_j)$ by $F_j = [a_j, b_j]$.) To see not smaller, replace closed F_j by open G_j where $\ell(G_j) = \ell(F_j) + \epsilon/2^j$.

(b) There is also something called inner measure. But it is not needed, so we won't use it.

2.0.3
For any subset E of $[0, 1]$,

(a) $0 \leq m^*(E) \leq 1$.

(b) $E \subset F \Rightarrow m^*(E) \leq m^*(F)$.

(c) $m^*(\emptyset) = m^*(\text{one point}) = 0$.

Proof.

(a) For any $\epsilon > 0$, consider the open interval $(-\frac{\epsilon}{2}, 1 + \frac{\epsilon}{2})$ that has length $1 + \epsilon$. This is an open cover of $[0, 1]$, and so it is also an open cover of $E \subset [0, 1]$. Now, since ϵ was arbitrarily chosen, $m^*(E) \leq 1$.

Also, since $m^*(E)$ is the infimum of a set of non-negative numbers, $m^*(E) \geq 0$.

So $0 \leq m^*(E) \leq 1$ for all $E \subset [0, 1]$.

(b) Let $H = \left\{ \sum_{j=1}^{\infty} \ell(G_j) : F \subset \bigcup_{j=1}^{\infty} G_j \right\}$ and let $G = \left\{ \sum_{j=1}^{\infty} \ell(G_j) : E \subset \bigcup_{j=1}^{\infty} G_j \right\}$. Since $E \subset F$, we know that $H \subset G$. Therefore $\inf(G) \leq \inf(H)$.

(c) For any $x \in [0,1]$, consider $(x-\epsilon/2, x+\epsilon/2)$ for any $\epsilon > 0$. This is an open cover of $\{x\}$ with length ϵ. Using the fact that ϵ was arbitrarily chosen, the infimum of sums of lengths of intervals must be less than or equal to ϵ, which implies that $m^*(\{x\}) \leq 0$.

Now, since $\emptyset \subset \{x\}$, we can use part (b), which gives $m^*(\emptyset) \leq m^*(\{x\}) \leq 0$.

□

2.0.4
If E is a countable set, then $m^*(E) = 0$. In particular, $m^*(\mathbb{Q}_0) = 0$ where $\mathbb{Q}_0 = \mathbb{Q} \cap [0,1]$ is the set of all rational numbers in $[0,1]$.

(Given $\epsilon > 0$ cover $E = \{x_k\}_{k=1}^{\infty}$ with $\bigcup_{j=1}^{\infty} G_j$ where $\ell(G_j) = \epsilon/2^j$.)

Proof. Suppose E is countable. Then we can write $E = \{x_i\}_{i=1}^{\infty}$. Then for any $\epsilon > 0$, define $G_i = (x_i - \frac{\epsilon}{2^{i+1}}, x_i + \frac{\epsilon}{2^{i+1}})$. Then $\ell(G_i) = 2\frac{\epsilon}{2^{i+1}} = \frac{\epsilon}{2^i}$. Therefore $\bigcup_{i=1}^{\infty} G_i$ is an open cover of E, and

$$\sum_{i=1}^{\infty} \ell(G_i) = \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} = \epsilon.$$

Therefore

$$m^*(E) \leq \inf \left\{ \sum_{i=1}^{\infty} \ell(G_i) : E \subset \bigcup_{i=1}^{\infty} G_i \right\} = 0.$$

and so $m^*(E) = 0$. □

Remark 2.0.5. *This is the first evidence that covering E with countably infinite unions rather than finite unions makes a big difference. This is, in fact exactly the innovation that will allow us to deal much more efficiently with infinite processes and with irregular sets.*

2.0.6
If $F = [a,b]$ is a closed interval in $[0,1]$, then $m^*(F) = b - a$. ($m^*(F) \leq b - a$ is clear from $G = (a - \epsilon, b + \epsilon)$. F is compact, so any cover has a finite subcover $\bigcup_{j=1}^{n} G_j$. May assume that no element of this subcover can be omitted (Explain why.) Order them by their left endpoints and show $\sum_{j=1}^{n} \ell(G_j) > b - a$.)

Proof. First we will show that $m^*(F) \leq (b-a)$ and then that $m^*(F) \geq (b-a)$.

Now, $m^*(F) = \inf \left\{ \sum_{j=1}^{\infty} \ell(G_j) : F \subset \bigcup_{j=1}^{\infty} G_j \right\}$, and so if we consider an arbitrary open cover $\{G_j\}_{j=1}^{\infty}$ of F, $m^*(F) \leq \sum_{j=1}^{\infty} \ell(G_j)$. In particular, Let us consider the open cover $(a - \epsilon/2, b + \epsilon/2)$ for any $\epsilon > 0$. This has length $(b-a) + \epsilon$. Thus $m^*(F) \leq (b-a) + \epsilon$ for all $\epsilon > 0$, and so

$$m^*(F) \leq b - a$$

Now, let $\{G_j\}_{j=1}^{n}$ be any finite subcover of F. Then if there are $G_k = (a_k, b_k), G_l = (a_l, b_l) \in [0,1]$ such that $G_k \subset G_l$, then we can throw out G_k.

Thus we can remove all extraneous G_k, and so if we reorder the elements of $\{G_j\}_{j=1}^{n}$ in increasing order such that $a_0 < a$ and $b < b_n$. With the increasing order for all j, $a_j < a_{j+1}$ and $a_{j+1} < b_j < b_{j+1}$, since otherwise we would have $(a_j, b_j) \subset (a_{j+1}, b_{j+1})$. Then

$$\sum_{j=1}^{n} \ell(G_j) = \sum_{j=1}^{n} (b_j - a_j)$$

This can be re-written as

$$\sum_{j=1}^{n} (b_j - a_j) = (b_n - a_1) + (b_{n-1} - a_n) + (b_{n-2} - a_{n-1}) + \cdots$$

Now, since for all j, $b_j > a_{j+1}$, this is

$$b_n - a_1 + \epsilon$$

for some $\epsilon > 0$. Now, since $b = b_n + \epsilon_b$ and $a_1 = a - \epsilon_b$ for some $\epsilon_a, \epsilon_b > 0$, we have

$$b_n + a_1 + \epsilon \geq b_n + a_1 = b - a + (\epsilon_b + \epsilon_a) \geq b - a.$$

Continuing this with $\sum_{j=1}^{n} \ell(G_j)$, we end with

$$\sum_{j=1}^{n} \ell(G_j) \geq b - a$$

Since this open cover was arbitrarily chosen, $(b-a)$ is a lower bound, which means that since $m^*(F)$ is the infimum,

$$m^*(F) = \inf \left\{ \sum_{j=1}^{\infty} \ell(G_j) : F \subset \sum_{j=1}^{\infty} G_j \right\} \geq (b-a)$$

$\square$

2.0.7
If $E, F \subset [0,1]$, $m^*(E \cup F) \leq m^*(E) + m^*(F)$.

Proof. Since the outer measure of any set m^* is an infimum, it is a cluster point. Therefore for any $\epsilon > 0$, there are open covers $\{G_j\}_{j=1}^{\infty}$ and $\{H_j\}_{j=1}^{\infty}$ such that

$$\sum_{j=1}^{\infty} \ell(G_j) = m^*(E) + \frac{\epsilon}{2} \quad \text{and} \quad \sum_{j=1}^{\infty} \ell(H_j) = m^*(F) + \frac{\epsilon}{2}$$

Now, since $E \subset \bigcup_{j=1}^{\infty} G_j$ and $F \subset \bigcup_{j=1}^{\infty} H_j$, $E \cup F \subset \left(\bigcup_{j=1}^{\infty} G_j\right) \cup \left(\bigcup_{j=1}^{\infty} H_j\right)$ and so

$$\begin{aligned} m^*(E \cup F) \quad &\leq \quad \sum_{j=1}^{\infty} \ell(G_j) + \sum_{j=1}^{\infty} \ell(H_j) \quad = \quad m^*(E) + \tfrac{\epsilon}{2} + m^*(F) + \tfrac{\epsilon}{2} \\ & \quad = \quad m^*(E) + m^*(F) + \epsilon \end{aligned}$$

Thus $m^*(E \cup F) \leq m^*(E) + m^*(F)$. □

2.0.8
For any subsets $\{E_j\}_{j=1}^{\infty}$ of $[0,1]$, $m^*\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} m^*(E_j)$. We refer to this property of outer measure as **countable subadditivity**.

Proof. Note that for any set of open covers of all of the E_j is an open cover of $\cup E_j$. That is, for $\left\{\{I_{j_k}\}_{k=1}^{\infty} : E_j \subset \bigcup_{k=1}^{\infty} I_{j_k}\right\}$, $\bigcup_{j=1}^{\infty} E_j \subset \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} I_{j_k}$.

For any $\{E_j\}_{j=1}^{\infty}$, consider any open covers $\{I_{j_k}\}_{k=1}^{\infty}$ of each E_j. Then since each $m^*(E_j)$ is a cluster point, then for any $\epsilon > 0$ there is $\{I_{j_k}\}_{k=1}^{\infty}$ such that

$$\sum_{k=1}^{\infty} \ell(I_{j_k}) \leq m^*(E_j) + \frac{\epsilon}{2^j}$$

And so for any $\epsilon > 0$, we can choose open covers of each of the E_j such that

$$m^*\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \ell(I_{j_k}) \leq \sum_{j=1}^{\infty} \left(m^*(E_j) + \frac{\epsilon}{2^j}\right) = \sum_{j=1}^{\infty} m^*(E_j) + \epsilon$$

Therefore by (5),

$$m^*\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} m^*(E_j)$$

□

Definition 2.0.9. *A subset E of $[0,1]$ is* ***measurable*** *if for each $A \subset [0,1]$,*

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c).$$

Notation 2.0.10. *If E is measurable, we write $m(E)$ for $m^*(E)$ and refer to this number as the* ***Lebesgue measure*** *of E.*

Remark 2.0.11.

(a) From the previous problem, to show that E is measurable one need only prove $m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c)$ since the other direction is automatic.

(b) In particular, taking $A = [0,1], 1 = m^*(E) + m^*(E^c)$ is a necessary condition for measurability. It turns out that this special case of the definition is also sufficient. The definition is the way it is (rather than this simpler version) just because it is a little more convenient to use.

2.0.12

(a) $\emptyset$ is measurable. $[0,1]$ is measurable.

(b) $m^*(E) = 0 \Rightarrow E$ is measurable.

Proof.

(a) In $[0,1]$, $\emptyset^c = [0,1]$, and $[0,1]^c = \emptyset$. Therefore for any $A \subset [0,1]$,

$$\begin{aligned} m^*(A) &= m^*(A \cap [0,1]) &&= m^*(A \cap [0,1]) + 0 \\ &= m^*(A \cap [0,1]) + m^*(\emptyset) &&= m^*(A \cap [0,1]) + m^*(A \cap \emptyset) \end{aligned}$$

Thus $[0,1]$ is measurable. Similarly,

$$\begin{aligned} m^*(A) &= 0 + m^*(A \cap [0,1]) \\ &= m^*(\emptyset) + m^*(A \cap [0,1]) \\ &= m^*(A \cap \emptyset) + m^*(A \cap [0,1]) \end{aligned}$$

Thus $\emptyset$ is measurable.

(b) If $m^*(E) = 0$, then for any set A, $A \cap E \subset E$, and since $A \cap E^c \subset A$, we have (by 21 (b)):

$$\begin{aligned} m^*(A \cap E^c) &\leq m^*(A) \\ 0 + m^*(A \cap E^c) &\leq m^*(A) \\ m^*(A \cap E) + m^*(A \cap E^c) &\leq m^*(A) \end{aligned}$$

Therefore, using the observation in 29(a), E is measurable.

□

Remark 2.0.13. *In particular, every subset of a set of outer measure 0 is measurable.*

2.0.14
E is measurable $\Leftrightarrow$ E^c is measurable.

Proof.

$$\begin{aligned}
E \text{ is measurable} &\Leftrightarrow \forall A \subset [0,1], m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)\\
&\Leftrightarrow \forall A \subset [0,1], m^*(A) = m^*(A \cap E^c) + m^*(A \cap E)\\
&\Leftrightarrow \forall A \subset [0,1], m^*(A) = m^*(A \cap E^c) + m^*(A \cap (E^c)^c)\\
&\Leftrightarrow E^c \text{ is measurable}
\end{aligned}$$

□

2.0.15
A closed interval $[a,b]$ is measurable with measure $b-a$.

(Show $m^*(A) \geq m^*(A \cap [a,b]) + m^*(A \cap [a,b]^c) - \epsilon$ for every ϵ. If $\bigcup_{j=1}^{\infty} G_j$ covers A with $\bigcup_{j=1}^{\infty} \ell(G_j)$ near $m^*(A)$, then the $G_j \cap [a,b]$ are nearly an open cover of $A \cap [a,b]$ by open intervals and similarly for $G_j \cap [a,b]^c$.)

Proof. Let $A \subset [0,1]$. Then for any $\epsilon > 0$, choose an open cover $\{G_j\}_{j=1}^{\infty}$ of A such that $\sum_{j=1}^{\infty} \ell(G_j) \leq m^*(A) + \frac{\epsilon}{2}$.

Define $F_1 = \{G_k \cap (a,b)\} \cup (a - \frac{\epsilon}{8}, a + \frac{\epsilon}{8}) \cup (b - \frac{\epsilon}{8}, b + \frac{\epsilon}{8})$, $F_2 = \{G_k \cap [a,b]^c\}$,

Then $A \cap [a,b] \subset F_1$ and $A \cap [a,b]^c \subset F_2$. (note that the elements of F_2 are either open intervals or else sets of the form $(x,a) \cup (b,y)$ for some $x, y \in \mathbb{R}$. Therefore the notion of length is still well-defined on F_2 and F_2 is still a collection of open intervals, and thus an open cover)

Since $\{O : O \subset F_1\}$ is an open cover of $A \cap [a,b]$ and $\{O : O \subset F_2\}$ is an open cover of $A \cap [a,b]^c$, Therefore, $m^*(A \cap [a,b]) \leq \sum_{O \in F_1} \ell(O)$ and $m^*(A \cap [a,b]^c) \leq \sum_{O \in F_2} \ell(O)$. Also, since the only intersection of F_1 and F_2 is in the $(a - \frac{\epsilon}{8}, a + \frac{\epsilon}{8})$, $(b - \frac{\epsilon}{8}, b + \frac{\epsilon}{8})$ terms, then

$$\sum_{O \in F_1 \cup F_2} \ell(O) \leq \sum_{j=1}^{\infty} \ell(G_j) + \ell(a - \frac{\epsilon}{8}, a + \frac{\epsilon}{8}) + \ell(b - \frac{\epsilon}{8}, b + \frac{\epsilon}{8}) = \sum_{j=1}^{\infty} \ell(G_j) + \frac{\epsilon}{2}$$

Putting this together with the upper bounds for measures of $A \cap [a,b]$ and $A \cap [a,b]^c$, we get

$$m^*(A \cap [a,b]) + m^*(A \cap [a,b]^c) \leq \sum_{O \in F_1 \cup F_2} \ell(O) = \sum_{k=1}^{\infty} \ell(G_k) + \frac{\epsilon}{2}$$

And since $\sum_{j=1}^{\infty} \ell(G_j) \leq m^*(A) + \frac{\epsilon}{2}$, adding $\frac{\epsilon}{2}$ to both sides of the inequality gives $\sum_{j=1}^{\infty} \ell(G_j) + \frac{\epsilon}{2} \leq m^*(A) + \epsilon$, which we can substitute into the previous equation to yield

$$m^*(A \cap [a,b]) + m^*(A \cap [a,b]^c) \leq \sum_{k=1}^{\infty} \ell(G_k) + \frac{\epsilon}{2} \leq m^*(A) + \epsilon$$

and so $m^*(A \cap [a,b]) + m^*(A \cap [a,b]^c) \leq m^*(A) + \epsilon$, which can be re-written

$$m^*(A \cap [a,b]) + m^*(A \cap [a,b]^c) - \epsilon \leq m^*(A)$$

For any $A \subset [0,1]$ for all $\epsilon > 0$. □

2.0.16
If E and F are measurable, then so are $E \cap F$, $E \cap F^c$, $E^c \cap F$, and $E^c \cap F^c$.

(To show $m^*(A) \geq m^*(A \cap E \cap F) + m^*(A \cap (E \cap F)^c)$ use the measurability of F with test set $A \cap E$. It is not necessary to go back to the level of open covers.)

Proof. Note first that For any sets A, B, C,

$$\{A \cap B \cap C^c\} \cup \{A \cap B^c \cap C\} \cup \{A \cap B^c \cap C^c\} = A \cap (B^c \cup C^c) = A \cap (B \cap C)^c$$

Now, since E is measurable, for any $A \subset [0,1]$,

$$\begin{aligned} m^*(A) &= m^*(A \cap E) + m^*(A \cap E^c) \\ &= m^*(A \cap E \cap F) + m^*(A \cap E \cap F^c) \\ &\quad + m^*(A \cap E^c \cap F) + m^*(A \cap E^c \cap F^c) \\ &\geq m^*(A \cap E \cap F) + m^*((A \cap E \cap F^c) \cup (A \cap E^c \cap F) \cup (A \cap E^c \cap F^c)) \\ &= m^*(A \cap E \cap F) + m^*(A \cap (E \cap F)^c) \end{aligned}$$

Now, since E is measurable, E^c is measurable, and similarly for F^c. Also, Since the intersection of two arbitrary measurable sets is measurable (just shown), then $E^c \cap F$, $E \cap F^c$ and $E^c \cap F^c$ are all measurable given that E and F are. □

2.0.17
If E and F are measurable, then so is $E \cup F$. If E and F are disjoint, then $m(E \cup F) = m(E) + m(F)$.

Proof. if E and F are measurable, then so are E^c and F^c, and so $E^c \cap F^c$ is also. Therefore $E^c \cap F^c = (E \cup F)^c$ is also measurable, so $E \cup F$ is also measurable.

Now, if E and F are measurable and disjoint, then taking the set $E \cup F$ together with the measurability of E, (and given that $E \cup F$ is measurable also)

$$\begin{aligned} m(E \cup F) &= m((E \cup F) \cap E) + m((E \cup F) \cap E^c) \\ &= m(E) + m(F) \end{aligned}$$

□

2.0.18
Any interval $\langle a, b\rangle$ with any choice of endpoints is measurable with measure $b - a$.

Proof. From problem 33, $[a, b]$ is measurable with measure $b-a$. On the other hand, if we consider the set (a, b), $m^*((a, b)) \leq m^*([a, b]) = b - a$

Now, from problem 21 (c), $m^*(\{a\}) = m^*(\{b\}) = 0$, and so by 30(b), $\{a\}$ and $\{b\}$ are both measurable sets with measure 0. Therefore, since (by 35) $(a, b) \cap \{a\} = (a, b) \cap \{b\} = \emptyset$, and

$$m^*((a, b)) = m^*((a, b) \cup \{a\} \cup \{b\}) = m^*([a, b]) = b - a$$

$$m^*([\ a, b\ )) = m^*((a, b) \cup \{a\}) = m^*((a, b)) + m^*(\{a\}) = m(a, b) = b - a$$

$$m^*((\ a, b\ ]) = m^*((a, b) \cup \{b\}) = m^*((a, b)) + m^*(\{b\}) = m(a, b) = b - a$$

□

2.0.19
Any finite union or intersection of measurable sets is measurable. If $\{E_k\}_{k=1}^n$ are measurable and disjoint, then $m\left(\bigcup\limits_{k=1}^n E_k\right) = \sum\limits_{k=1}^n m(E_k)$. (Induction).

Proof.
- Base case: proven in problem 35
- Induction step: Given that $m\left(\bigcup\limits_{k=1}^n E_k\right) = \sum\limits_{k=1}^n m(E_k)$, and given that E_{n+1} is measurable, then let $F = \bigcup\limits_{k=1}^n E_k$. Since E_{n+1} and F are both measurable, then by 35, $\bigcup\limits_{k=1}^{n+1} E_k = E \cup F$ is also measurable. If E_{n+1} and F are disjoint, then by 35,

$$m\left(\bigcup_{k=1}^{n+1} E_k\right) = m(E \cup F) = m(E) + m(F) = \sum_{k=1}^{n+1} m(E_k)$$

□

2.0.20
Let $E_1, E_2, ..., E_n$ be pairwise disjoint and measurable. Then for any A,

$$m^*(A) = \sum_{k=1}^{n} m^*(A \cap E_k) + m^*\left(A \cap \left(\bigcup_{k=1}^{n} E_k\right)^c\right).$$

Proof. Base Case: Let E_1, E_2 be pairwise disjoint and measurable. Then for any A,

$$\begin{array}{rcccc} m^*(A) & = & m^*(A \cap (E_1 \cup E_2)) & + & m^*(A \cap (E_1 \cup E_2)^c) \\ & = & m^*((A \cap E_1) \cup (A \cap E_2)) & + & m^*(A \cap (E_1 \cup E_2)^c) \\ & = & m^*(A \cap E_1) + m^*(A \cap E_2) & + & m^*(A \cap (E_1 \cup E_2)^c) \end{array}$$

Now, assume that $E_1, E_2, ..., E_n$ are pairwise disjoint and measurable, and also that E_{n+1} is measurable and $E_i \cap E_{n+1} = \emptyset$ for all $i \in \{1, ..., n\}$. Assume furthermore that for all A,

$$m^*(A) = \sum_{k=1}^{n} m^*(A \cap E_k) + m^*\left(A \cap \left(\bigcup_{k=1}^{n} E_k\right)^c\right).$$

Then $F = \bigcup_{k=1}^{n} E_k$ is a measurable set that is disjoint from E_{n+1}, and therefore for all A,

$$\begin{array}{rcl} m^*(A) & = & m^*(A \cap F) + m^*(A \cap E_{n+1}) + m^*(A \cap (F \cup E_{n+1})^c) \\ & = & \sum_{j=1}^{n} m^*(A \cap E_j) + m^*(A \cap E_{n+1}) + m^*\left(A \cap \left(\bigcup_{j=1}^{n} E_j \cup E_{n+1}\right)^c\right) \\ & = & \sum_{j=1}^{n+1} m^*(A \cap E_j) + m^*\left(A \cap \left(\bigcup_{j=1}^{n+1} E_j\right)^c\right) \end{array}$$

Note that the expansion of $m^*(A \cap F)$ into a sum of measures from the first to second line is due to the fact that all of the E_j are disjoint, and so the $A \cap E_j$ are also, and so 37 states that the measure of the union is therefore equal to the sum of the measures. ∎

2.0.21
Let $\{E_k\}_{k=1}^{\infty}$ be pairwise disjoint and measurable. Then for any A,

$$m^*(A) = \sum_{k=1}^{n} m^*(A \cap E_k) + m^*\left(A \cap \left(\sum_{k=1}^{n} E_k\right)^c\right).$$

$\left(\left\{\sum_{k=1}^{n} m^*(A \cap E_k)\right\}_{n=1}^{\infty}\right.$ is a non-decreasing sequence that converges to $\sum_{k=1}^{\infty} m^*(A \cap E_k)$. $\{s_n\}_{n=1}^{\infty} = \left\{m^*\left(A \cap \left(\sum_{k=1}^{n} E_k\right)^c\right)\right\}_{n=1}^{\infty}$ is a non-increasing sequence all of whose terms are at least $m^*\left(A\cap\left(\sum_{k=1}^{n}E_k\right)^c\right)$. These imply $m^*(A)\geq\sum_{k=1}^{\infty}m^*(A\cap E_k)+m^*\left(A\cap\left(\sum_{k=1}^{n}E_k\right)^c\right)$.)

Proof. First note that for every $n\in\mathbb{Z}^+$,

$$m^*\left(\bigcup_{j=1}^{n}(A\cap E_j)\right)=\sum_{j=1}^{n}m^*(A\cap E_j)$$

Therefore, when we consider the sequence $\left\{m^*\left(\bigcup_{j=1}^{n}(A\cap E_j)\right)\right\}_{n=1}^{\infty}$, this is equivalently written as $\left\{\sum_{j=1}^{n}m^*(A\cap E_j)\right\}_{n=1}^{\infty}$.

Since $\left\{\sum_{j=1}^{n}m^*(A\cap E_j)\right\}_{n=1}^{\infty}$ is a sequence of non-decreasing partial sums each of which is bounded by $[0,1]$, we know the limit of this sequence is bounded, and therefore (by 12)

$$\lim_{n\to\infty}m^*\left(\bigcup_{j=1}^{n}(A\cap E_j)\right)=\lim_{n\to\infty}\sum_{j=1}^{n}m^*(A\cap E_j)=\sum_{j=1}^{\infty}m^*(A\cap E_j)$$

Now, since $\{S_n\}_{n=1}^{\infty}=\left\{m^*\left(A\cap\left(\cup_{j=1}^{n}E_j\right)^c\right)\right\}_{n=1}^{\infty}$ is a non-increasing sequence, all of whose terms are at least $m^*\left(A\cap\left(\cup_{j=1}^{\infty}E_j\right)^c\right)$, then for every $n\in\mathbb{Z}^+$,

$$\sum_{j=1}^{\infty}m^*(A\cap E_j)+m^*\left(A\cap\left(\bigcup_{j=1}^{n}E_j\right)^c\right)\geq\sum_{j=1}^{\infty}m^*(A\cap E_j)+m^*\left(A\cap\left(\bigcup_{j=1}^{\infty}E_j\right)^c\right)$$

and so

$$\begin{aligned}&\lim_{n\to\infty}\left\{m^*\left(\bigcup_{j=1}^{n}(A\cap E_j)\right)+m^*\left(A\cap\left(\bigcup_{j=1}^{n}E_j\right)^c\right)\right\}\\&=m^*\left(\bigcup_{j=1}^{\infty}(A\cap E_j)\right)+\lim_{n\to\infty}m^*\left(A\cap\left(\bigcup_{j=1}^{n}E_j\right)^c\right)\\&=\sum_{j=1}^{\infty}m^*(A\cap E_j)+\lim_{n\to\infty}m^*\left(A\cap\left(\bigcup_{j=1}^{n}E_j\right)^c\right)\\&\geq\sum_{j=1}^{\infty}m^*(A\cap E_j)+m^*\left(A\cap\left(\bigcup_{j=1}^{\infty}E_j\right)^c\right)\end{aligned}$$

Therefore

$$m^*(A)\geq\sum_{j=1}^{\infty}m^*(A\cap E_j)+m^*\left(A\cap\left(\bigcup_{j=1}^{\infty}E_j\right)^c\right)$$

and so by the criterion in (25)

$$m^*(A) = \sum_{j=1}^{\infty} m^*(A \cap E_j) + m^*\left(A \cap \left(\bigcup_{j=1}^{\infty} E_j\right)^c\right)$$

□

2.0.22
Let $\{E_k\}_{k=1}^{\infty}$ be measurable. Then $\bigcup_{k=1}^{\infty} E_k$ is measurable. If the E_k are pairwise disjoint, then

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m(E_k).$$

(Unions can be written as disjoint unions.) This property is called **countable additivity**.

Proof. Let $\{E_j\}_{j=1}^{\infty}$ be measurable. Then by (4), we can define

$$F_1 = E_1 \qquad F_k = E_k \setminus \bigcup_{j=1}^{k-1} E_j$$
$$\text{where} \quad F \bigcup_{j=1}^{\infty} F_j \quad = \quad \bigcup_{j=1}^{\infty} E_j$$

and since for all n, F_n is a finite intersection of measurable sets, F_n is measurable. Therefore $\{F_j\}_{j=1}^{\infty}$ is a sequence of measurable sets that are pairwise disjoint, so by (39), for any A,

$$\begin{aligned}
m^*(A) &= \sum_{k=1}^{\infty} m^*(A \cap F_k) + m^*\left(A \cap \left(\bigcup_{k=1}^{\infty} F_k\right)^c\right) \\
&= m^*\left(\bigcup_{k=1}^{\infty} (A \cap F_k)\right) + m^*\left(A \cap \left(\bigcup_{k=1}^{\infty} F_k\right)^c\right) \\
&= m^*\left(A \cap \bigcup_{k=1}^{\infty} F_k\right) + m^*\left(A \cap \left(\bigcup_{k=1}^{\infty} F_k\right)^c\right) \\
&= m^*\left(A \cap \bigcup_{k=1}^{\infty} E_k\right) + m^*\left(A \cap \left(\bigcup_{k=1}^{\infty} E_k\right)^c\right)
\end{aligned}$$

So $\bigcup_{k=1}^{\infty} E_k$ is measurable.

Now, assume that the E_j are pairwise disjoint. Then letting $F = \bigcup_{k=1}^{\infty} E_k$, we can use the fact just proven that $\bigcup_{k=1}^{\infty} E_k$ is measurable together with the test set F and by (39),

$$\begin{aligned}
m^*(F) &= \sum_{k=1}^{\infty} m^*(F \cap E_k) + m^*\left(F \cap \left(\bigcup_{k=1}^{\infty} E_k\right)^c\right) \\
m^*(F) &= \sum_{k=1}^{\infty} m^*(F \cap E_k) + m^*\left(F \cap (F)^c\right) \\
m^*(F) &= \sum_{k=1}^{\infty} m^*(F \cap E_k) + m^*(\emptyset) \\
m^*(F) &= \sum_{k=1}^{\infty} m^*(F \cap E_k) \\
m^*\left(\bigcup_{k=1}^{\infty} E_k\right) &= \sum_{k=1}^{\infty} m^*\left(\left[\bigcup_{j=1}^{\infty} E_j\right] \cap E_k\right) \\
m^*\left(\bigcup_{k=1}^{\infty} E_k\right) &= \sum_{k=1}^{\infty} m^*(E_k)
\end{aligned}$$

□

2.0.23

Let $\{E_k\}_{k=1}^{\infty}$ be measurable and nested: $E_1 \subset E_2 \subset ...$ If $\bigcap_{k=1}^{\infty} E_k = \emptyset$, then $\lim_{k\to\infty} m(E_k) = 0$.

(E_1 is a disjoint union $\bigcup_{k=1}^{\infty} F_k$ where $F_1 = E_1 \setminus E_2$, $F_2 = E_2 \setminus E_3$ etc. $m(E_n) = \sum_{k=n}^{\infty} m(F_k)$.)

Proof. Since these sets are nested,

$$m(E_1) = m(\bigcup_{k=1}^{\infty} E_k).$$

Since $\{E_k\}_{k=1}^{\infty}$ is countable, let $F_k = E_k \setminus E_{k+1}$. Then by problem 40,

$$m(E_1) = m(\bigcup_{k=1}^{\infty} E_k) = m(\bigcup_{k=1}^{\infty} F_k) = \sum_{k=1}^{\infty} F_k$$

Now, let $\bigcap_{k=1}^{\infty} E_k = \emptyset$.

$$E_1 \setminus E_k = \bigcup_{j=1}^{k-1} F_j$$

and since the F_k are disjoint,

$$\bigcup_{j=1}^{k} E_j \setminus \bigcap_{j=1}^{k} E_j = \bigcup_{j=1}^{k-1} F_j = \bigcup_{j=1}^{k} F_j \setminus F_k$$

After rearranging the terms, we get

$$\bigcap_{j=1}^{k} E_j = \bigcup_{j=1}^{k} E_j \setminus \left(\bigcup_{j=1}^{k} F_j \setminus F_k\right)$$

Since these are countable unions and intersections of measurable sets, we have

$$m\left(\bigcap_{j=1}^{k} E_j\right) = m\left(\bigcup_{j=1}^{k} E_j \setminus \left(\bigcup_{j=1}^{k} F_j \setminus F_k\right)\right)$$

Taking the limit as k goes to infinity, we have

$$\begin{aligned} m(\emptyset) &= m(E_1) - m(E_1) + \lim_{k\to\infty} m(F_k) \\ 0 &= \lim_{k\to\infty} m(F_k) \\ &= \lim_{k\to\infty} m(E_k \setminus E_{k+1}) \end{aligned}$$

Since $\bigcap_{k=1}^{\infty} E_k = \emptyset$, $\lim\limits_{k\to\infty} m(E_k \setminus E_{k+1}) = 0$ can only be true if $\lim\limits_{k\to\infty} m(E_k) = 0$ therefore if $\{E_k\}_{k=1}^{\infty}$ are measurable and nested, then

$$E_1 \supset E_2 \cdots \text{ and } \bigcap_{k=1}^{\infty} E_k = \emptyset$$

then $\lim\limits_{k\to\infty} m(E_k) = 0$ □

Definition 2.0.24. *If $E \subset [0,1]$ and r is a real number, the n $E\dot{+}r$ is the set of $[0,1)$ consisting of all fractional parts of the set $\{x+r : x \in E\}$. Geometrically, add r to each element of E and then translate the elements by an integer so that they land back in $[0,1)$. (This can mean "breaking the set into two parts." For instance $[0,.4]\dot{+}\frac{3}{4} = [0,.15] \cup [3/4,1)$.)*

2.0.25
If E is measurable and r is a real number, then $E\dot{+}r$ is measurable and $m(E\dot{+}r) = m(E)$.
(I found it helpful (i) to show that if A is any subset of $[0,1]$, (not necessarily measurable) and $r \in \mathbb{R}$ is such that $A + r \subset [0,1]$, then $m^*(A) = m^*(A+r)$ and (ii) to remember that we have already seen in a previous problem that if A is any subset of $[0,1]$ and $s \in (0,1)$ then $m^*(A) = m^*(a \cap [0,s]) + m^*(A \cap (s,1])$.)

Proof. □

Remark 2.0.26. *We can sum up what we have done so far like this.*
The set $\mathfrak{M}$ of Lebesgue measurable subsets of $[0,1]$ has the following properties

1. $\emptyset \in \mathfrak{M}$
2. *Every interval is in* $\mathfrak{M}$ *with* $m(\langle a,b\rangle) = b-a$. *Moreover every open subset of* $[0,1]$ *is in* $\mathfrak{M}$.
3. $E \in \mathfrak{M} \Rightarrow E^c \in \mathfrak{M}$.
4. $E \in \mathfrak{M} \Rightarrow E\dot{+}r \in \mathfrak{M}$ *and* $m(E\dot{+}r) = m(E)$.
5. *If* $\{E_k\}_{k=1}^{\infty}$ *are in* $\mathfrak{M}$, *then* $\bigcup\limits_{k=1}^{\infty} E_k \in \mathfrak{M}$.
6. *If, in addition, the* E_k *are pairwise disjoint, then* $m\left(\bigcup\limits_{k=1}^{\infty} E_k\right) = \sum\limits_{k=1}^{\infty} m(E_k)$

Example 2.0.27. *ince the set* $\mathbb{Q}_0$ *of rational numbers in* $[0,1]$ *is countable,* $\mathbb{Q}_0$ *is measurable and* $m(\mathbb{Q}_0) = 0$. *Then* $\mathbb{Q}_0^c$*-the set of irrational numbers in* $[0,1]$*-is also measurable, and* $m(\mathbb{Z}_0^c) = m([0,1]) - m(\mathbb{Q}_0) = 1$. *Thus the "length" of the irrationals is 1, while the "length" of the rationals is 0. This is connected to the fact that* $\mathbb{Q}_0$ *is countable while* $\mathbb{Q}_0^c$ *is uncountable, but it would be wrong to assume that all uncountable subsets of* $[0,1]$ *have measure 1, or even nonzero measure. See the next example.*

Example 2.0.28. *Recall that the Cantor set* C *is the subset of* $[0,1]$ *that remains after removing* $\bigcup\limits_{k=1}^{\infty} I_k$ *where* $I_1 = \left(\frac{1}{3},\frac{2}{3}\right)$, $I_2 = \left(\frac{1}{9},\frac{2}{9}\right) \cup \left(\frac{7}{9},\frac{8}{9}\right)$ *and in general* I_k *is the union of the* 2^k *open middle thirds of the closed intervals remaining after* I_{k-1} *has been removed. Furthermore,* $m(I_k) = \frac{1}{3}\left(\frac{2}{3}\right)^{k-1}$ *so that* $m\left(\bigcup\limits_{k=1}^{\infty} I_k\right) = \frac{1}{3}\sum\limits_{k=0}^{\infty}\left(\frac{2}{3}\right)^k = 1$ *and* $m(C) = 0$.

Remark 2.0.29. *It is not immediately obvious from the construction above that* C *is uncountable. Some of you at least have seen an alternative characterization of* C *as the set of all numbers in* $[0,1]$ *with a ternary expansion containing only 0's and 2's. Since this set can be put in 1-1 correspondence with the set of all binary expansions of numbers in* $[0,1]$, *that is, with* $[0,1]$, *it is true that* C *is uncountable. Thus the Lebesgue measure of a set is not particularly connected to the cardinality of the set. These are two rather different versions of the "size" of an infinite set.*

2.0.30

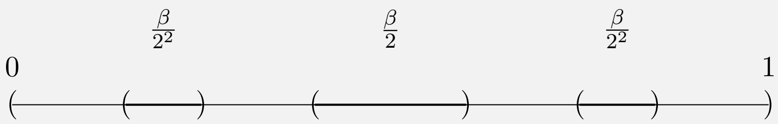
By altering the lengths of the removed intervals, construct a "fat Cantor set"-a Cantor set of positive measure. Ideally, show how to construct a Cantor set of measure α for any $\alpha < 1$.

(You can check that if you try to imitate the Cantor set construction by removing a constant fraction of what remains at each stage, the total length of removed intervals will be 1 no matter what the fraction removed is, so you will have to be a little cleverer than that. Try to make choices so that your calculations are reasonably easy.)

Proof. For any $\alpha < 1$, let $\beta = 1 - \alpha$. We will construct the complement C_f^c of a "fat Cantor set C_f" that has measure β so that the measure of C_f is $1 - \beta = \alpha$.

To construct this set, as with the Cantor set, we will use the intervals that are discarded from the center of a given interval at each step.

For the first step, remove the interval of length $\frac{\beta}{2}$ from the center of the interval $[0, 1]$. Then at each successive step, remove the intervals of length $\frac{\beta}{2^{2n}}$ from the center of each remaining open interval.



The intervals removed form a set that has a sum length of intervals of

$$\frac{\beta}{2} + 2\frac{\beta}{4^2} + \cdots = \beta \sum_{i=1}^{\infty} \frac{1}{2^i} = \beta$$

which means that the complement of this set has measure $1 - \beta = \alpha$. $\square$

Remark 2.0.31. *Properties (1),(3),(5) of #44 are really structural properties of the collection $\mathfrak{M}$ of measurable subsets of $[0, 1]$. How interesting they are depends on whether $\mathfrak{M}$ is just the set of all subsets of $[0, 1]$ (sometimes called the* ***power set*** *$\mathscr{P}([0, 1])$) or some proper collection of $\mathscr{P}([0, 1])$. We "construct" some non-measurable sets in order to see that $\mathfrak{M} \neq \mathscr{P}([0, 1])$. This makes the structural properties much more interesting.*

Given $x \in [0, 1]$, let $A_x = \{y \in [0, 1] : x - y \in \mathbb{Q}\}$. Each A_x is a countable set. Clearly $y \in A_x \Leftrightarrow x \in A_y$. If $x \in A_y$ and $y \in A_z$, then $x - z = (x + y) + (y - z)$ is rational so $z \in A_x$. It follows that the relation $x \sim y$ if $y \in A_x$ is reflexive, symmetric and transitive, that is, it is an ***equivalence relation****. Each distinct set A_x is an* ***equivalence class****. As you have probably verified somewhere else, distinct equivalence classes are disjoint, that is, if $x \neq y$ then either $A_x = A_y$ or A_x and A_y are disjoint. (Easy if you haven't done it before.)*

Now choose one element from each distinct equivalence class, and let E be the set of all such elements. E is an uncountable set; otherwise $[0, 1]$ would be a countable union of countable sets, and so countable. See the remark below about hte choice process.

Let $\{q_k\}_{k=1}^{\infty}$ be an enumeration of the set $\mathbb{Q}_0$ of rationals in $[0, 1]$, I claim the sets $\{E \dot{+} q_k : k = 1, 2, 3\}$ are disjoint and that their union is $[0, 1]$. First note that for any $x \in [0, 1]$, $x \in E \dot{+} q_k$ if for some $e \in E$, either $x = e + q_k$ (if $e + q_k < 1$) or $x = e + q_k - 1$ (in case $e + q_k \geq 1$). Now x is in the same equivalence class A_x as some element e of E. Thus $|x - e| = q_k$ for some k and we see that either $x \in E \dot{+} q_k$ (if $x \geq e$) or $x \in E \dot{+} (1 - q_k)$ (if $x < e$). Next we see that the $\{E \dot{+} q_k\}$ are pairwise disjoint. If $x = e + q_k = e^t + q_j$, then $e - e^t = q_j - q_k$, that is, e and e^t are in the same equivalence class. This is a contradiciton.

To summarize, $[0, 1] = \bigcup_{k=1}^{\infty} (E \dot{+} q_k)$ as a disjoint union. Further, by property 4 *of #44, either all these sets are measurable with the same measure, or all are non - measurable. But the first alternative is impossible by property 6 of #44. Thus we have a countably infinite collection of non-measurable sets.*

Remark 2.0.32. *The construction of E in the preceding example used the property that I can make a set by choosing one element out of a collection of non-empty sets. The assertion that one can do this is called the **Axiom of Choice**. In the early twentieth century when mathematicians and logicians were trying to construct all of mathematics from a precisely defined set of axioms, and to show that the mathematics so obtained would be consistent (no possibility that legal arguments would produce a contradiction), the relationship between the Axiom of Choice and other "usual axioms" was much studied, because the Axiom of Choice has surprising consequences (e.g. that it is possible to order the set of real numbers so that each subset of reals has a least element in the ordering-just like the positive integers with the usual ordering. This is called a Well-Ordering.) However in 1963 Paul Cohen completed the proof that AofC is independent of the other axioms of set theory. This marked the very end of the interest that most mathematicians have in the foundations of mathematics. (Most feel that it is new ideas and connections that are of interest, and Cohen's result showed that AofC, which is often very convenient, can't "make things worse" (introduce contradictions where there were none before.)).*

Definition 2.0.33. *A collection $\mathfrak{A}$ of subsets of a non-empty set X is a $\sigma-$**algebra** of subsets if $\mathfrak{A}$ has the properties*

(i) $\emptyset \in \mathfrak{A}$,

(ii) $E \in \mathfrak{A} \Leftrightarrow E^c \in \mathfrak{A}$

(iii) if $\{E_k\}_{k=1}^{\infty}$ *are in* $\mathfrak{A}$, *then* $\bigcup\limits_{k=1}^{\infty} E_k \in \mathfrak{A}$

Remark 2.0.34. *Thus $\mathfrak{M}$ is a proper σ-algebra of subsets of $\mathscr{P}([0,1])$, the power set of $[0,1]$. Another $\sigma-$algebra of subsets of $\mathscr{P}([0,1])$ is the collection $\mathfrak{B}$ of all **Borel subsets** of $[0,1]$. $\mathfrak{B}$ is defined to be the smallest $\sigma-$algebra of subsets of $\mathscr{P}([0,1])$ that contains the open sets. (Take the intersection of all such $\sigma-$algebras.) Clearly $\mathfrak{B} \subset \mathfrak{M}$. It can be shown that $\mathfrak{B}$ and $\mathfrak{M}$ are different by a cardinality argument-Since the Cantor set C can be put into 1-1 correspondence with $\mathscr{P}([0,1])$. All elements in $\mathscr{P}(C)$ are measurable, since they are subsets of a set of measure 0, thus $\mathfrak{M}$ has the same cardinality as $\mathscr{P}([0,1])$. On the other hand, $\mathfrak{B}$ can also be generated in a countable fashion from the set of open intervals with rational endpoints. Thus its cardinality is the same as that of $[0,1]$ and so strictly less than that of $\mathscr{P}([0,1])$.*

On the other hand, every measurable set is "nearly" an open set, and also "nearly" a closed set. This is the content of the next problem.

2.0.35
The following properties are equivalent for a subset E of $[0, 1]$

(i) $E \in \mathfrak{M}$

(ii) for each $\epsilon > 0$ there is an open set G such that $E \subset G$ and $m^*(G \setminus E) < \epsilon$.

(iii) there is a G_δ set H such that $E \subset H$ and $m^*(H \setminus E) = 0$.
(A G_δ set is a countable intersection of open sets. A G_δ is a Borel set but not open in general, for instance any closed interval is a G_δ. (Proof?-but not a problem))

(iv) for each $\epsilon > 0$ there is a closed set F such that $F \subset E$ and $m^*(E \setminus F) < \epsilon$.

(v) there is an F_σ set K (a countable union of closed sets, e.g. $\mathbb{Q}_0$) such that $K \subset E$ and $m^*(E \setminus K) = 0$.

(We will split this into two parts, from two different people, first (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i), second (ii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (i).)

Proof. $(i) \Rightarrow (ii)$: If E is a measurable set, then for each $\epsilon > 0$ there is an open cover $\{G_j\}_{j=1}^{\infty}$ of E such that

$$m(E) < \sum_{j=1}^{\infty} \ell(G_j) < m(E) + \epsilon$$

Now, let $G = \bigcup\limits_{i=1}^{\infty} G_j$. Then G, as a union of open sets, is itself open. Also, since the G_j were chosen as an open cover of E, $E \subset G$, and so (since $G \setminus E$ and E are mutually disjoint),

$$m^*(G \setminus E) + m^*(E) \leq m^*(G) \leq m^*(E) + \epsilon$$

which can be re-written

$$m^*(G \setminus E) + m^*(E) \leq m^*(E) + \epsilon \quad \Rightarrow \quad m^*(G \setminus E) \leq \epsilon$$

$(ii) \Rightarrow (iii)$: Let $H = \bigcup\limits_{k=1}^{\infty} H_i$ where $E \subset H_i$, $\forall i$ and (from part (ii)):

$$m^*(H_i \setminus E) = \frac{1}{i}$$

Then $E \subset H$ and $H = \bigcup_{k=1}^{\infty} H_i$ is a G_δ set such that $m^*(H \setminus E) = \lim\limits_{i \to \infty} \frac{1}{i} = 0$.
$(iii) \Rightarrow (i)$: Given a G_δ set H, such that $E \subset H$, and $m^*(H \setminus E) = 0$, Then H is measurable since $\mathfrak{M}$ is a σ-algebra.

$$E = (H \setminus E)^c \cap H.$$

Notice all the constituent sets are measurable, thus $E \in \mathfrak{M}$.

$(ii) \Rightarrow (iv)$: Let $\epsilon > 0$. Then there is an open set G such that $E \subset G$ and $m^*(G \setminus E) < \epsilon$.

Then E^c is also measurable and so there is an open set F^c such that $E^c \subset F^c$ and $m^*(F^c \setminus E) \leq \epsilon$. This implies $F \subset E$.

Then $m^*(F^c \setminus E^c) = m^*(F^c \cap E) = m^*(E \cap F^c) < \epsilon$.

$(iv) \Rightarrow (v)$: Let $K \subset E$ such that $K = \bigcup_{k=1}^{\infty} F_i$, and each $F_i \in E$ $m^*(E \cap F_i^c) < \frac{\epsilon}{2^k}$. Then $E \setminus K \subset F_i$. Furthermore,

$$m^*(E \setminus K) < m^*(E \setminus F_i) < \frac{\epsilon}{2^k} < \epsilon$$

for all $1 \leq k \leq \infty$

$(v) \Rightarrow (i)$: Let K be an F_σ set such that $K \subset E$ and $m^*(E \setminus K) = 0$. Then $m^*(E \setminus K)$ is measurable. Also, since K is an F_σ set, it is a countable union of closed sets and is therefore measurable. Then $E = K \cup E \setminus K$ is the union of measurable sets. Thus E is measurable.

$\square$

3 Measure on $\mathbb{R}$

Remark 3.0.1. *There are several ways that we could develop a theory of measure on the entire real line. One is to start over and just repeat the development we have already been through. (Actually we could just have started that way.) The only important difference is that now some sets will have infinite measure ($\mathbb{R}$ itself for instance), so that we would have to keep in mind that the infinite sums are not guaranteed to converge. What we will actually do is use our theory on* $[0,1]$ *to create a theory on* $\mathbb{R}$.

Definition 3.0.2. *(a) For any real number x, let $\mathfrak{M}_x$ be the collection of all subsets E of $[x, x+1]$ such that $E - x$ is a measurable subset of $[0,1]$. For $E \in \mathfrak{M}_x$ we set $m(E) = m(E - x)$. It is apparent that $\mathfrak{M}_x$ is a σ-algebra of subsets of $[x, x+1]$ on which all the properties of #44 hold.*

(b) For any subset E of $\mathbb{R}$, and each integer n, let $E_n = E \cap [n, n+1)$, We say that E is Lebesgue measurable if $E_n \in \mathfrak{M}_n$ for each n. We denote the collection of all Lebesgue measurable subsets of $\mathbb{R}$ by $\mathfrak{M}_{\mathbb{R}}$. If E is measurable, we define the Lebesgue measure of E by $m(E) = \sum_{n=-\infty}^{\infty} m(E_n)$.

Remark 3.0.3. *It is very easy to verify that the set $\mathfrak{M}_{\mathbb{R}}$ of Lebesgue measurable subsets of $\mathbb{R}$ is a σ-algebra of sets containing the Borel sets, and that the Lebesgue measure m has all the properties of #44 except that we may now interpret translation invariance (property 4) as ordinary translation invariance instead of "translation invariance mod 1," that is, for any $E \in \mathfrak{M}_{\mathbb{R}}$, and any real number c, $E + c \in \mathfrak{M}_{\mathbb{R}}$ and $m(E) = m(E + c)$.*

Of course we do now have the possibility that $m(E) = \infty$. For this purpose we regard ∞ as just another number, more or less. It is necessary however, to avoid expressions like $\infty - \infty$, so at times we will need to be careful about whether the measure of a set is finite.

3.0.4
if E is a measurable set of finite measure, then for any $\epsilon > 0$ there is a finite collection of open intervals $\{G_k\}_{k=1}^n$ so that

$$m\left(E \setminus \bigcup_{k=1}^{n} G_k\right) + m\left(\bigcup_{k=1}^{n} G_k \setminus E\right) < \epsilon$$

(Thus, every measurable set is "almost a finite union of open sets.")

Proof. By 53, there is a closed set $F \subset E$ such that $m(E \setminus F) < \frac{\epsilon}{4}$ Now, define $F_n = F \cap [-n, n]$. Then each F_n is compact, bounded by E and

$$F_1 \subset F_2 \subset \dots$$

Furthermore, $m(F) = \sum_{n=1}^{\infty} m(F_n \setminus F_{n-1})$ converges, so there is N such that

$$\sum_{n=N+1}^{\infty} m(F_n \setminus F_{n-1}) < \frac{\epsilon}{4}$$

So $m(E \setminus F_N) = m(E \setminus F) + m(F \setminus F_N) < \frac{\epsilon}{2}$.

Since F_N is measurable, there is an open G such that $F_N \subset G$, which we can write as a union of open intervals $G = \bigcup_{k=1}^{\infty} G_k$. Since F_N is compact, there is a finite subcover of G such that

$$F_N \subset \bigcup_{k=1}^{n} G_k$$

Therefore since $F_N \subset \bigcup_{k=1}^{n} G_k$,

$$m\left(E \setminus \bigcup_{k=1}^{n} G_k\right) \leq m(E \setminus F_N) < \frac{\epsilon}{2}$$

Also, since $F_N \subset \bigcup_{k=1}^{n} G_k$,

$$m\left(\bigcup_{k=1}^{n} G_k \setminus F_N\right) \leq m\left(G \setminus E\right) < \frac{\epsilon}{2}$$

and since $F_N \subset E$,

$$m\left(\bigcup_{k=1}^{n} G_k \setminus E\right) < m\left(G \setminus F_N\right) < \frac{\epsilon}{2}$$

and so putting the two results together we have

$$m\left(E \setminus \bigcup_{k=1}^{n} G_k\right) + m\left(\bigcup_{k=1}^{n} G_k \setminus E\right) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

□

4 Measurable Functions

4.0.1
A real-valued function f defined on a measurable subset E of $\mathbb{R}$ is **measurable** if for every real number α, $\{x \in E : f(x) < \alpha\}$ is measurable.

4.0.2
These are equivalent:

(a) f is measurable

(b) for every α, $\{x \in E : f(x) \geq \alpha\}$ is measurable

(c) for every α, $\{x \in E : f(x) > \alpha\}$ is measurable

(d) for every α, $\{x \in E : f(x) \leq \alpha\}$ is measurable

(e) for every open set G, $\{x \in E : f(x) \in G\}$ is measurable. (Show (a) $\leftrightarrow$ (b) $\Rightarrow$ (c) $\leftrightarrow$ (d) $\Rightarrow$ (a) and (a) & (c) $\Rightarrow$ (e), (e) $\Rightarrow$ (a))

Proof. $(a) \Leftrightarrow (b)$ Let f be a measurable function. Then for any α, the set $\{x \in E : f(x) < \alpha\}$ is measurable. Since the complement of a measurable set is measurable, this means that $\{x \in E : f(x) \geq \alpha\}$ is measurable for any α.
$(b) \Rightarrow (c)$ If f is measurable, then for every α, the set $\{x \in E : f(x) \geq \alpha\}$ is measurable. Consider the sets

$$\left\{x \in E : f(x) \geq \alpha + \frac{1}{i}\right\}_{i=1}^{\infty}$$

Each of these sets is measurable, and so as the countable union of measurable sets is measurable, this means that t

$$\bigcup_{i=1}^{\infty}\left\{x \in E : f(x) \geq \alpha + \frac{1}{i}\right\} = \{x \in E : f(x) > \alpha\}$$

is measurable.

$(c) \Leftrightarrow (d)$ Given that for all $\alpha \in \mathbb{R}$, $\{x \in E : f(x) > \alpha\}$ is measurable, then we also have the complement of a measurable set is measurable. Therefore for all real α, $\{x \in E : f(x) \leq \alpha\}$ is measurable.
$(d) \Rightarrow (a)$ Similarly to showing (b) implies (c), consider the sets

$$\left\{x \in E : f(x) \leq \alpha - \frac{1}{i}\right\}_{i=1}^{\infty}$$

As each set is measurable, the countable union is also, and so

$$\bigcup_{i=1}^{\infty} \left\{x \in E : f(x) \leq \alpha - \frac{1}{i}\right\} = \{x \in E : f(x) < \alpha\}$$

is measurable for any α. Thus f is a measurable function on E.
$(a), (c) \Rightarrow (e)$ For any open set G, we can write G as a union of open intervals about each of its points. Say, $G = \bigcup\limits_{i=1}^{\infty} F_i$ where each F_i is an open interval. If the number of points in G is some finite $n \in \mathbb{Z}^+$, we can take $F_{n+1}, F_{n+2} ... = \emptyset$.

Now, for each $F_k = (a_k, b_k)$, since f is measurable, we know that the intersection of the sets $\{x \in E : f(x) < b_k\}$ and $\{x \in E : f(x) > a_k\} = \{x \in E : f(x) \in F_k\}$ is measurable. Therefore, since the countable union of measurable sets (each of which is the intersection of two measurable sets) is measurable, then

$$\{x \in E : f(x) \in G\} = \bigcup_{i=1}^{\infty} \{x \in E : f(x) \in F_i\}$$

is measurable.
$(e) \Rightarrow (a)$ Now, assume that for all open G, the set $\{x \in E : f(x) \in G\}$ is measurable. For any $\alpha \in \mathbb{R}$, we can write

$$\begin{aligned} \{x \in E : f(x) < \alpha\} &= \left\{x \in E : f(x) \in \bigcup_{k=1}^{\infty} (\alpha - k, \alpha - (k-1))\right\} \\ &= \bigcup_{k=1}^{\infty} \{x \in E : f(x) \in (\alpha - k, \alpha - (k-1))\} \end{aligned}$$

which, as a countable union of measurable sets is measurable. Therefore f is measurable. □

Example 4.0.3. 1. *if E is a measurable set, the **characteristic function** of E is defined by*

$$\chi_E(x) = \begin{cases} 1, & x \in E, \\ 0, & x \notin E \end{cases}$$

is measurable.

2. *By part (e) of the previous problem, any continuous function with a measurable domain is measurable. (What property of continuous functions does this follow from?)*

4.0.4
If f is measurable with domain E, and $c \in \mathbb{R}$, then cf and $f + c$ are measurable.

Proof. If $c > 0$, then For any $\alpha \in \mathbb{R}$, $\alpha/c \in \mathbb{R}$, and

$$\{x \in E : cf(x) > \alpha\} = \{x \in E : f(x) > \alpha/c\}$$

is measurable, since f is measurable. Therefore cf is measurable for all $c > 0$.

If $c = 0$, then for any $\alpha \in R$, the set $\{x \in E : cf(x) > \alpha\}$ is either $\mathbb{R} \cap E$ (if $\alpha \leq 0$), which is measurable, or else $\emptyset$ (if $\alpha > 0$), which is also measurable.

If $c < 0$, then for any $\alpha \in \mathbb{R}$, the set $\{x \in E : cf(x) > \alpha\} = \{x \in E : f(x) \leq \alpha/c\}$ which is measurable, since f is measurable.

Thus $cf(x)$ is measurable for all $c \in \mathbb{R}$.

Now, if f is measurable over E, then for any $c \in \mathbb{R}$, for all $\alpha \in \mathbb{R}$,

$$\{x \in E : f(x) + c > \alpha\} = \{x \in E : f(x) > \alpha - c\}$$

which is measurable, since f is measurable.

Therefore $f(x) + c$ is measurable. □

4.0.5
If f and g are measurable with domain E, then $f + g$ is measurable. (Show $\{x \in E : f(x) + g(x) < \alpha\} = \bigcup_q (\{x \in E : f(x) < \alpha - q\} \cap \{x \in E : g(x) < q\})$ where the union is over all of $q \in \mathbb{Q}$.)

Proof. For each $q \in \mathbb{Q}$, the sets $\{x \in E : f(x) < \alpha - q\}$ and $\{x \in E : g(x) < q\}$ are both measurable, since f and g are both measurable functions. Also, as the intersection of two measurable sets is measurable, for any $\alpha \in \mathbb{R}$ and any $q \in \mathbb{Q}$,

$$\{x \in E : f(x) < \alpha - q\} \cap \{x \in E : g(x) < q\}$$

is measurable. Also, since the rationals are countable, the union

$$\bigcup_{q \in \mathbb{Q}} \{\{x \in E : f(x) < \alpha - q\} \cap \{x \in E : g(x) < q\}\}$$

is measurable. And so

$$\{x \in E : f(x) + g(x) < \alpha\} = \bigcup_{q \in \mathbb{Q}} \{x \in E : f(x) < \alpha - q\} \cap \{x \in E : g(x) < q\}$$

is measurable. □

4.0.6
if f and g are measurable with domain E, then $\max\{f, g\}$ and $\min\{f, g\}$ are measurable.

Proof. For any α, $\{x \in E : max(f, g) > \alpha\}$ is equal to

$$\{x \in E : f(x) > \alpha\} \cup \{x \in E : g(x) > \alpha\}$$

which, as the union of two measurable sets, is measurable. Therefore $max(f, g)$ is measurable.

Similarly for $min(f, g)$, for any α,

$$\{x \in E : min(f, g) < \alpha\} = \{x \in E : f < \alpha\} \cup \{x \in E : g < \alpha\}$$

is also measurable. □

Definition 4.0.7. *If f is measurable with domain E, the **positive part** of f is the function $f_+ = \max\{f, 0\}$, and the **negative part** of f is the function $f_- = \max\{-f, 0\}$.*

Remark 4.0.8. *Note that $f = f_+ - f_-$ and $|f| = f_+ + f_-$. It follows from the previous problems that if f is measurable, then so are all of f_+, f_-, and $|f|$.*

4.0.9
If f and g are measurable with domain E, then f^2 and fg are measurable.
(Do f^2 first. Then $fg = \frac{1}{2}\left((f + g)^2 - f^2 - g^2\right)$.)

Proof. If f is a measurable function, then for any α, if $\alpha \geq 0$,

$$\{x \in E : [f(x)]^2 < \alpha\} = \{x \in E : f(x) < \sqrt{\alpha}\} \cap \{x \in E : f(x) > -\sqrt{\alpha}\}$$

as the intersection of two measurable sets, is measurable. If on the other hand, $\alpha < 0$, then $\{x \in E : f^2 < \alpha\} = \emptyset$, which is measurable. Therefore for any α, $\{x \in E : [f(x)]^2 < \alpha\}$ is measurable, and so f^2 is measurable.

Now, as we have seen, the linear combination of measurable functions is measurable, and the square of a measurable function is measurable. Therefore

$$fg = \frac{1}{2}\left(f^2 - f^2 + 2fg + g^2 - g^2\right) = \frac{1}{2}\left((f + g)^2 - f^2 - g^2\right)$$

is measurable. □

Definition 4.0.10. *We say that a property holds **almost everywhere** (abbreviated **a.e.**) if the set where it does not hold has measure zero. In this context a set of measure zero is often called a **null set**. We also say that a property that holds in a set E except on a null set holds for **almost all** x in E.*

4.0.11
If f and g are defined on a measurable set E, f is a measurable function, and $f = g$ a.e., then g is measurable.

Proof. For all α, the sets $\{x \in E : f(x) < \alpha\}$ and $\{x \in E : f(x) \neq g(x)\}^c$ are measurable. Therefore

$$\{x \in E : f(x) < \alpha\} \cap \{x \in E : f(x) \neq g(x)\}^c$$

is measurable, and

$$\underbrace{\overbrace{\{x \in E : f(x) < \alpha\} \cap \{x \in E : f(x) \neq g(x)\}^c}^{\text{measurable}} \cup \overbrace{\{x \in E : f(x) \neq g(x) \& g(x) < \alpha\}}^{\text{measure } 0}}_{\{x \in E : g(x) < \alpha\}}$$

is measurable. □

Definition 4.0.12. *The* ***extended real numbers*** *consist of* $\mathbb{R}$ *together with* $+\infty$ *and* $-\infty$. *These have the property that* $-\infty < r < +\infty$ *for every real number* r, *but we try not to do arithmetic with* $\pm\infty$. *It is often convenient to allow measurable functions to have range in the extended real numbers. The preceding problem shows that as long as functions take finite values almost everywhere, we can do arithmetic with them without bothering about what happens on a null set.*

4.0.13
If f is a measurable extended real-valued function defined on a bounded measurable set E and f is finite a.e., then for any $\epsilon > 0$ there is M so that $|f| \leq M$ except on a set of measure less than ϵ. (So any measurable function on a bounded set is "almost bounded.")
(Remember 2.0.23)

Proof. Let $E_k = \{x \in E : |f(x)| > k\}$. Each of these sets is measurable, and $m(E_{k+1}) \geq m(e_k) \geq ...$ for all k. Therefore for any $\epsilon > 0$, there is some k such that

$$m(E_k) < \epsilon$$

which implies that for all $x \notin E_k$,

$$|f(x)| \leq k$$

□

4.0.14
Is it necessary to assume in the previous problem that the set E is bounded?

sort of answer: We need the measure of each of those E not to be infinite in the previous example, else 2.0.23 not apply.

4.0.15
If $\{f_n\}_{n=1}^{\infty}$ is any sequence of real-valued measurable functions defined on a measurable set E, then the function $F(x) = \sup_n f_n(x)$ is a measurable. (Note, for instance, that $F(x) = \infty$ if $f_n(x) \to \infty$ as $n \to \infty$.)

Proof. for any α,

$$\begin{aligned} \{x \in E : F(x) > \alpha\} &= \{x \in E : \sup\{f_n(x)\}_{n=1}^{\infty} > \alpha\} \\ &= \bigcup_{n=1}^{\infty} \{x \in E : f_n(x) > \alpha\} \in \mathfrak{M} \end{aligned}$$

□

Definition 4.0.16. *The* ***limit superior*** *of a sequence of real numbers* $\{a_k\}_{k=1}^{\infty}$ *is the extended real number* $E = \inf_n \sum_{k\geq n} a_k$. *We write* $A = \limsup a_k$.
Note that the sequence $\{\sum_{k\geq n} a_k\}_{n=1}^{\infty}$ *is non-increasing, so the inf is the same as the limit of this sequence, keeping in mind that all elements of this sequence may be* ∞.

Similarly the ***limit inferior*** *of* $\{a_k\}_{k=1}^{\infty}$ *is* $B = \sup_n \inf_{k\geq n} a_k$. *We write* $B = \liminf a_k$. *The sequence* $\{\inf_{k\geq n} a_k\}_{n=1}^{\infty}$ *is non-increasing, so the sup is the same as the limit of this sequence.*

4.0.17
For any sequence $\{a_k\}_{k=1}^{\infty}$ of real numbers, $\limsup a_k \geq \liminf a_k$. $\limsup a_k = \liminf a_k$ if and only if the sequence $\{a_k\}_{k=1}^{\infty}$ converges.

Proof. ($\Rightarrow$) Let $\limsup_{a_k} = \liminf_{a_k}$. Then for each a_k,

$$\begin{aligned} \inf_{k\geq n} a_k &\leq a_k \leq \sup_{k\geq n} a_k \\ \inf a_k - \inf a_k &\leq a_k - \inf a_k \leq \sup a_k - \inf a_k \\ 0 &\leq a_k - \inf a_k \leq \sup a_k - \inf a_k \\ \lim_{n\to\infty} 0 &\leq \lim_{n\to\infty} (a_k - \inf a_k) \leq \lim_{n\to\infty} \sup a_k - \lim_{n\to\infty} \inf a_k \\ 0 &\leq \lim_{n\to\infty} (a_k - \inf a_k) \leq 0. \end{aligned}$$

Therefore, $\lim_{n\to\infty} (a_k - \inf a_k) = 0$, so $\lim_{n\to\infty} a_k = \lim_{n\to\infty} \inf a_k = \lim_{n\to\infty} \sup a_k$. Therefore the limit as $n \to \infty$ of a_k exists (and is a real number), so the sequence converges.

($\Leftarrow$) Assume that $\{a_k\}_{k=1}^{\infty}$ converges to a limit. Then all of its subsequences converge to the same limit, nad since both $\sup a_k$ and $\inf a_k$ are subsequences of $\{a_k\}_{k=1}^{\infty}$, they must converge to the same limit. $\square$

4.0.18
If $\{f_n\}_{n=1}^{\infty}$ is any sequence of extended real-valued measurable functions defined on a measurable set E, and if $f(x) = \lim_{n\to\infty} f_n(x)$ exists for each $x \in E$, then f is an extended real-valued measurable function on E.

Proof. $\square$

Remark 4.0.19. *(i) The conclusion still holds in the previous problem if the sequence $\{f_n\}_{n=1}^{\infty}$ converges a.e. on E.*

(ii) That the pointwise limit of a sequence of measurable functions is measurable is essential to the theory of integration that we are about to create. It is the step that allows us to avoid the difficulty that the pointwise limit of a sequence of Riemann integrable functions may not be Riemann integrable.

Definition 4.0.20. *A **simple function** is a measurable function whose range is a finite subset of $\mathbb{R}$. A simple function s taking values a_j on measurable sets E_j can be written*

$$s = \sum_{j=1}^{n} a_j \chi_{E_j}$$

*Of course this representation is not unique. The **canonical** representation of a simple function is the sum where $\{E_j\}_{j=1}^{n}$ are disjoint and $j \neq k$ implies $a_j \neq a_k$. (In other words, each $E_j = s^{-1}(a_j)$ for the distinct values a_j in the range of s.)*

4.0.21
Let f be an extended real-valued measurable function defined on an interval $[a, b]$. Let $M > 0$. For any $\epsilon > 0$ there is a simple function s so that $|f(x) - s(x)| < \epsilon$ on the set where $|f| \leq M$.
(Partition $[-M, M]$ into slices and use the sets where f takes values in a slice to define s.)

Proof. For any $\epsilon > 0$, let N be the smallest integer such that $N \geq \frac{2M}{\epsilon}$. We will cover the interval $[-M, M]$ with a family of disjoint intervals

$$\{[-M + (j-1)\epsilon, -M + j\epsilon)\}_{j=1}^{N}$$

each of which has length ϵ. Let E_j denote the j^{th} interval, and e_j the midpoint of that interval.

Let $s(x) = \sum\limits_{j=1}^{N} e_j \chi_{E_j}(f(x))$. Then for every x, $s(x)$ is the midpoint of an ϵ-length interval containing $f(x)$, so

$$|f(x) - s(x)| \leq \epsilon/2, \quad \text{and so} \quad |f(x) - s(x)| < \epsilon$$

□

Definition 4.0.22. *A* ***step function*** *p is a simple function whose domain is a closed bounded interval $[a, b]$ and which can be written in the form $\sum_{j=1}^{n} a_j \chi_{\epsilon_j}$ where the E_j are intervals.*

4.0.23
Let s be a simple function whose domain is a closed bounded interval $[a, b]$. For every $\epsilon > 0$ there is a step function p so that $m(\{x \in [a, b] : s(x) \neq p(x)\}) < \epsilon$. (Remember #57).

Proof. As a simple function, we can re-write s as

$$s = \sum_{j=1}^{n} a_j \chi_{E_j}$$

for some measurable sets E_j and some $a_j \in \mathbb{R}$ such that $a_j \neq 0$.

For each E_j, let $\{G_{j_k}\}_{k=1}^{m_j}$ be a finite set of open intervals such that

$$m^*\left(E \setminus \bigcup_{k=1}^{m_j} G_{j_k}\right) + m^*\left(\bigcup_{k=1}^{m_j} G_{j_k} \setminus E\right) < \frac{\epsilon}{n}$$

And furthermore, let $G_j = \bigcup\limits_{k=1}^{m_j} G_{j_k}$ for each j. Then

$$p = \sum_{j=1}^{n} a_j \chi_{G_j}$$

is a step function such that for all $x \in [a, b]$,

$$s(x) \neq p(x)$$

implies that

$$x \in N = (E_j \setminus G_j) \cup (G_j \setminus E_j)$$

for at least one $j \in \{1, ..., n\}$, and therefore

$$m(\{x \in [a, b] : s(x) \neq p(x)\}) = m(N) < \sum_{j=1}^{n} \frac{\epsilon}{n} = \epsilon$$

□

4.0.24
If f is a measurable extended real-valued function defined on a closed bounded interval $[a, b]$ and f is finite a.e., then for any $\epsilon > 0$ there is a step function p and a continuous function g so that $|f - p| < \epsilon$ except on a set of measure less than ϵ, (So any measurable function is "almost continuous.")
(This is mostly assembling the information from the previous three problems.)

Proof. if f is a measurable extended real-valued function on a closed bounded interval $[a, b]$ and f is finite a.e., then for any $\epsilon > 0$, we know by 4.0.21, there is some simple function s such that

$$|f(x) - s(x)| < \epsilon$$

And, since s is simple, we know from 4.0.23 that there is some step function p such that $s(x) = p(x)$ except on a set of measure less than ϵ. Therefore

$$|f(x) - p(x)| = |f(x) - s(x)| < \epsilon$$

except on a set of measure less than ϵ. □

4.0.25
Let E be a measurable set of finite measure. Suppose that f and $\{f_n\}_{n=1}^{\infty}$ are measurable functions defined on E so that $f_n(x) \to f(x)$ for each $x \in E$. Then for every $\epsilon > 0$ and $\delta > 0$ there is a measurable set $A \subset E$ with $m(A) < \delta$ and an integer N so that for all $x \notin A$ and all $n \geq N$,

$$|f_n(x) - f(x)| < \epsilon.$$

(Consider the sets $G_n = \{x : |f_n(x) - f(x)| \geq \epsilon\}$ and the nested collection

$$E_N = \bigcup_{n=N}^{\infty} G_n = \{x : |f_n(x) - f(x)| \geq \epsilon \text{ for some } n \geq N\}.$$

Show $\bigcup\limits_{N=1}^{\infty} E_N = \emptyset$.)

Proof. since $f_n \to f$ for all x, then for any $\epsilon > 0$ and any $\delta > 0$, there exist $A \in \mathfrak{M}$ and $N \in \mathbb{Z}^+$ such that for all $x \notin A$ and all $n \geq N$,

$$|f_n(x) - f(x)| < \epsilon.$$

Consider the sets $G_n = \{x : |f_n(x) - f(x)| \geq \epsilon$ for all n. Then

$$E_N = \bigcup_{n=N}^{\infty} G_n = \{x : |f_n(x) - f(x)| \geq \epsilon \text{ for some } n \geq N\}$$

are a sequence of nested sets that, as countable unions of measurable sets, are also measurable.

$$E_N \supset E_{N+1} \supset \ldots$$

and $\lim_{n\to\infty} |f_n(x) - f(x)| = 0$ for each x, so

$$\bigcap_{N=1}^{\infty} E_N = \emptyset$$

and so $\forall \delta > 0$, there is some N such that $m(E_N) < \delta$ and

$$E_N = \{x : |f_n(x) - f(x)| \geq \epsilon \text{ for some } n \geq N\}$$

which implies that for all $x \notin E_N$,

$$|f_n(x) - f(x)| < \epsilon \quad \forall n > N.$$

$\square$

Remark 4.0.26. *It is clear that the conclusion of the preceding problem continues to hold if we assume only that $f_n(x) \to f(x)$ a.e. on E.*

Definition 4.0.27. *A sequence $\{f_n\}_{n=1}^{\infty}$ of functions **converges uniformly** to a function f on a set E if for every $\epsilon > 0$ there is an integer N so that $|f_n(x) - f(x)| < \epsilon$ for all $n \geq N$ and all $x \in E$.*

Of course the "uniform" part is that N does not depend on x. Note that the preceding problem has a conclusion that somewhat resembles uniform convergence, except that the set A may depend on ϵ. We fix this in the next problem.

4.0.28
Let E be a measurable set of finite measure. Suppose that f and $\{f_n\}_{n=1}^{\infty}$ are measurable functions defined on E so that $f_n(x) \to f(x)$ a.e. on E. Then for each $\alpha > 0$ there is a set $A \subset E$ with $m(A) < \alpha$ and $f_n \to f$ uniformly on $E \setminus A$.
(Apply the preceding problem repeatedly with $\epsilon_n = 1/n$ and $\delta_n = 2^{-n}\alpha$.)

Proof. Let $\alpha > 0$. Then let $\epsilon_n = \frac{1}{n}$, $\delta_n = \frac{\alpha}{2^n}$. Using 4.0.25 for each n, there exists a measurable set $A_n \subset E$ such that $m(A_n) < \frac{\alpha}{2^n}$ and a positive integer N_n such that for all $x \in E \setminus A_n$ and all $j \geq N_n$,

$$|f_j(x) - f(x)| < \frac{1}{n}.$$

Let $A = \bigcup_{i=1}^{\infty} A_i$. Then

$$m(A) \leq \sum_{i=1}^{\infty} A_k < \sum_{i=1}^{\infty} \frac{\alpha}{2^i} = \alpha \sum_{i=1}^{\infty} \frac{1}{2^i} = \alpha$$

So $m(A) < \alpha$.

Let $\epsilon > 0$. Then there exists some n such that $\frac{1}{n} < \epsilon$ (by the Archimedean property), so there exists some N_n such that

$$|f_j(x) - f(x)| < \epsilon_n < \epsilon \ \ \forall j \geq N_n \forall x \in E \setminus A_n.$$

This is therefore true for all $x \in E \setminus A$ as $E \setminus A$ is a subset of $E \setminus A_n$. $\square$

5 The Lebesgue Integral

Remark 5.0.1.

1. *Recall that in the Riemann theory of integration, for a bounded function f on a closed interval $[a,b]$ we define the* ***upper integral****, U, and* ***lower integral****, L as*

$$U = \inf\left\{\int_a^b s(x)dx\right\}, \quad L = \sup\left\{\int_a^b r(x)dx\right\}$$

where the inf and sup are over the set of all step functions s such that $s \geq f$ and the set of all step functions r such that $r \leq f$ respectively and we define the integral of a step function $s(x) = \sum_{j=1}^{n} c_j\chi_{[a_j,b_j]}(x)$ (the canonical representation) to be $\int_a^b s(x)dx = \sum_{j=1}^{n} c_j(b_j - a_j)$.(This is the usual definition expressed in an unusual way. The integrals of the step functions are usually called upper sums and lower sums.)

The function f is ***Riemann integrable*** *if $U = L$. and in that case we call the common value the* ***Riemann integral*** *of f over $[a,b]$, denoted $\int_a^b f(x)dx$.*

2. *Of course one requirement for the Lebesgue integral is that any Riemann integrable function should also be Lebesgue integrable, and that the values of the two integrals should be the same. Another practical requirement is that there should be enough Lebesgue integrable functions that are not Riemann integrable to make it worthwhile to go to the trouble of developing the Lebesgue integral.*

Definition 5.0.2.

1. *If s is a simple function with canonical representation $s = \sum_{j=1}^{n} c_j\chi_{E_j}$ (recall Def Definition 4.0.20) that vanishes outside a set of finite measure, the integral of s is*

$$\int s = \sum_{j=1}^{n} c_j m(E_j).$$

2. *If E is any measurable set, the integral of s over E is*

$$\int_E s = \int s\chi_E.$$

Remark 5.0.3. *Of course we think of this as "the signed area under the curve" as usual. One irritating fine point we need to deal with is to check that if we express the simple function as a linear combination of characteristic functions in a different way, then $\int s$ is also given by the corresponding sum.*

5.0.4

Suppose s is a simple function that vanishes outside a set of finite measure and $s = \sum_{j=1}^{n} c_j \chi_{A_j}$ where the sets $\{A_j\}_{j=1}^{n}$ are pairwise disjoint (but several different $A_j's$ may carry the same value c_j). Then $\int s = \sum_{j=1}^{n} c_j m(A_j)$.

(For any c in the range of s, the canonical representation of s includes $E_c = \cup\{A_j : c_j = c\}$. We know $m(E_c) = \sum_{c_j = c} m(A_j)$.)

Proof. For each $c_j = s(x)$ for some x in the set of finite measure, let $E_j = \cup\{A_i : c_i = c_j\}$. Therefore, there is a canonical representation of $s(x)$:

$$s = \sum_{j=1}^{m} c_j \chi_{E_j}$$

Where $E_j \cap E_k = \emptyset$ if $j \neq k$ (as they are unions of pairwise disjoint sets), and each c_i is distinct. Then applying the definition of the integral of a simple function,

$$\int s = \sum_{i=1}^{m} c_i m(E_i)$$

For any individual E_i, the measure of E_i is the sum of the measures of the A_j which make up the union of E_i. This is possible because all of the A_j's are pairwise disjoint, so countable subadditivity can be applied here. Therefore

$$c_i m[E_i] = c_i \sum_{k=1}^{l} m[A_k].$$

This turns (5.1) into:

$$\int s = \sum_{j=1}^{n} c_j m[A_j]$$

$\square$

5.0.5
Suppose that r and s are simple functions that vanish outside a set of finite measure and that a and b are real numbers.

1. $\int(ar+bs) = a\int r + b\int s$.
2. If $r \geq s$, then $\int r \geq \int s$.

(If $s = \sum_{j=1}^{n} c_j\chi_{E_j}$, $r = \sum_{j=1}^{m} d_j\chi_{F_j}$ are the canonical representations, consider the representations using all possible sets $E_j \cap F_k$. For (2), consider $r - s$.)

Proof.

1. Let $s = \sum_{j=1}^{n} c_j\chi_{E_j}$, $r = \sum_{j=1}^{m} d_j\chi_{F_j}$ be the canonical representations for r and s. Then let $\{G_k\}_{k=1}^{m\times n}$ be given by all of the intersections $E_j \cap F_k$ (where some intersections can be empty). Then we can re-write s and r as

$$s = \sum_{k=1}^{nm} c_j\chi_{G_k} \quad r = \sum_{k=1}^{nm} d_j\chi_{G_k}$$

Where, as the G_k are each a subset of one and only one E_j, and one F_j, so the c_j and d_j are not necessarily unique, but are the coefficients associated to the supersets E_j, F_j. respectively. Then

$$\begin{aligned}
\int(ar+bs) &= \int\left(a\sum_{k=1}^{mn} d_k\chi_{G_k} + b\sum_{k=1}^{mn} c_k\chi_{G_k}\right)\\
&= \sum_{k=1}^{mn}(ad_k + bc_k)m(G_k)\\
&= a\sum_{k=1}^{mn} d_k m(G_k) + b\sum_{k=1}^{mn} c_k m(G_k)\\
&= a\int r + b\int s
\end{aligned}$$

2. If $r \geq s$, then consider $\int(r-s)$. From part (1), we know that this is equal to $\int r - \int s$, and since $r \geq s$, then $r - s \geq 0$, and so we can write $r - s$ as a sum of non-negative values, and so $\int(r-s) = \int r - \int s \geq 0$, and so $\int r \geq \int s$.

$\square$

Remark 5.0.6. *By the preceding problem and induction, if $\{A_j\}_{j=1}^{n}$ is any collection of sets of finite measure, not necessarily disjoint, and if $s = \sum_{j=1}^{n} c_j\chi_{A_j}$, then $\int s = \sum_{j=1}^{n} c_j m(A_j)$ so that the restriction above to pairwise disjoint sets is unnecessary.*

5.0.7
Let f be a bounded function defined on a measurable set E with $m(E)$ finite. Then

$$\inf_{s \geq f} \int s = \sup_{r \leq f} \int r$$

if and only if f is measurable.
(If $|f| \leq M$ is measurable, partitions the range into $2n$ equal portions and use this to define simple functions r and s with $r \leq f \leq s$. For the other direction, let $\{r_n\}$, $\{s_n\}$ be sequences of simple functions whose integrals approach the sup and inf. Then $\sup r_n \leq f \leq \inf s_n$, $\sup r_n$ and $\inf s_n$ are measurable, and you can show $m\{x : \sup r_n \neq \inf s_n\} = 0$.)

Proof. $(\Rightarrow)$: Note first that if $m(E) = 0$, then f is measurable, as the set $\{x \in E : f(x) < \alpha\}$ has measure 0 for any α and so is measurable.

Let $\{r_n\}$, $\{s_n\}$ be sequences of step functions such that

$$\lim_{n\to\infty} \int s_n = \inf \quad \& \quad \lim_{n\to\infty} \int r_n = \sup$$

Now, since $r_n \leq f \leq s_m$ for all n and m, then

$$\sup_n r_n \leq f \leq \inf_n s_n$$

and for all $\epsilon > 0$, there exist integers N and M such that for all $n \geq N$ and $m \geq M$,

$$\left| \int r_n - \int f \right| < \frac{\epsilon}{2} m(E) \quad \text{and} \quad \left| \int s_m - \int f \right| < \frac{\epsilon}{2} m(E)$$

So

$$\left| \int s_p - \int r_p \right| < \epsilon \cdot m(E) \;\; \forall \; p \geq max(M, N)$$

and by 5.0.5, this implies that

$$\left| \int (s_p - r_p) \right| < \epsilon \cdot m(E)$$

and since $s_p \geq r_p$ for all p, this is the same as

$$\int (s_p - r_p) < \epsilon \cdot m(E)$$

Now, define $\tau - \epsilon \; \chi_E$. Then τ is a simple function with a value of ϵ everywhere on E, and 0 elsewhere. So, by 5.0.5(2),

$$\int (s_p - r_p) < \int \tau = \epsilon \cdot m(E) \implies (s_p - r_p) < \tau$$

Therefore $s_p - r_p < \epsilon$ for all $x \in E$, and so, by 4.0.18, f is measurable.

($\Leftarrow$) : let f be measurable and bounded on $E \in \mathfrak{M}$ where $m(E)$ is finite, and let $|f| \leq M$ for all $x \in E$. Then partition $[-M, M]$ into $2n$ equal intervals for any n, so that

$$E_j = \left[-M + \frac{M}{n}(j-1), -M + \frac{M}{n}j\right)$$

From 4.0.21, define r_n, s_n as

$$s_n = \sum_{j=1}^{2n}(-M + \tfrac{M}{n}j)\chi_{E_j}(f(x))$$
$$r_n = \sum_{j=1}^{2n}(-M + \tfrac{M}{n}(j-1))\chi_{E_j}(f(x))$$

Then

$$\int(s_n - r_n) = \sum_{j=1}^{2n}\left[\left(-M + \frac{M}{n}(j-1)\right)\frac{M}{n} + \left(-M + \frac{M}{n}j\right)\frac{M}{n}\right] = \frac{M^2}{n}$$

Now, since M is fixed, as n increases we have

$$\frac{2M^2}{n} \geq \int s_n - \int r_n \geq \inf s \geq f \int s - \sup_{r \leq f}\int r \geq 0$$

and so inf = sup. □

Remark 5.0.8. *The moral of the preceding problem is that, unlike the situation with the Riemann integral, we do not need to consider both upper integrals and lower integrals because we already have the correct class of functinos, the measurable functions. Thus we can say, forget about lower integrals and just approximate from above. (This is analagous to not needing inner measure to help define Lebesgue measure.)*

Definition 5.0.9. *If f is a bounded measurable function defined on a measurable set E with $m(E)$ finite, then the (Lebesgue) integral of f is*

$$\int f = \inf_{s \geq f}\int s$$

where the inf is over the class of all simple functions s with $s \geq f$. We sometimes write $\int_E f$. If $E = [a, b]$, we write $\int_a^b f$.
If A is a measurable subset of E, we define $\int_A f = \int f\chi_A$.

5.0.10
Let f be a bounded function defined on $[a, b]$. If f is Riemann integrable on $[a, b]$, then f is measurable and the Riemann integral $R\int_a^b f = \int_a^b f$.
(A step function is a simple function.)

Proof. □

5.0.11
The Lebesgue integral has the following properties for bounded measurable functions defined on a set E of finite measure.

1. $\int_E (af + bg) = a \int_E f + b \int_E g$,
2. If $f \leq g$ a.e., then $\int_E f \leq \int_E g$,
3. If $f = g$ a.e., then $\int_E f = \int_E g$,
4. If $\alpha \leq f(x) \leq \beta$ for almost all $x \in E$, then $\alpha\, m(E) \leq \int_E f \leq \beta\, m(E)$,
5. If A and B are disjoint measurable subsets of E, then $\int_{A \cup B} f = \int_A f + \int_B$.

(For (1) show $\int af = a \int f$ (easy) and $\int (f+g) = \int f + \int g$ (Requires some manipulation of infs. You may also find # 5.0.7 useful.) For (2) you can show $f \leq 0$ a.e. implies $\int_E f \leq 0$. (3), (4), and (5) follow from previous parts.)

Proof. □

Remark 5.0.12. *It follows from part (3) and the previous prbolem about measurability of functions equal almost everywhere that changing the values of a function in a bounded way on a countable subset of E does not affect $\int_E f$. Thus the Dirichlet example of a non-Riemann integrable function (the characteristic function of the set $\mathbb{Q}_0$ of rationals in $[0,1]$) is not a problem here. We have $\int_0^1 \chi_{\mathbb{Q}_0} = 0$.*

5.0.13
Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions on a measurable set E of finite measure such that for some $M > 0$, $|f_n| \leq M$ a.e. for each n. If $\lim\limits_{n\to\infty} f_n(x)$ exists a.e. on E, then the function defined by $\lim\limits_{n\to\infty} f_n(x)$ is equal almost everywhere on E to a bounded measurable function f and

$$\int_E f = \lim_{n\to\infty} \int_E f_n.$$

(Use # 4.0.25 or Egoroff's Thm # 4.0.28).

Proof. If $\lim\limits_{n\to\infty} f_n$ exists a.e. on E, then $f = \lim\limits_{n\to\infty} f_n$ is measurable by 4.0.18, and $|f| \leq M$ a.e.

Then by 4.0.28, for any $\epsilon > 0$, there is $A \subset E$ with $m(A) < \epsilon/4M$ and $f_n \to f$ uniformly on $E \setminus A$. So

$$\int_E (f - f_n) \leq \int_E |f - f_n| = \int_{E \setminus A} |f - f_n| + \int_A |f - f_n|$$

and since $f_n \to f$ uniformly on $E \setminus A$, there is $N \in \mathbb{Z}$ such that for all $n \geq N$,

$$\int_E (f - f_n) < \int_E |f - f_n| \leq \int_{E\setminus A} \epsilon/2m(E \setminus A) + \int_A |f - f_n|$$

and so

$$\begin{array}{rcll}
\int_E (f - f_n) & < & \frac{\epsilon}{2m(E\setminus A)} \cdot m(E \setminus A) + \int_A |f - f_n| & \forall n \geq N \\
 & < & \frac{\epsilon}{2} + \int_A 2M & \forall n \geq N \\
 & < & \frac{\epsilon}{2} + m(A)\ 2M & \forall n \geq N \\
 & < & \frac{\epsilon}{2} + \frac{\epsilon}{4M} 2M & \forall n \geq N \\
\int_E (f - f_n) & < & \frac{\epsilon}{2} + \frac{\epsilon}{2} & \forall n \geq N \\
\int_E f - \int_E f_n & < & \epsilon & \forall n \geq N \\
\int_E f - \lim\limits_{n\to\infty} \int_E f_n & < & \epsilon & \\
\Rightarrow \quad \int_E f & = & \lim\limits_{n\to\infty} \int_E f_n &
\end{array}$$

□

Remark 5.0.14.

1. *The preceding problem is a form of the Bounded Convergence Theorem-the first of the important convergence theorems. It asks more than pointwise convergence (the uniform boundedness of the functions), but much less than uniform convergence.*

2. *Now we'll extend things by dropping our conditions that the functions be bounded and defined on a set of finite measure. We'll start by defining the integral for non-negative functions only, and then writing an arbitrary function as a linear combination of these. The idea for non-negative functions is to take the sup over all bounded functions non-zero on a set of finite measure that are below the given function.*

Definition 5.0.15. *Let f be a non-negative measurable function on a measurable set E. Then we define*

$$\int_E f = \sup_{h \leq f} \int_E h$$

where the sup is over all bounded non-negative measurable functions $h \leq f$ such that $m\{x \in E : h(x) \neq 0\} < \infty$ and we interpret $\int_E h$ as $\int_{\{x\in E: h(x)\neq 0\}} h$.

5.0.16
If f and g are non-negative measurable functions defined on a measurable set E, then

1. $\int_E \alpha f = \alpha \int_E f$ for any $\alpha \geq 0$,

2. $\int_E (f + g) = \int_E f + \int_E g$,

3. If $f \geq g$ a.e. then $\int_E f \geq \int_E$. In particular, if $f \geq 0$ a.e. then $\int_E f \geq 0$.

Proof. □

5.0.17
Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of non-negative measurable functions on a measurable set E such that $f(x) = \lim_{n\to\infty} f_n(x)$ exists for almost all $x \in E$. Then f is measurable and

$$\int_E f \leq \liminf \int_E f_n.$$

Proof. □

Remark 5.0.18. *The preceding problem is known as Fatou's Lemma. You should be sure that you know of an example where equality does not hold.*

5.0.19
Let $\{f_n\}_{n=1}^{\infty}$ be a non-decreasing sequence of non-negative measurable functions on a measurable set E, and let $f(x) = \lim_{n\to\infty} f_n(x)$ for each $x \in E$. ($f(x) = \infty$ is allowed.) Then

$$\int_E f = \lim_{n\to\infty} \int_E f_n$$

where it is possible that both sides of the equation are $+\infty$.

Proof. consider $g = f - f_n$. Then g is positive and $g \leq f$.

WTS: $\lim_{n\to\infty} \int_E f_n \leq \int_E f$ and $\lim_{n\to\infty} \int_E f_n \geq \int_E f$.

$f_n \leq f$ for all n, and so

$$\lim_{n\to\infty} \int_E f_n \leq \int_E f$$

On the other hand, using Fatou's Lemma, we have

$$\int_E f \leq \liminf \int_E f_n = \lim_{n\to\infty} \int_E f_n$$

and so

$$\int_E f = \lim_{n\to\infty} \int_E f_n$$

□

Remark 5.0.20. *This is the monotone Convergence Theorem. It and Fatou's Lemma are the principal convergence theorems for non-negative functions.*

5.0.21

(a) Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of non-negative measurable functions on a measurable set E. Then

$$\int_E \left(\sum_{k=1}^{\infty} f_k\right) = \sum_{k=1}^{\infty} \left(\int_E f_k\right).$$

(b) Let f be a non-negative measurable function on a measurable set E, and let $\{A_n\}_{n=1}^{\infty}$ be a sequence of mutually disjoint measurable subsets of E. Then

$$\int_{\bigcup\limits_{n=1}^{\infty} A_n} f = \sum_{n=1}^{\infty} \int_{A_n} f.$$

Proof.

(a) Define $g_n = \sum\limits_{k=1}^{n} f_k$. Then $\{g_n\}$ is a nondecreasing sequence and so by 5.0.19,

$$\int_E \lim_{n\to\infty} g_n = \lim_{n\to\infty} \int_E g_n$$

and now, we will show that $\int_E \sum\limits_{k=1}^{n} f_k = \sum\limits_{k=1}^{n} \int_E f_k$ by induction.

Base case: $\int_E (f+g) = \int_E f + \int_E g$ by 5.0.11

Induction step: Assume that the hypothesis is true for some $n-1$, and then

$$\begin{aligned} \int_E \sum_{k=1}^{n} f_k &= \int_E \left[\sum_{k=1}^{n-1} f_k + f_n\right] \\ &= \int_E \sum_{k=1}^{n-1} f_k + \int_E f_n \end{aligned}$$

So

$$\int_E \sum_{k=1}^{\infty} f_k = \sum_{k=1}^{\infty} \int_E f_k$$

(b) Let $A = \bigcup\limits_{n=1}^{\infty} A_n$. Then

$$\begin{aligned} \int_A f &= \int_A \sum_{n=1}^{\infty} f\chi_{A_n} = \sum_{n=1}^{\infty} \int_A f\chi_{A_n} \\ \int_{\bigcup\limits_{n=1}^{\infty}} f &= \sum_{n=1}^{\infty} \int_{A_n} f. \end{aligned}$$

$\square$

5.0.22
Let f and g be non-negative measurable functions on a measurable set E such that $f(x) \geq g(x)$ on E. If $\int_E f < \infty$, then $\int_E g < \infty$, $\int_E (f-g) < \infty$, and

$$\int_E (f-g) = \int_E f - \int_E g.$$

Proof.

- If $\int_E f < \infty$, then by 5.0.16,

$$\int_E g \leq \int_E f < \infty \quad \Rightarrow \quad \int_E g < \infty$$

- Since $f \geq g$ on E, then $0 \leq f - g \leq f$ on E, and so

$$\int_E (f-g) \leq \int_E f < \infty$$

-

$$\underbrace{\int_E (f+g) + \int_E g}_{\text{both non-negative}} = \textstyle\int_E [(f-g)+g] = \int_E f \quad \Rightarrow \quad \int_E f = \int_E (f-g) + \int_E g$$

$$\Rightarrow \quad \textstyle\int_E f - \int_E g = \int_E (f-g)$$

□

5.0.23
Let f be a non-negative measurable function on a mesaurable set E such that $\int_E f < \infty$. Then for each $\epsilon > 0$ there is $\delta > 0$ so that for every $A \subset E$ with $m(A) < \delta$, it is the case that $\int_A f < \epsilon$.

Proof. □

Remark 5.0.24. *This looks like a sort fo continuity condition. As the proof demonstrates, it really says that an integrable function cannot have "delta function" parts-places of size zero but positive area (or positive mass if you prefer to think of f as representing density).*

Remark 5.0.25. *Let f be an extended real-valued function. The positive and negative parts of f, $f_+ = max\{f, 0\}$ and $f_- = max\{-f, 0\}$ can be defined as extended real-valued functions just as in 4.0.7 with the properties already developed. Notice that for nay x¡ at most one of f_+ and f_- is different from 0, and that, as before, $f = f_+ - f_-$ and $|f| = f_+ + f_-$.*

Definition 5.0.26. *1. A non-negative extended real-valued measurable function f defined on a measurable function f defined on a measurable set E is* ***integrable over*** *E if $\int_E f < \infty$.*

2. An extended real-valued measurable function f defined on a measurable set E is ***integrable over*** *E if both its positive part and its negative part, f_+ and f_-, are integrable over E. In that case, we define*

$$\int_E f = \int_E f_+ - \int_E f_-.$$

We denote the set of all functions integrable over E by $L^1(E)$.

Remark 5.0.27. *1. An integrable extended real-valued function f must have finite values almost everywhere.*

2. A measurable function f on E is integrable over E if and only if $|f| = f_+ + f_-$ is integrable over E. Thus $L^1(E)$ is usually defined as the set of all measurable functions on E such that $\int_E |f| < \infty$.

5.0.28

If f and g are integrable on E, and if α, β are real numbers, then

(a) $\int_E(\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g$.

(b) if $f \leq g$, then $\int_E f \leq \int_E g$,

(c) if A and B are disjoint measurable subsets of E, then $\int_{A\cup B} f = \int_A f + \int_B f$.

Proof.

(a) if $|f| = f_+ + f_-$, then, if $\alpha >= 0$,

$$\int_E \alpha f = \int_E \alpha(f_+ - f_-) = \int_E [\alpha f_+ - \alpha f_-]$$

and by 5.0.22, this is

$$\begin{aligned} \textstyle\int_E \alpha f = \int_E[\alpha f_+ - \alpha f_-] &= \textstyle\int_E \alpha f_+ - \int_E \alpha f_- \\ &= \textstyle\alpha \int_E f_+ - \alpha \int_E f_- \\ &= \textstyle\alpha \left[\int_E f_+ - \int_E f_-\right] \\ &= \textstyle\alpha \int_E f. \end{aligned}$$

If on the other hand, $\alpha < 0$, then

$$\begin{aligned} \textstyle\int_E \alpha f = \int_E(-\alpha)(-f) &= \textstyle(-\alpha) \int_E(-f) \\ &= \textstyle-\alpha \left[\int_E f_- - \int_E f_+\right] \\ &= \textstyle-\alpha \int_E f_- + \alpha \int_E f_+ \\ &= \textstyle\alpha \int_E f_+ - \alpha \int_E f_- \\ &= \textstyle\alpha \int_E f. \end{aligned}$$

Now, to show that $\int_E (f+g) = \int_E f + \int_E g$,

$$\begin{aligned}\textstyle\int_E (f+g) = \int_E [f_+ + g_+ - f_- - g_-] &= \textstyle\int_E f_+ + \int_E g_+ - \int_E f_- - \int_E g_- \\ &= \textstyle\int_E f_+ - \int_E f_- + \int_E g_+ - \int_E g_- \\ &= \textstyle\int_E (f_+ - f_-) + \int_E (g_+ - g_-) \\ &= \textstyle\int_E f + \int_E g.\end{aligned}$$

(b) If $g \geq f$, then $g - f \geq 0$, which (using the first part of this problem) implies that

$$0 \leq \int_E (g-f) = \int_E g - \int_E f \quad \Rightarrow \quad \int_E g \geq \int_E f$$

(c)

$$\int_{A\cup B} f = \int_{A\cup B} (f\chi_A + f\chi_B) = \int_{A\cup B} f\chi_A + \int_{A\cup B} f\chi_B = \int_A f + \int_B f$$

□

5.0.29
Let g be a non-negative integrable function on E, and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions such that $|f_n| \leq g$ a.e. on E for each n. If $f_n(x) \to f(x)$ for almost all of $x \in E$, then

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

Proof. If $|f_n| \leq g$ a.e. for all n, then $\{g - f_n\}_{n=1}^{\infty}$ and $\{g + f_n\}_{n=1}^{\infty}$ are both non-negative a.e. □

Remark 5.0.30. *The preceeding result is the Lebesgue Dominated Convergence Theorem. It is perhaps the most often used of the convergence theorems. Note that our first convergence theorem 5.0.13, is a special case.*

Remark 5.0.31. *Note that f being Lebesgue integrable over a set E is not the same as f having an improper Riemann integral over E. For instance, $\int_0^\infty \frac{\sin x}{x} dx$ exists, as you may see by observing that if we set $a_n = \int_{(n-1)\pi}^{n\pi} \frac{\sin x}{x} dx$ then $\int_0^\infty \frac{\sin x}{x} dx = \sum_{n=1}^{\infty} a_n$ converges by the Alternating Series Test. However the integral of the positive part of $\frac{\sin x}{x}$ is infinite (and the integral of the negative part also) since $|a_n| \approx \frac{k}{n}$. In other words, $\sum_{n=1}^{\infty} a_n$ converges conditionally, but not absolutely. f being Lebesgue integrable corresponds to absolute convergence-convergence via cancellation is not allowed. This makes working with Lebesgue integrable functions much simpler, at the expense of making the class of integrable functions somewhat smaller.*

6 The Classical Banach Spaces

Remark 6.0.1. *We explore briefly one major application of the Lebesgue integral. First I recall some familiar definitions.*

Definition 6.0.2. *A **metric space** is a pair (X, d) where X is a set and $d : X \times X \to \mathbb{R}$ is a function such that*

(a) $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0$ iff $x = y$.

(b) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(c) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Definition 6.0.3.

1. *A sequence $\{x_n\}_{n=1}^{\infty}$ **converges to the limit** x in the metric space (X, d) if for every $\epsilon > 0$ there is an integer N such that $d(x_n, x) < \epsilon$ whenever $n \geq N$.*

2. *A sequence $\{x_n\}_{n=1}^{\infty}$ is a **Cauchy sequence** in the metric space (X, d) if for every $\epsilon >)$ there is an integer N such that $d(x_n, x_m) < \epsilon$ whenever $m, n \geq N$. It is easy to see that every convergent sequence is Cauchy, but the converse is in general not true.*

3. *A metric space (X, d) is **complete** if every Cauchy sequence is convergent.*

Remark 6.0.4. *You will recall that the set of rationals $\mathbb{Q}$ with the usual distance on $\mathbb{R}$ is not complete, but $\mathbb{R}$ is complete, since "the holes have been filled in." A somewhat different example is that the open interval (0,1) with the usual distance is not complete, because sequences "converging to 0 or 1" do not have a limit. In each case we know how to find the complete metric space that appears to contain the given metric space most efficiently. You may have seen a general procedure to produce the "completion" of any metric space-the idea is that the points of the completion are equivalence classes of Cauchy sequences of points of the original space. Constant sequences may be identified with points of the original space, so we can think of this as a larger space. It is rather hard to visualize this abstract completion, so more concrete constructions (as in the two examples) are desirable. The Lebesgue integral will provide some of these.*

Example 6.0.5.

1. *Any non-empty subset of $\mathbb{R}^n$ together with the usual Euclidean distance is a metric space. It is also possible to introduce alternative distance functions,*

such as $d_1(x,y) = \sum_{j=1}^{n} |x_j - y_j|$ *for* $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n)$ *in* $\mathbb{R}^n$. *It turns out that these are not very different in the sense that a sequence converges with respect to any readsonable distance function in* $\mathbb{R}^n$ *(such as* d_1*) if and only if it converges to the same limit with respect to the Euclidean distance.*

2. *The set* $C[0,1]$ *of all real-valued continuous functions on* $[0,1]$ *is a metric space with the distance* $d(f,g) = \max_x |f(x) - g(x)|$. *(Since the set* $[0,1]$ *is compact, the supremum of the differences* $|f(x) - g(x)|$ *is actually a max.) Convergence with respect to this distance is uniform convergence. The theorem that the limit of a uniformly convergent sequence of continuous functions is continuous translates in this terminology to the statement that* $C[01,]$ *with this metric is a complete metric space. However for the set* $C[0,1]$ *it is easy to define other distance functions which are essentially different in the sense that they produce a form of convergence that is not equivalent to uniform convergence and with repsect to which* $C[01,]$ *is not a complete metric space. We will do this shortly.*

Definition 6.0.6. *A real* ***linear space*** *(or real* ***vector space****) is a triple* $(V, +, \cdot)$ *consisting of a set* V *of objects (called* ***vectors****) a mapping* $+ : V \times V \to V$ *(called* ***addition****) and a mapping* $\cdot : \mathbb{R} \times V \to V$ *(called* ***scalar multiplication****) such that*

1. $u + v = v + u$ *for all* $u, v \in V$,
2. $(u + v) + w = u + (v + w)$ *for all* $u, v, w \in V$,
3. *there is a vector* $0 \in V$ *(called the* ***zero vector****) such that* $u + 0 = u$ *for all* $u \in V$,
4. *for each* $u \in V$ *there is a vector* $-u \in V$ *such that* $u + (-u) = 0$,
5. $c(u + v) = cu + cv$ *for all* $u, v \in V$ *and all* $c \in \mathbb{R}$,
6. $(c + d)u = cu = du$ *for all* $u \in V$ *and all* $c, d \in \mathbb{R}$,
7. $c(du) = (cd)u$ *for all* $c, d \in \mathbb{R}$ *and all* $u \in V$,
8. $1u = u$ *for all* $u \in V$.

Definition 6.0.7. *A subset* W *of a vector space* V *is a* ***subspace*** *of* V *if it is a vector space in its own right with the same operations as* V.

Example 6.0.8.

1. *Once again* $\mathbb{R}^n$ *with the obvious component-wise operations is the basic example. This time, however, only certain subsets are subspaces of* $\mathbb{R}^n$. *(Those consisting of lines or planes through the origin, or just the zero element for* $n = 3$.*)*

2. $C[0,1]$ *is also a vector space with the usual operations of adding functions and multiplying a function by a real number. The set* $\mathscr{P}$ *of of polynomial functions is a subspace of* $C[0,1]$, *and for every positive integer* n, *the set* $\mathscr{P}_n$ *of polynomials of degree at most* n *is a subspace of* $\mathscr{P}$.

Definition 6.0.9. *A **normed linear space** is a pair $(V, \|\cdot\|)$ consisting of a real linear space V (strictly $(V, +, \cdot)$) and a mapping $\|\cdot\| : v \to \mathbb{R}$ called a **norm** on V such that*

1. $\|x\| \geq 0$ *for all* $x \in V$ *and* $\|x\| = 0$ *if and only if* $x = 0$ *(the zero element of* V*)*

2. $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in V$.

Remark 6.0.10. *A normed linear space is a metric space with the distance function* $d(x, y) = \|x - y\|$.

Definition 6.0.11. *A normed linear space that is a complete metric space with this distance function is called a **Banach space**.*

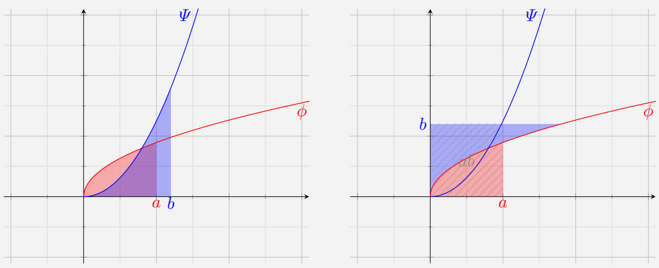
Example 6.0.12. $C[0,1]$ *with the norm* $\|f\| = \max_x |f(x)|$ *is a Banach space. So is* $\mathbb{R}^n$ *with the Euclidean norm or, for that matter, any other reasonable norm.*

6.0.13
Let ϕ be a continuous, increasing, real-valued function defined on $[0, \infty)$ such that $\phi(0) = 9$ and $\lim_{x\to\infty} \phi(x) = \infty$. Let $\mathit{\Psi}$ be the inverse of ϕ. Let $\Phi(x) = \int_0^x \phi$ and $\Psi(x) = \int_0^x \mathit{\Psi}$. Then for any positive real numbers a and b,

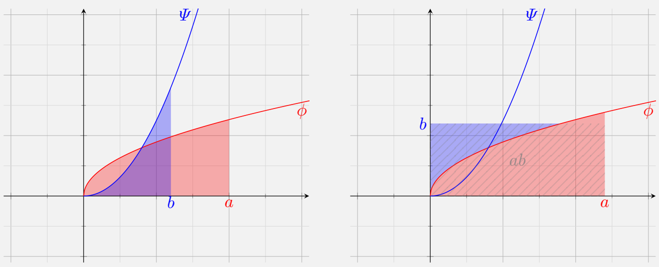
$$ab \leq \Phi(a) + \Psi(b)$$

with equality if and only if $b = \phi(a)$.
(interpret $\Phi(a)$ and $\Phi(b)$ as areas associated with the graph of ϕ. For $\Psi(b)$ you will need to remember that the graph of $\mathit{\Psi}$ is the "mirror image" of the graph of ϕ, so you can find $\Psi(b)$ as an area between the y-axis and the graph of ϕ. This result is called Young's Inequality.)

Proof. (not yet, but pretty pictures anyway)
Case 1: $a \leq b$:



Case 2: $b > a$:



□

6.0.14
Let $p > 1$. If $\phi(x) = x^{p-1}$, then the preceding inequality takes the form

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q$$

where q is the number such that $\frac{1}{p} + \frac{1}{q} = 1$. Equality holds if and only if $b = a^{p-1}$ or equivalently iff $b^q = a^p$.

Proof. □

6.0.15
Let $p > 1$ and let q be so that $\frac{1}{p} + \frac{1}{q} = 1$. Let f and g be measurable functions on a set E such that $\int_E |f|^p < \infty$ and $\int_E |g|^q < \infty$. Then

$$\int_E |fg| \leq \left(\int_E |f|^p\right)^{1/p} \left(\int_E |g|^q\right)^{1/q}.$$

In particular, the left side of the inequality is finite. Equality holds iff $|f|^p = |g|^q$ a.e. on E or $f = 0$ a.e. or $g = 0$ a.e.
(Suppose first that $\int_E |f|^p = \int_E |g|^q = 1$. Apply the preceding problem with $a = |f(x)|,\ b = |g(x)|,\ x \in E$ and integrate. For arbitrary f and g, consider $F = f/(\int_E |f|^p)^{1/p},\ G = g/(\int_E |g|^q)^{1/q}$. This is Hölder's Inequality. For $p = q = 2$ it is usually called the Cauchy or Cauchy-Schwarz Inequality.)

Proof. □

6.0.16
Let $p > 1$. If $\int_E |f|^p < \infty$ and $\int_E |g|^p < \infty$ then $\int_E |f+g|^p < \infty$ and

$$\left(\int_E |f+g|^p\right)^{1/p} \leq \left(\int_E |f|^p\right)^{1/p} + \left(\int_E |g|^p\right)^{1/p}.$$

(For the first assertion, for any $x \in E$,

$$\begin{aligned} |f(x)+g(x)|^p &\leq (2\max\{|f(x)|, |g(x)|\})^p \\ &\leq 2^p \max\{|f(x)|^p, |g(x)|^p\} \leq 2^p(|f(x)|^p + |g(x)|^p). \end{aligned}$$

Then show $|f+g|^p \leq |f+g|^{p-1}|f| + |f+g|^{p-1}|g|$ and apply Hölder's Inequality where you use $q(p-1) = p$. The result of this problem is called Minkowski's Inequality.)

Proof. □

Definition 6.0.17. *A measurable function f on a set E is* ***essentially bounded*** *if its* ***essential supremum***

$$\|f\|_\infty = \inf\{M : |f(x)| \leq M a.e.\}$$

is finite. Note that if f is continuous on E, then $\|f\|_\infty = \sup\{|f(x)| : x \in E$.

6.0.18
If f is a measurable function on a set E such that $\|f\|_\infty$ is finite, then there is a subset A of E such that $m(A) = 0$ and $|f(x)| \leq \|f\|_\infty$ for all $x \in E \setminus A$.
(If not, then for some n, there is $B, m(B) > 0$ and $|f(x)| \geq \|f\|_\infty + \frac{1}{n}$ on B.)

Proof. □

6.0.19
If $\|f\|_\infty$ and $\|g\|_\infty$ are finite, then $\|f+g\|_\infty$ is finite and $\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.

Proof. □

6.0.20
If $\int_E |f| < \infty$ and $\int_E |g| < \infty$, then $\int_E |f+g| \leq \int_E |f| + \int_E |g|$.

Proof. □

Remark 6.0.21. *We have now shown that the mapping $f \to \|f\|_p$ satisfies the triangle inequality for each p, $1 \leq p \leq \infty$, where $\|f\|_p = (\int_E |f|^p)^{1/p}$ for $1 \leq p < \infty$. These functions are, however, not quite norms on the sets of functions on which they are defined because it is not true that $\|f\|_p = 0$ implies that $f = 0$. What it does imply is that $f(x) = 0$ a.e. on E. We can fix this by regarding our objects as equivalence classes of functions, where $f \sim g$ means $f = g$ a.e. With this understanding we have now shown that for each p, $1 \leq p < \infty$, the set*

$$L^p(E) = \left\{ f : \int_E |f|^p < \infty \right\}$$

is a normed linear space with norm $\|f\|_p = (\int_E |f|^p)^{1/p}$. Also

$$L^\infty(E) = \{f : \|f\|_\infty < \infty\}$$

is a normed linear space.

Remark 6.0.22. *As normed linear spaces the L^p spaces are metric spaces the L^p spaces are metric spaces with the distance function $d_p(f,g) = \|f-g\|_p$. The next big question is whether they are complete with respact to this distance function. This is easier for L^∞ than for L^p with $P < \infty$.*

6.0.23
The set $C[0,1]$ can be considered a subset of $L^p[0,1]$ for each p. It is in fact dense in L^p for each finite p as we could see by using the problems from some time ago about approximatin measurable functions by continuous functions. (It is not dense in L^∞ as we will see in the next problem.) Consider the sequence $\{f_n\}_{n=3}^\infty$ of continuous functions defined by

$$f_n(x) = \begin{cases} 0, & 0 \le x \le \frac{1}{2} - \frac{1}{n} \\ \frac{n}{2}\left(x - \frac{1}{2} + \frac{1}{n}\right), & \frac{1}{2} - \frac{1}{n} \le x \le \frac{1}{2} + \frac{1}{n} \\ 1, & \frac{1}{2} + \frac{1}{n} \le x \le 1. \end{cases}$$

(a) Show that this sequence is a Cauchy sequence in $L^1[0,1]$. (Not a calc problem. Draw a picture and compute area.)

(b) Show that this sequence is a Cauchy sequence in $L^p[0,1]$ for $1 < p < \infty$. (Use $|f_n - f_m| < 1$ on $[0,1]$!)

(c) What is the limit f of this sequence in $L^p[0,1]$, $1 \le p < \infty$? Justify your conclusion.

Proof. □

6.0.24

(a) Show that the sequence $\{f_n\}_{n=3}^\infty$ of the preceding problem is not Cauchy in $L^\infty[0,1]$, or equivalently in $C[0,1]$ with the uniform norm.

(b) Show that the limit f from the preceding problem satisfies $\|f - f_n\|_\infty = 1/2$ for each n.

(c) Show that for any $g \in C[0,1]$, $\|f - g\|_\infty \ge 1/2$. (Thus $C[0,1]$ is not dense in $L^\infty[0,1]$.)

Proof. □

6.0.25
$L^\infty(E)$ is a Banach space.
(Let $\{f_n\}_{n=1}^\infty$ be a Cauchy sequence in $L^\infty(E)$ and choose a representative of each f_n, which we still denote by f_n. For each positive integer k there is n_k so that $\|f_n - f_m\|_\infty < 2^{-k}$ whenever $m, n \ge n_k$. Consider the sequence $\{f_{nk}\}_{k=1}^\infty$. Then $\ell > k$ implies $\|f_{n\ell} - f_{nk}\| < 2^{-k}$. There is a set A, $m(A) = 0$, such that the series $f_{n_1}(x) + \sum_{i=2}^{k}(f_{n_i}(x) - f_{n_{i-1}}(x))$ of real numbers converges absolutely for each

$x \in E \setminus A$. The function f defined as the pointwise limit of this series is in $L^\infty(E)$. Given $\epsilon > 0$ there is n_ϵ so that $m, n \geq n_\epsilon$ implies that $\|f_n - f_m\|_\infty < \epsilon$. Then for any such f_m and any $x \in E \setminus A$, $|f(x) - f_m(x)| = \lim_{k\to\infty} |f_{n_k}(x) - f_m(x)| \leq \epsilon$.)

Proof. □

6.0.26
For each $1 \leq p < \infty$, if $\{f_n\}_{n=1}^\infty$ is a Cauchy sequence in $L^p(E)$ and if we choose a representative of each f_n (we will still denote these functions by $\{f_n\}$), then there is a subsequence $\{f_{n_k}\}_{k=1}^\infty$ that converges pointwise a.e. on E to a measurable function f such that $\int_E |f|^p < \infty$. (For each positive integer k there is n_k so that $\|f_n - f_m\|_p < 2^{-k}$ whenever $m, n \geq n_k$. Why? Consider the sequence $\{f_{n_k}\}_{k=1}^\infty$. Then $\ell > k$ implies $\|f_{n_\ell} - f_{n_k}\|_p < 2^{-k}$. For each integer $k \geq 2$, set $g_k = |f_{n_1}| + \sum_{i=2}^{k} |f_{n_i} - f_{n_{i-1}}|$. Then $\{g_k\}_{k=1}^\infty$ is a non-decreasing sequence of non-negative functions such that $\|g_k\|_p \leq \|f_{n_1}\|_p + 1$ for each k, and the extended real-valued function $g(x) = \lim_{k\to\infty} g_k(x)$ satisfies $\int_E g^p = \lim_{k\to\infty} \int_E g_k^p < \infty$. For almost all $x \in E$, the series $f_{n_1}(x) + \sum_{i=2}^{\infty} (f_{n_i}(x) - f_{n_{i-1}}(x))$ of real numbers converges absolutely, and so defines a measurable function f there. This function is in $L^p(E)$.)

Proof. □

6.0.27
If $\{f_n\}_{n=1}^\infty$ and f are as in the preceding problem, then $f_n \to f$ in $L^p(E)$. Thus $L^p(E)$ is complete.
(Let $\epsilon > 0$ be given. For $m, n > n_\epsilon$ Then Fatou's Lemma implies

$$\int_E |f - f_m|^p \leq \liminf_k \int_E |f_{n_k} - f_m|^p < \epsilon^p.)$$

Proof. □

Remark 6.0.28. *The completeness of L^p is sometimes referred to as the Riesz-Fischer Theorem, although what Riesz actually proved in 1906 was rather different. It was similar to the example explored below.*

Remark 6.0.29. *There is another way to arrange the proof of the completeness of L^p that looks a little slicker, but perhaps changes the perception of what's really going on. One proves the abstract theorem that a normed linear space X is complete if and only if the following property is true: if a series $\sum_{n=1}^{\infty} x_n$* ***converges absolutely*** *(this* *means that the series* $\sum_{n=1}^{\infty} \|x_n\|$ *of norms converges in* $\mathbb{R}$*) then the series converges in* X *(this means that the sequence of partial sums* $\sum_{n=1}^{N} x_n$ *converges to an element of* X*). Then you show that* $L^p(E)$ *has this property. If you look carefully, you will find these elements in what we did.*

Definition 6.0.30. *The set* ℓ^p *(read "little* L^p*") is the set of real (or complex) sequences* $\{c_n\}_{n=1}^{\infty}$ *such that* $\sum_{n=1}^{\infty} |c_n|^p < \infty$*. It is a complete normed linear space-a Banach space-with the norm*

$$\|\{c_n\}\|_p = \left(\sum_{n=1}^{\infty} |c_n|^p\right)^{1/p}.$$

(That this is a norm with respect to which ℓ^p *is complete can be proved by methods roughly similar to what we have done.)*

Remark 6.0.31. *We can add more structure for* L^2 *and* ℓ^2*. An* ***inner product*** *on a vector space is a map* $(\cdot,\cdot) : V \times V \to \mathbb{R}$ *with the properties*

1. $(v, v) \geq 0$ *and* $(v, v) = 0$ *iff* $v = 0$ *(zero element in* V*, of course),*
2. $(v, w) = (w, v)$ *for all* $v, w \in V$*,*
3. $(av + bw, u) = a(v, u) + b(w, u)$ *for all* $u, v, w \in V$ *and all* $a, b \in \mathbb{R}$*.*

Given an inner product, $\|v\| = \sqrt{(v, v)}$ *is a norm on* V*, so every inner product space is a normed linear space and hence a metric space. (Topology from algebra!) If it is complete as a metric space, we call it a* ***Hilbert space****. However in an inner product space we can also define the angle between elements by analogy with the dot product. In particular* v *and* w *are* ***orthogonal*** *if* $(v, w) = 0$*. It is possible to extend "dot product geometry" to this context-orthogonal bases, expressing an arbitrary element as the sum of its projections onto the elements of an orthogonal basis and so forth. The so forth includes the equivalent of the Pythagorean Theorem: the square of the norm of an element is the sum of the squares of the lengths of its projections. However the extension must cope with the fact that bases are typically countably infinite sets, so the sums are infinite sums and there are questions of convergence.*

In ℓ^2 *the inner product is* $(\{c_n\}, \{d_n\}) = \sum_{n=1}^{\infty} c_n d_n$*. The equivalent of Hölder's Inequality guarantees that* $|(\{c_n\}, \{d_n\})| \leq \|\{c_n\}\|\|\{d_n\}\|$ *and so in particular that the sum converges absolutely. In* $L^2(E)$ *the inner product is* $(f, g) = \int_E fg$*. Again this converges by Hölder's Inequality.*

In $L^2[0, \pi]$ *the sequence* $\{\sin nx\}_{n=1}^{\infty}$ *has the properties* $\|\sin nx\|_2 = \sqrt{\frac{\pi}{2}}$ *and* $\sin nx \perp \sin mx$ *if* $m \neq n$*, that is,* $\int_0^{\pi} \sin nx \sin mx dx = 0$ *if mneqn. The projection of any function* $f \in L^2[0, \pi]$ *on nx is*

$$\frac{(f, \sin nx)}{(\sin nx, \sin nx)} \sin nx = \left(\frac{2}{\pi} \int_0^{\pi} f(t) \sin ntdt\right) \sin nx.$$

The projection p_n *of* $f \in L^2[0,\pi]$ *on* SPAN$\{\sin x, \sin 2x, \ldots \sin nx\}$ *is* $\sum_{k=1}^{n} c_k \sin kx$ *where* $c_k = \frac{2}{\pi}\int_0^\pi f(t)\sin kldt$. *It is easy to see, using the orthogonality of the different sine functions that* $\|p_n\|_2^2 = \sum_{k=1}^{n} c_k^2 \leq \|f\|_2^2$ *where the last inequality just comes from the fact that any projection of* f *has norm less than or equal to that of* f. *It follows that* $\{c_k\}_{k=1}^\infty$ *is in* ℓ^2, *that* $\{p_n\}_{n=1}^\infty$ *is Cauchy in* $L^2[0,\pi]$ *(since* $\|p_n - p_m\|_2^2 = \sum_{k=m+1}^{n} c_k^2$*), and then by completeness that* $\{p_n\}_{n=1}^\infty$ *converges in* $L^2[0,\pi]$ *to an element* $\sum_{k=1}^{\infty} c_k \sin kx$. *It turns out that* $f = \sum_{k=1}^{\infty} c_k \sin kx$ *and that*

$$\|f\|_2^2 = \sum_{k=1}^{\infty} c_k^2 = \|\{c_n\}\|^2.$$

(These turn out to be equivalent to the statement that the only element of $L^2[0,\pi]$ *that is orthogonal to every* $e\sin nx$ *is the zero element, that is, the set* $\{\sin nx\}_{n=1}^\infty$ *is* ***complete*** *(roughly, large enough to function as a basis).)*

The result of all of this is that the mapping $f \to \{c_n\}$ *is an isometry (linear mapping preserving the norm) of* $L^2[0,\pi]$ *onto* ℓ^2. *In particular, given any sequence in* ℓ^2, *there is an* L^2 *function* f *with that sequence of Fourier sine coefficients. This is approximately the content of the original Riesz-Fischer Theorem in 1906.*