

# Autoregressive Moving Average (ARMA) Models

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The material in this set of notes is based on S&S Chapter 3. We're finally going to define our first time series model! ☺ The first time series model we will define is the **autoregressive (AR)** model. We will then consider a different simple time series model, the **moving average (MA)** model. Putting both models together to create one more general model will give us the **autoregressive moving average (ARMA)** model.

## The Autoregressive (AR) Model

The first kind of time series model we'll consider is an **autoregressive (AR)** model. This is one of the most intuitive models we'll consider. The basic idea is that we will model the response at time  $t$   $x_t$  as a linear function of its  $p$  previous values and some independent random noise, e.g.

$$x_t = 0.5x_{t-1} + w_t, \tag{1}$$

where  $x_t$  is stationary and  $w_t \stackrel{i.i.d.}{\sim} \mathcal{N}(0, \sigma_w^2)$ . This kind of model is especially well suited to forecasting, as

$$\mathbb{E}[x_{t+1}|x_t] = 0.5x_{t-1}. \tag{2}$$

We explicitly define an **autoregressive model of order  $p$** , abbreviated as **AR** ( $p$ ) as:

$$(x_t - \mu_x) = \phi_1 (x_{t-1} - \mu_x) + \phi_2 (x_{t-2} - \mu_x) + \cdots + \phi_p (x_{t-p} - \mu_x) + w_t, \quad (3)$$

where  $x_t$  is stationary with mean  $\mathbb{E}[x_t] = \mu_x$  and  $w_t \stackrel{i.i.d.}{\sim} \mathcal{N}(0, \sigma_w^2)$ . For convenience:

- We'll often assume  $\mu_x = 0$ , so

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \cdots + \phi_p x_{t-p} + w_t. \quad (4)$$

- We'll introduce the **autoregressive operator** notation:

$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p, \quad (5)$$

where  $B^p x_t = x_{t-p}$  is the **backshift operator**. This allows us to rewrite (3) and (4) more concisely as  $\phi(B)(x_t - \mu_x) = w_t$  and

$$\phi(B)(x_t) = w_t, \quad (6)$$

respectively.

An **AR** ( $p$ ) model looks like a linear regression model, but the covariates are also random variables. We'll start building an understanding of the **AR** ( $p$ ) model by starting with the simpler special case where  $p = 1$ .

The **AR** (1) model with  $\mu_x = 0$  is a special case of (3)

$$x_t = \phi_1 x_{t-1} + w_t. \quad (7)$$

A natural thing to do is to try to rewrite  $x_t$  as a function of  $\phi_1$  and the previous values of the random errors. Then (7) will look more like a classical regression problem, as it will no longer have random variables as covariates. Furthermore, if we could rewrite  $x_t$  as a function of  $\phi_1$  and the random errors  $\mathbf{w}$ , then  $x_t$  would be a **linear process**.

A **linear process**  $x_t$  is defined to be a linear combination of white noise variates  $w_t$  and

is given by

$$x_t = \mu_x + \sum_{j=-\infty}^{\infty} \psi_j w_{t-j},$$

where the coefficients satisfy  $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$ ,  $w_t$  are independent and identically distributed with mean 0 and variance  $\sigma_w^2$ , and  $\mu_x = \mathbb{E}[x_t]$ . Importantly, it can be shown that the autocovariance function of a linear process is

$$\gamma_x(h) = \sigma_w^2 \sum_{j=-\infty}^{\infty} \psi_{j+h} \psi_j, \quad (8)$$

for  $h \geq 0$ , recalling that  $\gamma_x(h) = \gamma_x(-h)$ . This means that once we know the linear process representation of any time series process, we can easily compute its autocovariance (and autocorrelation) functions. We will often use the **infinite moving average operator** shorthand  $1 + \psi_1 B + \psi_2 B^2 + \dots \psi_j B^j + \dots = \psi(B)$ .

We can start rewriting  $x_t$  as follows:

$$\begin{aligned} x_t &= \phi_1^2 x_{t-1} + \phi_1 w_{t-1} + w_t \\ &= \phi_1^3 x_{t-2} + \phi_1^2 w_{t-2} + \phi_1 w_{t-1} + w_t \\ &= \underbrace{\phi_1^k x_{t-k}}_{(*)} + \sum_{j=0}^{k-1} \phi_1^j w_{t-j}. \end{aligned}$$

We can see that we can almost lagged values of  $\mathbf{x}$  out of the right hand side. Fortunately, when  $|\phi_1| < 1$ , then

$$\lim_{k \rightarrow \infty} \mathbb{E} \left[ \left( x_t - \sum_{j=0}^{k-1} \phi_1^j w_{t-j} \right)^2 \right] = \lim_{k \rightarrow \infty} \phi_1^{2k} \mathbb{E} [x_{t-k}^2] = 0,$$

because  $\mathbb{E}[x_{t-k}^2]$  is constant as long as  $x_t$  is stationary is assumed. This means that when  $|\phi_1| < 1$ , then we can write elements of the response  $x_t$  as a linear function the previous

values of the random errors:

$$x_t = \sum_{j=0}^{\infty} \phi^j w_{t-j}. \quad (9)$$

(9) is the **linear process** representation of an **AR**(1) model. It follows that the autocovariance function

$$\begin{aligned} \gamma_x(h) &= \sigma_w^2 \sum_{j=0}^{\infty} \phi^{j+h} \phi^j \\ &= \sigma_w^2 \phi^h \sum_{j=0}^{\infty} \phi^{2j} \\ &= \sigma_w^2 \phi^h \left( \frac{1}{1 - \phi^2} \right). \end{aligned} \quad (10)$$

and the autocorrelation function is

$$\rho_x(h) = \phi^h. \quad (11)$$

Note that this is not the only way to compute the values of autocovariance function. We could compute them directly from (4),

$$\begin{aligned} \gamma_x(h) &= \mathbb{E}[x_{t-h}x_t] \\ &= \mathbb{E}[x_{t-h}(\phi_1 x_{t-1} + w_t)] \\ &= \phi_1 \mathbb{E}[x_{t-1-(h-1)}x_{t-1}] + \mathbb{E}[x_{t-h}w_t] \\ &= \phi_1 \gamma_x(h-1). \end{aligned} \quad (12)$$

This gives us a recursive relation that we can use to compute the autocovariance function

$\gamma_x(h)$ , starting from  $\gamma_x(0)$ . We can compute  $\gamma_x(0)$  using substitution:

$$\begin{aligned}
\gamma_x(0) &= \mathbb{E}[x_t^2] \\
&= \mathbb{E}[x_t^2] \\
&= \mathbb{E}[(\phi_1 x_{t-1} + w_t)^2] \\
&= \mathbb{E}[\phi_1^2 x_{t-1}^2 + 2\phi_1 w_t x_{t-1} + w_t^2] \\
&= \phi_1^2 \mathbb{E}[x_{t-1}^2] + \sigma_w^2 \\
&= \sigma_w^2 \sum_{j=0}^{\infty} \phi_1^{2j} && \text{(follows from continued substitution)} \\
&= \frac{\sigma_w^2}{1 - \phi_1^2}, && \text{if } |\phi_1| < 1, \gamma_x(0) = \infty \text{ otherwise!}
\end{aligned} \tag{13}$$

Whether or not  $|\phi_1| < 1$  is closely related to whether or not  $\mathbf{x}$  is a stationary time series. If  $|\phi_1| < 1$ , then it is easy to see that the **AR**(1) time series  $\mathbf{x}$  is stationary because the mean of each  $x_t$  is zero and the autocovariance function  $\gamma_x(h) = \sigma_w^2 \phi_1^h \left( \frac{1}{1 - \phi_1^2} \right)$  depends only on the lag,  $h$ . What happens when  $\phi_1 > 1$ ?

When  $\phi_1 < 1$ , we can revisit (7) and note that  $x_{t+1} = \phi_1 x_t + w_{t+1}$ . Rearranging gives

$$x_t = \left( \frac{1}{\phi_1} \right) x_{t+1} - \left( \frac{1}{\phi_1} \right) w_{t+1}.$$

If  $\phi > 1$ , then  $\left( \frac{1}{\phi_1} \right) < 1$  and we can use the same approach we used previously to write

$$x_t = - \sum_{j=1}^{\infty} \left( \frac{1}{\phi_1} \right)^j w_{t+j}.$$

The problem, however, is that this requires that  $x_t$  is a function of *future* values, which may not be known at time  $t$ . We call such a time series **non-causal**. Using a **non-causal** model will rarely make sense in practice, and furthermore makes forecasting impossible - in the future, whenever we talk about **AR**( $p$ ) models we restrict our attention to **causal** models.

Understanding when a **AR**( $p$ ) model is causal is more difficult than understanding when an **AR**(1) model is causal. We figured out when an **AR**(1) model is causal by finding

the coefficients  $\psi_{-\infty}, \dots, \psi_{\infty}$  of its linear process representation as a function of the AR coefficient  $\phi_1$ , and showing that all of the coefficients  $\psi_{-\infty}, \dots, \psi_{-1}$  for future errors are exactly equal to zero.

The linear process representation is especially useful for an **AR**( $p$ ) time series when  $p > 1$ , because computing the autocovariance function  $\gamma_x(h)$  directly as we did in (12) and (13) gets much more cumbersome when  $p > 1$ . We can see this in the **AR**(2) case, where we have

$$x_t = \phi_2 x_{t-2} + \phi_1 x_{t-1} + w_t. \quad (14)$$

We can get a recursive relation for the autocovariance function  $\gamma_x(h)$  starting from  $\gamma_x(0)$  and  $\gamma_x(1)$  as follows:

$$\begin{aligned} \gamma_x(h) &= \mathbb{E}[x_{t-h}x_t] \\ &= \mathbb{E}[x_{t-h}(\phi_1 x_{t-1} + \phi_2 x_{t-2} + w_t)] \\ &= \phi_1 \mathbb{E}[x_{t-1-(h-1)}x_{t-1}] + \phi_2 \mathbb{E}[x_{t-2-(h-2)}x_{t-2}] + \mathbb{E}[x_{t-h}w_t] \\ &= \phi_1 \gamma_x(h-1) + \phi_2 \gamma_x(h-2). \end{aligned}$$

We can try to compute  $\gamma_x(0)$  and  $\gamma_x(1)$  using substitution:

$$\begin{aligned} \gamma_x(0) &= \mathbb{E}[x_t^2] \\ &= \mathbb{E}[(\phi_1 x_{t-1} + \phi_2 x_{t-2} + w_t)^2] \\ &= \mathbb{E}[\phi_1^2 x_{t-1}^2 + \phi_2^2 x_{t-2}^2 + 2\phi_1 \phi_2 x_{t-1} x_{t-2} + 2\phi_1 x_{t-1} w_t + 2\phi_2 x_{t-2} w_t + w_t^2] \\ &= \mathbb{E}[\phi_1^2 x_{t-1}^2 + \phi_2^2 x_{t-2}^2 + 2\phi_1 \phi_2 x_{t-1} x_{t-2}] + \sigma_w^2. \end{aligned}$$

However, this gets *very* complicated, even though we only have two lags!

Unfortunately, it's much harder to find the linear process representation of an **AR**( $p$ ) time series by simple substitution as we did with an **AR**(1) time series. Substituting according

to (14)

$$\begin{aligned}
x_t &= \phi_1 \phi_2 x_{t-3} + (\phi_2 + \phi_1^2) x_{t-2} + \phi_1 w_{t-1} + w_t \\
&= (\phi_2 + \phi_1^2) \phi_2 x_{t-4} + \phi_1 (2\phi_2 + \phi_1^2) x_{t-3} + (\phi_2 + \phi_1^2) w_{t-2} + \phi_1 w_{t-1} + w_t \\
&= (\phi_2 + \phi_1^2) \phi_2 x_{t-4} + \phi_1 (2\phi_2 + \phi_1^2) (\phi_2 x_{t-5} + \phi_1 x_{t-4} + w_{t-3}) + (\phi_2 + \phi_1^2) w_{t-2} + \phi_1 w_{t-1} + w_t \\
&= \phi_1 \phi_2 (2\phi_2 + \phi_1^2) x_{t-5} + (\phi_2^2 + \phi_1^2 \phi_2 + 2\phi_1 \phi_2^2 + \phi_1^3 \phi_2) x_{t-4} + \\
&\quad \phi_1 (2\phi_2 + \phi_1^2) w_{t-3} + (\phi_2 + \phi_1^2) w_{t-2} + \phi_1 w_{t-1} + w_t \dots
\end{aligned}$$

Again, this is *not* working out nicely!

Instead, we can find the values of  $\psi_1, \dots, \psi_\infty$  that satisfy  $\phi(B) \psi(B) w_t = w_t$ , which follows from substituting  $x_t = \sum_{j=-\infty}^{\infty} \psi_j w_{t-j}$  into (19). This is equivalent to finding the inverse function  $\phi^{-1}(B)$  that satisfies  $\phi(B) \phi^{-1}(B) w_t = w_t$ .

We can see how this method for finding the values of  $\psi_1, \dots, \psi_\infty$  works by returning to the **AR**(1) case. The values  $\psi_1, \dots, \psi_\infty$  that satisfy  $\phi(B) \psi(B) w_t = w_t$  solve:

$$1 + (\psi_1 - \phi_1) B + (\psi_2 - \psi_1 \phi_1) B^2 + \dots + \psi_j B^j + \dots = 1, \quad (15)$$

where (15) follows from expanding  $\phi(B)$  and  $\psi(B)$ . This allows us to recover the linear process representation of the **AR**(1) process in a different way, as (15) holds if all of the coefficients for  $B^j$  with  $j > 0$  are equal to zero, i.e.  $\psi_k - \psi_{k-1} \phi_1 = 0$  for  $k > 1$ .

Now let's try this approach for the **AR**(2) case. We have

$$\begin{aligned}
1 &= (1 - \phi_1 B - \phi_2 B^2) (1 + \psi_1 B + \psi_2 B^2 + \dots + \psi_j B^j + \dots) \\
&= 1 + (\psi_1 - \phi_1) B + (\psi_2 - \phi_2 - \phi_1 \psi_1) B^2 + (\psi_3 - \phi_1 \psi_2 - \phi_2 \psi_1) B^3 + \dots + \\
&\quad (\psi_j - \phi_1 \psi_{j-1} - \phi_2 \psi_{j-2}) B^j + \dots
\end{aligned}$$

We see that we can compute the values of  $\psi_1, \dots, \psi_\infty$  recursively,

$$\psi_1 = \phi_1$$

$$\psi_2 = \phi_2 + \phi_1^2$$

$$\psi_3 = 2\phi_1 (\phi_2 + \phi_1^2),$$

and so on.

It's also very tricky to figure out when  $\mathbf{AR}(p)$  model is **causal** for  $p > 1$  - we'll come back to this later.

## The Moving Average (MA) Model

Instead of assuming that elements of a time series  $x_t$  are linear function of previous elements of the time series  $x_1, \dots, x_{t-1}$  and independent, identically distributed noise  $w_t$ , we might assume that elements of a time series  $x_t$  are a linear function of all of the current and previous noise variates,  $w_1, \dots, w_{t-1}$ . The latter gives us the **moving average model of order  $q$** , abbreviated as **MA( $q$ )**. The **MA( $q$ )** model is explicitly defined as

$$x_t - \mu_x = w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2} + \dots + \theta_q w_{t-q}, \quad (16)$$

where  $\mathbb{E}[x_t] = \mu_x$ ,  $w_t \stackrel{i.i.d.}{\sim} \mathcal{N}(0, \sigma_w^2)$ , and

( $\star$ )  $\mathbf{w}$  has an infinite AR representation.

We'll come back to ( $\star$ ) later. For convenience:

- We'll often assume  $\mu_x = 0$ , so

$$x_t = w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2} + \dots + \theta_q w_{t-q}. \quad (17)$$

- We'll introduce the **moving average operator** notation:

$$\theta(B) = 1 + \phi_1 B + \phi_2 B^2 + \dots + \phi_p B^p, \quad (18)$$



which allows us to rewrite (16) and (17) more concisely as  $x_t - \mu_x = \theta(B) w_t$  and

$$x_t = \theta(B) w_t, \quad (19)$$

respectively.

Again, the  $\mathbf{MA}(q)$  model looks like a linear regression model. Importantly, the  $\mathbf{MA}(q)$  model is stationary for any values of the parameters  $\theta_1, \dots, \theta_q$ .

Like we did with the  $\mathbf{AR}(p)$  model, we'll start building an understanding of the  $\mathbf{MA}(q)$  by starting with the simpler special case where  $q = 1$ ,

$$x_t = \theta_1 w_{t-1} + w_t. \quad (20)$$

It is easy to see that this  $\mathbf{MA}(q)$  time series is mean zero. We can compute the autocovariance function as follows:

$$\begin{aligned} \gamma_x(h) &= \mathbb{E}[x_t x_{t-h}] \\ &= \mathbb{E}[(\theta_1 w_{t-1} + w_t)(\theta_1 w_{t-h-1} + w_{t-h})] \\ &= \mathbb{E}[\theta_1^2 w_{t-1} w_{t-h-1} + \theta_1 w_t w_{t-h-1} + \theta_1 w_{t-1} w_{t-h} + w_t w_{t-h}] \\ &= \mathbb{E}[\theta_1^2 w_{t-1} w_{t-h-1} + \theta_1 w_{t-1} w_{t-h} + w_t w_{t-h}] \\ &= \begin{cases} \sigma_w^2 (\theta_1^2 + 1) & h = 0 \\ \theta_1 & h = 1 \\ 0 & h > 1 \end{cases}. \end{aligned} \quad (21)$$

The corresponding autocorrelation function is

$$\rho_x(h) = \begin{cases} \frac{\theta_1}{\theta_1^2 + 1} & h = 1 \\ 0 & h > 1 \end{cases}. \quad (22)$$

The autocovariance and autocorrelation functions of the  $\mathbf{MA}(q)$  model are noteworthy in two ways:

- (•) The autocorrelation function  $\rho_x(h)$  is bounded,  $\rho_x(h) \leq 1/2$  for  $h > 1$ .

- (\*) The parameters of the **MA** ( $q$ ) model do not uniquely determine the autocovariance and autocorrelation function values.  $\theta_1$  and  $\sigma_w^2$  do not uniquely determine the value of the autocovariance function  $\gamma_x(h)$ , and  $\theta_1$  does not determine the value of the autocorrelation function.

It is easiest to understand (\*) via some examples. First, we compute  $\gamma_x(h)$  and  $\rho_x(h)$  for a **MA** (1) process with  $\theta_1 = 5$  and  $\sigma_w^2 = 1$ ,

$$\gamma_x(h) = \begin{cases} 5^2 + 1 = 26 & h = 0 \\ 5 & h = 1 \\ 0 & h > 1 \end{cases} \quad \text{and} \quad \rho_x(h) = \begin{cases} \frac{5}{5^2+1} = \frac{5}{26} & h = 1 \\ 0 & h > 1 \end{cases}.$$

Compare this to  $\gamma_x(h)$  and  $\rho_x(h)$  for a **MA** (1) process with  $\theta_1 = 1/5$  and  $\sigma_w^2 = 25$ ,

$$\gamma_x(h) = \begin{cases} 25 \left( \frac{1}{5^2} + 1 \right) = 25 \left( \frac{1+25}{25} \right) = 26 & h = 0 \\ 25 \left( \frac{1}{5} \right) = 5 & h = 1 \\ 0 & h > 1 \end{cases} \quad \text{and} \quad \rho_x(h) = \begin{cases} \frac{\frac{1}{5}}{\frac{1}{5^2}+1} = \frac{5}{26} & h = 1 \\ 0 & h > 1 \end{cases}.$$

Both sets of **MA** (1) parameters give the values of the autocovariance and autocorrelation functions! This is undesirable - it means that even if we know that our time series is mean zero with a specific autocovariance function  $\gamma_x(h)$  autocorrelation function  $\rho_x(h)$ , we can't find a **unique** pair of **MA** (1) parameter values  $(\theta_1, \sigma_w^2)$  that generated it. ☹