# Autoregressive Moving Average (ARMA) Models

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The material in this set of notes is based on S&S Chapter 3. We're finally going to define our first time series model! ☺ The first time series model we will define is the **autoregressive (AR)** model. We will then consider a different simple time series model, the **moving average (MA)** model. Putting both models together to create one more general model will give us the **autoregressive moving average** (ARMA) model.

## The Autoregressive (AR) Model

The first kind of time series model we'll consider is an **autoregressive (AR)** model. This is one of the most intuitive models we'll consider. The basic idea is that we will model the response at time $t$ $x_t$ as a linear function of its $p$ previous values and some independent random noise, e.g.

$$x_t = 0.5x_{t-1} + w_t, \tag{1}$$

where $x_t$ is stationary and $w_t \overset{i.i.d.}{\sim} \mathcal{N}\left(0, \sigma_w^2\right)$. This kind of model is especially well suited to forecasting, as

$$\mathbb{E}\left[x_{t+1}|x_t\right] = 0.5x_{t-1}. \tag{2}$$

We explicitly define an **autoregressive model of order** $p$, abbreviated as $\mathbf{AR}(p)$ as:

$$(x_t - \mu_x) = \phi_1 (x_{t-1} - \mu_x) + \phi_2 (x_{t-2} - \mu_x) + \cdots + \phi_p (x_{t-p} - \mu_x) + w_t, \tag{3}$$

where $\phi_p \neq 0$, $x_t$ is stationary with mean $\mathbb{E}[x_t] = \mu_x$, and $w_t \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma_w^2)$. For convenience:

- We'll often assume $\mu_x = 0$, so

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \cdots + \phi_p x_{t-p} + w_t. \tag{4}$$

- We'll introduce the **autoregressive operator** notation:

$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p, \tag{5}$$

  where $B^p x_t = x_{t-p}$ is the **backshift operator**. This allows us to rewrite (3) and (4) more concisely as $\phi(B)(x_t - \mu_x) = w_t$ and

$$\phi(B)(x_t) = w_t, \tag{6}$$

  respectively.

An $\mathbf{AR}(p)$ model looks like a linear regression model, but the covariates are also random variables. We'll start building an understanding of the $\mathbf{AR}(p)$ model by starting with the simpler special case where $p = 1$.

The $\mathbf{AR}(1)$ model with $\mu_x = 0$ is a special case of (3)

$$x_t = \phi_1 x_{t-1} + w_t. \tag{7}$$

A natural thing to do is to try to rewrite $x_t$ as a function of $\phi_1$ and the previous values of the random errors. Then (7) will look more like a classical regression problem, as it will no longer have random variables as as covariates. Furthermore, if we could rewrite $x_t$ as a function of $\phi_1$ and the random errors $\boldsymbol{w}$, then $x_t$ would be a **linear process**.

A **linear process** $x_t$ is defined to be a linear combination of white noise $w_t$ and is given by

$$x_t = \mu_x + \sum_{j=-\infty}^{\infty} \psi_j w_{t-j},$$

where the coefficients satisfy $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$, $w_t$ are independent and identically distributed with mean 0 and variance $\sigma_w^2$, and $\mu_x = \mathbb{E}[x_t] < \infty$. The condition $\sum_{j=-\infty}^{\infty} |\psi_j|$ ensures that $x_t = \mu_x + \sum_{j=-\infty}^{\infty} \psi_j w_{t-j} < \infty$. Importantly, it can be shown that the autocovariance function of a linear process is

$$\gamma_x(h) = \sigma_w^2 \sum_{j=-\infty}^{\infty} \psi_{j+h}\psi_j, \tag{8}$$

for $h \geq 0$, recalling that $\gamma_x(h) = \gamma_x(-h)$. This means that once we know the linear process representation of any time series process, we can easily compute its autocovariance (and autocorrelation) functions. We will often use the **infinite moving average operator** shorthand $1 + \psi_1 B + \psi_2 B^2 + \ldots \psi_j B^j + \cdots = \psi(B)$.

We can start rewriting $x_t$ as follows:

$$\begin{aligned} x_t &= \phi_1^2 x_{t-1} + \phi_1 w_{t-1} + w_t \\ &= \phi_1^3 x_{t-2} + \phi_1^2 w_{t-2} + \phi_1 w_{t-1} + w_t \\ &= \underbrace{\phi_1^k x_{t-k}}_{(*)} + \sum_{j=0}^{k-1} \phi_1^j w_{t-j}. \end{aligned}$$

We can see that we can almost lagged values of $\boldsymbol{x}$ out of the right hand side. Fortunately, when $|\phi_1| < 1$, then

$$\lim_{k\to\infty} \mathbb{E}\left[\left(x_t - \sum_{j=0}^{k-1} \phi_1^j w_{t-j}\right)^2\right] = \lim_{k\to\infty} \phi^{2k} \mathbb{E}\left[x_{t-k}^2\right] = 0,$$

because $\mathbb{E}\left[x_{t-k}^2\right]$ is constant as long as $x_t$ is stationary is assumed. This means that when $|\phi_1| < 1$, then we can write elements of the response $x_t$ as a linear function the previous values of the random errors:

$$x_t = \sum_{j=0}^{\infty} \phi^j w_{t-j}. \tag{9}$$

(9) is the **linear process** representation of an $\mathbf{AR}\,(1)$ model. It follows that the autocovariance function

$$\begin{aligned}\gamma_x\left(h\right) &= \sigma_w^2 \sum_{j=0}^{\infty} \phi_1^{j+h} \phi_1^j \\ &= \sigma_w^2 \phi_1^h \sum_{j=0}^{\infty} \phi_1^{2j} \\ &= \sigma_w^2 \phi_1^h \left(\frac{1}{1-\phi_1^2}\right). \end{aligned} \tag{10}$$

and the autocorrelation function is

$$\rho_x\left(h\right) = \phi^h. \tag{11}$$

Note that this is not the only way to compute the values of autocovariance function. We could compute them directly from (4),

$$\begin{aligned}\gamma_x\left(h\right) =& \mathbb{E}\left[x_{t-h} x_t\right] \qquad (12) \\ =& \mathbb{E}\left[x_{t-h}\left(\phi_1 x_{t-1} + w_t\right)\right] \\ =& \phi_1 \mathbb{E}\left[x_{t-1-(h-1)} x_{t-1}\right] + \mathbb{E}\left[x_{t-h} w_t\right] \\ =& \phi_1 \gamma_x\left(h-1\right). \end{aligned}$$

This gives us a recursive relation that we can use to compute the autocovariance function

$\gamma_x(h)$, starting from $\gamma_x(0)$. We can compute $\gamma_x(0)$ using substitution:

$$\begin{aligned}
\gamma_x(0) =& \mathbb{E}\left[x_t^2\right] && (13)\\
=& \mathbb{E}\left[x_t^2\right] \\
=& \mathbb{E}\left[(\phi_1 x_{t-1} + w_t)^2\right] \\
=& \mathbb{E}\left[\phi_1^2 x_{t-1}^2 + 2\phi_1 w_t x_{t-1} + w_t^2\right] \\
=& \phi_1^2 \mathbb{E}\left[x_{t-1}^2\right] + \sigma_w^2 \\
=& \sigma_w^2 \sum_{j=0}^{\infty} \phi_1^{2j} && \text{(follows from continued substitution)} \\
=& \frac{\sigma_w^2}{1-\phi_1^2}, && \text{if } |\phi_1| < 1,\ \gamma_x(0) = \infty \text{ otherwise!}
\end{aligned}$$

If $|\phi_1| < 1$, then it is easy to see that the $\mathbf{AR}(1)$ model $\boldsymbol{x}$ is stationary because the mean of each $x_t$ is zero and the autocovariance function $\gamma_x(h) = \sigma_w^2 \phi_h \left(\frac{1}{1-\phi^2}\right)$ depends only on the lag, $h$. What happens when $|\phi_1| > 1$? (9) does **not** give a linear process representation if $|\phi_1| > 1$, because $\sum_{j=0}^{\infty} |\phi_1|^j = +\infty$.

However when $|\phi_1| > 1$, we can revisit (7) and note that $x_{t+1} = \phi_1 x_t + w_{t+1}$. Rearranging gives

$$x_t = \left(\frac{1}{\phi_1}\right) x_{t+1} - \left(\frac{1}{\phi_1}\right) w_{t+1}.$$

If $\phi > 1$, then $\left(\frac{1}{\phi_1}\right) < 1$ and we can use the same approach we used previously to write

$$x_t = -\sum_{j=1}^{\infty} \left(\frac{1}{\phi_1}\right)^j w_{t+j}.$$

The problem, however, is that this requires that $x_t$ is a function of *future* values, which may not be known at time $t$. We call such a time series **non-causal**. Using a **non-causal** model will rarely make sense in practice, and furthermore makes forecasting impossible - in the future, whenever we talk about $\mathbf{AR}(p)$ models we restrict our attention to **causal** models.

Understanding when a $\mathbf{AR}(p)$ model is causal is more difficult than understanding when an $\mathbf{AR}(1)$ model is causal. We figured out when an $\mathbf{AR}(1)$ model is causal by finding the coefficients $\dots, \psi_{-j}, \dots, \psi_j, \dots$ of its linear process representation as a function of the AR coefficient $\phi_1$, and showing that all of the coefficients $\psi_{-\infty}, \dots, \psi_{-1}$ for future errors are exactly equal to zero.

The linear process representation is especially useful for an $\mathbf{AR}(p)$ model when $p > 1$, because computing the autocovariance function $\gamma_x(h)$ directly as we did in (12) and (13) gets much more cumbersome when $p > 1$. We can see this in the $\mathbf{AR}(2)$ case, where we have

$$x_t = \phi_2 x_{t-2} + \phi_1 x_{t-1} + w_t. \tag{14}$$

We can get a recursive relation for the autocovariance function $\gamma_x(h)$ starting from $\gamma_x(0)$ and $\gamma_x(1)$ as follows:

$$\begin{aligned}
\gamma_x(h) =& \mathbb{E}\left[x_{t-h} x_t\right] \\
=& \mathbb{E}\left[x_{t-h}\left(\phi_1 x_{t-1} + \phi_2 x_{t-2} + w_t\right)\right] \\
=& \phi_1 \mathbb{E}\left[x_{t-1-(h-1)} x_{t-1}\right] + \phi_2 \mathbb{E}\left[x_{t-2-(h-2)} x_{t-2}\right] + \mathbb{E}\left[x_{t-h} w_t\right] \\
=& \phi_1 \gamma_x(h-1) + \phi_2 \gamma_x(h-2).
\end{aligned}$$

We can try to compute $\gamma_x(0)$ and $\gamma_x(1)$ using substitution:

$$\begin{aligned}
\gamma_x(0) =& \mathbb{E}\left[x_t^2\right] \\
=& \mathbb{E}\left[\left(\phi_1 x_{t-1} + \phi_2 x_{t-2} + w_t\right)^2\right] \\
=& \mathbb{E}\left[\phi_1^2 x_{t-1}^2 + \phi_2^2 x_{t-2}^2 + 2\phi_1\phi_2 x_{t-1} x_{t-2} + 2\phi_1 x_{t-1} w_t + 2\phi_2 x_{t-2} w_t + w_t^2\right] \\
=& \mathbb{E}\left[\phi_1^2 x_{t-1}^2 + \phi_2^2 x_{t-2}^2 + 2\phi_1\phi_2 x_{t-1} x_{t-2}\right] + \sigma_w^2.
\end{aligned}$$

However, this gets *very* complicated, even though we only have two lags!

Unfortunately, it's much harder to find the linear process representation of an $\mathbf{AR}(p)$ model by simple substitution as we did with an $\mathbf{AR}(1)$ model. Substituting according to

(14)

$$
\begin{aligned}
x_t =&\phi_1\phi_2 x_{t-3} + \left(\phi_2 + \phi_1^2\right) x_{t-2} + \phi_1 w_{t-1} + w_t \\
=&\left(\phi_2 + \phi_1^2\right)\phi_2 x_{t-4} + \phi_1\left(2\phi_2 + \phi_1^2\right) x_{t-3} + \left(\phi_2 + \phi_1^2\right) w_{t-2} + \phi_1 w_{t-1} + w_t \\
=&\left(\phi_2 + \phi_1^2\right)\phi_2 x_{t-4} + \phi_1\left(2\phi_2 + \phi_1^2\right)\left(\phi_2 x_{t-5} + \phi_1 x_{t-4} + w_{t-3}\right) + \left(\phi_2 + \phi_1^2\right) w_{t-2} + \phi_1 w_{t-1} + w_t \\
=&\phi_1\phi_2\left(2\phi_2 + \phi_1^2\right) x_{t-5} + \left(\phi_2^2 + \phi_1^2\phi_2 + 2\phi_1\phi_2^2 + \phi_1^3\phi_2\right) x_{t-4} + \\
&\phi_1\left(2\phi_2 + \phi_1^2\right) w_{t-3} + \left(\phi_2 + \phi_1^2\right) w_{t-2} + \phi_1 w_{t-1} + w_t \dots
\end{aligned}
$$

Again, this is *not* working out nicely!

Instead, we can find the values of $\psi_1, \dots, \psi_j, \dots$ that satisfy $\phi(B)\psi(B)w_t = w_t$, which follows from substituting $x_t = \sum_{j=-\infty}^{\infty} \psi_j w_{t-j}$ into (20). This is equivalent to finding the inverse function $\phi^{-1}(B)$ that satisfies $\phi(B)\phi^{-1}(B)w_t = w_t$.

We can see how this method for finding the values of $\psi_1, \dots, \psi_j, \dots$ works by returning to the $\mathbf{AR}(1)$ case. The values $\psi_1, \dots, \psi_j, \dots$ that satisfy $\phi(B)\psi(B)w_t = w_t$ solve:

$$
1 + (\psi_1 - \phi_1) B + (\psi_2 - \psi_1\phi_1) B^2 + \dots + \psi_j B^j + \dots = 1, \tag{15}
$$

where (15) follows from expanding $\phi(B)$ and $\psi(B)$. This allows us to recover the linear process representation of the $\mathbf{AR}(1)$ process in a different way, as (15) holds if all of the coefficients for $B^j$ with $j > 0$ are equal to zero, i.e. $\psi_k - \psi_{k-1}\phi_1 = 0$ for $k > 1$.

Now let's try this approach for the $\mathbf{AR}(2)$ case. We have

$$
\begin{aligned}
1 =& \left(1 - \phi_1 B - \phi_2 B^2\right)\left(1 + \psi_1 B + \psi_2 B^2 + \dots + \psi_j B^j + \dots\right) \\
=&1 + (\psi_1 - \phi_1) B + (\psi_2 - \phi_2 - \phi_1\psi_1) B^2 + (\psi_3 - \phi_1\psi_2 - \phi_2\psi_1) B^3 + \dots + \\
&(\psi_j - \phi_1\psi_{j-1} - \phi_2\psi_{j-2}) B^j + \dots
\end{aligned}
$$

We see that we can compute the values of $\psi_1, \ldots, \psi_j, \ldots$ recursively,

$$\begin{aligned}
\psi_1 &= \phi_1 \\
\psi_2 &= \phi_2 + \phi_1^2 \\
\psi_3 &= 2\phi_1 \left(\phi_2 + \phi_1^2\right),
\end{aligned}$$

and so on.

It's also very tricky to figure out when **AR** $(p)$ model is **causal** for $p > 1$. An **AR** $(p)$ model is **causal** for $p > 1$ model is **causal** when all of the roots of the **AR polynomial**

$$\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p,$$

lie outside the unit circle, i.e. $\phi(z) \neq 0$ for $|z| \leq 1$. This condition ensures that the $\sum_{j=1}^{\infty} |\psi_j| < \infty$. This is not very intuitive. If we want to try to get a handle on why the roots of the AR polynomial need to lie outside the unit circle for a **AR**$(p)$ model to be **causal**, we need to take a look at the proof. You won't be tested on your understanding of this - we'll just go through it here in case you are curious following along the proof of Theorem 3.2 in Chan (2010).

Let's suppose that $\phi(z)$ has roots $r_1, \ldots, r_p$ that satisfy $1 < |r_1| \leq \cdots \leq |r_p|$, i.e. $\phi(r_j) = 0$ for $j = 1, \ldots, p$. Then this ensures that we can invert $\phi(z)$ when $z \leq |r_1|$. Recalling that $\psi(B)$ can be thought of as the inverse of $\phi(B)$, this means that

$$\frac{1}{\phi(z)} = \sum_{j=0}^{\infty} \psi_j z^j < \infty \text{ if } |z| \leq |r_1|,$$

where $\psi_0 = 1$. Then we can invert $\phi(z)$ at any value of $z < |r_1|$, e.g. at $z = 1 + \delta < |r_1|$, where $\delta > 0$. Writing this out, we have

$$\frac{1}{\phi(1+\delta)} = \sum_{j=0}^{\infty} \psi_j (1+\delta)^j < \infty. \tag{16}$$

If (16), then there must be some constant $M > 0$ that gives an upper bound for all $\left|\psi_j (1+\delta)^j\right|$, i.e. $\left|\psi_j (1+\delta)^j\right| \leq M$ for all $j = 0, 1, 2, \ldots$. Shifting things around, this is equivalent to $|\psi_j| \leq M(1+\delta)^{-j}$. Then

$$\begin{aligned}
\sum_{j=1}^{\infty} |\psi_j| \leq & M \sum_{j=1}^{\infty} \left(\frac{1}{1+\delta}\right)^j \\
= & M \left(\sum_{j=0}^{\infty} \left(\frac{1}{1+\delta}\right)^j - 1\right) \\
= & M \left(\frac{1}{1-\frac{1}{1+\delta}} - 1\right) && \text{(follows from } \frac{1}{1+\delta} < 1 \text{ if } \delta > 0) \\
= & M \left(\frac{1+\delta}{1+\delta-1} - 1\right) = M\left(\frac{1}{\delta}\right) < \infty.
\end{aligned}$$

# The Moving Average (MA) Model

Instead of assuming that elements of a time series $x_t$ are linear function of previous elements of the time series $x_1, \dots, x_{t-1}$ and independent, identically distributed noise $w_t$, we might assume that elements of a time series $x_t$ are a linear function of all of the current and previous noise variates, $w_1, \dots, w_{t-1}$. The latter gives us the **moving average model of order** $q$, abbreviated as $\mathbf{MA}(q)$. The $\mathbf{MA}(q)$ model is explicitly defined as

$$x_t - \mu_x = w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2} + \dots + \theta_q w_{t-q}, \tag{17}$$

where $\theta_q \neq 0$, $\mathbb{E}[x_t] = \mu_x$, and $w_t \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma_w^2)$. For convenience:

- We'll often assume $\mu_x = 0$, so

$$x_t = w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2} + \dots + \theta_q w_{t-q}. \tag{18}$$

- We'll introduce the **moving average operator** notation:

$$\theta(B) = 1 + \theta_1 B + \theta_2 B^2 + \dots + \theta_p B^p, \tag{19}$$

  which allows us to rewrite (17) and (18) more concisely as $x_t - \mu_x = \theta(B) w_t$ and

$$x_t = \theta(B) w_t, \tag{20}$$

respectively.

Again, the $\mathbf{MA}\,(q)$ model looks like a linear regression model. Importantly, the $\mathbf{MA}\,(q)$ model is stationary for any values of the parameters $\theta_1, \ldots, \theta_q$.

Like we did with the $\mathbf{AR}\,(p)$ model, we'll start building an understanding of the $\mathbf{MA}\,(q)$ by starting with the simpler special case where $q = 1$,

$$x_t = \theta_1 w_{t-1} + w_t. \tag{21}$$

It is easy to see that this $\mathbf{MA}\,(q)$ model is mean zero. We can compute the autocovariance function as follows:

$$\begin{aligned}
\gamma_x\left(h\right) &= \mathbb{E}\left[x_t x_{t-h}\right] \\
&= \mathbb{E}\left[\left(\theta_1 w_{t-1} + w_t\right)\left(\theta_1 w_{t-h-1} + w_{t-h}\right)\right] \\
&= \mathbb{E}\left[\theta_1^2 w_{t-1} w_{t-h-1} + \theta_1 w_t w_{t-h-1} + \theta_1 w_{t-1} w_{t-h} + w_t w_{t-h}\right] \\
&= \mathbb{E}\left[\theta_1^2 w_{t-1} w_{t-h-1} + \theta_1 w_{t-1} w_{t-h} + w_t w_{t-h}\right] \\
&= \begin{cases} \sigma_w^2\left(\theta_1^2 + 1\right) & h = 0 \\ \theta_1 & h = 1 \\ 0 & h > 1 \end{cases}.
\end{aligned} \tag{22}$$

The corresponding autocorrelation function is

$$\rho_x\left(h\right) = \begin{cases} \frac{\theta_1}{\theta_1^2 + 1} & h = 1 \\ 0 & h > 1 \end{cases}. \tag{23}$$

The autocovariance and autocorrelation functions of the $\mathbf{MA}\,(q)$ model are noteworthy in two ways:

(•) The autocorrelation function $\rho_x\left(h\right)$ is bounded, $\rho_x\left(h\right) \leq 1/2$ for $h = 1$.

(∗) The parameters of the $\mathbf{MA}\,(q)$ model do not uniquely determine the autocovariance and autocorrelation function values. $\theta_1$ and $\sigma_w^2$ do not uniquely determine the value of the autocovariance function $\gamma_x(h)$, and $\theta_1$ does not determine the value of the autocorrelation function.

It is easiest to understand $(*)$ via some examples. First, we compute $\gamma_x(h)$ and $\rho_x(h)$ for a **MA** (1) process with $\theta_1 = 5$ and $\sigma_w^2 = 1$,

$$\gamma_x(h) = \begin{cases} 5^2 + 1 = 26 & h = 0 \\ 5 & h = 1 \\ 0 & h > 1 \end{cases} \text{ and } \rho_x(h) = \begin{cases} \frac{5}{5^2+1} = \frac{5}{26} & h = 1 \\ 0 & h > 1 \end{cases}.$$

Compare this to $\gamma_x(h)$ and $\rho_x(h)$ for a **MA** (1) process with $\theta_1 = 1/5$ and $\sigma_w^2 = 25$,

$$\gamma_x(h) = \begin{cases} 25\left(\frac{1}{5^2} + 1\right) = 25\left(\frac{1+25}{25}\right) = 26 & h = 0 \\ 25\left(\frac{1}{5}\right) = 5 & h = 1 \\ 0 & h > 1 \end{cases} \text{ and } \rho_x(h) = \begin{cases} \frac{\frac{1}{5}}{\frac{1}{5^2}+1} = \frac{5}{26} & h = 1 \\ 0 & h > 1 \end{cases}.$$

Both sets of **MA** (1) parameters give the values of the autocovariance and autocorrelation functions! This is undesirable - it means that even if we know that our time series is mean zero with a specific autocovariance function $\gamma_x(h)$ autocorrelation function $\rho_x(h)$, we can't find a **unique** pair of corresponding **MA** (1) parameter values $(\theta_1, \sigma_w^2)$. ☹

We solve this problem by requiring that our **MA**(1) model be **invertible**, which means that it has a causal linear process representation $(1 + \pi_1 B + \pi_2 B^2 + \dots + \pi_j B^j + \dots) x_t = w_t$ with $\sum_{j=1}^{\infty} |\pi_j| < \infty$. We can find a **unique** pair of corresponding **MA** (1) parameter values $(\theta_1, \sigma_w^2)$ if we restrict our attention to the parameter values that give an **invertible** **MA** (1) model. What we mean by this is that we can rearrange (21) to resemble a **AR**(1) model for $w_t$,

$$
\begin{aligned}
w_t =& -\theta_1 w_{t-1} + x_t \\
=& \theta_1^2 w_{t-2} - \theta_1 x_{t-1} + x_t \\
=& -\theta_1^3 w_{t-3} + \theta_1^2 x_{t-2} - \theta_1 x_{t-1} + x_t \\
=& \left(-\theta_1\right)^k w_{t-k} + \sum_{j=0}^{k} \left(-\theta_1\right)^j x_{t-j},
\end{aligned}
$$

where $\lim_{k\to\infty} \left(-\theta_1\right)^k w_{t-k} + \sum_{j=0}^{k} \left(-\theta_1\right)^j x_{t-j} = \sum_{j=0}^{\infty} \left(-\theta_1\right)^j x_{t-j}$. Recalling the **AR**(1) model, this will be the case when $|\theta_1| < 1$. Going back to our example where we considered the **MA** (1) parameters $(\theta_1, \sigma_w^2) = (5, 1)$ and $(\theta_1, \sigma_w^2) = \left(\frac{1}{5}, 25\right)$, this means that only the latter pair $(\theta_1, \sigma_w^2) = \left(\frac{1}{5}, 25\right)$ satisfy our definition of a **MA** (1) model.

More generally, requiring that an **MA**($q$) model be **invertible** ensures that we can find a **unique** set of corresponding **MA** ($q$) parameter values $(\theta_1, \dots, \theta_q, \sigma_w^2)$ if we know that our time series is **MA**($q$) with mean zero, a specific autocovariance function $\gamma_x(h)$, and autocorrelation function $\rho_x(h)$. We introduce some additional notation for this; an **MA**($q$) model is **invertible** if we can write $w_t = \pi(B) x_t$, where $\pi(B) = 1 + \pi_1 B + \dots + \pi_j B^j + \dots$ and $\sum_{j=0}^{\infty} |\pi_j| < \infty$. This looks a lot like the problem of ensuring that a **AR** ($p$) model is **causal**, and it turns out that an **MA** ($q$) model is **invertible** if when all of the roots of the **MA polynomial**

$$
\theta(z) = 1 + \theta_1 z + \dots + \theta_q z^q,
$$

lie outside the unit circle, i.e. $\theta(z) \neq 0$ for $|z| \leq 1$.

# The Autoregressive Moving Average (ARMA) Model

The **autoregressive moving average (ARMA)** model combines the **AR** and **MA** models. We define an **ARMA**$(p, q)$ model as:

$$(x_t - \mu_x) = \phi_1 (x_{t-1} - \mu_x) + \cdots + \phi_p (x_{t-p} - \mu_x) + \theta_1 w_{t-1} + \cdots + \theta_q w_{t-q} + w_t, \tag{24}$$

where $w_t \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma_w^2)$, $x_t$ is stationary, $\phi_p \neq 0$, $\theta_q \neq 0$, $\sigma_w^2 > 0$, and the MA and AR polynomials $\theta(B)$ and $\phi(B)$ have no common roots. We refer to $p$ as the **autoregressive order** and $q$ as the **moving average order**. Again, for convenience we will usually assume $\mu_x = 0$, so

$$x_t = \phi_1 x_{t-1} + \cdots + \phi_p x_{t-p} + \theta_1 w_{t-1} + \cdots + \theta_q w_{t-q}. \tag{25}$$

Using operator notation becomes especially beneficial for **ARMA**$(p, q)$ models; we can just write $\phi(B) x_t = \theta(B) w_t$ instead of (25). Note that:

- Setting $p = 0$ gives a **MA** $(q)$ model;
- Setting $q = 0$ gives an **AR** $(p)$.

As with **AR** $(p)$ and **MA** $(q)$ models, we will need to figure out when an **ARMA**$(p, q)$ is **causal** and **invertible**. Fortunately, this is simple given the work we've already done for **MA** $(q)$ and **AR** $(p)$ models. An **ARMA**$(p, q)$ is:

- **Causal**, i.e. we can find $\psi_1, \ldots, \psi_j, \ldots$ such that $\psi(z) = \sum_{j=0}^{\infty} \psi_j z^j = \frac{\theta(z)}{\phi(z)}$ that satisfy $\sum_{j=0}^{\infty} |\psi_j| < \infty$ for $|z| < 1$, if $\phi(z) \neq 0$ for $|z| \leq 1$;
- **Invertible**, i.e. we can find $\pi_1, \ldots, \pi_j, \ldots$ such that $\pi(z) = \sum_{j=0}^{\infty} \pi_j z^j = \frac{\phi(z)}{\theta(z)}$ that satisfy $\sum_{j=0}^{\infty} |\pi_j| < \infty$ for $|z| < 1$, if $\theta(z) \neq 0$ for $|z| \leq 1$.

Returning to the definition of an **ARMA**$(p, q)$ model, it is not immediately obvious why we require that the moving average and autoregressive polynomials $\theta(B)$ and $\phi(B)$ have no common roots. Consider the following model, which resembles an $\mathbf{ARMA}(p,q)$ model:

$$x_t = 0.5x_{t-1} - 0.5w_{t-1} + w_t, \tag{26}$$

where $x_t$ is stationary and $w_t \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma_w^2)$. It's easy to see that the mean function $\mu_x = 0$. The autocovariance function $\gamma_x(h)$ satisfies:

$$\begin{aligned}
\gamma_x(h) &= \mathbb{E}[x_t x_{t-h}] \\
&= \mathbb{E}[(0.5x_{t-1} - 0.5w_{t-1} + w_t)\, x_{t-h}] \\
&= 0.5\mathbb{E}[x_{t-1}x_{t-h}] - 0.5\mathbb{E}[w_{t-1}x_{t-h}] + \mathbb{E}[w_t x_{t-h}] \\
&= \begin{cases} 0.5\gamma_x(0) - 0.5\sigma_w^2 & h = 1 \\ 0.5\gamma_x(h-1) & h > 1 \end{cases}
\end{aligned} \tag{27}$$

We just need to combine this with a starting value, $\gamma_x(0)$:

$$\begin{aligned}
\gamma_x(0) &= \mathbb{E}\left[x_t^2\right] \\
&= \mathbb{E}\left[0.5^2 x_{t-1}^2 + 0.5^2 w_{t-1}^2 + w_t^2 - (2)(0.5)^2 w_{t-1}^2\right] \\
&= 0.5^2\gamma_x(0) + \left(1 - 0.5^2\right)\sigma_w^2 \implies \gamma_x(0) = \sigma_w^2
\end{aligned}$$

Plugging this in to (27), we get

$$\gamma_x(h) = 0!$$

This means that (26) is equivalent to the white noise model, $x_t = w_t$!

If we examine the corresponding AR and MA polynomials, we see that they share the common factor $1 - 0.5B$, $\theta(B) = 1 - 0.5B$ and $\phi(B) = 1 - 0.5B$. Dividing each by the common factor yields $\theta(B) = 1$ and $\phi(B) = 1$, which gives us the familiar definition of the white noise model, $x_t = w_t$. This is why we require that the the moving average and autoregressive polynomials $\theta(B)$ and $\phi(B)$ have no common roots, otherwise we could mistake a white noise process for an $\mathbf{ARMA}(p,q)$ process with $p, q > 0$.

As with the $\mathbf{AR}(p)$ model, the linear process representation of an $\mathbf{ARMA}(p,q)$ model is especially useful for computing the autocovariance function of an $\mathbf{ARMA}\,(p,q)$ model. Using the same approach we used for the $\mathbf{AR}\,(p)$ model, the values of $\psi_1,\dots,\psi_j,\dots$ that satisfy $x_t = \psi\,(B)\,w_t$ with $\sum_{j=0}^{\infty}|\psi_j| < \infty$ can be computed by substituting $\psi\,(B)\,w_t$ into the equation that defines the $\mathbf{ARMA}\,(p,q)$ model, $\phi\,(B)\,x_t$, and matching the coefficients for each power of $B$ on each side, i.e.

$$\begin{aligned}
&\phi\,(B)\,\psi\,(B)\,w_t = \theta_z w_t \\
&\implies (1-\phi_1 B - \dots \phi_p B^p)\left(1+\psi_1 B + \dots \psi_j B^j\right) w_t = (1+\theta_1 B + \dots + \theta_q B^q)\,w_t.
\end{aligned}$$

This yields a sequence of equations that would start with

$$\begin{aligned}
\psi_1 - \phi_1 &= \theta_1 \\
\psi_2 - \phi_2 - \phi_1\psi_1 &= \theta_2,
\end{aligned}$$

and continue on for $\psi_3,\dots,\psi_j,\dots$. We will *not* be computing $\psi_1,\dots,\psi_j,\dots$ by hand in class - this requires a knowledge of differential equations that goes above and beyond the prerequisites for this course. However, statistical software like `R` will often include functions that can be used to compute the $\phi_1,\dots,\phi_K$ for some user specified value $K > 1$ given values for $\phi_1,\dots,\phi_p$ and $\theta_1,\dots,\theta_p$.