

Homework 2

Due: Thursday 2/7/19 by 12:00pm (noon)

An Overview of Level- α Tests

This homework is going to ask you to conduct a level- α tests of a null hypothesis, which requires that you combine a few bits of information from class.

Let's call the null hypothesis H and the alternative hypothesis K . Suppose we have a test statistic \hat{t} , that is a function of the data, and that we know either exactly or approximately what the distribution of \hat{t} is under the null H . Then we can construct a level- α test of the null hypothesis H by comparing \hat{t} to the $1 - \alpha/2$ and $\alpha/2$ quantiles of the distribution of \hat{t} under the null H . If \hat{t} is within those quantiles, we **accept** the null hypothesis H , otherwise we reject it.

This general idea is something that I expect have seen in your previous statistics classes, and we referenced it in our review of linear regression when we talked about testing the null hypothesis H that $\hat{\beta}_j$ is exactly equal to a specific value for some j . Consider testing the null hypothesis $H : \beta_1 = 0$ against the alternative $K : \beta_1 \neq 0$. For such a problem, our test statistic is

$$\hat{t} = \frac{\hat{\beta}_1}{s_w \sqrt{(\mathbf{Z}'\mathbf{Z})_{11}^{-1}}},$$

and we know that $\hat{t} \sim \mathcal{T}_{n-q}$ under H , where q is the number of covariates we have in our regression model (number of columns of \mathbf{Z}). We perform a level- α test of H by comparing \hat{t} to the $1 - \alpha/2$ and $\alpha/2$ quantiles of a \mathcal{T}_{n-q} distribution. If \hat{t} is within these quantiles, we accept $H : \beta_1 = 0$, otherwise we reject it.

To apply this idea to these problems, ask yourself: What is a sample quantity that we can calculate for a time series \mathbf{x} and use as a test statistic \hat{t} that:

- We talked about in class;
- Is relevant to testing a hypothesis about $\rho_x(1)$;
- We know the approximate or exact distribution of under the null that \mathbf{x} is a white noise time series, with $\rho_x(1) = 0$?

The AR(1) Model

1. This problem will ask you to work with the autoregressive (AR) model.
 - (a) Describe what R returns when you run `x <- arima.sim(n = 100, list(ar=1), sd = 1)`, and why this occurs.
 - (b) Simulate 1,000 **AR**(1) time series of length $n = 100$ with $\sigma_w^2 = 1$ for values of $\phi_1 = \{-0.5, -0.25, -0.125, 0, 0.125, 0.25, 0.5\}$. For each value of ϕ_1 , compute the percent of simulations in which a level-0.05 test of the null hypothesis that the time series is white noise, with $\rho_x(1) = 0$, rejects the null, using $\hat{\rho}_x(1)$ from class and 3.(h) in Homework 1. Plot the percent of simulations in which a test of the null hypothesis rejects the null against ϕ_1 .
 - (c) When $\phi_1 \neq 0$, the percent of simulations in which a test of the null hypothesis that $\rho_x(1) = 0$ rejects the null estimates the **power** of the test. When $\phi_1 = 0$, the percent of simulations in which a test of the null hypothesis that $\rho_x(1) = 0$ rejects the null estimates the **level** of the test. Is the estimated level 0.05, as we would expect from a level-0.05 test? If not, why not?
 - (d) Describe in at most two sentences how the power of the test relates to the true value ϕ_1 . Intuitively, does this make sense?

The MA(1) Model

2. This problem will ask you to work with the moving average (MA) model.
- (a) Without using the `arma.sim` function or any other third party function for simulating an MA time series, simulate a length-100 time series \mathbf{x} according to the MA model:

$$x_t = 0.5w_{t-1} + w_t, w_t \stackrel{i.i.d.}{\sim} \mathcal{N}(0, 1)$$

- (b) Using the code you wrote in (a) or `arma.sim`, simulate 1,000 **MA**(1) time series of length $n = 100$ with $\sigma_w^2 = 1$ for values of $\theta_1 = \{-1, -0.268, -0.127, 0, 0.127, 0.268, 1\}$. For each value of θ_1 , compute the percent of simulations in which a test of the null hypothesis that the time series is white noise, with $\rho_x(1) = 0$, rejects the null, using $\hat{\rho}_x(1)$ from class and 3.(h) in Homework 1. Plot the percent of simulations in which a test of the null hypothesis rejects the null against θ_1 .
- (c) When $\theta_1 \neq 0$, the percent of simulations in which a test of the null hypothesis that $\rho_x(1) = 0$ rejects the null estimates the **power** of the test. When $\theta_1 = 0$, the percent of simulations in which a test of the null hypothesis that $\rho_x(1) = 0$ rejects the null estimates the **level** of the test. Is the estimated level 0.05? If not, why not?
- (d) Describe in at most two sentences how the power of the test relates to the true value θ_1 . Intuitively, does this make sense?

Comparing AR(1) and MA(1) Models

3. This problem asks you to compare what you observed in 1. (b)-(d) to what you observed in 2. (b)-(d).
- (a) Combine the plots from 1. (b) and 2. (b) into a single plot.
- (b) Compute the true lag-one autocorrelation $\rho_x(1)$ under for an **AR**(1) model with $\phi_1 = \{-0.5, -0.25, -0.125, 0.125, 0.25, 0.5\}$.
- (c) Compute the true lag-one autocorrelation $\rho_x(1)$ under for an **MA**(1) model with $\theta_1 = \{-1, -0.268, -0.127, 0, 0.127, 0.268, 1\}$.
- (d) In one sentence, interpret what you observe in (a), taking what you find in (b) and (c) into account.