

# Homework 4

*Due: Thursday 2/28/19 by 12:00pm (noon)*

Although this homework assignment is not due until Thursday 2/28/19, there may be related material on your exam on Friday, 2/22/19.

## Forecasting

1. In class, we derived the **forecasting equation** for computing  $\hat{x}_{n+1} = \sum_{j=1}^n c_{nj}x_{n-j}$  based on  $x_1, \dots, x_n$  by minimizing

$$v_n = \mathbb{E} \left[ \left( x_{n+1} - \sum_{j=1}^n c_{nj}x_{n-j} \right)^2 \right].$$

- (a) Write down the quantity we should minimize if we want to forecast values further into the future, i.e. if we want to compute  $\hat{x}_{n+k}$  for any  $k > 0$ .
- (b) Write out the forecasting equation that corresponds to (a). Hint: In class we talked about minimizing  $v_n$ . Defining  $a_{n,ij} = \gamma_x(i-j)$  and  $b_{n,i} = \gamma_x(i)$ , the corresponding forecasting equation was:

$$\mathbf{A}_n \mathbf{c}_n = \mathbf{b}_n.$$

- (c) In class, I showed you the following function for computing the values  $\mathbf{c}_n$  and  $v_n$  for a **ARMA**(1,1) model.

```
solve.direct <- function(n, phi1 = 0, theta1 = 0, sig.sq.w = 1) {  
  A.n <- matrix(nrow = n, ncol = n)  
  b.n <- numeric(n)  
  for (i in 1:n) {  
    b.n[i] <- gamma.x(i, phi1 = phi1, theta1 = theta1, sig.sq.w = sig.sq.w)  
    for (j in 1:n) {  
      A.n[i, j] <- gamma.x(h = i - j, phi1 = phi1, theta1 = theta1, sig.sq.w = sig.sq.w)  
    }  
  }  
  c.n <- solve(A.n) %*% b.n  
  v.n <- gamma.x(0, phi1 = phi1, theta1 = theta1, sig.sq.w = sig.sq.w) +  
    t(c.n) %*% A.n %*% c.n - 2*t(c.n) %*% b.n  
  return(list("c.n" = c.n, "v.n" = v.n))  
}
```

Modify this function to take an additional argument,  $k$ , and return the coefficients that give the forecast  $\hat{x}_{n+k}$  as well as the expected squared error loss of  $\hat{x}_{n+k}$ . Note that you will need to use the `gamma.x` function from class.

```
gamma.x <- function(h, phi1, theta1, sig.sq.w) {  
  h <- abs(h)  
  if (h == 0) {  
    g.x <- (theta1^2 + 2*phi1*theta1 + 1)*sig.sq.w/(1 - phi1^2)  
  } else {  
    g.x <- sig.sq.w*phi1^(h - 1)*((1 + theta1*phi1)*(phi1 + theta1)/(1 - phi1^2))  
  }  
  return(g.x)  
}
```

(d) Use the function you wrote in (c) to plot the expected squared error loss of  $\hat{x}_{3+k}$  for the following models for  $k = 1, \dots, 5$ , all with  $\sigma_w^2 = 2$ :

- i.  $\phi_1 = 0.5, \theta_1 = 0$ ;
- ii.  $\phi_1 = -0.5, \theta_1 = 0$ ;
- iii.  $\phi_1 = 0, \theta_1 = 0.57735$ ;
- iv.  $\phi_1 = 0, \theta_1 = -0.57735$ .

Include a horizontal dashed line at  $\gamma_x(0)$  (Note:  $\gamma_x(0)$  is the same for all of the models).

- (e) In one sentence, interpret how the forecast error relates to the sign of  $\phi_1$  or  $\theta_1$ .
- (f) In one sentence, interpret how the forecast error changes as  $k$  increases for an **AR**(1) model versus an **MA**(1) model.
- (g) Add two more lines to your plot in (c) corresponding to the following models with  $\sigma_w^2 = 0.2533333$ :
  - v.  $\phi_1 = 0.9, \theta_1 = 0$ ;
  - vi.  $\phi_1 = 0, \theta_1 = 2.064742$ .
- (i) Is the variance  $\gamma_x(0)$  still the same as it was for the models first plotted in (c)?
- (j) In one sentence, interpret what you see in (g). How does increasing  $\phi_1$  or  $\theta_1$  affect how the forecast error changes as  $k$  increases?

## Estimation

Let's look back at the chicken data one more time. We're going to consider **ARMA**( $p, q$ ) time series models for the residuals from a linear regression of the chicken prices on an intercept and time. I've computed them for you here and named them **r**, to make sure everyone starts out on the same page.

```
library(astsa)
data(chicken)
r <- lm(chicken~time(chicken))$res
```

2. First, we're going to consider setting  $q = 0$  and fitting an **AR**( $p$ ) model to **r**.
  - (a) Using the **acf** function, plot the sample partial autocorrelations. Based on the sample partial autocorrelation function, what would you select for  $p$ ? Accompany your choice with at most one sentence of reasoning.
  - (b) Setting the order based on (a), fit the **AR**( $p$ ) model to **r** using the Yule-Walker equations. I do not want you to use the **ar.yw** function, but you can base your code on the **solve.direct** function. Give the **AR**( $p$ ) parameter estimates.
3. Now, we're going to consider setting  $p = 0$  and fitting an **MA**( $q$ ) model to **r**.
  - (a) Using the **acf** function, plot the sample autocorrelations. Based on the sample autocorrelations, what would you select for  $q$ ? Accompany your choice with at most one sentence of reasoning.
  - (b) Fit an **MA**(1) model to **r** by computing the innovation coefficients  $d_n$  using the function given in class, and treating the innovation coefficient estimates  $d_n$  as estimates of  $\psi_1, \dots, \psi_n$ . Give the **MA**(1) coefficient estimate.
4. Finally, let's to consider fitting an **ARMA**( $p, q$ ) model to **r**.
  - (a) Can we use the sample autocorrelations or sample partial autocorrelations to select  $p$  or  $q$ ? Just give a yes or no.

- (b) Using the `arma` function, fit **ARMA**( $p, q$ ) models with  $p = 0, \dots, 3$  and  $q = 0, \dots, 3$ , excluding the case where  $p = q = 0$ . Compute AIC, AICc, and BIC according to the equations that were given in class early on by hand.
- (c) Plot the residuals from the AIC-minimizing model.
- (d) Fit the AIC-minimizing model to `r` using the `arma` function. What algorithm does the `arma` function use to estimate  $\phi_1, \dots, \phi_p, \theta_1, \dots, \theta_p$  and  $\sigma_w^2$  by default? (Unconditional maximum likelihood, unconditional least squares, conditional least squares...)
- (e) On a plot with the  $x$ -axis ranging from  $2001 + 7/12$  to 2020 and  $y$ -axis ranging from 60 to 130, plot:
- The observed chicken time series.
  - The linear model fit of the chicken time series.
  - Predicted mean chicken prices from  $2016 + 6/12$  to 2020 based on the model you fit in (e).
  - Predicted mean chicken prices from  $2016 + 6/12$  to 2020 plus or minus one standard error based on the model you fit in (e). You can get the predictions and standard errors by applying the `predict` function to your `arma` object.