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LEBESGUE MEASURE AND INTEGRATION

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I. Preliminaries

1. DEFINITION: A set A is **finite** if there is a 1-1 mapping of some set $\{1,2,3,...n\}$ of positive integers onto A. A is **countably infinite** if there is a 1-1 mapping of the set $\mathbb{Z}^+$ of positive integers onto A. A is **countable** if either finite or countably infinite. Otherwise A is **uncountable**.

2. (deMorgan laws) If $\{E_\lambda : \lambda \in \Lambda\}$ is any indexed collection of subsets of some set E, then

$$\left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^c = \bigcap_{\lambda\in\Lambda} E_\lambda^c \text{ and } \left(\bigcap_{\lambda\in\Lambda} E_\lambda\right)^c = \bigcup_{\lambda\in\Lambda} E_\lambda^c$$

   where A^c denotes the complement of A in E.

3. REMARK: In the previous problem the index set Λ need not be countable. One could imagine indexing a collection of sets by the real numbers, for instance. (e.g. E_x is the interval of length 1 centered at x.)

4. Any countable union of sets of real numbers can be expressed as a disjoint union: $E\cup F = E\cup(F\backslash E)$ or $\bigcup_{k=1}^{\infty} E_k = \bigcup_{k=1}^{\infty} F_k$ where $F_1 = E_1, F_k = E_k\backslash \bigcup_{j=1}^{k-1} E_j$. Here $A\backslash B = A\cap B^c$.

5. If a and b are real numbers, then $a \le b$ iff for every $\varepsilon > 0, a \le b+\varepsilon$. Similarly, $a \ge b$ iff for every $\varepsilon > 0, a \ge b-\varepsilon$.

6. REMARK: This problem may seem trivial, but we will use it over and over again during the quarter. It says that we can give a little something away, and then take it back.

7. DEFINITIONS:

   (a) A set E of real numbers is **bounded above** if there exists a real number u such that for every $x \in E, x \le u$. We call u an **upper bound** for E.Similarly, E is **bounded below** if there exists a real number w such that for every $x \in E, w \le x$, and then w is a **lower bound** for E.. E is **bounded**, if bounded both above and below.

   (b) If E is bounded above and nonempty, the **supremum** (or least upper bound) of $E, \sup E$, is the unique real number s such that (i) s is an upper bound for E and (ii) $s < u$ for any other upper bound u for E. Similarly, if E is bounded below and nonempty, the **infimum** (or greatest lower bound) of $E, \inf E$, is the unique real number t such that (i) t is a lower bound for E and (ii) $t > w$ for any other lower bound w for E. (Recall that the Least Upper Bound Axiom guarantees the existence of the supremum and infimum.)

(c) A real number c is a **cluster point** of a set E of real numbers if for every $\varepsilon > 0$ there is $y \in E$ such that $0 < |c - y| < \varepsilon$.

8. Let v be a lower bound for a set E of real numbers, where $v \notin E$. Then $v = \inf E$ if and only if v is a cluster point of E.

9. DEFINITIONS:

   (a) A **sequence** of real numbers $\{a_k\}_{k=1}^{\infty}$ is a function from the set of positive integers $\mathbb{Z}^+$ into the real numbers.(More generally, the domain of a sequence can be any set of the form $\{k : k \geq k_0\}$ for some integer k_0.) The sequence is **bounded** (or bounded above or ...) if its range is bounded (or ...).

   (b) The sequence $\{a_k\}_{k=1}^{\infty}$ **converges** to the **limit** a if for each $\varepsilon > 0$ there is a positive integer K such that $|a_k - a| < \varepsilon$ for all $k \geq K$.

   (c) A **subsequence** $\left\{a_{k_j}\right\}_{j=1}^{\infty}$ of a sequence $\{a_k\}_{k=1}^{\infty}$ is the composition of $\{a_k\}_{k=1}^{\infty}$ with an increasing sequence $\left\{k_j\right\}_{j=1}^{\infty}$ of integers in the domain of $\{a_k\}_{k=1}^{\infty}$.

10. Let $\{a_k\}_{k=1}^{\infty}$ be a non-decreasing sequence of real numbers. Then $\{a_k\}_{k=1}^{\infty}$ is bounded above iff $\{a_k\}_{k=1}^{\infty}$ converges to the least upper bound of the set $\{a_k : k \in \mathbb{Z}^+\}$. (Of course there is a symmetric result for non-increasing sequences.)

11. REMARKS:
    (1) Note the distinction between the sequence $\{a_k\}_{k=1}^{\infty}$, which as a function is a set of ordered pairs of real numbers, and the set $\{a_k : k \in \mathbb{Z}^+\} \subset \mathbb{R}$ which is the range of that function.
    (2) If $\{a_k\}_{k=1}^{\infty}$ is non-decreasing and not bounded above, we often write $\lim_{k\to\infty} a_k = \infty$ and speak as though ∞ were a number "way out there at the end of the number line." We do not, however, do arithmetic with ∞.

12. DEFINITION:
    (1) $\{a_k\}_{k=1}^{\infty}$ be a sequence of real numbers. The numbers $s_n = \sum_{k=1}^{n} a_k$ are the **partial sums of the infinite series** $\sum_{k=1}^{\infty} a_k$. The series **converges** if $\lim_{n\to\infty} s_n$ exists (as a real number). Otherwise the series **diverges**. If the series converges, the number $s = \lim_{n\to\infty} s_n$ is called the **sum of the**

infinite series and is denoted $\sum_{k=1}^{\infty} a_k$.

(2) More generally, if S is any set of positive integers, and for any positive integer $n, S_n = \{k \in Z^+ : k \in S \text{ and } k \leq n\}$, we define the finite sums $s_n = \sum_{k \in S_n} a_k$ to be the **partial sums** of the series $\sum_{k \in S} a_k$, and say that the series **converges** if $\lim_{n\to\infty} s_n$ exists (as a real number). Otherwise the series **diverges**. If the series converges, the number $s = \lim_{n\to\infty} s_n$ is called the **sum of the series** and is denoted $\sum_{k \in S} a_k$.

13. Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of non-negative reals. Let $\{F_j\}_{j=1}^{\infty}$ be any nested sequence of sets of positive integers such that $F_1 \subset F_2 \subset \ldots$ and $\bigcup_{j=1}^{\infty} F_j = \mathbb{Z}^+$. For each j let $f_j = \sum_{k \in F_j} a_k$ Then $\sup_j f_j = \sum_{k=1}^{\infty} a_k$. (This means that if either number is finite, then they both are and then they are equal. Notice that it is not assumed that the sets F_j are finite, so the f_j need not be finite.)

    Remark: This is a strong form of saying that a series with non-negative terms can be summed in any order without affecting the sum.

14. Consider the double indexed set $\{a_{j,k}\}_{j,k=1}^{\infty}$ of non-negative numbers. Set $S_n = \sum_{j,k=1}^{n} a_{j,k}$. (This means the sum of the n^2 terms $a_{j,k}$ where $1 \leq j \leq n$ and $1 \leq k \leq n$.) Let $A_1 = \lim_{n\to\infty} S_n, A_2 = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{j,k}, A_3 = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} a_{j,k}$. Then $A_1 = A_2 = A_3$. This means that if any one of these three numbers is finite, then all of them are and they are equal. (This is, of course, the equivalent for series of the fact that a double integral can be evaluated as an iterated integral. For instance, A_2 if it exists, is the limit as $n \to \infty$ of $\sum_{j=1}^{n}\sum_{k=1}^{\infty} a_{j,k}$.)

15. DEFINITIONS:

    (a) An **open ball** about a real number x of radius r is the set $B_r(x) = (x-r, x+r)$.

    (b) A set G of real numbers is **open** if it contains an open ball about each of its points.

(c) A set F of real numbers is **closed** if every cluster point of F is contained in F.

16. Every open subset G of real numbers can be expressed as a countable union of pairwise disjoint open intervals. (Think about the rationals in G.)

17. DEFINITION: The set E of real numbers is **compact** if every sequence $\{a_k\}_{k=1}^{\infty}$ of elements of E has a subsequence that converges to an element of E.

18. FACT: These properties are equivalent for a set E of real numbers.

    (a) E is compact.

    (b) E is closed and bounded.

    (c) Every infinite subset of E has a cluster point in E.

    (d) If $\{G_\lambda : \lambda \in \Lambda\}$ is a collection of open sets such that $E \subset \bigcup_{\lambda \in \Lambda} G_\lambda$ (an **open cover** of E), then some finite subcollection is also an open cover of E.

II. Measure on [0,1]

19. DEFINITION: For any subset E of $[0,1]$, the **outer measure** of E is
$$m^*(E) = \inf\left\{\sum_{j=1}^{\infty} \ell\left(G_j\right) : E \subset \bigcup_{j=1}^{\infty} G_j\right\}$$
where each $G_j = \left(a_j, b_j\right)$ is an open interval in $\mathbb{R}$ and $\ell\left(G_j\right) = b_j - a_j$.(Note: this includes finite sums by setting all $G_j = \emptyset$ from some point on. Also note the G_j need not be contained in $[0,1]$,e.g. in considering $m^*(\{1\})$).

20. REMARKS (a) We could use coverings by closed intervals and get the same result. It is clear that we would get numbers not larger than we have. (Replace $G_j = \left(a_j, b_j\right)$by $F_j = \left[a_j, b_j\right]$.) To see not smaller, replace closed F_j by open G_j where $\ell\left(G_j\right) = \ell\left(F_j\right) + \varepsilon/2^j$.
(b) There is also something called inner measure. But it is not needed, so we won't use it.

21. For any subset E of $[0,1]$,
    (a) $0 \le m^*(E) \le 1$.
    (b) $E \subset F \Rightarrow m^*(E) \le m^*(F)$.
    (c) $m^*(\emptyset) = m^*(\text{one point}) = 0$.

22. If E is a countable set, then $m^*(E) = 0$. In particular, $m^*(\mathbb{Q}_0) = 0$ where $\mathbb{Q}_0 = \mathbb{Q} \cap [0,1]$ is the set of all rational numbers in $[0,1]$.
(Given $\varepsilon > 0$ cover $E = \{x_k\}_{k=1}^{\infty}$ with $\bigcup_{j=1}^{\infty} G_j$ where $\ell\left(G_j\right) = \varepsilon/2^j$.)

23. REMARK: This is the first evidence that covering E with countably infinite unions rather than finite unions makes a big difference. This is, in fact exactly the innovation that will allow us to deal much more efficiently with infinite processes and with irregular sets.

24. If $F = [a,b]$ is a closed interval in $[0,1]$, then $m^*(F) = b - a$.
($m^*(F) \le b - a$ is clear from $G = (a - \varepsilon, b + \varepsilon)$. F is compact, so any cover has a finite subcover $\bigcup_{j=1}^{n} G_j$. May assume that no element of this subcover can be omitted. (Explain why.) Order them by their left endpoints and show $\sum_{j=1}^{n} \ell\left(G_j\right) > b - a$.)

25. If $E, F \subset [0,1], m^*(E \cup F) \le m^*(E) + m^*(F)$.

26. For any subsets $\{E_j\}_{j=1}^{\infty}$ of $[0,1], m^*\left(\bigcup_{j=1}^{\infty} E_j\right) \le \sum_{j=1}^{\infty} m^*(E_j)$.
We refer to this property of outer measure as **countable subadditivity**.

27. DEFINITION: A subset E of $[0,1]$ is **measurable** if for each $A \subset [0,1]$,
$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c).$$

28. NOTATION: If E is measurable, we write $m(E)$ for $m^*(E)$ and refer to this number as the **Lebesgue measure** of E.

29. REMARKS (a) From the previous problem, to show that E is measurable one need only prove $m^*(A) \ge m^*(A \cap E) + m^*(A \cap E^c)$ since the other direction is automatic.
(b) In particular, taking $A = [0,1], 1 = m^*(E) + m^*(E^c)$ is a necessary condition for measurability. It turns out that this special case of the definition is also sufficient. The definition is the way it is (rather than this simpler version) just because it is a little more convenient to use.

30. (a) $\emptyset$ is measurable. $[0,1]$ is measurable.
(b) $m^*(E) = 0 \Rightarrow E$ is measurable.

31. REMARK: In particular, every subset of a set of outer measure 0 is measurable.

32. E is measurable $\Longleftrightarrow E^c$ is measurable.

33. A closed interval $[a,b]$ is measurable with measure $b-a$.
(Show $m^*(A) \ge m^*(A \cap [a,b]) + m^*(A \cap [a,b]^c) - \varepsilon$ for every ε. If $\bigcup_{j=1}^{\infty} G_j$ covers A with $\sum_{j=1}^{\infty} \ell(G_j)$ near $m^*(A)$, then the $G_j \cap [a,b]$ are nearly an open cover of $A \cap [a,b]$ by open intervals and similarly for $G_j \cap [a,b]^c$.)

34. If E and F are measurable, then so are $E \cap F, E \cap F^c, E^c \cap F$, and $E^c \cap F^c$.
(To show $m^*(A) \ge m^*(A \cap E \cap F) + m^*(A \cap (E \cap F)^c)$ use the measurability of F with test set $A \cap E$. It is not necessary to go back to the level of open covers.)

35. If E and F are measurable, then so is $E \cup F$. If E and F are disjoint, then $m(E \cup F) = m(E) + m(F)$.
(For the second assertion you can use the measurability equation with test set $E \cup F$.)

36. Any interval $\langle a, b \rangle$ with any choice of endpoints is measurable with measure $b - a$.

37. Any finite union or intersection of measurable sets is measurable. If $\{E_k\}_{k=1}^{n}$ are measurable and disjoint, then $m\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} m(E_k)$.
(Induction).

38. Let $E_1, E_2, \ldots, E_n$ be pairwise disjoint and measurable. Then for any A,
$$m^*(A) = \sum_{k=1}^{n} m^*(A \cap E_k) + m^*\left(A \cap \left(\bigcup_{k=1}^{n} E_k\right)^c\right).$$

39. Let $\{E_k\}_{k=1}^{\infty}$ be pairwise disjoint and measurable. Then for any A,
$$m^*(A) = \sum_{k=1}^{\infty} m^*(A \cap E_k) + m^*\left(A \cap \left(\bigcup_{k=1}^{\infty} E_k\right)^c\right).$$
($\left\{\sum_{k=1}^{n} m^*(A \cap E_k)\right\}_{n=1}^{\infty}$ is a non-decreasing sequence that converges to $\sum_{k=1}^{\infty} m^*(A \cap E_k)$.
$\{s_n\}_{n=1}^{\infty} = \left\{m^*\left(A \cap \left(\bigcup_{k=1}^{n} E_k\right)^c\right)\right\}_{n=1}^{\infty}$ is a non-increasing sequence all of whose terms are at least $m^*\left(A \cap \left(\bigcup_{k=1}^{\infty} E_k\right)^c\right)$. These imply $m^*(A) \geq \sum_{k=1}^{\infty} m^*(A \cap E_k) + m^*\left(A \cap \left(\bigcup_{k=1}^{\infty} E_k\right)^c\right)$.)

40. Let $\{E_k\}_{k=1}^{\infty}$ be measurable. Then $\bigcup_{k=1}^{\infty} E_k$ is measurable. If the E_k are pairwise disjoint, then
$$m\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m(E_k).$$

(Unions can be written as disjoint unions.) This property is called **countable additivity**.

41. Let $\{E_k\}_{k=1}^{\infty}$ be measurable and nested: $E_1 \supset E_2 \supset \ldots$. If $\bigcap\limits_{k=1}^{\infty} E_k = \emptyset$, then $\lim\limits_{k\to\infty} m(E_k) = 0$.
(${E_1}$ is a disjoint union $\bigcup\limits_{k=1}^{\infty} F_k$ where $F_1 = E_1 \backslash E_2, F_2 = E_2 \backslash E_3$ etc. $m(E_n) = \sum\limits_{k=n}^{\infty} m(F_k)$.)

42. DEFINITION: If $E \subset [0,1]$ and r is a real number, then $E \dotplus r$ is the subset of $[0,1)$ consisting of all fractional parts of the set $\{x+r : x \in E\}$. Geometrically, add r to each element of E and then translate the elements by an integer so that they land back in $[0,1)$. (This can mean "breaking the set into two parts." For instance $[0,.4] \dotplus \frac{3}{4} = [0,.15] \cup [3/4,1)$.)

43. If E is measurable and r is a real number, then $E \dotplus r$ is measurable and $m(E \dotplus r) = m(E)$.
(I found it helpful (i) to show that if A is any subset of $[0,1]$, (not necessarily measurable) and $r \in \mathbb{R}$ is such that $A + r \subset [0,1]$, then $m^*(A) = m^*(A+r)$ and (ii) to remember that we have already seen in a previous problem that if A is any subset of $[0,1]$ and $s \in (0,1)$ then $m^*(A) = m^*(A \cap [0,s]) + m^*(A \cap (s,1]))$.)

44. REMARK. We can sum up what we have done so far like this.
The set $\mathfrak{M}$ of Lebesgue measurable subsets of $[0,1]$ has the following properties
1. $\emptyset \in \mathfrak{M}$.
2. Every interval is in $\mathfrak{M}$ with $m(\langle a,b \rangle) = b - a$. Moreover every open subset of $[0,1]$ is in $\mathfrak{M}$.
3. $E \in \mathfrak{M} \Rightarrow E^c \in \mathfrak{M}$.
4. $E \in \mathfrak{M} \Rightarrow E \dotplus r \in \mathfrak{M}$ and $m(E \dotplus r) = m(E)$.

5. If $\{E_k\}_{k=1}^{\infty}$ are in $\mathfrak{M}$, then $\bigcup\limits_{k=1}^{\infty} E_k \in \mathfrak{M}$.

6. If, in addition, the E_k are pairwise disjoint, then $m\left(\bigcup\limits_{k=1}^{\infty} E_k\right) = \sum\limits_{k=1}^{\infty} m(E_k)$.

45. EXAMPLE. Since the set $\mathbb{Q}_0$ of rational numbers in $[0,1]$ is countable, $\mathbb{Q}_0$ is measurable and $m(\mathbb{Q}_0) = 0$. Then $\mathbb{Q}_0^c$–the set of irrational numbers in $[0,1]$–is also measurable, and $m\left(\mathbb{Q}_0^c\right) = m([0,1]) - m(\mathbb{Q}_0) = 1$. Thus the "length" of the irrationals is 1, while the "length" of the rationals is 0. This is connected to the fact that $\mathbb{Q}_0$ is countable while $\mathbb{Q}_0^c$ is uncountable, but it would be wrong to assume that all uncountable subsets of $[0,1]$ have measure 1, or even nonzero measure. See the next example.

46. EXAMPLE. Recall that the Cantor set C is the subset of $[0,1]$ that remains after removing $\bigcup\limits_{k=1}^{\infty} I_k$ where $I_1 = \left(\frac{1}{3}, \frac{2}{3}\right), I_2 = \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right)$ and in general I_k is the union of the 2^k open middle thirds of the closed intervals remaining after I_{k-1} has been removed. Furthermore $m(I_k) = \frac{1}{3}\left(\frac{2}{3}\right)^{k-1}$ so that $m\left(\bigcup\limits_{k=1}^{\infty} I_k\right) = \frac{1}{3}\sum\limits_{k=0}^{\infty}\left(\frac{2}{3}\right)^k = 1$ and $m(C) = 0$.

47. REMARK: It is not immediately obvious from the construction above that C is uncountable. Some of you at least have seen an alternative characterization of C as the set of all numbers in $[0,1]$ with a ternary expansion containing only 0's and 2's. Since this set can be put in 1-1 correspondence with the set of all binary expansions of numbers in $[0,1]$, that is, with $[0,1]$, it is true that C is uncountable. Thus the Lebesgue measure of a set is not particularly connected to the cardinality of the set. These are two rather different versions of the "size" of an infinite set.

48. By altering the lengths of the removed intervals, construct a "fat Cantor set"–a Cantor set of positive measure. Ideally, show how to construct a Cantor set of measure α for any $\alpha < 1$.
(You can check that if you try to imitate the Cantor set construction by removing a constant fraction of what remains at each stage, the total length of removed intervals will be 1 no matter what the fraction removed is, so you will have to be a little cleverer than that. Try to make choices so that your calculations are reasonably easy.)

49. REMARK and EXAMPLE: Properties (1),(3),(5) of #44 are really structural properties of the collection $\mathfrak{M}$ of measurable subsets of $[0,1]$. How interesting they are depends on whether $\mathfrak{M}$ is just the set of all subsets of $[0,1]$ (sometimes called the **power set** $\mathscr{P}([0,1])$) or some proper collection of $P([0,1])$. We "construct" some non-measurable sets in order to see that $\mathfrak{M} \neq \mathscr{P}([0,1])$. This makes the structural properties much more interesting.

Given $x \in [0,1]$, let $A_x = \{y \in [0,1] : x - y \in \mathbb{Q}\}$. Each A_x is a countable set. Clearly $x \in A_x$ and $y \in A_x \Leftrightarrow x \in A_y$. If $x \in A_y$ and $y \in A_z$, then $x - z = (x - y) + (y - z)$ is rational so $z \in A_x$. It follows that the relation $x \sim y$ if $y \in A_x$ is reflexive, symmetric, and transitive, that is, it is an **equivalence relation**. Each distinct set A_x is an **equivalence class.** As you have probably verified somewhere else, distinct equivalence classes are disjoint, that is, if $x \neq y$, then either $A_x = A_y$ or A_x and A_y are disjoint. (Easy if you haven't done it before.)

Now choose one element from each distinct equivalence class, and let E be the set of all such elements. E is an uncountable set; otherwise $[0,1]$ would be a countable union of countable sets, and so countable. See the remark below about the choice process.

Let $\{q_k\}_{k=1}^{\infty}$ be an enumeration of the set $\mathbb{Q}_0$ of rationals in $[0,1]$. I claim the sets $\{E \dot{+} q_k : k = 1,2,3....\}$ are disjoint and that their union is $[0,1]$. First note that for any $x \in [0,1]$, $x \in E \dot{+} q_k$ if for some $e \in E$, either $x = e + q_k$ (if $e + q_k < 1$) or $x = e + q_k - 1$ (in case $e + q_k \geq 1$). Now x is in the same equivalence class A_x as some element e of E. Thus $|x - e| = q_k$ for some k and we see that either $x \in E \dot{+} q_k$ (if $x \geq e$) or $x \in E \dot{+} (1 - q_k)$ (if $x < e$). Next we see that the $\{E \dot{+} q_k\}$ are pairwise disjoint. If $x = e + q_k = e' + q_j$, then $e - e' = q_j - q_k$, that is, e and e' are in the same equivalence class. This is a contradiction.

To summarize, $[0,1] = \bigcup_{k=1}^{\infty} (E \dot{+} q_k)$ as a disjoint union. Further, by property 4 of #44, either all these sets are measurable with the same measure, or all are non-measurable. But the first alternative is impossible by property 6 of #44. Thus we have a countably infinite collection of non-measurable sets.

50. REMARK: The construction of E in the preceding example used the property that I can make a set by choosing one element out of a collection of non-empty sets. The assertion that one can do this is called the **Axiom of Choice**. In the early twentieth century when mathematicians and logicians were trying to construct all of mathematics from a precisely defined set of axioms, and to show that the mathematics so obtained would be consistent (no possibility that legal arguments would produce a contradiction), the relationship between the Axiom of Choice and the other "usual axioms" was much studied, because the Axiom of Choice has surprising consequences (e.g. that it is possible to order the set of real numbers so that each subset of reals has a least element in the ordering–just like the positive integers with the usual ordering. This is called a Well-Ordering.) However in 1963 Paul Cohen completed the proof that AofC is independent of the other usual

axioms of set theory. This marked the very end of the interest that most mathematicians have in the foundations of mathematics. (Most feel that it is new ideas and connections that are of interest, and Cohen's result showed that AofC, which is often very convenient, can't "make things worse" (introduce contradictions where there were none before.)).

51. DEFINITION. A collection $\mathfrak{A}$ of subsets of a non-empty set X is a **σ-algebra** of subsets if $\mathfrak{A}$ has the properties
(i) $\emptyset \in \mathfrak{A}$,

(ii) $E \in \mathfrak{A} \Leftrightarrow E^c \in \mathfrak{A}$,

(iii) if $\{E_k\}_{k=1}^{\infty}$ are in $\mathfrak{A}$, then $\bigcup_{k=1}^{\infty} E_k \in \mathfrak{A}$.

52. REMARK: Thus $\mathfrak{M}$ is a proper σ-algebra of subsets of $\mathscr{P}([0,1])$, the power set of $[0,1]$. Another σ-algebra of subsets of $\mathscr{P}([0,1])$ is the collection $\mathfrak{B}$ of all **Borel subsets** of $[0,1]$. $\mathfrak{B}$ is defined to be the smallest σ-algebra of subsets of $\mathscr{P}([0,1])$ that contains the open sets. (Take the intersection of all such σ-algebras.) Clearly $\mathfrak{B} \subset \mathfrak{M}$. It can be shown that $\mathfrak{B}$ and $\mathfrak{M}$ are different by a cardinality argument–Since the Cantor set C can be put into 1-1 correspondence with $[0,1]$, $\mathscr{P}(C)$ can be put into 1-1 correspondence with $\mathscr{P}([0,1])$. All elements in $\mathscr{P}(C)$ are measurable, since they are subsets of a set of measure 0. Thus $\mathfrak{M}$ has the same cardinality as $\mathscr{P}([0,1])$. On the other hand, $\mathfrak{B}$ can also be generated in a countable fashion from the set of open intervals with rational endpoints. Thus its cardinality is the same as that of $[0,1]$ and so strictly less than that of $\mathscr{P}([0,1])$.

On the other hand, every measurable set is "nearly" an open set, and also "nearly" a closed set. This is the content of the next problem.

53. The following properties are equivalent for a subset E of $[0,1]$.

(i) $E \in \mathfrak{M}$,

(ii) for each $\varepsilon > 0$ there is an open set G such that $E \subset G$ and $m^*(G \backslash E) < \varepsilon$.

(iii) there is a G_δ set H such that $E \subset H$ and $m^*(H \backslash E) = 0$.

(A G_δ set is a countable intersection of open sets. A G_δ is a Borel set but not open in general, for instance any closed interval is a G_δ. (Proof?–but not a problem.)

(iv) for each $\varepsilon > 0$ there is a closed set F such that $F \subset E$ and $m^*(E \backslash F) < \varepsilon$.

(v) there is an F_σ set K (a countable union of closed sets, e.g. $\mathbb{Q}_0$) such that $K \subset E$ and $m^*(E \backslash K) = 0$.

(We will split this into two parts, from two different people, first $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$; second $(ii) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (i)$.)

III. Measure on $\mathbb{R}$

54. REMARK: There are several ways that we could develop a theory of measure on the entire real line. One is to start over and just repeat the development we have already been through. (Actually we could just have started that way.) The only important difference is that now some sets will have infinite measure ($\mathbb{R}$ itself for instance), so that we would have to keep in mind that the infinite sums are not guaranteed to converge. What we will actually do is use our theory on $[0,1]$ to create a theory on $\mathbb{R}$.

55. DEFINITION: (a) For any real number x, let $\mathfrak{M}_x$ be the collection of all subsets E of $[x, x+1]$ such.that $E - x$ is a measurable subset of $[0,1]$. For $E \in \mathfrak{M}_x$, we set $m(E) = m(E - x)$. It is apparent that $\mathfrak{M}_x$ is a σ-algebra of subsets of $[x, x+1]$ on which all the properties of #44 hold.

    (b)For any subset E of $\mathbb{R}$, and each integer n, let $E_n = E \cap [n, n+1)$. We say that E is Lebesgue measurable if $E_n \in \mathfrak{M}_n$ for each n. We denote the collection of all Lebesgue measurable subsets of $\mathbb{R}$ by $\mathfrak{M}_\mathbb{R}$. If E is measurable, we define the Lebesgue measure of E by $m(E) = \sum_{n=-\infty}^{\infty} m(E_n)$.

56. REMARK: It is very easy to verify that the set $\mathfrak{M}_\mathbb{R}$ of Lebesgue measurable subsets of $\mathbb{R}$ is a σ-algebra of sets containing the Borel sets, and that Lebesgue measure m has all the properties of #44 except that we may now interpret translation invariance (property 4) as ordinary translation invariance instead of "translation invariance mod 1," that is, for any $E \in \mathfrak{M}_\mathbb{R}$, and any real number c, $E + c \in \mathfrak{M}_\mathbb{R}$ and $m(E) = m(E + c)$.

    Of course we do now have the possibility that $m(E) = \infty$. For this purpose we regard ∞ as just another number, more or less. It is necessary however, to avoid expressions like $\infty - \infty$, so at times we will need to be careful about whether the measure of a set is finite.

57. If E is a measurable set of finite measure, then for any $\varepsilon > 0$ there is a finite collection of open intervals$\{G_k\}_{k=1}^n$ so that

$$m\left(E \backslash \bigcup_{k=1}^{n} G_k\right) + m\left(\bigcup_{k=1}^{n} G_k \backslash E\right) < \varepsilon.$$

(Thus, every measurable set is "almost a finite union of open sets.")

IV. Measurable Functions

58. DEFINITION: A real-valued function f defined on a measurable subset E of $\mathbb{R}$ is **measurable** if for every real number $\alpha, \{x \in E : f(x) < \alpha\}$ is measurable.

59. These are equivalent:
(a) f is measurable
(b) for every $\alpha, \{x \in E : f(x) \geq \alpha\}$ is measurable
(c) for every $\alpha, \{x \in E : f(x) > \alpha\}$ is measurable
(d) for every $\alpha, \{x \in E : f(x) \leq \alpha\}$ is measurable
(e) for every open set $G, \{x \in E : f(x) \in G\}$ is measurable.
(Show $(a) \leftrightarrow (b) \Rightarrow (c) \leftrightarrow (d) \Rightarrow (a)$ and $(a)\&(c) \Rightarrow (e), (e) \Rightarrow (a)$)

60. EXAMPLES: (1) if E is a measurable set, the **characteristic function** of E defined by
$$\chi_E(x) = \begin{cases} 1, & x \in E, \\ 0, & x \notin E \end{cases}$$
is measurable.
(2) By part (e) of the previous problem, any continuous function with a measurable domain is measurable. (What property of continuous functions does this follow from?)

61. If f is measurable with domain E, and $c \in \mathbb{R}$, then cf and $f + c$ are measurable.

62. If f and g are measurable with domain E, then $f + g$ is measurable.
(Show $\{x \in E : f(x) + g(x) < \alpha\} = \bigcup_q (\{x \in E : f(x) < \alpha - q\} \cap \{x \in E : g(x) < q\})$
where the union is over all $q \in \mathbb{Q}$.)

63. If f and g are measurable with domain E, then $\max\{f, g\}$ and $\min\{f, g\}$ are measurable.

64. DEFINITION: If f is measurable with domain E, the **positive part** of f is the function $f_+ = \max\{f, 0\}$, and the **negative part** of f is the function $f_- = \max\{-f, 0\}$.

65. REMARKS. Note that $f = f_+ - f_-$ and $|f| = f_+ + f_-$. It follows from the previous problems that if f is measurable, then so are all of f_+, f_-, and $|f|$.

66. If f and g are measurable with domain E, then f^2 and fg are measurable.
(Do f^2 first. Then $fg = \frac{1}{2}\left((f+g)^2 - f^2 - g^2\right)$.)

67. DEFINITION: We say that a property holds **almost everywhere (**abbreviated **a.e.**) if the set where it does not hold has measure zero. In this context a set of measure zero is often called a **null set.** We also say that a property that holds in a set E except on a null set holds for **almost all** x in E.

68. If f and g are defined on a measurable set E, f is a measurable function, and $f = g$ a.e., then g is measurable.

69. DEFINITION and REMARK: The **extended real numbers** consist of $\mathbb{R}$ together with $+\infty$ and $-\infty$. These have the property that $-\infty < r < +\infty$ for every real number r, but we try not to do arithmetic with $\pm\infty$. It is often convenient to allow measurable functions to have range in the extended real numbers. The preceding problem shows that as long as functions take finite values almost everywhere, we can do arithmetic with them without bothering about what happens on a null set.

70. If f is a measurable extended real-valued function defined on a bounded measurable set E and f is finite a.e., then for any $\varepsilon > 0$ there is M so that $|f| \le M$ except on a set of measure less than ε. (So any measurable function on a bounded set is "almost bounded.")
(Remember #41)

71. QUESTION: Is it necessary to assume in the previous problem that the set E is bounded?

72. If $\{f_n\}_{n=1}^{\infty}$ is any sequence of extended real-valued measurable functions defined on a measurable set E, then the function $F(x) = \sup_n f_n(x)$ is a measurable extended real-valued function. Similarly, $G(x) = \inf f_n(x)$ is measurable. (Note, for instance, that $F(x) = \infty$ if $f_n(x) \to \infty$ as $n \to \infty$.)

73. DEFINITION: The **limit superior** of a sequence of real numbers $\{a_k\}_{k=1}^{\infty}$ is the extended real number $A = \inf_n \sup_{k\ge n} a_k$. We write $A = \limsup a_k$.
Note that the sequence $\left\{\sup_{k\ge n} a_k\right\}_{n=1}^{\infty}$ is non-increasing, so the inf is the same as the limit of this sequence, keeping in mind that all elements of this sequence may be ∞.

Similarly the **limit inferior** of $\{a_k\}_{k=1}^{\infty}$ is $B = \sup_n \inf_{k \geq n} a_k$. We write $B = \liminf a_k$. The sequence $\left\{ \inf_{k \geq n} a_k \right\}_{n=1}^{\infty}$ is non-decreasing, so the sup is the same as the limit of this sequence.

74. For any sequence $\{a_k\}_{k=1}^{\infty}$ of real numbers, $\limsup a_k \geq \liminf a_k$. $\limsup a_k = \liminf a_k$ if and only if the sequence $\{a_k\}_{k=1}^{\infty}$ converges.

75. If $\{f_n\}_{n=1}^{\infty}$ is any sequence of extended real-valued measurable functions defined on a measurable set E, and if $f(x) = \lim_{n \to \infty} f_n(x)$ exists for each $x \in E$, then f is an extended real-valued measurable function on E.

76. REMARKS: (i) The conclusion still holds in the previous problem if the sequence $\{f_n\}_{n=1}^{\infty}$ converges a.e. on E.
(ii) That the pointwise limit of a sequence of measurable functions is measurable is essential to the theory of integration that we are about to create. It is the step that allows us to avoid the difficulty that the pointwise limit of a sequence of Riemann integrable functions may not be Riemann integrable.

77. DEFINITION: A **simple function** is a measurable function whose range is a finite subset of $\mathbb{R}$. A simple function s taking values a_j on measurable sets E_j can be written

$$s = \sum_{j=1}^{n} a_j \chi_{E_j}.$$

Of course this representation is not unique. The **canonical** representation of a simple function is the sum where $\{E_j\}_{j=1}^{n}$ are disjoint and $j \neq k$ implies $a_j \neq a_k$. (In other words each $E_j = s^{-1}(a_j)$ for the distinct values a_j in the range of s.)

78. Let f be an extended real-valued measurable function defined on an interval $[a,b]$. Let $M > 0$. For any $\varepsilon > 0$ there is a simple function s so that $|f(x) - s(x)| < \varepsilon$ on the set where $|f| \leq M$.
(Partition $[-M, M]$ into slices and use the sets where f takes values in a slice to define s.)

79. DEFINITION: A **step function** p is a simple function whose domain is a closed bounded interval $[a,b]$ and which can be written in the form $\sum_{j=1}^{n} a_j \chi_{E_j}$ where the E_j are intervals.

80. Let s be a simple function whose domain is a closed bounded interval $[a,b]$. For every $\varepsilon > 0$ there is a step function p so that $m(\{x \in [a,b] : s(x) \neq p(x)\}) < \varepsilon$.

    (Remember #57).

81. If f is a measurable extended real-valued function defined on a closed bounded interval $[a,b]$ and f is finite a.e., then for any $\varepsilon > 0$ there is a step function p and a continuous function g so that $|f - p| < \varepsilon$ except on a set of measure less than ε, and $|f - g| < \varepsilon$ except on a set of measure less than ε. (So any measurable function is "almost continuous.")

    (This is mostly assembling the information from the previous three problems.)

82. Let E be a measurable set of finite measure. Suppose that f and $\{f_n\}_{n=1}^{\infty}$ are measurable functions defined on E so that $f_n(x) \to f(x)$ for each $x \in E$. Then for every $\varepsilon > 0$ and $\delta > 0$ there is a measurable set $A \subset E$ with $m(A) < \delta$ and an integer N so that for all $x \notin A$ and all $n \geq N$,

    $$|f_n(x) - f(x)| < \varepsilon.$$

    (Consider the sets $G_n = \{x : |f_n(x) - f(x)| \geq \varepsilon\}$ and the nested collection

    $$E_N = \bigcup_{n=N}^{\infty} G_n = \{x : |f_n(x) - f(x)| \geq \varepsilon \text{ for some } n \geq N\}.$$

    Show $\bigcap_{N=1}^{\infty} E_N = \emptyset$.)

83. REMARK: It is clear that the conclusion of the preceding problem continues to hold if we assume only that $f_n(x) \to f(x)$ a.e. on E.

84. DEFINITION and REMARK: A sequence $\{f_n\}_{n=1}^{\infty}$of functions **converges uniformly** to a function f on a set E if for every $\varepsilon > 0$ there is an integer N so that $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq N$ and all $x \in E$.

    Of course the "uniform" part is that N does not depend on x. Note that the preceding problem has a conclusion that somewhat resembles uniform convergence, except that the set A may depend on ε. We fix this in the next problem.

85. Let E be a measurable set of finite measure. Suppose that f and $\{f_n\}_{n=1}^{\infty}$ are measurable functions defined on E so that $f_n(x) \to f(x)$ a.e. on E. Then for each $\alpha > 0$ there is a set $A \subset E$ with $m(A) < \alpha$ and

$f_n \to f$ uniformly on $E \setminus A$.

(Apply the preceding problem repeatedly with $\varepsilon_n = 1/n$ and $\delta_n = 2^{-n}\alpha$.)

V. The Lebesgue Integral

86. REMARKS: (1) Recall that in the Riemann theory of integration, for a bounded function f on a closed interval $[a,b]$ we define the **upper integral**, U, and **lower integral**, L, as

$$U = \inf\left\{\int_a^b s(x)\,dx\right\}, L = \sup\left\{\int_a^b r(x)\,dx\right\}$$

where the inf and sup are over the set of all step functions s such that $s \geq f$ and the set of all step functions r such that $r \leq f$ respectively and we define the integral of a step function $s(x) = \sum_{j=1}^{n} c_j \chi_{[a_j,b_j]}(x)$ (the canonical representation) to be $\int_a^b s(x)\,dx = \sum_{j=1}^{n} c_j \left(b_j - a_j\right)$. (This is the usual definition expressed in an unusual way. The integrals of the step functions are usually called upper sums and lower sums.)
The function f is **Riemann integrable** if $U = L$, and in that case we call the common value the **Riemann integral of** f **over** $[a,b]$, denoted $\int_a^b f(x)\,dx$.
(2) Of course one requirement for the Lebesgue integral is that any Riemann integrable function should also be Lebesgue integrable, and that the values of the two integrals should be the same. Another practical requirement is that there should be enough Lebesgue integrable functions that are not Riemann integrable to make it worthwhile to go to the trouble of developing the Lebesgue integral.

87. DEFINITIONS. (1) If s is a simple function with canonical representation $s = \sum_{j=1}^{n} c_j \chi_{E_j}$ (recall Def. 77) that vanishes outside a set of finite measure, the **integral** of s is

$$\int s = \sum_{j=1}^{n} c_j m(E_j).$$

(2) If E is any measurable set, the **integral of** s **over** E is

$$\int_E s = \int s\chi_E.$$

88. REMARK: Of course we think of this as “the signed area under the curve” as usual. One irritating fine point we need to deal with is to

check that if we express the simple function as a linear combination of characteristic functions in a different way, then $\int s$ is also given by the corresponding sum.

89. Suppose s is a simple function that vanishes outside a set of finite measure and $s = \sum_{j=1}^{n} c_j \chi_{A_j}$ where the sets $\{A_j\}_{j=1}^{n}$ are pairwise disjoint (but several different $A_j's$ may carry the same value c_j). Then $\int s = \sum_{j=1}^{n} c_j m(A_j)$.

    (For any c in the range of s, the canonical representation of s includes $E_c = \cup \{A_j : c_j = c\}$. We know $m(E_c) = \sum_{c_j=c} m(A_j)$.)

90. Suppose that r and s are simple functions that vanish outside a set of finite measure and that a and b are real numbers.

    (1)
    $$\int (ar + bs) = a \int r + b \int s.$$

    (2) If $r \geq s$, then $\int r \geq \int s$.

    (If $s = \sum_{j=1}^{n} c_j \chi_{E_j}, r = \sum_{j=1}^{m} d_j \chi_{F_j}$ are the canonical representations, consider the representations using all possible sets $E_j \cap F_k$. For (2), consider $r - s$.)

91. REMARK: By the preceding problem and induction, if $\{A_j\}_{j=1}^{n}$ is any collection of sets of finite measure, not necessarily disjoint, and if $s = \sum_{j=1}^{n} c_j \chi_{A_j}$, then $\int s = \sum_{j=1}^{n} c_j m(A_j)$ so that the restriction above to pairwise disjoint sets is unnecesary.

92. Let f be a bounded function defined on a measurable set E with $m(E)$ finite. Then
    $$\inf_{s \geq f} \int s = \sup_{r \leq f} \int r$$
    if and only if f is measurable.

    (If $|f| \leq M$ is measurable, partition the range into $2n$ equal portions and use this to define simple functions r and s with $r \leq f \leq s$. For the other direction, let $\{r_n\}, \{s_n\}$ be sequences of simple functions whose integrals approach the sup and inf. Then $\sup r_n \leq f \leq \inf s_n$, $\sup r_n$ and $\inf s_n$ are measurable, and you can show $m\{x : \sup r_n \neq \inf s_n\} = 0$.

93. REMARK: The moral of the preceding problem is that, unlike the situation with the Riemann integral, we do not need to consider both upper integrals and lower integrals because we already have the correct class of functions, the measurable functions. Thus we can, say, forget about lower integrals and just approximate from above. (This is analogous to not needing inner measure to help define Lebesgue measure.)

94. DEFINITION: If f is a bounded measurable function defined on a measurable set E with $m(E)$ finite, then the (Lebesgue) **integral** of f is
$$\int f = \inf_{s \geq f} \int s$$
where the inf is over the class of all simple functions s with $s \geq f$. We sometimes write $\int_E f$. If $E = [a,b]$, we write $\int_a^b f$.
If A is a measurable subset of E, we define $\int_A f = \int f \chi_A$.

95. Let f be a bounded function defined on $[a,b]$. If f is Riemann integrable on $[a,b]$, then f is measurable and the Riemann integral $R\int_a^b f = \int_a^b f$.
(A step function is a simple function.)

96. The Lebesgue integral has the following properties for bounded measurable functions defined on a set E of finite measure.
(1) $\int_E (af + bg) = a\int_E f + b\int_E g$,
(2)If $f \leq g$ a.e., then $\int_E f \leq \int_E g$,
(3) If $f = g$ a.e., then $\int_E f = \int_E g$,
(4) If $\alpha \leq f(x) \leq \beta$ for almost all $x \in E$, then $\alpha m(E) \leq \int_E f \leq \beta m(E)$,
(5) If A and B are disjoint measurable subsets of E, then $\int_{A \cup B} f = \int_A f + \int_B f$.
(For (1) show $\int af = a\int f$ (easy) and $\int (f+g) = \int f + \int g$ (Requires some manipulation of infs. You may also find #92 useful.). For (2) you can show $f \leq 0$ a.e. implies $\int_E f \leq 0$. (3), (4) and (5) follow from previous parts.)

97. REMARK: It follows from part (3) and the previous problem about measurability of functions equal almost everywhere that changing the values of a function in a bounded way on a countable subset of E does not affect $\int_E f$. Thus the Dirichlet example of a non-Riemann integrable function (the characteristic function of the set $\mathbb{Q}_0$ of rationals in $[0,1]$) is not a problem here. We have $\int_0^1 \chi_{\mathbb{Q}_0} = 0$.

98. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions on a measurable set E of finite measure such that for some $M > 0, |f_n| \leq M$ a.e. for each n. If $\lim_{n\to\infty} f_n(x)$ exists a.e. on E, then the function defined by $\lim_{n\to\infty} f_n(x)$ is equal almost everywhere on E to a bounded measurable function f and

$$\int_E f = \lim_{n\to\infty} \int_E f_n.$$

(Use # 82 or Egoroff's Thm # 85.)

99. REMARKS: (1) The preceding problem is a form of the Bounded Convergence Theorem–the first of the important convergence theorems. It asks more than pointwise convergence (the uniform boundedness of the functions), but much less than uniform convergence.

(2) Now we'll extend things by dropping our conditions that the functions be bounded and defined on a set of finite measure. We'll start by defining the integral for non-negative functions only, and then writing an arbitrary function as a linear combination of these. The idea for non-negative functions is to take the sup over all bounded functions non-zero on a set of finite measure that are below the given function.

100. DEFINITION. Let f be a non-negative measurable function on a measurable set E. Then we define

$$\int_E f = \sup_{h \leq f} \int_E h$$

where the sup is over all bounded non-negative measurable functions $h \leq f$ such that $m\{x \in E : h(x) \neq 0\} < \infty$ and we interpret $\int_E h$ as $\int_{\{x\in E:h(x)\neq 0\}} h$.

101. If f and g are non-negative measurable functions defined on a measurable set E, then

(1) $\int_E \alpha f = \alpha \int_E f$ for any $\alpha \geq 0$,

(2) $\int_E (f+g) = \int_E f + \int_E g$,

(3) if $f \geq g$ a.e., then $\int_E f \geq \int_E g$. In particular, if $f \geq 0$ a.e., then $\int_E f \geq 0$.

(For (2), if $h \leq f$ and $k \leq g$ then $h+k \leq f+g$ and $\int_E h + \int_E k = \int_E (h+k) \leq \int_E (f+g)$. Take sup on the left. If $h \leq f+g$ and $A = \{x : h(x) \neq 0\}$ has finite measure, then $h_f = \min\{f,h\}$ and $h_g = h - h_f$ are bounded, non-negative, measurable and vanish outside A.)

102. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of non-negative measurable functions on a measurable set E such that $f(x) = \lim_{n\to\infty} f_n(x)$ exists for almost all $x \in E$. Then f is measurable and

$$\int_E f \leq \liminf \int_E f_n.$$

(For $h \leq f$ vanishing outside a set A of finite measure, show $\int_E h \leq \liminf \int_E f_n$ by taking $h_n = \min\{f_n, h\}$ and showing that $h_n \to h$ a.e., and that $\int_E h = \lim_{n\to\infty} \int_E h_n \leq \liminf \int_E f_n$.)

103. REMARK: The preceding problem is known as Fatou's Lemma. You should be sure that you know of an example where equality does not hold.

104. Let $\{f_n\}_{n=1}^{\infty}$ be a non-decreasing sequence of non-negative measurable functions on a measurable set E, and let $f(x) = \lim_{n\to\infty} f_n(x)$ for each $x \in E$. ($f(x) = \infty$ is allowed.) Then

$$\int_E f = \lim_{n\to\infty} \int_E f_n$$

where it is possible that both sides of the equation are $+\infty$.
($\lim_{n\to\infty} \int_E f_n \leq \int_E f$ is easy. And so is the other direction from Fatou's Lemma.)

105. REMARK: This is the Monotone Convergence Theorem. It and Fatou's Lemma are the principal convergence theorems for non-negative functions.

106. (a) Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of non-negative measurable functions on a measurable set E. Then

$$\int_E \left(\sum_{k=1}^{\infty} f_k\right) = \sum_{k=1}^{\infty} \left(\int_E f_k\right).$$

(b) Let f be a non-negative measurable function on a measurable set E, and let $\{A_n\}_{n=1}^{\infty}$ be a sequence of mutually disjoint measurable subsets of E. Then

$$\int_{\bigcup_{n=1}^{\infty} A_n} f = \sum_{n=1}^{\infty} \int_{A_n} f.$$

107. Let f and g be non-negative measurable functions on a measurable set E such that $f(x) \geq g(x)$ on E. If $\int_E f < \infty$, then $\int_E g < \infty$, $\int_E (f-g) < \infty$ and

$$\int_E (f-g) = \int_E f - \int_E g.$$

(Use #101.)

108. Let f be a non-negative measurable function on a measurable set E such that $\int_E f < \infty$. Then for each $\varepsilon > 0$ there is $\delta > 0$ so that for every $A \subset E$ with $m(A) < \delta$, it is the case that $\int_A f < \varepsilon$.

(If not for some ε, there is a sequence A_n with $m(A_n) < 2^{-n}$ and $\int_{A_n} f \geq \varepsilon$. Set $B_n = \bigcup_{k=n}^{\infty} A_k$. and $f_n = f - f\chi_{B_n}$. Show that $\{f_n\}$ is a non-decreasing sequence that converges pointwise to f a.e. on E. This leads to a contradiction.)

109. REMARK: This looks like a sort of continuity condition. As the proof demonstrates, it really says that an integrable function cannot have "delta function" parts–places of size zero but positive area (or positive mass if you prefer to think of f as representing density).

110. REMARK: Let f be an extended real-valued function. The positive and negative parts of f, $f_+ = \max\{f, 0\}$ and $f_- = \max\{-f, 0\}$ can be defined as extended real-valued functions just as in # 64 with the properties already developed. Notice that for any x, at most one of f_+ and f_- is different from 0, and that, as before, $f = f_+ - f_-$ and $|f| = f_+ + f_-$.

111. DEFINITION: (1) A non-negative extended real-valued measurable function f defined on a measurable set E is **integrable over** E if $\int_E f < \infty$.

(2) An extended real-valued measurable function f defined on a measurable set E is **integrable over** E if both its positive part and its negative part, f_+ and f_-, are integrable over E. In that case, we define

$$\int_E f = \int_E f_+ - \int_E f_-.$$

We denote the set of all functions integrable over E by $L^1(E)$.

112. REMARKS: (1) An integrable extended real-valued function f must have finite values almost everywhere.

(2) A measurable function f on E is integrable over E if and only

if $|f| = f_+ + f_-$ is integrable over E.(Why?) Thus $L^1(E)$ is usually defined as the set of all measurable functions on E such that $\int_E |f| < \infty$.

113. If f and g are integrable on E, and if α, β are real numbers, then
(a) $\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g$,
(b) if $f \le g$, then $\int_E f \le \int_E g$,
(c) if A and B are disjoint measurable subsets of E, then $\int_{A \cup B} f = \int_A f + \int_B f$.
(For (a), show $\int_E (\alpha f) = \alpha \int_E f$ and $\int_E (f+g) = \int_E f + \int_E g$. For (b), use (a) and #101.)

114. Let g be a non-negative integrable function on E, and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions such that $|f_n| \le g$ a.e. on E for each n. If $f_n(x) \to f(x)$ for almost all $x \in E$, then

$$\lim_{n \to \infty} \int_E f_n = \int_E f.$$

(Apply Fatou's Lemma to the non-negative sequences $\{g - f_n\}$ and $\{g + f_n\}$.)

115. REMARK: The preceding result is the Lebesgue Dominated Convergence Theorem. It is perhaps the most often used of the convergence theorems. Note that our first convergence theorem #98, is a special case.

116. REMARK: Note that f being Lebesgue integrable over a set E is not the same as f having an improper Riemann integral over E. For instance, $\int_0^\infty \frac{\sin x}{x} dx$ exists, as you may see by observing that if we set $a_n = \int_{(n-1)\pi}^{n\pi} \frac{\sin x}{x} dx$ then $\int_0^\infty \frac{\sin x}{x} dx = \sum_{n=1}^{\infty} a_n$ converges by the Alternating Series Test. However the integral of the positive part of $\frac{\sin x}{x}$ is infinite (and the integral of the negative part also) since $|a_n| \approx \frac{k}{n}$. In other words, $\sum_{n=1}^{\infty} a_n$ converges conditionally, but not absolutely. f being Lebesgue integrable corresponds to absolute convergence–convergence via cancellation is not allowed. This makes working with Lebesgue integrable functions much simpler, at the expense of making the class of integrable functions somewhat smaller.

VI. The Classical Banach Spaces

117. REMARK. We explore briefly one major application of the Lebesgue integral. First I recall some familiar definitions.

118. DEFINITION: A **metric space** is a pair (X,d) where X is a set and $d : X \times X \to \mathbb{R}$ is a function such that
(a) $d(x,y) \geq 0$ for all $x,y \in X$ and $d(x,y) = 0$ iff $x = y$,
(b) $d(x,y) = d(y,x)$ for all $x,y \in X$,
(c) $d(x,z) \leq d(x,y) + d(y,z)$ for all $x,y,z \in X$.

119. DEFINITIONS: (1) A sequence $\{x_n\}_{n=1}^{\infty}$ **converges to the limit** x in the metric space (X,d) if for every $\varepsilon > 0$ there is an integer N such that $d(x_n,x) < \varepsilon$ whenever $n \geq N$.
(2) A sequence $\{x_n\}_{n=1}^{\infty}$ is a **Cauchy sequence** in the metric space (X,d) if for every $\varepsilon > 0$ there is an integer N such that $d(x_n,x_m) < \varepsilon$ whenever $m,n \geq N$. It is easy to see that every convergent sequence is Cauchy, but the converse is in general not true.
(3) A metric space (X,d) is **complete** if every Cauchy sequence is convergent.

120. REMARK. You will recall that the set of rationals $\mathbb{Q}$ with the usual distance on $\mathbb{R}$ is not complete, but $\mathbb{R}$ is complete, since "the holes have been filled in." A somewhat different example is that the open interval $(0,1)$ with the usual distance is not complete, because sequences "converging to 0 or to 1" do not have a limit. In each case we know how to find the complete metric space that appears to contain the given metric space most efficiently. You may have seen a general procedure to produce the "completion" of any metric space–the idea is that the points of the completion are equivalence classes of Cauchy sequences of points of the original space. Constant sequences may be identified with points of the original space, so we can think of this as a larger space. It is rather hard to visualize this abstract completion, so more concrete constructions (as in the two examples) are desirable. The Lebesgue integral will provide some of these.

121. EXAMPLES. (1) Any non-empty subset of $\mathbb{R}^n$ together with the usual Euclidean distance is a metric space. It is also possible to introduce alternative distance functions, such as $d_1(x,y) = \sum_{j=1}^{n} \left| x_j - y_j \right|$ for $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$. It turns out that these are not very different in the sense that a sequence converges with respect to any reasonable distance function in $\mathbb{R}^n$ (such as d_1) if and only if it converges to the same limit with respect to Euclidean distance.
(2) The set $C[0,1]$ of all real-valued continuous functions on $[0,1]$ is a metric space with the distance $d(f,g) = \max_x |f(x) - g(x)|$. (Since the

set $[0,1]$ is compact, the supremum of the differences $|f(x)-g(x)|$ is actually a max.) Convergence with respect to this distance is uniform convergence. The theorem that the limit of a uniformly convergent sequence of continuous functions is continuous translates in this terminology to the statement that $C[0,1]$ with this metric is a complete metric space. However for the set $C[0,1]$ it is easy to define other distance functions which are essentially different in the sense that they produce a form of convergence that is not equivalent to uniform convergence and with respect to which $C[0,1]$ is not a complete metric space. We will do this shortly.

122. DEFINITION: A real **linear space** (or real **vector space**) is a triple $(V,+,\cdot)$ consisting of a set V of objects (called **vectors**) a mapping $+: V\times V\to V$ (called **addition**) and a mapping $\cdot:\mathbb{R}\times V\to V$ (called **scalar multiplication**) such that
(1) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ for all $\mathbf{u},\mathbf{v}\in V$,
(2) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$ for all $\mathbf{u},\mathbf{v},\mathbf{w}\in V$,
(3) there is a vector $\mathbf{0}\in V$ (called the **zero vector**) such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$ for all $\mathbf{u}\in V$,
(4) for each $\mathbf{u}\in V$ there is a vector $-\mathbf{u}\in V$ such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$,
(5) $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$ for all $\mathbf{u},\mathbf{v}\in V$ and all $c\in\mathbb{R}$,
(6) $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$ for all $\mathbf{u}\in V$ and all $c,d\in\mathbb{R}$,
(7) $c(d\mathbf{u})=(cd)\mathbf{u}$ for all $c,d\in\mathbb{R}$ and all $\mathbf{u}\in V$,
(8) $1\mathbf{u}=\mathbf{u}$ for all $\mathbf{u}\in V$.

123. DEFINITION. A subset W of a vector space V is a **subspace** of V if it is a vector space in its own right with the same operations as V.

124. EXAMPLES: (1) Once again $\mathbb{R}^n$ with the obvious component-wise operations is the basic example. This time, however, only certain subsets are subspaces of $\mathbb{R}^n$. (Those consisting of lines or planes though the origin, or just the zero element for $n=3$.)
(2) $C[0,1]$ is also a vector space with the usual operations of adding functions and multiplying a function by a real number.. The set $\mathscr{P}$ of polynomial functions is a subspace of $C[0,1]$, and for every positive integer n, the set $\mathscr{P}_n$ of polynomials of degree at most n is a subspace of $\mathscr{P}$.

125. DEFINITION: A **normed linear space** is a pair $(V,\|\cdot\|)$ consisting of a real linear space V (strictly $(V,+,\cdot)$) and a mapping $\|\cdot\|:V\to\mathbb{R}$ called a **norm** on V such that
(1) $\|x\|\geq 0$ for all $x\in V$ and $\|x\|=0$ if and only if $x=0$ (the zero element of V)
(2) $\|x+y\|\leq\|x\|+\|y\|$ for all $x,y\in V$.

126. REMARK: A normed linear space is a metric space with the distance function $d(x,y) = \|x-y\|$.

127. DEFINITION. A normed linear space that is a complete metric space with this distance function is called a **Banach space**.

128. EXAMPLES. $C[0,1]$ with the norm $\|f\| = \max_x |f(x)|$ is a Banach space. So is $\mathbb{R}^n$ with the Euclidean norm or, for that matter, any other reasonable norm.

129. Let ϕ be a continuous, increasing, real-valued function defined on $[0,\infty)$ such that $\phi(0) = 0$ and $\lim_{x\to\infty} \phi(x) = \infty$. Let ψ be the inverse of ϕ. Let $\Phi(x) = \int_0^x \phi$ and $\Psi(x) = \int_0^x \psi$. Then for any positive real numbers a and b,
$$ab \leq \Phi(a) + \Psi(b)$$
with equality if and only if $b = \phi(a)$.
(Interpret $\Phi(a)$ and $\Psi(b)$ as areas associated with the graph of ϕ. For $\Psi(b)$ you will need to remember that the graph of ψ is the "mirror image" of the graph of ϕ, so you can find $\Psi(b)$ as an area between the y-axis and the graph of ϕ. This result is called Young's Inequality.)

130. Let $p > 1$. If $\phi(x) = x^{p-1}$, then the preceding inequality takes the form
$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q$$
where q is the number such that $\frac{1}{p} + \frac{1}{q} = 1$. Equality holds if and only if $b = a^{p-1}$ or equivalently iff $b^q = a^p$.

131. Let $p > 1$ and let q be so that $\frac{1}{p} + \frac{1}{q} = 1$. Let f and g be measurable functions on a set E such that $\int_E |f|^p < \infty$ and $\int_E |g|^q < \infty$. Then
$$\int_E |fg| \leq \left(\int_E |f|^p\right)^{1/p} \left(\int_E |g|^q\right)^{1/q}.$$
In particular, the left side of the inequality is finite. Equality holds iff $|f|^p = |g|^q$ a.e. on E or $f = 0$ a.e. or $g = 0$ a.e.
(Suppose first that $\int_E |f|^p = \int_E |g|^q = 1$. Apply the preceding problem with $a = |f(x)|, b = |g(x)|, x \in E$ and integrate. For arbitrary f and g, consider $F = f/\left(\int_E |f|^p\right)^{1/p}, G = g/\left(\int_E |g|^q\right)^{1/q}$. This is Hölder's Inequality. For $p = q = 2$ it is usually called the Cauchy or Cauchy-Schwarz Inequality.)

132. Let $p > 1$. If $\int_E |f|^p < \infty$ and $\int_E |g|^p < \infty$ then $\int_E |f+g|^p < \infty$ and

$$\left(\int_E |f+g|^p\right)^{1/p} \leq \left(\int_E |f|^p\right)^{1/p} + \left(\int_E |g|^p\right)^{1/p}.$$

(For the first assertion, for any $x \in E$,

$$\begin{aligned} |f(x)+g(x)|^p &\leq (2\max\{|f(x)|, |g(x)|\})^p \\ &\leq 2^p \max\{|f(x)|^p, |g(x)|^p\} \leq 2^p(|f(x)|^p + |g(x)|^p). \end{aligned}$$

Then show $|f+g|^p \leq |f+g|^{p-1}|f| + |f+g|^{p-1}|g|$ and apply Hölder's Inequality where you use $q(p-1) = p$. The result of this problem is called Minkowski's Inequality.)

133. DEFINITION. A measurable function f on a set E is **essentially bounded** if its **essential supremum**

$$\|f\|_\infty = \inf\{M : |f(x)| \leq M \text{ a.e}\}$$

is finite. Note that if f is continuous on E, then $\|f\|_\infty = \sup\{|f(x)| : x \in E\}$.

134. If f is a measurable function on a set E such that $\|f\|_\infty$ is finite, then there is a subset A of E such that $m(A) = 0$ and $|f(x)| \leq \|f\|_\infty$ for all $x \in E \backslash A$.

(If not, then for some n, there is $B, m(B) > 0$ and $|f(x)| \geq \|f\|_\infty + \dfrac{1}{n}$ on B.)

135. If $\|f\|_\infty$ and $\|g\|_\infty$ are finite, then $\|f+g\|_\infty$ is finite and $\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.

136. If $\int_E |f| < \infty$ and $\int_E |g| < \infty$, then $\int_E |f+g| \leq \int_E |f| + \int_E |g|$.

137. REMARK: We have now shown that the mapping $f \to \|f\|_p$ satisfies the triangle inequality for each $p, 1 \leq p \leq \infty$, where $\|f\|_p = (\int_E |f|^p)^{1/p}$ for $1 \leq p < \infty$. These functions are, however, not quite norms on the sets of functions on which they are defined because it is not true that $\|f\|_p = 0$ implies that $f = 0$. What it does imply is that $f(x) = 0$ a.e. on E. We can fix this by regarding our objects as equivalence classes of functions, where $f \sim g$ means $f = g$ a.e. With this understanding we have now shown that for each $p, 1 \leq p < \infty$, the set

$$L^p(E) = \left\{ f : \int_E |f|^p < \infty \right\}$$

is a normed linear space with norm $\|f\|_p = \left(\int_E |f|^p\right)^{1/p}$. Also

$$L^\infty(E) = \{f : \|f\|_\infty < \infty\}$$

is a normed linear space.

138. REMARK: As normed linear spaces the L^p spaces are metric spaces with the distance function $d_p(f,g) = \|f-g\|_p$. The next big question is whether they are complete with respect to this distance function. This is easier for L^∞ than for L^p with $p < \infty$.

139. The set $C[0,1]$ can be considered a subset of $L^p[0,1]$ for each p. It is in fact dense in L^p for each finite p as we could see by using the problems from some time ago about approximating measurable functions by continuous functions. (It is not dense in L^∞ as we will see in the next problem.) Consider the sequence $\{f_n\}_{n=3}^\infty$ of continuous functions defined by

$$f_n(x) = \begin{cases} 0, & 0 \le x \le \frac{1}{2} - \frac{1}{n}, \\ \frac{n}{2}\left(x - \frac{1}{2} + \frac{1}{n}\right), & \frac{1}{2} - \frac{1}{n} \le x \le \frac{1}{2} + \frac{1}{n}, \\ 1, & \frac{1}{2} + \frac{1}{n} \le x \le 1. \end{cases}$$

(a) Show that this sequence is a Cauchy sequence in $L^1[0,1]$. (Not a calc problem. Draw a picture and compute area.)
(b) Show that this sequence is a Cauchy sequence in $L^p[0,1]$ for $1 < p < \infty$. (Use $|f_n - f_m| < 1$ on $[0,1]$!)
(c) What is the limit f of this sequence in $L^p[0,1]$, $1 \le p < \infty$? Justify your conclusion.

140. (a) Show that the sequence $\{f_n\}_{n=3}^\infty$ of the preceding problem is not Cauchy in $L^\infty[0,1]$, or equivalently in $C[0,1]$ with the uniform norm.
(b) Show that the limit f from the preceding problem satisfies $\|f - f_n\|_\infty = 1/2$ for each n.
(c) Show that for any $g \in C[0,1]$, $\|f-g\|_\infty \ge 1/2$. (Thus $C[0,1]$ is not dense in $L^\infty[0,1]$.)

141. $L^\infty(E)$ is a Banach space.

(Let $\{f_n\}_{n=1}^\infty$ be a Cauchy sequence in $L^\infty(E)$ and choose a representative of each f_n, which we still denote by f_n. For each positive integer k there is n_k so that $\|f_n - f_m\|_\infty < 2^{-k}$ whenever $m,n \ge n_k$. Consider the sequence $\{f_{n_k}\}_{k=1}^\infty$. Then $\ell > k$ implies $\|f_{n_\ell} - f_{n_k}\|_\infty < 2^{-k}$. There is a set $A, m(A) = 0$, such that the series $f_{n_1}(x) + \sum_{i=2}^{k} \left(f_{n_i}(x) - f_{n_{i-1}}(x) \right)$ of real numbers converges absolutely for each $x \in E \backslash A$. The function f defined as the pointwise limit of this series is in $L^\infty(E)$. Given $\varepsilon > 0$ there is n_ε so that $m, n \geq n_\varepsilon$ implies that $\| f_n - f_m \|_\infty < \varepsilon$. Then for any such f_m and any $x \in E \backslash A, |f(x) - f_m(x)| = \lim_{k \to \infty} |f_{n_k}(x) - f_m(x)| \leq \varepsilon$.)

142. For each $1 \leq p < \infty$, if $\{f_n\}_{n=1}^\infty$ is a Cauchy sequence in $L^p(E)$ and if we choose a representative of each f_n (we will still denote these functions by $\{f_n\}$), then there is a subsequence $\{f_{n_k}\}_{k=1}^\infty$ that converges pointwise a.e. on E to a measurable function f such that $\int_E |f|^p < \infty$.
(For each positive integer k there is n_k so that $\| f_n - f_m \|_p < 2^{-k}$ whenever $m, n \geq n_k$. Why? Consider the sequence $\{f_{n_k}\}_{k=1}^\infty$. Then $\ell > k$ implies $\| f_{n_\ell} - f_{n_k} \|_p < 2^{-k}$. For each integer $k \geq 2$, set $g_k = |f_{n_1}| + \sum_{i=2}^{k} \left| f_{n_i} - f_{n_{i-1}} \right|$. Then $\{g_k\}_{k=1}^\infty$ is a non-decreasing sequence of non-negative functions such that $\|g_k\|_p \leq \| f_{n_1} \|_p + 1$ for each k, and the extended real-valued function $g(x) = \lim_{k \to \infty} g_k(x)$ satisfies $\int_E g^p = \lim_{k \to \infty} \int_E g_k^p < \infty$. For almost all x in E, the series $f_{n_1}(x) + \sum_{i=2}^{\infty} \left(f_{n_i}(x) - f_{n_{i-1}}(x) \right)$ of real numbers converges absolutely, and so defines a measurable function f there. This function is in $L^p(E)$.)

143. If $\{f_n\}_{n=1}^\infty$ and f are as in the preceding problem, then $f_n \to f$ in $L^p(E)$. Thus $L^p(E)$ is complete.
(Let $\varepsilon > 0$ be given. For $m, n > n_\varepsilon, \| f_n - f_m \|_p < \varepsilon$. In particular, $\| f_{n_k} - f_m \|_p < \varepsilon$ if $m, n_k > n_\varepsilon$ Then Fatou's Lemma implies

$$\int_E |f - f_m|^p \leq \liminf_k \int_E |f_{n_k} - f_m|^p < \varepsilon^p.)$$

144. REMARK: The completeness of L^p is sometimes referred to as the Riesz-Fischer Theorem, although what Riesz actually proved in 1906 was rather different. It was similar to the example explored below.

145. REMARK: There is another way to arrange the proof of the completeness of L^p that looks a little slicker, but perhaps changes the perception of what's really going on. One proves the abstract theorem that a normed linear space X is complete if and only if the following property

is true: if a series $\sum_{n=1}^{\infty} x_n$ **converges absolutely** (this means the series $\sum_{n=1}^{\infty} \|x_n\|$ of norms converges in $\mathbb{R}$) then the series converges in X (this means the sequence of partial sums $\sum_{n=1}^{N} x_n$ converges to an element of X). Then you show that $L^P(E)$ has this property. If you look carefully, you will find these elements in what we did.

146. DEFINITION: The set ℓ^p (read "little L^p") is the set of real (or complex) sequences $\{c_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} |c_n|^p < \infty$. It is a complete normed linear space–a Banach space–with the norm $\|\{c_n\}\|_p = \left(\sum_{n=1}^{\infty} |c_n|^p\right)^{1/p}$. (That this is a norm with respect to which ℓ^p is complete can be proved by methods roughly similar to what we have done.)

147. REMARK: We can add more structure for L^2 and ℓ^2. An **inner product** on a vector space is a map $(\cdot,\cdot) : V \times V \to \mathbb{R}$ with the properties
(1) $(v,v) \geq 0$ and $(v,v) = 0$ iff $v = 0$ (zero element in V, of course),
(2) $(v,w) = (w,v)$ for all $v,w \in V$,
(3) $(av + bw, u) = a(v,u) + b(w,u)$ for all $u,v,w \in V$ and all $a,b \in \mathbb{R}$.
Given an inner product, $\|v\| = \sqrt{(v,v)}$ is a norm on V, so every inner product space is a normed linear space and hence a metric space. (Topology from algebra!) If it is complete as a metric space, we call it a **Hilbert space**. However in an inner product space we can also define the angle between elements by analogy with the dot product. In particular v and w are **orthogonal** if $(v,w) = 0$. It is possible to extend "dot product geometry" to this context–orthogonal bases, expressing an arbitrary element as the sum of its projections onto the elements of an orthogonal basis and so forth. The so forth includes the equivalent of the Pythagorean Theorem: the square of the norm of an element is the sum of the squares of the lengths of its projections. However the extension must cope with the fact that bases are typically countably infinite sets, so the sums are infinite sums and there are questions of convergence.

In ℓ^2 the inner product is $(\{c_n\},\{d_n\}) = \sum_{n=1}^{\infty} c_n d_n$. The equivalent of Hölder's Inequality guarantees that $|(\{c_n\},\{d_n\})| \leq \|\{c_n\}\| \, \|\{d_n\}\|$ and so in particular that the sum converges absolutely. In $L^2(E)$ the inner product is $(f,g) = \int_E fg$. Again this converges by Hölder's Inequality.

In $L^2[0,\pi]$ the sequence $\{\sin nx\}_{n=1}^{\infty}$ has the properties $\|\sin nx\|_2 = \sqrt{\frac{\pi}{2}}$ and $\sin nx \perp \sin mx$ if $m \neq n$, that is, $\int_0^{\pi} \sin nx \sin mx dx = 0$ if $m \neq n$. The projection of any function $f \in L^2[0,\pi]$ on $\sin nx$ is

$$\frac{(f, \sin nx)}{(\sin nx, \sin nx)} \sin nx = \left(\frac{2}{\pi} \int_0^{\pi} f(t) \sin nt dt \right) \sin nx.$$

The projection p_n of $f \in L^2[0,\pi]$ on $\text{SPAN}\{\sin x, \sin 2x, ..., \sin nx\}$ is $\sum_{k=1}^{n} c_k \sin kx$ where $c_k = \frac{2}{\pi} \int_0^{\pi} f(t) \sin kt dt$. It is easy to see, using the orthogonality of the different sine functions that $\|p_n\|_2^2 = \sum_{k=1}^{n} c_k^2 \leq \|f\|_2^2$ where the last inequality just comes from the fact that any projection of f has norm less than or equal to that of f. It follows that $\{c_k\}_{k=1}^{\infty}$ is in ℓ^2, that $\{p_n\}_{n=1}^{\infty}$ is Cauchy in $L^2[0,\pi]$ (since $\|p_n - p_m\|_2^2 = \sum_{k=m+1}^{n} c_k^2$), and then by completeness that $\{p_n\}_{n=1}^{\infty}$ converges in $L^2[0,\pi]$ to an element $\sum_{k=1}^{\infty} c_k \sin kx$. It turns out that $f = \sum_{k=1}^{\infty} c_k \sin kx$ and that

$$\|f\|_2^2 = \sum_{k=1}^{\infty} c_k^2 = \|\{c_n\}\|^2.$$

(These turn out to be equivalent to the statement that the only element of $L^2[0,\pi]$ that is orthogonal to every $\sin nx$ is the zero element, that is, the set $\{\sin nx\}_{n=1}^{\infty}$ is **complete** (roughly, large enough to function as a basis).)

The result of all this is that the mapping $f \to \{c_n\}$ is an isometry (linear mapping preserving the norm) of $L^2[0,\pi]$ onto ℓ^2. In particular, given any sequence in ℓ^2, there is an L^2 function f with that sequence of Fourier sine coefficients. This is approximately the content of the original Riesz-Fischer Theorem in 1906.

VII. Abstract Measure Spaces

148. REMARK: We will proceed in several stages. First we will define an abstract measure space and remark that the class of measurable functions and integration with respect to a measure can be developed exactly as before. Then we will turn to the problem of generating a measure from a more primitive concept such as an outer measure.

149. DEFINITION: A **measure space** is a triple $(X, \mathfrak{B}, \mu)$ where X is a non-empty set, $\mathfrak{B}$ is a σ-algebra of subsets of X and $\mu : \mathfrak{B} \to [0, \infty]$ is an extended real valued function on $\mathfrak{B}$ with the properties
(1) $\mu(\emptyset) = 0$, and
(2) $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n)$ for every sequence $\{E_n\}$ of pairwise disjoint elements of $\mathfrak{B}$.

150. DEFINITIONS: We call μ a **measure** on X (really it is a measure on $\mathfrak{B}$, but terminology is often sloppy). The elements of $\mathfrak{B}$ are **measurable** sets (or **μ-measurable** sets if there is any possibility of confusion). μ is a **finite** measure if $\mu(X) < \infty$, and is a **σ-finite** measure if $X = \bigcup_{n=1}^{\infty} X_n$ where each $\mu(X_n) < \infty$. $(X, \mathfrak{B}, \mu)$ is a **complete** measure space if $\mathfrak{B}$ contains every subset of every set E with $\mu(E) = 0$.

151. EXAMPLES. (1) $([0,1], \mathfrak{M}, m)$ is a complete finite measure space; $(\mathbb{R}, \mathfrak{M}_{\mathbb{R}}, m)$ is a complete σ-finite measure space. If we restrict either measure to the Borel subsets of $[0,1]$ or $\mathbb{R}$, we have a finite or σ-finite measure space that is no longer complete.

(2) Let E be a non-empty measurable subset of $\mathbb{R}$, $\mathfrak{M}_E$ the σ-algebra of measurable subsets of E, and f a non-negative measurable function defined on E. Then $\mu(A) = \int_A f \ (= \int_A f\,dm)$ for $A \in \mathfrak{M}_E$ defines a measure on E. If $\int_E f < \infty$, μ is a finite measure.

(3) Let E be a non-empty measurable subset of $\mathbb{R}$, and $\mathfrak{M}_E$ the σ-algebra of measurable subsets of E. For each point $x \in E$, let $\mu(\{x\}) = 1$ and in general $\mu(A)$ is the number of points in A. (Thus $\mu(A) = \infty$ for all infinite subsets of E.) This is the **counting measure** on E.

(4) Alternatively, with the same σ-algebra of sets, fix one element x of E. Define $\mu(A) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \neq A. \end{cases}$ This is the **point mass** at x.

(5) For a strange example, let $X = [0,1]$, let $\mathfrak{B}$ be the collection of all subsets of $[0,1]$ which are either countable or the complement in $[0,1]$ of a countable set. For $E \in \mathfrak{B}$, define $\mu(E) = 0$ if E is countable and $\mu(E) = 1$ if E is the complement of a countable set.

(6) The most trivial example is: Let X be any non-empty set, let $\mathfrak{B} = \{\emptyset, X\}$ and set $\mu(X) = c$, any non-negative extended real number.

152. (a) If A and B are measurable sets with $A \subset B$, then $\mu(A) \leq \mu(B)$.
(b) If $\{E_n\}_{n=1}^{\infty}$ is any sequence of measurable sets, then

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu(E_n).$$

(For (b), let $A_n = E_n \backslash \bigcup_{k=1}^{n-1} E_k$. Elements of the sequence $\{A_n\}$ are pairwise disjoint.)

153. If $\{E_n\}_{n=1}^{\infty}$ is any sequence of measurable sets, then $\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \lim_{n\to\infty} \mu\left(\bigcup_{k=1}^{n} E_k\right)$.
(Use same sequence $\{A_n\}$.)

154. If $\{E_n\}_{n=1}^{\infty}$ is a nested sequence of measurable sets, $E_1 \supset E_2 \supset \ldots$ and if $\mu(E_1) < \infty$, then $\lim_{n\to\infty} \mu(E_n) = \mu\left(\bigcap_{n=1}^{\infty} E_n\right)$.

155. REMARK: In general all the results we proved for m on $[0,1]$ remain true in the abstract setting except for those that used specific facts about $\mathbb{R}$–for instance the result that every measurable set is "almost a finite union of open intervals."

156. REMARK: If $(X, \mathfrak{B}, \mu)$ is a measure space, an extended real-valued function f defined on X is **measurable** if for each real number α, $\{x \in X : f(x) < \alpha\} \in \mathfrak{B}$. All the properties of measurable functions, including the alternate forms of the definition, go through just as before.

157. REMARK: As long as $(X, \mathfrak{B}, \mu)$ is not too bizarre, we can develop a theory of integration on X with respect to the measure μ just as before, with the same theorems holding. "Not too bizarre" means that there should be enough measurable sets whose measure is positive and finite for the first stage of our previous procedure–defining the integral of a bounded measurable function f on a set of finite measure by taking the infimum of integrals of simple functions $s \geq f$–to seem reasonable. This is certainly true on any σ-finite measure space. We obtain the same theorems, including the same convergence theorems as before.

158. REMARK: The next step is to see how to define a measure space by starting with something more primitive. We will proceed in two stages.

159. DEFINITION. Let X be a non-empty set. An **outer measure** μ^* on X is an extended real-valued function on the collection $\mathscr{P}(X)$ of all subsets of X with the properties that
(1) $\mu^*(\emptyset) = 0$,

(2) if $A \subset B$, then $\mu^*(A) \leq \mu^*(B)$

(3) if $E \subset \bigcup_{k=1}^{\infty} E_k$, then $\mu^*(E) \leq \sum_{k=1}^{\infty} \mu^*(E_k)$.

160. DEFINITION and REMARK: Let μ^* be an outer measure on X. A set E is **measurable** (or μ^*-measurable) if

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

for every $A \subset X$.

It follows exactly as in our development of Lebesgue measure on $[0,1]$ that the collection $\mathfrak{B}$ of all measurable sets is a σ-algebra of subsets of X containing all sets of outer measure zero and that the restriction μ of μ^* to $\mathfrak{B}$ is countably additive. Thus $(X, \mathfrak{B}, \mu)$ is a complete measure space.

161. REMARK: What requires a little more work is this: how do we know what sets are in $\mathfrak{B}$? In particular, in any concrete situation we want all "reasonable sets" to be in $\mathfrak{B}$. For instance, if X is a metric space, we generally want all open sets (and therefore all Borel sets) to be in $\mathfrak{B}$. Clearly this will require some appeal to the other more specific properties of X. However we can abstract some of it.

162. DEFINITION: An **algebra**, $\mathscr{A}$, of subsets of a non-empty set X is a collection of subsets that is closed under finite unions and the taking of complements. Note that then any algebra of subsets of X contains both X and the empty set, and is also closed under finite intersections.

163. DEFINITION: A function μ_0 defined on an algebra $\mathscr{A}$ of subsets of a non-empty set X with non-negative extended real values is a **measure on an algebra** if

(1) $\mu_0(\emptyset) = 0$, and

(2) if $\{E_k\}_{k=1}^{\infty}$ is a sequence of pairwise disjoint elements of $\mathscr{A}$ such that $\bigcup_{k=1}^{\infty} E_k \in \mathscr{A}$, then $\mu_0\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu_0(E_k)$.

164. EXAMPLE: Of course we are mimicking our original construction of Lebesgue measure on $[0,1]$ starting from the concept of length defined on intervals. Note that the collection of finite unions of open intervals is not an algebra because it is not closed under complementation. However, if we regard the basic set as $(0,1]$, (or the other way around) and the intervals as open on the left and closed on the right (or the other way around), then we do have an algebra, and the properties of

the previous definition do hold for length, though the second one requires a proof generally similar to the proof that the outer measure of an interval is its length.

165. Let μ_0 be a measure on an algebra $\mathscr{A}$ of subsets of X. Define an extended real-valued function μ^* on the collection $\mathscr{P}(X)$ of all subsets of X by

$$\mu^*(E) = \inf\left\{ \sum_{k=1}^{\infty} \mu_0(B_k) : E \subset \bigcup_{k=1}^{\infty} B_k \text{ and each } B_k \in \mathscr{A} \right\}.$$

Then μ^* is an outer measure on X.

166. If $E \in \mathscr{A}$ and $\mu^*(E)$ is defined as in the previous problem, then $\mu^*(E) = \mu_0(E)$.
($\mu^*(E) \leq \mu_0(E)$ is clear. For the other direction, use yet again that every countable union is a countable disjoint union.)

167. In the same situation, every element of $\mathscr{A}$ is μ^*-measurable.
(If $E \in \mathscr{A}$ and A is any subset of X, we need $\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c)$. May assume $\mu^*(A) < \infty$ and, given $\varepsilon > 0$ choose $\{B_k\}_{k=1}^{\infty} \in \mathscr{A}$ so that $A \subset \bigcup_{k=1}^{\infty} B_k$ and $\mu^*(A) + \varepsilon > \sum_{k=1}^{\infty} \mu_0(B_k)$. Then $\mu_0(B_k) = \mu_0(B_k \cap E) + \mu_0(B_k \cap E^c)$ for each k can be used to show $\sum_{k=1}^{\infty} \mu_0(B_k) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c)$.

168. REMARK: We have now proved the following: If μ_0 is a measure on the algebra $\mathscr{A}$ of subsets of X then there is an extension μ of μ_0 to a complete measure on a σ-algebra $\mathfrak{B}$ of subsets of X containing $\mathscr{A}$. This is called the Hahn Extension Theorem. It is fairly easy to show that if μ_0 is σ-finite, then the extension is unique. The part of the construction that says you can extract a measure from an outer measure is called the Carathéodory Extension Theorem. This is just an abstract version of our original construction of Lebesgue measure starting from the lengths of intervals. Next we will look briefly at two important specific examples.

169. EXAMPLE: Let $\mathscr{A}$ be the collection of all finite unions of intervals on $\mathbb{R}$ that are open on the left and closed on the right. (This includes all intervals unbounded to the right, or to the left, or both.) Let g be any non-decreasing function on $\mathbb{R}$ that is continuous from the right. (Formally, $g(a) = \lim_{h \to 0^+} g(a+h)$; informally, at any jump point a, the

value of g at the jump point goes with the part of the graph extending to the right from a.) Define

$$\begin{aligned}\mu_g((a,b]) &= g(b)-g(a)\\ \mu_g((-\infty,b]) &= g(b)-\lim_{x\to-\infty} g(x)\\ \mu_g((a,\infty)) &= \lim_{x\to\infty} g(x)-g(a)\\ \mu_g(\mathbb{R}) &= \lim_{x\to\infty} g(x)-\lim_{x\to-\infty} g(x)\end{aligned}$$

and extend μ_g linearly on finite disjoint unions of these sets. More or less the same argument as with intervals shows that μ_g is a σ-finite measure on the algebra $\mathscr{A}$. The complete measure obtained via the Hahn Extension Theorem is called the **Lebesgue-Stieltjes measure generated by** g. If we restrict the σ-algebra of measurable sets to the Borel sets, we call the associated measure the **Borel-Stieltjes measure generated by** g. We usually write $\int_E f dg$ in place of $\int_E f d\mu_g$. Note that μ_g has a point mass at any point of discontinuity of g, that is, $\mu_g(\{a\}) = g(a) - \lim_{h\to 0^+} g(a-h)$. μ_g is a finite measure if and only if g is a bounded function.

This process can be reversed. If μ is a finite Borel measure on $\mathbb{R}$, we can define the cumulative distribution function F of μ by

$$F(x) = \mu((-\infty,x]).$$

It is obvious that F is non-decreasing, $F(-\infty) = 0$ (this is really a statement about a limit as $x \to -\infty$, of course) and $F(\infty) = \mu(\mathbb{R})$. It is not difficult to see that F is continuous from the right at all points. F is continuous at a if and only if $\mu(\{a\}) = 0$.

170. EXAMPLE: Let $\mathscr{A}$ be the collection of all finite unions of "half-open intervals" in $\mathbb{R}^n$, that is, all finite unions of sets of the form

$$I = \left\{(x_1,\ldots,x_n) : a_j < x_j \le b_j \text{ for } j = 1,2,\ldots,n\right\}$$

where $a_1,\ldots,a_n,b_1,\ldots,b_n$ are extended real numbers. (Thus the sets need not be bounded. Define

$$\mu(I) = (b_1-a_1)(b_2-a_2)\ldots(b_n-a_n).$$

Then μ is a measure on the algebra $\mathscr{A}$. Verifying property (2) of a measure is rather tedious. The essential step is to show that if one finite union of intervals is contained in another finite union of intervals, then the sum of the volumes of the first collection is less than or equal to the sum of the volumes of the second collection. After (2) has been verified, applying the extension theorems yields a complete measure

m_n defined on a σ-algebra containing the Borel sets in $\mathbb{R}^n$ which agrees with volume on "intervals." This measure is, of course, n-dimensional Lebesgue measure.

171. Complete the verification of property (2) (of Def. 162) for the preceding example, assuming the statement about finite unions of intervals.

(It follows immediately from the statement that if $I = \bigcup_{j=1}^{\infty} I_j$ as a disjoint union, then $\mu(I) \geq \sum_{j=1}^{\infty} \mu(I_j)$ (since this is true for each partial sum). For the other direction, assume $\sum_{j=1}^{\infty} \mu(I_j) < \mu(I)$."Blow up" each I_j slightly into an open interval $I_j^{'}$ so that $\bigcup_{j=1}^{\infty} I_j^{'}$ contains the closure of I, but still $\sum_{j=1}^{\infty} \mu\left(I_j^{'}\right) < \mu(I)$ and use compactness. Finally, we can "blow up" each $I_j^{'}$ again to an element $I_j^{''}$ which is back in the collection $\mathscr{A}$ of half-open intervals, but still $\sum_{j=1}^{n} \mu\left(I_j^{''}\right) < \mu(I)$.)

172. REMARK: The construction of the preceding example can be made more abstract as follows: let $(X, \mathfrak{M}, \mu)$ and $(Y, \mathfrak{N}, \nu)$ be measure spaces. We will construct a product measure on the set $X \times Y = \{(x,y) : x \in X, y \in Y\}$. The collection $\mathscr{A}$ of finite unions of measurable rectangles $A \times B$, where $A \in \mathfrak{M}$ and $B \in \mathfrak{N}$ is an algebra of subsets of $X \times Y$. Define $\pi_0(A \times B) = \mu(A)\nu(B)$, and extend linearly to finite disjoint unions of measurable rectangles. (We interpret $\infty \cdot 0$ as 0.) π_0 is a measure on the algebra A provided we can verify property (2). In this more abstract setting we will have to do something different from the previous problem.

173. In the context of the previous remark, if a measurable rectangle $A \times B = \bigcup_{j=1}^{\infty} (A_j \times B_j)$ as a disjoint union, then $\mu(A)\nu(B) = \sum_{j=1}^{\infty} \mu(A_j)\nu(B_j)$.

(Rewrite the equality between sets as an equality between characteristic functions: $\chi_A(x)\chi_B(y) = \sum_{j=1}^{\infty} \chi_{A_j}(x)\chi_{B_j}(y)$. Fix x. Integrate with respect to ν and use the Monotone Convergence Theorem, noting for instance that $\int_Y \chi_A(x)\chi_B(y)\,d\nu(y) = \nu(B)\chi_A(x)$. Then integrate with respect to μ.)

174. REMARK: The result is that starting with $(X, \mathfrak{M}, \mu)$ and $(Y, \mathfrak{N}, \nu)$ there is a product measure π on a σ-algebra $\mathfrak{L}$ of subsets of $X \times Y$ containing the measurable rectangles. This measure is complete and has the property that $\pi(A \times B) = \mu(A)\nu(B)$. Next we want to imitate multivariate calculus and relate integration with respect to the product measure to iterated integration with respect to μ and then ν (or ν and then μ). The critical part is to see that if we start with an $\mathfrak{L}$-measurable function of two variables x and y, and then create a function of, say x alone by fixing y, we get an $\mathfrak{M}$-measurable function of x for ν-almost every value of y. We will need some lemmas.

175. Let μ_0 be a measure on an algebra $\mathscr{A}$ of subsets of X. Let $\mathscr{A}_\sigma$ denote the collection of countable unions of elements of $\mathscr{A}$, and let $\mathscr{A}_{\sigma\delta}$ denote the collection of all countable intersections of elements of $\mathscr{A}_\sigma$. Let μ^* be the induced outer measure. Then for any $E \subset X$ with $\mu^*(E) < \infty$ and any $\varepsilon > 0$ there is $A \in \mathscr{A}_\sigma$ with $E \subset A$ and $\mu^*(A) < \mu^*(E) + \varepsilon$. Moreover there is $A \in \mathscr{A}_{\sigma\delta}$ with $E \subset A$ and $\mu^*(A) = \mu^*(E)$.

176. NOTATION: Let $\mathscr{R}$ denote the collection of finite unions of measurable rectangles in $X \times Y, \mathscr{R}_\sigma$ the collection of countable unions of measurable rectangles, and $\mathscr{R}_{\sigma\delta}$ the collection of countable intersections of elements of $\mathscr{R}_\sigma$.

177. DEFINITION: If $E \subset X \times Y$, $x \in X$, and $y \in Y$ then

$$E_x = \{y \in Y : (x,y) \in E\} \text{ and}$$
$$E^y = \{x \in X : (x,y) \in E\}$$

are **cross-sections** of E. Note that $E_x \subset Y$ and $E^y \subset X$. Moreover the set relationship can be expressed in terms of characteristic functions as

$$\chi_{E_x}(y) = \chi_E(x,y) \text{ and}$$
$$\chi_{E^y}(x) = \chi_E(x,y).$$

178. If $E \in \mathscr{R}_\sigma$ and $x \in X$, then $E_x \in \mathfrak{N}$ or equivalently, $\chi_{E_x \text{ is}}$ $\mathfrak{N}$-measurable.

(First consider $E \in \mathscr{R}$. If $E = \bigcup_{j=1}^{\infty} E_j$, then $\chi_{E_x}(y) = \chi_E(x,y) = \sup_j \chi_{E_j}(x,y) = \sup_j \chi_{(E_j)_x}(y)$.)

179. If $E \in \mathscr{R}_{\sigma\delta}$ and $x \in X$, then $E_x \in \mathfrak{N}$.

(Similar, but use infs.)

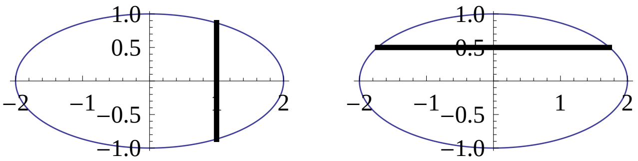


Figure 1: Left Figure: $E_x \subset Y$ Right Figure: $E^y \subset X$

180. Let π be the product measure on $X \times Y$ defined above, and let $E \in \mathscr{R}_{\sigma\delta}$ with $\pi(E) < \infty$. Then the function $g(x) = \nu(E_x)$ is μ-measurable on X and

$$\int_X g d\mu = \pi(E).$$

(Pretty clear if $E \in \mathscr{R}$. If $E = \bigcup_{j=1}^{\infty} E_j$, with the $E_j \in \mathscr{R}$ and disjoint (we can assume this), and $g_j(x) = \nu\left(\left(E_j\right)_x\right)$, then $g = \sum_{j=1}^{\infty} g_j$. Use countable additivity. If $E = \bigcap_{j=1}^{\infty} E_j$, with the $E_j \in \mathscr{R}_\sigma$, we may assume $E_1 \supset E_2 \supset \ldots$and $\pi(E_1) < \infty$. The sequence $g_j(x) = \nu\left(\left(E_j\right)_x\right)$ is non-increasing and g_1 is finite μ-almost everywhere. Use the Dominated Convergence Theorem.)

181. REMARK: In other notation the previous problem shows that if $E \in \mathscr{R}_{\sigma\delta}$ with $\pi(E) < \infty$ and $f(x,y) = \chi_E(x,y)$, then

$$\int_X \left(\int_Y f(x,y)\, d\nu(y)\right) d\mu(x) = \int_{X \times Y} f(x,y)\, d\pi.$$

This is a very special case of what we are aiming at. Actually many theorems about classes of functions are proved by first considering characteristic functions of sets, then simple functions, and then the general case.

182. Let E be any π-measurable subset of $X \times Y$ such that $\pi(E) = 0$, where $(Y, \mathfrak{N}, \nu)$ is a complete measure space. Then for μ-almost all $x \in X$, E_x is ν-measurable and $\nu(E_x) = 0$.

(Use #175 and #180).

183. Let E be any π-measurable subset of $X \times Y$ such that $\pi(E) < \infty$. Then for μ-almost all $x \in X$, E_x is a ν-measurable subset of Y. The function $g(x) = \nu(E_x)$ is defined almost everywhere and μ-measurable on X and
$$\int_X g d\mu = \pi(E).$$
In particular, g is finite μ-almost everywhere on X.
($E = F \backslash G$ where $F \in \mathscr{R}_{\sigma\delta}$ and $\pi(G) = 0$. Then $E_x = F_x \backslash G_x$.

184. Let $(X, \mathfrak{M}, \mu)$ and $(Y, \mathfrak{N}, \nu)$ be complete measure spaces and let $(X \times Y, \mathfrak{L}, \pi)$ be the product measure space. Let $f \in L^1(X \times Y, \pi)$. Then
(1) for μ-almost all x the function f_x defined by $f_x(y) = f(x,y)$ is ν-integrable on Y,
(2) for ν-almost all y the function f_y defined by $f_y(x) = f(x,y)$ is μ-integrable on X,
(3) $F(x) = \int_Y f_x d\nu$ is an integrable function on X,
(4) $G(y) = \int_X f_y d\mu$ is an integrable function on Y,
(5) $\int_X \left[\int_Y f_x d\nu\right] d\mu = \int_{X\times Y} f d\pi = \int_Y \left[\int_X f_y d\mu\right] d\nu$.
(Enough by symmetry to do (1),(3),(5). If true for $f \geq 0$, then true for general f by $f = f^+ - f^-$. Known for $f = \chi_E$, and hence for simple functions. If f is non-negative and integrable, f is the pointwise limit of a non-decreasing sequence of simple functions. Use the convergence theorems. Finally, write an arbitrary $f \in L^1(X \times Y)$ in the form $f = f_+ - f_-$.)

185. REMARK: The preceding result is Fubini's Theorem. It says that sections of "nice" functions are nice a.e. and that double integrals can be computed as iterated integrals, just as in multivariable calculus

VIII. Differentiation and Related Matters

186. DEFINITION: A real-valued function f defined on $\mathbb{R}$ is **monotonic** if it is either **nondecreasing** ($x \leq y$ implies $f(x) \leq f(y)$) or **nonincreasing** ($x \leq y$ implies $f(x) \geq f(y)$).

187. Every monotonic function is measurable.

188. DEFINITION: If f is defined on $\mathbb{R}$, the **right-hand limit of** f **at** x_0, denoted $f(x_0+)$, exists if for every $\varepsilon > 0$, $\exists \delta > 0$ such that $x_0 < x < x_0 + \delta$ implies $|f(x) - f(x_0)| < \varepsilon$. The **left-hand limit of** f **at** x_0, denoted $f(x_0-)$ is defined similarly. f **is continuous from the right (left)** at x_0 if $f(x_0) = f(x_0+)$ $(f(x_0) = f(x_0-)$.

189. REMARK: f is continuous at x_0 iff $f(x_0-)$ and $f(x_0+)$ both exist and $f(x_0-) = f(x_0) = f(x_0+)$. If f is monotonic, then the right-hand and left-hand limits both exist at every point, and f is continuous at every point at which they are equal. If they are not equal at x_0, then $f(x_0+) - f(x_0-)$ is the **jump** of f at x_0.

190. Let f be monotonic on $[a,b]$. Then f is continuous except at an at most countable set of points and if $\{x_k\}$ are the points of discontinuity,

$$\sum_{k=1}^{\infty} (f(x_k+) - f(x_k-)) \leq f(b) - f(a).$$

(For any positive integer n, $f(x_0+) - f(x_0-) \geq \frac{1}{n}$ at no more than $n(f(b) - f(a))$ points. For the last statement, consider the partial sums.)

191. DEFINITION: Let $\{x_k\}_{k=1}^{\infty}$ be any sequence of points in $[a,b]$ and let $\{h_k\}_{k=1}^{\infty}$ be any sequence of positive numbers such that $\sum_{k=1}^{\infty} h_k < \infty$. The nondecreasing function

$$j(x) = \sum_{x_k < x} h_k$$

is called a **jump** function.

192. REMARK: If $x_1 < x_2 < \ldots$, then the jump function is a step function. However jump functions can be more complicated than that. Let $\{x_k\}_{k=1}^{\infty}$ be all the rationals in $[0,1]$ whose denominators are powers of 2, that is, the **dyadic fractions**, ordered by increasing denominator and numerator, that is, $\frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \frac{1}{16}, \ldots$. Let $h_1 = \frac{1}{2}, h_2 = h_3 = \frac{1}{8}, h_4 = \ldots = h_7 = \frac{1}{2^5}$, and in general $h_{2^n} = \ldots = h_{2^{n+1}-1} = \frac{1}{2^{2n+1}}$. Then $\sum_{k=2^n}^{2^{n+1}-1} h_k = \frac{1}{2^{n+1}}$ for each positive integer n so that $\sum_{k=1}^{\infty} h_k = 1$. The jump function with jump h_k at x_k has $f(0) = 0$ and $f(1) = 1$, but is hard to draw! (By symmetry, $f\left(\frac{1}{2}-\right) = \frac{1}{4}, f\left(\frac{1}{2}+\right) = \frac{3}{4}, f\left(\frac{1}{4}-\right) = \frac{1}{16}, f\left(\frac{1}{4}+\right) = \frac{3}{16}, f\left(\frac{1}{8}-\right) = \frac{1}{64}, f\left(\frac{1}{8}+\right) = \frac{3}{64}$, and so forth. For the first equality, note that the list of x_k's with $x_k < \frac{1}{2}$ comprise exactly half of the list of all x_k's if $x_1 = \frac{1}{2}$ is excluded, and that the corresponding h_k's comprise half of a set of numbers that add to $\frac{1}{2}$.)

193. A jump function j is continuous from the left at each point and continuous at each point not in the set $\{x_k\}_{k=1}^{\infty}$ of jump points. Its jump at x_k is h_k, that is, $j(x_k+) - j(x_k-) = j(x_k+) - j(x_k) = h_k$.
($\{x_k : x_k < x\} = \bigcup_{n=1}^{\infty} \left\{x_k : x_k < x - \frac{1}{n}\right\}$. $j(x+) = \sum_{x_k \leq x} h_k$.)

194. Let f be nondecreasing and continuous from the left on $[a,b]$. Then $f = g + j$ where g is continuous and nondecreasing, and j is a jump function.
(Let f have jumps h_k at $x_k, k = 1,2,\ldots$. Define $j(x) = \sum_{x_k < x} h_k$, and $g(x) = f(x) - j(x)$. If $x' > x$, then $g(x') - g(x) = f(x') - f(x) - (j(x') - j(x))$. Use #190 on $[x, x']$.)

195. Let f be any nondecreasing function on $[a,b]$. Then $f = g + j + u$ where where g and j are as in the previous problem and u is a nonnegative function equal to 0 except possibly at the jump points x_k.

196. DEFINITION: Let f be continuous on $[a,b]$. The point $x \in [a,b]$ is **invisible from the right (invisible from the left)** if $\exists w \in [a,b], w > x$ $(w < x)$ such that $f(x) < f(w)$.

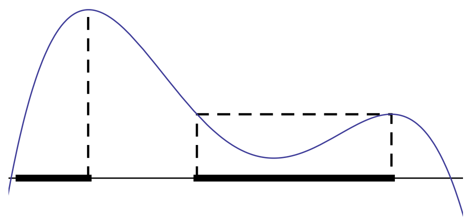


Figure 2: Thick Line indicates ‘Invisible from the Right’.

197. REMARK: The set of points in (a,b) invisible from the right is open and so a disjoint union of at most countably many open intervals (a_k, b_k). Note that each b_k is not invisible from the right. Similarly, the set of points in (a,b) invisible from the left is open.

198. With the notation of the previous remark, $f(x) \leq f(b_k)$ for all $x \in [a_k, b_k]$. In particular, $f(a_k) \leq f(b_k)$. Similarly, for the set of points invisible from the left, $f(a_k) \geq f(b_k)$ for each open interval (a_k, b_k).

(Suppose for some $x_0 \in (a_k, b_k)$, $f(x_0) > f(b_k)$. Then the set of all $z \in [x_0, b_k]$ such that $f(z) = f(x_0)$ has a greatest element x_m. Then x_m is not invisible from the right, which is a contradiction. ($x \in [x_m, b_k]$ implies $f(x) < f(x_m)$ by the Intermediate Value Theorem, and also $x \in [b_k, b]$ implies $f(x) < f(x_m)$.)

199. REMARK: The previous problem, due to the famous Hungarian mathematician Frigyes Riesz (1880-1956, one of the principal early figures in functional analysis), is the key lemma that we will use in proving a number of facts about differentiation. It is often called the Rising Sun Lemma, and substitutes for the use, more commonly seen in real analysis texts, of the more complicated Vitali Covering Theorem.

200. REMARK: The definition of invisibility from the right can be extended as follows. Assume only that f defined on $[a, b]$ has one-sided limits at each point, so that any discontinuity is due to a jump. Then $x \in [a, b]$ is invisible from the right with respect to f if there exists $w \in [a, b]$, $w > x$, such that

$$\max\{f(x-), f(x), f(x+)\} < f(w).$$

The conclusion of #198 must be modified from $f(a_k) \leq f(b_k)$ to

$$f(a_k+) \leq \max\{f(b_k-), f(b_k), f(b_k+)\}.$$

This extended version leads to the differentiability a.e. of any monotonic function, without the need for continuity.

201. DEFINITION: If f is a real valued function defined on an interval $(a, a+h)$, then

$$\limsup_{x \searrow a} f(x) = \lim_{\varepsilon \to 0} \{\sup(f(x) : x \in (a, a+\varepsilon))\}$$
$$\liminf_{x \searrow a} f(x) = \lim_{\varepsilon \to 0} \{\inf(f(x) : x \in (a, a+\varepsilon))\}$$

and if f is defined on some $(a-h, a)$ then

$$\limsup_{x \nearrow a} f(x) = \lim_{\varepsilon \to 0} \{\sup(f(x) : x \in (a-\varepsilon, a))\}$$
$$\liminf_{x \nearrow a} f(x) = \lim_{\varepsilon \to 0} \{\inf(f(x) : x \in (a-\varepsilon, a))\}.$$

Note that all four of these numbers must exist since each of the infs and sups is a monotonic function of ε. For instance, $\sup(f(x) : x \in (a, a+\varepsilon))$ is nonincreasing as $\varepsilon \to 0$.

202. DEFINITION: Let f be defined on an interval $(a-h, a+h)$. Then the

four **Dini derivates** of f at a are

$$D^{+}f(a)=\limsup_{h\searrow 0}\frac{f(a+h)-f(a)}{h} \text{ the \textbf{upper right derivative at} } a,$$

$$D^{-}f(a)=\liminf_{h\searrow 0}\frac{f(a+h)-f(a)}{h} \text{ the \textbf{lower right derivative at} } a,$$

$$D_{+}f(a)=\limsup_{h\searrow 0}\frac{f(a)-f(a-h)}{h} \text{ the \textbf{upper left derivative at} } a,$$

$$D_{-}f(a)=\liminf_{h\searrow 0}\frac{f(a)-f(a-h)}{h} \text{ the \textbf{lower left derivative at} } a.$$

Clearly, for any $a, D_{-}f(a)\le D_{+}f(a)$ and $D^{-}f(a)\le D^{+}f(a)$. f is **differentiable from the left at** a if $D_{-}f(a)=D_{+}f(a)$, f is **differentiable from the right at** a if $D^{-}f(a)=D^{+}f(a)$, and d**ifferentiable at** a if all four derivates are equal.

203. EXERCISE: Let $f(x)=x\sin(1/x)$ for $x\neq 0$, $f(0)=0$.
Find $D^{+}f(0), D^{-}f(0), D_{+}f(0), D_{-}f(0)$. Draw a picture!

204. REMARK: Our first main goal in this chapter is to prove that for a continuous nondecreasing function f defined on an interval $[a,b]$, all four derivates are equal almost everywhere on $[a,b]$ with respect to Lebesgue measure.

205. Let f be continuous and nondecreasing on $[a,b]$. Then $D^{+}f(x)$ is finite almost everywhere on $[a,b]$.
(Let G_C be the set of x in (a,b) where $D^{+}f(x)>C$. If $x_0\in G_C$, then there exists $z>x_0$ such that $\dfrac{f(z)-f(x_0)}{z-x_0}>C$ or $f(z)-Cz>f(x_0)-Cx_0$ so that x_0 is invisible from the right in $[a,b]$ with respect to $f(x)-Cx$. By #198 and the remark preceding it, the set of points invisible from the right is a countable disjoint union of intervals (a_k,b_k) such that $f(a_k)-Ca_k\le f(b_k)-Cb_k$. Rearrange to show

$$m(G_C)\le\sum_{k=1}^{\infty}(b_k-a_k)\le\sum_{k=1}^{\infty}\frac{f(b_k)-f(a_k)}{C}\le\frac{f(b)-f(a)}{C}.$$

Use #154. Explain why the estimate for $m(G_C)$ is naively reasonable– what if G_C were an interval?)

206. Let f be continuous and nondecreasing on $[a,b]$. Let $c<C$ be positive numbers. Set

$$E_{c,C}=\left\{x\in(a,b): D_{-}f(x)<c \text{ and } D^{+}f(x)>C\right\}.$$

Then for every open interval $(\alpha,\beta)\subset[a,b]$,

$$m\left(E_{c,C}\cap(\alpha,\beta)\right)\le\frac{c}{C}(\beta-\alpha).$$

(Suppose $x_0 \in (\alpha, \beta)$, and $D_- f(x_0) < c$. Then there is $z \in (\alpha, \beta), z < x_0$ such that $\dfrac{f(z) - f(x_0)}{z - x_0} < c$. Thus x_0 is invisible from the left in (α, β) with respect to the function $f(x) - cx$. By #198 and the remark preceding it, the set of such x_0 is a countable disjoint union of open intervals (α_k, β_k) where for each k, $f(\alpha_k) - c\alpha_k \geq f(\beta_k) - c\beta_k$ or $f(\beta_k) - f(\alpha_k) < c(\beta_k - \alpha_k)$. Let G_k be the set of points in (α_k, β_k) for which $D^+ f(x) > C$. The argument from the preceding problem gives that $G_k \subset \bigcup_{n=1}^{\infty} (\alpha_{kn}, \beta_{kn})$ (a disjoint union) where for each n, $\beta_{kn} - \alpha_{kn} \leq \dfrac{f(\beta_{kn}) - f(\alpha_{kn})}{C}$. Then

$$\begin{aligned} m\left(E_{c,C} \cap (\alpha, \beta)\right) &\leq \sum_{k,n=1}^{\infty} (\beta_{kn} - \alpha_{kn}) \leq \frac{1}{C} \sum_{k,n=1}^{\infty} f(\beta_{kn}) - f(\alpha_{kn}) \\ &\leq \frac{1}{C} \sum_{k=1}^{\infty} f(\beta_k) - f(\alpha_k) \leq \frac{c}{C} \sum_{k=1}^{\infty} (\beta_k - \alpha_k) \leq \rho (\beta - \alpha). \end{aligned}$$

207. Let f be continuous and nondecreasing on $[a, b]$. Then $D_- f(x) \geq D^+ f(x)$ for almost all $x \in (a, b)$.
(The set where $D_- f(x) < D^+ f(x)$ is a countable union of sets $E_{c,C}$ as defined in the previous problem. Fix a set $E_{c,C}$ and set $t = m\left(E_{c,C}\right)$. For any $\varepsilon > 0$ there is a countable collection of open intervals (a_k, b_k) whose union contains $E_{c,C}$ such that $\sum_{k=1}^{\infty} (b_k - a_k) < t + \varepsilon$. Set $t_k = m\left(E_{c,C} \cap (a_k, b_k)\right)$. Then $t_k \leq \dfrac{c}{C}(b_k - a_k)$ which implies

$$t \leq \frac{c}{C}(b_k - a_k) < \frac{c}{C}(t + \varepsilon).$$

If ε is small enough (how small?), this forces $t = 0$.)

208. Let f be continuous and nondecreasing on $[a, b]$. Then f has a finite derivative at almost all points of $[a, b]$.
($f^*(x) = -f(a + b - x)$ is also continuous and nondecreasing on $[a, b]$. (Graph f and f^*!) Show $D_- f^* = D^- f$, and $D^+ f^* = D_+ f$. Then the preceding problem forces all four Dini derivates of f to be equal almost everywhere.)

209. REMARK: Of course the result also holds for continuous nonincreasing functions f, since if f is nonincreasing, then $-f$ is nondecreasing. As mentioned in Remark 200, the assumption of continuity can be dropped if the more general version of the Rising Sun Lemma is used.

210. REMARK: The derivative is a measurable function since it is the pointwise limit almost everywhere of the sequence of difference quotients $\Phi_n(x) = \dfrac{f(x+1/n) - f(x)}{1/n}$.

211. Let $f \in L^1[a,b], f \geq 0$. Then $F(x) = \int_a^x f$ is continuous and nondecreasing.

212. Let $f \in L^1[a,b]$. Then $F(x) = \int_a^x f$ is differentiable a.e. on $[a,b]$.

213. REMARK: Of course we would like $F'(x) = f(x)$ a.e., but this will take some effort. More generally, if F is any function differentiable a.e. on $[a,b]$, we would like

$$F(x) = F(a) + \int_a^x F',$$

but this is not always true, even for monotonic functions. One of the two directional inequalities does, however, hold for monotonic functions.

214. Let F be a continuous nondecreasing function on $[a,b]$. Then

$$\int_a^b F' \leq F(b) - F(a).$$

(Extend the definition of F to $[b, b+1]$ by setting $F(x) = F(b)$ there. Note $F'(x) = \lim_{n\to\infty} \Phi_n(x)$ a.e. on $[a,b]$ where Φ_n is as in #210. Show that for each n

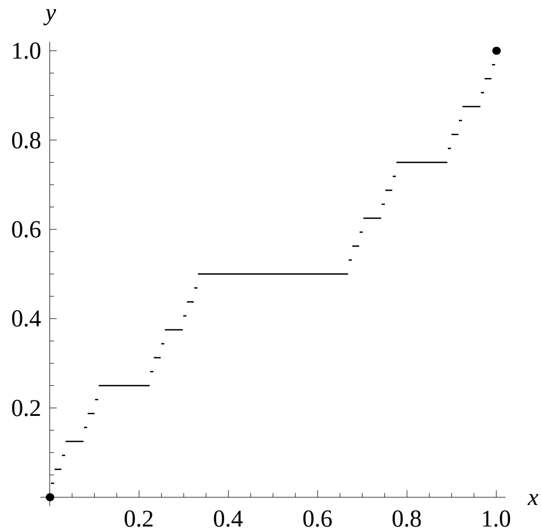
$$\int_a^b \Phi_n = n\left(\int_b^{b+1/n} F - \int_a^{a+1/n} F\right) \leq F(b) - F(a) \tag{1}$$

and remember Fatou's Lemma.)

215. If F is a nonconstant, nondecreasing step function on $[a,b]$, then F is differentiable a.e. on $[a,b]$, but $\int_a^b F' < F(b) - F(a)$.

216. REMARK and EXAMPLE: It can happen that $\int_a^b F' < F(b) - F(a)$ even when F is a continuous nondecreasing function. **Lebesgue's singular function** S (sometimes called the Cantor function) is a nondecreasing continuous mapping of $[0,1]$ onto $[0,1]$ defined using the ternary expansion definition of the Cantor set C as follows. Recall that C consists of the numbers in $[0,1]$ with a ternary expansion consisting of 0's and 2's. The map $0.a_1a_2\ldots_3 \to 0.\frac{a_1}{2}\frac{a_2}{2}\ldots_2$ (where the subscripts denote ternary and binary expansions respectively) maps C onto $[0,1]$ so that the two endpoints of any removed open interval are carried to

the same real number. For instance $\frac{1}{3} = .0222\ldots_3 \to .0111\ldots_2 = \frac{1}{2}$ and $\frac{2}{3} = .2_3 \to .1_2 = \frac{1}{2}$. Note that all of the numbers in the range of these endpoints are **dyadic fractions** (fractions whose denominator is a power of two). The singular function S is defined on $[0,1]$ by extending the map just described to be constant on each of the removed intervals. Part of the graph of S (sometimes called the Devil's Staircase), looks like this:



S is nondecreasing, continuous on $[0,1]$ (clearly no jumps), and differentiable (since locally constant) at every point not in the Cantor set with derivative 0. Thus $\int_0^1 S' = 0 < S(1) - S(0) = 1$.

217. REMARK: We will now study when equality does hold , that is, when a differentiable function can be reconstructed from its derivative. To do this I will introduce two new classes of functions–functions of bounded variation and absolutely continuous functions.

218. DEFINITION: Let f be a real-valued measurable function on $[a,b]$. The extended real number

$$V_a^b f = \sup\left\{\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| : a = x_0 < x_1 < \ldots < x_n = b\right\}$$

where the supremum is over all finite partitions of $[a,b]$ into n subintervals for all positive integers n, is the **total variation** of f on $[a,b]$. If $V_a^b f < \infty$, we say f is of **bounded variation**. We denote the set of all functions of bounded variation on $[a,b]$ by $BV[a,b]$.

219. REMARK: Clearly a monotonic function is of bounded variation with $V_a^b f = |f(b) - f(a)|$. Here is a sort of converse.

220. A real-valued measurable function on $[a,b]$ is of bounded variation if and only if it is the difference of two nondecreasing measurable functions.
($f = g - h$ implies $V_a^b f \le V_a^b g + V_a^b h$. For the other direction, if $f \in BV[a,b]$, the function $g(x) = V_a^x f$ is a nondecreasing measurable function, where we take $g(a) = V_a^a f = 0$. $g(x) - f(x)$ is also nondecreasing.)

221. EXAMPLE: The function $f(x) = x\sin(1/x)$ for $x \ne 0$, $f(0) = 0$ is not BV on any interval containing 0. (Consider the intervals $\left[\frac{1}{(n+1/2)\pi}, \frac{1}{n\pi}\right]$.)

222. $BV[a,b]$ is a vector space. Every continuous function in $BV[a,b]$ is differentiable almost everywhere.

223. REMARK: As already noted in #200 and #209, the assumption of continuity in #208 can be avoided. Thus in fact every function in $BV[a,b]$ is differentiable almost everywhere.

224. If $f \in BV[a,b]$ and $f \in BV[b,c]$, then $f \in BV[a,c]$ and $V_a^c f = V_a^b f + V_b^c f$.
(The sum $\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|$ increases if you add points to the partition.)

225. REMARK: It can be shown that $\|f\| = V_a^b f$ is a norm on $BV_0[a,b] = \{f \in BV[a,b] : f(a) = 0\}$ and that $BV_0[a,b]$ is complete with respect to this norm.

226. EXAMPLE: Let $f \in L^1[a,b]$ and $F(x) = \int_a^x f(t)\,dt$. Then $F \in BV[a,b]$ and $V_a^b F \le \int_a^b |f|$.

227. REMARK: If f is a step function, then it is clear that $V_a^b F = \int_a^b |f|$. That this is always true follows from approximating f by step functions.

228. Let $f \in L^1[a,b]$ and $F(x) = \int_a^x f(t)\,dt$. Then F is uniformly continuous on $[a,b]$, that is, for each $\varepsilon > 0$ there is $\delta > 0$ such that $|F(x) - F(y)| \le \int_I |f| < \varepsilon$ whenever $|x - y| < \delta$, where I is the interval with endpoints x and y.
(Recall #108.)

229. Suppose that for every $f \in L^1[a,b]$, $f(x) \ge F'(x)$ a.e. on $[a,b]$, where $F(x) = \int_a^x f$. Then for every $f \in L^1[a,b]$, $f(x) = F'(x)$ a.e. on $[a,b]$.
(Use the assumed inequality for f and $-f$.)

230. Let $f \in L^1[a,b]$ and $F(x) = \int_a^x f(t)\,dt$. The set $E = \{x : f(x) < F'(x)\}$ is a countable union of sets $E_{\alpha\beta} = \{x : f(x) < \alpha < \beta < F'(x)\}$, where α and β may be chosen to be rational.

231. With the notation of the previous problem, for each α and β it is the case that $m\left(E_{\alpha\beta}\right) = 0$.
(Given $\varepsilon > 0$, there is $\delta > 0$ as in #228 and an open set G with $E_{\alpha\beta} \subset G$ and $m(G) < m\left(E_{\alpha\beta}\right) + \delta$. (Why?) G is a countable disjoint union of open intervals (a_k, b_k). If $x_0 \in G_k = (a_k, b_k) \cap E_{\alpha\beta}$, then

$$\frac{F(x) - F(x_0)}{x - x_0} > \beta$$

for all x close to x_0. Hence (why?) x_0 is invisible from the right with respect to the function $F(x) - \beta x$. The set S_k of points in (a_k, b_k) invisible from the right with respect to $F(x) - \beta x$ is a countable disjoint union, and $G_k \subset S_k = \bigcup_n (a_{kn}, b_{kn})$ where

$$\beta\left(b_{kn} - a_{kn}\right) \le F\left(b_{kn}\right) - F\left(a_{kn}\right) = \int_{a_{kn}}^{b_{kn}} f$$

for each n. Then $E_{\alpha\beta} \subset S = \bigcup_k S_k \subset G$ and

$$\beta m(S) = \beta \sum_{k,n} \left(b_{kn} - a_{kn}\right) \le \sum_{k,n} \int_{a_{kn}}^{b_{kn}} f = \int_S f.$$

Moreover, $m\left(S \backslash E_{\alpha\beta}\right) < \delta$, so

$$\beta m\left(E_{\alpha\beta}\right) \le \beta m(S) \le \int_S f = \int_{E_{\alpha\beta}} f + \int_{S \backslash E_{\alpha\beta}} f < \alpha m\left(E_{\alpha\beta}\right) + \varepsilon.$$

Thus $m\left(E_{\alpha\beta}\right) < \dfrac{\varepsilon}{\beta - \alpha}$.)

232. Let $f \in L^1[a,b]$ and $F(x) = \int_a^x f(t)\,dt$. Then $F'(x) = f(x)$ a.e. on $[a,b]$.

233. REMARK: You may recall the calculus theorem that $F'(x) = f(x)$ at each point of continuity of f. (It is a good exercise to prove this– $F(x+h) - F(x) = \int_x^{x+h} f(t)\,dt$ is an area; $\dfrac{F(x+h) - F(x)}{h}$ is the average height of the region. If f is continuous at x, then the average height of the region must be nearly $f(x)$ for h near 0.) However a function in $L^1[a,b]$ need not be continuous at any point– for instance let f be the characteristic function of the irrationals in $[0,1]$. Problem 232 says that nevertheless at almost every point x the average height of the region bounded on the sides by x and $x+h$ and on the top by the graph of f approaches $f(x)$ as $h \to 0$. (For the characteristic function of the irrationals, $F(x) = x$ for all $x \in [0,1]$ (why?) so $F'(x) = 1$ everywhere and $F'(x) = f(x)$ at each irrational.)

234. REMARK: We can be a little more explicit about identifying points at which $\int_a^x f(t)\,dt$ is differentiable.

235. DEFINITION: Let $f \in L^1[a,b]$. A point x is a **Lebesgue point** of f if

$$\lim_{h\to 0} \frac{1}{2h}\int_{x-h}^{x+h} |f(t)-f(x)|\,dt = 0.$$

This is a statement that $f(x)$ is an average of the values of f near x. It is easy to see that it holds at every point of continuity of f. (Exercise!) We will see that it is true for almost all x for any L^1 function. Lebesgue points are of some importance in classical real analysis, especially in classical Fourier analysis.

236. EXERCISE: What are the Lebesgue points for the characteristic function of the irrationals in $[0,1]$? The characteristic function of the rationals?

237. Let $f \in L^1[a,b]$. Then

$$\lim_{h\searrow 0} \frac{1}{h}\int_{x}^{x+h} |f(t)-\alpha|\,dt = \lim_{h\searrow 0} \frac{1}{h}\int_{x-h}^{x} |f(t)-\alpha|\,dt = |f(x)-\alpha|$$

for all real numbers α and all x in a set E where $m([a,b]\setminus E)=0$.
(Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence of real numbers dense in $\mathbb{R}$. For each n, let $g_n(t)=|f(t)-\alpha_n|$, and $G_n(x)=\int_a^x g_n(t)\,dt=\int_a^x |f(t)-\alpha_n|$. By #232 each G_n is differentiable a.e. and $G_n'(x)=|f(x)-\alpha_n|$ on a set E_n with $m([a,b]\setminus E_n)=0$. If $E=\bigcap_{n=1}^{\infty} E_n$, then $m([a,b]\setminus E)=0$. Fix $\alpha\in\mathbb{R}$ and $x\in E$ and let $\varepsilon>0$ be given. Choose α_n so that $|\alpha-\alpha_n|<\frac{\varepsilon}{3}$. Then

$$||f(t)-\alpha|-|f(t)-\alpha_n||<\frac{\varepsilon}{3} \text{ for all } t\in[a,b].$$

Then

$$\begin{aligned}\frac{1}{h}\int_x^{x+h}|f(t)-\alpha|\,dt-|f(x)-\alpha| &= \frac{1}{h}\int_x^{x+h}|f(t)-\alpha|\,dt-\frac{1}{h}\int_x^{x+h}|f(t)-\alpha_n|\,dt\\ &+\frac{1}{h}\int_x^{x+h}|f(t)-\alpha_n|\,dt-|f(x)-\alpha_n|\\ &+|f(x)-\alpha_n|-|f(x)-\alpha|\end{aligned}$$

where the first and third terms on the right are each less than $\frac{\varepsilon}{3}$ in magnitude for all h and the second term is also for sufficiently small h. The proof of the other equality is similar.)

238. Let $f \in L^1[a,b]$. Then almost every point of $[a,b]$ is a Lebesgue point for f.
(For any x in the set E of the previous problem, let $\alpha=f(x)$.)

239. Let $f \in L^1[a,b]$. Then $F(x) = \int_a^x f(t)\,dt$ is differentiable at every Lebesgue point of f with $F'(x) = f(x)$.
(Compare $\dfrac{F(x+h) - F(x)}{h}$ to the expression for a Lebesgue point. You may want to consider $h > 0$ and $h < 0$ separately, since the notation must be adapted slightly when $h < 0$.)

240. DEFINITION: A measurable function f on $[a,b]$ is **absolutely continuous** on $[a,b]$ if for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon$$

for every finite set of disjoint subintervals $\{[a_k, b_k]\}_{k=1}^{n}$ of $[a,b]$ such that $\sum\limits_{k=1}^{n} (b_k - a_k) < \delta$.

241. EXAMPLE: Lebesgue's singular function S defined in Example 216 is a continuous nondecreasing function that is not absolutely continuous, since the complement of the first $2^n - 1$ open intervals removed to construct the Cantor set "by subtraction" consists of 2^n closed intervals $[a_k, b_k]$, the sum of whose lengths is $\left(\dfrac{2}{3}\right)^n$, and such that $\sum\limits_{k=1}^{2^n} |f(b_k) - f(a_k)| = 1$. In fact, as the following definition formalizes, S is the opposite of an absolutely continuous function.

242. DEFINITION: A function f of bounded variation on $[a,b]$ whose derivative is equal to 0 almost everywhere on $[a,b]$ is a **singular** function.

243. The set $AC[a,b]$ of absolutely continous functions on $[a,b]$ is a vector space.

244. If f is absolutely continuous on $[a,b]$, then f is of bounded variation on $[a,b]$, that is, the set $AC[a,b]$ is a proper subspace of $BV[a,b]$.
(Partition $[a,b]$ into short intervals and use #224.)

245. If f is absolutely continuous on $[a,b]$, then $g(x) = V_a^x f$ is absolutely continuous, and f is the difference of two nondecreasing absolutely continuous functions.
(For any $(\alpha, \beta) \subset [a,b]$, $|f(\beta) - f(\alpha)| \le V_\alpha^\beta f = g(\beta) - g(\alpha)$ so that $g - f$ is nondecreasing.. Given ε, choose $\delta > 0$ so $\sum\limits_{k=1}^{n} |f(b_k) - f(a_k)| <$ $\varepsilon/2$ whenever $\sum\limits_{k=1}^{n} (b_k - a_k) < \delta$. For each k, there are disjoint subin-

tervals $\left\{\left(c_{k_j},d_{k_j}\right)\right\}_{j=1}^{n_k}$ of (a_k,b_k) so that

$$g(b_k)-g(a_k)=V_{a_k}^{b_k}f<\sum_{j=1}^{n_k}\left|f\left(d_{k_j}\right)-f\left(c_{k_j}\right)\right|+\frac{\varepsilon}{2n}.$$

$\sum\limits_{j,k}\left(d_{k_j}-c_{k_j}\right)\leq\sum\limits_{k=1}^{n}(b_k-a_k)<\delta$. Then

$$\sum_{k=1}^{n}|g(b_k)-g(a_k)|\leq\sum_{j,k}\left|f\left(d_{k_j}\right)-f\left(c_{k_j}\right)\right|+n\frac{\varepsilon}{2n}<\varepsilon.)$$

246. If $f\in L^1[a,b]$, then $F(x)=\int_a^x f(t)\,dt$ is absolutely continuous. (Recall #108.)

247. REMARK: The Lebesgue singular function shows that a continous function can have $f'(x)=0$ a.e. without being constant. However this is not possible for an absolutely continous function.

248. If F is a nondecreasing absolutely continuous function on $[a,b]$ and $F'(x)=0$ a.e., then F is constant on $[a,b]$.
(Denote the set where $F'=0$ by E, so that $Z=[a,b]\backslash E$ has $m(Z)=0$. For any $\varepsilon>0$ there is δ as in the definition of absolute continuity, and a countable collection of disjoint open intervals (a_k,b_k) such that $Z\subset\bigcup\limits_{k=1}^{\infty}(a_k,b_k)$. Then $F[Z]=\{y:y=F(z)\text{ for some }z\in Z\}\subset\bigcup\limits_{k=1}^{\infty}(F(a_k),F(b_k))$ and $\sum\limits_{k=1}^{\infty}(F(b_k)-F(a_k))\leq\varepsilon$ (true for every partial sum), so $m(F[Z])\leq\varepsilon$. Thus $m(F[Z])=0$. Let $x_0\in E$. For any $\varepsilon>0$, $\dfrac{F(x)-F(x_0)}{x-x_0}<\varepsilon$ for all $x>x_0$ close to x_0. Then x_0 is invisible from the right on $[a,b]$ with respect to $\varepsilon x-F(x)$. Then $E\subset\bigcup\limits_{k=1}^{\infty}(\alpha_k,\beta_k)$, a disjoint union, with $F(\beta_k)-F(\alpha_k)<\varepsilon(\beta_k-\alpha_k)$ for each k. Then $m(F[E])\leq\sum\limits_{k=1}^{\infty}(F(\beta_k)-F(\alpha_k))<\varepsilon\sum\limits_{k=1}^{\infty}(\beta_k-\alpha_k)\leq\varepsilon(b-a)$. Thus also $m(F[E])=0$ so $m(F[a,b])=0$, that is, $F[a,b]$ is an interval of length 0.)

249. If F is an absolutely continuous function on $[a,b]$, then

$$F(x)=F(a)+\int_a^x F'(t)\,dt.$$

(First suppose F is nondecreasing. If $g(x)=\int_a^x F'(t)\,dt$, then $F'=g'$ a.e, so $F-g$ is constant. Every absolutely continuous function is of bounded variation.)

250. REMARK: Thus a function is absolutely continuous if and only if it is a constant plus an indefinite integral. This is the Lebesgue form of the Fundamental Theorem of Calculus.

IX. Signed Measures, Decomposition of Measures

251. EXAMPLE and REMARK: In Example 151 (2) I noted that if f is a non-negative measurable function on an interval $[a,b]$ of $\mathbb{R}$ and if $\mathfrak{M}$ is the σ-algebra of Lebesgue measurable subsets of $[a,b]$, then $\nu(E) = \int_E f\,dm$ defines ν as a measure on $\mathfrak{M}$. Slightly more generally, f $f \in L^1[a,b]$, then $\nu(E) = \int_E f\,dm$ defines ν as a countably additive set function on $\mathfrak{M}$. Here "countably additive set function" means that if $\{E_n\}_{n=1}^{\infty}$ is a sequence of pairwise disjoint measurable sets, then $\nu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \nu(E_n)$. Since we can write f as the difference of non-negative functions, $f = f_+ - f_-$, we can write ν as a difference of measures, $\nu = \nu_+ - \nu_-$ where $\nu_+(E) = \int_E f_+\,dm, \nu_-(E) = \int_E f_-\,dm$. This suggests that we should perhaps attempt to develop a theory of "signed measures" or even complex-valued measures. In fact, if we can show that every signed measure is a difference of (non-negative) measures, then we can read the properties of signed measures from what we already know about measures.Moreover, the set of signed measures would have the enormous structural advantage that it would be a vector space, which the set of measures is not. We could even make it into a normed vector space, though we will not do that in these notes. We may as well proceed in the context of an abstract measure space, since all the arguments of the basic theory are the same.

252. DEFINITION: Let $\mathfrak{M}$ be a σ-algebra of subsets of a set X. A function $\nu : \mathfrak{M} \to \mathbb{R}$ is a **finite signed measure** if
(1) $\nu(\varnothing) = 0$, and
(2) $\nu\left(\bigcup_{n=1}^{\infty} E_k\right) = \sum_{n=1}^{\infty} \nu(E_n)$ for every sequence $\{E_n\}$ of pairwise disjoint elements of $\mathfrak{M}$.

253. REMARK: So ν is just like a measure as already defined, except that it is now allowed to have both positive and negative values. It is possible to allow ν to take one of the values ∞ or $-\infty$, but in the interests of simplicity, we will stick to finite values.

254. EXAMPLE and REMARK: As already essentially pointed out in Example 251, if μ is a measure on $\mathfrak{M}$ (that is, in the familiar sense of taking non-negative values), and and if $f \in L^1(X,\mu)$, then we have proved (actually only in the context of Lebesgue measure) that $\nu(E) =$

$\int_E f\,d\mu$ is a signed measure. In this chapter we will first show that every signed measure is a difference of non-negative measures, and then we will characterize the measures that are of the form $\int_E f\,d\mu$.

255. DEFINITION: Let ν be a finite signed measure on $\mathfrak{M}$. A measurable subset A of X is a **positive set** for ν if $\nu(E) \geq 0$ for each measurable $E \subset A$. Similarly, a subset B of X is a **negative set** for ν if $\nu(E) \leq 0$ for each measurable $E \subset B$. A set N is a **null set** for ν if $\nu(E) = 0$ for each measurable $E \subset N$.

256. EXAMPLE: If $f \in L^1[a,b]$, then $A = \{x \in [a,b] : f(x) \geq 0\}$ is a positive set for the signed measure $\nu(E) = \int_E f\,dm$ and $B = \{x \in [a,b] : f(x) < 0\}$ is a negative set for ν such that A and B are disjoint and $A \cup B = [a,b]$. Note that the null set $N = \{x \in [a,b] : f(x) = 0\}$ could be put into either A or B. In particular, if $f(x) = x$ on $[-1,1]$, then $A = [0,1]$, and $B = [-1,0)$. Note that $\nu([-a,a]) = 0$ for any $0 \leq a \leq 1$, but these sets are not null sets because they have subsets whose ν-measure is not 0.

257. Every subset of a positive set is positive. A countable union of positive sets is positive. Similarly for negative sets.
(A countable union is a countable disjoint union.)

258. If E is a measurable set such that $\nu(E) > 0$, then E contains a positive set A with $\nu(A) > 0$.
(If E is not already a positive set, let n_1 be the smallest positive integer such that there is a measurable subset E_1 of E with $\nu(E_1) < -\dfrac{1}{n_1}$. If $n_1, \ldots, n_k$ have been constructed, let n_{k+1} be the smallest positive integer such that there is a measurable subset E_{k+1} of $E \setminus \bigcup_{j=1}^{k} E_j$ with $\nu(E_{k+1}) < -\dfrac{1}{n_{k+1}}$. Either the process terminates, or $\nu\left(\bigcup_{j=1}^{\infty} E_j\right) > -\infty$ implies that $\dfrac{1}{n_k} \to 0$ as $k \to \infty$. Either way, $A = E \setminus \bigcup_{j=1}^{\infty} E_j$ is a positive set.)

259. Let ν be a finite signed measure on $\mathfrak{M}$. Then X contains a positive set A and a negative set B such that $X = A \cup B$.
(Let $s = \sup \nu(E)$ where the sup is over all positive subsets E of X. If $\{A_k\}$ is a sequence of positive sets such that $\nu(A_k) \to s$, then $A = \bigcup_{k=1}^{\infty} A_k$ and $B = A^c$ work.)

260. REMARKS:
(1) In the previous problem, $A \cap B$ is a null set. By assigning it to either

A or B we can assume A and B are disjoint.
(2) This result is called the Hahn Decomposition Theorem, and the pair $\{A,B\}$ is sometimes called a **Hahn Decomposition** for ν.
(3) If $\{A,B\}$ is a Hahn Decomposition for ν, and we define $\nu_+(E) = \nu(E \cap A), \nu_-(E) = -\nu(E \cap B)$, then ν_+ and ν_- are (nonnegative) measures on $\mathfrak{M}$ such that $\nu = \nu_+ - \nu_-$. The nonnegative measure $|\nu| = \nu_+ + \nu_-$ is called the **total variation** of ν. Note that $|\nu|(E) = 0$ if and only if E is a null set for the signed measure ν in the sense defined above.

261. REMARK: In what follows "measure" means a nonnegative measure as in the notes preceding this chapter. A countably additive set function that can take both positive and negative values will always be called a signed measure.

262. DEFINITION: Two measures μ and ν on $\mathfrak{M}$ are **mutually singular** (we write $\mu \perp \nu$) if there are disjoint measurable sets A and B such that $X = A \cup B, \mu(B) = 0$ and $\nu(A) = 0$. (We may also say that ν is **singular with respect to** μ.) We call B the **support** of ν.

263. DEFINITION: If μ and ν are measures on $\mathfrak{M}$, we say ν is **absolutely continuous with respect to** μ if $\mu(E) = 0$ implies $\nu(E) = 0$. We write $\nu \ll \mu$.

264. EXAMPLE: Let $(X, \mathfrak{M}, \mu)$ be a measure space, and let f be a nonnegative μ-integrable function. Then $\nu(E) = \int_E f \, d\mu$ defines a measure on $\mathfrak{M}$ that is clearly absolutely continuous with respect to μ. It is the purpose of the following problems to prove the converse of this statement–if ν is absolutely continuous with respect to μ, then there is such a function f.

265. Let $(X, \mathfrak{M}, \mu)$ be a finite measure space (that is, $\mu(X) < \infty$) and let ν be a finite measure on $\mathfrak{M}$ that is absolutely continuous with respect to μ.
(a) For any nonnegative number $\alpha, \nu - \alpha\mu$ is a signed measure on $\mathfrak{M}$.
(b) Let $\{\alpha_1, \alpha_2, ...\}$ be an enumeration of the non-negative rational numbers, for each α_k let $\{A_{\alpha_k}, B_{\alpha_k}\}$ be a fixed Hahn Decomposition for $\nu - \alpha_k\mu$ with $A_{\alpha_k} \cap B_{\alpha_k} = \varnothing$, and let $A_\infty = \bigcap_{k=1}^{\infty} A_{\alpha_k}$. Then A_∞ is measurable and $\mu(A_\infty) = 0$.
($A_\infty \subset A_{\alpha_k}$ for each k, so (why?) $\nu(A_\infty) \geq \alpha_k \mu(A_\infty)$ for each k)

266. Define a nonnegative function f on A_∞^c, (that is, μ-almost everywhere on X by the preceding problem) by $f(x) = \inf\{\alpha_k : x \in B_{\alpha_k}\}$. Set $f = \infty$ on A_∞. Then

(a) for any real number $s > 0$,

$$\{x : f(x) < s\} = \bigcup_{\alpha_k < s} B_{\alpha_k}.$$

(b) f is measurable with respect to the σ-algebra $\mathfrak{M}$.

267. If $E \subset \{x : f(x) < s\}$, then $\nu(E) < s\mu(E)$.
($E \subset \bigcup_{\alpha_k < s} (E \cap B_{\alpha_k})$ and as usual this can be made into a disjoint union.)

268. If $E \subset \{x : f(x) \geq t\}$, then $\nu(E) \geq t\mu(E)$
(#5 is useful here.)

269. Fix a measurable set E. Fix a positive integer N. For $k \geq 1$, set $E_k = \left\{x \in E : \frac{k-1}{N} \leq f(x) < \frac{k}{N}\right\}$. Then
(a) $\frac{k-1}{N}\mu(E_k) \leq \nu(E_k) < \frac{k}{N}\mu(E_k)$ and $\frac{k-1}{N}\mu(E_k) \leq \int_{E_k} f\, d\mu \leq \frac{k}{N}\mu(E_k)$.
(b) $\left|\nu(E_k) - \int_{E_k} f\, d\mu\right| \leq \frac{1}{N}\mu(E_k)$

270. Let $(X, \mathfrak{M}, \mu)$ be a finite measure space, and let ν be a finite measure on $\mathfrak{M}$ that is absolutely continuous with respect to μ. Let f be defined as in #266. Then f is measurable and for any measurable subset E of X,

$$\nu(E) = \int_E f\, d\mu.$$

271. DEFINITION: The function f from the previous problem is called the **Radon-Nikodym derivative** of ν with respect to μ. It is sometimes denoted $\frac{d\nu}{d\mu}$. The result is the Radon-Nikodym Theorem.

272. Let μ and ν be finite measures on $(X, \mathfrak{M})$. Then $\nu = \nu_a + \nu_s$ where ν_a is absolutely continuous with respect to μ and ν_s is singular with respect to μ.
(Let $\tau = \mu + \nu$.Then both μ and ν are absolutely continuous with respect to τ so there is a τ-integrable non-negative function f so that for any $E \in \mathfrak{M}$,

$$\mu(E) = \int_E f\, d\tau\ .$$

Let $A = \{x : f(x) > 0\}$, $B = \{x : f(x) = 0\}$. Then $\mu(B) = 0$. Set $\nu_s(E) = \nu(E \cap B)$. Set $\nu_a(E) = \nu(E \cap A)$.)

273. REMARK: If we specialize to the Lebesgue measure space $([a,b], \mathfrak{M}, m)$, then we can use the differentiation result of the previous chapter to construct the Radon-Nikodym derivative in a much simpler way.

274. If μ is a measure on the σ-algebra $\mathfrak{M}$ of Lebesgue measurable sets, and if μ is absolutely continuous with respect to m, then for every $\varepsilon > 0$ there is $\delta > 0$ such that $\mu(E) < \varepsilon$ whenever $m(E) < \delta$.
(If not then there is $\varepsilon > 0$ and sets A_n with $m(A_n) < 2^{-n}$ and $\mu(A_n) > \varepsilon$. Setting $E_n = \bigcup_{k=n+1}^{\infty} A_k$, we may also assume $E_1 \supset E_2 \supset \ldots$.Use #154.)

275. REMARK: This generalizes #108, at least if we take the view that we have temporarily forgotten the Radon-Nikodym Theorem and are constructing a new proof. The result also looks very much like the definition of absolute continuity for functions (#240).

276. Let μ be absolutely continuous with respect to m. Define $f(x) = \mu([a,x])$ for $a \leq x \leq b$. Then f is a nondecreasing absolutely continuous function.

277. Let μ be absolutely continuous with respect to m. Define $f(x) = \mu([a,x])$ for $a \leq x \leq b$. Then $\mu(E) = \int_E f'\, dm$ for every measurable set E.
(True for every interval (why?) and so for every open set (why?). Use #53.)

278. In this more specialized situation we can refine the decomposition of measures given in #272. (Really all that is required is that one point sets be measurable.)

279. DEFINITION: Let μ, defined on the collection $\mathfrak{M}$ of Lebesgue measurable subsets of $[a,b]$, be singular with respect to Lebesgue measure m. If $\mu(\{x\}) = 0$ for each $x \in [a,b]$, then μ is a **continuous** singular measure.

280. Let μ be a finite measure defined on the collection $\mathfrak{M}$ of Lebesgue measurable subsets of $[a,b]$ Then $\mu = \mu_a + \mu_s + \mu_p$ where μ_a is absolutely continuous with respect to m, μ_s is a continous singular measure, and μ_p is a sum of point masses.
(Construct ν_a and ν_s as in #272. There are at most a countably infinite number of points x such that $\nu_s(\{x\}) > 0$.(Why?) If these are $\{x_n\}_{n=1}^{\infty}$, set $\mu_p = \sum_{n=1}^{\infty} \mu(x_n)\, \delta_{n_n}$ where δ_{n_n} is the point mass at x_n : $\delta_{n_n}(E) = \begin{cases} 1, & x_n \in E \\ 0, & x_n \notin E. \end{cases}$ Then $\mu_s = \nu_s - \mu_p$ is a continuous singular measure.)